IDENTIFYING LATENT CLASSES OF MIDDLE GRADES TEACHERS BASED ON REASONING ABOUT FRACTION ARITHMETIC

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The purpose of this study was to examine distinct latent classes of middle grades mathematics teachers with respect to reasoning about fractions. Survey response data came from a nationwide sample of 990 in-service middle grades mathematics teachers. The survey focused on four components of reasoning about fractions in terms of quantities: referent unit, partitioning and iterating, appropriateness, and reversibility. The mixture Rasch model analysis detected three latent classes, each with strengths and weaknesses. Chi-square tests indicated significant relationships between latent class membership and various teacher characteristics such as gender, mathematics credential, grade-level experience, and highest grade-level certification. The results extend recent advances in measuring mathematical knowledge of teachers.

Keywords: Data Analysis and Statistics, Rational Numbers, Teacher Knowledge

Fractions are core content in the upper elementary and middle grades mathematics curriculum and are highly interconnected to whole-number multiplication and division, and ratios and proportional relationships (e.g., Vergnaud, 1988). Additionally, fractions are necessary for algebraic reasoning and further study in mathematics (Hackenberg & Lee, 2015). Although most teachers can multiply or divide two fractions by computing correctly, many studies acknowledged the difficulties that teachers experience in reasoning about products or quotients of fractions when they are embedded in problem situations (e.g., Ball, Lubienski, & Mewborn, 2001; Lee, 2017). Despite strong emphasis by recent curriculum standards such as the Common Core State Standards for Mathematics (CCSS-M; e.g., National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) on the necessity of developing reasoning about fraction arithmetic when solving problems embedded in situations, two main challenges exist in mathematics education. One main challenge is to foster teachers' reasoning about fractions in terms of quantities. A second main challenge is to understand how to use psychometric models for measuring teachers' fine-grained mathematical knowledge. Many recent applications of psychometric models to measure teachers' mathematical knowledge have relied on traditional item response theory (IRT) models. These efforts include the Learning Mathematics for Teaching (LMT) project (e.g., Hill, 2007), the Diagnostic Mathematics Assessments for Middle School Teachers (DTAMS) project (Saderholm, Ronau, Brown, & Collins, 2010), and the Knowledge of Algebra for Teaching (KAT) project (Senk, 2010). Traditional IRT models rely on the assumption that all examinees in a given sample belong to a single population. Some recent studies (e.g., Izsák, Orrill, Cohen, & Brown, 2010; Izsák, Jacobson, de Araujo, & Orrill, 2012), however, have demonstrated the existence of distinct latent classes of middle grades teachers. The presence of distinct latent classes violates local independence, a key assumption of traditional IRT models. To address this issue, the present study employs the mixture Rasch model (Rost, 1990), a combination of a latent class model and a traditional IRT model. When applying the mixture Rasch model, one can examine model fit for different numbers of latent classes. Each class is characterized by a distinct pattern of item responses, and differences in response patterns are thought to indicate different underlying cognitive strategies (Bolt, Cohen, & Wollack, 2001). For each examinee, the mixture Rasch model provides an estimate of ability (as does a unidimensional traditional IRT model) and a probability of class membership.

The present study used responses from a sample of 990 in-service middle grades teachers across the U.S. to the Diagnosing Teachers' Multiplicative Reasoning (DTMR) Fractions survey (Bradshaw, Izsák, Templin, & Jacobson, 2014). The survey measures teachers' capacities to reason about multiplication and division of fractions in terms of quantities. The purpose of this study was to identify distinct latent classes of middle grades teachers on reasoning about fractions and investigate the relationships between class membership and teacher characteristics such as gender, mathematics credential, grade-level experience, highest grade-level certification, and years of teaching experience. The following research questions were addressed:

1. Do distinct latent classes exist in the national sample of middle grades teachers?
2. If so, what areas of strength and weakness on reasoning about fractions distinguish the distinct latent classes?
3. Are there significant relationships between latent class membership and teacher characteristics including gender, mathematics credential, grade-level experience, highest grade-level certification, and years of teaching experience?

Theoretical Framework

The theoretical framework for this study considers middle grades teachers' reasoning about quantities and focuses on using drawings (e.g., area models and number lines) to learn and teach fraction arithmetic. We consider fine-grained components of reasoning as the property of an individual by following the constructivist perspective, in which the individual dynamically stores each component in his/her mind. We think that reasoning, which goes beyond computational fluency, requires a teacher to make sense of quantities in fraction arithmetic problems using drawings. From this standpoint, a teacher's capacity to reason about quantities with drawings can be increased by paying more consistent attention to distinct fine-grained components such as *referent units*, *partitioning and iterating*, *appropriateness*, and *reversibility* — the importance of which have been established in past research (e.g., Bradshaw et al., 2014; Izsák, Jacobson, & Bradshaw, in press) and explained later. Moreover, we take the stance that reasoning depends on context. That is, a teacher's performance about the use of one component in one situation might be appropriate, but his/her performance in another situation might be problematic, depending on the wording of the situation, the arithmetic operations required, or the representations provided. Hence, it is important to examine teachers' capacities to employ particular components across a range of problem situations.

Methods

Participants

The data consisted of survey responses from a sample of 990 in-service middle grades teachers across the U.S. (Table 1).

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Table 1: Demographic information

Variables	%
Gender (*N = 976*)	
Female	75.0
Mathematics credential (*N = 974*)	
Yes	67.0
Grade-level experience (*N = 972*)	
K-5	46.0
Grades 6-8	29.0
Grades 9-12	25.0
Highest grade-level certification (*N = 971*)	
K-6	8.0
Grades 7-9	57.0
Grades 10-12	35.0
Years of teaching experience (*N = 966*)	
0-4 years	17.1
5-14 years	52.9
> 14 years	30.0

Instruments

The DTMR Fractions survey consists of two parts. The first part has 27 items (19 multiple choice and 8 constructed response) that measure four distinct components of reasoning about fractions including *referent unit*, *partitioning and iterating*, *appropriateness*, and *reversibility*. *Referent unit* deals with reasoning about units when numbers are embedded in problem situations and consists of three sub-components. *Norming* refers to the formation of standard units for measurement and occurs either in case of selecting a standard unit from alternate choices or in case of making at least two choices for a measurement unit in a given situation (i.e., renorming). *Referent unit for multiplication* and *referent unit for division* concern the problem situations that can be modeled by the equation $M \bullet N = P$ where M and N refer to different units. The second component, *partitioning and iterating*, refers to dividing a quantity into equal-sized pieces and concatenating unit fractions. It consists of three sub-components. *Partitioning in stages* refers to making a repartition to obtain a desired partition. *Partitioning using common denominators* and *partitioning using common numerators* refer to using common denominators or numerators to obtain common partitions. The third component, *appropriateness*, concerns identifying situations that can be modeled by multiplication and division and includes three sub-components: *identifying multiplication*, *identifying partitive division*, and *identifying quotitive division*. The fourth component, *reversibility*, deals with returning to a starting point after making some process. We conjecture that proficiency in use of these four components of reasoning across different problem situations enables teachers to solve fraction arithmetic problems in terms of reasoning with quantities.

Because the DTMR Fractions survey items are secure, we present one example item similar to an actual survey item (Figure 1). This item measures *referent unit* and *partitioning and iterating*. The correct choice is (b). A teacher who chose (a) or (c) would indicate confusion about the referent unit for 1/8. A teacher who chose (e) would indicate an incorrect partition (5 groups of 6 pieces that create 30ths). A teacher who chose (b) would demonstrate both the correct referent unit (the 1 meter) and the correct partition (5 groups of 8 pieces that create 40ths). The rest of the multiple-choice items were also constructed so that the different choices provided information about the four components of reasoning. A correct choice provided evidence for the components of reasoning intended for that item; incorrect choices simply indicated lack of evidence for the components of reasoning intended for that item. Constructed response items were also scored using rubrics for evidence of intended components of reasoning.

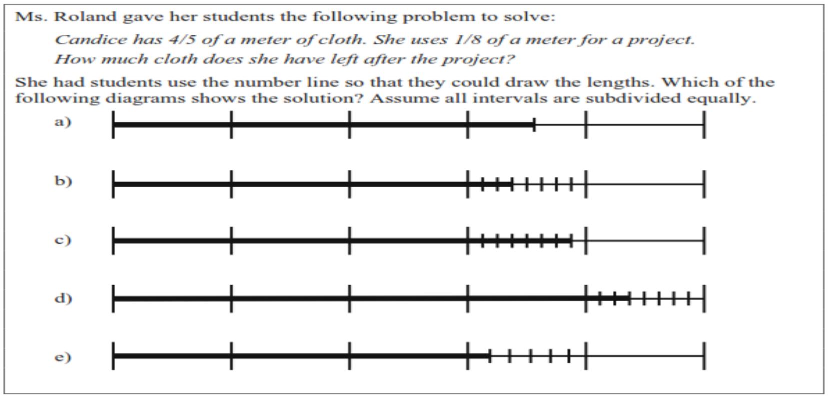

Figure 1: An item that measures referent unit and partitioning using common multiples of denominators. From Izsák et al. (2010). All rights reserved.

The second part of the survey consists of a questionnaire to obtain information about various teacher characteristics including gender, mathematics credential, grade-level experience, highest grade-level certification, and years of teaching experience (see Table 1).

Data Analysis

We analyzed the data using the mixture Rasch model implemented in the computer program WINMIRA (von Davier, 2001). First, we estimated the mixture Rasch model with one, two, three, four, five, and six latent classes. Second, we compared three information indices to select the best fitting model: Akaike’s information criterion (AIC), Bayesian information criterion (BIC), and consistent AIC (CAIC). With each of these criterion, smaller values indicate better fit. We selected the model with the smallest BIC values as the best fitting model (Li, Cohen, Kim, & Cho, 2009). Next, we analyzed the reasoning characteristics of each latent class by examining raw response data. In addition, we evaluated the relationships between latent class membership and teacher characteristics using analysis of variance (ANOVA) and chi-square tests across the latent classes.

Results

Checking Dimensionality

An exploratory factor analysis using maximum likelihood estimation as implemented in the SPSS 16.0 software (SPSS Inc., 2007) indicated eigenvalues of the first three factors as 5.1, 1.5, and 1.3, and the total variance explained by the first factor was 19%. Because the first eigenvalue was relatively large (Lord, 1980), a unidimensional model could be fit to the data.

Model Selection

Values for the three information indices are given in Table 2. Minimum values for BIC (30226.79) and CAIC (30312.79) indicated a three-class solution in the data.

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Table 2: Model fit indices of the mixture Rasch model

Model	AIC	BIC	CAIC
One class	30526.03	30663.16	30691.16
Two classes	30011.27	30290.43	30347.43
Three classes	29805.59	**30226.79**	**30312.79**
Four classes	29706.11	30269.35	30384.35
Five classes	29672.67	30377.94	30521.94
Six classes	**29579.19**	30426.49	30599.49

Note. AIC = Akaike information criterion; BIC = Bayesian information criterion; CAIC = Consistent Akaike information criterion; the smallest information criterion index is bold.

Table 3 presents the descriptive statistics about raw scores for each of the three latent classes. Based on Table 3, Class-C is the least proficient latent class with the average score of 7.948 over the total score of 27, and 39% of the teachers (385 over 990 teachers) are members of this class. Moreover, Class-B is the middle proficient latent class with 19% of the teachers (184 over 990 teachers) in this class. And, Class-A is the most proficient latent class with 42% of the teachers (421 teachers over 990 teachers) in this class.

Table 3: Descriptive statistics for the latent classes.

	Class-A	Class-B	Class-C
Raw scores			
M	15.399	10.750	7.948
SD	4.165	4.265	3.529
N (%)	421 (42.5)	184 (18.7)	385 (38.8)

Analysis of Raw Response Data

To get the clearest view of the reasoning characteristics associated with each latent class, we narrowed analysis to the raw response data of the 649 teachers who were assigned to a latent class with a probability of .9 or higher. Table 4 lists the characteristics of the latent classes based on the percentage of teachers in each class who answered the survey items correctly.

Table 4: Characteristics of the latent classes (N = 649)

Component	Sub-component	Characteristic	Class-A	Class-B	Class-C
RU	Norming	Choosing a standard unit from alternate choices	Strong	Partial	Weak
RU	Norming	Renorming in the presence of proper fractions	Strong	Strong	Weak
RU	Norming	Renorming in the presence of improper fractions	Weak	Weak	Weak
RU	RU for Multiplication	Distinguishing part-of-a-part from part-of-a-whole	Partial	Partial	Weak

RU	RU for Multiplication	Reasoning when the whole is not present visually	Partial	Partial	Weak
RU	RU for Division	When quotient as a whole number	Partial	Partial	Partial
RU	RU for Division	When quotient as a fraction	Partial	Partial	Weak
PI	Partitioning in Stages	Partitioning in stages	Strong	Weak	Weak
PI	Common Denominator	Using common denominators	Strong	Weak	Weak
PI	Common Numerator	Using common numerators	Strong	Weak	Weak
APP	Identifying Multiplication	Identifying multiplication	Strong	Partial	Weak
APP	Identifying Partitive Division	Identifying partitive division	Strong	Strong	Strong
APP	Identifying Quotitive Division	Identifying quotitive division	Strong	Strong	Weak
REV	Reversibility	Reversibility	Strong	Strong	Weak

Note. RU=Referent Units; PI=Partitioning & Iterating; APP=Appropriateness; REV=Reversibility.

Finally, an exploratory examination of the latent classes obtained from the mixture Rasch analysis and an examination of the raw response data of the 649 teachers revealed the reasoning characteristics of each latent class (Figure 2). Based on this analysis, Class-C teachers perform well only in *identifying partitive division* (*Appropriateness*) problems, but have trouble in the remaining three components of fraction arithmetic (i.e., *referent units*, *partitioning and iterating*, and *reversibility*). On the other hand, Class-B teachers are found to perform well in *identifying multiplication* and *identifying quotitive division* (*Appropriateness*) problems, in addition to problems that involve *identifying partitive division* (*Appropriateness*), and in using *reversibility*. However, similar to Class-C teachers, Class-B teachers struggle with items that measure *referent units*, and *partitioning and iterating* such as *partitioning using common denominators* and *partitioning using common numerators*. In addition to having the strengths of Class-B teachers, Class-A teachers perform well in *partitioning using common denominators*, *partitioning using common numerators* and *partitioning in stages* (*Partitioning and iterating*). On the other hand, Class-A teachers have partial difficulty in *renorming* and *distinguishing part-of-a-part from part-of-a-whole* (*Referent units*). In this component, Class-C and Class-B teachers experience much more difficulty than Class-A teachers.

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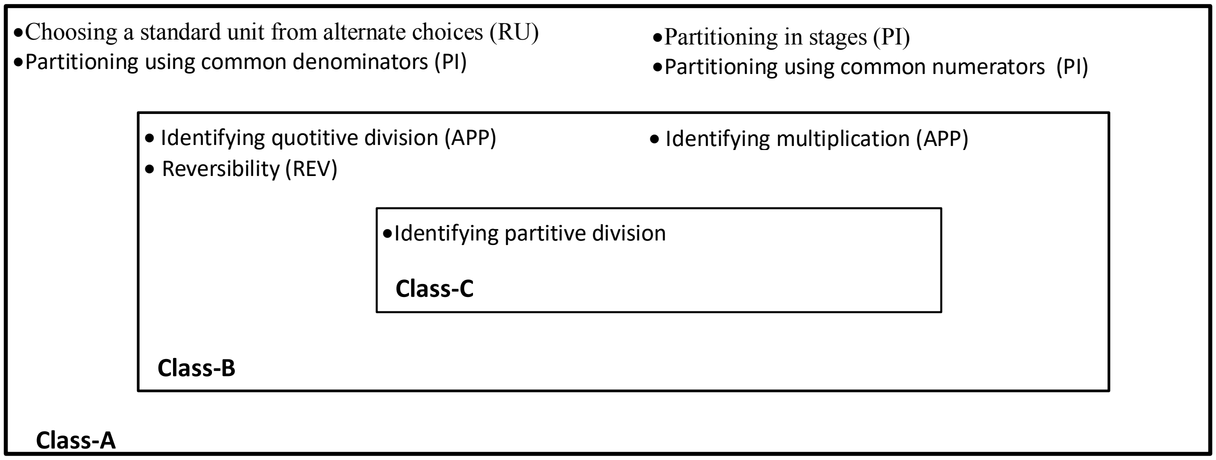

Figure 2: Identifying latent classes based on mixture Rasch analysis (N = 649).

The Relationships between Latent Class Membership and Teacher Characteristics

We also examined the relationships between latent class membership and various teacher demographic and professional history characteristics. First, we found significant differences for total mean raw scores based on ANOVA (F (2, 987) = 363.557, p = .00). The effect size (η^2) of the main effect was .42, indicating that the latent classes explained 42% of the variance in the total mean raw scores. Post hoc analyses using Scheffé's test showed significant differences of the three latent classes from each other (p = .00 for each comparison). This indicated teachers in Class-A scored significantly higher than those in Class-B, and teachers in Class-B scored significantly higher than those in Class-C. Second, a chi-square test for gender was significant ($\chi^2(2) = 25.40$, $p < .001$), but Crámer's V statistic was .16, indicating a weak association between latent class membership and gender. Third, a chi-square test for mathematics credential was significant ($\chi^2(2) = 15.27$, $p < .001$), indicating a relationship between having a mathematics credential and latent class membership. Fourth, the relationship between latent class membership and grade-level experience was significant ($\chi^2(4) = 25.07$, $p < .001$). Fifth, the author(s) found a significant relationship between latent class membership and highest grade-level certification ($\chi^2(4) = 29.20$, $p < .001$). Finally, we found a significant relationship between latent class membership and years of teaching experience ($\chi^2(4) = 11.92$, $p = .018$). These results indicate that teachers who achieved higher scores, those who had a mathematics credential, those who had a high-school credential, and those who had more teaching experience tended to be in Class-A as opposed to other latent classes.

Discussion

Results of the present study demonstrate how combining research in mathematics education with psychometric models can reveal patterns in middle grades teachers' fine-grained reasoning about fractions. We used the mixture Rasch model to characterize differences in reasoning about fractions of middle grades teachers. Results for the first research question revealed three distinct latent classes. Results for the second research question indicated that teachers in the three latent classes were distinguished by their attention to norming and referent units for multiplication (i.e., *referent unit*), using common numerators (i.e., *partitioning and iterating*), identifying multiplication and division (i.e., *appropriateness*), and *reversibility*. These results extend those reported by Izsák et al. (2010, 2012). For instance, Izsák et al. (2010) found two latent classes in a convenience sample of 201 in-service middle grades teachers where teachers in one class were more proficient in *referent unit* than those in the second class. The present study used a more refined instrument and a much larger sample to refine the earlier results, found three latent classes instead of two, and identified differences across the latent classes based not only on *referent unit* but also on other components of reasoning including *partitioning and iterating*, *appropriateness*, and *reversibility*. For the third research question, we found significant relationships between latent class membership and the teacher characteristics such as mathematics credential, grade-level experience, highest grade-level certification, and years of teaching experience, as similar to the results of Hill (2007) and Izsák et al. (in press). Future studies should continue examining teachers' mathematical knowledge using innovative measures and applications of diverse psychometric models, including the mixture Rasch model.

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