SCALING CONTINUOUS COVARIATION: SUPPORTING MIDDLE SCHOOL STUDENTS' ALGEBRAIC REASONING

Amy Ellis
University of Georgia
amyellis@uga.edu

Rob Ely
University of Idaho
ely@uidaho.edu

Brandon Singleton
University of Georgia
bksingleton@uga.edu

Halil Ibrahim Tasova
University of Georgia
halil.tasova25@uga.edu

Middle school is a critical time when students begin formal study of functional relationships in algebra. However, many students struggle in understanding functions as relationships between quantities that change according to a dependency relationship. We report on the influence of Scaling Continuous Covariation in fostering productive ideas about graphical relationships and rates of change. Scaling Continuous Covariation entails the ability to imagine a re-scaling to any increment for x and coordinate that scaling with associated values for y. We present findings from two students, one who reasoned with Scaling Continuous Covariation and one who did not, and report on how Scaling Continuous Covariation supported students' reasoning in three ways: (a) sense making about graphs, (b) forming constant rates of change, and (c) understanding constantly-changing rates of change.

Keywords: Algebra and Algebraic Thinking, Cognition, Middle School Education

Introduction: Supporting Function Understanding in Middle School

Functions and relations comprise a critical aspect of secondary mathematics, with recommendations for supporting students' algebraic reasoning emphasizing an early introduction to functional relationships in late elementary and middle school (Stephens, Ellis, Blanton, & Brizuela, 2017). Middle school in particular represents a key time when students enter a formal investigation of function and begin to develop the algebraic tools to express and represent different functional relationships. However, students' difficulty in acquiring the function concept is well documented (e.g., Stephens et al., 2017; Thompson & Carlson, 2017). In particular, students struggle to use functions to model real-world contexts that require a conceptualization of quantities and how they change together (Carlson et al., 2002; Monk & Nemirovsky, 1994).

One potentially fruitful approach to better support students' understanding of function and rates of change is an instructional emphasis on variational and covariational reasoning (Carlson, Smith, & Peterson, 2003; Kaput, 1994; Thompson & Carlson, 2017). Early research suggests that providing students with opportunities to reason covariationally can position them to make meaningful sense of functions (e.g., Ellis, 2007, 2011; Johnson, 2012; Moore, 2014), as well as the ideas in calculus (Thompson & Carlson, 2017). We report on a study investigating the reasoning of two middle school students who explored linear and quadratic growth within the context of continuously co-varying quantities. We found that a particular form of covariational reasoning, scaling continuous covariation, supported a robust understanding of graphical relationships and rates of change.

Background and Theoretical Framework

One contribution to the challenges in building and supporting a robust understanding of function is the lack of attention to variation in algebra curricula. Thompson and Carlson (2017) reviewed seventeen U.S. secondary textbooks, ranging from Algebra I through Precalculus, and found that all relied on a correspondence definition of function. Under this definition, y is a function of x if each value of x has a unique value of y associated with it (Farenga & Ness, 2005). This static view underlies much of school mathematics, with the associated set theoretic meaning of variable becoming the foundation for school definitions of function (Cooney & Wilson, 1996). Students' function concepts are consequently dominated by static images of arithmetic computations used to evaluate outcomes at individual values (Carlson & Moore, 2015). This results in students viewing functions through the lens of symbolic manipulations rather than as a mapping (Carlson, 1998).

Integrating the mathematics of change into students' investigation of functional relationships provides opportunities to interpret how a function's output values can change in relation to its input values, which is an essential component of making sense of dynamic situations (Carlson & Moore, 2015). These opportunities are typically reserved for introductory calculus courses, thus effectively restricting access to these forms of reasoning to the minority of students who will reach the highest level of high-school mathematics (Roschelle, Kaput, & Stroup, 2000). Thinking about functions covariationally, however, can support students' abilities to make sense of linear (Ellis, 2007; Johnson, 2012), quadratic (Ellis, 2011), exponential (Ellis et al., 2015), and trigonometric (Moore, 2014) functions.

Covariational Reasoning

Researchers have addressed covariational reasoning in a number of ways, but for the purposes of this paper we draw on work that considers the possible imagistic foundations that can support students' abilities to think covariationally (e.g., Castillo-Garsow, Johnson, & Moore, 2013; Thompson & Carlson, 2017). These researchers describe covariational thinking as the act of holding in mind a sustained image of two quantities' values varying simultaneously. One can imagine how one quantity's value changes while imagining changes in the other. A person thinking covariationally can couple two quantities in order to form a multiplicative object (Thompson & Carlson, 2017); once such an object is formed, one can then track either quantity's value with the immediate understanding that the other quantity also has a value at every moment.

Castillo-Garsow (2013) distinguished between two types of continuous variation, chunky and smooth. Chunky continuous variation entails thinking about values varying discretely, except that one has a tacit image of a continuum between successive values. One imagines that a change in values occurs in completed chunks, without imagining that variation occurs within the chunk. In contrast, smooth continuous variation entails an image of a quantity changing in the present tense; one can map from one's own experiential time to a time period within the mathematical context, thinking about a value varying as its magnitude increases in bits while simultaneously anticipating smooth variation within each bit (Thompson & Carlson, 2017). Building on these distinctions, Thompson and Carlson (2017) created a covariational reasoning framework that attends to students' images of quantities' values varying. They stressed that smooth continuous variational and covariational reasoning necessarily involves thinking about motion.

Ely and Ellis (in press) subsequently introduced a related but distinct form of reasoning they call *scaling continuous covariation*, which entails imagining that at any scale, the continuum is still a continuum and a variable takes on all values on the continuum. One can conceive of the continuum as arbitrarily or even infinitely "zoomable", in which the process of zooming will never reveal any holes or atoms. Thus, one can imagine a re-scale to any arbitrarily small increment for x and coordinate that scaling with associated values for y. Importantly, unlike smooth continuous covariation, this way of thinking does not fundamentally rely on an image of

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motion.

Just like chunky continuous covariation, scaling continuous covariation relies on partitioning a domain and then reasoning about the corresponding chunks of the covarying quantity. A chunky reasoner can re-chunk the domain to a different sized chunk, but then must re-run his or her chunking scheme from scratch. A person using scaling continuous reasoning, on the other hand, can abstract the chunking process and imagine the result of this chunking scheme for chunks at all scales. This enables one to generalize properties of the covarying quantity's chunks at all of these scales as well, enabling him or her to coordinate the covariation of the two quantities on increments of any scale. With this in mind, we can distinguish between students' use of chunky continuous and scaling continuous covariation and study how these reasoning types support their understanding of rates of change.

Methods

We conducted a 10-day, 15-hour videotaped teaching experiment (Steffe & Thompson, 2000) with two 7th-grade students in general general mathematics (neither had yet taken algebra). The first author was the teacher-researcher. We assigned gender-preserving pseudonyms to each student. The aim of the teaching experiment was to support the students' emerging understanding of linear, piece-wise linear, quadratic, and higher-order polynomial functions from a rate-of-change perspective.

Building on the literature emphasizing the importance of continuously-covarying quantities, we developed tasks to support a conception of linear growth as a representation of a constant rate of change, and quadratic growth as a representation of a constantly-changing rate of change. The tasks emphasized these ideas within the contexts of speed and area. The area tasks presented "growing rectangles", "growing stair steps", and "growing triangles" via dynamic geometry software, in which the students could manipulate the figure by extending the length and observing the associated growth in area (Figure 1).

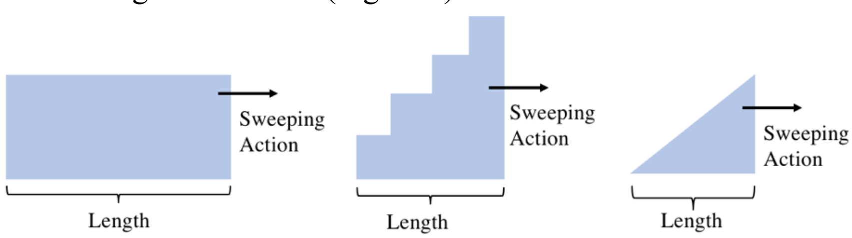

Figure 1. Growing rectangle, stair step, and triangle tasks.

Relying on the videos and transcripts of each teaching session and copies of the students' work, we used the constant-comparative method (Strauss & Corbin, 1990) to analyze the teaching-experiment data in order to identify (a) students' forms of covariational reasoning, and (b) the students' conceptions of constant and changing rates of change. For the first round of analysis we drew on Thompson and Carlson's (2017) framework of variational and covariational reasoning. We used open coding to infer categories reasoning based on students' talk, drawings and graphs, gestures, and task responses. The first round led to an initial set of codes, which then guided subsequent rounds of analysis in which the project team met regularly to refine and adjust the codes in relation to one another. This iterative process continued until no new codes emerged. The final round of analysis was descriptive and supported the development of an emergent set of relationships between students' covariational reasoning and their conceptions of constant, changing, and instantaneous rates of change.

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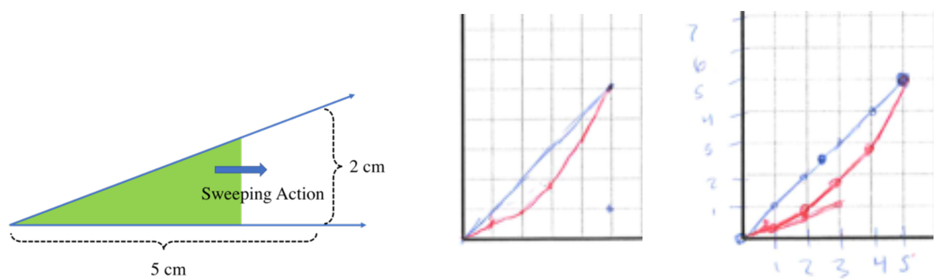
Results

One student, Wesley, exhibited evidence of scaling continuous covariation, while the other student, Olivia, exhibited evidence of chunky continuous covariation. We report three ways in which the form of covariational reasoning influenced students' sense-making about functional relationships: (a) reasoning about graphs, (b) constant rates of change, and (c) constantly changing rates of change. We address each in turn.

Reasoning about Graphs

Wesley and Olivia interpreted and discussed their graphs of quadratic phenomena differently. Wesley conceived of his graphs as smooth continuous curves that, for any given increment, regardless of its size, did not reduce to a straight line within the increment. In contrast, Olivia conceived of curved graphs as being composed of line segments. For instance, the students graphed the relationship between the total accumulated area and the length swept for a shaded cm by 5 cm triangle in which the area and the length swept out together (Figure 2).



(a) (b) (c)

Figure 2. Wesley (b) and Olivia's (c) graphs for the growing triangle (a).

Both students plotted five points in order to draw their graphs, and also drew the "average journey" graph that would be represented by a rectangle sweeping out the same total area for a length of 5 cm. The teacher-researcher (TR) asked the students whether their triangle graphs were curved everywhere, or whether they were piecewise linear. Wesley's response was that the graph was curved everywhere, explaining, "I think in between these points [indicates two points on his graph], if you added a bunch of little points in between, it would make a curve." Wesley understood that the total accumulated area increased with respect to each additional centimeter swept, but he also understood that this relationship would hold even for a smaller increment. He did not need an increment to be any particular size, such as 1 cm, in order to claim that the graph would be curved in between any two points; he already had an image of growth occurring with each increment, no matter how small. In contrast, Olivia could imagine re-sizing an increment, but within each increment, the graph would be a straight line:

> I think, because, if you made the increments even smaller like into 0.1 as your first point then I think it'd be, all the little lines together, I think they'd make a very subtle curve but relatively straight. So, when I did it with the increments as 1, I see them as straight, but if they were smaller they might *look* as if they were curved to make one big curve.

In order to better probe the students' thinking, the teacher-researcher asked the students to consider what the graph would look like if a perfectly precise robot could construct the graph with almost imperceptibly tiny increments:

TR: Would it be curved in between the points or straight in between the points?
Wesley: I believe it would be curved.

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TR: What do you [Olivia] think?

Olivia: I think [long pause]. Well I mean, I think it'd be small enough to the point because the way I think of curves is a whole bunch of straight lines together to make a curve. So, I think if it was to the smallest possible thing even if it could go to infinity, but if it had to be down to the smallest possible things I think it'd be straight lines.

Olivia's remarks suggest that she still saw the graph as composed of straight segments for infinitesimal increments. She could explain the change in area with respect to each additional centimeter, but she viewed the rate of change in area with respect to length swept within a given increment to be constant. This imagery is consistent with chunky continuous covariation, in which one does not imagine change within a given increment. Wesley, however, remained consistent in his belief that such a graph would be everywhere curved, and provided supporting remarks such as: "There's tiny points *in between* those tiny points", "It goes on infinitely, kind of, the points", and "In between those, there's still more points, and it goes on forever."

Wesley appealed to a scaling image in his explanations. He described zooming to smaller and smaller scales, a process that could go on forever and never ground out at an atomic level. In this iterative process, he treated each re-scaled increment as being similar to the bigger increments. This image of scaling continuous *variation* supported his scaling continuous *covariation*, because he generalized across scales a property he noticed about the covariation of area and length: Namely, because the area grows at a changing rate over a large increment, it must also grow at a changing rate over increments at each smaller scale, and thus be curved everywhere. This generalization could extend to an infinitesimal scale just as his image of scaling continuous variation appeared to.

Constant Rates of Change

Wesley and Olivia could both discuss length and area growing together. However, their conceptions of the ratio of area to length differed. Olivia conceived of this relationship as a static ratio, whereas for Wesley, it was a rate of change. For instance, at one point the students investigated the way the area changed as a rectangle with a constant height of 4 cm grew in length (Figure 1). Both students produced a number of equivalent ratios to represent the rate at which the area accumulated relative to length. Olivia could not produce a ratio for a length less than 1 cm, whereas Wesley generated equivalent ratios such as 2 cm^2:0.5 cm, 0.4 cm^2:0.1 cm, and $4x cm^2$:x cm. Wesley explicitly referenced both quantities, but Olivia's descriptions appealed to an image of breaking the area into parts. We believe that Wesley's ratio of 4 cm^2 for 1 cm of length represented a rate. Thompson and Thompson (1992) described a rate as a reflectively abstracted constant ratio. A ratio is a multiplicative comparison of two taken-as-unchanging quantities, whereas a rate is a conception of a constant ratio variation as being a single quantity. It symbolizes the ratio structure as a whole while giving prominence to the constancy of the result of the multiplicative comparison. In order to understand the ratio as a rate, Wesley needed to have an image of change such that 4 cm^2:1 cm represented an equivalence class. Thus, he would need to understand that the unit ratio was simply a convenient measure of expressing the growth in area for a standard unit of length, and was just one of infinitely many equivalent ratios. A scaling continuous covariation image could enable this understanding, as Wesley would be able to mentally zoom in and out for different length increments, generalizing that any arbitrarily-sized increment of length would imply an associated amount of area adhering to the 4:1 ratio.

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Both students also produced drawings of a 4 cm-high rectangle, but only Wesley believed that the shape of the rectangle would not change if the provided rate of 4 cm^2:1 cm were represented as 8 cm^2:2 cm. In justifying his belief, Wesley explained, "The height doesn't – like, it's not a different shape, it's the same. So, it [the rectangle] would be the same, I think." Wesley recognized that all of the ratios were instantiated in the same rectangle *height*; he appeared to understand the height as a representation of the rate of change of area, which does not depend on an amount swept. In contrast, Olivia could only justify the sameness of the picture for any equivalent ratio by converting the new ratio to the original 4 cm^2:1 cm ratio and comparing.

Constantly Changing Rates of Change

The students were also asked a series of questions about situations in which area was constantly changing, such as when the growing area was bounded by a line slanting upward at 45º-angle, or a slope of 1 (Figure 1). Wesley expressed his answer using the quantities length and area: "Every time you increase by 1 in. in length, the area for that will grow by [an additional] 1 in^2." Wesley's care in expressing the change in the growth in area for a specific length increase is additional evidence that he understood the unit ratio as a convenient representation of the constantly-increasing rate that depended on a particular increment. In contrast, Olivia had to draw pictures to visually determine the amount of increase from one increment to the next (Figure 3). She came to the same conclusion, but conceived of the 1-inch increments as "columns". When asked whether the size of the length increment mattered in terms of determining that the rate of change was constantly increasing, Olivia was uncertain and had to check with a new column size of 2 cm. She found that the new constantly-increasing rate was 4 cm^2:2cm for each of the columns. Wesley knew without having to check that the rate of the rate of change would remain invariant for any given increment, even though that value was dependent on the increment size. Scaling-continuous reasoning enabled him to generalize this observed property across all possible scales.

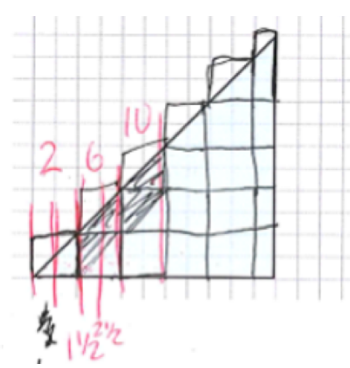

Figure 3. Olivia's drawing to find the amount of area accumulated for each 1-cm increment.

In a second example, when investigating the constantly-increasing rate of change of the area for a 3 cm:2 cm triangle, Olivia again relied on a visual strategy, explaining, "I counted it out." Wesley did not have to draw increments or make any calculations. He instead wrote "1.5", and then explained, "It [the rate of change] increased by the slope." Olivia's drawing, partitioning, and counting strategy was sensible given her image of growth across completed chunks. It enabled her to make calculations for each column and then compare the increases from one column to the next. Wesley's reliance on the slope of the triangle indicates a different conceptual foundation. He could conceive of the slope as a convenient way to express a unit ratio while also understanding that it was dependent on a particular chosen increment of 1 cm. For Wesley, the

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slope was a multiplicative object (Thompson & Carlson, 2017), which was produced by mentally uniting accumulated area and accumulated length simultaneously. Scaling continuous covariation could potentially support the development of such a multiplicative object because an image of re-scaling the continuum to any increment for x, including infinitesimal increments, while simultaneously coordinating that scaling with associated values for y provides a set of operations conducive to creating a new conceptual object "that is, simultaneously, one *and* the other" (Thompson & Carlson, 2017, p. 433, emphasis original). Scaling-continuous covariation enables the generalization of the ratio over any elapsed increment.

Discussion

Our findings indicate that scaling continuous covariational reasoning has the potential to support a meaningful understanding of constant and constantly-changing rates of change. In particular, it affords productive generalization of covariational properties across arbitrarily small, even infinitesimal, scales. Thompson and Carlson (2017) noted that the idea of a non-constant rate of change "is actually constituted by thinking of the function having constant rates of change over small (infinitesimal) intervals of its argument, but different constant rates of change over different infinitesimal intervals of the argument" (p. 452). As evidenced by Wesley's language, this image is compatible with scaling continuous covariation, in which one can imagine zooming to any scale, even an infinitesimal one, to visualize a tiny interval over which the function's rate of change is constant. Wesley recognized that the changes in the changes of area under a sloping line were uniform no matter the scale, which is precisely the constant second-differences characteristic that is unique to quadratic growth. Because he could imagine this at arbitrarily small scales, he could also connect this idea to the curvature of the area graphs he made.

In addition, scaling continuous covariational reasoning has the potential to support an image of instantaneous rate of change and other foundational ideas in calculus. A student who reasons with chunky continuous covariation may struggle to think about a rate of change that is not dependent on an elapsed amount of swept length. This was the case for Olivia, who needed to imagine a completed increment and an associated amount of area in order to create comparisons across same-size increments (Ely & Ellis, 2018). Wesley, however, was able to construct the height of a figure at any given instant as a multiplicative object representing the rate of change of the area compared to the length swept. This enabled him to conceive of the height as a potentiality; once it would sweep out, it would turn the potential rate into an amount of area depending on how much length has been swept. Alternatively, one could imagine the rate of change at a point to be an average rate of change over an infinitesimal interval, which offers a natural motivation for the limit definition of the derivative.

The case of Olivia and Wesley offers evidence that middle school students can develop powerful ideas about constant and changing rates, and that scaling continuous covariation could potentially offer a foundation for building sophisticated ideas about instantaneous rates of change. Situating students' exploration of functional relationships within contexts that foster images of covariation and address ideas of infinitesimal increments is critical for providing this foundation. Given the potential of scaling continuous covariation for supporting important ideas about function, we advocate for additional research to better understand the nature of this form of reasoning and its affordances for algebraic thinking.

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