LEARNING FROM NAEP RELEASED ITEMS: U.S. ELEMENTARY STUDENTS' GRASP OF MULTIPLICATIVE RELATIONSHIPS

John P. Smith III
Michigan State University
jsmith@msu.edu

Curricular analysis indicates that the U.S. students are introduced to multiplication in additive terms (as the replication of equal groups and repeated addition). But the virtue of this introduction for supporting students' understanding of the full range of multiplicative relationships is unclear. This paper reports an analysis of all grade 4 released NAEP items that expressed a multiplicative relationship, focusing on the range of relationship types, related quantities, and item difficulty. Results show that multiplicative items (1) frequently presented discrete quantities and equal group situations, (2) were easier when they involved discrete versus continuous quantities, and (3) within discrete items, equal groups and array items were easier than other types. These results provide measured support for the conjecture the additive introduction to multiplication may limit the development of elementary students' understandings.

Keywords: Number Concepts and Operations, Elementary School Education, Assessment and Evaluation

Objectives of the Study

Can analyses of released items from the National Assessment of Educational Progress (NAEP)—the "nation's report card"—and performance on those items shed light on questions of interest to mathematics educators, beyond what has been reported in summary volumes (e.g., Klosterman & Lester, 2007)? This analysis addressed that question for the broad content area of multiplicative relationships. "Multiplicative relationships" designates a set of tasks and situations, numerical and quantitative, that engage students in multiplicative reasoning and in carrying out numerical operations of multiplication or division. Given the introduction of multiplication and multiplicative relationships in grades 2 and 3 in U.S. classrooms, the analysis examined released items from grade 4.

Multiplicative relationships are significantly more diverse and challenging for students to master than additive relationships (Nunes & Bryant, 1996; Vergnaud, 1983, 1988). Students work on multiplicative relationships for many years, and this work is intensive in the upper elementary through middle school years. They are introduced to the operations of multiplication and division and some "applied" situations in grades 2 and 3. In the U.S., this introduction is essentially additive in nature. Quantitatively, multiplication is presented as involving the replication of equal-sized groups of discrete objects; numerically, it is presented as repeated addition (Smith, 2017). Research on students' understanding of different multiplicative relationships raises questions about whether this instructional foundation effectively supports students' access to and understanding of the full range of multiplicative relationships. This analysis of released NAEP items is one small step in a larger effort to address that critical question. The NAEP analysis complements the results of many prior studies that have assessed students' understanding of multiplicative relationships (as summarized in Greer [1992] and Harel & Confrey [1994]).

Two main questions focused the analysis: (1) what multiplicative situations (type and frequency) appear in released Grade 4 items and (2) how well do the performance results align with the additive introduction to multiplication? For example, are items presenting the

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.

replication of equal groups easier than other types of situations that are less amenable to replication and repeated addition?

Theoretical Perspective

Broadly, the larger inquiry that motivated this analysis was framed in constructivist terms. If learning is a social and psychological process of adapting prior understandings to cope with new and problematic mathematical situations (e.g., Wood, Cobb, Yackel, & Dillon, 1993), then how we introduce students to multiplicative relationships matters greatly for their subsequent work to engage the full multiplicative conceptual field (Vergnaud, 1983). Understanding multiplication and division means knowing where and why situations encountered in the world are "multiplicative," not additive. Mastery of the numerical aspects multiplication and division (i.e., basic facts, algorithms, and properties of operations) may contribute to understanding multiplicative relationships but is neither sufficient nor central.

A framework of types of multiplicative reasoning and quantitative situations that typically elicit such reasoning framed the analysis of the released items. *Quantities* are countable or measureable attributes of objects or collections of objects that are constituents of situations that student encounter and reason about in school and the everyday world (Smith & Thompson, 2008). To use their mathematical knowledge effectively in reasoning about and resolving these situations, students must consider the quantities involved and how they are related. Table 1 presents (and relates) different types of multiplicative reasoning and types of situations. But situations do not determine students' reasoning about them. Rather, the correspondence below reflects how prior research has characterized situations in relation to multiplicative reasoning. "Replication," an additive form of reasoning, has been included for "coverage" of the released items and because U.S. curricula treat replication as multiplicative.

Table 1: Types of Multiplicative Reasoning and Situations

Types of Multiplicative Reasoning	Types of Quantitative Situations
Replication	Equal groups; rectangular arrays; some area and volume situations
One-to-many	Unit conversion
Scaling	Comparison; price (discrete); rate/cost (continuous)
Successive partitioning	Folding; splitting
Composition	Cartesian product (discrete); area; volume (continuous)

Central to the analysis of situation types is the distinction between *discrete* and *continuous* quantities. Discrete quantities are sets of objects; their numerical value can be determined by counting. Continuous quantities are initially attributes of unsegmented objects (e.g., distances or lengths, areas, time periods between two events). Their measurement requires the selection and iteration of a unit (a smaller piece of the target quantity). Their numerical value is the number of such units that collectively fill up or "exhaust" the initial quantity.

Rectangular arrays and some area and volume situations are listed in Table 1 along with equal groups because (1) array situations support replication reasoning (when a row or column of objects is interpreted as a group) and (2) area and volume situations are often presented as arrays of squares or stacks of cubes that similarly support replication. Price indicates situations where some discrete number of items has been purchased. It is related to, but a special case of the more general set of rate/cost situations that accept measured quantities (e.g., "9.45 gallons of gas"). Cartesian product and area/volume are both multiplicative compositions, where the product differs from both factor quantities. In the former, "pants" and "shirts" are different quantities than "outfits." Similarly, both rectangular area and prism volume are different quantities than the lengths from which they are composed.

The entries in Table 1 suggest that understanding multiplication involves grasping fundamentally different forms of multiplicative relationship that many students may see as conceptually distinct, especially early in their mathematical experience.

Methods

Items from nine Mathematics assessments (administered in 1990, 1992, 1996, 2003, 2005, 2007, 2009, 2011, and 2013) have been released for public examination (https://nces.ed.gov/nationsreportcard/nqt/). NAEP characterizes items by content area—*Number properties and operations* (NPO), *measurement* (M), *geometry* (G), *data, statistics and probability* (DSP), and *algebra* (A) and by format—multiple-choice, short constructed response, and extended constructed response. Released NAEP items are characterized by difficulty, as "easy" (performance ≥ 60% correct), "medium" (performance is between 40% and 59% correct), or "hard" (performance < 40% correct).

All 388 released grade 4 items were examined and coded by the author as either additive, multiplicative, or other. Multiplicative items presented one of the situation types listed above in Table 1 (including replication). *Additive* items presented one of three types of additive relationship—combine, separate, or compare. Additive items included area and volume/capacity items that presented collections of squares and cubes that supported counting and (additive) comparison. *Other* items presented content topics such as place value, ordering, estimating, fractions, graphing, and stating probability, where neither an additive nor multiplicative relationship between numbers or quantities was expressed.

Multiplicative items were found in all five content areas but were most common in NPO and M domains. All multiplicative items were first coded as "numerical" or "quantitative." Numerical items presented written numerals and operations with minimal prose. "Quantitative" items were primarily expressed in words, where numerals were associated with quantities. Some items were presented entirely in prose (as "word problems"); others presented tables or figures with the written text, and the tabular or figural information was necessary for solving the item. Quantitative items were then coded for the type of quantities involved, discrete or continuous. Discrete items were further distinguished according to the type of relationship presented. Eight types proved sufficient for coding all discrete items: *Equal groups*, *equal shares*, *rectangular arrays*, *money*, *price*, *multiplicative comparison*, *unit conversion*, and *Cartesian product*. Continuous items presented length, area, or volume/capacity measurement situations where the quantity could not be evaluated with additive reasoning. Four types were sufficient for coding continuous items: *Unit conversion, equal shares, multiplicative comparison*, and *computation*.

Results

For a broad overview, Table 2 presents an overview of all 388 released grade 4 items by year. Column 3 lists the total number of multiplicative items; column 4 lists the total number of additive items; and columns 5–8 characterize the multiplicative items, first as numerical or quantitative items and then for quantitative items, as discrete and continuous.

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.

Table 2: General Character of Grade 4 Items (1990-2013)

Year	All	Mult	Add	M; Number	M; Quan	M, Quan; Discrete	M, Quan; Continuous
2013	46	14	9	6	8	6	2
2011	49	20	14	7	13	10	3
2009	31	6	8	1	5	3	2
2007	54	14	12	3	11	9	2
2005	32	5	11	2	3	3	0
2003	59	23	9	3	20	16	4
1996	25	9	6	1	8	6	2
1992	59	20	13	6	14	12	2
1990	33	10	7	6	4	3	1
Total	388	121	89	35	86	68	18

The number of released items and the number of multiplicative items varied substantially across the nine assessments, but multiplicative items were generally more frequent than additive items. In most years, multiplicative quantitative items outnumbered multiplicative numerical items, often dramatically. Among quantitative items, those presenting discrete quantities were at least twice as frequent as those presenting continuous quantities.

More substantively, the distribution of discrete items across the eight types listed above was not uniform (Table 3).

Table 3: Frequency of Grade 4 Discrete Multiplicative Items by Type (1990-2013)

Year	Disc	Grps	Share	Array	Comp	Unit	Money	Price	C.P.
2013	6	2	0	0	2	1	0	0	1
2011	10	3	2	0	2	3	0	0	0
2009	3	1	0	0	0	1	1	0	0
2007	9	3	0	0	1	3	1	1	0
2005	3	2	0	0	1	0	0	0	0
2003	16	5	2	1	1	1	0	5	1
1996	6	2	0	2	0	1	0	1	0
1992	12	5	0	0	0	0	1	5	1
1990	3	0	0	1	0	0	0	2	0
Total	68	23	4	4	7	10	3	14	3

Note: “Grps” = equal groups, “Comp” = comparison, “Unit” = unit conversion, “C.P.” = Cartesian product

Equal groups was by far the most frequent type of situation presenting discrete quantities in multiplicative relationship, followed by price and unit conversion. Equal groups and unit conversion items always involved quantities with whole number values, where price items involved some whole number of items at a cost represented as decimal (e.g., $0.87 or $2.79).

Many authors have argued that multiplicative relationships are intrinsically more difficult for students to master than additive relationships (e.g., Vergnaud, 1983, 1988). Is this claim reflected in the NEAP results? Overall, the entries in Table 4 indicate an affirmative answer.

Table 4: Relative Difficulty of Grade 4 Additive and Multiplicative Items (1990-2013)

Year	All	Mult.	Add.	Avg % Corr; A	Avg % Corr; M	%A – %M
2013	46	14	9	46.8	43.6	3.2
2011	49	20	14	58.2	48.8	9.4
2009	31	6	8	57.4	52.8	4.6
2007	54	14	12	61.3	46.0	15.3
2005	32	5	11	53.3	58.2	-4.9
2003	59	23	9	57.8	50.8	7.0
1996	25	9	6	54.2	42.0	12.2
1992	59	20	13	45.2	40.4	4.8
1990	33	10	7	44.6	44.2	0.4
Total	388	121	89			

On seven of nine assessments, multiplicative items were more difficult than additive items, and in three (1996, 2007, and 2011) significantly so. The difference was negligible in 1990 and had the opposite sign in 2005.

But item difficulty across types of multiplicative relationships was the central focus of this analysis. Table 5 below reports percent correct for items presenting each type of discrete multiplicative relationship. The first value in right-most column lists the average percent correct for all items of that type. The values in parentheses are the average percent correct for a meaningful subset of those items, as explained below. The other columns list the number of items rated "easy," "medium," and "hard" and the percent correct for each item in those three categories.

Table 5: Difficulty of Grade 4 Discrete Multiplicative Items by Type (1990-2013)

Discrete sub-type	N	N Easy	%	N Med	%	N Hard	%	Avg %
Equal groups	23	4	75, 61, 80, 70	11	50, 53, 59, 53, 56, 46, 57, 50, 55, 47, 48	8	38, 35, 23, 21, 36, 39, 37, 37	49.0 (56.4)
Price	14	2	70, 62	4	58, 58, 53, 48	8	4, 35, 39, 31, 17, 8, 9, 21	39.3 (53.2)
Unit Conversion	10	6	66, 75, 65, 85, 78, 61	2	53, 44	2	17; 39	58.3 (62.9)
Compare	7	1	72	2	47, 47	4	32, 34, 24, 34	43.0 (36.3)
Array	4	1	79	2	50, 48	1	35	53.0
Money	3	1	60	1	58	1	20	46.0
Equal shares	4			2	47, 51	2	23; 38	39.8
Cartesian product	3			1	48	2	24; 28	33.3
Total	68	15		25		28		45.9

Items in three discrete types were somewhat easier overall (Unit Conversion, Array, and Equal Groups, in descending order), with average correct at or slightly above 50%. Six of the 10

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.

Unit Conversion items provided the conversion ratio explicitly in the item (e.g., 1 qt. = 2 cups). The two "hard" conversion items (39% and 17% correct) both presented 3:1 ratios, where the other eight items presented conversion ratios of 2:1, 5:1, 10:1, or 100:1. Arrays were either represented directly in diagram (n = 1) or described in words (n = 3). The "easy" item (79% correct) described a particularly familiar array—two rows of six cookies on a cookie sheet.

As shown, Equal Groups and Price items were often difficult; eight items of both types were "hard." But within both types, hard items often involved two or more steps, where a multiplicative relationship was involved in at least one step. For example, some two-step Price items asked for the change received for the purchase of a set of items at a given price when a specific bill was given for payment—requiring both multiplicative and additive reasoning. Some two-step Equal Groups items introduced more than one group (e.g., students in a class and buses with maximum capacity for students). Most multi-step items were more difficult than single step items of the same type. The average percent correct for the thirteen single-step Equal Groups items was 56.4% (as shown), where the corresponding average for the ten multi-step items was 39.2%. Similarly, the six single-step Price items were considerably easier (average 53.2% correct) than the eight multi-step items (average 28.9% correct).

By contrast, Compare, Equal Shares, and Cartesian Product items were more challenging, at 43%, 39.8%, and 33.3% average correct, respectively. The majority of Compare and Cartesian Product items were "hard," even when six of the seven Compare items involved a 2:1 ratio. Only one, presenting ten stars and five triangles in a 3 by 5 array and four possible ratios, was "easy" (72% correct). Without that item, average correct fell to 36.3%. Of the four Equal Shares items, none called for simply distributing some discrete quantity equally to a given number of recipients. One "hard" item (38% correct) asked students to distribute 24 wheels to bikes and wagons in two different ways; one "medium" item (47% correct) required interpreting the remainder after equal sharing. Cartesian Product items were difficult even though support was provided for solving two of the three (i.e., items provided the solution for smaller factors).

Finally, Table 6 presents the performance on the N = 18 multiplicative items that presented length, area, or volume/capacity quantities and within each quantity, the type of multiplicative relationship involved.

Table 6: Difficulty of Grade 4 Continuous Multiplicative Items by Type (1990-2013)

Continuous Sub-type	N	N Easy	%	N Med	%	N Hard	%	Avg %
Length	7			1		6		30.6
compare	2						27, 33	30.0
equal shares	4				47		27, 26, 23	30.8
unit convert	1						31	31
Area	6	1		1		4		36.5
compute rect	4						23, 24, 24, 19	22.5
compare	2		78		51			64.5
Volume/capacity	5	2		1		2		47.4
unit convert	3		67, 61				32	53.3
compute stack	1				56			56
compare	1						21	21
Total	18	3		3		12		37.2

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.

Overall, the results show that multiplicative items presenting continuous quantities were generally more difficult than those with discrete quantities. Six of seven length items were “hard,” as were four of six area items—even when the area items involved a single-step. However, the two items comparing areas— both presenting sectors of partitioned circles—were markedly easier (average 64.5% correct) than the corresponding discrete comparison items (n = 7; average 43.0% correct). The three volume/capacity unit conversion items were somewhat more difficult (average 53.3% correct) than the discrete unit conversion items (n = 10; average 58.3% correct). As was true for the discrete items, all three continuous items stated the conversion ratio in the item text.

Discussion

The analysis was revealing in two principal ways. First and generally, the careful examination of released NAEP items supported a finer-grained analysis of U.S. grade 4 students’ successes and challenges—as a proxy measure of national understanding—than the published volumes have thus far (Klosterman & Lester [2007] and prior volumes in that series). The released item set provided greater access to the items as presented to students, their difficulty, and the details of item performance. Second and more specific to this inquiry, the analysis provided a measure of empirical support for the concern that the additive introduction of multiplication may present challenges for the growth of students’ understanding beyond equal groups of discrete objects and repeated addition. However, that support was mixed and complicated by many factors outside the frame of the analysis. Since many factors other than quantity and relationship type likely contribute to item difficulty, the analysis shows the difficulty inherent in establishing what makes an item easy (or challenging) for students.

Some of the results are consistent with (a) the curricular introduction of multiplication as the replication of equal groups of discrete quantities and repeated addition and (b) the concern that that an additive foundation likely makes extension to a wider set of situations problematic. Additive items were on average easier than multiplicative items. Second, items posing multiplicative relationships among discrete quantities were much more frequent and generally easier than item involving continuous quantities. Third, situations involving Equal Groups were the most frequent of type of discrete multiplicative relationship and were easier than other discrete item types that are less amenable to interpretation as equal groups—specifically, Comparison and Cartesian Product. But this was not always the case; Unit Conversion items were easier on average than both Equal Groups and Array items. Overall, the results are consistent with and do not remove the concern that the additive introduction to multiplication and multiplicative relationships may support initial access to multiplicative relationships, but at the cost of later conceptual challenges as application extends both quantitatively and numerically (e.g., to fractions, decimals, and negative numbers).

There are numerous limitations to this analysis and more broadly to using NAEP released items to address questions about the effects of curricular approaches on student learning. Perhaps the most important is that the released item set stands in uncertain relationship to the larger corpus of items where NAEP has performance data. The conditions under which a NAEP item is released to the public are unknown. Also, where the type of multiplicative item and the type of related quantities may well influence the item difficulty, other factors do so as well, including the numerical values of the quantities (even within the set of whole numbers), the length and clarity of item prose, and the nature of support provided (e.g., stating conversion ratios or not). Third, multi-step problems present challenges for characterizing items as additive or multiplicative and for judging the sources of difficulty. Fourth, though the capacity to explain one’s reasoning may

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.

be a better indicator of understanding than producing the right numerical answer, the number of released NAEP items requiring explanation (that is, both short and extended constructed-response items) have been small in number, in this target content area and likely in most others. The smaller the item set, the more perilous any conclusion drawn from such data becomes. Last, the analysis has been completed by a single person and the coding scheme must be shown to be reliable.

Despite these limitations, released NAEP items are an underused resource for mathematics education researchers who wish to address questions of learning at a national level. Where it is unlikely that any similar analysis of items selected for a given topic or response type (e.g. constructed-response) will resolve questions about student understanding, they may contribute to inquiries that draw on multiple sources of evidence. NAEP released item data, reflecting such a large and nationally-representative sample of students, is a unique source of evidence. Its analysis can provide either general support (as in this case), no support, or contradictory evidence for a given hypothesis.

References

Davis, B. & Simmt, E. (2006). Mathematics-for-teaching: An ongoing investigation of the mathematics that teachers (need to) know. *Educational Studies in Mathematics, 61*, 293–319.

Greer, B. (1992). Multiplication and division as models of situations. In D. A. Grouws (Ed.), *Handbook of Research in Mathematics Teaching and Learning* (pp. 276–295). New York, NY: MacMillan Publishing.

Harel, G. & Confrey, J. (Eds.). (1994). *The development of multiplicative reasoning in the learning of mathematics*. Albany, NY: SUNY Press.

Klosterman, P. & Lester, F. K. (Eds.). (2007). *Results and interpretations of the 2003 Mathematics Assessment of the National Assessment of Educational Progress*. Reston, VA: National Council of Teachers of Mathematics.

Nunes, T. & Bryant, P. (1996). *Children doing mathematics*. Oxford, UK: Blackwell Publishers.

Schwartz, J. L. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. Behr (Eds.), Number concepts and operations in the middle grades (pp. 41-52). Reston, VA: NCTM.

Smith, J. P. (2017). Conceptual deficits in curricular introductions to multiplication. In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 156–159). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Smith, J. P. & Thompson, P. W. (2008). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 95–132). Mahwah, NJ: Lawrence Erlbaum Associates.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127–174). New York, NY: Academic Press.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 141–161). Reston, VA: National Council of Teachers of Mathematics.

Vergnaud, G. (1994). Multiplicative conceptual field: What and why? In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 41–59). Albany, NY: SUNY Press.

Wood, T., Cobb, P., Yackel, E., & Dillon, D. (Eds.) (1993). Rethinking elementary school mathematics: Insights and issues (selections). *Journal for Research in Mathematics Education Monograph*, no. 6. Reston, VA: National Council of Teachers of Mathematics.

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.