GRADE 5 STUDENTS' NEGATIVE INTEGER MULTIPLICATION STRATEGIES

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Twenty-four Grade 5 students participated in clinical interviews where they solved integer multiplication number sentences. Drawing on the theoretical perspective of strategies that students use with whole number multiplication and integer addition and subtraction, we describe the strategies that students employ when negative integers are incorporated with multiplication. The students, although drawing on similar strategies for whole number multiplication (e.g., repeated addition, direct modeling), used these strategies differently (e.g., using Unifix cubes to represent -1). The students also used unconventional strategies for solving integer multiplication, such as analogies and invented procedures. The results highlight the important constructions of students prior to formal instruction on integer multiplication, where prior research has been mainly situated in thinking about integer addition and subtraction.

Keywords: Number Concepts and Operations, Elementary School Education, Cognition

Investigations of strategies that students invent, and even struggle with, for integer multiplication number sentences, will provide teachers and researchers with insight into students' thinking about integers. With this understanding, we can begin to develop instructional strategies that support building on students' thinking about integer multiplication, a neglected topic in our field. In order to improve instructional approaches, we must first investigate students' constructions and reasoning.

Children invent sophisticated and robust ways of reasoning about integers and integer addition and subtraction (e.g., Bofferding, 2014; Bishop et al., 2014). As children approach addition and subtraction of integers for the first time, they use different strategies (Bofferding, 2010), ways of reasoning (e.g., Bishop et al., 2014; Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2016), and conceptualizations (e.g., Aqazade, Bofferding, & Farmer, 2017; Bofferding & Wessman-Enzinger, 2017; Wessman-Enzinger, 2015). Although there has been an increased focus on children's reasoning about integers (e.g., Aquazade et al., 2017; Bofferding, Aqazade, & Farmer, 2017; Bishop et al., 2016), investigations into integer multiplication remain overlooked.

The goal of this research report is to present an inaugural framework of strategies students created as they engaged with integer multiplication number sentences for the first time. Our research question focuses on students' invented strategies for integer multiplication number sentences (e.g., $-2 \times 3 = \square$): What strategies do Grade 5 students use as they solve integer multiplication number sentences?

Theoretical Perspective

Because children often build on their whole number knowledge and extend this to integer reasoning (Bofferding, 2014), looking towards strategies that children employ with whole number multiplication may provide insight into how children may begin to reason about integer multiplication. Multiplication and division problems are often approached by children through a variety of invented strategies, such as repeated addition or direct modeling with grouping collections of countable objects (Carpenter, Fennema, Franke, Levi, & Empson, 2015).

Carpenter et al. (2015) and Baek (1998) demonstrate that children are able to understand multiplication when they can invent their own strategies. Some of the strategies for single-digit (Carpenter et al., 2015) and multi-digit (Baek, 1998) multiplication with whole number include: direct modeling strategies, counting strategies, repeated addition, and derived fact strategies. The extent to which students will use similar strategies with negative integer multiplication is an open question.

With *direct modeling*, students model groups using manipulatives (e.g., Unifix cubes) or drawings. When students use *counting* strategies they may skip count accounting for groups, sometimes using fingers or choral counting. Students draw on *repeated addition* or *doubling* (e.g., $4 \times 3 = 3 + 3 + 3 + 3$). *Derived facts* strategies include drawing on factual knowledge and creating a new algorithm based on previously known facts (e.g., 2×3 may be solved by know that $2 \times 2 = 4$ and then 2 more added to that product is 6).

From the integer addition and subtraction literature, we know that students use a variety of strategies different from the CGI frameworks. These include using *computations* or *procedures* (Bishop et al., 2014), drawing on *recalled facts* (Bofferding & Wessman-Enzinger, in press), and making *comparisons* or *analogies* (Bishop et al., 2016; Bofferding, 2011; Wessman-Enzinger, 2017; Whitacre et al., 2017).

As we began our study, we drew on both single-digit and double-digit strategies for multiplication with whole numbers and strategies for integer addition and subtraction. We thought these strategies would provide insight into the ways that students may solve multiplication problems involving negative integers.

Methods: Participants, Interviews, and Analysis

We conducted clinical interviews (Clement, 2000) with 24 Grade 5 students from the rural Pacific Northwest. We selected Grade 5 students that did not have formal school experiences with integers; *Common Core State Standards* recommendations include integer operations in Grade 7 (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). We interviewed each student once, using the following integer multiplication number sentences (see Figure 1). Students solved the integer multiplication number sentences, with each number sentence provided on a singular piece of paper. Manipulatives and tools provided during this interview included: Unifix cubes, two-colored chips, empty number lines, and markers. We asked prompting questions throughout the interviews, without giving the students the answer or additional information. These types of questions included: “How did you come up with that?”; “Can you explain your thinking?”

$$3 \times 5 = \square$$

$$-2 \times 3 = \square$$

$$3 \times -4 = \square$$

$$-4 \times -2 = \square$$

Figure 1. Integer multiplication number sentences provided to students.

We videotaped and transcribed each interview. Our unit of data included the video clip, drawings, and transcripts associated with each integer multiplication number sentence. We began coding with the framework delineated in the analytical framework (Baek, 1998; Carpenter et al., 2015). For instance, we looked for the use of manipulatives and drawings for direct modeling

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strategies. We looked for choral counting, skip counting, and use of fingers for counting strategies. Using the definitions established for these various strategies, we employed constant comparative methods (Merriam, 1998). We modified the strategies to include the ways that students used negative integers, not previously captured with only positive integer multiplication strategies or integer addition and subtraction strategies. We met to compare codes and negotiated any disagreements. In the results section, we highlight the integer multiplication strategies that the students in our study used.

Results: Strategies for Integer Multiplication

We highlight the descriptions of the strategies, rather than focusing on correctness or incorrectness. Because the students have powerful strategies paired with some correctness, this is provides a space to understand children's thinking as a vehicle for leveraging discourse in the classroom in the future.

Direct Modeling

Example of direct modeling. Edie solved -2 × 3 = □, using a direct modeling strategy, that resulted in a solution of -6 (see Figure 2). Edie assigned the value of -1 to each Unifix cube. She constructed three groups of two blocks (see white blocks in Figure 2). Because she attributed the value of -1 to each of the white Unifix cubes, she modeled -2 × 3 = □, instead of 2 × 3 = □.

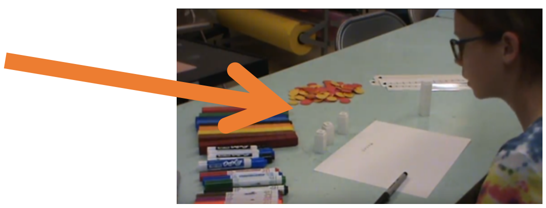

Figure 2. Example of direct modeling strategy for -2 × 3 = □.

The following transcript excerpt illustrates how Edie shared her strategy:

> (Reaches for Unifix cubes) I'm going to pretend this is negative… okay this is negative 2 (pulls off 2 white Unifix cubes) negative plus a negative would be a negative… so if these are negatives then that would 3 times the 2 negatives which would equal 6 negative (writes "-6" on paper).

Description of direct modeling. Students used a direct modeling strategy when they illustrated integer multiplication with physical tools (e.g., Unifix cubes, two-colored chips, pictures)—modeling (number of groups) × (number of things in each group) = total. The students who used direct modeling strategies determined the solutions to integer multiplication through physically manipulating and modeling with these objects.

The students used two-colored chips (one yellow side, one red side) to be a physical representation of the difference between a negative number and a positive number. Notably, the students flexibly used the colors. Sometimes, red chips represented negative integers and yellow chips represented positive integers; other times, red chips represented positive integers and yellow negative integers.

Using Unifix cubes, the students used the cubes to model multiplication as groups of the same amount of quantities. The students who used the cubes mapped values of -1 to each of the cubes. The cubes represented a way to account for groups of negative quantities and provided a physical way to add the groups together in order to determine their solutions.

Use of direct modeling. The students used direct modeling strategies seven times for 3 × 5 = ☐. For the number sentences with negative integers, the students used direct modeling strategies five times for -2 × 3 = ☐ and four times for 3 × -4 = ☐. Direct modeling was used only once for -4 × -2 = ☐, which is not surprising given the physical limitations of negative amounts of groups.

Repeated Addition and Subtraction

Example of repeated addition. Eliza solved 3 × -4 = □ using repeated addition (see Figure 3) and obtained the solution, -12. Eliza demonstrated repeated addition as she repeatedly added -3 four times in order to get her product of -12. Notably, she added -3 four times, instead of adding -4 three times; her strategy actually aligns to 4 × -3 = □ instead of 3 × -4 = □. Essentially, Eliza implicitly recognized the equality of 4 × -3 and 3 × -4, without commenting on it. In Figure 3, the black writing illustrates her final computed product. However, the red writing illustrates her repeated addition, which she wrote first.

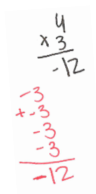

Figure 3. Example of repeated addition strategy.

Description of repeated addition and subtraction. Repeated addition, as a strategy, describes multiplication with adding positive integers repeatedly (Baek, 1998; Carpenter et al., 2015). The students in our study drew on repeated addition with negative integers. However, they also used *repeated subtraction* of positive integers.

Use of repeated addition and subtraction. The students used repeated addition strategies ten times for 3 × 5 = ☐. For the number sentences with negative integers, the students used repeated addition and subtraction four times for -2 × 3 = ☐ and three times for 3 × -4 = ☐. A student used repeated addition and subtraction only once for -4 × -2 = ☐, which is also not surprising given the challenges of adding -4 “negative two” times.

Recalled Fact

Example of recalled fact. Zoe first solved -2 × 3 = □ and obtained -6 as a recalled fact, even though it was her first time engaging with integer multiplication. She quickly stated the answer, -6, before the interviewers even completely finished reading the multiplication number sentence, -2 × 3 = □. Zoe relied on her factual knowledge of the product of 2 × 3 = 6, when questioned. With probing she justified her solution with a procedure “you just do 2 times 3 and then you make it a negative,” which will be discussed later.

Description of recalled fact. Within the CGI strategy framework, students often draw on facts to make derived facts (e.g., Carpenter et al., 2015). In our study with integer multiplication, students did not seem to use derived facts, but did use their factual knowledge about whole number multiplication quickly for integer multiplication without verbal explanation. Students used recalled facts when they stated their solutions to integer multiplication as a fact, likely memorized from whole numbers. Or, they drew on their memory so much that it did not require any form of deliberation. Students stated their solution quickly with an often “just is” explanation.

Use of recalled fact. The students used recalled fact four times for -2 × 3 = □, six times for 3 × -4 = □, and five times for -4 × -2 = □. The students demonstrated confidence with single digit whole number multiplication (e.g., fifteen stated the answer of 3 × 5 = □ as a recalled fact).

Procedure

Example of procedure. Lia solved 3 × -2 = □ with a solution of 4, using a procedure as a strategy. In this example, Lia solved the integer multiplication sentence by using a "negative integer as a singular subtrahend" procedure (see Figure 4). She first computed 3 × 2 by solving 3 + 3. Then, Lia incorporated the singular integer in the number sentence, -2, by subtracting 2 from the product of 3 × 2. This procedure is one of various types used in this study by the students.

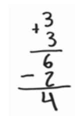

Figure 4. Example of procedure strategy.

Description of procedure. When students used an algorithm or created an invented procedure to find the solution they used the procedure strategy. Although this represents an addition to existing CGI framework for multiplication strategies (e.g., Baek, 1998), many integer researchers have stated that students use computational reasoning (Bishop et al., 2016) or procedures (Wessman-Enzinger, 2015; Bofferding & Wessman-Enzinger, in press) as they solve integer addition and subtraction problems. Thus, it seems to be a natural extension that students would also use computational and procedural strategies with integer multiplication.

The students in this study used different types of procedures (e.g., appending a negative sign to the solution, negative numbers as equivalent to zero, exclusive negativity). Describing the extensive use of procedures is beyond the realm of this research report. But, Zoe used the "appending a negative sign" procedure in her justification of derived fact strategy -2 × 3 = -6 when she stated that the negative sign is just "added on." Other students said that number sentences, such as -4 × -2 = □, needed to be "all negative," concluding that -4 × -2 = -8 based on a procedure of "exclusive negativity."

Use of procedure. Students used or invented various procedures for dealing with integer multiplication throughout the study (e.g., eleven times for -2 × 3 = ☐, sixteen times for 3 × -4 = ☐, and fifteen times for -4 × -2 = ☐). The students did not use a procedure for 3 × 5 = ☐ and used procedures only for multiplication number sentences with negative integers (e.g., -2 × 3 = ☐).

Counting

Example of counting. Cittie used counting on a number line to solve -2 × 3 = □, obtaining a solution of 7. Figure 5 illustrates Cittie's number line. She reasoned that she could start at -2 and counted in sequential order on the number line, moving right, to her destination, 7; she skip counted by 3, three times. Although this does not represent a correct solution, Cittie ordered the negative and positive numbers correctly and started her counting at -2, which represents beginning, ordered integer reasoning necessary for integer multiplication.

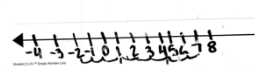

Figure 5. Example of counting strategy.

Description of counting. Students, sometimes with the help of a number line, counted in sequential order when solving multiplication number sentences. Students often described their strategies as "skip counting." The students sometimes drew on the number line, as a tool for facilitating their skip counting. In addition to number lines, students also used fingers and even cubes as number paths (Bofferding, 2010). Challenges with negative integer multiplication for students included deciding which direction to count in, as directions are not fixed like they are with whole number numbers, and determining the quantities of the skip counts (see, e.g., Cittie counting amounts of 3 three times).

Use of counting. The students used counting strategies three times for $3 \times 5 = \square$, four times for $-2 \times 3 = \square$, three times for $3 \times -4 = \square$, and eight times for $-4 \times -2 = \square$. When the students used counting strategies with integer multiplication, they sometimes did so with a number line. It is likely that students used counting strategies the most for $-4 \times -2 = \square$ because of challenges in physically representing this number sentence with strategies like direct modeling.

Analogy

Example of analogy. Jaxon first solved $-2 \times -4 = \square$, with analogy, determining that $-2 \times -4 = -8$. The following transcript highlights Jaxon's reasoning:

> Well, because it wouldn't really make as much sense for a negative multiplied by a negative to equal a positive. It's like, um, I'm not sure how to … it just wouldn't make as much sense. Because if a positive multiplied by a positive would equal a positive, then I would assume that it would be the same for a negative. And, it would be a negative times a negative would equal a negative.

In this excerpt, Jaxon compared $-2 \times -4 = \square$ to $2 \times 4 = \square$. Reasonably, he concluded that because a positive number times a positive number is positive (e.g., $2 \times 4 = 8$), then a negative number times a negative number is another negative number (e.g., $-2 \times -4 = -8$). Again, like the previous example where we highlighted a strategy with an incorrect solution, there is still powerful reasoning embedded in Jaxon's strategy. Jaxon connected his reasoning about whole numbers in a logical way (albeit not a culturally/mathematically correct way).

Description of analogy. Students used analogy when they connected previous knowledge about whole numbers to integers and compared it to a whole number multiplication number sentence for constructing or justifying new claims when solving the integer multiplication number sentences. Although this is an addition to the CGI framework for multiplication strategies, there is evidence that students use analogies with integer addition and subtraction (Bishop et al., 2016; Bofferding, 2011; Wessman-Enzinger, 2017; Whitacre et al., 2017). We distinguish this from recalled facts or procedures in that the students made explicit comparisons, with reasoning focused on these comparisons.

Use of analogy. The students did not use an analogy strategy for $3 \times 5 = \square$, $-2 \times 3 = \square$, or $3 \times -4 = \square$. But, students used analogy twice for $-4 \times -2 = \square$. Although not used often in this study, students use analogies frequently with integer addition and subtraction (e.g., Whitacre et al., 2017). We conjecture that if we gave more "negative number multiplied by negative number" number sentences we would have seen analogy strategies employed more—analogies seem like potentially productive strategies for these types of integer multiplication number sentences.

Counter Movement

Example of counter movement. Warren solved $-2 \times -4 = \square$, determining a solution of 8. The following transcript excerpt highlights Warren's solution of 8.

> Um. So. I… this subtraction, this negative symbol. Scratch out this one and this one … counters this one too. So it takes both these out and then it's just 4 times 2. … I just thought that since they're both negative numbers and that one's whole, it's basically, kinda like dividing except you're multiplying. And… you just kinda just… I just thought that you counter them. … It's kind of like dividing because instead of making this uh… -8, you just make it normal 8, which means that the number … if these numbers were whole numbers … well no since their negative numbers… if like cause they're not equal to ... they're not equal to 0 they're this way (left of 0) and the whole numbers are this way (right of 0) it's kind of like since these numbers were divided …these ones were dividing they would get smaller or go the opposite way. Like if these ones they would go this way.

At first, Warren's strategy sounds procedural because he talks about "scratching" out symbols, referencing the negative symbols in front of -2 and -4. Then, Warren references a "countering" of movements in the negative and positive direction (e.g., $-1 \times -1 = 1$ or $-1\ \boxed{v?}\ -1 = 1$), making -8 "a normal 8." He discusses how we can treat this multiplication problem as "whole numbers" since multiplying the negative integers "counter" the directions of each other. Multiplying by -1 moves a number "this way (left of 0)" and multiplying it by -1 changes the direction.

Description of counter movement. Students use the counter movement strategy when they employ continuous movement or motion that "counters" each other. Use of this strategy includes a reference to changing directions, where the movement is countered or balanced. Consider this equation: $-2 \times -4 = (-1 \times 2) \times (-1 \times 4) = (-1 \times -1) \times (2 \times 4)$. Multiplying by negative one refers to a movement or translation in one direction and multiplying by the other negative one is a movement or translation in the other direction—consequently *countering* the overall movement.

Use of counter movement. Of the twenty-four students we interviewed, only one student constructed this strategy. Although this may not warrant the creation of a new category, the strategy uniquely helped Warren construct a correct solution to $-2 \times -4 = \square$, a notoriously challenging problem type. Honoring the student's use of the word "counter" and the use of continuous movement for constructing meaning, we called this strategy "counter movement."

Discussion and Final Remarks

The results of this work are significant in that we provide an inaugural framework for integer multiplication strategies that students use prior to school instruction, modified from CGI multiplication strategies frameworks and integer addition and subtraction literature. Previous research has focused on thinking and strategies of integer addition and subtraction (e.g., Bofferding, 2010; Bishop et al., 2014) and we extend the scholarly discussion on students' thinking about integers by describing their invented strategies for integer multiplication.

Our focus is on the powerful ways that students, prior to formal school instruction, solved integer multiplication number sentences, whether correct or incorrect. If we wish to support student inventions and discourse in the mathematics classroom, we must first understand their sophisticated reasoning (Carpenter et al., 2015). Jaxon, for example, obtained $-2 \times -4 = -8$. Although we know this to be an incorrect solution, it is rooted in a logical analogy (e.g., if $2 \times 4 = 8$, then $-2 \times -4 = -8$). As teachers and researchers, how do we promote conceptual change when students invent strategies that are logical, but not mathematically correct? We might consider pairing number sentences like $2 \times -4 = \square$, where students had success in correct answers, with number sentences like $-2 \times -4 = \square$, where students had more difficulty, to leverage growth or change in the students' sophisticated reasoning (Bofferding et al., 2017).

Empowering students in the classroom requires building on their thinking. Consequently, we

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must first learn about the ways students enter the mathematics classrooms and the invented strategies they construct; then, we can draw on their reasoning to build future instructional interventions.

Acknowledgments

This research is supported by the Richter Grant at George Fox University funded by Bank of America.

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