STRATEGIES USED BY MEXICAN STUDENTS IN SEEKING STRUCTURE ON EQUIVALENCE TASKS

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This paper presents strategies identified in Mexican rural school students in seeking structure on equivalence tasks that involve the equal sign and tasks that do not. The results arise from the pilot study of a research project on the structure of numbers and numerical operations – a key aspect of early algebraic thinking.

Keywords: Algebra and Algebraic Thinking, Elementary School Education.

Research on algebraic thinking in young students is a trending topic worldwide (Kieran, 2018; Singh & Kosko, 2017). In this and other works, different approaches to early algebra that recognize in arithmetic a strong algebraic feature can be identified (Carraher & Schliemann, 2007). Studies on equivalence in numerical sentences have centered on relational thinking (Carpenter, Franke, & Levi, 2003; Molina & Ambrose, 2008), as well as on generalization, as main aspects of algebraic thinking. However, another important aspect is that of structure in arithmetic (Kieran, 2018).

Recent studies show some of the relationships between expressing the structural and the operational on equivalence tasks. For instance, Asghari and Khosroshahi (2016), with tasks that do not involve the equal sign, propose the existence of an operational approach in developing algebraic thinking in the context of the associative property. According to these researchers, mathematical thinking in elementary school may involve both an operational and a structural conception. Hence, the authors identify the development of algebraic thinking as operationally experienced in the ability to transform a numerical structure.

Schifter (2018) states that an important feature of early algebra includes observation, development, and justification of structural properties in numerical operations expressed in students' computational strategies. This is seen, for instance, in students' verbalizations of the property that in an addition, adding and subtracting the same amount does not affect the value of the expression. Thus, it is important to engage students in discussions on their strategies to determine the veracity or not of numerical sentences such as 57 + 89 = 56 + 90. If students indicate a relational mode of thinking, it suggests that they are focused on the structure of such equalities. Schifter also analyzed students' thinking when they explored structural properties in related expressions in a sequence of expressions not involving the equal sign (14+1, 13+2, 12+3, 11+4).

In another work, Pang and Kim (2018) using sentences such as 67 + 86 = 68 + 85 reported that participants tended to use computational strategies; however, they also showed their ability to use a structural approach. According to Pang and Kim, one structural strategy consists in observing that an addend increases by one and the other decreases also by one. In Schwarzkopf, Nührenbörger, and Mayer (2018), however, it is considered that describing patterns in a sequence of expressions such as 30 + 20 = , 31 + 19 = , 32 + 18 = , … is not actually structural reasoning, even if they point out the importance of such thinking in patterns or regularities. These researchers agree with Mason, Stephens, and Watson (2009) in the sense that structural thinking is much more than only observing patterns.

It is clear, then, that there are different perspectives regarding the structural in arithmetic as a

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component of algebraic thinking, as well as the importance of its development. From here, the purpose of our work is to research the reasoning of Mexican students from rural schools in seeking structure on equivalence tasks that involve the equal sign and tasks that do not. The research question was: What are the strategies used by Mexican students from rural schools regarding structure on equivalence tasks?

Theoretical Framework

Structure in Numbers and Numerical Operations

One of the key aspects in developing algebraic thinking is the notion of structure; however, there are different perspectives on this notion. As mentioned in Kieran (2018), several researchers have worked in this area in the teaching and learning of algebra. Linchevski and Livneh (1999), for instance, recognize that students experience difficulties with structure in algebra and that these difficulties are due to lack of understanding of structure in arithmetic. Mason et al. (2009) have suggested that working with tasks that focus on relations rather than on procedures strengthens students' attention to the structural aspect of arithmetic. They refer to this as *structural thinking* and propose that it allows students to move away from the particular in a situation.

According to Kieran (2018) generalization-oriented activities encompass a structural aspect, but more attention is needed to the process that is complementary to generalizing, that is, the process of *seeing through mathematical objects,* decomposing and recomposing them in several structural ways. Kieran (2018, pp. 80-81) argues that to observe the structure of mathematical objects is to see through them. This means being aware of possible and different ways to structure number and numerical operations, for example, observing that 989 may be decomposed as $9 \times 109 + 8$, as $9 \times 110 - 1$, or as $9 \times 10^2 + 8 \times 10^1 + 9 \times 10^0$. According to this researcher, the generalization of mathematical ideas in arithmetic is linked to the idea of expressing structure. So generalization involves identifying the structural, and the structural involves identifying the general. Kieran (2018, p. 82) states that structure in numbers and numerical operations may be explained, firstly, by drawing on Freudenthal (1983, 1991, quoted in Kieran, 2018). That is, that the system of whole numbers constitutes an order structure, where addition is based on the order of this structure: in the addition structure, to each pair of whole numbers a third number (its sum) can be assigned. It is emphasized that, in Freudenthal's discussions of structure, there is not just one all-encompassing structure. He refers, for example, to order structure, additive structure, multiplicative structure, structure according to divisors, structure according to multiples, etc.

Based on the literature regarding perspectives on mathematical structure, specifically arithmetical structure, Kieran (2018) suggests promoting student experiences with equivalence through decomposition, recomposition, and substitution. Following Freudenthal, she points out that the structure in numbers and operations involves different *means of structuring*, according to factors, multiples, powers of 10, evens and odds, decomposition of primes, etc. Such structures expressed through decomposition, in other words, uncalculated forms, have properties. This perspective on structure constitutes a wider conceptualization of the fundamental aspect of structure in number and numerical operations as a means to develop early algebraic thinking. Taking into account the points made by Kieran (2018), as well as the suggestion of Schifter (2018) that structural properties can be implicit in students' procedures, this work will explore Mexican students' structure sense in equivalence tasks as evidenced through their strategies.

Methodological Considerations

Included in this report we present the preliminary results from an ongoing qualitative research aimed at investigating the strategies that students use in equivalence tasks.

Initial Task Design

Three tasks were designed in order to explore students' strategies; two of these did not include the equal sign. Task 1 aims at identifying the way in which students relate two numbers *a* and *b* with a third one *c* (i.e., its sum) and the rationale they use. It is a 4-item task with a main question: Can number *c* be written from numbers *a* and *b*? (*a* and *b* being specific numbers). Also, the task includes a generalization question: Can any number be written from other numbers? For all the questions, students were asked to provide an explanation.

Task 2 was based on the sequence proposed by Schifter (2018) and includes seven items. The aim is to observe the regularities students find in the proposed sequence based on this first item:

14+1
13+2
12+3
11+4
10+5

The rest of the items focus on two particular expressions from the sequence (e.g., 14+1 and 13+2). Students are asked to explain how to write an expression from the other. Another set of items focuses on discussing the equivalence of expressions without computation. The task ends with a question where a sequence of the same type is proposed, but involves subtracting; here we want to observe if students extrapolate from the discussion involving the case of adding.
Task 3 involves the use of the equal sign to show the equivalence of expressions, for instance, 4 + 5 = 4 + 3 + 2. The aim is to explore if students indicate relational thinking based on the structure of such equalities. The main goal in the task is to determine if the numeric sentences are true, as well as the possibility of rewriting them in an equivalent form. Task 3 also included numerical sentences with "big numbers".

Participants

Six sixth graders, ages 10 and 11, from a public Mexican school participated. This grade level was chosen because such students are finishing primary school and have been exposed to the official Mexican public education curricula. In the curriculum for the elementary school (SEP, 2016) the equivalence of numerical expressions is not mentioned. However, several tasks from the official textbooks have the potential to promote students' early algebraic thinking (Cabañas, Salazar, & Nolasco, 2017).

Data Collection

Prior to the unfolding of the designed activity, the teacher in charge of the group reviewed it. In her opinion, the students had never solved similar tasks; they had only worked with the use of the equal sign in an operational sense. The data collection technique was that of the Group Interview, so that students could verbalize their rationales. Data were obtained during three sessions, one session per task, with sessions lasting 30-40 minutes each. All six students participated in each of the three sessions.

Results and Discussion

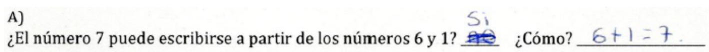
The preliminary results of an ongoing study are herein reported. The analysis focuses on the work of three students (S1, S2, and S3), those who participated most fully in the group interviews. Data for these results come from students' worksheets, videotaped footage of the sessions, and researcher's field notes.

Results from Task 1

Task 1 does not include the equal sign so as to see whether students use it spontaneously and, if so, in which way. Three of the items were the following:

1. *May number 7 be written from numbers 6 and 1? If so, how?*
2. *May number 19 be written from numbers 14 and 5? If so, how?*
3. *Is it correct to write number 7 as 3 + 4? As 8 + 2? Explain.*

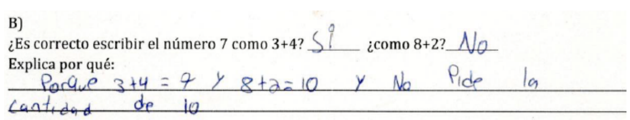
The students answered affirmatively items 1 and 2, their explanation being based on what in the literature is known as an operational use of the equal sign. For example, see S1's work in Fig. 1.


A)
¿El número 7 puede escribirse a partir de los números 6 y 1? Si ¿Cómo? 6+1=7.

May number 7 be written from numbers 6 and 1? *Yes* If so, how? *6 + 1 = 7*

Figure 1. S1's operational form of justification.

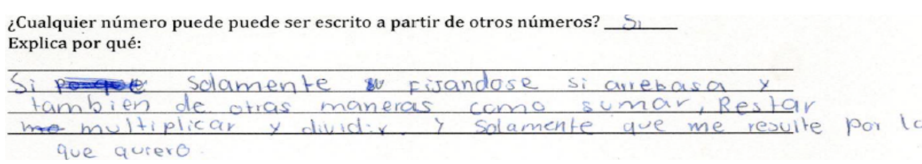
In his explanation, S1 relates 7 with 6 and 1 in an *operational sense*: 6 + 1 = 7, through a computational strategy. None of the students write, for example, 7 = 6 + 1, which would be recognized as a not strictly operational response. From a structural point of view, 6 and 1 can be interpreted as a decomposition of the number 7, which can then be recomposed from these numbers. In item 3, students answer in the same sense (Fig. 2) based on *the result they must obtain*.


B)
¿Es correcto escribir el número 7 como 3+4? Si ¿como 8+2? No
Explica por qué:
Porque 3+4 = 7 y 8+2=10 y No Pide la cantidad de 10

B)
Is it correct to write number 7 as 3+4? *Yes* As 8+2? . *No*
Explain
Because 3+ 4 = 7 and 8 +2 = 10 and [3+4] does not ask for a 10

Figure 2. S3's justification in terms of the result.

The same idea is present in the answers involving a generalization. *Can any number be written from other numbers?* Students identify the generality in terms of the response that they must arrive at. This is observed in S1's final explanation (Fig. 3) where he states *"…only if I get what I want"*.


¿Cualquier número puede puede ser escrito a partir de otros números? Si
Explica por qué:
Si Solamente fijandose si quebasa y también de otras maneras como sumar, Restar multiplicar y dividir y Solamente que me resulte por lo que quiero.

Can any number be written from other numbers?
Explain why
Yes, "*…only if I get what I want*

Figure 3. S1's general statement.

Students' answers show a lack of relational thinking by their use of the equal sign as a symbol that indicates the result. In other words, their strategy doesn't match with a notion of number decomposition, but with the idea of *operating with numbers* in order *to get a result*. The way in which they justify their answers – according to their teacher – shows how they have been systematically exposed to this way of thinking. In order to test whether the way in which Task 1 was designed led to the strategy that students used, Tasks 2 and 3 were designed differently.

Results from Task 2

The analysis of data from Task 2 (involving the sequence from Schifter, 2018) focuses on the

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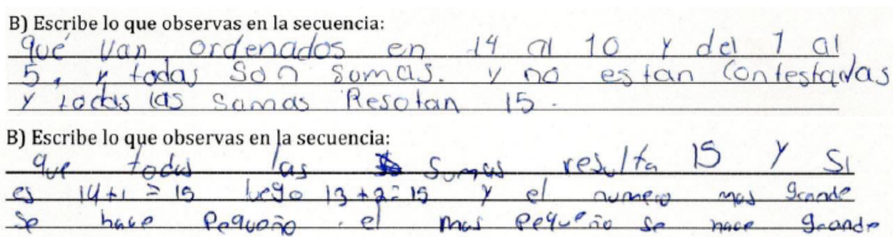
features of the $a + b$ form of expressions that were observed by the students, as well as on the possibility of transforming one expression of the sequence into another expression of the same sequence. Regarding the first aspect, students' answers show that they identify the regularity in the sequence. For instance, they write *about the involved operation (addition)*, *the sum (the result)* and the existence of an *order* in the sequence (*increase* and *decrease* of the addends). See S2's and S3's answers to the first item (Fig. 4).

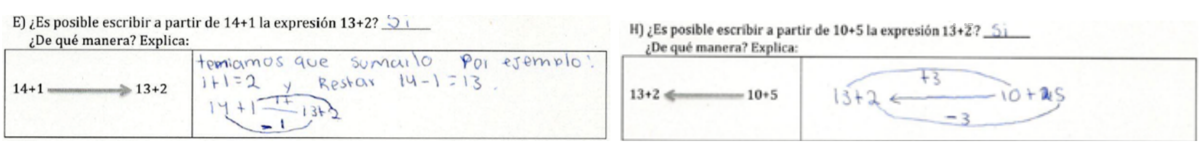


B) Write what you observe on the sequence
There is an order, in the 14 to 10 and in the 1 to 5, all of them are additions, they are not answered and all the additions result in 15.

In all the sums the result is 15, you have 14+1=15 then 13+2=15 and the biggest number becomes small, the smaller becomes big.

Figure 4. Features of the sequence as indicated by S2 (above) and S3 (below).

The kind of answers students produce regarding the presented sequence relates, on the one hand, to the kind of thinking they show throughout Task 1. That is, they identify the expression as an operation that must be carried out in order *to obtain a result*. This feature is clear in S2's response when he writes: *"…all of them are additions and they are not answered and all the additions result in 15"*. This suggests that these students do not see the expression as a mathematical object in itself, reflecting what is described in the literature as the *lack of closure* dilemma. On the other hand, there is some evidence of a train of thought that could be associated with *the structural*. According to Pang and Kim (2018), to identify patterns such as "increases by one and decreases by one" is a part of structural sense. This can be seen in S3's work (the lower half of Fig. 4) when he states: *"…and the biggest number becomes small, the smaller becomes big"*. However, he does not relate the feature he observes to the equivalence of the expressions. In the second part of Task 2, the students were asked how to obtain one expression in the sequence (e.g., 13 + 2) from another (e.g., 14 + 1). In these cases, all the students use an *additive compensation strategy*, as observed in Fig. 5.



E) Is it possible to write from 14+1 the expression 13+2? Yes
In which way? Explain
We have to add. For instance 1+1=2 and subtract 14 -1=13

H) Is it possible to write from 10+5 the expression 13+2? Yes
In which way? Explain
[Student illustrates an additive compensation strategy]

Figure 5. S1's additive compensation strategy.

The aim of our research was to study how students move from one expression to another, if they decompose and recompose the involved numbers. It was noticed that they identify the parts of the expressions, but not as a mathematical object that can be decomposed and recomposed to transform one expression into another. It was also seen that students use a *compensation strategy*: adding and subtracting the same amount to and from the involved addends in order to obtain the second expression. The following is an extract from an interview when, in the course of presenting the task, the interviewer asked for a generalization of the student's strategy:

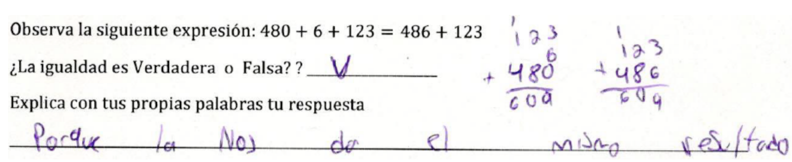
> *Researcher*: …Can you do it [referring to his strategy] in any case? Is there a rule for it? [After other students offer suggestions, S2 answers]
>
> *S2*: It's only a matter of adding and subtracting, depending on the required numbers.

Despite such structurally-related responses as S2 produced, there is not enough evidence, however, to determine if students identify an equivalence relationship among the expressions (e.g., 14+1 and 13+2). Nor is there enough evidence, with respect to the additive compensation strategy, to determine if they are aware that their strategy is generalizable to all additions (e.g., that 27 + 15 can be converted to, say, 30 + 12 or that 44 + 19 can be converted to 43 + 20) or simply applicable to the set of additive expressions provided in Task 2. If the latter, then — as suggested in Schifter (2018) — this would be an *ad hoc* strategy aimed at getting the numbers needed for the second expression from the first one, and viceversa.

Results from Task 3

This task includes the equal sign – in expressions such as $a + b = c + d$. As mentioned, Task 3 involves "big" numbers to see if this deters the use of computational strategies.

On the one hand, students accept expressions such as $a + b = c + d$; however, they justify the equality of both sides by calculating the result on each side. Again, this computational strategy demonstrates that students are not relying on relational thinking. Their computational strategy is called upon in both cases, whether with "small" or "big" numbers (Fig. 6).



Observe the following expression: 480 + 6 + 123 = 486 + 123
Is the equality True or False? _T_
Explain with your own words
Because we get the same result

Figure 6. S3's computational strategy.

On the other hand, students rewrite the equalities in the form of other equivalent equalities according to two strategies. In the first of these strategies, they decompose each of the addends, but not in a way that shows a clear relationship between one side and the other of the equality (see S1's work in Fig. 7). In the second, which is based on calculating the *total (the result) for each side* without first decomposing the involved addends, students then look for two or more numbers for which they could obtain the same total (see S3's work in Fig. 8).

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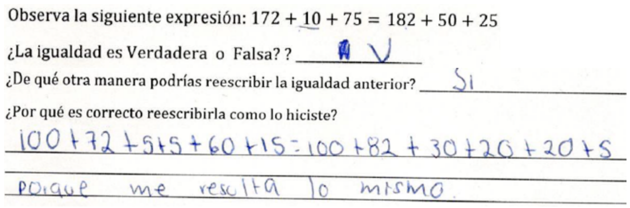
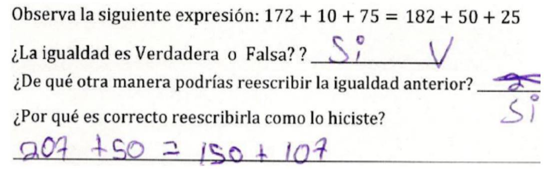


Observe the following expression: 172 + 10 + 75 = 182 + 50 +25
Is the equality True or False? _T_
In which other way could you re-write the previous equality? _Yes._
Why it is correct re-write the expression in such a way?
100 + 72 + 5 + 5 + 60 + 15 = 100 + 82 + 30 + 20 +20 + 5
Because I get the same result

Figure 7. S1's equality rewriting.



Observe the following expression: 172 + 10 + 75 = 182 + 50 +25
Is the equality True or False? _Yes_
In which other way could you re-write the previous equality? _Yes._
Why it is correct re-write the expression in such a way?
207 + 50 = 150 + 107

Figure 8. S3's equality rewriting.

This task shows that students accept equalities in the form of $a + b = c + d$. However, they transform them without relating the right and left sides except according to their totals. Their strategy is to maintain the equivalence through the two above-mentioned strategies. Hence, as observed in Figs. 7 and 8, there is not a natural inclination in students to re-express the equalities in such a way that both sides of the equalities look alike; for instance, 172 + 10 + 50 + 25 = 172 + 10 + 50 + 25, or 170 + 2 + 10 + 50 + 25 = 170 + 2 + 10 + 50 + 25, or even as 182 + 75 = 182 + 75. However, S1's work shows some structural sense according to the reviewed literature. Even when S1 and S3 write correct equalities, each side is considered on an individual basis. The left side is decomposed in one fashion and the right side in a different way, without showing explicitly the equality of both sides. S1 (Fig. 7) does not explain that both sides look more or less the same, he only mentions that the result (on both sides) is the same.

Conclusions

From the strategies students used, only one can be considered to illustrate a structural approach (S1 in Task 3, as shown in Fig. 7), even though the accompanying explanation refers to the result of both sides of the equality. The rest of the students' strategies are clearly computational, referring to the expected result, whether it involves operating with the numbers of an expression so as to calculate the result on both sides of an equality (the strategy observed in Tasks 1 and 3), or operating with the addends of one expression to obtain the addends of the other expression (the strategy observed in Task 2). In this sense, the presence or absence of the equals sign in the tasks seems not to influence the students in their chosen strategy.

Our results coincide with those reported by Pang and Kim (2018), in the sense that students tend to use computational strategies. This means that they show a strong operational sense, even when they accept equalities in the form of $a + b = c + d$. Nevertheless, this acceptance could be used as a base to promote the development of structural sense within algebraic thinking by designing tasks in such a way that students are explicitly requested not to pass through the intermediate step of computing the total for each expression in their work on judging the equivalence of the component expressions. Accepting expressions as bona fide numerical objects, and operating with and on these objects, is essential to seeking and expressing structure within the domain of arithmetic and thereby fostering the development of algebraic thinking.

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