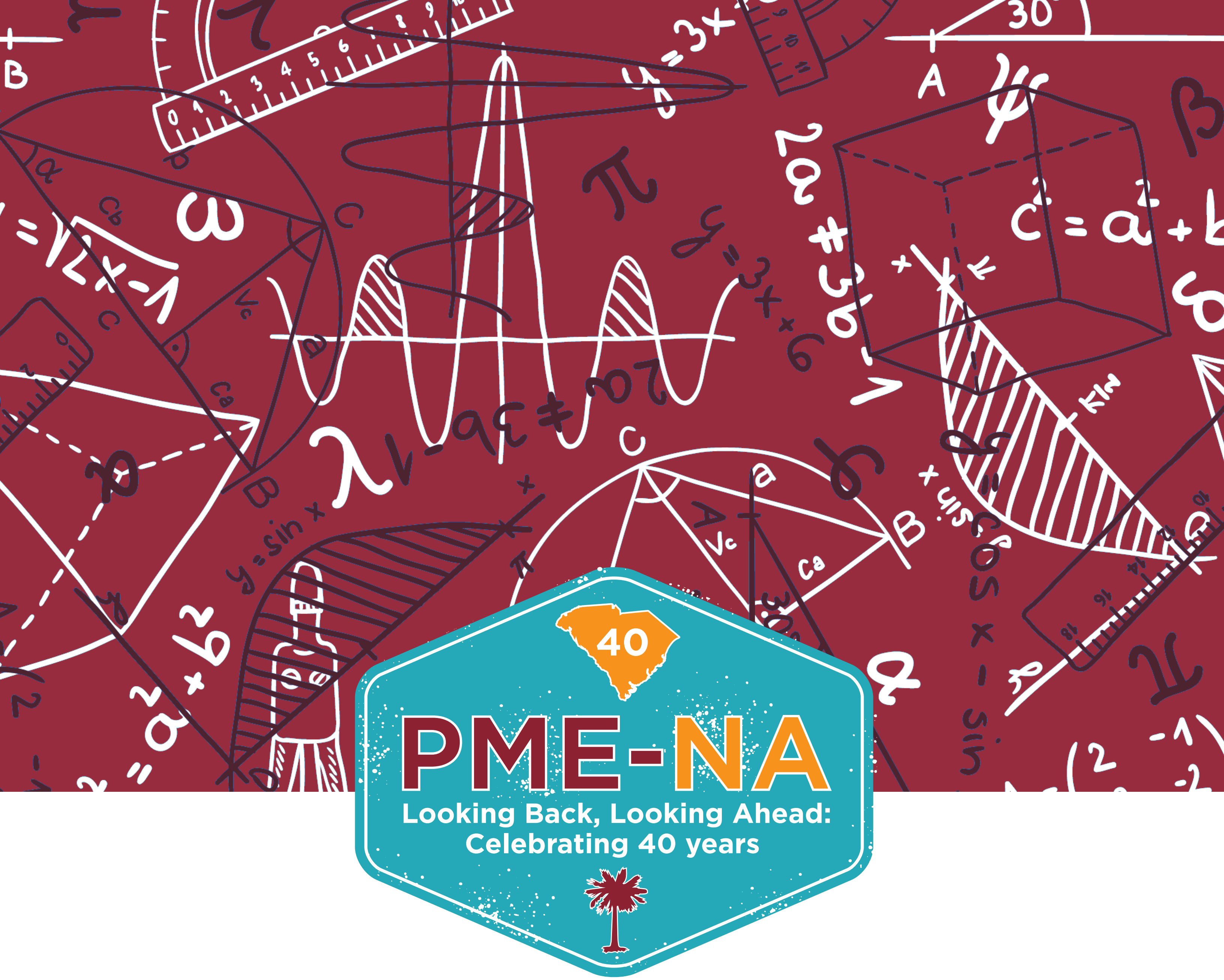

PME-NA^{40}
Looking Back, Looking Ahead

Editors: Thomas E. Hodges, George J. Roy, & Andrew M. Tyminski

Proceedings of the North American Chapter of the International Group for the Psychology of Mathematics Education

Proceedings of the Fortieth Annual Meeting of the North American Chapter of the International Group for the

Psychology of Mathematics Education

Looking Back, Looking Ahead: Celebrating 40 Years

Greenville, SC USA
November 15-18, 2018

Editors

Thomas E. Hodges
George J. Roy
Andrew M. Tyminski

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The major goals of the International Group and the North American Chapter are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
2. To promote and stimulate interdisciplinary research in the aforesaid area, with the cooperation of psychologists, mathematicians, and mathematics teachers;
3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

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Preface

Dear Colleagues,

On behalf of the 2018 PME-NA Steering Committee, the 2018 PME-NA Local Arrangements Committee, the College of Education at the University of South Carolina, and the College of Education at Clemson University, we would like to welcome you to South Carolina and the 40th Annual Meeting of the International Group for the Psychology of Mathematics Education – North American Chapter, held at the Hyatt Regency in Greenville, South Carolina.

As part of our celebration of 40 years of PME-NA, this year's conference will focus on enduring challenges in mathematics education research and look ahead to emerging opportunities in the field. Plenary sessions will address our major conference themes: mathematical and pedagogical demands for P-16 education, equitable mathematics teaching and research, leveraging new technologies, and perspectives on the nature of mathematics and research.

Elham Kazemi will present the opening plenary session on Thursday evening, *How Can Understanding Student Experiences in the Mathematics Classroom Enrich, Challenge, and Help us Improve our Own Learning as Teacher Educators and Researchers?* The talk explores the ways in which learning more about research on students' experiences in mathematics classrooms has the potential to transform the work we do with teachers in teacher preparation, professional development, and research settings. Corey Drake will serve as the discussant. On Friday, Marta Civil will present and Laurie Rubel will serve as discussant on the plenary session entitled *Looking Back, Looking Ahead: Equity in Mathematics Education*, which explores funds-of-knowledge, participation, and valorization of knowledge orientations towards mathematics education. Rubel extends our understandings of the political dimensions of equity in mathematics education, as well as articulates a vision for future work around equity in mathematics education. Saturday's plenary session features Margaret Niess with discussant Jeremy Roschelle in a session entitled *Transforming Teachers' Knowledge for Teaching Mathematics with Technology through Online Knowledge-Building Communities*. Their session describes the development and transformation of Technological Pedagogical Content Knowledge among inservice teachers via online technology courses designed with intentional opportunities to explore and discuss reform-based instructional strategies for teaching with technologies within teachers' communities. Finally, Anderson Norton and Julie Sarama present *Perspectives on the Nature of Mathematics and Research* during Sunday's plenary session. Norton and Sarama provide unique and compelling notions of mathematics as a body of knowledge, helping to ground our understandings of what it means to learn and do mathematics. Each presenter provides powerful implications for the teaching, learning and research of mathematics education.

The concurrent and poster presentations serve to advance our field through a firm grounding in where we have been, our current understandings of mathematical learning, and with an eye towards the work ahead of us - truly, we are *Looking Back, Looking Ahead*. This year's conference will be attended by more than 500 researchers, faculty and graduate students from around the world including Mexico, Canada, Australia, Republic of Korea, and 42 states from the United States of America. The acceptance rate was 37% for research reports as research reports, 69% for brief research reports as brief research reports, 94% for posters as posters, and 100% for working groups. The accepted proposals included 69 research reports, 137 brief research reports, 133 posters, and 17 working groups.

We would like to thank the many people who generously volunteered their time over the past year in preparation for this conference. In particular, we wish to thank Jessica Allen, Graduate Assistant at the University of South Carolina, for her tireless efforts in facilitating planning, communicating with various stakeholders and formatting the conference proceedings.

Professional regards,

Thomas E. Hodges	George J. Roy	Andrew M. Tyminski
University of South Carolina	University of South Carolina	Clemson University
2018 PME-NA	2018 PME-NA	2018 PME-NA
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HOW CAN UNDERSTANDING STUDENT EXPERIENCE IN THE MATHEMATICS CLASSROOM ENRICH, CHALLENGE, AND HELP US IMPROVE OUR OWN LEARNING AS TEACHER EDUCATORS AND RESEARCHERS?

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In this paper, we explore the ways in which learning more about research on students' experiences in mathematics classrooms has the potential to transform the work we do with teachers in teacher preparation, professional development, and research settings. We focus in particular on questions of student access to and participation in mathematics and highlight studies of the racialized and gendered experiences of students and the connections between these experiences and broader narratives about race, gender, and ability/disability. We conclude with questions and possibilities raised by these studies for our individual and collective efforts to support and understand teacher learning and changes in teacher practice.

Keywords: Teacher learning, Student experience, Equity

The title of our paper is a question, not a statement, which we hope will provoke conversation among us. To begin, we would like to introduce ourselves and then explain how we came to ask the organizing question for this paper and use selected studies as cases to help us dive into discussion.

How Did We Come to the Focus of this Paper?

We decided to use this occasion as an opportunity to further our own learning by highlighting recent work in the field that we think can inform our work with teachers. To that end, we are not attempting a comprehensive review of any kind. Instead, we selected articles, be they written for researchers or practitioners, that would shed light into how particular students have experienced the mathematics classroom. You will notice that some of these accounts and narratives are first-person accounts, while others were generated through close collaboration between researchers and students.

The ways we think about teacher practice and teacher learning in research and teacher education focus heavily on teacher performance - along a variety of dimensions, with a variety of foci. This focus can be seen in the many studies of changes in teachers' practices, in recent practice-based teacher education efforts, and in the variety of observation protocols used to observe, understand, and sometimes evaluate teaching. Both personally and as a field, we have learned a lot in recent decades from thinking about teaching in this way. In our own work, we have learned about ways in which teaching is difficult and complex for teachers, particularly novice teachers; approaches to supporting prospective teachers in developing ambitious teaching practices; the roles of tools such as student work and frameworks of children's mathematical thinking in advancing changes in teachers' practice; and how to design learning environments for teachers to learn together. We have also learned from work that has examined relational aspects of teaching, though primarily from the perspective of teachers, about the importance of teacher care and productive relationships with students (e.g., Bartell, 2011; Jansen & Bartell, 2013).

Studies of teaching can benefit from more attention to the nature of student experience in mathematics classrooms as a lens for understanding teacher practice and teacher learning. When we make this claim, we want to be sure to note that attending to student experience is not the same as attending to student outcomes/achievement; many studies have tried to link teacher practice and student achievement. Instead, we are interested in understanding teaching in terms of student experience, which is broader than student achievement or even student learning, and also takes into account students' experiences of mathematics in relation to identity, participation, motivation, and agency (Aguirre, Mayfield-Ingram, and Martin, 2013).

Our interest in this paper is in deepening our understanding of teaching by focusing on students and their experiences because although we have learned many important ideas from focusing on teacher performance and practice, we have yet to deeply understand the link between teaching and student experiences or ways in which teaching might disrupt persistently inequitable patterns in student experiences. Jansen & Bartell (2013) note that, "A teacher may intend to enact care, but unless the care has been received by a student, the student will not feel cared for." (p. 36) This points to a key limitation of research on teacher education that focuses on teacher performance without also considering student experience. In order to address this limitation, we, as teacher educators and researchers of teaching, we are going to foreground research on students' experiences. Connecting and expanding the literatures we put into conversation together can further our efforts to prepare teachers who can create transformative and inclusive classroom and support students' access to and participation in mathematics.

What do we Know about Student Experience that Might be Helpful for Thinking about Teacher Practice?

In the cases that follow, we focus first on understanding individual students' motivations and experiences related to participation in mathematics discussions. We then move to cases of studies that explored the racialized, gendered, and networked nature of students' participation in mathematics classes. Finally, we explore cases of participation in and access to mathematics in relation to broader racialized, gendered, and ability-related narratives.

Understanding Students' Experiences as Listeners and Speakers in the Mathematics Classroom

Hintz (2011) studied the experiences fourth-grade students had in two classrooms during classroom discussions. She sat with them as they replayed video from a recent discussion and asked the students to share with her what was happening for them during those segments. In research on classroom discussions, studies have examined how discussions unfold, what children say or do, and what decisions teachers make for discussions to be mathematically productive. Hintz applies a different lens to understanding classroom discussions, as she carefully examines the demands that these discussions place on students both as listeners and speakers and illuminates how students experience those demands. In her 2011 article, Hintz presents the complexities one student, Norah, experienced in a common discussion structure, called strategy-reporting, during which students share the different ways they thought about a problem.

Hintz sat with Norah to look at a particular time when Norah shared her answer to a multiplication problem while also making a hand gesture to indicate that she was not confident it was right. During strategy reporting, teachers commonly ask questions about a student's strategy and often work through any errors that arise. But Norah did not like to share when her solution had a mistake because she anticipated being asked to talk about it in front of the other students. Hintz recounts what unfolded in the classroom and how Norah experienced it:

> During one particular interview, Norah and I rewatched the videotape of an episode during a lesson when she shared a mistake and I listened to her talk about that experience. After solving the problem 14×5 mentally, she had offered up her answer saying, "I think it is 120."

> As she said her answer she was grimacing with a look of uncertainty and concern. She turned to her neighbor and gestured her hand back and forth in a flip-flop motion showing that she was unsure about her answer. Later in the discussion, as a different answer was decided to be correct, Norah raised her hand and said with a shy smile and shrugging downward, "I counted wrong." The teacher responded, "You counted wrong. And that's OK, we do that all the time don't we? That's part of our life," and moved on with the discussion.
>
> As Norah narrated her experience during this episode, she shared,
>
> *Since I did 14 × 5 = 120, I messed up because I added differently than I should have. I did 4 × 5 is 20 and then I put the one from the 10 right on as 100. I should have thought more about it instead of going right to it. It is kind of embarrassing.*
>
> What she felt was embarrassing was "getting it wrong," and she added, "But if you make a mistake then you can keep practicing that problem and it will become a fact that you know. Next time I would still start with 4 × 5 = 20 but then I would do a different step." This comment reveals that she sees the potential for learning from your mistakes and continued practice. Yet the social consequences of making a mistake publicly weighed heavy on her mind and shaped how she chose to take on the roles of sharer and listener. An important part of why Norah did not like to share when her solution had a mistake was because of how she may have been asked to engage in talk about the mistake with her teacher in front of the other students. It is common during strategy reporting for a teacher to ask questions about a student's strategy when there is a mistake in an effort to uncover a misconception and work through an error. In talking about this experience, Norah said, "Sometimes I don't like to make mistakes because it's kind of embarrassing when you thought you got it right and then you got it wrong and then you have to keep working out loud" (Hintz, 2011, p. 268).

Norah explained she was happier not to be called on when her thinking was incorrect. It does not seem, in the way she recounts her experience, that she does not like revising her thinking. But, she feels badly doing it in front of others. The teacher, like many of us, tries to normalize mistake making. Classrooms benefit when a norm is established that it is okay to be wrong. And certainly we do not have evidence other students made Norah feel badly. Still, her worry was real and impacted whether or not she wanted to participate. And Norah's feelings about sharing and listening are not unidimensional. She also told Hintz that she liked hearing and using other students' strategies, and that she was comfortable trying out multiple strategies until one worked for her. If the teacher also understood the complexities of how Norah felt, what kind of dialogue between them could help Norah process these experiences and grow from them?

Amanda Jansen's work (e.g., Jansen, 2006, 2008) similarly focuses on students' experiences with participating in classroom mathematics discussions. In this work, she identifies relationships among middle school students' beliefs, motivation, and participation in whole-class discussions. Through in-depth interviews with students and many hours of classroom observation, she identified student beliefs that both constrained and supported student participation in whole-class discussions. Some students she interviewed echoed Norah's feelings about the risks of verbal participation and the benefits of listening during mathematics discussions.

Students in her study also demonstrated motivation to participate more actively in order to meet social or behavioral goals, such as helping others. In particular, these students believed in the value of participation for supporting their own learning and understanding of mathematics. These views were exemplified by Becky:

> Becky: And when you have to do problems, don't just sit there. You have to get into the conversation in order to actually get it yourself and make you understand it, don't just understand it like how other kids do it. (Jansen, 2006, p. 417)

Consider Becky's perspective and Norah's side by side. One common practice in mathematics classrooms is to collect answers and then ask students to share those answers. While many conceptions of and research on ambitious pedagogy focus on the importance of discussion for learning, it is important to also recognize the risks of participation that can be inherent in participation for some students. Further, while teachers may work on class-wide strategies for encouraging participation, we need to understand that these strategies will be received and understood differently by different students, depending on their prior experiences, beliefs, and goals. How often do we learn about what our students are experiencing in the classroom? How can we create the time and gain the trust for students to tell us? Taken together, Hintz's and Jansen's work remind us that participation in mathematics discussion can take many forms and that understanding the ways in which listening, questioning, and other forms of participation support learning is important work. Finally, Jansen's findings illustrate the deep interconnectedness of students' social and academic goals and beliefs in shaping students' access to and participation in mathematics.

Understanding Students' Participation and Positioning as Racialized and Socially Networked

In the next section, we continue to learn from and about students' experiences in mathematics classrooms with a focus on the social aspect of participation as racialized, gendered, networked, and closely connected to mathematics access and achievement.

Maisie Gholson's and Danny Martin's work (2014) takes a "microsociological (e.g., Shalin, 1978) approach" to understanding Black girls' experiences in a mathematics classroom, "using the girls' voices in this study to make sense of the emergent social structures that organize access to mathematics participation and learning." (Gholson and Martin, 2014, p. 19). Through this approach, they identify the shifting roles, identities, and social networks within a 3rd-grade classroom that not only affect the girls' social identities, but also shape their access to and success in the classroom mathematics. They find that even those students who identified as "competent mathematics students" found their access to participation in mathematics class "mediated" by their positioning within the social network of the classroom, especially their positioning in relation to the "high-status cluster" within the network. (p. 30). The close connection between social positioning and access to classroom content is illustrated through the story of Shawna, a strong mathematics student who identified as good in mathematics, but often found herself outside the "high-status cluster" of the classroom girls' social network:

> M(Gholson): Like if Ms. Robinson calls everybody to the rug, sometimes you'll sit at your desk or sit at the very back. Do you think that's true?
>
> S: [Nods affirmatively.] When she calls us to the rug, I'll stay at my desk sometimes.
>
> M: Why do you like to stay at your desk sometimes?
>
> S: Because I don't like to go to the rug.
>
> M: Is it good to sit away from the rug and get away from people sometimes?
>
> S: Yes.

M: How does it make you feel when you sit away from the group? Does it make you feel good? Does it make you feel bad?

S: Good.

M: It does? And why does it make you feel good?

S: Cause some of the people mess with me.

(Interview 03/01) (Gholson & Martin, 2014, p. 28)

Gholson and Martin go on to say that, "On the rare occasions when Shawna was included by one or more of the girls in a classroom activity, she was highly engaged." (Gholson & Martin, p. 28) Gholson and Martin conclude that, "It is not uncommon for reports of studies of mathematics learning to state explicitly that any talk not related to mathematics was excluded from the analyses. However, this necessarily dismissed children's social worlds as unimportant and misunderstands the intimate connection between children's learning of disciplinary content, such as mathematics, and their social relationships." (p. 31).

In more recent work (Gholson & Martin, under review), the authors use a performative framework to analyze classroom video and student interview data in order to understand how one student, a Black middle school girl, positions herself within mathematics class. They focus in particular on the movement of bodies within the mathematics classroom space and illustrate how the performative lens allows us to see the ways in which "mathematics learning is a contextualized performance, requiring and enabling children to simultaneously negotiate race, class, and gender." (p. 4) In doing so, they illuminate the ways in which mathematical practices, described abstractly in documents such as the CCSSM, are realized through embodied performances.

Gholson and Martin's work helps us understand how we can miss salient aspects of students' engagement with and access to mathematics when we focus only on overt teacher and student behaviors and only on students' interactions with and identities in relation to content. Understanding students' positioning with respect to one another, as well as to the content, provides a lens for making sense of patterns of student participation. At the same time, Gholson's and Martin's long-term and deep interactions with students lead to compelling narratives of the personal, social, familial, and community contexts in which students' mathematical development is situated. We wonder if and how teachers can engage in similar long-term and deep interactions leading to the co-construction of student narratives. Finally, Gholson and Martin point to the ways in which students' classroom and network positioning is situated in broader narratives of race, gender, and ability in mathematics classrooms. These narratives are the focus of the next set of studies described below, by Shah, Lewis, and Rubel.

Understanding Students' Experiences of Narratives of Race, Gender, and Ability/Disability in Mathematics Classrooms

The classroom, while a community onto itself, is of course constitutive of the outside world. Shah (2017) interviewed 35 high school students across 4 classrooms who went to the same high school in Northern California. They identified as Asian, Black or African American, Latinx, Polynesian, White, and mixed race. He was interested in how students invoked racial narratives when they talked about mathematics learning, how these narratives worked in relation to one another, and what these relationalities meant for how students were positioned with respect to race. A primary interest in his work is to study how students make sense of these racial narratives and how it contributes to the shaping of racial ideologies and students' own identify formation.

It is not surprising that the adolescents he talked to reported hearing numerous racial narratives at school and that they were in line with what Danny Martin (2009) has called a racial hierarchy of who is good at mathematics. Within his data corpus, students invoked 98 different racial narratives over a broad range of topics from intelligence, academic performance, and ability to body type, personality, cultural practices and career paths. Shah examines how various racial narratives and their interconnections impacted their everyday experience in the mathematics classroom. For example, at the time of the study, Troi was a higher performing senior in an advanced course of Precalculus. Troi's statements about how a substitute teacher's might react to him conveys how he is aware of the racial hierarchy in mathematics with respect to how Indians and Samoans are perceived in mathematics, "'Yeah, so say a substitute teacher would come in [to class] and she'll see the Indian kid and think, 'Oh he must be the best one here in math,' and she'll look at me and think, 'How did he get into this class? What the heck is he doing here?'" (p. 23). Moreover, as Shah explains in this next excerpt, for Troi, narratives about intelligence were linked to narratives about physicality and mathematical ability:

> Polynesian students at Eastwood High were a small but prominent population on campus. Samoan and Tongan cultural practices were well represented in school events, and the Polynesian male students in particular were known for their participation in contact sports, such as football and rugby. Several of the faculty I spoke with viewed them as "troublemakers" and found them difficult to manage in their classes. In the excerpt below, Troi (Samoan, 12th grade) elaborates on how perceptions of Polynesian bodies and personalities contributed to their being positioned as mathematically, academically, and intellectually inferior:
>
>> Other students just see me as big and mean…and here [at Eastwood High], the Polynesian kids are seen as like we're big, that we do whatever we want. Like we're not very intellectual, and like we're not smart. But once they meet me they'll know that I'm actually very intelligent, and I can do math, I know how to do English, I can do science…all that kind of stuff. I think when I come in they just see me as someone who's going to hurt them or beat them up or someone who freaking wants to kill. They're not going to take time out to sit and talk with me, and actually greet me and actually get to know me.
>
> In this excerpt, Troi draws connections among multiple categories of racial narratives. Initially, he connects a narrative about Polynesians being "big" to narratives about Polynesians being seen as "mean" and "someone who's going to hurt them or beat them up or someone who freaking wants to kill." The relations among these narratives evoke an image of Polynesian students as angry and violent people that others should fear. Indeed, Troi implies that classmates tend to avoid him, and do not attempt to "greet me and actually get to know me." But Troi perceives these narratives to be consequential in ways that go beyond his social standing. They also matter for how Troi is positioned from an intellectual standpoint. After invoking narratives about Polynesians' body type and personality, he notes that people view Polynesians as being "not very intellectual." (Shah, 2017, p. 27-28)

Reading Troi's experience and the many others in Shah's article provides us with a compelling window into how students feel about the way they are read by others and the very real consequences for the educational opportunities they pursue or not. Implicit in Troi's description is a rather segregated social space. If students do not have genuine opportunities to develop friendships across cultural and racial groups, how does the lack of relationships figure into the ways they are asked to interact in the academic setting of the classroom? How do teachers check their own assumptions and views of particular racial and cultural groups in the school? Where do the perceptions about troublemakers get challenged?

Understanding how students with learning disabilities developing systems to compensate for cognitive differences. Katie Lewis studies the characteristics of mathematics learning disabilities. Drawing on sociocultural frameworks, she analyzes how students make sense of mediational tools such as symbols and representation when doing mathematics. Importantly, she tries to understand the resources that students use not what they seem to lack. One important turn in her recent work is to marry the Vygotskian notion of compensation with a critical disability studies frame. By collaborating with individuals with mathematics learning disabilities, she has been documenting the intentional actions that they take to gain access to spaces, context, and mediational tools in mathematics that would otherwise be inaccessible to them. This emancipatory research inverts the typical power dynamic between researcher and researched.

In a recent paper, Lewis and Dylan Lynn (2018) discuss the significant and persistent challenges Dylan, who graduated with a major in statistics from UC Berkeley, encountered when doing mathematics and how she compensated for them in order to succeed. Together they documented eight distinct compensatory strategies by analyzing videos of interactions between and another college student with mathematics learning disabilities and interviews with Katie. Dylan's challenges included inverting numbers, distinguishing symbols, making sense of dense notations, and understanding the impact of operations on values. She had developed her own system of addressing these challenges which included the use of mathematical tools such as graph paper, particular colored pens and pencils in ways that helped her navigate notations and solution processes.

We will give you one particular example of these compensatory strategies. An important aspect of understanding Dylan's experience was how our education system's policies were set up to exclude her and actively discourage her from pursuing mathematics. When she was diagnosed with a disability in college, the university's response was to waive her mathematics requirement. There was no real way for her to be supported to continue with mathematics, and she had to find her own way to persevere. She learned what to ask her tutors to do, and she had to persist through numerous course graders who complained about the length and verboseness of her assignments. In the excerpt below, she explains how one strategy of rewriting mathematical symbols into words supported her understanding of new concepts and notational system.

> Dylan: "This is calculus, but you can see it illustrated with this notation [*writes f(x) = x + 4x3; see Figure 2*] this notation, the way people say this is "f of x" which is also terrible. It's the *function* of *x* equals this [*as writing "function of x = x + 4x3; see Figure 2*]. This little notation here [*points to f(x)*] would throw me off really badly in my classes, because f times x? No, it's a notation that is basically applying this function to the variable x. I would sometimes write out something like this [*writes bracket underneath "function of x"*] right underneath whatever it was and again, this is really verbose, [but] it might be helpful."

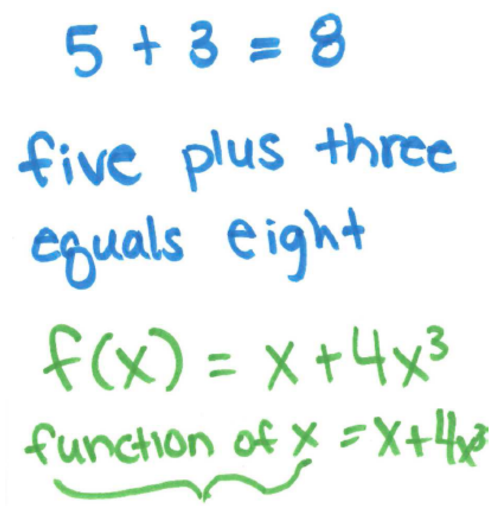

Figure 2. Dylan's examples of rewriting mathematical symbols with words, which helped her translate and make sense of the meaning of the notation.

> Dylan noted that she often used this kind of translation in her notes to help her decode the meaning of the symbols, and described it as her creation of metadata for the notation. ...
>
> This translation of symbols into words took Dylan extra time both when writing notes and when solving problems. Because this kind of translation was not available in her classes or textbooks, she paid tutors to provide this kind of support. She explained that "I would force tutors to give me the English words for the symbols, which was always funny because these are grad students who haven't thought about this stuff in years. 'How would you use this in a sentence? I haven't thought about it that way.'" The kinds of supports that Dylan needed were not something that the tutors were skilled at providing. Although this compensatory strategy provided her with a way of understanding the mathematical symbols, it placed additional demands upon her requiring that she spend more time and money than her peers to have access to mathematics. (Lewis & Lynn, 2018, p. 6-7)

One theme that is beginning to emerge in the selections we have made is how much students are doing and thinking about that is not available to the teacher. The full study documents many more strategies that Dylan generated to help her understand and advance in her mathematics coursework. Her brilliance and ingenuity are so clear, even though it appears that her college instructors are not empathetic to how much more work she does in order to be sure she understands.

Understanding students' gendered experience in the mathematics classroom. There are many issues with respect to gender and sexuality that we need to consider as mathematics educators. Gender narratives around mathematics have been typically associated with masculinity. In a brief article written for practitioners (Rubel, 2016), Laurie Rubel recounts her own experience as a teacher in a professional development session led by a mathematician who was engaging participants in how problems could be modeled with graph theory. He chose a regularly used problem because its purportedly binary categories would simplify the mathematics for learning purposes. But that is not how she experienced it, as she explains in this next excerpt.

> When I was a beginning teacher and in my twenties, I attended a professional development course for mathematics teachers, in which the Stable Marriage problem was explored as an example of a problem that can be modeled with graph theory. The facilitator, a professor of mathematics, led an activity similar to the one described above. He handed pink cards with fictitious names to the people he identified as women in the room and blue cards to the men. He told the women, holding the pink cards, to create rankings of their marriage preferences. If you were a woman, you were holding a pink card, and you were allowed only to rank your choices of men as spouses.
>
> I remember feeling uncomfortable with this arrangement. When I voiced an objection to this constraint, I was told that this is the set-up of the problem. In other words, this problem is not really about marriages. The problem refers to a particular kind of mathematical pairing between set A and set B. The story about men and women and marriages is just a story to lead us to a particular mathematical model. The story is supposed to help clarify the parameters of the mathematical model. "Just focus on the mathematics," I was told, even though I was being handed a pink card and thereby being placed in a particular location on a gender binary. Not only that, but heteronormativity was being reinforced with the statement that, in this model, all women have to want to marry men. (Rubel, 2016, p. 438)

There are several important ideas here that are important for our work with teachers. As a student in this context, Laurie, tried to speak out but was rebuffed. The teacher responded to her by admonishing her to just focus on the mathematics, making her own reaction to the problem irrelevant. So even though the algorithm they were studying had been applied to settings where college applicants are matched with colleges or medical students with residencies, the context of this problem was set in marriage between heterosexual couples, and students were not given a choice in their gender assignment or whether they wanted to use the marriage context for an extended discussion. Through her writing, Rubel pushes us to take up Rands' (2013) idea about gender-complex education, directly acknowledging gender diversity by making our curriculum and pedagogy reflect the existence of transgender and gender nonconforming people. If mathematics is a way of making sense of our world, it seems impossible to discount our world to just focus on the mathematics.

What Theoretical Frameworks are Researchers Drawing on to Study Student Experience?

A rich array of critical theoretical perspectives are used across these studies to help us interpret student experiences. Noting these are important for what theories we study in teacher preparation, doctoral preparation, and our own ongoing learning. Psychological, cognitive, and sociocultural theories of learning are likely to be insufficient in helping us understand students' mathematical learning. In this small collection of articles, scholars are drawing on theories that help us attend more to relations of power and how race, gender, class, ability, and sexuality shape these relations. Social theorists, philosophers, critical race theorists, disability studies, black feminist scholars, and poststructural theorists, to name a few, are being used to bring depth and complexity to our understanding of teaching, learning, schooling in how they shape students' learning, their identification with mathematics, their experiences as learners and the meanings of their education.

How Did Researchers Learn about Student Experience?

Narrative plays a central role in the studies we have highlighted here. Spending time with students over a long period of time to come to understand their varied experiences is common to them all. Interviews aided by video enabled Lewis and Hintz to have moment-by-moment interpretations of what students were thinking, and doing, and feeling. In his interview protocols, Shah asked students to comment on cartoons that conveyed narratives that broadly circulate in society. Rubel and Dylan use personal biography tell us their own stories, albeit in different ways. Rubel shares her own stories as a way of speaking out and speaking up. Lewis, in her collaboration with Dylan, provides an example of emancipatory research, which aims to transform the relationship between researcher and researched. Many of these stories are not readily shared by students with one another or with their teachers. So in some respects we can expect that many of us through our teaching would not necessarily have access to these stories. They demand then, that we think about the relationships we need to foster, and the kinds of interactions we need to have with students in order to better understand what is happening for them as they try to learn with one another.

What are the Implications of Thinking about Teacher Practice in Terms of Student Experiences?

Students' participation in and access to mathematics is not solely or even primarily about the student's mathematical competence or the teacher's moves. Instead, it is about the individual, social, and cultural narratives within which the student is positioned and positions her/himself. Therefore, any study of or work with teachers should include attention to ways in which teachers can learn about these individual, social, and cultural factors, along with how the work of teaching can respond to and/or disrupt their effects in ways that provide opportunities for greater access and participation for all students, particularly those who find themselves marginalized in classroom communities.

The cases we have selected here are not full of joy and delight and liberation while our goals for teaching and education purportedly are. Instead they are filled with tension, with challenge, with being unknown and unseen. Like Shah, we wonder, “Do all students have the opportunity to be seen for what they are truly capable of doing in a classroom?” (p. 36). But we think it is worth pondering how that can be difficult and perhaps not normative. The work discussed above has important implications for the work of teaching and our work with teachers. In our work as teacher educators, we must intentionally build time and space for teachers not to just reflect on their own teaching, what went well and what did not, what did they observe and notice in their classrooms, but also what did their students directly teach them. It leads us to ask how teachers learn about student experiences and how they respond to what they have learned. What did students think and feel and experience during a lesson? How did they feel they were treated? What was challenging to do and what was not? What enables students to develop enough trust to be in honest conversation with their teachers and their peers about how they are experiencing the classroom community?

It would be an understatement to say coming to learn about how students are experiencing the classroom takes a lot of skill and empathy on behalf of teachers and the ability to step outside of one’s own worldview, to suspend judgement. What do teachers need to know and be able to do to understand and engage with student experience? How can teacher educators support teachers in learning these things? Is it possible for teacher and teacher educators, given their positioning and authority in classroom spaces, to elicit and learn about student experiences in the ways that researchers do? In many of the studies described above, teachers were not aware of the student perspectives and experiences uncovered by researchers. Students' experiences remained hidden from teachers, either because the experiences are happening in students' worlds, where teachers may not belong, and/or because teachers and students are not having the kinds of conversations that researchers and students have had. Even if we acknowledge that it is not fully possible for teachers to access students' experiences in the ways that researchers are able to, how do we take what we are learning from these researchers about the emotional, motivational, social, structural, political, and identity-related aspects of students' experiences in mathematics and use these understandings to build mathematics classrooms and teaching practices that pull students in and increase access to mathematics rather than pushing them out and preventing access?

Despite all of these questions, we can begin to imagine how this work related to student experiences might transform the ways we work with teachers in teacher education and professional development settings. For instance, what if the focus of a mathematics methods course assignment or professional development experience was for teachers to deeply understand the experiences of a student who was different from them along one or more dimensions of identity? The goal would be to understand not just their knowledge and ways of thinking about mathematics, not just their home and community-based funds of knowledge, as has been explored in other projects (e.g., Aguirre et al., 2013), but the ways they experience participation, their positioning in the networks of the classroom and community, their relationship to broader racial, gendered, and ability-based narratives? What would this understanding motivate teachers to want to know and be able to do in relation to mathematics teaching? What further questions would they want to ask? What if teachers collected video and sat with a few students to get their take on what was happening in the classroom?

The work on students' experiences also has implications for studying teacher learning and practice. In fact, it was in the context of a project studying novice teacher practice that Elham (a member of the project's advisory board) suggested the focus of this paper. As we explored various protocols for studying (or measuring) teaching practice, we (the project team) asked Elham what we might be missing when viewing teaching through the lens of these protocols. She suggested that student experience was notably absent and asked the question at the center of this paper – What if we focused on student experience as well as teacher performance when studying teaching practice? How would that change the ways we study teaching or work with teachers to improve teaching? Some research projects have begun to move in this direction and we will be interested to follow the extent to which they are able to move the field forward in understanding teaching in terms of student experience. For example, the work of several researchers and partner districts on "practical measures of instruction" builds on ideas related to improvement science (Bryk et al., 2015) to incorporate quick and actionable measures of student experiences in class discussions into professional development and the improvement of teaching. Another example might be the work of Reinholz and Shah (2018) on "equity analytics" - quantitative measures of who is getting access to the mathematics and mathematical discourse during classroom instruction. While neither of these examples fully capture the richness of student experiences in the ways described above, they do provide tools and processes for teachers and researchers to gain some understanding of student experiences as they unfold in the context of instruction. These examples also suggest the importance of research-practice partnerships in both understanding and improving teaching through a focus on student experience.

In conclusion, we wonder about directions for our own learning. How do we continue learning about student experiences in mathematics classrooms and the ways in which those experiences are related to learning, to power and participation, and to dimensions of identity including race, gender, and class? How do we teach each other about these ideas and how do we support teachers in learning about, responding to, and enhancing student experiences in ways that promote access to rigorous mathematics for all students? An important aspect of the studies described above is that they each draw on theories that go well beyond the theories of learning we learned in graduate school. How will we and our doctoral students learn about these theories and/or work together to bring multiple theoretical lenses to these questions? By addressing these questions in collaboration with one another and with teachers, we can begin to make progress in understanding and supporting teaching that disrupts inequitable patterns of participation and provides access to mathematics for all students.

References

Aguirre, J. M., Mayfield-Ingram, K., Martin, D. B. (2013). The impact of identity in K-8 mathematics: Rethinking equity-based practices. Reston, VA: National Council of Teachers of Mathematics.

Aguirre, J. M., Turner, E. E., Bartell, T. G., Kalinec-Craig, C., Foote, M. Q., Roth McDuffie, A., & Drake, C. (2013). Making connections in practice: How prospective elementary teacher connect to children's mathematical thinking and community funds of knowledge in mathematics instruction. *Journal of Teacher Education 64*(2), 178-192.

Ball, D. L & Cohen, D. K. (1996). Reform by the book: what is - or might be - the role of curriculum materials in teacher learning and instructional reform? *Educational Researcher, 25*(6)*,* 8-14.

Bartell, T. G. (2011). Caring, race, culture, and power: A research synthesis toward supporting mathematics teachers in caring with awareness. *Journal of Urban Mathematics Education, 4*(1), 50-74.

Bryk, A. S., Gomez, L. M., Grunow, A., & LeMahieu, P. G. (2015). *Learning to improve: How America's schools can get better at getting better.* Cambridge, MA: Harvard Education Press.

Davis, E. A. & Krajcik, J. (2005). Designing educative curriculum materials to promote teacher learning. *Educational Researcher, 34*(3), 3-14.

Gholson, M. & Martin, D. B. (2014). Smart girls, Black girls, mean girls, and bullies: At the intersection of identities and the mediating role of young girls' social network in mathematical communities of practice. *Journal of Education, 194*(1), 19-33.

Gholson, M. & Martin, D. B. (under review). Blackgirl face: Racialized and gendered performativity in mathematical contexts.

Hintz, A. B. (2011). Understanding students' experiences as listeners during mathematical discussion. *Canadian Journal of Science, Mathematics and Technology Education*, *11*(3), 261–272. https://doi.org/10.1080/14926156.2011.595883

Jansen, A. (2006). Seventh-graders' motivations for participating in two discussion-oriented mathematics classrooms. *The Elementary School Journal, 106*(5), 409-428.

Jansen, A. (2008). An investigation of relationships between seventh-grade students' beliefs and their participation during mathematics discussions in two classrooms. *Mathematical Thinking and Learning, 10*(1), 68-100.

Jansen, A. & Bartell, T. (2013). Caring mathematics instruction: Middle school students' and teachers' perspectives. *Middle Grades Research Journal, 8*(1), 33-49.

Lewis, K. E., & Lynn, D. M. (2018). Against the odds: Insights from a statistician with dyscalculia. *Education Sciences*, *8*. https://doi.org/10.3390/educsci8020063

Martin, D. B. (2009). Researching race in mathematics education. *Teachers College Record*, *111*(2), 295–338.

Rands, K. (2013). Supporting transgender and gender-nonconforming youth through teaching mathematics for social justice. *Journal of LGBT Youth, 10,*106–26.

Reinholz, D. L. & Shah, N. (2018). Equity analytics: A methodological approach for quantifying participation patterns in mathematics classroom discourse. *Journal for Research in Mathematics Education, 49*(2), 140-177.

Rubel, L. H. (2016). Speaking up and speaking out about gender in mathematics. *The Mathematics Teacher*, *109*(6), 434–439.

Shah, N. (2017). Race, ideology, and academic ability: A relational analysis of racial narratives in mathematics. *Teacher College Record*, *119*(7), 1–42.

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LOOKING BACK, LOOKING AHEAD: EQUITY IN MATHEMATICS EDUCATION

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Drawing on the conference theme of looking back and looking ahead, in this paper I first look at the placement of equity throughout PME-NA as a way to set the stage for where to go next. I then focus on aspects of my work on the affordances and tensions around out-of-school and in-school mathematics to discuss equity as opportunity to participate in mathematically rich and socio-culturally grounded experiences. I argue for the need to develop classroom environments where students' mathematical funds of knowledge are brought to the forefront but also where students can use their cultural ways of being and acting as resources for their learning of mathematics.

Keywords: Equity and Diversity

At the suggestion of the conference organizers we (Laurie Rubel and I) decided to collaborate on the elaboration of the plenary and discussion / response papers. While we each wrote our own paper, we exchanged drafts and had online discussions on what we wanted to convey. We share some common interests and concerns in terms of equity and mathematics education, yet our personal histories, trajectories and approaches are different. In particular, in one of our first conversations I remember telling Laurie that sometimes in my own work I felt a tension between socio-cultural and socio-political approaches. I can see my work clearly fitting the socio-cultural framework, and I am aware that I am working with political issues (e.g., language policy, immigrant families), yet I do not necessarily see my work as "fitting" the political category. It is clear to me that issues of power and privilege permeate my attention to valorization of knowledge and participation in the mathematics classroom, but I feel like I leave these issues somewhat implicit. In her response, Laurie's section on "making the political explicit", drawing on one the examples from funds of knowledge that I present in this paper, was particularly inspiring for me. This is just one example of the back and forth exchange of ideas that Laurie and I engaged in as we each wrote our pieces.

In what follows I first provide some background to indicate my positionality with respect to the idea of "equity in mathematics education." Then, I provide a brief historical account of my experience with PME-NA, given that the theme of the conference is "Looking back, looking ahead: celebrating 40 years." This is my personal (and I admit, incomplete) attempt at tracing equity across PME-NA since the first conference I attended. The rest of the paper looks at some aspects of my research around in-school and out-of-school mathematics.

Some Background

I was drawn to issues related to equity through my work with preservice elementary teachers. I have always had an interest in how children and adults make sense of mathematics, how they think about mathematics. To this end, I like to use tasks that may lead to cognitive conflict. In listening to preservice teachers talking about mathematics, I noticed that some of them brought their everyday experiences to the discussion. I also noticed that oftentimes, those who sought to make sense of the mathematics (by connecting it to their life experiences) had had less "successful" trajectories with school mathematics than their peers who basically played by the rules and did not seem to be concerned about whether mathematics made sense outside (or even inside) the classroom. What are we doing in our teaching (K-16+) that leads to this lack of connection between in-school and out-of-school? This lack of connection is what drives much of my work. Shortly after I moved to the University of Arizona in 1990, I was fortunate to join the Funds of Knowledge for Teaching project (González, Moll, & Amanti, 2005), which was a perfect fit for me. The project took me into working-class communities and schools attended largely by students of Mexican origin, many of whom spoke Spanish as their home language, yet another fit for me as that is my home language too. I was able to work with teachers dedicated to developing learning experiences that built on students' and their families' funds of knowledge; teachers who invited family members to come to the classroom to share their expertise. I saw children engaged in discussions, participating in rich classroom activities. This is where my definition of equity developed. For me equity is about the opportunity to participate in mathematical experiences that are both rich from a "mathematics for the sake of mathematics" point of view ("reform / standards-based") and at the same time reflect the socio-cultural experiences of the participants (culturally responsive / sustaining). It is about participants maintaining their cultural identity while also engaging as doers of mathematics. Over the years, I think that what has most influenced my approach is my work and friendship with immigrant families. Learning from them and seeing their enjoyment and sense of humor in mathematical discussions (whether it is a group of mothers or a group of seventh graders) constitute uplifting experiences and are constant reminders of why I do this work. To me, this is particularly important currently, given the stressful and depressing reality that many immigrant families are experiencing.

Looking Back at PME-NA

I attended my first PME-NA when I was a graduate student in 1989. In looking at those proceedings, these were the topics: Affective and cultural factors in mathematics learning (2 papers, 1 on cognitive and affective aspects with 2 prospective elementary teachers; the other reports on a study done in Ciskei (South Africa, though an independent state at the time of the study); that paper mentions "socio cultural" and "lack of continuity between the cultural world of the family and that of school" (p. 13). The author was from a university in South Africa; Algebra/Algebraic Thinking (4 papers); Calculus (3 papers); Computer environments in mathematics learning (3 papers); number concepts (5 papers); geometry, measurement, and spatial visualization (6 papers); multiplicative structures (8 papers); representations, metacognition, and problem solving (6 papers); teacher beliefs (4) (my paper, "prospective elementary teachers' conceptions about the teaching and learning of mathematics in the context of working with ratios", was in that section); teacher education and teacher development (8 papers). There were two plenary lectures (and responses) (one on mathematical processes, the other one around understanding of numbers (the authors have a section on the research on out-of-school mathematics, in terms of how "non-schooled" children and adults understand numbers). There were five Symposia: realistic mathematics education; sex differences in mathematics ability; clinical investigations in mathematics teaching; assessment and function graphing tools; probability.

The next PME-NA I attended was in 1993. In that one, there were several strands including one on equity, which had a panel and discussion sessions. The panel has one Australian researcher (Gilah Leder), one US researcher (Walter Secada), and one respondent from Brazil (Ubiratan D'Ambrosio). There was one paper in the section on language and mathematics (by Judit Moschkovich). And there was a section on social and cultural factors affecting learning (5 papers), where one of my first papers on funds of knowledge was located ("Household Visits and Teachers' Study Groups: Integrating Mathematics to a Socio-Cultural Approach to Instruction").

There were 14 discussion groups; one of them was on “cultural support for mathematics understanding” (related to relationship between culture, language and numerical systems).

The strand on social and cultural factors (renamed later on as “sociocultural issues”) remained till 2007. Then in 2009 through 2011, there was a strand called “equity and diversity” (I am not including 2008 because that was a joint PME / PME-NA meeting). The strands we currently have (with no specific strand for equity or similar terms) were implemented in 2012. One possible argument for not having a strand on equity is that equity should permeate the work we do (Aguirre et al., 2017). Having a separate strand seems to imply that some researchers do equity work and others do not, or that equity is being addressed in one part of the conference and we do not need to worry about it elsewhere. This is something that calls for further reflection.

Finally, I took a more in-depth look at the most recent PME-NA proceedings (2017) to see how equity was featured. There was one plenary talk that focused on equity. There was also a response to another plenary (not-equity focused) that looked at that talk with an equity lens. Finally, one of the papers in the technology panel also addressed equity. Three of the 13 working groups had something to do with equity, with one of them being explicitly about equity, one on critical perspectives on disability, and the third one on special education. Two of the other working groups mention equity a few times in the write-up. I then went through all the strands and searched for the keyword equity. In some cases, where the term equity did not appear, I used my judgment to classify some of the papers as pertaining to equity based on other terms (e.g., culturally relevant; social justice). This is not a scientific analysis and I am aware that I may have missed some papers that are about equity. At the same time, there were some papers that had the keyword equity, but it was not obvious to me why that keyword was there. Here is what I found out: there were 75 research reports (RR) presented; 8 of them had something to do with equity. I counted 134 brief research reports (BRR), with 26 of them mentioning equity. Finally, I counted 141 posters, with 22 mentioning equity.

While over the years I have attended quite a few PME-NA conferences, I have also skipped several of them here and there. For a while, I felt that PME (rather than PME-NA) was more my community. At PME, I always seemed to find several presentations, working groups, discussion groups that related to my research interests in equity, while that was less the case with PME-NA. And yet, even in that more international arena, I should note that there was dissatisfaction with the attention to equity, in particular to social and political issues. In 1996, at PME in Valencia, I recall a fascinating AGM (Annual General Meeting) that discussed dropping the “P” from PME to reflect the fact that many research papers had moved away from the Psychology focus. Shortly after that, in 1998 Mathematics Education and Society (MES) was created in great part as a counter-space to PME (Gates & Jorgensen (Zevenbergen), 2015). In 2000, Lerman’s influential chapter for the field, “the social turn in mathematics education research” was published (Lerman, 2000). In 2004, the book edited by Valero and Zevenbergen (Jorgensen) on the socio-political dimensions in mathematics education research was published (Valero & Zevenbergen, 2004). Of course there are several other researchers who have written on these issues since then. But for me those are two pivotal pieces. For my own work, Lerman’s chapter is particularly relevant as it refers to the influence of Vygostky’s work, which is central to the program of research around Funds of Knowledge (González et al., 2005); that chapter also discusses situated cognition and mentions ethnomathematics, all of which are at the center of my long term interest in studying the affordances and tensions around out-of-school and in-school mathematics. In what follows, I turn my attention to this topic.

Navigating Out-of-school and In-school Mathematics

In the 2006 PME-NA plenary (Civil, 2006), I trace my trajectory from mostly a cognitive preparation to a sociocultural approach in my work, where concerns for equity became central. I am bringing this up here because the cognitive aspect is still very present in my work, but at the same time, I cannot interpret data (a video, students' work) without wondering about sociocultural elements (who is involved? What are their stories?). I argued then and I still argue now that we need these two perspectives (and most likely others too) to make sense of the teaching and learning of mathematics. In fact, what I wrote then is still very present in my thinking now:

> Sometimes I wonder if I have moved away from my initial cognitive-based interest in research in mathematics education to address issues that focus largely on the social and cultural context, with mathematics playing a very peripheral role. As I look over my writing from the last few years, I notice that I often raise the question "where is the mathematics?" Mathematics plays a central role in my work and recently, in our current project, I find myself pushing for the mathematics in our activities and research discussions. (p. 30)

Thus, in this paper I am continuing this thread by focusing on three key elements in my work: funds of knowledge; valorization of knowledge; and participation. As I wrote in 2006, "A concern for those who are being left out of the mathematical journey seems to guide my work" (Civil, 2006, p. 30). This concern has not changed.

Funds of Knowledge

Since the terms "funds of knowledge" is now so widely used in mathematics education research, I thought that providing some history may be useful. Anthropologists Vélez-Ibañez and Greenberg (1992) are credited to have introduced this term, as they write, "strategic and cultural resources, which we have termed *funds of knowledge*, that households contain" (p. 313). Through their collaboration with educational researchers (in particular, Luis Moll and Norma González), the project Funds of Knowledge for Teaching (FKT) was developed in Tucson in the 80s (see González, at al., 2005, for a detailed account of this project). When we bring these ideas to mathematics education, what we are saying is that all communities and families have mathematical funds of knowledge. Children come to school with mathematical funds of knowledge. Yet, as we well know, whether these funds of knowledge are recognized and used as resources for learning varies greatly. How did Alberto (Civil 2016; Civil & Andrade, 2002), a recent immigrant, see himself as a mathematical learner in his fifth-grade class, as he kept largely to himself and was not encountering success? A concerned teacher did not leave it at this and sought to learn more about Alberto and his family, through a funds of knowledge household visit. In that visit she learned about Alberto's unwillingness to leave Mexico and move to the US with his family. He left behind places, people, and activities he enjoyed, including actively helping out with his family's bakery business. He had his set of customers and was in charge of all monetary and goods transactions, yet at school he was struggling with "basic" arithmetic? Alberto's case reminds me of the studies on street mathematics (e.g., Nunes, Schliemann, & Carraher, 1993), which were very influential in my work. If we do not see the relevance in what we are being asked to do, if we do not have an affective connection, is it surprising that we may not do as well?

What about the several children (mostly in grades 5-8) who told me in interviews that they were learning things in mathematics that they had already learned in prior years in Mexico, yet I saw no evidence of them being given more challenging tasks, and in fact sometimes they seemed to be placed at a lower level because they did not know English well yet? This school knowledge (e.g., different algorithms) and learning habits (e.g., use of a notebook for each subject to record their work; bringing their own school tools instead of the school providing them) that they bring from other contexts are part of their funds of knowledge, but schools do not necessarily know of these. The lack of communication between families and school and the differing understandings of each other's roles and expectations are two of the most common findings in my work with immigrant parents.

An example of funds of knowledge in a parents' workshop. Elsewhere (Civil, 2002; 2007) I have discussed examples of applications from funds of knowledge to the mathematics classroom. Here I want to share a brief example (Menéndez & Civil, 2009) of how an activity on comparing fractions became more meaningful when a father participant suggested a connection to wrenches. While I do not know if in this case their children were familiar with how wrenches work (in terms of the different sizes), based on my experience of many years working with families, I would not be surprised if indeed several students at that school had a familiarity with wrenches. Comparing fractions is a typical school activity that can be challenging for children (and for adults, as we have seen in the Math For Parents courses that we have run for several years). In this scenario, the facilitator had asked the participants (most of them mothers and fathers of students at that middle school) to compare ¾ and 6/8. One of the men (Isidoro) successfully drew some pictures to show that they were equal. Another man (Marcos) then mentioned something about wrenches and how they have different measures. The facilitator encouraged both men to bring the wrenches to the next meeting. Isidoro brought "a few wrenches" (people laughed when he said that he had only brought a few of them, as he had about 15 wrenches on display) (see Figure 1). Isidoro very confidently explained what the standard measures are for the wrenches and the facilitator recorded those on chart paper, as Isidoro was mentioning them: 1/4, 5/16, 3/8, 7/16, 1/2, 9/16, 5/8, 11/16, 7/8, 13/16, 15/16, 1) (the facilitator noticed that 7/8 and 13/16 were switched but did not mention anything at that poing); Isidoro commented that there were other wrenches but that these were the most commonly used. Marcos noticed that the ¾ was missing and Isidoro told the facilitator to put it between the 11/16 and the 7/8. Isidoro did not look at the wrenches to see the size, he seemed to have those visualized and knew their ordering (despite the error in the list). The facilitator then said, "I'm not completely convinced that these (the fractions on the chart paper) go like this, in this order. It's just that I have to believe it because I don't know (participants laugh). Or is there a way to find out?"

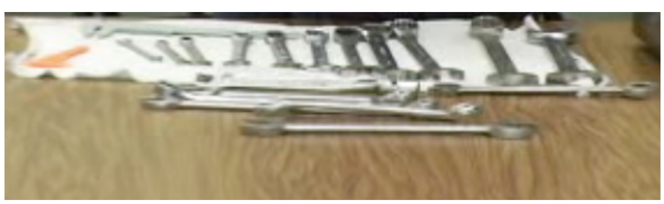

Figure 1. Isidoro's wrenches

The facilitator then encouraged the group to come up with a visual way to help him see how to compare ¼ and 3/8 and 7/8 and 13/16. The participants had grid paper and used this to come up with a visual approach to compare the fractions and resolve the issue with the ordering of 7/8 and 13/16. At this point the activity is a typical school task with the participants representing the different fractions on graph paper and comparing them. But the familiarity with the wrenches provided a context for this activity. The participants remained engaged, and while there were clear gender aspects with the men appearing as experts, some of the women asked questions, probed, and made comments, indicating they were engaged in the task too. There are probably several other mathematical explorations that could use these participants' knowledge of wrenches as contexts. For example, Isidoro and Marcos referred to their knowledge of the metric wrenches. Familiarity with the metric system is one aspect that has come up in other contexts (e.g., recipes). I have noticed students bringing in a knowledge of the metric system either from home or from having lived in Mexico and yet teachers teaching it as if it was new material for everybody.

It could be argued that the wrenches' example is a superficial or contrived context to promote a deep understanding of the mathematics behind comparison of fractions. This is a tension that I have expressed before in my work on funds of knowledge and mathematics (Civil, 2007; González, Andrade, Moll, & Civil, 2001). In Civil (2007) I refer to this tension as preserving the purity of the funds of knowledge at the expense, maybe, of the mathematics. This tension is related to what we decide counts as mathematics.

Valorization of Knowledge

I argue that questions around which mathematics for whom and for what purpose are at the center of any equity debate. I am encouraged by the current discussions around school detracking and, at the college level, around the efforts to create more avenues for students' access to mathematics beyond the standard college algebra (in a traditional sense), precalculus, and calculus path. But I am also aware of the potential obstacles to those initiatives, obstacles mostly based on what we count as a "good" mathematics education and who can "really" do it. I work in a mathematics department and this discourse permeates how we talk about the content of the courses, what needs to absolutely be in these courses to be a mathematics course, and who takes which courses.

For me, this is a personal issue because in my own teaching and research I find myself going back and forth between the need to engage students in activities that are culturally relevant (e.g., funds of knowledge based) and the need to engage students in rich mathematical learning experiences. I do not mean to imply that this is an "either – or" situation. Obviously, both can happen. But I do not find that easy to accomplish. Researchers engaged in social justice mathematics teaching report a similar tension between the mathematical and the social justice goals (Atweh & Ala'i, 2012; Bartell, 2013; Rubel, 2017).

My recent conversations with teachers and colleagues around after school projects with middle school students and mathematical modeling of culturally relevant contexts with teachers bring up these questions for me: is the context taking over the mathematics? Is the mathematics superficial or contrived? In a chapter on modeling and culturally relevant pedagogy, we write, "Teachers may encounter some tension between incorporating authentic cultural knowledge into the modeling process while staying true to the goals and modes of analysis of the discipline of mathematics" (Anhalt, Staats, Cortez, & Civil, 2018, p. 326). Elsewhere (e.g., Civil, 2002; 2016), I comment on the difficulty in seeing mathematics in culturally-based activity when our only lens may be that of "academic / formal" mathematics. In these cases we may not be able to appreciate the mathematics in the activity or we may risk trivializing both the mathematics and the activity.

Throughout my current work and teaching, I often bring up the famous question, "where is the math?" Yes, I do ask this question in part probably due to my own view of what counts as mathematics, but in part too because of the many classrooms I have visited where students are not being challenged in mathematics and are subjected to what I would consider quite dry and uninteresting tasks. Most likely, tasks that are related to their funds of knowledge would be more engaging, but they need to also be mathematically engaging. I also raise this question because I have seen these same students engaging in mathematical discussions of tasks that are largely what I would describe as rich tasks, but not necessarily culturally relevant. In the next section I discuss the concept of participation and in particular the importance of developing classroom environments that let students use their cultural ways of being (which includes the use of their home language(s)) as they do mathematics. So, while the tasks themselves may not have been culturally relevant, developing trusting relationships and letting the students use their cultural resources (language, humor, interaction style) seem to support their engagement with mathematics (Civil & Hunter, 2015).

Participation

The idea of participation has been central to my work for many years. For example, in Civil and Planas (2004) we look at "the effects of social and organizational structures on students' participation in the mathematics classroom" (p. 8) in two different contexts, Tucson and Barcelona. In other pieces, I have looked at the effect of language policies on the participation of emergent bilinguals (English Learners (ELs)) in the mathematics classroom (e.g., Civil, 2011) and of immigrant parents in their children's schooling (Acosta-Iriqui, Civil, Díez-Palomar, Marshall, & Quintos-Alonso, 2011). In Civil (2012, 2014), I look at participation and issues around what language gets privileged? Whose experiences are represented in the tasks? Whose knowledge and approaches get valued. Finally, in Civil and Hunter (2015) we look at immigrant students' participation in argumentation in the mathematics classroom through lenses of culture and language in two different geographic contexts, New Zealand and the US.

In this section I present yet another example from the same classroom discussed in Civil, (2011, 2012), and Civil and Hunter (2015) to illustrate how having an atmosphere where students can basically be themselves, can lead to rich mathematical discussions and students' participation. The setting is a small seventh grade class composed of only 8 students, most of whom were recent immigrants from Mexico (within the previous two years) and all classified as ELs. Elsewhere I have discussed the restrictive language policy in Arizona (Civil, 2011; Civil & Menéndez, 2011) that places ELs in basically segregated classrooms for most of the school day. This was the case for these students. I worked with the teacher (an EL herself) and the students for close to a year. We videotaped 30 class sessions from February through May. It is important to note that three of the students' mothers regularly attended the mathematics sessions for parents (and their children) we had at the school (Civil & Menéndez, 2011). The teacher also attended those sessions. Thus, we had developed rapport not only with the students but with some of the parents too. I have been arguing for quite some time for the importance of developing stronger and trusting relationships between home and school, particularly in the communities where my work is located, where families may be less familiar with the school system or worse, where sadly, they have reasons to feel insecure and less trusty of organizations.

By the time formal data collection began (videotaping) the students were starting to become used to the idea of discussing their work and having to justify their thinking to others. We let the argumentation develop naturally, that is we did not use any norms or roles. We basically relied on tasks that would create situations that promote argumentation. For example, in interpreting a distance / time graph of a bike trip, students engaged in spirited discussions arguing about which part of the graph showed the most progress made by the bike rider (Civil, 2012). When showing the video clips to varied audiences, while some do appreciate the level of engagement and mathematical argumentation that is taking place, others are somewhat surprised by the loudness and "chaotic" looking and sounding discussions. Yet, this "chaos" and "loudness" allowed for a student like Octavio to find his mathematical voice. Octavio was somewhat quiet and did not seem very interested in engaging with mathematics. But it turns out that he liked to argue and that gave him an entry into the mathematics. As we encouraged students to talk about mathematics, we discovered a new (to us) Octavio. As he said in an interview towards the end of the year:

> **Marta:** What is it that you like most about math class, well if there is something that you like, of course?
> **Octavio**: To argue
> **Marta:** And why do you like to argue?
> **Octavio**: Because I feel, I feel like a good student when I think that the answer is right.
> **Marta**: What other things do you like about the class?
> **Octavio**: To chat and do work with my group.
> **Marta**: Did you work in groups last year in math class?
> **Octavio**: No, we worked individually.

(All the transcripts in this paper come from exchanges that took place in Spanish. For reasons of space I have only included the English translation. I know that this is unfortunate as we miss the idiomatic turns and the richness of the speakers' home language.)

The example below shows different features of how students (and us) engaged with mathematics, such as use of humor and teasing and use of their home language (Spanish). It was a relaxed atmosphere. Students had been working in small groups on a problem on planning a class party with three options (going to a pizza place and movie theater; going to a water park; or to a skate ring, (Preston & Garner, 2003)). They were given some information on the cost of the three options and the students were to decide which option may be best and why. Carlos and Larissa are at the board to explain how they used equations to find how the cost of the water park ($W = 100 + 5 P$) and the cost of the skate ring compare ($S = 200 + 2 P$). So what they are going to solve is: $100 + 5 P = 200 + 2P$. They have just subtracted 2P from both sides and have: $100 + 3P = 200$.

1. **Carlos**: There it is. Here we take minus one hundred.
2. **Octavio:** But why?
3. **Marta**: Octavio is asking why
4. **Carlos**: Why did we take minus hundred? [smiling]
5. **Octavio**: Yes.
6. **Marta**: Yes, he is the one asking it. I wasn't asking it, he's the one who asked it
7. **Carlos**: Because that's what we have to take away. Because here we subtracted minus one hundred, and here, we also subtracted minus one hundred.
8. **Marta**: No, it's a very valid question.
9. **Octavio**: Ah, yes, okay, it's fine, it's fine.
10. **Carlos**: Then here you get one hundred and here you get three P.
11. **Octavio**: Three P? [with a tone of surprise]
12. **Simón**: Why?
13. **Octavio**: Why?
14. **Lucas**: Why?
15. **Carlos**: Because I subtract one hundred.
16. **Larissa**: Because we subtracted one hundred.
17. **Carlos**: And you get three P.
18. **Ms. Adams**: But what was the reason for subtracting one hundred?

19. **Larissa**: We wanted (incomprehensible) to know for--
20. **Octavio**: Positive. Why negative?
21. **Carlos:** Be quiet. [Softly and smiling in the direction of Octavio]

In this brief exchange we hear Octavio probing four times (lines 2, 11, 13, 20). He is following the explanation and wants to make sure that he understands what they are doing. This is quite different from other cases where students are presenting at the board but the rest of the students are not really engaged. After that, Larissa and Carlos deciding how to solve for P because that is one thing they had not done prior to coming to the board to explain. They end up with P = 33.333…There is also some joking around because the teacher and I ask them to erase what is in blue from the white board (from a previous exercise) so that they can have more room to show their work and Larissa points to the top of the board where part of the date is written in blue. This is sort of an inside joke because from when I started coming to their class I was asking them not to erase their work so that I could see how they were thinking, but it took a while for students to let go of their attachment to the eraser. And this time I was telling them to erase, so Larissa picks up on that.

Next, the teacher and I asked Larissa and Carlos about the meaning of having found P to be 33.333.

1. **Carlos**: That one P is equal to 33.33333.
2. **Marta**: Yes, but what does that mean?
3. **Carlos**: Because we divided it.
4. **Marta**: No, no, what does it mean in the problem?
5. **Ms. Adams**: What is P? What is P? People…
6. **Marta**: What does P represent?
7. **Larissa**: People.
8. **Carlos**: One person.
9. **Ms. Adams**: Okay—
10. **Carlos**: One person is going to pay 33…for the--
11. **Octavio**: Why 33 dollars if it's two dollars per person?
12. **Carlos**: It's cause, nosy [*mitotero*] [to Octavio, as implying stay out of it, in a joking way].
13. **Larissa**: It's cause it's wrong [slightly laughing]
14. **Marta**: It's a good thing that Octavio is, is, really on the ball, eh?
15. **Larissa**: Yes.
16. **Marta**: He's absolutely right
17. **Larissa**: This is the number of people that can go.
18. **Carlos**: That's why!
19. **Octavio**: Ah [incomprehensible; some laughter]
20. **Carlos**: That's what I was saying. [smiling]
21. **Larissa**: That's why, it's not the price! [smiling]

Larissa and Carlos are confused about what the P represents. As soon as Carlos says that it is the cost of one of the activities, Octavio jumps at that (line 11), since the cost of the activities are \$2 per person for the skating and \$5 per person for the water park. In line 12, Carlos uses a cultural term "mitotero" in a joking way to basically tell Octavio to stay out of it ("mitotero" in Sonora, Mexico (which is where many of the families in my context come from) means "gossipy" / "nosy").

Finally, once they have agreed that P is the number of people, and they decide on P=33, the question of how much it is going to be comes up.

1. **Ms. Adams**: And how much would it cost?
2. **Larissa**: I mean we wanted to see, I mean we wanted, cause I mean we wanted to see--
3. **Carlos**: The exact point.
4. **Ernesto**: How much would it cost?
5. **Octavio**: So how much would 33 people cost? (pause- someone else says something) How much? Tell me, well!
6. **Carlos**: How mu—33 people?
7. **Marta**: Do it, do it, all of you --
8. **Carlos**: Ah well, here it is [looks at his notes again]
9. **Ms. Adams**: Let's see, that table--
10. **Marta**: You can all do it, eh?
11. **Carlos**: In the graph.
12. **Marta**: How much does it cost for 33 people?
13. **Carlos**: You can all do it. [repeating what Marta has said / teasing]
14. **Ms. Adams**: How much does it cost for 33 people to go?
15. **Marta**: All of you, yes, yes.

Once again, we see Octavio engaged and asking in a challenging tone to tell him how much it would cost for 33 people (line 5). I then turn it over to all of them and suggest that they all figure it out (line 7 and again line 10). In line 13, Carlos repeats my saying "you can all do it" in a teasing tone. Perhaps some could interpret his repeating what I said as mimicking me and not being respectful, but that is not how I took it at all because it was part of our interaction style. I had known Carlos since the year before; he and his two siblings came regularly to the mathematics workshops for parents with their mother. We had developed a rapport over the two years.

The concepts of "confianza" (trust) and family feeling are often mentioned in research with Latinx communities (González, et al., 2005; Rodríguez-Brown, 2010). The importance of building relationships among students and teachers and more broadly, among school personnel and families has been extensively documented in educational research with "diverse" students (e.g., Gay, 2000; Nieto, 2013). This importance has also been documented in the teaching of mathematics in non-dominant communities (e.g., Berlin & Berry, 2018; Guerra & Lim, 2017; Id-Deen, 2017; Kitchen, 2007; LópezLeiva, Celedón-Pattichis, Pattichis, & Morales, 2017; Martin, 2009; Musgrove & Willey, 2018). What I just presented is one more example of something we have known for a long time: relationships matter. If my view of equity is about the opportunity to participate, we need to develop an environment where students are going to want to participate. I saw this happening in that 7th grade class, and I also saw it in another school, first in a 4th/5th grade combination and then the year after (with the same teacher) in a 6th grade class. While the two settings were different, one aspect in common was a feeling of being a family and to a certain degree, the teachers acted almost as if being a family member. Allowing students to be themselves, to walk around the room and see what others were doing when working in groups, to tease each other (including me), all of this seemed to contribute to developing a safe and supportive environment where students were willing to take mathematical risks.

I have been wondering for quite some time whether what we are missing when working with minoritized students in school is to bring in their home ways of being and acting. While in the work in the 7th grade class, I may have seen glimpses of that, I am certainly not claiming that we succeeded, it was just that, glimpses. In Civil (2016), I argue for the need to gain a better understanding of how people engage in out-of-school settings in practices that are potentially mathematically rich and see how these ways of engaging relate to how, for example, mathematicians engage in the practice of mathematics. Nasir, Rosebery, Warren, and Lee (2006) argue that as learners we navigate through a variety of repertoires of practice as we move through different settings (e.g., home, school, clubs, groups that we may belong to). In looking at "the intersections between everyday practices and important disciplinary knowledge" they claim that "educators can use the varied and productive resources youth develop in their out-of-school lives to help them understand content-related ideas" (p. 493). I wonder, is school helping or hindering connecting these two practices, everyday practices and disciplinary (e.g., mathematics) practices? When we develop learning activities that build on students' funds of knowledge but also engage them in rich mathematics, is it school mathematics that we are working with? Is it disciplinary (mathematicians') mathematics? Should it be something else? In discussing the possible connections between different forms of mathematics (e.g., everyday mathematics, school mathematics, mathematicians' mathematics), Nemirovsky, Kelton, and Civil (2017) point out that schools can only bring in "real world" problems to a certain point since school has its own constraints and after all the students are not really engaged in that real world problem that often serves mostly as a scenario to address school mathematics.

I started this section discussing participation but the questions I just raised relate back to funds of knowledge and valorization of knowledge, thus bringing me back full circle. I want to close with some further thoughts on these ideas in part inspired by my current work as well as what I still see as challenges in the field when it comes to equity research.

Next Steps?

There are quite a few people now doing work in mathematics education building on the concept of funds of knowledge. This is certainly very different from the early 90s. Some of this work is informed by a variety of theoretical frameworks, which makes for a more robust account of the research efforts. In this sense, we are making progress, as researchers build on the concept and take it into different directions, contributing to the deepening of "the field's knowledge base related to equity-based research" (Aguirre et al., 2017, p. 125). I think we can all agree that there seems to be more attention given to equity in mathematics education in recent years. Whether it is because some conference proposals ask for an explicit connection, or whether it is an expectation of funding agencies, or whether it is that there seem to be more researchers coming out of doctoral programs (and some NSF-funded Centers for Learning and Teaching) where equity is central to their preparation, I think that there are more people engaged in equity-related research activity than when I started my career. In Aguirre et al., we argue for the need to make equity part of research in mathematics education, no matter what our main topic of research may be. That is, we call for the need to make equity part of our research as an "intentional collective professional responsibility" (p. 128). In looking at the four political acts discussed by Aguirre et al., I see equity as central to my work (Political Act 1). As for Political Act 2, I have occasionally felt tokenized as the "equity expert" on projects but I would rather have that than the second approach described in that political act, which is having researchers who are not grounded in equity work provide superficial attention to equity issues. I believe that as researchers whose expertise is in equity, we have a responsibility to support others who want to do this work but may not feel knowledgeable. In looking at some of the recent PME-NA proceedings as well as listening to a variety of talks in diverse conferences, I wonder about how widely "equity research" can be interpreted and whether we may be going in the direction that everything we do addresses equity somehow. Is that the case? Do we risk watering down equity?

Where is the mathematics and what counts as mathematics are discussed in Political Acts 3 and 4 (Aguirre et al., 2017) and are the center of my work, as I have addressed earlier in this paper. While I agree that to challenge equity work with the question of where is the mathematics can be problematic and creates a separation that is not (or should not) be there, the reality is that I have raised that question myself quite a few times, about my own work and about those of others, as I mentioned earlier in this paper. Is it mostly related to my views of what counts as mathematics? Are we doing enough in our writing and in our work to bring the centrality of equity to mathematics and the centrality of mathematics to equity?

In my current work, I continue to explore questions centered on equity as opportunity to participate and I wonder about the potential for K-16+ "formal" learning (as in a school / college class) of looking at how people learn in everyday life or in informal mathematics settings (see Nemirovsky et al., 2017 for more on the distinction between mathematics in school, in everyday life, and in informal settings). In particular, as I have discussed elsewhere (Civil, 2007; 2016) learning in everyday settings often takes place through participation in the practice, often by observing first and then engaging with the activity. Lipka, Sharp, Brenner, Yanez, and Sharp (2005) describe the case of a Yup'ik teacher, Nancy Sharp, using a Yup'ik approach to learning (apprenticeship; observing and participating in the practice) to work with her students in mathematics. Similarly in their work with teachers of Maori and Pāsifika students in New Zealand, Hunter and Anthony (2011) refer to how teachers draw on their "students' concepts of collectivism to develop communal responsibility" (p. 6) and build on these strengths to engage students in mathematical discussions.

I am intrigued by the potential for mathematics teaching of building on students' cultural ways of being and acting, as I illustrated briefly with the 7th grade example earlier and elsewhere (Civil, 2011; 2012; Civil & Hunter, 2015). Rogoff (2012; Rogoff et al., 2017) has been studying how children of Indigenous origin learn in communities in Guatemala and other places in the Americas, including some Mexican-origin children in the US, and contrasting these ways of learning to those of children from middle-class families, mostly of European origin. An example of this contrast is captured below:

> The toddlers [in a Mayan community] observed keenly and engaged in multi-way interaction with the group. In contrast, middle-class European American mothers' approach resembled Assembly-Line Instruction, with mock excitement and praise to engage the little one in mini language lessons. These toddlers were less broadly attentive and seldom engaged with the group as a whole. (Rogoff, 2012, p. 236)

Rogoff et al. (2017) discuss the concept of "sophisticated collaboration" that they have seen among Indigenous children, including children in the US of Indigenous Mexican origin. Sophisticated collaboration implies a form of working together that is fluid and coordinated. They noted that "rural Mexican children were more likely to cooperate in a game than were urban children in the United States, who competed with each other even at the expense of any of them winning" (p. 880). Children engaged in sophisticated collaboration think and work together. On the other hand, children from middle-class families tend to split a task and do less sharing, less thinking together, and take bossy roles. In a study of children engaged in computer programming, they note, "pairs from Indigenous-heritage U.S. Mexican backgrounds collaborated twice as much as did pairs from highly schooled European American backgrounds" (p. 880).

One of the key components of equity as opportunity to participate involves productive group work, in particular along the lines of Complex Instruction (Cohen & Lotan, 1997; Featherstone, et al., 2011). Do immigrant origin students from certain communities (e.g., in Mexico) bring strengths along the lines of sophisticated collaboration that we could be tapping onto? My work with families of Mexican origin points to interactions between parents and children in the mathematics workshops that range from direct instruction on the part of the parents to more collaborative, joint meaning construction (Civil, Díez-Palomar, Menéndez-Gómez, & Acosta-Iriqui, 2008; Menéndez, Civil, & Mariño, 2009). Certainly, more work needs to be done to gain a better understanding of the strengths that immigrant-origin students, such as the ones in the 7th grade class I describe, bring with them. Children from different backgrounds are likely to bring cultural ways of participation that may be different from the ones expected by the school. This is an asset, as the example of sophisticated collaboration shows. I think it is worth noting that students who belong to non-dominant groups often have rich experiences and skills such as knowing more than one language (important in a global world), knowing how to collaborate (important for teamwork, a trait that is valued in many professions), learning at home through participation in the activity rather than through direct teaching, contributing to the household functioning (e.g., helping out with the home economy; language brokers). What are the implications of this richness of skills, knowledge, and experience for teachers and researchers in mathematics education?

Why do I do this work? The words of Adrienne Rich (1986) say it much better than I could ever say it:

> When those who have power to name and to socially construct reality, choose not to see you or hear you, whether you are dark-skinned, old, disabled, female, or speak with a different accent or dialect than theirs, when someone with the authority of a teacher, say, describes the world and you are not in it, there is a moment of psychic disequilibrium as if you looked into a mirror and saw nothing (p. 199)

As we reflect on the work we do with teachers, students, communities, I hope that we can provide accounts that counter these words. I would hope that the students and families with whom we work see themselves in the mathematics worlds that we share with them in our classrooms. I close with the words from a Pāsifika student, as a reminder that this is indeed possible: "When the maths is about us and our culture, it makes me feel normal, and my culture is normal" (Hunter & Hunter, in press, p. 16).

References

Acosta-Iriqui, J., Civil, M., Díez-Palomar, J. Marshall, M., & Quintos-Alonso, B. (2011). Conversations around mathematics education with Latino parents in two Borderland communities: The influence of two contrasting language policies. In K. Téllez, J. Moschkovich, & M. Civil (Eds.), *Latinos/as and mathematics education: Research on learning and teaching in classrooms and communities* (pp. 125-147). Charlotte, NC: Information Age Publishing.

Aguirre, J., Herbel-Eisenmann, B., Celedón-Pattichis, S., Civil, M., Wilkerson, T., Stephan, M., Pape, S., & Clements, H. D. (2017). Equity within mathematics education research as a political act: Moving from choice to intentional collective professional responsibility. *Journal for Research in Mathematics Education, 48*(2),124-147.

Anhalt, C. O., Staats, S., Cortez, R., & Civil, (M). (2018). Mathematical modeling and culturally relevant pedagogy. In Y. J. Dori, Z. Mevarech, & D. Baker (Eds.). *Cognition, metacognition, and culture in STEM education* (pp. 307-330). New York: Springer.

Atweh, B. & Ala'i, K. (2012). Socially reponse-able mathematics education: Lessons from three teachers. In J. Dindyal, L. P. Cheng, & S. F. Ng (Eds.), *Mathematics education: Expanding horizons- Proceedings of the 35th annual conference of the Mathematics Education Research Group of Australasia* (pp. 98-105). Singapore: MERGA.

Bartell, T. G. (2013). Learning to teach mathematics for social justice: Negotiating social justice and mathematical goals. *Journal for Research in Mathematics Education*, *44*(1), 129-163.

Berlin, R., & Berry, R. Q. III (2018). Confronting the lies I tell myself. In S. Crespo, S. Celedón-Pattichis, & M. Civil (Eds.), *Access and equity: Promoting high-quality mathematics in grades 3-5* (pp. 9-20). Reston, VA: National Council of Teachers of Mathematics.

Civil, M. (2002). Culture and mathematics: A community approach. *Journal of Intercultural Studies*, *23*(2), 133-148.

Civil, M. (2006). Working towards equity in mathematics education: A focus on learners, teachers, and parents. In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez (Eds.), *Proceedings of the twenty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 30-50). Mérida, Mexico: Universidad Pedagógica Nacional.

Civil, M. (2007). Building on community knowledge: An avenue to equity in mathematics education. In N. Nasir & P. Cobb (Eds.), *Improving access to mathematics: Diversity and equity in the classroom* (pp. 105-117). New York: Teachers College Press.

Civil, M. (2011). Mathematics education, language policy, and English language learners. In W. F. Tate, K. D. King, & C. Rousseau Anderson (Eds.), *Disrupting tradition: Research and practice pathways in mathematics education* (pp. 77-91). Reston, VA: NCTM.

Civil, M. (2012). Opportunities to Learn in Mathematics Education: Insights from Research with "Non-Dominant" Communities. In T.Y. Tso (Ed.), *Proceedings of the 36th conference of the International Group for the Psychology of Mathematics Education* (Vol.1, pp. 43-59). Taipei, Taiwan.

Civil, M. (2014). Musings around participation in the mathematics classroom. *The Mathematics Educator*, *23*(2), 3-22.

Civil, M. (2016). STEM learning research through a funds of knowledge lens. *Cultural Studies of Science Education*, *11*(1), 41-59. DOI 10.1007/s11422-014-9648-2

Civil, M., & Andrade, R. (2002). Transitions between home and school mathematics: Rays of hope amidst the passing clouds. In G. de Abreu, A. J. Bishop, & N. C. Presmeg (Eds.), *Transitions between contexts of mathematical practices* (pp. 149-169). Boston, MA: Kluwer.

Civil, M., Díez-Palomar, J., Menéndez-Gómez, J. M., & Acosta-Iriqui, J. (2008). Parents' interactions with their children when doing mathematics. *Adults Learning Mathematics: An International Journal*, 3(2a), 41-58.

Civil, M. & Hunter, R., (2015). Participation of non-dominant students in argumentation in the mathematics classroom. *Intercultural Education*, *26*, 296-312. DOI 10.1080/14675986.2015.1071755.

Civil, M., & Menéndez, J. M. (2011). Impressions of Mexican immigrant families on their early experiences with school mathematics in Arizona. In R. Kitchen & M. Civil (Eds.), *Transnational and borderland studies in mathematics education* (pp. 47-68). New York, NY: Routledge.

Civil, M., & Planas, N. (2004). Participation in the mathematics classroom: Does every student have a voice? *For the Learning of Mathematics*, *24* (1), 7-12.

Cohen, E.G., & Lotan, R.A. (Eds.) (1997). *Working for equity in heterogeneous classrooms: Sociological theory in practice*. New York, NY: Teachers College Press.

Featherstone, H., Crespo, S., Jilk, L., Oslund, J., Parks, A., & Wood, M. (2011). *Smarter together! Collaboration and equity in the elementary math classroom*. Reston, VA: National Council of Teachers of Mathematics.

Gates, P. & Jorgensen (Zevenbergen), R. (2015). Mapping the field and documenting the contributions. In P. Gates & R. Jorgensen (Zevenbergen) (Eds.), *Shifts in the field of mathematics education: Stephen Lerman and the turn to the social* (pp. 1-14). New York: Springer.

Gay, G. (2000). *Culturally responsive teaching: Theory, research, and practice*. New York: Teachers College Press.

González, N., Andrade, R., Civil, M., & Moll, L.C. (2001). Bridging funds of distributed knowledge: Creating zones of practices in mathematics. *Journal of Education for Students Placed at Risk, 6,* 115-132.

González, N., Moll, L, & Amanti, C. (Eds.) (2005). *Funds of knowledge: Theorizing practice in households, communities, and classrooms*. Mahwah, NJ: Lawrence Erlbaum.

Guerra, P., & Lim, W. (2017). Latinas, their parents, and mathematics schooling: Nurturing relationships with families and their communities. In A. Fernandes, S. Crespo, & M. Civil (Eds.), *Access and equity: Promoting*

high-quality mathematics in grades 6-8 (pp. 121-130). Reston, VA: National Council of Teachers of Mathematics.
Hunter, R., & Anthony, G. (2011). Forging mathematical relationships in inquiry-based classrooms with Pasifika students. *Journal of Urban Mathematics Education*, *4*(1), 98-119.
Hunter, R. & Hunter, J. (in press). Maintaining a cultural identity while constructing a mathematical disposition as a Pāsifika learner. In E. A. McKinley & L. Tuhiwai Smith (Eds.), *Handbook of Indigenous Education*. New York: Springer.
Id-Deen, L. (2017). "It will help me do better in mathematics": Building trusting relationships with Black students. In A. Fernandes, S. Crespo, & M. Civil (Eds.), *Access and equity: Promoting high-quality mathematics in grades 6-8* (pp. 131-141). Reston, VA: National Council of Teachers of Mathematics.
Kitchen, R. S. (2007). The importance of building relationships. In R.S. Kitchen, J. DePree, S. Celedón-Pattichis, & J. Brinkerhoff (Eds.), *Mathematics education at highly effective schools that serve the poor: Strategies for change* (pp. 115-129). Mahwah, NJ: Lawrence Erlbaum.
Lerman, S. (2000). The social turn in mathematics education research. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 19-44). Westport, CT: Ablex.
Lipka, J., Sharp, N., Brenner, B., Yanez, E. & Sharp, F. (2005). The relevance of culturally based curriculum and instruction: The case of Nancy Sharp. Journal of American Indian Education, 44(3), 31-54.
LópezLeiva, C., Celedón-Pattichis, S., Pattichis, M. S., & Morales, J. (2017). Teaching and learning number systems for computational thinking: Underrepresented students accessing high-quality STEM practices. In A. Fernandes, S. Crespo, & M. Civil (Eds.), *Access and equity: Promoting high-quality mathematics in grades 6-8* (pp. 89-103). Reston, VA: National Council of Teachers of Mathematics.
Martin, D. (2009) (Ed.). *Mathematics teaching, learning, and liberation in the lives of Black children*. New York: Routledge.
Menéndez, J. M. & Civil, M. (2009). The role of cognition and affect on adults' participation in a nonformal setting for learning mathematics. In K. Safford-Ramus (Ed.), *Proceedings of the 15th International Conference of Adults Learning Mathematics - A Research Forum* (pp. 147-164). Philadelphia, PA: Chestnut Hill College, Philadelphia.
Menéndez, J.M., Civil, M., & Mariño, V. (2009, April). *Latino parents as teachers of mathematics: Examples of interactions outside the classroom*. Presented at the annual meeting of the American Educational Research Association (AERA), San Diego, CA.
Musgrove, J., & Willey, C. (2018). Problem-based mathematics learning in urban spaces: Exploring mathematical circles in youth's community circles. In D. Y. White, A. Fernandes, & M. Civil (Eds.), *Access and equity: Promoting high-quality mathematics in grades 9-12* (pp. 13-31). Reston, VA: National Council of Teachers of Mathematics.
Nasir, N., Rosebery, A., Warren, A., & Lee, C. D. (2006). Learning as a cultural process: achieving equity through diversity. In R. K. Sawyer (Ed.), *The Cambridge handbook of the learning sciences* (pp. 489-504). Cambridge: Cambridge University Press.
Nemirovsky, R., Kelton, M. L., & Civil, M. (2017). Towards a vibrant and significant informal mathematics education. In J. Cai (Ed.), *First Compendium for Research in Mathematics Education* (pp. 968-980). Reston, VA: NCTM.
Nieto, S. (2013). *Finding joy in teaching students of diverse backgrounds: Culturally responsive and socially just practices in U.S. classrooms*. Porstmouth, NH: Heinemann
Nunes, T., Schliemann, A., & Carraher, D. (1993). *Street mathematics and school mathematics*. New York: Cambridge University Press.
Preston, R. V. & Garner, A. S. (2003). Representation as a vehicle for solving and communicating. *Mathematics Teaching in the Middle School*, *9*(1), 38-43.
Rich, A. (1986). *Blood, bread, and poetry: Selected prose, 1979-1985*. New York: W. W. Norton.
Rodríguez-Brown, F. V. (2010). Latino Families: Culture and Schooling. In E. G. Murillo Jr., S. A. Villenas, R. Trinida Galván, J. Sánchez Muñoz, C. Martínez, & M. Machado-Casas (Eds.), *Handbook of Latinos and Education: Theory, Research, and Practice* (pp. 350-360). New York: Routledge.
Rogoff, B. (2012). Learning without lessons: Opportunities to expand knowledge. *Infancia y Aprendizaje: Journal for the Study of Education and Development*, *35*(2), 233-252. http://dx.doi.org/10.1174/021037012800217970
Rogoff, B., Coppens, A. D., Alcalá, L., Aceves-Azuara, I., Ruvalcaba, O., López, A., Dayton, A. (2017). Noticing learners' strengths through cultural research. *Perspectives on Psychological Science*, *12*(5), 876-888.

Rubel, L. H. (2017). Equity-directed instructional practices: Beyond the dominant perspective. *Journal of Urban Mathematics Education. 10,* (2), 66–105.
Valero, P. & Zevenbergen, R. (2004) (Eds.). *Researching the socio-political dimensions of mathematics education*. Boston, MA: Kluwer.
Vélez-Ibañez, C. G., & Greenberg, J. B. (1992). Formation and transformation of funds of knowledge of among U.S.-Mexican households. *Anthropology and Education Quarterly*, *23*(4), 313-335.

LOOKING BACK, AHEAD, AND IN NEW DIRECTIONS

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In this paper, I respond to and expand on Civil's plenary address (this volume), in which she articulates key themes related to funds-of-knowledge orientations to mathematics education. I review the themes selected by Civil: funds of knowledge, participation, and valorization of knowledge, and provide additional analysis. Next, I accentuate the often absent but inherent political dimensions of work in mathematics education that emphasizes the cultural, as well as the often absent but inherent cultural dimensions of work in mathematics education that emphasizes the political. Finally, I contribute additional perspectives to future directions for the research community around equity in mathematics education.

Keywords: Equity and Diversity

Introductions

I trace the origins of my work around equity in mathematics education to my involvement as an undergraduate student in a spin-off project prompted by the then-contemporary study, in which Treisman and colleagues (see Treisman & Fullilove, 1990) attributed difficulties in college mathematics for African American students to their social and academic isolation. In 1991, the mathematics department at the college where I was a student, inspired by Treisman's findings, created a co-curricular Calculus "workshop" for first-year students of color. I was tapped as a senior student facilitator, and this experience contributed towards my extension of experience in informal and Jewish education towards mathematics teaching as a profession.

I continue to benefit from the power and privilege that come with whiteness in the U.S., in opportunities around education, employment, and housing. Yet white privilege only goes so far for me, as a queer, gender non-conforming, Jewish woman. In general, through my intersecting life experiences of Otherness (Du Bois, 1903) and my struggles from these marginalized positions, I feel solidarity and identify with people and groups who are being othered, objectified, or oppressed. Once I became a high school mathematics teacher in the mid-1990s, for example, I discovered a system in the elite school in New York City where I taught that was supposedly ability oriented but had produced its lowest track with nearly all of the school's students of color. After sizing up their brilliance, the students and I turned what was supposed to be a low-track senior math class to preparing for and taking the AP Calculus AB exam. The impetus was certainly inspired by my achieved standpoint (Harding, 1993) as a queer Jew, and likely also in part by the then-contemporary *Stand & Deliver*.

In that popular Hollywood rendering of real-life Jaime Escalante's classroom, we saw Latinx students face racial discrimination as, presumed by the College Board to have been cheating, they were forced to retake their AP Calculus exams. Though my students and I were spared the attention of The College Board, they schooled me about the range of everyday racial discrimination and microaggressions they experienced, from regularly being called one another's names by white teachers, to being tailed inside stores by local shopkeepers, to being targeted by city truancy officers on route to school and then missing class while being "processed" by that system, to having to absorb the feeling of being feared on the streets by white women. My love for them as their teacher connected me in an emotional way to what was their marginalization, their pain, their disappointment. I wanted to teach mathematics that would feel relevant, to support their and my own understandings of processes in the world, especially the structural and systemic processes that were designed to keep power and material resources from them.

In my subsequent and ongoing work as a mathematics education researcher and teacher educator, I continue to learn about the importance of providing students with windows through which to use mathematics to understand the outside world but also of mirrors, through which to see themselves and connections to their pasts and potential futures (Gutiérrez, 2007). As Gutiérrez (2007) elaborates: "The goal is not to replace traditional mathematics with a pre-defined 'culturally relevant mathematics,' but rather to strike a balance between the number of windows and mirrors provided to any given student in his/her math career" (p. 3). Otherwise, like the poet Adrienne Rich (1986) warns, without such "mirrors," there is the potential for "psychic disequilibrium" as the teacher, the school, the curriculum, and their aggregated power is describing the world, but without you in it, and it "as if you looked in the mirror and saw nothing" (p.199).

Since the 1980s, mathematics education research and teacher education has considered an array of related sociocultural perspectives about mathematics and its teaching and learning -- ethnomathematics (e.g., D'Ambrosio, 1985), funds of knowledge for mathematics (e.g., Aguirre et al., 2013; Civil, 1998), and culturally relevant or responsive mathematics teaching (e.g., Gutstein, Lipman, Hernández & de los Reyes, 1997). A commonality among these perspectives is around "centering" (Tate, 1995) mathematics on students' experiences, their affiliations with various cultural or social groups, or the everyday practices of those groups, by creating opportunities for hybridity between the mathematical thinking in everyday practices or other out-of-school domains and the formal school mathematics curriculum. Unlike an incremental or vertical development of mathematical expertise typical to school-based learning, horizontal expertise develops through coordination across the diverse set of contexts through which one traverses (K. Gutiérrez, 2008).

In this volume, Civil (2018) reflects on such a perspective about mathematics teaching and learning equity in mathematics education and identifies three interrelated themes that she views as central in her own "looking back": funds of knowledge; participation; and valorization of knowledge. I have been greatly influenced by Marta's corpus of scholarly contributions, and I begin by addressing each of these themes. Next, I follow Marta's lead and present ideas about how we might "look forward," or really, blaze new trails in mathematics education, in vision, mission, and action.

Civil's "Look Back"

Funds of Knowledge

Across the breadth of Civil's work, she emphasizes a baseline premise that every community, family, and person possess mathematical funds of knowledge, and that these funds of knowledge can be leveraged as intellectual resources for school success. This approach to equity in mathematics is considered "asset-based" (Celedón-Pattichis et al., 2018), in how it avoids a deficit construction of minoritized youth, their families, communities, and material spaces. Civil's research includes studies of children's mathematical thinking, mathematics teaching, and parents' mathematical thinking and perceptions, and draws on a blend of cognitive and sociocultural perspectives about learning. One of her most significant contributions to date is her work on immigrant parents' mathematical thinking, their cultural and linguistic funds of knowledge, their views about their children's mathematics education in the U.S., and implications for equity in mathematics education (e.g., Civil & Andrade, 2002; Civil & Bernier, 2006; Civil & Menéndez, 2011).

Much of the related scholarship around funds of knowledge in mathematics education is practice-oriented, around processes or implications for teacher education. Beginning with Moll, Amanti, Neff and Gonzalez' (1992) outline of a process for teachers as co-researchers to conduct home visits with the goal of identifying funds of knowledge, there exists a growing set of studies that present potential protocols for, and demonstrate the effectiveness of, similar approaches in teacher education (Aguirre et al., 2013; Civil & Andrade, 2002; Foote et al., 2013; Rubel, 2012; Turner & Drake, 2015; Turner et al., 2016). In aggregate, these studies guide teacher education in terms of how to foster orientations to teaching that value the funds of knowledge that youth bring to the classroom and in so doing, deflect prevalent deficit framings of minoritized peoples. In addition, these studies identify and analyze ways of supporting teachers in developing and improving instructional practices that leverage those knowledge funds as intellectual resources.

A related area of research comprises curriculum design fueled by a funds of knowledge orientation. The Math in Cultural Context project is an example, in which teachers, researchers, and Yup'ik elders co-designed elementary school mathematics curriculum around mathematics of Yup'ik cultural practices (Kisker et al., 2012). Less well-known examples around curriculum design based on community funds of knowledge can be seen, for example, in Katsap and Silverman's (2015) work with geometry curriculum using traditional Bedouin weaving and embroidery, or Massarwe, Verner and Bshouty's (2010) example of plane geometry curriculum with a focus on Arab art and design. Civil has endowed our field with a variety of such examples, such as elementary mathematics curriculum around the theme of construction (Civil, 2002) or garden-design (Civil & Kahn, 2001).

Curricular design that builds on students' funds of knowledge typically relies on iterative processes of studying one's students to identify funds of knowledge domains, identifying mathematics embedded in those everyday practices, and building curriculum that negotiates the connections and tensions between the mathematical thinking inherent to this domain and the desired school mathematics. Although such processes are productive as equity-directed instructional practices, they extend beyond the normative scope of a teacher's workload. As our research and collaborations with teachers accumulate evidence that such processes are essential, we must concurrently adjust the scale of teacher workload and advocate for teachers as our collaborators in this endeavor so that there is not a disconnect between the necessary and the realizable.

In this volume, Civil (2018) showcases an example of how a funds of knowledge domain emerged in her mathematics workshop for parents of school students. Civil learned from the parents that American wrench sizes are sized in inches and expressed as fractions. Ordering wrenches by increasing size is, therefore, equivalent to ordering fractions. Civil presents the comparison of wrench sizes as an authentic context that can elicit or support mathematical thinking about fraction comparison. Here, with respect to this wrench dimension example, Civil reiterates an essential tension that she has earlier described (e.g., Civil, 2007), between "preserving the purity of the funds of knowledge" and the mathematical goals of instruction, in terms of how a curricular focus on a real-world context might de-prioritize, limit, and constrain mathematical exploration and mathematical content.

This tension is derived, in part, by how lesson planning in mathematics is traditionally driven by a predefined set of mathematical definitions, concepts, and skills, and then so-called "real-world" examples are provided as afterthought applications. Curricular design around a social practice or a social theme takes the opposite starting-point, which in and of itself is new for teachers or designers (Nicol, 2002; Wager, 2012). I have negotiated this tension in my work with mathematics teachers around a binary of teaching the real-world context in service of learning mathematics or teaching mathematics in service of learning about the world. In navigating that false binary, we lose sight of how we are teaching children: our children, the children of our neighbors, and young people all. Their collective well-being, their sense of being cared for, and how mathematics can support the development, cultivation, and refinement of their empathy, creativity, curiosity, and capacity to care for one another and the natural world could be our primary priorities.

Participation in Mathematics

Centering mathematics instruction on students using a funds of knowledge approach typically implies designing thematic curriculum around a selected socio-cultural domain. An alternative way that funds of knowledge can be leveraged in mathematics is in terms of using analogies during instruction that relate a mathematical object or process to students' existing knowledge but without focusing the curriculum thematically around that knowledge domain. For example, in one of my projects, I observed a teacher using one of her student's breakdancing hobby in relation to the mathematical concept of a triangle's center of mass (Rubel & Chu, 2012). The lesson was not organized around breakdancing, but the breakdancing funds of knowledge was used as an analogy to bridge students' experiences with a mathematical object and its definition. Similarly, the example in Moses and Cobb (2001) about using students' experiences with the public transit system as a means of learning integers was absent a thematic focus on the trains themselves.

Here, Civil (2018) presents an additional alternative, in her reminder that students from minoritized groups can engage in mathematical content that is devoid of thematic connection to specific lived experiences. She stresses that building on students' experiences and their funds of knowledge does not need to be limited to building a curriculum focused on a particular everyday practice, real-world artifact or process, but could be implemented through the ways that teachers organize their classroom for participation -- not necessarily what mathematical questions the students are considering, but what kinds of participation are being made available to them, and if or how those kinds of participation are in synch with or in opposition to their ways of participation in other aspects of their lives. For example, apprenticeship models of gaining expertise or assumptions of competence are endemic to various contexts and could be leveraged as resources for classroom learning (Civil & Khan, 2001; Kisker et al., 2012; Nasir, 2005).

I would like to draw attention to this point, especially in the context of current educational policies and mantras. We know that teachers' beliefs about students and about learning underlie how they think what doing mathematics is supposed to look like, and which forms of participation they will make available for their students (Hand, 2012; Rubel & Stachelek, 2018). The common perception of order and silence as prerequisites to learning mathematics constrains individual teachers' views about participation, can lead toward over-interpretation of student participation as off-task or disruptive, and typically results in didactic teaching (Hand, 2010). In this volume, Civil (2018) draws attention to a 7th grade classroom vignette in which she was the instructor and her students engage deeply with evaluating algebraic expressions. She credibly ascribes significance to the blend of humor, use of the students' home language, and *confianza* (trust) among participants.

Civil's observations remind me of my own, recent observations of two accomplished teachers of color, Ms. Hudson and Ms. Garcia (pseudonyms), who collaborated with me on a recent project. Civil's description of the family-like environment in that 7th grade classroom is reminiscent for me of the my sense of the atmosphere in Hudson's and Garcia's classrooms, notably different from the classrooms of the 9 white teachers in that group of teachers. Hudson created a family-life atmosphere through her playfulness and use of youth or informal language. For instance, she would ask the class, "Can I mess with you now?" as a way of initiating a more complicated exercise. This playfulness initiated a kind of lightness and communicated both rapport, support, and a sense of challenge. Garcia also connected to her students through language, but in her case, she drew on strategically using Spanish (her and their first language), to translate mathematical terms, to check in with students about their emotional state, and to redirect behavior. Garcia positioned herself as a loving caretaker for her students by donning an apron around her waist, its pockets filled with pencils, erasers, and calculators for their use. Civil's description of her experience in that 7th grade class corresponds with my observations of Hudson and Garcia, and of the Lee example described in Clark, Badertscher, and Napp (2013). In sum, this suggests that we expand our understandings about fostering hybridity in mathematics classes through participation processes and language and that we could follow the guidance of expert teachers of color and their community cultural wealth (Burciaga & Kohli, 2018). Other kinds of "disruptions" of traditional learning environments, like situating formal learning in places outside of schools and classrooms, are likely productive as well in creating opportunities for hybridity (Ma, 2016).

Valorization of Knowledge

The research literature describes interventions and provides teacher education modules designed around supporting teachers about how to develop mathematics curriculum that builds on students' experiences or everyday practices (Aguirre et al., 2013; Rubel, 2012; Taylor, 2012). As part of this process, we are seemingly led to search in a cultural practice for what Noss, Hoyles and Pozzi (200) have called "visible mathematics," meaning recognizable as school mathematics. Here and in other papers, Civil draws attention to the valorization of knowledge, in terms of the issue of whose knowledge is being valued, meaning that oftentimes we do not sufficiently valorize the mathematical thinking endemic to various cultural practices as mathematics. For example, one of Civil's exemplar funds of knowledge examples relates to a parent's geometric design and measurement work as a dressmaker (Civil & Andrade, 2002). As Civil has discussed, there is a distinction between valorizing the dressmaker's knowledge as itself mathematical or doing so through potential connections to existing school mathematics.

At the same time, Civil cautions about organizing lessons around a real-world context and then have the context "take over" the mathematics. Civil cautions that the social context might be superficial or contrived and draws attention to the oft stated concern "where is the math?" Elsewhere (Rubel & McCloskey, under review), I have written about how the "where is the math?" critique is at times employed to protect Western mathematics as if it were universal and as if success in school mathematics were inherently fair. Culturally relevant pedagogy, or pedagogy organized around funds of knowledge, is then positioned as communicating a "watering-down" of mathematics, for those seen as unable or unwilling to engage otherwise. Here, in reiterating her "where is the math?" concern, Civil comes close to positioning instruction and pedagogy that is fundamentally oriented around building on students' experiences as potentially at odds with rigorous mathematics instruction. When we position culturally relevant pedagogy as threatening mathematics, we falsely re-inscribe Western mathematics as neutral and deny that school and academic mathematics are historically, socially, spatially, and culturally bounded.

Indeed, while ethnomathematics could focus on the mathematics of *any* social or cultural group, it is typically used as a catch-all term for mathematics among "identifiable cultural groups" (D'Ambrosio, 1985), and those groups are typically limited to subordinated social groups (Knijnik, 1997). Western mathematics is then understood as separate from and in opposition to ethnomathematics (Gutiérrez, 2017). But as Knijnik (2002) explains:

> by considering the form of other, non-hegemonic ways of knowing and producing mathematics, ethnomathematics relativises the "universality" of (academic) mathematics and, moreover, questions its very nature. … In problematising academic mathematics, ethnomathematics emphasizes not only that mathematics is a social construction but, more than this, that such a construction takes place in a terrain shaped by political dispute around what will be seen as mathematics, around which will be considered the legitimate way of reasoning, and therefore, around which groups are those that can legitimately produce science…..Thus, it is not a matter of talking naively about different mathematics, but of considering that these mathematics are, in terms of power, unequally different. (p.13)

Hence, if we legitimize mathematical funds of knowledge only in terms of connections to Western mathematics, we are ignoring this political terrain. Using a Western gaze onto our students' funds of knowledge is different than valorizing cultural expertise, curiosity, and other ways of knowing as avenues that will yield new mathematical questions, ideas, representations, and ways of knowing. I call attention to the work of R. Gutiérrez (2002) in which she outlines how making space for people "under the tent" of mathematics will necessarily change, expand, and improve mathematics -- people pose mathematical questions and develop mathematical solutions informed by their experiences. Mathematics is necessarily enriched by participation of a wider variety of people from a broader range of experiences. As Gutiérrez teaches, it is not only that marginalized people need mathematics. Of course, they can use mathematics in a variety of ways to evade their marginalization. However, for mathematics to stay relevant and to be able to solve many problems that remain unsolved, it is mathematics that needs a diversity of peoples.

The Cultural is Political

Although mathematics is an ongoing human creation, only some mathematics is recognized in the Western cannon and included in school curriculum. For students in the U.S., from the White, Christian power-majority, this is not experienced as exclusion, since this is the normative culture. Teachers (as well as textbooks and test writers), who are largely from that power-majority, tend to draw on their own funds of knowledge in selecting, creating, and implementing mathematical tasks. In sum, this means that white students already have their funds of knowledge reflected in typical mathematics instruction and curriculum, without need of intervention and unacknowledged as such. Even when trying to connect to students' funds-of knowledge, mathematics teachers have been found to contextualize mathematics using contexts related to sports or consumer activities of adults like home remodeling, shopping, banking, or budgeting (Bright, 2015; Watson, 2012), domains that are seemingly "safe," or germane for everyone in the same way. However, these domains are neither arbitrary nor neutral, but correspond to the interests, life experiences, and values of communities who already benefit from being part of the power-majority. Moreover, the positioning of the importance of learning mathematics in school as preparation for roles as marketplace consumers or corporate employees demonstrates how mathematics in schooling is being used to support ideologies of individualism, competition, and capitalism, ideologies that benefit that same white power-majority. How might our word problems in mathematics read if they were guided instead around commitments to collective well-being or to the health of our natural environment?

Making the Political Explicit

There is a known tendency to avoid political questions like these in the teaching of mathematics (e.g., Simic-Miller, Fernandes, & Felton-Koestler, 2015), which can be explained by a combination of a lack of sociopolitical awareness among teachers, fear among teachers for job security especially in the current nationalistic climate, or a result of the common misperception that mathematics is apolitical. In one of my projects, for example, I observed teachers struggle with how to take up issues of power and oppression in their teaching of mathematics for all of these reasons. One white teacher with whom I collaborated, for example, created interesting curriculum around using geometric loci as a language with which to analyze or design home or facility location, relative to constraints like environmental pollution. Even though her school was in a low-income area with high asthma rates, presumed to be related to more extreme and multiple sources of environmental pollution, she built the lesson in hypothetical, generic terms. In conversation afterwards, she agreed that she could have provided the students the opportunity to use mathematics to articulate their own physical geometries in the context of actual existing environmental hazards and disease rates. She reflected that she tended to have "blinders on" about specifically engaging in hybridity between relevant political concerns and her teaching of geometry (see Rubel, 2017). I, too, as a teacher educator struggle with how to better support teachers in this regard.

Indeed, despite a vision of mathematics as socially constructed with political ambitions and consequences, funds of knowledge projects in the literature are often described in cultural or linguistic terms absent explicit socio-political contextualization. For example, taking Civil's (2018) new example about wrench sizing, she has demonstrated that this is an interesting and likely productive context in which to explore comparison of fractions. The wrenches, and what they are used for, as well as who is using them is seemingly put aside to focus on the wrench dimensions themselves, because fractions and their relative magnitudes explicitly reside in those dimensions. But what of the unasked question as to the mathematics of why these people, in this moment of time, and in this place possess this specific knowledge about hand tools? What is the mathematics of how this knowledge and expertise is capitalized on by those in power? What is the mathematics of who owns the construction or tool companies, who gives the building permits, and what the zoning processes value? How is this knowledge acquired and how is it shared? Who invented this tool, for what purpose, and how is mathematics used to produce it? What fraction of construction profit goes to those who are doing the actual back-breaking and dangerous construction labor? What is the mathematics of labor unions in the U.S. and their support for construction workers? What else are these tools used for? And further, what is the mathematics of American persistence to have a unique measurement system?

The Political Crowding Out or Denigrating the Cultural

It is not only that funds-of-knowledge curriculum modules are often presented absent political context. There are examples in the literature of projects that are explicitly organized around political questions or themes but that do not honor or sufficiently leverage students' funds of knowledge. In my work with colleagues, as part of the City Digits project, for example, we identified widespread participation in state lottery games as a cultural fund of knowledge in which to explore probability, combinations, scale, measurement, data analysis with youth (see Rubel, Lim, Hall-Wieckert, & Sullivan, 2016c). In a second example, we drew on rampant participation in a local array of alternative financial institutions like pawn shops and check cashers as an entry-point to evaluate spatial distribution of these alternative financial institutions in the context of other social variables (Rubel, Lim, Hall-Wieckert, & Katz, 2016b). In both cases, students built new or more developed mathematical understandings, which contributed to furthering their political formation (Rubel, Hall-Wieckert, & Lim, 2017). Their interest in and curiosity about spatial justice engaged them in furthering their mathematical understandings (Rubel, Lim, Hall-Wieckert, in press). Participating students reflected that contextualizing mathematics through political issues that they identified as connected to their lives and the places in which they live -- their funds of knowledge -- linked their learning to their sense of being agents of change in their families in terms of lottery spending, or decisions about financial institutions (Rubel et al., 2016c).

At the same time, along with these important successes, these curricular modules and their associated maps did not make sufficient use of funds of knowledge as resources and instead, likely reinforced and highlighted common deficit notions about students and their families, at least for some students. For example, central to these modules were data and map representations that could have been interpreted as suggesting that participation in the lottery or loan-taking from alternative financial institutions among low-income people are produced by a lack of mathematical understanding, instead of as products of spatial injustice organized to maintain the status-quo of white supremacy (Rubel, Hall-Wieckert, & Lim, 2016a). The knowledge that the youth accessed in the community about people's sense of hopelessness in the context of their care for loved ones remained largely in the shadows of the project's focus on navigating probability and combinatorics ideas or on the mathematics of loans. How could mathematics be directed at questions or problems related to hope or love instead of only on analysis of or strategies about capital gain amd loss?

As another example, consider Cirillo, Bartell, and Wager's (2016) presentation of a mathematical modeling investigation around the theme of soda pricing techniques used by fast food restaurants. By presenting data showing drink volume and corresponding price per ounce, the finding is that over-sized drinks are sold at a cheaper rate per ounce than smaller, recommended sizes, effectively enticing consumers to buy larger drinks. Since these drinks are heavily caloric but without other dietary benefits, this becomes significant relative to obesity, diabetes, and other health factors. Cirillo et al. (2016) direct readers to contextualize this finding in terms of the density of fast food restaurant locations relative to demographic variables around income, and speculate, indeed, that students in low-income areas saturated with fast food restaurants might surmise that there is not concern for their health. Educating people about their miniscule probability of winning the lottery can be seen as using mathematics to promote abstinence from the lottery. This is akin to how teaching that interest rates charged by pawn shops or check cashers are higher than other financial institutions might lead people to borrow money elsewhere. Similarly, educating people about this value pricing of soft-drinks technique relative to the health risks of drinking soda is intended to guide learners towards abstinence from indulging in those drinks. A commonality across these types of mathematical investigations is that they do not directly challenge why we accept a society that allows, supports, and even encourages these kinds of predatory systems.

Fundamental to this distinction is our current paradigm of democracy, which is a power-over system (Guinier & Torres, 2002), meaning that competition for power yields some who dominate and more who are dominated. Even if one assumes that the current democratic systems in the U.S. are fair and meritocratic, still, these systems are designed to generate inequalities, to yield rewards only to some, and mathematics supports these systems that are designed to rank and order of people (see Valero, 2017). One could argue that teaching young people to understand the lottery as a social project, for example, which necessitates various mathematical understandings, is a way to bolster them in this competitive "power-over" notion of democracy. It is one thing to support youth in navigating the world and its systems as they are, but it is another thing to challenge this underlying conception of democracy. The power-over paradigm could be replaced with a "power-with" democracy (Guinier & Torres, 2002), in which the focus on the individual were shifted to a focus on the collective, wherein an essential value around competition among individuals were replaced with solidarity, resistance, and collective struggle. What would happen to our state-supported lottery systems, our loaning institutions, our fast-food restaurants and their pricing techniques, for example, if our democracy were "power-with" instead of "power-over"? How can mathematics help us to advocate for such a paradigm shift?

Next Steps

In summary of next steps, Civil (2018) reiterates the set of political acts from Aguirre et al. (2017): "1) enhance mathematics education research with an equity lens, 2) acquire the knowledge necessary to do genuine equity work, 3) challenge the false dichotomy between equity and mathematics, and 4) expand the view of what counts as mathematics." Civil's reflections, as well as my comments above, are largely concentrated around the latter two political acts. Inspired by Political Acts #3 and #4 in particular, Civil asks: "Are we doing enough in our writing and in our work to bring the centrality of equity to mathematics and the centrality of mathematics to equity?"

I opened this piece by citing perspectives about learning that guide us in terms of giving students opportunities to draw on their cultural and linguistic funds of knowledge as resources in mathematics. I cited Gutiérrez' (2002) call that we need to offer mathematics as a window through which students can look out onto the world, but also as a mirror in which students can see themselves, their families, their pasts, and their futures in mathematics. And yet, the notion of mathematics as a mirror through which to see self seems not yet fully possible. For example, I have bumped up against the gender binary in and with mathematics, by mathematics teachers, by mathematics exercises and theories that indicate and support a set of untruths (see Rubel, 2016). School mathematics tells me that I was born either a boy or a girl, and that this is a fixed state. Schooling in the U.S. prescribes that liking or doing math is doing masculinity, and that girls can excel at math but at the cost of sacrificing femininity. School mathematics decries sexuality, race, and ethnicity as irrelevant, and that success in school mathematics is determined by a fundamentally meritocratic system. In these and other ways, mathematics, even as a mirror, distorts reality. I never felt safe enough to come out as queer as a classroom teacher, and my physical and material vulnerability remain an issue, especially in the current political context. Mathematics, windows or mirrors, does not protect me, and if anything, my work and interest in mathematics likely make my ideas *more* threatening and put me in greater danger than if I worked in education research around another school discipline. In the U.S., where mathematics is intertwined with white supremacy, patriarchy, and heteronormativity, my scholarly critiques, my publicly funded salary, and even my basic existence pose a challenge to that constellation of hegemonic forces.

And so, what of the centrality of mathematics to equity and equity to mathematics? This circles back to the first two Political Acts from Aguirre et al., 2018, which speak to our participation as mathematics education researchers in the current political context. There is the constant evidence of hegemony across mathematics education in the U.S. of review panels, editorial boards, plenary panels, and faculty rosters comprised largely or exclusively of white, CIS, straight, Christian, gender-normative people. We must acknowledge that by limiting who participates in knowledge-building, including through entrenched processes of institutional elitism in the academy, we limit the knowledge building itself. Peruse the research that I have cited in and across this paper as well as the references cited in this volume by Civil - in nearly all cases, cited works are written by people from minoritized groups. Of course, it is reasonable and appropriate that these scholars lead the field in thinking about equity -- after all, we build knowledge in part on our own life experiences, the standpoints we have achieved, and we draw from our own marginalization to recognize when, where, and how it happens to others (Harding, 1993). Those of you who are part of the power majority, either by being male, white, Christian, straight, gender-normative, or tenured faculty at a research university: how might the workings of our field be different and the knowledge we produce improve if we made space for other bodies and voices, listened to critiques, were better allies and advocates, or if we ceded, or at least shared, power with others who do not share our privileges?

Instead of token nods of inclusion, we should blaze new trails for our research community by shaping our research agendas, methodologies, avenues for sharing knowledge, and ways of collaborating around a priority of *tikkun olam* (Hebrew for "repair the world.") and collective productivity. We could refashion our national and international leadership around a vision for mathematics education that focuses on cultivating kindness, empathy, curiosity, creativity, and collaboration among people in a power-with democracy. That way, we would create mathematics that helps in posing and solving pressing questions about our natural environment and our wellness. As importantly, we would know when to put aside mathematics for other, better suited tools or ways of knowing. Perhaps instead of, or at least in parallel to, the oft-asked "where's the math?" challenge, thinly veiled as a question, we ought to invoke the inverse challenge: "where's the justice?" After all, if a research project does not engage equity, privilege, power or justice, and is not contributing toward repairing our world, then how can such research be relevant to mathematics education?

References

Aguirre, J. M., Turner, E. E., Bartell, T. G., Kalinec-Craig, C., Foote, M. Q., Roth McDuffie, A., & Drake, C. (2013). Making connections in practice: Developing prospective teachers' capacities to connect children's mathematical thinking and community funds of knowledge in mathematics instruction. *Journal of Teacher Education, 64*(2), 178-192.

Aguirre, J., Herbel-Eisenmann, B., Celedón-Pattichis, S., Civil, M., Wilkerson, T., Stephan, M., Pape, S., & Clements, H. D. (2017). Equity within mathematics education research as a political act: Moving from choice to intentional collective professional responsibility. *Journal for Research in Mathematics Education, 48*(2), 124-147.

Burciaga, R. & Kohli, R. (2018). Disrupting whitestream measures of quality teaching: The community cultural wealth of teachers of color. *Multicultural Perspectives, 20*(1), 5-12.

Bright, A. (2015). Lift every voice and sing. A response to "Mathematics for what? High school students reflect on mathematics as a tool for social inquiry." *Democracy and Education, 23*(1), Article 8.

Celedón-Pattichis, S., Peters, S., Borden, L., Males, J., Pape, S., Chapman, O., Clements, D., Leonard, J. (2018). Asset-based approaches to equitable mathematics education research and practice. *Journal for Research in Mathematics Education, 49*(4), 373–389.

Cirillo, M., Bartell, T., Wager, A. (2016). Teaching mathematics for social justice through mathematical modeling. In C. Hirsch & A.R. MacDuffie (Eds.) *Mathematical modeling and modeling mathematics* (pp. 87-96). Reston, VA: NCTM.

Civil, M. (1998). Bridging in-school mathematics and out-of-school mathematics, presented at the *Annual Meeting of the American Educational Research Association*, San Diego, CA. Accessed at https://files.eric.ed.gov/fulltext/ED425066.pdf

Civil, M. (2002). Culture and mathematics: A community approach. *Journal of Intercultural Studies*, *23*, 133-148.

Civil, M. (2006). Working towards equity in mathematics education: A focus on learners, teachers, and parents. In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez (Eds.), *Proceedings of the twenty-eighth annual meeting of the*

North American Chapter of the International Group for the Psychology of Mathematics Education (Vol. 1, pp. 30-50). Mérida, Mexico: Universidad Pedagógica Nacional.
Civil, M. (2007). Building on community knowledge: An avenue to equity in mathematics education. In N. Nassir & P. Cobb (Eds.), *Improving Access to Mathematics: Diversity and Equity in the Classroom* (pp. 105-117). New York, NY: Teachers College Press.
Civil, M., & Andrade, R. (2002). Transitions between home and school mathematics: Rays of hope amidst the passing clouds. In G. de Abreu, A.J. Bishop, and N.C. Presmeg (Eds.), *Transitions between Contexts of Mathematical Practices* (pp. 149-169). Boston, MA: Kluwer.
Civil, M., & Bernier, E. (2006). Exploring images of parental participation in mathematics education: Challenges and possibilities. *Mathematical Thinking and Learning, 8*, 309-330.
Civil, M., & Kahn, L. (2001). Mathematics instruction developed from a garden theme. *Teaching Children Mathematics, 7*, 400-405.
Civil, M., & Menéndez, J. M. (2011). Impressions of Mexican immigrant families on their early experiences with school mathematics in Arizona. In R. Kitchen & M. Civil (Eds.), *Transnational and Borderland Studies in Mathematics Education* (pp. 47-68). New York, NY: Routledge
Clark, L., Badertscher, E. M., & Napp, C. (2013). African American mathematics teachers as agents in their African American students' mathematics identity formation. *Teachers College Record, 115*(2), 1–36.
D'Ambrosio, U. (1985). Ethnomathematics and its place in the history and pedagogy of mathematics. *For the Learning of Mathematics, 5*(1), 44– 48.
DuBois, W.E.B. (1903). *The souls of Black folk*. Chicago: A.C. McClurg & Co.
Foote, M. Q., Roth McDuffie, A., Turner, E. E., Aguirre, J. M., Bartell, T. G., & Drake, C. (2013). Orientations of prospective teachers towards students' families and communities. *Teaching and Teacher Education, 35*, 126-136
Gutiérrez, K. (2008). Developing a sociocritical literacy in the Third Space. *Reading Research Quarterly, 43*(2), 148–164.
Gutiérrez, R. (2002). Toward a new equity research agenda. *Mathematical Thinking and Learning, 4* (2&3), 145–87.
Gutiérrez, R. (2007). Context matters: Equity, success, and the future of mathematics education. In T. Lamberg & L. R. Wiest (Eds.), *Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1– 18). Stateline (Lake Tahoe), NV: University of Nevada, Reno.
Gutiérrez, R. (2017). Living mathematx: Towards a vision for the future. In Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 2-26).. Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.
Gutstein, E., Lipman, P., Hernández, P., & de los Reyes, R. (1997). Culturally relevant mathematics teaching in a Mexican American context. *Journal for Research in Mathematics Education, 28*, 709–737.
Hand, V. M. (2010). The co-construction of opposition in a low-track mathematics classroom. *American Educational Research Journal, 47*(1), 97–132.
Hand, V.M.. (2012). Seeing culture and power in mathematical learning: Toward a model of equitable instruction. *Educational Studies in Mathematics, 80*(1&2), 233–247.
Harding, Sandra. (1993). Rethinking standpoint epistemology: What is "strong objectivity"? In L. Alcoff & E. Potter (Eds.), *Feminist epistemologies* (pp. 49–82). New York: Routledge.
Katsap, A. & Silverman, F. (2015). *Ethnomathematics of Negev Bedouins' Existence in Forms, Symbols and Geometric Patterns*. Rotterdam: Sense Publishers.
Knijnik, G. (1997). Ethnomathematics: Culture and politics of knowledge in mathematics education. *For the Learning of Mathematics, 22*(1), 11-14.
Kisker, E. E., Lipka, J., Adams, B. L., Rickard, A., Andrew-Ihrke, D., & Yanez, E. E. (2012). The potential of a culturally based supplemental mathematics curriculum to improve the mathematics performance of Alaska native and other students. *Journal for Research in Mathematics Education, 43*(1), 75–113.
Ma, J. (2016). Designing disruptions for productive hybridity: The case of walking scale geometry. *Journal of the Learning Sciences, 25*(3), 335–371.
Massarwe, K.,Verner, I., & Bshouty, D. (2010). An Ethnomathematics exercise in analyzing and constructing ornaments in a geometry class. *Journal of Mathematics & Culture, 5*(1).
Moses, R. P., & Cobb, C. E. (2001). *Radical equations: Math literacy and civil rights*. Boston, MA: Beacon Press.
Nasir, N. S. (2005). Individual cognitive structuring and the sociocultural context: Strategy shifts in the game of dominoes. *Journal of the Learning Sciences, 14*(1), 5–34.

Nicol, C. (2002). Where's the math? Prospective teachers visit the workplace. *Educational Studies in Mathematics, 50,* 289–309.
Noss, R., Hoyles, C. and Pozzi, S.: (2000). Working knowledge: Mathematics in use. In A. Bessot and J. Ridgway (Eds.), *Education for Mathematics in the Workplace (pp. 17-35).* Dordrecht: Kluwer.
Rich, A. (1986). *Blood, Bread, and Poetry: Selected Prose, 1979-1985*. New York: W. W. Norton.
Rosebery, A., Warren, B., & Conant, F. (1992). Appropriating scientific discourse: Findings from language minority classrooms. *Journal of the Learning Sciences, 2*(1), 61–94.
Rubel, L. (2012). Centering the teaching of mathematics on urban youth: Learning together about our students and their communities. In J. Bay-Williams & R. Speer (Eds.), *Professional Collaborations in Mathematics Teaching and Learning: Seeking Success for All* (NCTM 2012 Yearbook). Reston, VA: National Council of Teachers of Mathematics.
Rubel, L. (2016). Speaking up and speaking out about gender. *Mathematics Teacher, 109*(6): 434-439.
Rubel, L. (2017). Equity-directed instructional practices: beyond the dominant perspective. *Journal of Urban Mathematics Education, 10*(2), 66-105.
Rubel, L., & Chu, H. (2012). Reinscribing urban: High school mathematics teaching in low-income communities of color. *Journal of Mathematics Teacher Education, 12*(1), 39–52.
Rubel, L., Hall-Wieckert, M., & Lim, V. (2016a). Teaching mathematics for spatial justice in urban schools: Beyond a victory narrative. *Harvard Educational Review, 86*(4), 556–579.
Rubel, L., Hall-Wieckert, M., & Lim, V. (2017). Making space for place: the role of mapping tools in learning as political formation. *Journal of the Learning Sciences, 26*(4). 643-687.
Rubel, L., Lim, V., & Hall-Wieckert, M. (In press). "So we only have one we share with more, and then they have way more and they share with less": Mathematics and spatial justice. In T. Bartell & K. Bieda (Eds.) *Toward equity and social justice in mathematics education.* New York: Springer.
Rubel, L., Lim, V., Hall-Wieckert, M., & Katz, S. (2016b). Cash across the city: Participatory mapping & teaching for spatial justice. *Journal of Urban Learning, Teaching & Research, 12*(1), 4–14.
Rubel, L., Lim, V., Hall-Wieckert, M., & Sullivan, M. (2016c). Teaching mathematics for spatial justice: An investigation of the lottery. *Cognition & Instruction, 34*(1), 1–26.
Rubel, L., & Stachelek, A. J. (2018). Tools for rethinking classroom participation in secondary mathematics. *Mathematics Teacher Educator, 6*(2), 8-23.
Simic-Muller, K., Fernandes, A., & Felton-Koestler, M. (2015). "I just wouldn't want to get as deep into it": Preservice teachers' beliefs about the role of controversial topics in mathematics education. *Journal of Urban Mathematics Education, 8*(2), 53–86.
Tate, W. F. (1995). Returning to the root: A culturally relevant approach to mathematics pedagogy. *Theory Into Practice, 34*(3), 166–173.
Treisman, U. & Fullilove, R. (1990) Mathematics achievement among African American undergraduates at the University of California, Berkeley: An evaluation of the mathematics workshop program. *Journal of Negro Education, 59* (3), 463-478.
Turner, E. E., & Celedón-Pattichis, S. (2011). Mathematical problem solving among Latina/o kindergarteners: An analysis of opportunities to learn. *Journal of Latinos and Education* 10(2), 146–169.
Turner, E. & Drake, C. (2015). A review of research on prospective teachers' learning about children's mathematical thinking and cultural funds of knowledge. *Journal of Teacher Education*, 67, 1, 32-46.
Turner, E. E., Foote, M. Q., Stoehr, K. J., Roth McDuffie, A., Aguirre, J. M., Bartell, T. G., & Drake, C. (2016). Learning to leverage children's multiple mathematical knowledge bases in instruction. *Journal of Urban Mathematics Education, 9*(1), 48-78.
Valero, P. (2017). Mathematics for all, economic growth, and the making of the citizen-worker. In T. S. Popkewitz, J. Diaz, & C. Kirchgasler (Eds.), *A political sociology of educational knowledge: Studies of exclusions and difference* (pp. 117-132). New York: Routledge.
Wager, A. (2012). Incorporating out-of-school mathematics: From cultural context to embedded practice. *Journal of Mathematics Teacher Education*, 15(1), 9–23.
Watson, D. (2012). Norming suburban: How teachers talk about race without using race words. *Urban Education, 47*(5), 983–1004.

TRANSFORMING TEACHERS' KNOWLEDGE FOR TEACHING MATHEMATICS WITH TECHNOLOGIES THROUGH ONLINE KNOWLEDGE-BUILDING COMMUNITIES

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Mathematics teacher educators are faced with designing teacher in-service professional development experiences for developing and transforming Technological Pedagogical Content Knowledge (TPACK) towards integrating digital technologies as mathematics learning tools. Online environments provide opportunities to a broad range of teachers, yet, the asynchronous nature presents communication and collaboration challenges. A researcher-conjectured, empirically-supported learning trajectory guides this online TPACK program for engaging teachers in knowledge-building communities. Three online technology education courses provide teachers with experiences as students, learning about the technologies while confronting challenges to their thinking about teaching with the technologies. The fourth course provides the teachers with key experiences through blended instruction. Through online explorations and discourses in their communities, they examine reform-based instructional strategies for teaching with technologies. Concurrently, they design, implement, analyze and reflect on their teaching experiences through their designed five-day unit in their mathematics classrooms. Four TPACK components reveal how this experience in knowledge-building communities transforms their TPACK.

Keywords: Learning Trajectory, Teacher Education-Inservice/Professional, Teacher Knowledge, Technology

Introduction

Mathematics teachers are challenged to actively engage students with current, more effective technologies as learning tools. A recent handbook chapter (Roschelle, Noss, Jackiw & Blikstein., 2017) highlights three important research-based categories of effective learning digital tools: tools like graphing calculators that can do some of the detail work and students can focus on concepts; tools for providing guidance and feedback to students as they practice mathematics; and tools that help students visualize concepts and develop understanding. With the rapid pace of the development of digital tools, teachers cannot fully realize the value of these tools without teacher professional development. While teachers may have heard about the technologies, this simple knowledge is not sufficient for guiding students in learning mathematics with the technologies. Now recognizing twenty-first century learning, teachers must engage their students in developing four key skills (the 4 C's) in preparation for effectively connecting with a global society: Critical thinking, Communication, Collaboration and Creativity (Partnership for 21st Century Learning, 2015; Thoughtful Learning Organization, 2016). Through these skills, students are prepared to effectively engage in the more complex social, cultural, and educational environments that depend on the advantages offered through the reliance on multiple technological resources. Taking advantage of these 4C's, students participate through various thinking and engagement strategies as they concurrently learn mathematics:

- Critical thinking: Students use analysis as they engage in problem solving where they make comparisons, contrast ideas, analyze ideas, categorize data, and evaluate results of trials.
- Creative thinking: Students engage in open-ended invention and discovery of myriad solution possibilities as they design, improvise, innovate, problem solve and ask questions.
- Communication: Students interact with others, connecting through multiple modes (e.g., text, social media, cell phones, email, Internet and other avenues) where they examine the messages with respect to the purpose, sender, receiver, medium and the context of various communications.
- Collaboration: Student work together toward a common goal, brainstorming ideas, making decisions as a group, delegating, evaluating, goal setting, managing time, resolving conflicts, and team building.

As mathematics teachers consider integrating technologies in their instruction, they must not only determine how the technologies support learning the mathematics, they must consider which pedagogical strategies effectively engage students in learning the mathematics with the tools as they incorporated the 4 C's (Roschelle & Leinwand, 2011). Teachers must identify, orchestrate, and manage different pedagogical strategies and learning tasks for integrating the technologies in new and perhaps different mathematical topics. The challenge involves far more than their understanding of the mathematics content. The experience ultimately challenges their technological pedagogical content knowledge and reasoning (TPACK) with an array of technologies. "Quality teaching requires developing a nuanced understanding of the complex relationships between technology, content, and pedagogy, and using this understanding to develop appropriate, context-specific strategies and representations" (Mishra & Koehler, 2006, p. 1029).

Today's mathematics teachers must continue learning about teaching mathematics beyond their learning in their pre-service teacher preparation programs and typically this learning happens as they are actively teaching. Transforming in-service teachers' TPACK requires more than participation in short-term professional development experiences. Teachers need experiences to actively engage them in the process of "working toward a more complete and coherent understanding," otherwise referred to as knowledge-building experiences (Scardamalia & Bereiter, 1993, p. 39). Knowledge-building communities (e.g., Scardamalia & Bereiter, 1993, 2003; Bereiter, 2002) integrated with classroom teaching experiences are more likely to engage teachers in relearning, rethinking, and redefining teaching and learning to take advantage of new and emerging technologies and methods for teaching mathematics. Through such communities combined with practical teaching experiences, teachers have opportunities to confront their current pedagogical conceptions for integrating technologies as useful learning tools in their content areas (Loughran, 2002) in the process of developing reformed understandings for teaching in the twenty-first century with multiple technologies.

In this paper, we specifically consider the potential benefits but also the potential challenges of providing teachers with learning experiences in online or blended formats that intend to establish online knowledge-building communities. On one hand, online or blended learning is an increasingly recognized educational setting for teachers' professional learning experiences. It can allow teachers more choice about how, when and where they learn, reduce cost, and provide increased access for many more teachers across a broader geographical area. Yet, the primarily asynchronous nature of online learning poses additional challenges for the design of these needed educational experiences (Means, Toyama, Murphy, & Bakia, 2013).

When considering these points, mathematics teacher educators are confronted with a primary and critical question in the design of knowledge-building communities for transforming teachers' TPACK:

What experiences are essential for guiding in-service teachers as they learn about the technologies as well as about teaching mathematics with the technologies?

Building on this broad question, important sub-questions in the design of online professional development instructional programs emerge:

1. What are the key features for online learning for guiding teachers in reframing their current teacher knowledge?
2. What online learning trajectories are not only useful for engaging teachers in knowledge-building communities but also for providing them with an understanding of the pedagogical challenges in their classrooms?
3. How might teachers gain classroom-based learning experiences for applying their theoretical ideas about teaching with technologies?

Through this paper, we report on the first author's research and development project, addressing these questions in the process of developing ad analyzing the outcomes of a new online in-service teacher TPACK program containing four graduate courses. The effort used a researcher-conjectured and empirically-supported learning trajectory (Niess & Gillow-Wiles, 2013, 2014) to frame the experiences and pedagogical strategies to engage the in-service teacher participants in online knowledge-building communities blended with practical teaching experiences when teaching mathematics with technologies. The second author adds comments in the Discussion section.

Theoretical Framework

Teachers' knowledge for teaching with technologies requires far more than just understanding the subject matter. It ultimately necessitates a strong pedagogical knowledge merged with the knowledge for teaching mathematics using a vast array of technological innovations. This task calls for Technological Pedagogical Content Knowledge (Angeli & Valanides, 2009; Mishra & Koehler, 2006; Niess, 2005), otherwise referred to as TPACK (called 'tee-pack', Thompson & Mishra, 2007). TPACK, as shown in Figure 1, describes this teacher knowledge through the intersection of content knowledge, pedagogical knowledge and technological knowledge for guiding their strategic thinking of when, where, and how to guide students' learning of the content such as mathematics with technologies.

TPACK is the composite of the intersecting multiple domains (technological knowledge (TK), pedagogical knowledge (PK), content knowledge (CK), technological pedagogical knowledge (TPK), pedagogical content knowledge (PCK), technological content knowledge (TCK) and TPACK) within the Contexts. Further, the center subset is also described as TPACK. Mishra and Koehler (2008) described this center subset as:

> The representations of concepts using technologies; pedagogical techniques that apply technologies in constructive ways to teach content in differentiated ways according to students' learning needs; knowledge of what makes concepts difficult or easy to learn and how technology can help redress conceptual challenges; knowledge of students' prior

content-related understanding and epistemological assumptions; and knowledge of how technologies can be used to build on existing understanding to develop new epistemologies or strengthen old ones. (p. 3)

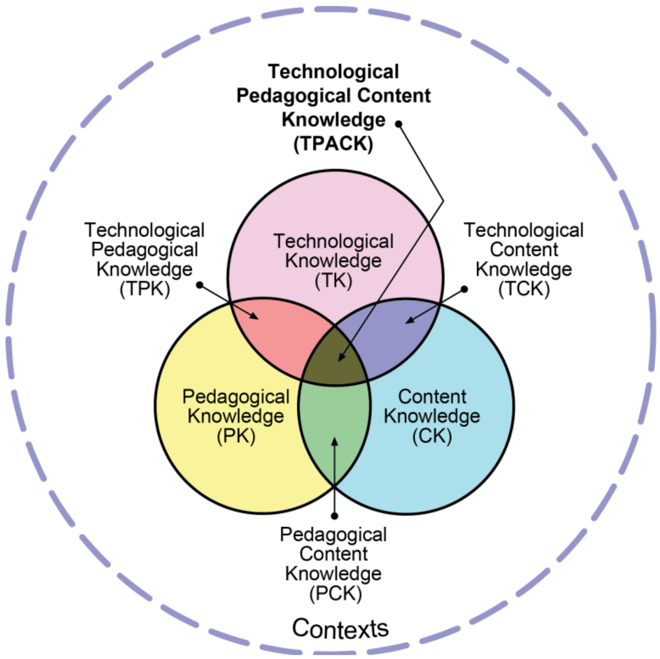


Figure 1. Representation of Technological Pedagogical Content Knowledge (TPACK) as teachers' transformed knowledge. Reproduced by permission of the publisher, © 2012 by tpack.org

Elaborating on TPACK

The TPACK construct is recognized and supported by extensive research and scholarly work. Elaborating on TPACK, Niess (2005) extended from Grossman's (1989, 1991) description of the four components of PCK. The four TPACK components incorporate the influence of technology in teachers' instruction, considering the teachers':

1. Overarching conceptions about the purposes for incorporating technology in teaching mathematics topics;
2. Knowledge of students' understandings, thinking and learning in mathematics topics with technology;
3. Knowledge of instructional strategies and representations for teaching and learning mathematics topics with technologies;
4. Knowledge of curriculum and curricular materials that integrate technology in learning and teaching mathematics topics.

Building on these components, Niess, Sadri and Lee (2007) described TPACK development as a process in transforming teachers' TPACK. They linked Rogers' (1995) five-step process in the ultimate decision of whether to accept or reject a particular innovation with the analysis of extensive observations of teachers' learning about spreadsheets and how to integrate spreadsheets as learning tools in their mathematics classrooms. Through this effort, they found the teachers at different stages in their TPACK transformations:

1. Recognizing (knowledge), where teachers are able to use the technology and recognize the alignment of the technology with mathematics content yet do not integrate the technology in teaching and learning of mathematics.
2. Accepting (persuasion), where teachers form a favorable or unfavorable attitude toward teaching and learning mathematics with an appropriate technology.
3. Adapting (decision), where teachers engage in activities that lead to a choice to adopt or reject teaching and learning mathematics with an appropriate technology.
4. Exploring (implementation), where teachers actively integrate teaching and learning of mathematics with an appropriate technology.
5. Advancing (confirmation), where teachers evaluate the results of the decision to integrate teaching and learning mathematics with an appropriate technology. (Niess et al., 2009, p. 9)

Here's a scenario: Mr. D is a middle school mathematics teacher with a degree in mathematics who was excited as he learned to design dynamic spreadsheets for exploring algebraic problems. However, he was constrained in his acceptance of students using spreadsheets for exploring algebraic changes when thinking about constants and dependent versus independent variables. He believed that students needed to create multiple graphs with paper and pencil in order to identify changes in the constants and variables of different functions. He was willing to use the spreadsheet for his summarization of the results, where he would demonstrate the changes for the students; however, he was not willing to engage the students in spreadsheet explorations. While he was considered to be at the *recognizing* level, even with more work with spreadsheets, he resisted the idea of adding spreadsheets to his mathematics classes.

In a contrasting case, Mrs. A, a teacher with a mathematics education degree, was excited with the ease and visualizations that resulted as she worked with her students in designing dynamic spreadsheets in their explorations. She wanted her students to have this experience as they worked in groups to explore changes in constants and variables. She felt that this more visual approach helped them gain a better understanding than if they had to individually graph each of the problems. As she worked with her students with the spreadsheets, she envisioned additional experiences for using spreadsheets in her mathematics classes and was thus viewed at the *exploring* stage.

These stages were proposed as an iterative process in the development of TPACK rather than a strictly linear process. In essence, some aspects of what is learned about teaching a particular topic with one technology may provide a disposition toward the acceptance of another technology. But teachers need to explore different topics with each new technology, considering its applicability for supporting learning mathematics with that technology

Developing In-service Teachers' TPACK

Reconstructing in-service teachers' knowledge to reflect the ideas as described in TPACK requires teacher engagement in systematic inquiries about teaching, learning, subject matter and curriculum, and schooling, much like that described in Cochran-Smith and Lytle's (2001) conception of "knowledge-of-practice" as a "transformed and expanded view of what 'practice' means" (p. 276). Such a reformed conception assumes that knowledge is "socially constructed by teachers who work together and also by teachers and students as they mingle their previous experiences, their prior knowledge, their cultural and linguistic resources, and the textual resources and materials of the classroom" (p. 280). Transforming teachers' knowledge suggests their involvement in an inquiry knowledge-building community where reflection is a central component (Cochran-Smith & Lytle, 2001; Loughran, 2002; Schön, 1983).

Borko (2004) describes a process for identifying high–quality teacher change programs as called for in transforming teachers' TPACK, as beginning with studies in a single site, exploring relationships between teachers as learners in a specific learning trajectory. Confrey and Maloney (2010) expand this thinking, describing the need for identifying a learning trajectory as a "researcher-conjectured, empirically-supported" description of an "ordered network of experiences" where teachers as students move from "informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time" (Confrey & Maloney, 2010, p. 968). In this manner, teachers are engaged in instruction designed to move from informal ideas through successive refinements toward a transformed knowledge for teaching with technology. Through this purposeful learning trajectory, they develop knowledge through their experiences as they are engaged in instructional strategies that ultimately model teaching with technologies.

Teachers as learners are, thus, charged with becoming aware and critical of their own and others' assumptions about teaching to achieve a paradigm shift that transforms their thinking and actions toward the ideas embedded in TPACK. McGonigal (2005) outlines five conditions and processes for fostering such a transformative learning experience for enhancing teachers' TPACK:

1. Teachers need an activating event to expose the limitations of their current knowledge.
2. Teachers need opportunities to identify and articulate underlying assumptions in their teaching knowledge.
3. Teachers need to engage in critical self-reflection, specifically considering the origin of underlying assumptions, and how these assumptions have influenced or limited their understandings about teaching.
4. Teachers need to engage in critical discourse with other teachers and the adult teacher educator in the process of examining alternative ideas and approaches.
5. Teachers need opportunities to test and apply their new perspectives.

Online TPACK Learning Trajectory

We investigated an educational setting where in-service teachers' professional learning experience is provided through online programs. The first author's research group designed an approach based on Niess and Gillow-Wiles (2013, 2014) empirically-supported learning trajectory, to frame online TPACK learning experiences to engage teachers in knowledge-building communities designed toward transforming the teachers' TPACK. The trajectory recognized key instructional strategies through a social metacognitive constructivist instructional framework that identified key tools and processes for organizing the TPACK content development in online asynchronous, text-based inquiry learning experiences.

Tools. Two tools support the online professional development: (1) a community of learners and (2) reflection. With the challenge for establishing connections among the learners and the instructor in online learning, establishing a community of learners provides an important tool for supporting the learners in communicating and interacting through discussions about the tasks and ideas being developed. This tool provides a ***social*** presence such that the community functions as a knowledge-building community. Through this community, the construction of knowledge is a "social activity, with new information and ideas brought into the discourses of a community that shares goals for knowledge advancement and recognizes contribution" (Scardamalia & Bereiter, 1993, p. 38-39). Such a social presence establishes community member participation and educational experiences for meaningful learning, open communication, and group cohesion as the learners engage in active roles to make sense of new information and ideas (Bereiter, 2002; Garrison, Anderson, & Archer, 1999; Garrison & Cleveland-Innes, 2005; Hill, Song, & West, 2009; Kinsel, Cleveland-Innes, & Garrison, 2005; Rourke, Anderson, Garrison, & Archer, 1999; Scardamalia & Bereiter, 1993; Sung & Mayer, 2012; Swan & Shih, 2005).

Reflection was the second important tool in the online learning trajectory. Critical reflection supported the cognitive presence in the online learning trajectory. Learner engagement through reflection happened in multiple ways, such as having learners prepare content reflective essays, reflective essays on the community engagement and peer reviews of another learners' work. The online portfolio provided a consistent way for each teacher to capture, reflect on and share their progress. Basically, the community of learners dynamically integrated the social, cognitive, and teaching presences in the online environment, supporting higher order learning through the reflective actions that result in deep approaches to learning (Garrison & Cleveland-Innes, 2005).

Processes. The online learning trajectory also includes two key processes for incorporating the tools: (1) shared/individual knowledge development and (2) inquiry learning. As the learners participates in the community of learners' activities, they share their understandings of how they are interpreting the ideas. As the discussions evolve through their interactions, the learner's individual knowledge matures. The learners move between group and individual knowledge-building so as to ultimately create an understanding that more clearly reflects a world view with respect to the learning experiences (Dunlap & Lowenthal, 2014; Rienties, Tempelaar, & Lygo-baker, 2013; Swan, 2001). As a result, the learners' individual knowledge expands beyond that which they were able to develop independently.

The second key process with the knowledge-building communities relies on inquiry-based activities to provide the learners with tasks, opportunities and experiences where they negotiated their understandings of the content. The inquiry process immerses them in constructing their understandings, where they take ownership of their learning, beginning with questions and their explorations where they investigated worthy questions, issues, problems or ideas. They ask questions, gathers and analyzes information, generates solutions, make decisions, and justifies their conclusions. The resulting actions interweaves multiple technologies, instructional approaches, and content topics through multiple units. Throughout the process, the participants consistently engage in thinking and reflecting about the dynamic interactions among content, pedagogy and technology that emerges from the tasks in their online learning experiences (Roberts, 2002; Wheatley, 1992).

An Online TPACK Instructional Program

Our program for transforming mathematics teachers' TPACK for teaching with technologies developed four new university graduate level courses. Three of the courses are fully online and focused on technology education. These courses are combined with a blended course that incorporates teachers' practical experiences where they implement their newly developed technological knowledge in their own classrooms with an online experience through a community of learners' inquiry and discourse about reformed-based pedagogical strategies.

Technological Education Courses

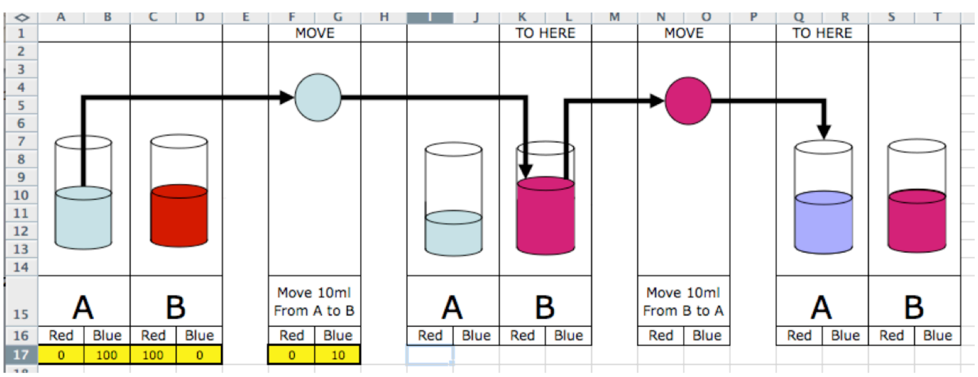
Three courses (SED 520, SED 521, and SED 522) engaged the teachers in experiences as students where they learn about and with some technologies in ways that challenge and advance their thinking about learning mathematics with these technologies. During these experiences, the teachers interact in small communities of learners in discourse, responding to the experiences as students and then as teachers. They use critical reflection in considering the dilemmas, their personal experiences and their discussions to inform their TPACK understandings.

For each unit in these courses, *inquiry* tasks challenge the teachers to examine and explore specific technologies. For example, in SED 520 one unit challenges them to examine and explore the temperature probe, a technology with which they are typically unfamiliar. They are provided with 10 laboratory experiments to guide their explorations as they learn how to collect temperature data and use the accompanying software to analyze the data, both graphically and numerically. As they work with the experiments, they also engage in questioning and discussion with their assigned small community of learner groups, discussing key questions as they examine the readings and their experiences in learning about and with the probeware. The unit emphasizes higher order thinking and inquiry learning to shape their ideas for integrating the technology as a learning tool in mathematics. To conclude the unit, the students prepare a critical reflection on their experiences with the technology, on the key questions and the inquiry processes and how their shared/individual experiences through the community of learner's discussions as well as their individual experiences form their understandings in this unit.

SED520. Integrating Technology and Literacy in Learning Mathematics (SED 520) is a course focused on multimedia technologies for twenty-first century mathematical literacy incorporating the 4 C's in teaching and learning mathematics. The course is arranged in five, two-week unit experiences. The technologies are purposefully selected and organized with the content. Unit 1 begins with presentation software, a technology with which teachers typically have experience. They are charged with using a Google presentation in a cooperative, collaborative experience to inquiry and examine new Web 2.0 technologies (Diigo, Wordle, Blabberize, Glogster, Voki, and Popplet for example). Each class member has responsibility for creating one slide that describes one specific technology term and how it can be used in education and a second slide that introduces them as new members of the class. Unit 2 promotes the teachers as learners exploring a technology with which they are unfamiliar – the temperature probe, as previously described. This experience engages them as students, where they must become familiar with the technology and consider how it might be used in learning mathematics; additionally, in their roles as teachers they consider what instructional strategies are needed to engage students in learning with this technology. The Web Inquiry Unit 3 pairs teachers in the design of a web presentation to guide students in specific mathematical explorations. This unit uses a framework similar to that in WebQuests (http://webquest.org/) to guide students in mathematical inquiries through web experiences (see http://webinquiry.org/ for sample inquiries). Unit 4 expects the teachers to work individually in designing three lessons that require use of a specific web application for learning mathematics. Teachers typically consider applications such as those available through the National Library of Virtual Manipulatives (http://nlvm.usu.edu/en/nav/vlibrary.html) or other similar libraries of interactive mathematics experiences. Finally, Unit 5 organizes the teachers as small cooperative groups for group analysis and writing experiences. These last two units expect the teachers to think about the comparison of knowledge gained as individuals (in Unit 4) versus that gained through cooperative and collaborative sharing with technologies (in Unit 5). The culminating product for this course is a web portfolio that presents the various products produced in each of the units of the course, demonstrating their knowledge of multimedia mathematical literacy.

SED 521. The second course, SED 521 (Teaching Mathematics With Digital and Video Technologies), uses inquiry experiences with digital images and videos to engage the teachers as students in higher order thinking and inquiry in mathematics. The major course difference between this course and SED 520 is the organization of the technology experiences into two major units, one focused on teaching mathematics with digital images and the second on incorporating videos as mathematics learning tools. Weekly, specific inquiry tasks confront the teachers. For example, during the first unit, they are challenged to gather digital images that afford opportunities for engaging students in twenty-first century learning of mathematics. The teachers are challenged to engage in their communities of learners, exploring these images to respond to this question: How might students' selection, acquisition, and presentation of digital images be beneficial for their learning of mathematics topics? The teachers engage in discourse with their assigned communities of learners and complete critical reflections through this question and the accompanying tasks and experiences with multiple digital images. Over the course of all the units they discuss the key questions and how their shared/individual experiences as well as their individual experiences influence their developing understandings. The culminating course product is a collection of digital images and videos with lessons for using them as mathematics learning tools. The collection also includes a video that portrays an example of what students might develop to communicate their higher order thinking when solving mathematics problem.

SED 522. The third course (SED 522), Dynamic Spreadsheets as Learning Tools in Mathematics, provides teachers with opportunities to explore algebraic reasoning when engaging students in learning with spreadsheets. Throughout the 10 units, inquiry tasks that involve developing skills with spreadsheets, the teachers' understanding for designing dynamic and dependable spreadsheets as a mathematics learning technology develops. The units model specific problems for gaining knowledge through access to spreadsheets. For example, Figure 2 presents a problem the teachers as students are to solve using relative cell referencing.



One beaker (A) has 100 ml of a blue liquid and the other beaker (B) has 100 ml of red liquid. First move 10 ml from A to B. Then move 10 ml from B to A. How many exchanges are needed before the liquid in both bottles contains the same concentration of blue and red liquids?

Figure 2. Mixture spreadsheet problem for SED 522

After designing a solution, they must develop a graphical representation (Figure 3) of their design and engage with their communities of learners' knowledge-building communities to explore multiple TPACK-related questions:

1. Where might this problem be useful in the mathematics curriculum?
2. What spreadsheet skills do students need for designing such a spreadsheet solution?
3. How should the students be organized for solving this problem?

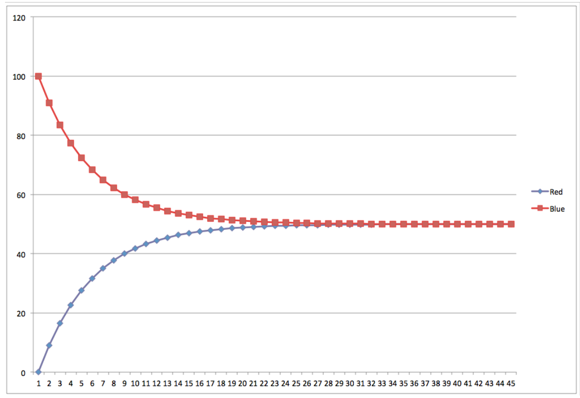


Figure 3. Graphical representation of the solution

In completing this spreadsheet course, the teachers' final project is a portfolio containing their collection of mathematics problems with solutions, curriculum plan for integrating spreadsheets as tools for exploring mathematics problems, and a final reflection on preparing their students to use the spreadsheet as a mathematical tool.

Blended Online and Practical Course: SED 594

Given these new online courses, the program designers questioned whether the experiences actually translated to the teachers' classrooms. For this reason, a fourth course, SED 594 (Advanced Teaching Strategies in Mathematics), was added to the TPACK program. As a blended course, this course combines online discussions and explorations about reform-based instructional strategies with practical teaching experiences in the teachers' own classrooms. Through the online explorations and discussions in their knowledge-building communities, they examine reform-based instructional strategies. Concurrently with their communities of learners' discussions, the teachers individually design, implement, analyze and extensively reflect on their personal, practical teaching experiences in their five-day unit in their mathematics classrooms. Essentially, they engage in action research about their own teaching as they gather artifacts to describe their instructional strategies, tools, and processes for engaging their students in learning mathematics with the selected technologies.

They incorporate extensive reflections in their electronic portfolios. These portfolios incorporate critical reflections from throughout the course to demonstrate how they implement their knowledge for teaching in their classrooms as they integrate technologies in teaching/learning mathematics. They gather two videos of their classroom instruction. They gather student products, examining whether the students demonstrate a strong understanding, average understanding or weak understanding of the concepts or processes. They complete multiple critical reflections throughout their instructional experiences (on the lesson designs before teaching, after teaching the lessons, about their students' work in the lessons, after watching videos of their instruction, and at the completion of the instruction). Weekly, they also reflect on their community of learners' explorations and discussions about the reformed-based instructional strategies. They peer review another teachers' portfolio, providing recommendations for enhancing the communication of the ideas. And, they complete a final in-depth analysis and reflection of the experiences they have had in their classroom teaching and their community of learner's discussions and interactions.

While they engage in the practical teaching experiences, they use the online community of learner groups to explore and discuss different instructional strategies, tools, and processes for teaching with technologies. Throughout their discussions, they cooperatively explore and examine reform-based instructional strategies – visible thinking, student discourse, grouping structures, and multiple representations for motivation and engagement. They peer-review another's electronic portfolio with the goal of improving the communication of the events and thinking. This review prepares them for the final critical reflection on the entire blended course experience, including the instructional experiences, video analyses, plans for improving their future instruction, and discourse in their community of learners where they discuss and consider instructional strategies for teaching with technologies. This final critical reflection reveals how their TK, TPK, and TCK merges with their PCK in the process of transforming their TPACK.

Teachers' TPACK Transformations

To investigate the influence of the online TPACK instructional program on teachers' knowledge transformations, we analyzed observations using the teachers' artifacts, expressions and reflections describe. The analysis reveals their knowledge, thinking and reflection with respect to the four TPACK components (Niess, 2005). Three representative cases (using pseudonyms Janis, Judy, and Lucy) were purposefully selected to display the patterns in the diversity of classroom situations and teaching levels. Janis, an elementary teacher, designed a fraction unit combining concrete and virtual manipulatives and games and activities she found on various websites for building students' understandings for representing, comparing and ordering fractions. Judy, a ninth-grade mathematics teacher chose to incorporate graphing calculators as a technology for her geometry course as they explored properties of transversals and the created angles, parallel lines, and perpendicular lines. Lucy, a high school mathematics teacher, incorporated the temperature probe with spreadsheet software in her pre-calculus class to engage her students in examining exponential, logarithmic and logistic functions.

Overarching Conceptions

As the teachers used technology in their own classrooms in the culminating SED 594 course, they were challenged to evaluate the value of the mathematical learning experiences for their students. In essence they were confronted with the question of whether or not the technology would support student learning in a purposeful and useful way.

Janis had previously taught her unit using only concrete manipulatives to build the students' understanding of fractions. "Using the virtual manipulatives was new and did cause me to change the way that students completed some of the tasks from previous years. The virtual manipulatives were nice because they didn't take up space like the concrete ones, and students were able to work with partners more instead of their whole table group." Janis saw the virtual manipulatives as adding to learning in an important way: "It allowed the students to make an instant connection between the manipulative and the mathematical concept because the symbolic notation was also shown on the screen." Reflecting on the value of the virtual manipulatives, she revised her original conceptions since the technologies provided a larger variety of learning experiences that she found important for student understanding of the concepts.

Judy's conception of graphing calculators as learning tools was intertwined with the importance of having students in working groups. She concluded that having the students work in groups led to solving problems and exploring ideas through technology tasks and deepened their learning because they were developing their own plans for solving the problems.

> When students worked in the groups with the graphing calculators, they helped each other when they were stuck. The groups also allowed students to explore many different equations of lines at the same time. They were exposed to many more possibilities of equations of lines than they would have been if I were leading the whole group.

Lucy designed her instruction to integrate technology and science in her mathematics instruction "The learners were supported in their learning through the computers and the Go!Temp Probe which collected data modeling a logistic curve. They were able to make real-world connections to their learning." She concluded "The connection to science makes the learning richer and fuller."

Students' Understandings, Thinking and Learning

Another concern for the teachers involved the question of how the technology aided in students' understandings, thinking and learning of specific mathematics topics. They were confronted with assessing the students' understandings and whether that understanding and thinking supported a stronger knowledge.

Janis noted that her students struggled to represent a given fraction three different ways but with the addition of the virtual manipulatives and other applications, she noted that when they built their fractions "on the fraction bar site they were forced to re-examine their ideas." She often talked about students' thinking and how their understandings and learning were impacted:

> Many students are visual learners, and it would not be enough for them to simply listen to another students' [explanation]. The combination of visual and auditory increased the number of students…engaged during group sharing and discussions and [increased] the chances that students [would] effectively process and retain the information in a meaningful way.

Judy gained an appreciation for the use of small group work with technologies for understanding students' thinking and understandings. During a Think-Pair-Share activity, she observed many conversations and heard many students mention slope. "It helped that they did not have to share among the larger class first ... more students [were] willing to share than usual, and I feel it is due to sharing with a neighbor first." She noted how using technology in the group work helped her understand the students' thinking. "Students were very engaged within the group work today. Perhaps the most engaged I have seen them in [the] group work. I feel the use of technology was a key factor."

Lucy realized her students' knowledge and thinking in one class did not necessarily transfer to other content areas. After the second day of her unit, she reflected that even though students had used microscopes in their biology class, they had a hard time translating that knowledge into the mathematics classroom. With a little guidance, "they got on track nicely and I anticipate this task to go faster as the days go on."

Instructional Strategies and Representations

Since the teachers were expected to design and implement a five-day unit, they were confronted with the task of identifying and scaffolding multiple instructional strategies to support students in learning the mathematics topics. They recognized the additional challenge for also scaffolding the instruction in such a manner that it supported student engagement with the technologies.

Janis connected the online discussions with her community of learners with her ideas for instructional strategies: "I incorporated some of the research-based instructional strategies that we have been discussing...Primarily, I used a lot of discourse, questioning, and collaborative learning…I tried to provide multiple representations…whenever possible, allowing students to connect their ideas with a visual." She gained a deeper understanding of strategies for integrating technologies through her video analysis. She saw the need to provide more direct instruction with the technologies. "What went really well was engaging students in the lessons and giving them opportunities to share their thinking as they worked collaboratively on the tasks." She also recognized that with students' ages and the novelty of using the technology, "I had to make sure that the tasks we did in the computer lab were very structured."

For Judy, the emphasis in the online discussions and her experiences with her instruction aligned her thinking about instructional strategies with grouping, problem solving, questioning and student discourse because these strategies "allow[ed] the learning environment to be centered on students who are actively engaged." From her work with these various strategies, she indicated that "Overall I want my classroom to be more student-centered and less teacher-centered."

The strategies Lucy implemented included discourse, questioning, inquiry, motivation, multiple representations, compare and contrast, cooperative and collaborative learning, providing feedback, and technology and science integration. Throughout the various science lab lessons, she was a guide. "I did not measure any pH, nor did I tell them which number it best matched. They asked each other and worked very collaboratively during this lesson…I know the more consistent I am with being a guide rather than the sole provider of information, the sooner it will become a habit."

Curriculum and Curricular Materials

A critical concern for the teachers in this final work was to identify a topic that was best suited for the technologies they wanted to include. They each want to make sure they were teaching the expected curriculum in order to support students as they advanced to future topics.

Janis' knowledge and thinking about her curriculum and curriculum materials changed as she reflected on the experiences in the lessons and what she saw in her lesson videos. She saw clear evidence about the importance of allowing students time to learn about and become familiar with the technology tools. "The students really enjoyed using the virtual manipulatives and they were motivated to complete the relevant tasks. However, I simply had not factored in enough time to let them explore with the manipulatives so that they could effectively utilize them during the lessons."

Judy indicated she needed to look for more ways to integrate technology: "Technology can aid in their learning and provide opportunities for them to be engaged." However, she noted that adding technologies to the learning required extending the curriculum and the need to "find activities and tasks" that would support students in learning with the technology.

A main objective in Lucy's unit was to "connect mathematics and science with the help of technology." She claimed that a promising practice might be for team teaching with the science teacher, where they would co-teach, and she would have the responsibility for the mathematics. "We plan to team teach using real-world data for mathematics…[making] an even better connection [of] math, science, and technology."

In Sum: Teachers' TPACK Transformations

After designing the online TPACK instructional program, our research effort focused on identifying the influence on the teachers' TPACK. The challenge was to identify how the teachers' thinking was transformed as a result of their engagement in their experiences. The most significant shift in their knowledge and thinking for all the teachers was a shift in their thinking about instruction. The majority of the teachers' primary approach prior to this program involved teacher-centered instruction. Yet, through the program, they shifted to valuing student-centered instructional strategies for teaching with technologies. This shift was reflected in the examination of the TPACK components.

Their overarching conceptions of the purposes for incorporating technology in teaching mathematics expanded, identifying the importance and value of multiple technologies, considering more than just content and thinking about the pedagogical strategies and the technological features. For example, they now considered technologies for communication, collaboration, and inquiry as important technologies for displaying mathematical ideas. In other words, their overarching conceptions shifted to a broader range of technologies for teaching in ways that were more student-centered.

Their reflections about student-centered strategies caused them to think more about student thinking and understanding. They reflected more on students' thinking with the technologies and what strategies were more supportive of their learning of mathematics with the technologies. They reflected on how students needed time to practice and work with the technologies before moving to more in-depth problems. They discussed and reflected on the scaffolding that they needed to do in their lessons as they integrated the technologies. They needed to allow students opportunities to discover and explore ideas with the technologies. Based on their experiences of working in groups and engaging in discourse with their communities of learners, they tested more student-centered strategies as they taught their lessons. After observing their videos, they saw how they needed to be "a guide rather than the sole provider of information."

The expanded vision on technologies in mathematics instruction also influenced their thinking about the curriculum. They shifted to the importance of expanding the curriculum to include teaching about the technology before expecting students to automatically gain understanding of mathematics immediately from working with the technology. During their work with various probeware technologies, they were engaged as students; through these experiences they identified the importance of learning about the technologies within content-specific explorations. As they designed and taught their lessons, they tried to incorporate explorations of mathematical ideas during the time in which their students were learning about the technologies.

In essence, the most obvious transformation in the teachers' TPACK was toward student – centered instructional strategies and the integration of multiple technologies for teaching in their classrooms. As teacher educators consider the design of programs to support the transformation of teachers' TPACK, they need to remember the importance of practical experiences. This work also identified the importance of combining practical experiences with opportunities for the teachers to engage in discourse through the knowledge-building communities of learners and extensive critical reflection on teaching in their classrooms. The combination provides teachers with opportunities to rethink, unlearn and relearn in ways that result in changing, revising and adapting their mathematical content and pedagogical strategies in light of the affordances of the multiple technologies – a TPACK transformation that reveals a deeper understanding of the role of technology and meaningful integration into their mathematics classrooms.

Discussion

From an outside perspective (that of the second author), this paper is a case study of how ambitious and laudatory teacher professional development goals can be achieved in an online or blended environment. The dual challenge it focuses on – of helping teachers learn to use technology effectively, through the medium of online and blended coursework – is also a key challenge of our time. Here I reflect on what general lessons and open issues emerge from thinking about this program as an example of where we should head in the future.

Transforming, Not Only Translating

Studies of successful online and blended learning highlight that when it works, it is not simply a matter of putting existing courses online (Bakia, Means & Murphy, 2004). Here, the dramatic shift is from a set of *courses* to a *knowledge building community.* Clearly, essential content from the courses was important to the firm foundations of the knowledge-building community. A knowledge-building community, however, implies a long-term structure of engagements, activities and supports that is much different from a typical course. For example, the engagements may start much more from teachers' own problems of practice. The activities may involve elements closer to that of lesson study (Hurd & Lewis, 2011) than to case studies in conventional courses. The supports involve more peer reflection and coaching, and less feedback from an authority. The concept of a knowledge-building community is also aspirational and would go beyond the results reported here. This report emphasized progress on individualized problems of practice, but a yet-richer knowledge-building community would also define community-level challenges and community-level knowledge about them.

From Courses to Competencies

The shift in this case also evokes a broader shift latent in professional learning more widely, a shift from counting course credit hours to recognizing new competencies (Nodine, 2016). Clearly, courses provided initial structure to the online experience. And yet, it is not clear that giving course credit is the most appropriate way to give teachers' recognition – course credit is normally tied to how many hours a teacher spent in activity and completion of assignments, but course credit much less frequently recognizes what a teacher now can actually do in a classroom. The focus here is nicely balanced, with clear opportunities to both shape teacher's experience based on course content but also to focus on and recognize classroom practice changes – not just seat time and assignment completion.

An emerging framework that would recognize this shift is the framework of educator micro-credentials (Berry, Airhart & Byrd, 2016). In this framework, complex shifts in teaching practice are broken down into smaller component competencies, and teachers have an opportunity to be recognized as they demonstrate the competencies. Importantly, there is no requirement to spend a fixed number of hours learning the competency, or to follow a set path in doing so. It seems in this example, and in many to follow in the future, online and blended learning will offer teachers lots of choices in how they learn and develop – and teachers clearly are not only showing a commitment in time and assignment completion, they are also reporting what they really do differently in their classrooms. This challenges us to think harder about how we will recognize and reward teachers for their learning, and the micro-credential movement gives one clear alternative (still much in development) to think with.

Coherent and Cumulative Frameworks, Leveraging Personal Experiences

A positive feature of online and blended learning is that the asynchronous and non-linear nature can make learning more relevant and timely given immediate challenges. It is clear these features can help teachers too, but if not balanced might lead to fragmentary knowledge. For example, the specific cases give a good sense of how this environment allowed teachers to effectively pursue personal goals. At the same time, this case also makes clear the value of a framework that helps organizing teacher learning to be coherent and cumulative – not just a reversion to the feels good, low calorie "take away" that old fashioned workshops have been critiqued for. In this case, the framework is TPACK, and the case shows how TPACK can organize the kinds of advances in individual teachers' practice into a more coherent whole. For example, TPACK can help us to see when teachers are making advances on many fronts – in their content knowledge, their technology knowledge, and pedagogical knowledge, and also in the relationships among these. Clearly, we need teachers who are learning on all these fronts.

And yet, TPACK is far from the only high-quality framework available. What TPACK appears best at – at least in this paper, is organizing qualitative observations of what teachers learned into categories. It is less obvious that TPACK is powerful in giving teacher's a sense of coherence and direction in their journeys. For example, if the journey is primarily towards student-centered learning, one might like a broad theoretical frame to organize the why, how, and what of that transformation. What might be the role of a framework like the one from *How People Learn* (National Research Council, 2000), which describes the best learning environments as learner-centered (one of the characteristics here), but also knowledge-centered, assessment-centered, and community-centered? There are also recommendations like the TRU framework (Schoenfeld and the Teaching for Robust Understanding Project, 2016), which is more specific to mathematics, and focuses on content, cognitive demand, equitable participation, agency, and formative assessment – and a framework like this might give coherence and direction to what teachers are learning overall. Overall, TPACK is clearly a valuable guiding framework, but for the field as a whole, it is not clear if focusing on the categorical differences among types of knowledge gives enough of a normative of what great math teaching with technology looks like to direct and make coherent teachers' long-term growth.

Further Challenges of Digital Professional Development Futures

One huge advantage that is apparent in this case is that the presenting problem is that teachers do not have enough prior experience as *digital learners.* And thus, it seems highly appropriate that professional development position the teachers themselves as digital learners. For example, as the case shows, teachers can more directly experience learning with the types of virtual manipulatives, online spreadsheets, and algebraic tools that their students can use. This is valuable because as mathematics is represented in digital form, how conceptual understanding arises shifts – new paths to learning emerge. For example, in the case of Dynamic Geometry, construction becomes more powerful in digital form than with compass and straight edge and a fuller counterpoint to proof in the learning experience (De Villiers, 2004). Social learning is also transformed through digital possibilities, such as the possibility to contribute to share mathematical constructions and experiences (Stroup et al., 2002). Teachers need to experience this first hand and be supported to reflect on their experience as learners and what implications this can have for their teaching, in a process of *Instrumental Genesis* (Drijvers et al., 2010).

Finally, challenges of access and equity arise constant in a digital world, and careful work is needed to understand who participates and how the online experience can produce beneficial or harmful experiences. We cannot pretend naively that online experiences will be safe and positive for all. There are also likely major challenges of data sharing as teachers share their classroom experience not just in a closed course classroom, but also now in an online environment where there is no limit to where the content they share may end up. There are also the positive and potentially negative implications of teachers' online activities producing data. On the positive side, we can potentially track and learn much more about the trajectories of teacher professional growth by studying the trajectories of their online activities (through the data those activities leave behind). Rather than a single university course progression, we can learn in detail about the multiple varied paths that teachers take towards professional knowledge; the different resources they use; the nature and range of social interactions that support them. But on the challenging side, there is always the possibility of such data being exploited in inappropriate ways, and the need to develop guards against this.

Conclusion

As more and more technologies become available as mathematics learning tools, teacher educators will be challenged to identify professional development avenues for supporting teachers in transforming their skills for teaching with the technologies. The online TPACK instructional program provides only one example of an online professional development program focused as a knowledge-building community to engage teachers in experiences that influence how their TPACK is transformed as they think about and learn from their experiences while learning as students and as they are implementing their accumulated knowledge in classroom practices. The combination of online learning with practical experiences provides a context within which these experiences were provided. The impact supports them in recognizing and valuing shared knowledge for expanding and enhancing their individual knowledge about learning with technologies. A significant teacher knowledge shift from their personal experiences found a change in their beliefs about their primary instruction to actively engaging students for learning with technologies – toward more cooperative and collaborative inquiry activities where students engage in discourse and reflection. The impact of the collective transformative learning experiences provides an important direction for in-service teacher professional development in the twenty-first century.

References

Angeli, C., & Valanides, N. (2009). Epistemological and methodological issues for the conceptualization, development, and assessment of ICT-TPCK: Advances in technological pedagogical content knowledge (TPCK). *Computers and Education, 52*(1), 154-168.

Bakia, M., Means, B., & Murphy, R. (2014). Learning online: What research tells us about whether, when and how. Routledge.

Bereiter, C. (2002). *Education and mind in the knowledge age.* Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Berry, B., Airhart, K. M., & Byrd, P. A. (2016). Microcredentials: Teacher learning transformed. Phi Delta Kappan, 98(3), 34-40.

Borko, H. (2004). Professional development and teacher learning: Mapping the terrain. *Educational Researcher, 33*(3), 3-15.

Cochran-Smith, M., & Lytle, S. L. (2001). Beyond certainty: Taking an inquiry stance on practice. In A. Liberman & L. Miller (Eds.), *Teachers caught in the action: professional development that matters* (pp. 45-58). NY: Teachers College Press.

Confrey, J., & Maloney, A. (2010). The construction, refinement, and early validation of the equipartitioning learning trajectory. *Proceedings of the 9th International Conference of the Learning Sciences – Volume 1*, 968-975.

De Villiers, M. (2004). Using dynamic geometry to expand mathematics teachers' understanding of proof. *International Journal of Mathematical Education in Science and Technology*, *35*(5), 703-724.

Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in mathematics*, *75*(2), 213-234.

Dunlap, J., & Lowenthal, P. (2014). The power of presence: Our quest for the right mix of social presence in online courses. In A. P. Mizell & A. A. Pina (Eds.), *Real Life Distance Education: Case Studies in Practice*. Charlotte, NC: Information Age Publishing.

Garrison, D. R., Anderson, T., & Archer, W. (1999). Critical inquiry in a text-based environment: Computer conferencing in higher education. *Internet and Higher Education, 2*(2–3), p 87-105

Garrison, D. R., & Cleveland-Innes, M. (2005). Facilitating cognitive presence in online learning: Interaction is not enough. *American Journal of Distance Education, 19*(3), p 133-148. doi: 10.1207/s15389286ajde1903_2

Grossman, P. L. (1989). A study in contrast: Sources of pedagogical content knowledge or secondary English. *Journal of Teacher Education, 40*(5), 24-31.

Grossman, P. L. (1991). Overcoming the apprenticeship of observation in teacher education coursework. *Teaching and Teacher Education, 7*, 245-257.

Hill, J., Song, L., & West, R. (2009). Social learning theory and web-based learning environments: A review of research and discussion of implications. *American Journal of Distance Education, 23*(2), p 88-103. doi: 10.1080/08923640902857713

Hurd, J., & Lewis, C. (2011). *Lesson study step by step: How teacher learning communities improve instruction.* Portsmouth, NH: Heinemann.

Kinsel, E., Cleveland-Innes, M., & Garrison, D. R. (2005). *Student Role Adjustment in Online Environments: From the Mouths of Online Babes*. uwex.edu, p 1-5.

Loughran, J. (2002). Effective reflective practice: In search of meaning in learning about teaching. *Journal of Teacher Education, 53*, 33-43.
McGonigal, K. (2005). Teaching for transformation: From learning theory to teaching strategies. *Speaking of Teaching: The Center for Teaching and Learning, Stanford University Newsletter, 12* (2).
Means, B., Toyama, Y., Murphy, R., & Bakia, M. (2013). The effectiveness of online and blended learning: A meta-analysis of the empirical literature. *Teachers College Record, 115(3),* 1-47.
Mishra, P., & Koehler, M. J. (2006). Technological pedagogical content knowledge: A framework for teacher knowledge. *Teachers College Record, 108*(6), 1017-1054. https://dx.doi.org/10.1111/j.1467-9620.2006.00684.x
Mishra, P., & Koehler, M. J. (2008, March). *Introducing Technological Pedagogical Content Knowledge*. Paper Presented at the Annual Meeting of the American Educational Research Association, New York City: NY.
National Research Council (2000). How people learn: Brain, mind, experience, and school: Expanded edition. Washington DC: National Academies Press.
Niess, M. L. (2005). Preparing teachers to teach science and mathematics with technology: Developing a technology pedagogical content knowledge. *Teaching and Teacher Education, 21*(5), 509-523.
Niess, M. L., & Gillow-Wiles, H. (2013). Developing asynchronous online courses: Key instructional strategies in a social metacognitive constructivist learning trajectory. *The Journal of Distance Education, 27*(1), 1-23.
Niess, M. L. & Gillow-Wiles, H. (2014). Transforming science and mathematics teachers' technological pedagogical content knowledge using a learning trajectory instructional approach. *Journal of Technology and Teacher Education (JTATE), 22*(4), 497-520.
Niess, M. L., Ronau, R. N., Shafer, K. G., Driskell, S. O., Harper S. R., Johnston, C., Browning, C., Özgün-Koca, S. A., & Kersaint, G. (2009). Mathematics teacher TPACK standards and development model. *Contemporary Issues in Technology and Teacher Education* [Online serial], *9*(1). Retrieved from http://www.citejournal.org/vol9/iss1/mathematics/article1.cfm
Niess, M. L., Sadri, P., & Lee, K. (2007, April). *Dynamic spreadsheets as learning technology tools: Developing teachers' technology pedagogical content knowledge (TPCK).* Paper presented at the meeting of the American Educational Research Association Annual Conference, Chicago, IL.
Nodine, T. R. (2016). How did we get here? A brief history of competency-based higher education in the United States. *The Journal of Competency-Based Education*, *1*(1), 5-11.
Partnership for 21st Century Learning. (2015). Retrieved from https://www.P21.org//our-work/p21-framework
Rienties, B., Tempelaar, D. T., & Lygo-baker, S. (2013). Redesigning teaching presence in order to enhance cognitive presence, a longitudinal analysis. In Z. Akyol & D. Garrison (Eds.), *Educational Communities of Inquiry: Theoretical Framework, Research and Practice* (pp. 109-132), Hershey, PA: IGI Global.
Roberts, B. (2002). Interaction, reflection and learning at a distance. *Open Learning, 17*(1), 39-55.
Roschelle, J., & Leinwand, S. (2011). Improving student achievement by systematically integrating effective technology. *NCSM Journal of Mathematics Education Leadership, 13(2),* 3–9.
Roschelle, J., Noss, R., Blikstein, P., & Jackiw, N. (2017). Technology for learning mathematics. In J. Cai (Ed.), *Compendium for Research in Mathematics Education.* Reston, VA: National Council of Teachers of Mathematics. 273-296.
Rogers, E. (1995). Diffusion of innovations. New York, NY: The Free Press of Simon & Schuster Inc.
Rourke, L., Anderson, T., Garrison, D.R., Archer, W. (1999). Assessing social presence in asynchronous, text-based computer conferencing. J. Distance Educ. 14(3), p 51–70
Scardamalia, M., & Bereiter, C. (1993). Technologies for knowledge-building discourse. *Com-munications of the ACM*, *36*(5), 37-41.
Scardamalia, M., & Bereiter, C. (2003). Knowledge building environments: Extending the limits of the possible in education and knowledge work. In A. DiStefano, K.E. Rudestam, & R.Silverman (Eds.), *Encyclopedia of distributed learning*. Thousand Oaks, CA: Sage Publications.
Schoenfeld, A. H. & the Teaching for Robust Understanding Project. (2016). *An Introduction to the Teaching for Robust Understanding (TRU) Framework.* Berkeley, CA: Graduate School of Education. Retrieved from http://tru.berkeley.edu.
Schön, D. A. (1983). *The reflective practitioner: How professionals think in action*. New York: Basic Books.
Stroup, W. M., Kaput, J., Ares, N., Wilensky, U., Hegedus, S. J., Roschelle, J., et al. (2002). *The nature and future of classroom connectivity: The dialectics of mathematics in the social space.* Paper presented at the Psychology and Mathematics Education North America conference, Athens, GA.
Sung, E. & Mayer, R. E. (2012). Five facets of social presence in online distance education. *Computers in Human Behavior*, 28(5), 1738-1747.

Swan, K. (2001). Virtual interaction: Design factors affecting student satisfaction and perceived learning in asynchronous online courses. *Distance Education, 22*(2), 306–331.

Swan, K., & Shih, L. F. (2005). On the nature and development of social presence in online course discussions. *Journal of Asynchronous Learning Networks, 9*(3), 115-136.

Thompson, A. D., & Mishra, P. (2007). Breaking news: TPCK becomes TPACK! *Journal of Computing in Teacher Education*, *24*(2), 38, 64.

Thoughtful Learning Organization. (2016). Retrieved from https://k12.toughtfullearning.com/FAQ/what-are-learning-skills

Wheatley, G. H. (1992). The role of reflection in mathematics learning. *Educational Studies in Mathematics, 23*(5), 529-541.

PERSPECTIVES ON THE NATURE OF MATHEMATICS

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As mathematics educators, we teach and research a particular form of knowledge. However, in reacting to Platonic views of mathematics, we often overlook its unique characteristics. This paper presents a Kantian and Piagetian perspective that defines mathematics as a product of psychology. This perspective, based in human activity, unites mathematical objects, such as shape and number, while explaining what makes mathematics unique. In so doing, it not only privileges mathematics as a powerful form of knowledge but also empowers students to own its objects as their own constructions. Examples and interdisciplinary research findings (e.g., neuroscience) are provided to elucidate and support the perspective.

Keywords: embodied cognition, mathematical epistemology, neuroscience, radical constructivism, reflective abstraction

Mathematics has been described as both a science and a language. Most sources define it as a collection of abstract sciences, but its objects of study are so varied that, according to Wikipedia, “mathematics has no generally accepted definition” (“Definitions of mathematics,” n.d.) Consider the following attempts:

- “the abstract science of number, quantity, and space” (“Mathematics,” n.d.)
- “the science of numbers and their operations, interrelations, combinations, generalizations, and abstractions and of space configurations and their structure, measurement, transformations, and generalizations. Algebra, arithmetic, calculus, geometry, and trigonometry are branches of mathematics” (“Mathematics,” n.d.)
- “a group of related sciences, including algebra, geometry, and calculus, concerned with the study of number, quantity, shape, and space and their inter-relationships by using a specialized notation” (“Mathematics,” n.d.)

What do number, quantity, space, and the various branches of mathematics have in common?

Mathematics is a unique body of knowledge owing to its apparent infallibility. Across millennia, continents, and cultures, mathematics has produced stubborn facts, so much so that we confidently assume that any alien life form, if intelligent enough, would recognize the prime numbers (Sagan, 1975). Students often appreciate the way that mathematics builds on itself, such as the way real numbers build on rational numbers, which build on integers. Scientists marvel at the “unreasonable effectiveness of mathematics in the natural sciences” (Wigner, 1960), such as when mathematical models predicted the existence and location of Neptune before it was discovered (see Norton, 2015). No wonder Platonism still holds sway in society and scientific communities alike.

As a mathematics education community, we often confront Platonist ideals, which position mathematics as something that lies beyond human experience. We understand the cultural influences and psychological roots of mathematical development and mathematics itself. We challenge mathematical myths but rarely acknowledge their persistent epistemological basis. For example, we cite Kline’s (!982) “Loss of Certainty” to break down perceptions of mathematics as a collection of immutable truths because such perceptions disinvite students to participate in creating mathematics (e.g., Chazan, 1990). However, we overlook the apparent certainty of mathematics as a feature that garners students' interests to begin.

Popular characterizations of mathematics do have a valid basis. There is a sense in which mathematics is infallible and builds upon itself, and mathematics holds a privileged position of predictive power among the sciences. However, these characterizations require psychological explanation rather than a Platonic dodge. Moreover, we need a definition that presents mathematics as a unified field of study rather than a collection of abstract sciences. What unifies mathematics? What are its objects of study? What is the basis for its reliability, utility, and ubiquity?

This paper presents a Kantian/Piagetian response—one grounded in cognitive psychology and buttressed by recent findings from neuroscience. Kline (1982) summarized the Kantian position as follows: "mathematics is not something independent of and applied to phenomena taking place in an external world but rather an element in our way of conceiving the phenomena" (p. 341). Piaget (1942), with Inhelder (1967), built upon this position by specifying children's development of mathematical structures used to organize the world, such as space and number. These structures depend on operations that, at once, demonstrate the unity and power of mathematics.

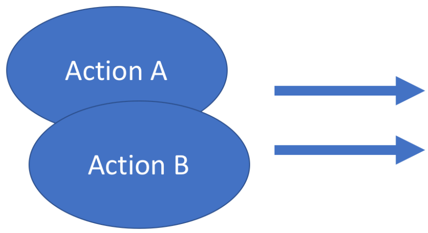
Mathematical Objects

Mathematical objects arise from our own activity within the worlds we experience. This is a view espoused by social constructivists, radical constructivists, and embodied cognitionists alike (Núñez, Edwards, & Matos, 1999; Vygotsky, 1986). The distinguishing feature of the Piagetian perspective concerns the role of abstraction, particularly *reflective abstraction*, in constructing those objects. Reflective abstraction is a psychological process that is notoriously difficult to grasp. As Chomsky lamented during a debate with Piaget, "my uneasiness with reflective abstraction is … that I do not know what the phrase means, to what processes it refers, or what are its principles" (Piattelli-Palmarini, 1980, p. 323). Here, we will attempt to specify the process of reflective abstractions and its principles, as well as its role in constructing mathematical objects.

We find Piaget's plainest description of reflective abstraction in *Genetic Epistemology* (1970). There, he describes the sensorimotor basis for logic and mathematics: "the roots of logical thought are not to be found in language alone, even though language coordinations are important, but are to be found more generally in the coordination of actions, which are the basis of reflective abstraction" (p. 19). He goes on to describe how actions become coordinated with one another, through reflective abstraction; but reflective abstraction does not apply to any and all actions—only those that are reversible.

Reversibility is another distinguishing feature of mathematics. Addition-subtraction, greater than-less than, integration-differentiation all form inverse pairs. However, Piaget (1970) refers primarily to reversibility of the mental actions that constitute these formalized operations, rather than the formalized operations themselves. For example, a student might know the sum of 10 and 5 but not know the sum of 9 and 6, even if she also knows that 9 is 1 less than 10 and 6 is 1 more than 6. In other words, she has not yet coordinated the actions of iterating (repeatedly integrating) units of 1 and disembedding them (separating units of 1 within the whole). Such a coordination relies upon organizing the actions of iterating and disembedding within a structure for composing and reversing them (compared to 5, 6 has an extra iterated unit of 1, which can be disembedded and composed with 9 to make 10). As educators, we might think about these as strategies, but through reflective abstraction, strategically coordinated actions become structures for assimilating mathematical situations so that the two sums become the same mathematical object, 15. Of course, numbers do not exist in isolation, so the coordination of mental actions like iterating and disembedding reorganize the child's entire number sequence, including 5, 6, 9, 10, and 15 (see Steffe, 1992).

Figure 1 illustrates the process of reflective abstraction. Several mathematics educators have recognized this process, or similar processes, as essential for mathematical development: Sfard (1992) referred to it as reification; Dubinski (2002) as interiorization; and Tall as encapsulation (Tall, Thomas, Davis, Gray, & Simpson, 2000). They all describe a process by which existing mental actions become coordinated to constitute a new mathematical object.



	i	A	B
i	i	A	B
A	A	B	i
B	B	i	A

Figure 1. Reflective abstractions and the construction of mathematical objects.

The table on the right side of Figure 1 represents the organization of actions within a mathematical group. This is a researcher's model for describing how the students' mental actions become coordinated with one another as composible and reversible operations; it does not imply that students are aware of a group structure (Piaget, 1970). Figure 1 illustrates the simplest example of coordinating mental actions, wherein two actions (A and B) are coordinated as inverses of one another and where i represents the identity element of the group (note that Piaget also referred to group-like structures that do not satisfy all conditions of a mathematical group but nonetheless model the coordination of reversible mental actions). The two arrows in Figure 1 represent the two aspects of objects noted by Piaget and Garcia (1986): "First of all, it is 'what can be done with them' either physically or mentally… (2) The meaning of object is also 'what it is made of,' or how it is composed. Here again, objects are subordinate to actions" (pp. 65-66).

The coordination of mental actions within group-like structures explains many of the unique features of mathematics. In particular, the reversibility of mental actions within the structure explains the reliability of mathematics. In the sciences, reliability is repeatability. The natural sciences never attain perfect repeatability because the initial conditions of a situation cannot be precisely reproduced. However, in mathematics, reversing one action with another action (e.g., A and B compose to form i, in Figure 1) returns one to the same exact starting point every time.

Coordinations of action also explain the ubiquity of mathematics because they become structures for organizing experience. For example, when I see seven cars in a parking lot, nothing in the parking lot imposes 7 upon me. Instead, I assimilate my perceptual experience by coordinating mental actions of unitizing (separating out each perceived car and treating it as a unit identical to the others) and iterating resulting units in one-to-one correspondence with my number sequence.

Furthermore, coordinations of action explain how mathematics builds upon itself, because the process of reflective abstraction does not end with the construction of the first structures. Rather, those structures, as objects, become material for further operating. For example, I can consider any multiplicity of 7 by acting upon one copy of my number sequence with another copy of it (Steffe, 1992). Such structures explain the subjectivity of mathematical experience when, for example, I see three rows of seven cars and assimilate them as three 7s, whereas a young child might see a spatial pattern but not the numerical structure of 3 times 7.

Evidence from Neuroscience

As noted in the introduction, definitions of mathematics generally refer to the study of a collection of objects, usually including number and space. As mathematics educators, the construction of number may seem more familiar, but Piaget and Inhelder (1967) used space (along with number; Piaget, 1942) as a primary example of a mathematical object. They demonstrated that space does not exist as an innate construct, as Kant had assumed, but that children construct it through the coordination of displacements within a group for composing and reversing them. In this section, we will see tight connections between space and number as psychological and neurological phenomena that depend on coordinated actions, beginning with sensorimotor activity.

One early connection concerns object permanence and the onset of self-locomotion (crawling). Developmental psychologists take object permanence as a critical marker in early child development, whereby children learn that objects persist in space even when removed from the child's perceptual field. Bell and Fox (1996, 1997) conducted studies on 76 eight-month-old infants, separated into four groups: pre-crawlers, crawlers with 1-4 weeks of experience, 5-8 weeks of experience, and 9 or more weeks of experience. Greater experience in crawling was associated with the development of object permanence.

Piaget and Inhelder (1967) had tied object permanence to children's construction of sensorimotor space, wherein objects would have residence. More recently, psychologists have associated object permanence with "spatial working memory", wherein children coordinate spatial transformations, such as displacements and rotations (e.g., Bell, 2001). Together with the findings from Bell and Fox (1996, 1997), the collective literature suggests that crawling provides sensorimotor experience that is critical to the construction of space as a coordination of displacements. After all, crawling enables children to transform their perceptual fields through voluntary movement, which (from the child's perspective) amounts to a displacement of space itself, similar to the transformations of space described by a vector field (or the group of vectors, under addition).

We find similar connections between embodied/sensorimotor experience and the child's construction of number. In particular, manual and numerical digits go hand-in-hand, in a manner that transcends etymology (see Norton, Ulrich, Bell, & Cate, 2018). For example, as mathematics educators, we know that children generally learn to count with the aid of their fingers as manipulatives, but recent neuroimaging studies indicate that the connection persists into adulthood. Specifically, neural substrates for finger recognition and finger use (e.g. pointing) overlap with those for counting and arithmetic, even among adults (Soylu, Lester, & Newman, 2018)—so much so that researchers now hypothesize that areas of the brain that evolved for manual dexterity (e.g., tool use) have been re-purposed to support mathematical development (Penner-Wilger & Anderson, 2013). In considering these neural substrates, the intraparietal sulcus (IPS) stands out.

Figure 2 presents a diagram of the neo-cortex—the outer layer of the human brain—and a few of its main regions. The frontal lobe lies above the eyes and plays the leading role in executive function (working memory, inhibitory control, and decision making). The parietal lobe rests toward the back of the brain and is generally associated with spatial reasoning, including hand-eye coordination. Between those two lobes sits the sensorimotor cortex, which initiates voluntary movement. The IPS aligns with the sensorimotor area associated with hand movement and runs between the upper and lower halves of the parietal lobe. This positioning would suggest that the IPS plays an important role in the manipulation of objects in space, which it does.

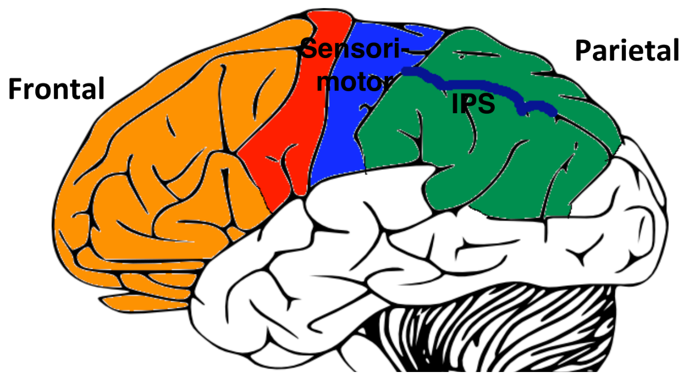


Figure 2. Neo-cortex.

In addition to its role in tool use and other coordinated actions involving the hand (e.g., Mruczek, von Loga, & Kastner, 2013), the IPS is implicated in virtually every neuroimaging study of numerical and spatial reasoning (e.g., Dehaene, 1997; Kucian et al., 2007). In and around the IPS we find the common neural substrate for the two primary objects of the mathematical sciences: space and number. There we also find their common link to coordinated sensorimotor activity, especially involving the hands.

The IPS exists as part of a network that includes the frontal lobe and the angular gyrus—an area in the lower part of the parietal lobe associated with memorized tables of information (e.g., multiplication tables). Studies of mathematical development generally show a shift, from frontal to parietal areas of the network, as children learn: "Solving a new multiplication problem involves the IPS bilaterally and also the frontal lobes, while dealing with the same problem a second time shifts the focus of activity to the angular gyrus in the left parietal lobe" (Butterworth & Walsh, 2011, pp. 19-20). Other studies (e.g., Ansari, 2008), show a similar frontal-parietal shift associated with age.

As we have mentioned, executive function is a primary role of the frontal lobe. It directs limited working memory resources (including spatial working memory) to solve novel problems. As children learn—either rotely through memorizing multiplication tables or through the development of conceptual structures—working memory is offloaded so that the same task becomes less demanding. We posit that areas in and around the IPS serve as the neural substrate for spatial-numerical structures. This view, too, is supported by neuroimaging studies (Hubbard, Piazza, Pinel, & Dehaene, 2005) and implies that the IPS is heavily involved in assimilating mathematical experiences. Resources from the frontal lobe are recruited when the assimilated experience becomes problematic. As such, frontal-parietal coherence (areas within the two lobes working in tandem, as indicated by brain wave frequencies) would be the neural correlate of mathematical development.

Returning to Bell and Fox's (1996) study of crawling, infants with 1-4 weeks of crawling experience exhibited greater frontal-parietal coherence than pre-crawling infants and infants with more crawling experience. Thus, self-locomotion appears to provide a sensorimotor foundation for the development of object permanence and the construction of space—the play space for subsequent geometric construction. In the next section, we consider the case of Euclidean geometry.

Euclidean Objects

Euclid was the first mathematician to formalize mathematics on an axiomatic basis. The purpose of the Elements, Book I, was to prove the Pythagorean (sic) theorem from common notions and postulates (axioms). This is evidenced by the appearance of the Pythagorean theorem and its converse as the final two propositions in the book (Propositions 47 & 48). But what was the basis for these axioms and formal arguments? Evidence appears in the axioms and arguments themselves. The first three axioms (Postulates 1-3) are Plato's rules for straight edge and compass constructions, indicating their sensorimotor basis within Greek culture. The chain of propositions leading from those axioms to the Pythagorean theorem indicates the kinds of mental actions behind Euclid's intuitions. Here, we demonstrate how coordinations of spatial transformations, like sweeping, shearing, and rotating, form the psychological basis for geometric objects.

Figure 3 illustrates the diagram Euclid used to support his arguments for the Pythagorean theorem. Essentially, he argued that the areas of the yellow and blue squares were equivalent to the areas of the yellow and blue rectangles, respectively (Proposition 47). The argument depended on previous propositions demonstrating that shearing triangles and parallelograms does not affect their areas (Propositions 35-38). In Figure 3, triangle DAC is the result of shearing triangle DAG (half of the blue rectangle) along segment FC. Likewise, triangle ABE is the result of shearing triangle ACE (half of the blue square) along segment HB. Because these triangles are congruent (Euclid relied on Proposition 4—a side-angle-side argument, which he demonstrated through Common Notion 4, displacing those elements from one triangle onto another and showing that the remaining sides must also coincide), the areas of the blue rectangle and blue square (each having twice the area) are equivalent. The same argument works for the yellow regions, thus proving the Pythagorean theorem.

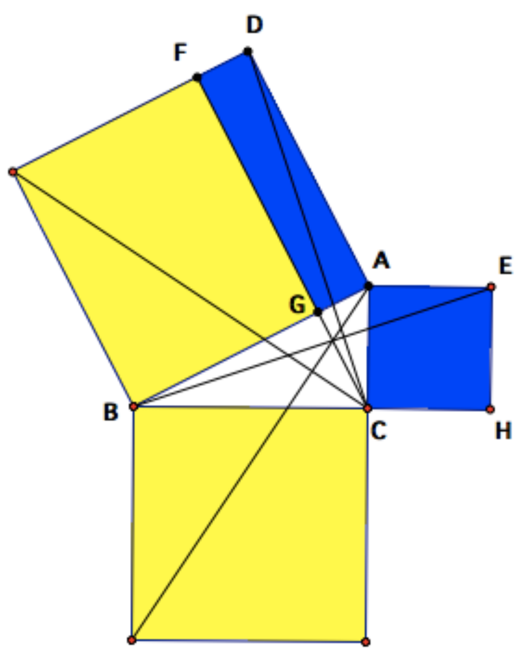


Figure 3. Euclid's proof of the Pythagorean theorem.

In sum, Euclid proved the Pythagorean theorem by transforming mathematical objects (e.g., squares) through mental actions of bisecting, displacing, and shearing. The mathematical objects being transformed are themselves the result of mental actions, such as sweeping (sweeping a point to make a line segment and sweeping a line segment to make a square, as in Proposition 46). Constructing and transforming mathematical objects in this way fits Piaget's descriptions of mathematical objects as coordinations of mental actions, as indicated by the two arrows in Figure 1: (1) mathematical objects arise through the coordination of actions and (2) can be subsequently transformed through further action. Thus, mathematical objects are characterized by both the actions that constitute them and the manner in which actions transform them, particularly aspects of the objects that remain invariant under transformation (e.g., the area of a parallelogram as invariant under the transformation of shearing).

Consider the simpler example of angle sums within a triangle (Proposition 32). Like number, children have to construct triangles: "children are able to recognize and especially to represent, only those shapes which they can actually reconstruct through their own actions" (Piaget & Inhelder, 1967, p. 43). Understanding triangles as mathematical objects requires children to move beyond the figurative material that represents or symbolizes them and to focus on the underlying mental actions that constitute them. The perfect triangle does not exist as a Platonic ideal, but rather as a coordination of actions, including sweeping and rotating.

To demonstrate that the angles in a triangle sum to a straight angle (pi, or 180 degrees), consider the construction of the triangle itself. It begins with a segment (side) swept from one vertex to another. Each pair of adjacent sides forms an angle, which measures the degree of openness, or rotation, between them (Moore, 2013). Figure 4 illustrates the three rotations (A, B, and C) that occur between pairs of adjacent sides. Each rotation is a transformation of one side onto the adjacent side, preserving the property of being a straight segment (a sweep from one vertex to another) but transforming its length and direction. After three such transformations, the original segment has been transformed back onto itself but in the reverse direction. In other words, the combined effect of composing the three angle rotations is a rotation of 180 degrees.

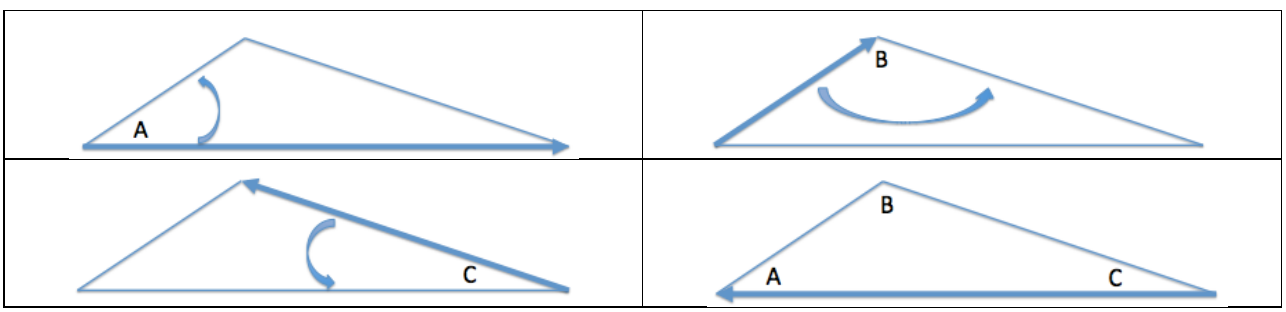


Figure 4. Sum of angles in a triangle.

What we see in the *Elements* is the historical trace of Euclidean geometry from sensorimotor activity all the way up to the first axiomatic system. Thus, we can trace formal mathematical objects, such as right triangles with all of their properties, all the way back to their psychological roots. Like numbers, shapes and their properties (e.g., the Pythagorean theorem) depend upon the coordination of mental actions. For the Greeks, those mental actions were derived from the sensorimotor activity of playing in the sand with compass and straight-edge. Reflective abstraction provides the mechanism for moving from each stage to the next: from sensorimotor activity to mental actions, to the construction and transformation of triangles, to the formal demonstration of the Pythagorean theorem.

Summary

From the Kantian/Piagetian perspective described here, mathematics can be defined as the study of reversible mental actions and the structures that organize them (Piaget, 1970). This unifying definition applies to shape as well as number, both of which arise from the coordination of actions that have a sensorimotor basis (Piaget, 1942; Piaget & Inhelder, 1967). The definition also explains unique features of mathematics while empowering students to construct, transform, and study mathematical objects on the basis of their own activity. The infallibility of students' constructions owes to the reversibility of the mental actions that undergird them (Piaget, 1970). For example, if a child has defined a triangle on the basis of its three planar angles (rotations), composing those rotations with one another inevitably leads to the conclusion that they form a straight angle—a single 180-degree rotation that can be partitioned into the three angles that constitute it. The trick is to find a way to compose all three rotations without appealing to the drawn figure itself but rather to the organization it represents. This process of organizing rotations within a structure for composing and reversing them is the process of reflective abstraction.

When we focus on students' available mental actions and their engagement in sensorimotor activity, we are valuing students' mathematics as they construct new mathematical objects—objects that empower students to model and structure the worlds they experience. Thus, the appeal to students' mental actions is an appeal for equity in mathematics education. Building from the work of Noddings (1999), Hackenberg (2010) has described the appeal in terms of *mathematical caring relations*, wherein the teacher builds models of the students' available mental actions and engages the student with tasks likely to foster new coordinations.

Although Kant and Piaget set the stage for investigating students' mathematical constructions, researchers have just begun the work of describing those constructions as coordinated mental actions. The task before us is compounded when we consider the entire body of formal mathematics, ultimately entailing an account of the sensorimotor basis of the mental actions that undergird it. For example, can we account for the development of geometry from the onset of crawling to the Pythagorean theorem? This work too is mathematical because it requires us to explicitly identify the structures that organize reversible mental actions. Only then will we fully understand mathematics as a human construction rather than a Platonic ideal.

References

Ansari, D. (2008). Effects of development and enculturation on number representation in the brain. *Nature Reviews Neuroscience, 9*(4), 278.

Bell, M. A. (2001). Brain electrical activity associated with cognitive processing during a looking version of the A-not-B task. *Infancy, 2,* 311–330.

Bell, M.A., & Fox, N.A. (1996). Crawling experience is related to changes in cortical organization during infancy: Evidence from EEG coherence. *Developmental Psychobiology, 29,* 551-561.

Bell, M. A., & Fox, N. A. (1997). Individual differences in object permanence performance at 8 months: Locomotor experience and brain electrical activity. *Developmental Psychobiology, 31,* 287–297.

Butterworth, B., & Walsh, V. (2011). Neural basis of mathematical cognition. *Current Biology*, *21*(16), R618–R621. doi:10.1016/j.cub.2011.07.005

Chazan, D. (1990). Quasi-empirical views of mathematics and mathematics teaching. *Interchange*, *21*(1), 14-23.

Definitions of mathematics. (n.d.) In *Wikipedia*. Retrieved June 5, 2018, from https://en.wikipedia.org/wiki/Definitions_of_mathematics

Dehaene, S. (1997). *The number sense.* Oxford University Press.

Dubinsky, E. (2002). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 95-123). Netherlands: Kluwer Academic Publishers.

Hackenberg, A. J. (2010). Mathematical caring relations in action. *Journal for Research in Mathematics Education*, 236-273.

Hubbard, E. M., Piazza, M., Pinel, P., & Dehaene, S. (2005). Interactions between number and space in parietal cortex. *Nature Reviews Neuroscience*, *6*(6), 435.
Kline, M. (1982). *Mathematics: The loss of certainty* (Vol. 686). Oxford Paperbacks.
Kucian, K., Von Aster, M., Loenneker, T., Dietrich, T., Mast, F. W., & Martin, E. (2007). Brain activation during mental rotation in school children and adults. *Journal of Neural Transmission*, *114*(5), 675-686.
Mathematics. (n.d.) In *Collins English Dictionary.* Retrieved June 5, 2018, from https://www.collinsdictionary.com/us/dictionary/english/mathematics
Mathematics. (n.d.) In *Oxford Dictionaries.* Retrieved June 5, 2018, from https://en.oxforddictionaries.com/definition/mathematics
Mathematics. (n.d.) In *Merriam Webster Online.* Retrieved June 5, 2018, from https://www.merriam-webster.com/dictionary/mathematics
Mruczek, R. E., von Loga, I. S., & Kastner, S. (2013). The representation of tool and non-tool object information in the human intraparietal sulcus. *Journal of Neurophysiology*, *109*(12), 2883-2896. doi: 10.1152/jn.00658.2012
Noddings, N. (1999). Care, justice, and equity. *Justice and caring: The search for common ground in education*, 7-20.
Norton, A., Ulrich, C., Bell, M. A., & Cate, A. (2018). Mathematics at hand. *The Mathematics Educator*.
Moore, K. C. (2013). Making sense by measuring arcs: A teaching experiment in angle measure. *Educational Studies in Mathematics*, *83*(2), 225-245.
Núñez, R. E., Edwards, L. D., & Matos, J. F. (1999). Embodied cognition as grounding for situatedness and context in mathematics education. *Educational Studies in Mathematics, 39*, 45-65.
Penner-Wilger, M., & Anderson, M. L. (2013). The relation between finger gnosis and mathematical ability: why redeployment of neural circuits best explains the finding. *Frontiers in Psychology*, *4*. doi: 10.3389/fpsyg.2013.00877
Piaget, J. (1942). *The child's conception of number*. London: Routledge & Kegan Paul.
Piaget, J. (1970). *Genetic epistemology* (E. Duckworth, Trans.). New York: Norton.
Piaget, J., & Garcia, R. (1986). *Toward a logic of meanings*. Hillsdale, NJ: Erlbaum.
Piaget, J., & Inhelder, B. (1967). *The child's conception of space* (Trans. F. J. Langdon & J. L. Lunzer). New York: Norton (Original work published in 1948).
Piattelli-Palmarini, M. (ed.). (1980) Language and learning: the debate between Jean Piaget and Noam Chomsky, Cambridge, MA, Harvard University Press.
Sagan, C. (1975). The recognition of extraterrestrial intelligence. *Proc. R. Soc. Lond. B*, *189*(1095), 143-153.
Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification – the case of function. In E. Dubinsky & G. Harel (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy* (pp. 59-84). Washington, Mathematical Association of America.
Soylu, F., Lester, F. K., & Newman, S. D. (2018). You Can Count on Your Fingers: The Role of Fingers in Early Mathematical Development. *Journal of Numerical Cognition, 4*(1), 107–135. doi:10.5964/jnc.v4i1.85
Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences, 4*(3), 259-309. doi: 10.1016/1041-6080(92)90005-Y
Tall, D., Thomas, M, Davis, G., Gray, E., & Simpson, A. (2000). What is the object of an encapsulation of a process? *Journal of Mathematical Behavior, 18*(2), 223-241.
Vygotsky, L. (1986). *Thought and language* (A. Kozulin, Ed.). Cambridge: Massachusetts Institute of Technology.

PERSPECTIVES ON THE NATURE OF MATHEMATICS AND RESEARCH

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When people address early mathematics education, commonly they write or reference policies, standards, "scope and sequences" and curriculum, or documents on instructional strategies. These are important; however, we believe that the core consideration should be the nature of mathematics and the development of mathematics *in children*.

An 80-year-old incident illustrates what we mean. A mother attending parents' night asked Fawcett (1938), "How is Willie doing in mathematics?" Fawcett replied: "Madam, you ask the wrong question. You should ask, 'How is mathematics doing in Willie?'"

This is what we mean by the "nature of mathematics and the development of mathematics in children": The mathematics that *does well in Willie* and all other children. We develop this position by describing *learning trajectories* and our theoretical framework for them, *Hierarchic Interactionalism*.

Learning Trajectories: Construct and Theory

Learning trajectories are a device whose purpose is to support the research-grounded development of a curriculum or other unit of instruction, as well as to conduct rigorous research in learning and teaching. The term "curriculum" stems from the Latin word for race course, referring to the course of experiences through which children grow. Thus, the notion of a path, or trajectory, has always been central to curriculum development and study. Simon stated that a "hypothetical learning trajectory" included "the learning goal, the learning activities, and the thinking and learning in which the students might engage" (Simon, 1995, p. 133). Building on Simon's definition, emphasizing a cognitive science perspective and a base of empirical research, "we conceptualize learning trajectories as descriptions of children's thinking and learning in a specific mathematical domain, and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain" (Clements & Sarama, 2004, p. 83).

The name "learning trajectory" reflects its roots in a constructivist perspective. That is, although the name emphasizes learning over teaching, both these definitions clearly involve teaching and instructional tasks. Some appropriations of the learning trajectory construct emphasize only the "developmental progressions." Although studying either psychological developmental progressions or instructional sequences separately can be valid research goals, and studies of each can and should inform mathematics education, we believe the power and uniqueness of the learning trajectories construct stems from the inextricable interconnection between these all three components, goal, developmental progression, and correlated instructional tasks.

Our learning trajectories base *goals* on both the expertise of mathematicians and research on students' thinking about and learning of mathematics (Clements, Sarama, & DiBiase, 2004; Fuson, 2004; National Governor's Association Center for Best Practices & Council of Chief State School Officers, 2010; Sarama & Clements, 2009). This results in goals that are organized into the "big" or "focal" ideas of mathematics: overarching clusters and concepts and skills that are mathematically central and coherent, consistent with students' (often intuitive) thinking, and generative of future learning. Our goals also include productive dispositions, including, curiosity, imagination, inventiveness, risk-taking, creativity, and persistence (National Research Council, 2001). With that in mind, we turn to the question of *how* children think about and learn mathematics.

Research is reviewed to determine if there is a natural developmental progression (at least for a given age range of students in a particular culture) identified in theoretically- and empirically-grounded models of children's thinking, learning, and development (Carpenter & Moser, 1984; Griffin & Case, 1997). That is, researchers build a cognitive model of students' learning that is sufficiently explicit to describe the processes involved in the construction of the mathematical goal across several qualitatively distinct structural levels of increasing sophistication, complexity, abstraction, power, and generality.

The issue of what is meant by a *natural* developmental progression is sure to arise. We believe the research supports a synthesis of aspects of previous theoretical frameworks that we call *Hierarchic Interactionalism* (for a full explication, see Sarama & Clements, 2009). The term indicates the influence and interaction of global and local (domain specific) cognitive levels and the interactions of innate competencies, internal resources, and experience (e.g., cultural tools and teaching). Mathematical ideas are represented intuitively, then with language, then metacognitively, with the last indicating that the child possesses an understanding of the topic and can access and operate on those understandings. The tenets of Hierarchic Interactionalism therefore lay the foundation for the creation of both the developmental progression and instructional tasks of research-based learning trajectories.

1. *Developmental progression.* Most content knowledge is acquired along developmental progressions of levels of thinking. These progressions play a special role in children's cognition and learning because they are particularly consistent with children's intuitive knowledge and patterns of thinking and learning at various levels of development,
2. *Domain specific progression.* These developmental progressions often are most propitiously characterized within a specific mathematical domain or topic. Children's knowledge, that is, the objects and actions they have developed in that domain, are the *main determinant of the thinking within each progression, although hierarchic interactions occur at multiple* levels within and between topics, as well as with general cognitive processes (e.g., executive, or metacognitive processes, potentialities for general reasoning and learning-to-learn skills, and some other domain general developmental processes). See Figure 1 for an illustration.
3. *Hierarchic development.* Development is less about the emergence of entirely new processes and products and more an interactive interplay among specific existing components of knowledge and processes. Also, each level builds hierarchically on the concepts and processes of the previous levels. The learning process is more often incremental and gradually integrative than intermittent and tumultuous. A critical mass of ideas from each level must be constructed before thinking characteristic of the subsequent level becomes ascendant in the child's thinking and behavior. Successful application leads to the increasing use of a particular level. However, under conditions of increased task complexity, stress, or failure this probability level decreases and an earlier level serves as a fallback position.
4. *Co-mutual development of concepts and skills.* Concepts constrain procedures, and concepts and skills develop in constant interaction.

5. *Initial bootstraps.* Children have important, but often inchoate, premathematical and general cognitive competencies and predispositions at birth or soon thereafter that support and constrain, but do not absolutely direct, subsequent development of mathematics knowledge.
6. *Different developmental courses.* Different developmental courses are possible within those constraints, depending on individual, environmental, and social confluences.
7. *Progressive hierarchization*. Within and across developmental progressions, children gradually make connections between various mathematically-relevant concepts and procedures, weaving ever more robust understandings that are hierarchical in that they employ generalizations while maintaining differentiations.
8. *Consistency of developmental progressions and instruction.* Instruction based on learning consistent with natural developmental progressions is more effective, efficient, and generative for the child than learning that does not follow these paths.
9. *Learning trajectories.* A particularly fruitful instructional approach is based on hypothetical learning trajectories. Curriculum developers design instructional tasks that include external objects and actions that mirror the hypothesized mathematical activity of children as closely as possible. These tasks are sequenced, with each corresponding to a level of the developmental progressions, to complete the hypothesized learning trajectory. *Specific learning trajectories are the main bridge that connects the "grand theory" of hierarchic interactionalism to particular theories and educational practice.*
10. *Instantiation of hypothetical learning trajectories.* Hypothetical learning trajectories must be interpreted by teachers and are only realized through the social interaction of teachers and children around instructional tasks.

For example, consider one *goal* regarded as important in all standards documents: young children should learn to be competent in whole number, including meaningful verbal and object counting and the application of counting to solve a variety of arithmetic problem types. The *developmental progressions* for each of these learning trajectories are sampled in the left column of Figure 1. The second column provides an example of children’s behavior and thinking for each level. The third column presents an example of an instructional task designed to catalyze that level of thinking.

In summary, learning trajectories describe the goals of learning, the developmental progression through which children pass, and the learning activities in which students might engage. The *source* of the developmental progressions— the thinking and learning processes of children at various levels—are extensive research reviews and empirical work that cannot be presented here due to space constraints. Also beyond the scope of this chapter are the complex, cognitive actions-on-objects that underlie the LTs (see Sarama & Clements, 2009). Here we will provide one illustration of both cognitive actions-on-objects and how different trajectories grow not in isolation, but interactively.

Consider learning a critical competence—counting on, used especially at the *Counting Strategies* level in Figure 1b. Children need to develop competencies from three trajectories: counting (Fig. 1a), subitizing (not shown, but see Clements & Sarama, 2009; Sarama & Clements, 2009), and the addition and subtraction trajectory (Fig. 1b) to learn to count on meaningfully. From the counting trajectory, they learn to count forward from any number. Then they learn to understand explicitly and apply the idea that each number in the counting sequence includes *the number before, hierarchically*. That is, 5 includes 4, which includes 3, and so forth. From the subitizing trajectory they quickly learn to recognize the number of—not just visual sets, but also *rhythmic patterns.* From the addition and subtraction trajectory, children learn to interpret situations mathematically, such as interpreting a real-world problem as a "part-part-whole" situation. They also learn to use counting to determine what is missing. The creative combination of these developments allows them to solve *meaningfully* problems such as, "You have three green candies and six orange candies. How many candies do you have in all?" by counting on. They understand that these numbers are two parts and that they need to find the whole. They also understand that the order of numbers does not matter in addition. They know, in practice, that the sum is the number that results by, starting at the first number and counting on a number of iterations, equal to the second number. They can use counting to solve this, starting by saying "siiiiix…" because they understand that word can stand for the counting acts from 1 to 6 (because 6 includes 5…). They know *how many more* to count because they use the subitized "rhythm of three" "Du de **Du" ("Doo – Day – Doo")** "seven (du…), eight (day…), nine (*du*)—*nine!*"

Consider Justin, who participated in the successful scale-up of the learning-trajectories-based Building Blocks curriculum (Clements, Sarama, Spitler, Lange, & Wolfe, 2011). At pretest, he operated at the Reciter level of counting, as he verbally counted correctly but when counting toy bananas, broke one-to-one correspondence as he counted a space between the bananas. He did not solve any arithmetic problems. After 7 months moving through the learning trajectories for counting, subitizing, and the counting-based addition and subtraction trajectories (among others), Jason showed remarkable growth. He counted up to 30 randomly-arranged objects accurately and could verbally count up or down from any number in that range. In arithmetic, he solved a variety of problems. For "…you have 3 candies and I gave you 2 more; how many do you have?" Justin put out 3 fingers, then 2 more, and then said, "Five. I was just counting but no words" (i.e., he didn't count out loud). Later, , shown 6 blocks, which were then covered with a cloth, and 4 secretly removed, leaving 2, he said "Two. There were six." "So, how many am I hiding?" Justin quickly counted the two and then counted, pointing to the table and said, "Four." These solutions suggest he was now operating at the Counting Strategies level of arithmetic.

a. Counting

Developmental Progression	Example Behavior	Instructional Tasks
Reciter Verbally counts with separate words, not necessarily in the correct order.	Count for me. "one, two, three, four, six, seven."	Provide repeated, frequent experience with the counting sequence in varied contexts.
Corresponder Keeps one-to-one correspondence between counting words and objects (one word for each object), at least for small groups of objects placed in a line.	Counts: ☆ ☆ ☆ ☆ "1, 2, 3, 4" But answers the question, "How many?" by re-counting the objects or naming any number word.	*Kitchen Counter* Students click on objects one at a time while the numbers from one to ten are counted aloud. 

Counter (Small Numbers) Accurately counts objects in a line to 5 and answers the "how many" question with the last number counted.	Can you count these? ☆ ☆ ☆ ☆ "1, 2, 3, 4… four!"	*How Many?* Tell students you have placed as many cubes (3, hidden) in your hand as you can hold. Ask them to count with you to see how many. Take out one at a time as you say the number word (so, when they say "two" they *see* two). Repeat the last counting number, "three," gesturing in a circular motion to all the cubes, and say "That's how many there are in all."
Counter and Producer (10+) Counts and counts out objects accurately to 10, then beyond.	Counts a scattered group of 19 chips, keeping track by moving each one as they are counted.	*Road Race* Board game.
Counter from N (N+1, N-1) Counts verbally and with objects from numbers other than 1 (but does not yet keep track of the *number* of counts).	Asked to "count from 5 to 8," counts: "5, 6, 7, 8!" Determines numbers just after or just before immediately.	*One more!.* Have the children count two objects. Add one and ask, "How many now?" Have children count on to answer. Add another and so on, until they have counted to ten.

b. Arithmetic

Developmental Progression	Example Behavior	Instructional Tasks
Small Number +/- Finds sums for joining problems up to 3 + 2 by counting-all with objects.	Asked, "You have 2 balls and get 1 more. How many in all?" counts out 2, then counts out 1 more, then counts all 3: "1, 2, 3, 3!."	*Finger Word Problems* Tell children to solve simple addition problems with their fingers.
Find Result +/- Finds sums *by direct modeling, counting-all, with objects.*	Asked, "You have 2 red balls and 3 blue balls. How many in all?" counts out 2 red, then counts out 3 blue, then counts all	*Places Scenes (Addition)—Part-part-whole, whole unknown problems.* Children play with toy on a background scene and combine groups.

	5.	
Counting Strategies +/- Finds sums for joining (you had 8 apples and get 3 more…) and part-part-whole (6 girls and 5 boys…) problems with finger patterns and/or by counting on.	Counting-on. "How much is 4 and 3 more?" "Fourrrrr…five, six, seven [uses rhythmic or finger pattern to keep track]. Seven!"	*How Many Now?* Have the children count objects as you place them in a box. Ask, "How many are in the box now?" Add one, repeating the question, then check the children's responses by counting all the objects. Repeat, checking occasionally.

Figure 1. Selected Levels/Descriptions from the Learning Trajectories for Counting and counting-based Arithmetic (these and other figures adapted from Clements & Sarama, 2014)

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References

Carpenter, T. P., & Moser, J. M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education*, 15(3), 179–202. doi:10.2307/748348

Clements, D. H., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning*, 6, 81–89. doi:10.1207/s15327833mtl0602_1

Clements, D. H., & Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach*. New York, NY: Routledge.

Clements, D. H., & Sarama, J. (2014). *Learning and teaching early math: The learning trajectories approach (2nd ed.)*. New York, NY: Routledge.

Clements, D. H., Sarama, J., & DiBiase, A.-M. (2004). *Engaging young children in mathematics: Standards for early childhood mathematics education*. Mahwah, NJ: Erlbaum.

Clements, D. H., Sarama, J., Spitler, M. E., Lange, A. A., & Wolfe, C. B. (2011). Mathematics learned by young children in an intervention based on learning trajectories: A large-scale cluster randomized trial. *Journal for Research in Mathematics Education*, 42(2), 127–166. doi:10.5951/jresematheduc.42.2.0127

Fawcett, H. P. (1938). *The nature of proof (Vol. 13th)*. New York, NY: Columbia University Teachers College Bureau of Publications.

Fuson, K. C. (2004). Pre-K to grade 2 goals and standards: Achieving 21st century mastery for all. In D. H. Clements, J. Sarama & A.-M. DiBiase (Eds.), *Engaging young children in mathematics: Standards for early childhood mathematics education* (pp. 105–148). Mahwah, NJ: Erlbaum.

Griffin, S., & Case, R. (1997). Re-thinking the primary school math curriculum: An approach based on cognitive science. *Issues in Education*, 3, 1–49.

National Governor's Association Center for Best Practices & Council of Chief State School Officers. (2010). Common core state standards initiative K-12 standards development teams. from https://www.nga.org/files/live/sites/NGA/files/pdf/2010COMMONCOREK12TEAM.PDF

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press. doi:10.17226/9822.

Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. New York, NY: Routledge.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 26(2), 114–145. doi:10.2307/749205

Chapter 2

Curriculum and Related Factors

Research Reports

Brief Research Reports

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.

Posters

THE INITIAL TREATMENT OF THE AREA MEASUREMENT IN THE SELECTED ELEMENTARY MATHEMATICS TEXTBOOKS FROM US AND KOREA

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This study compared area lessons from Korean textbooks and US standards-based textbooks to understand differences and similarities among these textbooks, as well as how these textbooks address known learning challenges in area measurement. Several well-known challenges have been identified in previous studies, such as covering, array structure, and linking array structure to area formula. We were interested in knowing if textbooks addressed these issues in their treatments of area measurement, and in doing so provided students with opportunities to overcome or become familiar with known challenges. The results show that both countries textbooks demonstrated similar limitations, only few area and area related lessons are covered and three important learning challenges in area measurement are not covered well, which need to be informed to practicing teachers.

Keywords: Area, Textbooks, Learning Challenges

For the last few decades, mathematics education researchers have been interested in how students in different countries learn mathematics. Such interests are the result of studies such as the Trends in International Mathematics and Science Study (TIMSS) and Programme for International Student Assessment (PISA), which indicate that students in East Asian countries perform consistently well. One important area we can examine among various opportunities to learn (OTL) is what textbooks offer to students for their learning, as textbooks play an important role in lesson enactment process, teachers use textbooks and other resources to select and modify tasks to prepare their lessons (Remillard & Heck, 2014). Among many mathematical topics, US students' performance in measurement is weaker than any other content area on TIMSS (Mullis, Martin, Foy, & Hooper, 2016). While researchers conducting international comparative studies of textbooks have examined various mathematical topics (Cady, Hodges, & Collins, 2015; Son & Hu, 2016), area lessons have not been examined and compared often. The purpose of this study was to examine and compare area lessons in US Common Core-aligned textbooks and Korean textbooks, and explore how textbooks address well-known learning challenges in area measurement.

Related Literature

Textbooks in the curriculum enactment process

Although not all contents in textbooks will automatically be transformed to mathematics lessons directly (teachers will likely modify textbook contents), examining the treatment of mathematical topics in textbooks can tell us how much attention is given to that specific topic. In the curriculum enactment process, teachers select and possibly modify mathematical tasks and activities from textbooks and other curriculum materials (Remillard & Heck, 2014). Textbook content and how teachers enact their lessons jointly influence what students experience in their classrooms (Smith, Males, Dietiker, Lee, & Mosier, 2013). For example, if textbooks do not address well-known challenges, we can interpret that as a possible reason for students' struggle because it is possible that those challenges are not well reflected in the teacher's lesson plans and decrease students' OTL to learn and get familiar with those challenges (Smith, Males, & Gonulates, 2016).We cannot say textbooks are the only reason for students' struggle in learning area measurement. However, limited coverage of topics will limit opportunities to learn for students and is a possible explanation for their performances.

How Students Learn Area Measurement

Foundational concepts for area include covering a region without gaps or overlaps, counting unit measures, iterating, understanding array structure and linking the number of squares to length and width (Battista, 2004). Being able to use same – sized units repeatedly to cover a region and iterating are fundamental skills to understand measurement in general (Smith et al., 2016). However, studies have shown that it is challenging for students to develop a good conceptual understanding of area. Students are not able to cover a two dimensional region with equal-sized units. Instead, they often use unequal unit or leave gaps or overlaps (Battista, 2004; Outhred & Mitchelmore, 2000). Without having conceptual understanding, students often use the area formula *length width* without understanding why and for the wrong figures (Zacharos, 2006). However, when students learn using the conceptual approaches of partitioning and covering (partitioning, filling a given space and seeing array structure), they are more likely to develop a better understanding of area measurement (Huang, 2017; Na, 2012; Outhred & Mitchelmore, 2000). Here are research questions that we attempt to answer.

- How do US and Korean textbooks distribute attention to area and area-related lessons?
- In what order do the curricula present concepts related to measuring area, and do the sequences differ significantly between textbooks?
- How well do the curricula address well-known students' challenges in learning area measurement?

Methods

Data Sources

Three textbooks series - *enVisionMath*, *Go Math*, and *MyMath* - are Common Core-aligned textbooks, which were not examined in recent study (Smith et al., 2016). A total of 9 US textbooks were examined, three textbooks from grade 1 through grade 3 from each publisher. Two recent studies examined length and area lessons in other non - Common Core American textbooks and also several other studies examined textbooks that were developed before the introduction of the Common Core State Standards (CCSS) (Hong & Choi, 2014, 2018; Smith et al., 2013; Smith et al., 2016; Son & Senk, 2010). Therefore, examining these common core aligned textbooks series may expand our understanding of how more current American textbooks treat volume measurement. In all, 431 (*enVision Math*), 441 (*My Math*) and 550 (*Go Math*) items were analyzed.

For Korea, the textbooks examined in this study, Elementary School Mathematics, were published by the government and the only mathematics textbooks used in elementary schools in Korea. For Korean textbooks, we examined 129 items from grades 1, 3 and 5. Table 1 describes the number of pages and lessons examined for this study.

Table 1: Textbooks used in the study

Textbook Series	Publisher	Publication Date	Pages	Items	Lessons
envisionMath series	Pearson	2015	86	431	18
Go Math series	Houghton Mifflin	2015	133	550	15
MyMath series	McGraw-Hill	2014	69	441	12
Korean Textbooks	The Ministry of Education in Korea	2014, 2015	31	129	6

Framework for Analyzing Textbooks

When textbooks are analyzed, exposition (e.g. introductory paragraphs, text boxes with definitions, formulas, or theorems), worked examples (problems presented together with an explained solution) and exercise problems (mathematical items students are expected to solve) should be examined because they can provide potentially different OTL for students.

We also searched for studies that examined measurement lessons in textbooks. Smith and his colleagues (2013, 2016) examined length and area measurement in US textbooks. Their framework specifically targets how textbooks address challenges in learning length and area. We adopted and modified Smith and colleagues' framework to analyze area measurement lessons. Table 2 describes our framework for this study.

Table 2: Analysis framework of content and problems

Area of Focus

• Number of area and area related lessons
• Timing and topic sequence
• Procedural and conceptual knowledge
• Known challenges in learning area measurement
  o Covering with equal-sized units
  o Row and column array structure
  o Area formula and definition
• Response type

Coding Procedures and Examples

Each exposition, worked example, and exercise problem in textbooks has its own instructional purpose (potentially different OTL). Each exposition introduces mathematics content including definitions, formulas, and procedures; each worked example demonstrates how certain problems are solved; and each exercise problem gives students opportunities to engage in problem solving. Thus, when we discussed our unit of analysis, we first considered each worked example, exercise problem, and exposition as one unit of analysis as each item provides OTL to teachers and students. Figure 1 shows examples of how we coded each item.

Translation:
• This is the floor plan for Chul Soo's home. Find out the area of each room. Parents' Room Chul Soo's Room Brother's Room

Figure 1. Coding examples from a Korean textbook (The Ministry of Education in Korea, 2015, p. 139)

First, we decided there are three items (three exercise problems or three units) to be coded. Students are asked to use the area formula to compute the area of each room. In terms of topic, these items are coded as area formula because students just need to multiply numbers to get the correct answers. In terms of procedural and conceptual knowledge types, these were coded as procedural (only multiplying two numbers is required). Finally for response type, these were coded as short response (only numbers are required). For all other textbook pages, we used the same method to identify expositions, worked examples, and exercise problems to count the number of units to be coded from each page.

Reliability

Each textbook included exposition, worked examples and exercise problems. After discussing the established codes, two authors coded approximately 20% of the textbook items to check inter-rater reliability. After comparing codes for sample items and finding an acceptable high inter-rater reliability, the authors coded all textbook items jointly to produce a final set of tables for analysis, resolving coding differences of individual items when they arose. To determine reliability, we applied a generalizability theory D study (Alkharusi, 2012). This technique produced a reliable coefficient of 0.964.

Results

Area Measurement Lessons in Textbooks

Table 3: Number and Percentage of Area and Area Related Lessons to the Total Lessons

	Grade	Area and Area-Related Lessons	Total
enVision Math	1 2 3	5 (4.5%) 8 (6.9%) 5 (4.2%)	110 116 119
Go Math	1 2 3	3 (3.0%) 4 (3.6%) 8 (7.6%)	101 110 105
MyMath	1 2 3	3 (3.2%) 2 (2.2%) 7 (6.2%)	95 92 113
Korean Textbooks	1 3 5	1 (1.4%) 1 (1.5%) 4 (5.7%)	70 68 70

Table 3 indicated that little curricula attention was given to area and area-related topics in these textbook series, supporting earlier findings by Smith et al. (2016). These percentages were less than the 1% to 12% range found by Smith et al. (2016).

Time and Sequence

With regards to the sequencing of topics, Korean and US textbooks differed. The Korean textbook series first introduced area in grade 1, with a lesson titled "Comparing Area". This was a short lesson that asked students to compare and make visual judgments between two objects. These items were not found in US textbooks. After a brief introduction to area in grade 1, Korean textbooks had one lesson about partitioning shapes into equal parts in grade 3. Lessons on area then began in grade 5, where the textbook introduced unit squares, area of rectangles and the area formula. In contrast, all three US textbook series included several lessons in grades 1 and 2 about partitioning regions, including rectangles and circles. Then, the area formula was introduced in grade 3 US textbook series. In terms of timing, it appeared that Korean textbooks introduced area first in grade 1, but ideas of partitioning, understanding and finding area of rectangles were found earlier in US textbook series.

Procedural, Conceptual and Conventional Knowledge

The majority of items (83 to 100 % in all textbooks) in both countries' textbooks were procedural. This finding supports previous work examining American textbooks, where procedural items accounted for more than 87% of items in US textbooks (Smith et al., 2016). Such findings imply that the focus of area lessons in both American and Korean textbooks are more about procedures than concepts. Again, this can be one way to lead both countries' students to a more procedural understanding of area.

Knowledge Needed in Understanding Area Measurement

Covering. Covering may be introduced with drawing, using tiles or iterating. Only limited number of covering items were included in textbooks (less than 17 % area items in each grade). Compared to US textbooks, Korean textbooks included covering items much later, only introducing them in the fifth grade. Such a lack of inclusion is problematic, as previous studies have shown that it was challenging to second graders to cover a region completely without gaps or overlaps (Battista, Clements, Arnoff, Battista, & Caroline Van Auken, 1998; Lee, 2010). We also noticed that when textbooks include covering items, the terms "gaps" and "overlaps" are used only few times: less than 10 times in each American textbook series and never in the Korean textbooks. Since students often struggle with covering a region without gaps and overlaps (Outhred & Mitchelmore, 2000; Sarama & Clements, 2009), careful attention to covering and explicit remarks about why gaps and overlaps are important will lead students to a more conceptual understanding of what it means to measure area.

Array structure. Items in this group included drawing, tiling or partitioning a region into rows or columns (the terms "rows" or "columns" needed to be included, or students needed to have opportunities to show array structure) and then counting them or using partial array structure. In US textbook series, partitioning a rectangle into rows and columns appeared first in grade 2, and grade 3 but coverage of the topic was brief and limited (less than 7 % area items in each grade). In Korean textbooks, the topic of array structures appeared in grade 5 but limited number of items were included (about 11 % of its area-related items).

Area definition and formula. Items were coded as area formula if they showed that multiplying two numbers gives the area, or if they used the *length width* formula to compute the area. Both countries' textbooks used array structure to introduce the area formula. Korean textbook lets students derive the formula after working on tasks counting unit squares in grade 5 textbook. US textbook series also used an array structure in grade 3. As previously mentioned, with limited opportunities to explore how array structure and the area formula are related, students are likely to resort to more procedural approaches to area. Such a tendency was reinforced by every textbook, as they all moved quickly to items, where students were only required to use a procedure, multiplying length and width.

Other area topics

One of the frequent topics in both countries' textbooks is counting unit squares in a shape without focusing on array structure (ranging from 23% to 41% of items in grade 3 textbooks and about 26 % in grade 5 Korean textbook). With such a heavy focus on counting squares, paired with limited experience with array structures in previous years, it will be challenging for students to connect array structure to the area formula. Prior studies have shown that even with drawn lines and squares, the connection between counting unit squares and area is not apparent to elementary students (Battista, 2004; Battista et al., 1998). Students may count unit squares procedurally without seeing array structure or understanding the purpose of not having gaps or overlaps. With limited opportunities to experience covering and array structure in grades 1 and 2, it will be challenging for students to see array structure when they are trying to count unit squares.

Summary and Conclusion

This study compared area lessons from Korean textbooks and US textbooks to understand differences and similarities among these textbooks, as well as how these textbooks address known learning challenges in area measurement. Our results indicated that textbooks from both countries paid modest or limited attention to area measurement lessons. In terms of timing, US textbook series introduced area related topics, partitioning, covering, array structure and area formula much earlier than Korean textbooks. In terms of sequence, US textbooks progressed through partitioning, tiling and then presenting the area definition and formula. Studies showed, 2nd and 3rd graders often struggled with covering and array structure (Battista, 2004; Sarama & Clements, 2009). However, textbooks from both countries introduced such topics either later or not at all. Such findings indicate that both timing and sequencing were an issue for both countries' textbook.

Both countries textbooks placed strong focus on procedures rather than concepts. Also, the most frequent items were partitioning regions without array structure. Compounded with issues of sequencing and timing, a procedural focus, and limited coverage of important conceptual area ideas are highly like to lead to students' challenges in learning area measurement. What may we conclude from our findings? As we mentioned previously, although textbooks do not provide mathematics lessons directly (content will likely be modified by teachers), they are one of the main resources teachers use when planning lessons. With the issues identified in these textbook series thus far, it is possible that limitations in textbooks can lead to area lesson plans that do not reflect challenges and important concepts of area measurement. In turn, limited coverage may lead to limiting elementary students' learning opportunities and they may be inclined to adopt a more procedural understanding of area, without attaining a conceptual understanding. With our findings, it will be important to provide teachers with additional supports so that they can attempt to modify tasks in these textbooks to properly address students' learning challenges in area measurement.

In terms of international comparative studies, we cannot say that the learning opportunities these textbooks offered are directly related to US students' performances in TIMSS. However, our results showed that how these textbooks treat area measurement could be one of the reasons for US students' TIMSS results. Despite Korean students' high performances in measurement, we did find that Korean textbooks shared many of the same conceptual limitations as US textbooks. As Smith and colleagues (2016) noted, further studies are required to examine the link between curriculum use and students' performances in assessments in order to make more distinct claims about influence of textbooks on students' performances.

References

Battista, M. (2004). Applying Cognition-Based Assessment to Elementary School Students' Development of Understanding of Area and Volume Measurement. *Mathematical Thinking and Learning, 6*(2), 185-204. doi:10.1207/s15327833mtl0602_6

Battista, M., Clements, D., Arnoff, J., Battista, K., & Caroline Van Auken, B. (1998). Students' Spatial Structuring of 2D Arrays of Squares. *Journal for Research in Mathematics Education, 29*(5), 503-532. doi:10.2307/749731

Cady, J. A., Hodges, T. E., & Collins, R. L. (2015). A Comparison of Textbooks' Presentation of Fractions. *School Science and Mathematics, 115*(3), 105-116. doi:10.1111/ssm.12108

Hong, D. S., & Choi, K. M. (2014). A comparison of Korean and American secondary school textbooks: the case of quadratic equations. *Educational Studies in Mathematics, 85*(2), 241-263. doi:10.1007/s10649-013-9512-4

Hong, D. S., & Choi, K. M. (2018). A comparative analysis of linear functions in Korean and American standards-based secondary textbooks. *International Journal of Mathematical Education in Science and Technology*, 1-27. doi:10.1080/0020739X.2018.1440327

Huang, H.-M. E. (2017). Curriculum Interventions for Area Measurement Instruction to Enhance Children's Conceptual Understanding. *International Journal of Science and Mathematics Education, 15*(7), 1323-1341. doi:10.1007/s10763-016-9745-7

Lee, J. (2010). Children's strategies for measurement estimation of rectangular covering tasks. *Journal of the Korean Society of Mathematical Education series A, 49*(3), 375-487.

Mullis, I. V. S., Martin, M. O., Foy, P., & Hooper, M. (2016). *TIMSS 2015 International Results in Mathematics*. Retrieved from Boston College: http://timssandpirls.bc.edu/timss2015/international-results/

Na, G. (2012). Examining Students' Conceptions about the Area of Geometric Figures. *Journal of Elementary Mathematics Education in Korea, 16*(3), 451-469.

Outhred, L., & Mitchelmore, M. (2000). Young Children's Intuitive Understanding of Rectangular Area Measurement. *Journal for Research in Mathematics Education, 31*(2), 144-167. doi:10.2307/749749

Remillard, J. T., & Heck, D. J. (2014). Conceptualizing the curriculum enactment process in mathematics education. *ZDM, 46*(5), 705-718. doi:10.1007/s11858-014-0600-4

Sarama, J., & Clements, D. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. New York: Routledge.

Smith, J. P., Males, L. M., Dietiker, L. C., Lee, K., & Mosier, A. (2013). Curricular Treatments of Length Measurement in the United States: Do They Address Known Learning Challenges? *Cognition and Instruction, 31*(4), 388-433. doi:10.1080/07370008.2013.828728

Smith, J. P., Males, L. M., & Gonulates, F. (2016). Conceptual Limitations in Curricular Presentations of Area Measurement: One Nation's Challenges. *Mathematical Thinking and Learning, 18*(4), 239-270. doi:10.1080/10986065.2016.1219930

Son, J.-W., & Hu, Q. (2016). The initial treatment of the concept of function in the selected secondary school mathematics textbooks in the US and China. *International Journal of Mathematical Education in Science and Technology, 47*(4), 505-530. doi:10.1080/0020739X.2015.1088084

Son, J.-W., & Senk, S. L. (2010). How reform curricula in the USA and Korea present multiplication and division of fractions. *Educational Studies in Mathematics, 74*(2), 117-142. doi:10.1007/s10649-010-9229-6

Zacharos, K. (2006). Prevailing educational practices for area measurement and students' failure in measuring areas. *The Journal of Mathematical Behavior, 25*(3), 224-239. doi:https://doi.org/10.1016/j.jmathb.2006.09.003

Textbooks Analyzed

enVision Math Common Core edition 2.0, Grade 1 (2015 a). New York, NY: Pearson/Scott Foresman– Addison Wesley.

enVision Math Common Core edition 2.0, Grade 2 (2015 b). New York, NY: Pearson/Scott Foresman– Addison Wesley.

enVision Math Common Core edition 2.0, *Grade 3* (2015 c). New York, NY: Pearson/Scott Foresman– Addison Wesley.
Go math! Common Core edition, Grade 1 (2015 a) (student edition e-book). Orlando, FL: Houghton Mifflin Harcourt
Go math! Common Core edition, Grade 2 (2015 b) (student edition e-book). Orlando, FL: Houghton Mifflin Harcourt
Go math! Common Core edition, Grade 3 (2015 c) (student edition e-book). Orlando, FL: Houghton Mifflin Harcourt
MyMath, Grade 1. (2014 a). New York, NY: Macmillan/McGraw-Hill.
MyMath, Grade 2. (2014 b). New York, NY: Macmillan/McGraw-Hill.
MyMath, Grade 3. (2014 c). New York, NY: Macmillan/McGraw-Hill.
The Ministry of Education in Korea (2014a) Mathematics 1. Seoul, Korea.
The Ministry of Education in Korea (2014b) Mathematics 3. Seoul, Korea.
The Ministry of Education in Korea (2015) Mathematics 5. Seoul, Korea.

TENDENCIAS EN MODELACIÓN MATEMÁTICA EN LATINOAMÉRICA

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A lo largo de los años la modelación matemática ha ganado gran atención internacional, no sólo en la investigación sino también en el desarrollo de propuestas curriculares y en su implementación en salones de clases. Pero, a pesar de la muy extensa cantidad de publicaciones, existen pocas revisiones de literatura sistemáticas. En este reporte presentamos algunos resultados de una revisión de 485 publicaciones internacionales sobre modelación matemática. Nos centramos en las publicaciones que reportan estudios realizados en América Latina y encontramos que, en términos de la cantidad de publicaciones, la producción es relativamente modesta; sin embargo, la vitalidad de los temas actualmente discutidos y la innovación de sus perspectivas dan cuenta de la importancia que tiene el trabajo desarrollado en la región.

Keywords: Modelación matemática, Investigación documental, Investigaciones en América Latina

La Necesidad de una Revisión de la Literatura

La incorporación de la modelación matemática a los currículos de varios países tiene sus raíces en un movimiento de finales de los años 1950's, cuando defensores de la modelación pugnaron por restaurar el foco en la utilidad de las aplicaciones de las matemáticas en escuelas y universidades (Niss, Blum y Galbright, 2007).

Un momento clave para el movimiento internacional de la modelación matemática fue la inauguración de la *Conference on the Teaching of Mathematical Modelling and Applications*, organizada en 1983 por la *International Community of Teachers of Mathematical Modelling and Applications* (ICTMA). Otro momento importante tuvo lugar en el año 2004 con la realización del estudio sobre Modelación Matemática organizado por la *International Commission on Mathematical Instruction* (ICMI). La publicación derivada de este ICMI Study (Blum, Galbraith, Henn y Niss, 2007) sigue siendo una referencia básica para la investigación en este tema. Desde entonces, los estudios en modelización se han incrementado significativamente ampliando los métodos e intereses más allá de las aproximaciones tradicionales (Stillman, Blum y Kaiser, 2017). No obstante, a pesar de la gran cantidad de publicaciones en educación sobre modelación matemática, son escasas las revisiones sistemáticas del estado del arte sobre el tema.

En la revisión efectuada por Kaiser y Sriraman (2006) se propone una clasificación de las investigaciones en la que se identifica seis perspectivas en modelación matemática: la *perspectiva realista* que tiene por objetivo resolver problemas de la vida real, más allá de las matemáticas; la *perspectiva epistemológica* que se centra en el desarrollo de teorías matemáticas e incluye modelos intra-matemáticos que son usados en la teoría matemática avanzada; la *perspectiva educacional* que considera que la modelación debe servir armoniosamente a propósitos prácticos, científicos y matemáticos; la *perspectiva contextual*, también llamada enfoque de modelación provocada (*model-eliciting*) que se centra actividades de resolución de problemas usando principios de diseño instruccional específicos; la *perspectiva socio-crítica* que enfatiza la necesidad de desarrollar una postura crítica frente al rol y la naturaleza de los modelos matemáticos y su impacto en las problemáticas sociales; y la *perspectiva cognitiva* que es transversal a las anteriores y se centra en aspectos cognitivos de los procesos de modelación matemática.

A pesar de que la clasificación de Kaiser y Sriraman sigue siendo vigente, la gran cantidad de trabajos y de perspectivas producidos en la última década demanda actualizarla y ajustarla. En nuestra revisión de la literatura, retomamos esta clasificación para identificar y contrastar temas contemporáneos sobresalientes, para ello analizamos sistemáticamente algunas de las publicaciones internacionales más importantes, tanto en revistas como en libros, como describimos a continuación.

Metodología de la Revisión de Literatura

Nuestra revisión es resultado del trabajo realizado en un seminario interinstitucional en el que participamos investigadores, profesores y estudiantes de grado y de posgrado de universidades de México y Canadá. Identificamos una vasta cantidad de literatura y gran variedad de perspectivas en modelación matemática. También notamos que las revisiones sistemáticas, como la realizada por Frejd (2013), son muy escasas. Esta escasez definió nuestro interés en realizar nuestra propia revisión, la cual comenzamos a en el primer semestre del año 2017.

Desarrollamos nuestra revisión en dos etapas. En la primera, recurrimos a la base de datos *SpringerLink* y buscamos las palabras “modeling” o “modelling” en el título. Filtramos la búsqueda usando “education” como disciplina y “mathematics education” como subdisciplina. Los artículos que no se relacionaban con modelación matemática fueron eliminados. De esta manera obtuvimos una lista de 73 artículos. A pesar de las restricciones impuestas en esta primera búsqueda, consideramos que la lista es representativa de los artículos de modelación publicados en Springer ya que la búsqueda centrada en los títulos sugiere que la modelación matemática es el principal foco de interés de estos artículos. El análisis de los artículos de esta lista nos ayudó a clarificar y refinar nuestras categorías de análisis; además, nos permitió identificar otras publicaciones clave en libros y artículos de otras editoriales.

En la segunda etapa de nuestra revisión incluimos: (a) los artículos provenientes de números especiales de revistas de Springer y completamos la revisión de artículos publicados en 2017, que no habían sido incluidos antes (debido al traslape de tiempos de edición, sólo fueron consideradas las versiones preliminares publicadas en línea de los dos números especiales sobre modelación publicados por ZDM en 2018); (b) los artículos del *Journal for Research in Mathematics Education* (JRME); (c) los cinco libros derivados del ICTMA y publicados por Springer y el libro correspondiente al 14th *ICMI Study* (Blum et al, 2007); (d) los artículos de algunas revistas internacionales de investigación en español; (e) un libro publicado recientemente que reporta investigación latinoamericana sobre modelación matemática (Arrieta y Díaz, 2016); y (f) los reportes de investigación, presentaciones de grupos de trabajo y conferencias plenarias de los últimos 10 años de las memorias del PME-NA.

Elegimos incluir JRME porque, de acuerdo con Toerner and Arzarello (2012), es la revista de mayor importancia a nivel internacional en el área de la Educación Matemática, junto con *Educational Studies in Mathematics*. Incluimos las revistas en español y el libro sobre investigación latinoamericana para ampliar nuestra mirada más allá de las publicaciones realizadas en lengua inglesa. Las revistas en español seleccionadas fueron *Revista Educación Matemática* y *Revista Latinoamericana de Investigación en Matemática Educativa*, debido a que se especializan en Educación Matemática, son relevantes en la región y pertenecen a índices internacionales prestigiosos (Scopus y JCR). Para la búsqueda de artículos en estas revistas se procedió con los mismos criterios que con las revistas en inglés.

En nuestro análisis nos concentramos en identificar aspectos metodológicos y teóricos de las investigaciones. Específicamente, ubicamos: propósitos de las investigaciones, población, instrumentos de levantamiento de datos, tipo de diseño metodológico (cualitativo, cuantitativo, mixto, etc.), perspectiva teórica y perspectiva de modelación, país donde se realiza la investigación y país de los autores, así como contenidos matemáticos abordados. Es necesario señalar, que el trabajo fue arduo pues muchas veces la definición de los aspectos teórico-metodológicos no está hecha explícitamente por los autores y hay que inferirlos de lo que se dice y hace en el artículo. Por ejemplo, para la definición de la perspectiva de modelización, partimos de las características señaladas por Kaiser y Sriraman (2006), identificamos los propósitos y elementos teóricos señalados en los artículos, cruzamos con las conclusiones y, finalmente, revisamos a los autores citados como referencias clave. Los artículos fueron revisados en paralelo por dos investigadores. En caso de diferencias en la determinación de las características, los investigadores se juntaron a revisar los detalles y tomar una decisión; los casos más difíciles se discutieron en reuniones plenarias. Los criterios de análisis fueron ajustados continuamente a lo largo de los meses que duró la revisión.

Un total de 485 documentos fueron revisados: 111 artículos de revistas de investigación, 341 capítulos de libros y 33 contribuciones del PME-NA. En el presente artículo reportamos los resultados de nuestro análisis sobre producción de América Latina en modelación matemática encontrados en estos documentos.

Tendencias y Producción en Modelación Matemática en América Latina

En contraste con la afirmación de Blum y Niss's (1991) de que la modelación matemática fue inicialmente desarrollada en regiones como Alemania y el Reino Unido, nuestra revisión revela actividad significativa en modelación matemática en América Latina desde los años 90's. De hecho, de acuerdo con Biembengut (2016) en el caso de Brasil esta actividad puede rastrearse hasta los años 70's. En nuestra revisión identificamos dos tendencias en las publicaciones sobre modelación matemática en América Latina: por una parte, la cantidad de producción es relativamente pequeña; pero, por otra parte, las aproximaciones y problemáticas abordadas presentan aproximaciones innovadoras para la investigación educativa en modelación matemática.

La Cantidad de Publicaciones

Identificamos en total 66 publicaciones de autores latinoamericanos del total de 485 que ubicamos en nuestra base de datos, es decir, aproximadamente el 13% de las publicaciones. En las publicaciones de lengua inglesa (las cuales fueron en total 454) identificamos 47 contribuciones de autores latinoamericanos (aproximadamente el 10%). La mayor parte de ellas pertenecen a capítulos de libro de la serie *International Perspectives on the Teaching and Learning of Mathematical Modelling* (41 contribuciones). Estos capítulos representan aproximadamente 14% del total de los capítulos de la serie. Hay que señalar que una cantidad significativa de las investigaciones fueron reportadas en el volumen 2013 de la serie ICTMA (11 capítulos). Es posible que esto se deba a que en ese año la conferencia del ICTMA fue realizada en Brasil.

Respecto a las aproximaciones y problemáticas abordadas, encontramos que gran parte de las investigaciones latinoamericanas son de tipo cualitativo (21 publicaciones), incluyendo metodologías de estudios de caso y etnográficas; además, la mayor parte de las investigaciones se realiza en los niveles de secundaria, post-secundaria y formación inicial de profesores. También es importante señalar que la perspectiva socio-crítica aglutina al conjunto más numeroso investigaciones latinoamericanas publicadas en lengua inglesa (8 publicaciones). La presencia de investigaciones realizadas bajo la perspectiva contextual también es significativa (6 publicaciones).

En las publicaciones realizadas en lengua española identificamos en total 15 contribuciones de autores latinoamericanos, la mayor parte de las cuales involucra a autores mexicanos (10 publicaciones). La mayoría de estas investigaciones se realizó en los niveles de secundaria y post-secundaria (10 investigaciones) y prácticamente todas son de tipo cualitativo (14 publicaciones). El conjunto más numeroso de investigaciones latinoamericanas (8 publicaciones) se ubica en las perspectivas socio-crítica y la educacional. Es necesario señalar que hay 4 investigaciones que no pudimos clasificar debido a que los autores no mencionan su adherencia a alguna de las perspectivas, ni encontramos las características que permiten clasificarlas (Kaiser y Sriraman, 2006; Preciado-Babb et al., 2018).

Finalmente, respecto a nuestra revisión de las memorias de los 10 últimos años de PME-NA, es importante señalar que con nuestros criterios de búsqueda ubicamos únicamente 4 contribuciones que involucran a autores latinoamericanos: dos del grupo de trabajo en "Models and Modeling" (2016 y 2017), el foro de investigación "Mathematical Modeling in School Education" (2014) y un reporte de investigación (2009).

Aproximaciones, Propósitos y Temas Innovadores en Modelación Matemática

Aunque numéricamente son pocas, las investigaciones latinoamericanas que identificamos tienen tendencias innovadoras en problemáticas y aproximaciones; en particular, enfatizan los aspectos sociales y culturales de la educación en modelación.

Al respecto, Stillman, Blum y Biembengut (2015) identificaron elementos de "a unique Latin American perspective to modelling" en el trabajo del autor brasileño Ubiratan D'Ambrosio, quien discute la generación de conocimiento (cognición), su organización individual y social (epistemología) y los modos en que es confiscado, institucionalizado y devuelto a las personas que lo generan (política). Su aproximación a la modelación matemática extiende la perspectiva socio-crítica y constituye una estrategia para la construcción de sistemas de conocimiento en contextos culturales diversos. D'Ambrosio señala:

> Through models, humans try to give explanations of myths and mysteries, and these explanations are organized as arts, techniques, theories, as strategies to explain and deal with facts and phenomena. These strategies, have been historically organized, in different groups, in different spatial and temporal contexts, which are the support of cultures, as systems of knowledge. (D'Ambrosio, 2015, p. 43)

Otra tendencia de la modelación en América Latina corresponde a las investigaciones reportadas como socio-epistemológicas (ver por ejemplo Arrieta Vera y Díaz Moreno, 2016). Esta aproximación comprende la modelación matemática en términos de prácticas sociales, tanto escolares como en las matemáticas formales.

Es importante señalar también que entre las investigaciones mexicanas reportadas resultan significativas las realizadas en el nivel de educación superior y específicamente en la formación de ingenieros (Domínguez, de la Garza y Zavala, 2015; Rodríguez, 2015).

Sobre los Propósitos de la Modelación Matemática

La revisión de los artículos y capítulos de libro permitió identificar propósitos de la modelación que extienden la lista propuesta por Kaiser y Sriraman (2006), ver Tabla 1 (la discusión detallada de la extensión de estos propósitos se hace en un reporte de investigación presentado por Preciado-Babb et al., (2018). En la parte inferior de esta tabla presentamos los propósitos característicos de las perspectivas epistemológicas y critica-social, notables en publicaciones de América Latina.

Tabla 1: Propósitos de la modelación matemática

Propósitos	**Ejemplos**
Aprender contenidos matemáticos	Álgebra, Geometría, Cálculo, Estadística
Aplicar matemáticas	Resolución de problemas
Aprender otras disciplinas	Química, Biología, Economía, Ciencias de la salud
Investigar sobre la conducta	Investigación sobre aprendizaje en ambientes virtuales
Diseñar ambientes de aprendizaje	Diseño de simuladores y ambientes virtuales de aprendizaje
Desarrollo de competencias de modelación	Elementos de modelación, criterios para determinar la calidad de la modelación matemática
Desarrollar habilidades de aprendizaje	Generalizar la solución de un problema a problemas similares
Propósitos específicos de los estudios latinoamericanos	
Generar teoría matemática	Comprensión conceptual y prueba matemática
Desarrollar pensamiento crítico	Juzgar modelos usados en la vida diaria; cuestionar los propósitos y los supuestos de diferentes modelos
Comprender las matemáticas como una disciplina	Aspectos históricos, sociales y políticos de las matemáticas como una disciplina
Desarrollar conciencia de problemas sociales y globales	Crear y criticar modelos usados para predecir crecimiento económico, calentamiento global, recaudación fiscal
Promover actitudes participativas	Comprometerse en la solución de problemas reales y la toma de decisiones dentro de la comunidad
Promover la cultura de la innovación	Crear soluciones para clientes, programar software para distintas audiencias
Participación en estrategias de emancipación	Descolonización, prácticas culturales en matemáticas y modelación matemática

De esta tabla se puede destacar que varios de los propósitos característicos en las perspectivas latinoamericanas buscan modificar entornos específicos, y promueven culturas de innovación y actitudes participativas. Por ejemplo, Orey y Rosa (2017) reportan tareas que abordan problemáticas reales de tarifas de transporte público con lo cual los autores no solo abordan un problema auténtico, sino que también buscan proponer soluciones que orienten la toma de decisiones de la comunidad afectada.

Contribuciones a Temas en Debate

Uno de los temas en que los estudios latinoamericanos han contribuido ampliamente es el relacionado con el reciente debate sobre las nociones de “autenticidad” y “mundo real”. Desde la llamada perspectiva realista, Kaiser y Sriraman (2006) afirman que los procesos de modelación se realizan como harían los matemáticos y científicos en la práctica. En este sentido, la autenticidad del conocimiento generado en el salón de clases puede considerarse a partir de su analogía con la actividad científica. Sin embargo, Jablonka (2007) ha sugerido que la recontextualización de prácticas de modelación de fenómenos científicos es problemática porque "causes a transformation of the unmediated discourses found in out-of-school practices of mathematical modelling, even though a modelling perspective overcomes the philosophy of naive realism encapsulated in traditional word problems" (Jablonka, 2007, p. 196).

Respecto a este debate, han surgido varias posiciones a las que se han sumado las investigaciones latinoamericanas. Por ejemplo, una propuesta consiste en considerar los elementos de la modelación auténtica dentro de algunas tareas específicas, y no para todo el ciclo de modelación. Al respecto, Silva Soares (2015) considera la modelación como una perspectiva de enseñanza en la cual los estudiantes analizan modelos ya existentes en lugar de crear algún modelo a partir de datos reales; es decir, no se comienza con el "mundo real".

Más recientemente, Carreira y Baioa (2017) participan en este debate introduciendo el concepto de "credibilidad" de las tareas matemáticas. Mientras que otros autores consideran que las situaciones de la "vida real" tienen potencial para convertirse en situaciones de aprendizaje atractivas, el enfoque de la credibilidad coloca la relevancia a nivel personal para los estudiantes. Finalmente, otros trabajos en Latinoamérica han usado simulaciones computacionales como modelos para enseñar contenidos matemáticos y científicos específicos (por ejemplo, Gomes Neves, Carvalho y Duarte, 2011). Aunque quizás los estudiantes no se involucren con datos reales, en estos casos sí pueden hacer experimentos con el comportamiento de los modelos y aprender tanto contenidos matemáticos como extra-matemáticos.

Reflexiones Finales

Este reporte complementa otros estados del arte y se centra en las contribuciones que sobre modelación matemática se han realizado por investigadores de América Latina. Encontramos que, en términos de la cantidad de publicaciones, la producción sobre modelación realizada en Latinoamérica es relativamente modesta: aproximadamente el 10% de las revistas de investigación en lengua inglesa revisadas y 12% de las contribuciones en las memorias de los últimos 10 años del PME-NA (reportes de investigación, grupos de trabajo y conferencias plenarias). Sin embargo, de nuestra revisión se deriva también que hay una riqueza de temas y perspectivas innovadoras en los estudios sobre modelación matemática que se realizan en América Latina. Cabe destacar en particular, la tendencia a realizar estudios que enfatizan las influencias sociales y culturales de la educación en modelación. Esta tendencia se refleja tanto en la adopción de perspectivas innovadoras (como la de U. D'Ambrosio), como en la identificación de propósitos comunes a un amplio número de investigaciones latinoamericanas.

Consideramos que las cantidades modestas de las publicaciones sobre modelación no se deben únicamente a un problema de lengua (publicaciones en inglés en relación a publicaciones en español), pues la mayor cantidad de publicaciones realizadas por investigadores latinoamericanos está hecha en publicaciones en inglés (aproximadamente el 77%). Además, la vitalidad de los temas actualmente discutidos en América Latina da cuenta de la importancia que está cobrando el trabajo en modelación matemática en la región. Pensamos que es necesario promover la realización de más investigaciones en los temas y perspectivas identificadas, pero también difundir el importante trabajo regional ya desarrollado.

LATINAMERICAN TRENDS IN MATHEMATICAL MODELING

Throughout the years, Mathematical Modeling has gained international attention, not only in research but also in the development of curricula and its applications in the classroom. However, systematic literary surveys are scarce. In this paper, we present some findings from a survey of 485 international publications related to different aspects of mathematical modeling. While the number of Latin-American publications is rather small compared to the number of international publications, the vitality of the currently discussed themes in this region and their innovative perspectives testify to the international relevance of this developed work on mathematical modeling.

Need for a Survey

The incorporation of mathematical modeling into the curricula has its roots in a movement in the late 1950s, when modeling advocates attempted to restore focus on the utility and applications of mathematics in schools and universities (Niss, Blum y Galbright, 2007).

A key moment for this international movement was the inauguration of the biennial *Conference on the Teaching of Mathematical Modelling and Applications* in 1983, organized by the *International Community of Teachers of Mathematical Modelling and Applications* (ICTMA). Another important moment took place in 2004 with the study on Mathematical Modeling organized by the *International Commission on Mathematical Instruction* (ICMI). The publication derived from this ICMI Study (Blum, Galbraith, Henn y Niss, 2007) continues to be a basic reference for research in this subject. Since then, international research has increased significantly, and research methods and focuses have extended beyond traditional approaches (Stillman, Blum y Kaiser, 2017). Despite the large number of publications on mathematical modeling in education, systematic reviews of the literature are scarce.

In their review, Kaiser and Sriraman (2006) proposed a classification identifying six perspectives on mathematical modeling research: the *realistic* perspective aims to solve real-life problems beyond mathematics; the *epistemological* perspective focuses on the development of mathematical theories, and includes intra-mathematical models that are used to advance theory in mathematics; the *educational* perspective considers different aims for modeling that serve scientific, mathematical and pragmatic purposes harmoniously; the *contextual* perspective, also called the model-eliciting approach, focuses on problem-solving activities constructed using specific instructional design principles; the *socio-critical* perspective emphasizes the need to develop a critical stance towards the role and nature of mathematical models, as well as their impact on social issues; and the *cognitive* perspective on modeling is transversal to the previous five and focuses on cognitive aspects of the mathematical modeling process.

Although Kaiser and Sriraman's classification is still in use, the large amount of work and perspectives produced in the last decade demands updating and adjusting it. In our review of the literature, we retake this classification to identify and contrast outstanding contemporary issues for which we systematically analyze some of the most important international publications, both in journal and book form, as described below.

Literature Review Methodology

This survey is the result of a seminar consisting of graduate and undergraduate students and educators from universities in Mexico and Canada. We identified multiple perspectives on modeling and found the literature on this topic to be vast. We also noticed that systematic reviews, such as the one conducted by Frejd (2013), were scarce. This influenced our decision to conduct our own review, which we started at the beginning of the year 2017.

We conducted the review in two stages. In the first stage, we searched peer-reviewed articles with ‘modeling’ or ‘modelling’ in the title through the SpringerLink database. Then, we refined the search using ‘Education’ as discipline and ‘Mathematics Education’ as subdiscipline. Articles that did not relate to mathematical modeling were excluded, resulting in a list with 73 articles. Despite the restrictions imposed in this first search, we consider that this list is representative of the modeling articles published in Springer because the use of these key words in the titles suggests that mathematical modeling was a main focus for the selected articles. The analysis of the articles in this list helped us to clarify and refine the categories that guided the review, and it also allowed us to identify key publications in books and articles from other publishers.

In the second stage of the review we included: (a) articles from the special issues on mathematical modeling, as well as articles published in 2017 not included previously (due to the overlap of editing times, we only considered Online First versions of the two special issues on modeling published by ZDM in 2018); (b) articles from the *Journal for Research in Mathematics Education* (JRME); (c) five books related to ICTMA and the 14th ICMI Study (Blum et al., 2007); (d) articles from journals on mathematics education published in Spanish; (e) a recent Latin-American book addressing research on mathematical modeling (Arrieta & Díaz, 2016); and (f) research reports, presentations of working groups, and plenary conferences from the last 10 years in the proceedings of PME-NA.

We chose *JRME* because it is at the top of the list of journals identified by Toerner and Arzarello (2012), along with *Educational Studies in Mathematics*. The same search criteria for the titles used in the first stage was followed to search articles in this journal. We included the Spanish journals and the Latin-American book to extend the scope of the review beyond publications in English. The selected journals were *Revista Educación Matemática* and *Revista Latinoamericana de Investigación en Matemática Educativa*, because they are specialized in mathematics education, are relevant among the Spanish journals, and appear in prestigious international indexes (Scopus and JCR). For the search in these journals, we proceeded with the same criteria as with the English journals.

In our analysis we focused on identifying methodological and theoretical aspects of the research. Specifically, we identified: research objectives, target populations, data collection instruments, type of methodological design (qualitative, quantitative, mixed, etc.), theoretical and modeling perspectives, country where research was carried out and authors’ country of residence, and mathematical themes.

It is necessary to point out that the work was arduous: often, authors did not explicitly state the theoretical-methodological underpinnings of their work and we had to infer these from the content. For example, for the definition of the perspective of modeling, we started from the characteristics pointed out by Kaiser and Sriraman (2006); then, we identified the purposes and theoretical elements indicated in the articles, triangulated this information with the conclusions, and, finally, considered the cited authors as key indicators.

Papers were also reviewed in parallel by two researchers. In case of differences in their analysis, the researchers reviewed the details together in order to come to a consensual decision; the most difficult cases were discussed in plenary meetings. The analysis criteria were adjusted continuously throughout the months when the review was conducted.

A total of 485 documents were included for this paper: 111 journal articles, 341 book chapters, and 33 PME-NA papers. Here, we report the results of our analysis of Latin American contributions to mathematical modeling found in these documents.

Latin American Trends in and Contributions to Mathematical Modeling Literature

In contrast to Blum and Niss's (1991) assertion that mathematical modeling was initially developed in regions such as Germany and the United Kingdom, our review reveals significant activity in mathematical modeling in Latin America since the 1990s. In fact, according to Biembengut (2016), this activity can be traced back to the 1970s in the case of Brazil. In our review, we identified two trends in publications on mathematical modeling in Latin America: on the one hand, the number of publications is relatively small; but, on the other hand, the approaches and problems addressed present innovative approaches for educational research in mathematical modeling.

Number of Publications

We identified a total of 66 publications by Latin American authors out of 485 documents in our database — approximately 13% of all the publications.

In the English publications (454 in total), we identified 47 contributions from Latin American authors (approximately 10%). Most were book chapters in the series *International Perspectives on the Teaching and Learning of Mathematical Modelling* (41 contributions). These represent approximately 14% of the total chapters that constitute the series. It should be noted that a significant number of studies was reported in the 2013 ICTMA volume (11 chapters). This is possibly related to the fact that the ICTMA conference was held in Brazil that year.

Regarding the approaches and problems addressed in the surveyed documents, we find that a large amount of Latin American research is qualitative (21 publications), including case study and ethnographic methodologies. In addition, most of the research deals with secondary and post-secondary education, and initial teacher training. It is also important to note that the socio-critical perspective brings together the largest group of Latin American research published in English (8 publications). The presence of research carried out under the contextual perspective is also significant (6 publications).

Within the category of articles published in Spanish, we identified 15 contributions from Latin American authors, most involving Mexican authors (10 publications). The majority of these studies focus on secondary and post-secondary levels (10 publications) and practically all are qualitative (14 publications). Most of the research (8 publications) assume socio-critical and educational perspectives. It is necessary to note that we could not classify four publications because the authors did not mention their adherence to any of the perspectives, and we could not identify particular classification characteristics (Kaiser & Sriraman, 2006; Preciado-Babb et al., 2018).

Finally, regarding our review of the last 10 years of the PME-NA proceedings, our search criteria only allowed us to find four contributions involving Latin American authors: two from the working group in “Models and Modeling” (2016 and 2017); a research forum “Mathematical Modeling in School Education” (2014); and a research report (2009).

Approaches, Purposes and Innovative Themes on Mathematical Modeling

Despite the reduced number of publications, the Latin American studies identified in our survey propose innovative ideas regarding addressed issues and approaches. They particularly emphasize the social and cultural aspects of education in modeling.

Stillman, Blum, and Biembengut (2015) identified elements of a unique Latin American perspective to modelling in the work of Brazilian author, Ubiratan D’Ambrosio, who discusses knowledge generation (cognition), its individual and social organization (epistemology), and the way it is confiscated, institutionalized and given back to the people who generated it (politics). This approach to mathematical modeling extends the socio-critical perspective and is a strategy for building up systems of knowledge in different cultural environments. As D'Ambrosio notes:

> Through models, humans try to give explanations of myths and mysteries, and these explanations are organized as arts, techniques, theories, as strategies to explain and deal with facts and phenomena. These strategies, have been historically organized, in different groups, in different spatial and temporal contexts, which are the support of cultures, as systems of knowledge. (D'Ambrosio, 2015, p. 43)

Another Latin American modeling trend corresponds to research reported as socio-epistemological (see, for instance, Arrieta & Díaz, 2016). This approach understands mathematical modeling in terms of social practices, both in school and in formal mathematics. The studies carried out at institutions of higher education, specifically in the training of engineers, stand out among the Mexican publications (Domínguez, de la Garza y Zavala, 2015; Rodríguez, 2015).

Purposes of Mathematical Modeling

Our review of articles and book chapters allowed us to identify some modeling purposes that extend the list proposed by Kaiser and Sriraman (2006) as presented in Table 1 (a detailed discussion of this extended list is presented by Preciado-Babb et al. (2018). In the lower part of Table 1 we present the characteristic purposes of the epistemological and critical-social perspectives notable in Latin American publications.

Table 1: Purposes of mathematical modeling

Purposes To:	**Examples**
Learn mathematics content	Algebra, Geometry, Calculus, Statistics
Apply mathematics	Problem solving
Learn other disciplines	Chemistry, Biology, Finances, Heath Care
Conduct research	Research on learning in virtual environments
Design learning environments	Design simulators and virtual environments for learning purposes
Develop modeling competencies	Elements of modeling; criteria for quality in mathematical modeling
Develop learning skills	Generalize the solution of a problem to other similar problems
Specific purposes of Latin American studies To:	
Generate mathematical theory	Conceptual understanding; mathematical proof
Develop critical thinking skills	Judge models used in daily life; question purpose and assumptions of different models

Understand mathematics as a discipline	Historical, social and political aspects of mathematics as a discipline
Develop awareness of social and global issues	Create and critique models used to predict economic growth, global warming, tax revenue, etc.
Promote a participatory attitude	Engage in addressing real problems and decision-making within the community
Promote a culture of innovation	Create something for a customer; create program software for an audience
Engage in emancipation strategies	Decolonization initiatives; cultural practices in mathematics and mathematical modeling

From Table 1, it is worth mentioning that several of the Latin American purposes seek to modify specific environments and promote cultures of innovation and participative attitudes. For example, Orey and Rosa (2017) reported a task addressing a real issue of tariffs in public transportation. Through their study, the authors not only addressed a real problem but also sought to propose solutions that guide the decision-making of the affected community.

Emergent Discussion Themes

Latin American studies have contributed extensively to the recent debate on the notions of ‘authenticity’ and ‘real world’. Regarding the realistic perspective, Kaiser and Sriraman (2006) claimed that modeling processes are carried out in a similar way to what mathematicians and scientists would do in practice. In this sense, the authenticity of the knowledge generated in the classroom can be considered based on its analogy with the scientific activity. However, Jablonka (2007) has suggested that recontextualization of modeling practices of scientific phenomena is problematic because it “causes a transformation of the unmediated discourses found in out-of-school practices of mathematical modeling, even though a modeling perspective overcomes the philosophy of naive realism encapsulated in traditional word problems” (Jablonka, 2007, p. 196).

Regarding this debate, several positions have emerged — and Latin American research has contributed to the discussion. For instance, one position is to consider the elements of authentic modeling within some specific tasks, not including the whole process of modeling. In this respect, Silva Soares (2015) suggested *model analysis* as a teaching approach in which students analyze an already existing model instead of creating a model from real data — in other words, it is not necessary to begin with the ‘real world.’

Most recently, Carreira and Baioa (2017) participated in this debate introducing the concept of ‘credibility’ of mathematical tasks. While authors have argued that real life situations have the potential to make the learning experience more attractive, this focus on credibility places the relevance at a personal level for students.

Finally, other Latin American studies have focused on computer simulations as models to teach specific mathematical and scientific content (e.g. Gomes, Carvalho & Duarte, 2011). While students may not engage with real data when using a simulator, they can experiment within the model and learn both mathematical and extra-mathematical content.

Final Remarks

This report complements other reviews of the state of the art focusing on the contributions made by Latin American researchers to mathematical modeling. We found that, in terms of number of publications, Latin American production on modeling is relatively modest:

approximately 10% of the contributions in English research journals and 12% of the reports from last 10 years of PME-NA (research reports, working groups and plenary conferences). However, we also found richness of innovative themes and perspectives in Latin American studies. It is particularly noteworthy the tendency to conduct studies that emphasize the social and cultural influences of modeling education. This trend is reflected both in the adoption of innovative perspectives (such as that of D'Ambrosio), and in the identification of the common purposes of a large number of Latin American research projects.

We consider that the modest number of publications is not solely due to language issues (English publications compared to Spanish publications), since we found that the largest number of Latin American research is published in English (approximately 77%). In addition, the vitality of the currently discussed themes in Latin America reveals the importance that mathematical modeling is gaining in this region. We think that it is necessary not only to promote more research in the identified topics and perspectives, but also to disseminate the significant regional work already developed.

References

Arrieta Vera, J. & Díaz Moreno, L. (Eds.) (2016). *Investigaciones latinoamericanas en modelación: Matemática educativa*. Barcelona, Spain: Gedisa.

Blum, W., Galbraith, P. L., Henn, H., & Niss, M. (Eds.) (2007). *Modelling and applications in mathematics education: The 14th ICMI study*. New York, NY: Springer-Verlag.

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects: State, trends and issues in mathematics instruction. *Educational Studies in Mathematics, 22*(1), 37-68.

Campbell, S. R. (2013). Mathematical modeling and virtual environments. In R. Lesh, P. L. Galbraith, C. R. Haines, & A. Hurford (Eds.), *Modeling students' mathematical modeling competencies: ICTMA 13* (pp. 583-583). New York, NY: Springer.

Carreira, S., & Baioa, A. M. (2017). Mathematical modelling with hands-on experimental tasks: On the student's sense of credibility. *ZDM Mathematics Education*. Retrieved from https://doi.org/10.1007/s11858-017-0905-1

D'Ambrosio, U. (2015). Mathematical Modelling as a Strategy for Building-Up Systems of Knowledge in Different Cultural Environments. In G. A. Stillman, W. Blum, & M. S. Biembengut (Eds.), *Mathematical modelling in education research and practice: Cultural, social and cognitive influences* (pp. 36-4). New York, NY: Springer.

Domínguez, A., de la Garza, J., y Zavala, G. (2015). Models and Modelling in an Integrated Physics and Mathematics Course. In G. A. Stillman, W. Blum, & M. S. Biembengut (Eds.), *Mathematical modelling in education research and practice: Cultural, social and cognitive influences* (pp. 513-522). New York, NY: Springer.

Frejd, P. (2013). Modes of modelling assessment—A literature review. *Educational Studies in Mathematics, 84*(3), 413-438.

Gomes Neves, R., Carvalho Silva, J., and Duarte Teodoro, V. (2011). Improving Learning in science and mathematics with exploratory and interactive computational modelling. In Kaiser, G., Blum, W., Borromeo Ferri, R., & Stillman, G. A. (Eds.), *Trends in teaching and learning of mathematical modelling: ICTMA 14* (pp. 331-339). New York, NY: Springer.

Jablonka, E. (2007). The relevance of modelling and applications: Relevant to whom and for what purposes? In W. Blum, P. L. Galbraith, H.W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education* (pp. 193-200). New York: Springer.

Kaiser, G., & Sriraman, H. (2006). A global survey of international perspectives on modelling in mathematics education. *ZDM Mathematics Education*, 38(3), 302-310.

Niss, M., Blum, W., & Galbraith, P. (2007). Introduction. In W. Blum, P. Galbraith, H.W. Henn & M. Niss. (Eds.) (2007), *Modelling and applications in mathematics education. The 14th ICMI study* (pp. 3-32). New York, NY: Springer.

Orey D.C. & Rosa, M. (2017). Mathematical modelling in a long-distance teacher education in Brazil: Democratizing mathematics. In G. A. Stillman, W. Blum, & G. Kaiser (Eds.), *Mathematical modelling and applications: Crossing and researching boundaries in mathematics education* (pp. 577-586). New York, NY: Springer.

Perrenet, J., Zwaneveld, B., Overveld, K., & Borghuis, T. (2017). Quality criteria for mathematical models in relation to models' purposes: Their usefulness in engineering education. In G. A. Stillman, W. Blum, & G. Kaiser (Eds.), *Mathematical modelling and applications: Crossing and researching boundaries in mathematics education*, (pp. 141-152). New York, NY: Springer.

Preciado-Babb, A.P., Solares-Rojas, A., Peña, F., Ortiz, A., Sandoval, M., Soriano, R., Carrión, V. & Farrugia, M. (2018). Exploring perspectives on mathematical modelling: a literature survey. In *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education*. Umeå, Sweden: PME.

Rodríguez Gallegos, R. (2015). A Differential Equations Course for Engineers Through Modelling and Technology. In G. A. Stillman, W. Blum, & M. S. Biembengut (Eds.), *Mathematical modelling in education research and practice: Cultural, social and cognitive influences* (pp. 545-555). New York, NY: Springer.

Salett Biembengut, M. (2016). Modelaje matemático en la educación brasileña: Historia de la ideas e ideas de las historias. In J. Arrieta Vera & L. Díaz Moreno, (Eds.), *Investigaciones latinoamericanas en modelación: Matemática educativa* (pp. 89-108). Barcelona, Spain: Gedisa.

Silva Soares, D. (2015). Model analysis with digital technology: A "hybrid approach." In G. Kaiser & G. A. Stillman (Eds.), *International perspectives on the teaching and learning of mathematical modelling* (pp. 453-463). New York, NY: Springer.

Stillman, G. A., Blum, W., & Biembengut, M. S. (2015). Cultural, social, cognitive and research influences on mathematical modelling education. In G. A. Stillman, W. Blum, & M. S. Biembengut (Eds.), *Mathematical modelling in education research and practice: Cultural, social and cognitive influences* (pp. 1-32). New York, NY: Springer.

Stillman, G. A., Blum, W., & Kaiser, G. (Eds.). (2017). Mathematical modelling and applications: Crossing and researching boundaries in mathematics education. In G. Kaiser & G. A. Stillman (Eds.), *International perspectives on the teaching and learning of mathematical modelling*. New York, NY: Springer.

Toerner, G., & Arzarello, A. (2012, December). Grading mathematics education research journals. *European Mathematical Society Newsletter* (86), 52-54. Zürich, Switzerland: European Mathematical Society.

Vos, P. (2011). What is 'authentic' in the teaching and learning of mathematical modeling? In G. Kaiser, W. Blum, R. Borromeo Ferri, & G. A. Stillman (Eds.), Trends in teaching and learning of mathematical modelling: ICTMA 14 (pp. 712-722). New York, NY: Springer.

THE COMMON CORE MORAL PANIC

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The Common Core State Standards (CCSS) Initiative was initially met with great enthusiasm from politicians and education experts. However, as states began rolling out the standards, backlash against the Common Core became widespread, and several states ended up pulling out of the initiative. To explore and better understand why there was such a negative reaction to the Common Core, we make use of the sociology construct of moral panics, and present an argument that the response to the CCSS was indeed a moral panic.

Keywords: Standards, Curriculum, Policy Matters

For decades, mathematics educators and politicians have called for a uniform, national set of mathematics standards as a way of improving mathematical instruction in the US. In 1989, the National Research Council, in the book *Everybody Counts* (National Research Council, 1989), asserted that "America needs to reach consensus on national standards for school mathematics" (p. 46). In 2009, President Barrack Obama called for states to work together to set higher standards and combat the disparities that arise from the fifty states' different sets of educational benchmarks (Montopoli, 2009). More recently, mathematics educators Deborah Ball, Mark Thames, and James Hiebert (Hiebert, 2013; Thames & Ball, 2013) argued "that the lack of a central or a common [national] curriculum is a major impediment" (Thames & Ball, 2013, p. 34) to improving the mathematical education of children in the US, and suggest that by unifying the nation on the mathematical learning goals for our students, we can greatly improve the teaching of mathematics.

With so many advocating for a set of national mathematics standards as a way to improve mathematics education in the US, it is no wonder that the Common Core State Standards Initiative ("Common Core State Standards Initiative," 2015), was initially met with great enthusiasm. The Common Core State Standards (CCSS) ("Common Core State Standards for English Language Arts," 2010; "Common Core State Standards for Mathematics," 2010) were developed by a state-led initiative in 2009, that included

> … governors and state commissioners of education from 48 states, two territories and the District of Columbia, through their membership in the National Governors Association Center for Best Practices (NGA Center) and the Council of Chief State School Officers (CCSSO). State school chiefs and governors recognized the value of consistent, real-world learning goals and launched this effort to ensure all students, regardless of where they live, are graduating high school prepared for college, career, and life ("Standards in Your State," 2015).

Through examining the best state standards then in existence, consulting with teachers, educational leaders, state political leaders, and leading thinkers, and eliciting and reviewing feedback from the public, the CCSS were developed to achieve the goal of unifying the states' learning goals for students across the nation.

Many in the mathematics education community had largely positive outlooks at the time the CCSS were unveiled. The 2010 Critical Issues in Mathematics Education conference at the Mathematical Sciences Research Institute (Rehmeyer, 2010) called the CCSS "an unprecedented

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.

opportunity to promote reasoning and sense-making across the United States," (p. 5), and further described the standards as "coherent: build[ing] the mathematical concepts in a logical, orderly way, introducing new ideas only when students have had a chance to master the concepts they are built on" (p. 5). The National Council for Teachers of Mathematics (NCTM) President Diane J. Briars also described how political leaders and educators were enthusiastic about the CCSS "and how having common, rigorous, world-class college- and career-ready standards would benefit both their students and the nation" (Briars, 2014). It seemed the CCSS would be a major step in improving the mathematical education of students in the US, unifying the mathematical learning goals of a majority of the states, and fulfilling a dream of many mathematics educators.

Despite the overwhelming optimism that the CCSS originally generated, these standards grew hugely controversial after their initial implementation. Views among large numbers of parents, educators, and politicians soured, and many began acting for the repeal of the CCSS, citing numerous negatively charged reasons for disposing of them.

As we examine how hostility towards the CCSS grew and became more widespread, it makes sense to ask: What happened? Why did a unified, research-based, national set of mathematics standards designed to improve mathematics education in the US become so controversial? To explore these questions and better understand the nation's reaction to the Common Core, we make use of a sociology construct originally developed by Stanley Cohen (2002), namely that of a *moral panic*.

The Moral Panic

Cohen (2002) defined moral panics as moments when

> A condition, episode, person or group of persons emerges to become defined as a threat to societal values and interests; its nature is presented in a stylized and stereotypical fashion by the mass media; the moral barricades are manned by editors, bishops, politicians, and other right-thinking people; socially accredited experts pronounce their diagnoses and solutions; ways of coping are evolved or (more often) resorted to; the condition then disappears, submerges or deteriorates … Sometimes the panic passes over and is forgotten, except in folklore and collective memory; at other times it has more serious and long-lasting repercussions and might produce such changes as those in legal and social policy or even in the way the society conceives itself. (p. 1).

These moral panics arise quite suddenly when a large, empowered sector of society labels something as deviant and threatening to the norms of that society, and urges that steps must be taken to repair the damage and prevent the perpetrators from further destruction of the moral order (Goode & Ben-Yehuda, 2009). These perpetrators become "folk devils" (Cohen, 2002), and are seen as "legitimate and deserving targets of self-righteous anger, hostility, and punishment" (Goode & Ben-Yehuda, 2009, p. 35).

Elements of a Moral Panic

Goode and Ben-Yehuda (2009) elaborated on the initial definition of the moral panic given by Cohen (2002) by identifying five crucial elements that appear within moral panics. First, there must be a heightened level of concern over the actions of a certain group or category of society whose behavior supposedly endangers other sectors of society. This concern should be manifest through a variety of outlets, such as public opinion polls, public commentary, media attention, and proposed legislation.

Second, there must be an increased level of hostility towards the deviant group, for the harm thought to be caused by them. In other words, "not only must the condition, phenomenon, or behavior be seen as threatening, but a clearly identifiable group in or segment of the society must be seen as responsible for the threat" (Goode & Ben-Yehuda, 2009, p. 38). As society begins to see a split between "us" – good, honest members of society – and "them" – the troublemakers or deviants – stereotyping of these outsiders occurs. This results in the creation of "folk devils" or villains whom society can blame for the phenomenon, and fight against, in order to maintain societal order.

Third, there must be substantial agreement or consensus that an actual threat exists, and is being caused by the behavior of the deviant group. "This sentiment must be fairly widespread, although the proportion of the population who feels this way need not be universal or, indeed, even make up a literal majority" (Goode & Ben-Yehuda, 2009, p. 38). Thus, if only a few scattered, separate individuals believe that a threat exists, then there is no moral panic, despite these individuals heightened emotions and concerns.

Fourth, as implicitly assumed in the term moral panic, the reaction to the supposedly harmful occurrence or behavior needs to be disproportionate. This disproportion is manifest through: (1) a belief that a more sizeable number of individuals are engaged in the deviant behavior than actually are; (2) a belief that the threat, danger, or damage is far more extensive than what is warranted; and (3) the wild exaggeration of numbers and figures, such as the number of deaths, violent acts, crimes committed, injuries, and dollars of damage caused by the behavior of the misbehaving group. "In short, the term moral panic conveys the implication that public concern is in excess of what is appropriate if concern were directly proportional to objective harm" (Goode & Ben-Yehuda, 2009, p. 40).

Fifth, moral panics are volatile, erupting suddenly onto the scene of social conscience, and, almost as suddenly, subsiding from the awareness of the concerned members of society. Some moral panics become institutionalized, that is, the concerns that arose during the moral panic lead to the creation of social movement organizations, legislation, or rules of enforcement. Other moral panics simply fade away, with little to no trace or effect on society. The volatility of moral panics does not mean that the issues involved had no structural or historical antecedents; in fact, "the specific issue that generates a particular moral panic may have done so in the past, perhaps even the not-so-distant past" (Goode & Ben-Yehuda, 2009, p. 42). However, the degree of hostility generated during a moral panic flares up quickly and is not typically sustainable, hence its volatile nature.

Actors in a Moral Panic

Goode and Ben-Yehuda (2009) further point out that within a moral panic, there are usually five main "actors" or groups of participants. The first and most important actor is the press or mass media. As noted by Cohen (2002), within industrialized societies, the information that society uses to build ideas about societal norms, and who or what should be labeled as deviants, is received second hand. In other words, what the general public receives as "news" has already been processed by the mass media, who determine what is newsworthy and how it should be presented, based upon the commercial and political constraints within which the media outlets operate. Thus, one of the main instigators and sustainers of a moral panic is the press, who plays a large part in defining and displaying to the public the groups or episodes that should be considered as "the enemy," by exaggeration and over-reporting of events, as well as stereotyping characters and behaviors (Goode & Ben-Yehuda, 2009). The second actor is the public. A moral panic cannot erupt unless the issues pointed out by the media strike a responsive chord within the general public, where the perpetrators identified can be focused on and vilified as a symbol for larger problems plaguing society. The third actor in a moral panic is the social control culture, or those who are responsible for keeping order within society. Within most moral panics studied in sociology, this group consists of police and the courts, or law enforcement. Public concern about a supposed threat within a moral panic leads to the creation of public attitudes about what law enforcement should be doing to quash the problems "the enemy" is producing. The fourth actor within a moral panic is the group consisting of politicians and legislators. In past moral panics, when politicians and legislators recognized that the media and general public identified a threat to society, they "… aligned themselves against the devil and on the side of angels; the fact is, they picked an 'easy target' … What counted was not the nature of the target but what side they were on and what they were against." (Goode & Ben-Yehuda, 2009, p. 26). These politicians and legislators within a moral panic push for immediate action within government and the law to end the disturbances to society caused by the groups or events branded as deviants. The fifth actor comprises "action groups," or members of the public who come together to advocate for solutions to the problems created by the deviants to society, usually claiming that existing remedies are insufficient.

While some of the actors involved in the reaction to the Common Core fall into the above categories (e.g. the press, the public, politicians, and action groups), there are some differences. First, since the Common Core does not involve crime or criminal acts, the police, courts, and law enforcement are not relevant. However, the analog of law enforcement within education, those who are part of the social control culture that are expected to uphold and enforce societal norms and improve the quality of the education of children locally, are state school boards. Thus, in examining the reaction to the CCSS, it is important to study how school boards reacted to the standards. Second, in the past few years, social media has vastly changed (1) the way people interact with one another, (2) the ease and accessibility of creating groups that call for action and change, and (3) how the public receives and interprets news. In fact, a recent study by the Pew Research Center found that about 30% of US citizens get their news on Facebook (Anderson & Caumont, 2014). Hence, in studying the nation's reaction to the CCSS, we also must pay particular attention to the public opinions, "news" displayed, and action groups active on social media.

Methods

Data sources for this paper include social media posts, news outlets, school board meeting notes, and legislative action. In particular, four states were chosen to examine in depth: Indiana, Massachusetts, South Carolina, and Utah. Data analysis consisted of descriptive statistics to examine volatility and consensus, and qualitative coding and thematic analysis (Braun & Clarke, 2006) of various sources to identify concerns, hostility, and disproportionate reactions of the public towards the common core. In the next section, we present evidence that all of the elements of a moral panic existed in the reaction against the Common Core.

Evidence of the Common Core Moral Panic

Concern

A wide variety of concerns arose as the Common Core was implemented, as illustrated by a popular picture circulated on Facebook, presented in Figure 1. These concerns manifested through Facebook posts, school board meetings, legislation, and news outlets, and include loss of state and local control over education decisions, developmentally and age inappropriate standards, relentless testing that students will be subjected to, and the privatization of education.

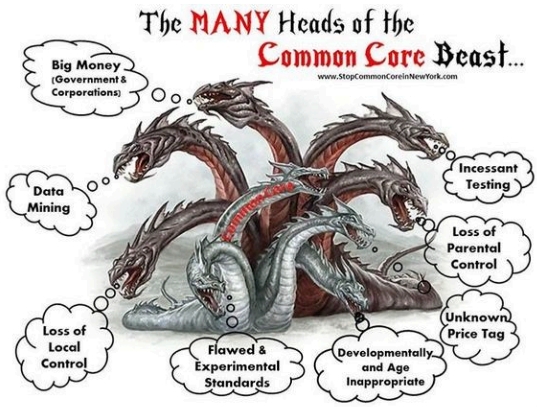

Figure 1. Widely circulated Facebook illustration from www.StopCommonCoreinNewYork.com.

Hostility and Folk Devils

Through examination of hundreds of Facebook posts of anti-Common Core groups, we were able to identify two categories of folk devils that were vilified as those responsible for the CCSS. The first category consisted of those who could be blamed for taking over state and local control of educational decisions, such as President Barrack Obama, Education Secretary Arne Duncan, or the federal government in general. The second category consisted of people or companies that people felt would benefit financially from adoption of the standards, including Bill Gates, Pearson, and other educational "big businesses" or test developers. In each post, the identified folk devil was blamed for the problems that the CCSS were believed to cause, including concerns mentioned in the previous section, as well as criticized for trying to make money through exploiting children's education.

Consensus

We identified three types of evidence for consensus that the CCSS were problematic. First, hundreds of anti-Common Core Facebook groups were created, with some consisting of thousands of members. Second, numerous articles about the CCSS were produced by national news outlets in 2013 and 2014. Third, because the voice of opposition became so loud, and had such a large number of people behind it, several states were forced to legislative action, such as Governor Herbert of Utah organizing a team to review the quality and legality of the standards (Herbert, 2014), or states (e.g. Indiana, Oklahoma, South Carolina) choosing to completely pull out of the standards altogether (Strauss, 2014).

Disproportion

Disproportionate claims that the CCSS were detrimental and causing great harm were not uncommon. Klein (2015) reported on five of the extreme claims, including that the Standards (a) "turn kids gay," (b) "indoctrinate kids under a Nazi society," or (c) "turn kids into communists or socialists." Considering that the CCSS are only a set of learning goals and not a brainwashing effort, these claims were clearly unfounded and over-reactive.

As another example of the disproportionate reaction towards the CCSS, consider what happened in the state of Indiana. On May 11, 2013, Governor Mike Pence signed a bill that paused implementation of the CCSS (Castleman, 2013), and by March of 2014, Pence signed legislature that completely withdrew Indiana from the Common Core Initiative (Nicks, 2014). Despite clear warnings from the Obama administration that fines would be enacted if the state's new standards were subpar (Lucas, 2015), Indiana pushed forward trumpeting their actions as an act of support toward state's rights. "By signing this legislation, Indiana has taken an important step forward in developing academic standards that are written by Hoosiers, for Hoosiers, and are uncommonly high," Governor Pence said in a statement (Calvert, 2015). The committee for creating new state standards met in October 2013, but after a draft of these new standards was released, the Indiana Department of Education found that 70 percent of the standards were exactly the same as the CC with another 20 percent of the content simply a modified version of the CC ("Open the Floodgates", 2014). These similarities to the CCSS continued on through to the final draft, with many seeing the new standards as little more than a "rebrand" (Kurtz, 2014) or a "warmed-over version of Common Core's standards" (Calvert, 2015). Hence, it seems that the negative reaction towards the standards themselves was unwarranted, and most likely an overreaction, as the state found it difficult to write high-quality standards that were different from the Common Core.

Volatility

Despite the CCSS development in 2009-2010, and the adoption of the standards by a large majority of states from 2010-2012, news coverage about the CCSS was fairly minimal until midway through 2012, when the number of news articles mentioning the CCSS grew rapidly. According to a LexisNexis Newspaper article search analysis of US publications conducted by the American Enterprise Institute (Hess & McShane, 2014), this rapid growth in articles reporting on the Common Core rose quickly from August 2012 to a peak in the number of articles in August 2013 (see Figure 1).

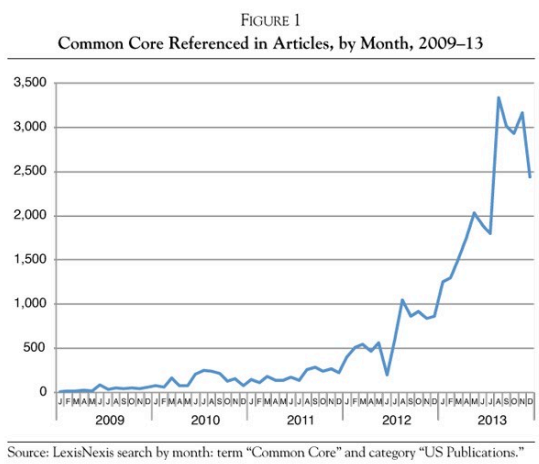

Figure 2. Number of US newspaper articles mentioning "Common Core" per month. From Hess & McShane (2014). Adapted with permission.

As additional evidence of the volatility of the reaction towards the CCSS, Figure 2 presents the number of anti-Common Core Facebook groups created by month, from a sample of 191 of the largest of these groups across the US:

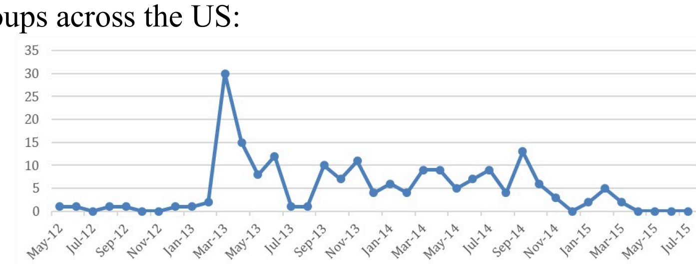

Figure 3. Number of anti-Common Core Facebook groups created per month from a sample of 191 of the largest groups in the US.

From these two figures, it is clear that public and press awareness of, and concern with, the CCSS erupted during the spring and summer of 2013, but over the past five years, concern and calls for repealing the CCSS have lost their prominence in the public square. However, this is not to say that distress over the Common Core will not reach these levels again, as some moral panics can become "a conceptual grouping of a series of more or less discrete, more or less localized, more or less short-term panics" (Goode & Ben-Yehuda, 2009, p. 42). Thus, other spikes in concern about the CCSS may still occur in the future.

Discussion

As education is a highly politicized, highly charged discussion within the US, it might not seem surprising that such opposition to the CCSS arose during their implementation. However, the sociological construct of moral panic gives us a more detailed description of the negative reaction towards the Common Core, helping us see why it was so easy for the public and media to paint the CCSS as a terrible idea.

Intriguingly, the Common Core moral panic afforded both sides of the political spectrum concerns and arguments against the Common Core. For those on the Left, the idea of big businesses profiting from the education of the nation's children was maddening. For those on the Right, Federal Government taking over local control was a large overstep of power. In both cases, calls for the repeal of the CCSS ran rampant across the country.

This leads us to end with the following question: if more educational reforms are desired, how can we avoid another moral panic in the future? Sadly, because education is so politicized, the answer may be that a moral panic cannot be avoided; nevertheless, it could be minimized. Better communication with parents and teachers, and a slower implementation of standards could have benefitted the CCSS greatly. Future educational reformers, including educators and politicians, should look closely at the Common Core Moral Panic, and consider carefully how mistakes made in its implementation could be avoided in implementing future reform.

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References

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*, 77-101.

Briars, D. J. (2014). Core truths. *President's Messages*. Retrieved from: http://www.nctm.org/News-and-Calendar/Messages-from-the-President/Core-Truths/

Calvert, K. (2014, March 25). In an About-Face, Indiana Decides to Drop Common Core. *PBS Newshour*. Retrieved from http://www.pbs.org/newshour/rundown/face-indiana-decides-drop-common-core-education-standards

Castleman, A. (2013). Pence signs bill 'pausing' reading, math standards. *WTHR 13 News*. Retrieved from https://www.wthr.com/article/pence-signs-bill-pausing-reading-math-standards.

Cohen, S. (2002). *Folk devils and moral panics* (Third ed.). New York, NY: Routledge.

Common Core State Standards for English Language Arts. (2010). *Common Core State Standards Initiative.* Retrieved from http://www.corestandards.org/wp-content/uploads/ELA_Standards.pdf

Common Core State Standards for Mathematics. (2010). *Common Core State Standards Initiative.* Retrieved from http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

Common Core State Standards Initiative. (2015). Retrieved from http://www.corestandards.org/

Goode, E., & Ben-Yehuda, N. (2009). *Moral panics: The social construction of deviance.* (Second ed.). Oxford: Blackwell.

Herbert, G. (2014). *Governor remarks - Strengthening Utah education for our students*. Salt Lake City, UT Retrieved from https://www.utah.gov/governor/standards/.

Hess, F. M., & McShane, M. Q. (2014). Flying under the radar? Analyzing Common Core media coverage. Retrieved from American Enterprise Institute website: https://www.aei.org/publication/flying-under-the-radar-analyzing-common-core-media-coverage/

Hiebert, J. (2013). The constantly underestimated challenge of improving mathematics instruction. In K. R. Leatham (Ed.), *Vital Directions in Mathematics Education Research* (pp. 45-56). New York: Springer.

Klein, R. (2015). 5 of the most extreme claims made against Common Core in the last 5 years. *Huffington Post*. Retrieved from https://www.huffingtonpost.com/2015/06/04/common-core-birthday_n_7506018.html

Kurtz, S. (2014, May 1). Pence's Common Core Miscalculation. *The Corner*. Retrieved from http://www.nationalreview.com/corner/376984/pences-common-core-miscalculation-stanley-kurtz

Lucas, F. (2014, May 7). Federal Gov't Issues Warning to State That Wants to Opt Out of Common Core. *The Blaze*. Retrieved from http://www.theblaze.com/stories/2014/05/07/federal-govt-issues-warning-to-state-that-wants-to-opt-out-of-common-core/

Montopoli, B. (2009, March 10, 2009). Obama's Remarks on Education. *CBS News*. Retrieved from http://www.cbsnews.com/news/obamas-remarks-on-education/

National Conference of State Legislatures. (2015). Common Core status map. *College and Career Readiness Standards Legislation.* Retrieved from http://www.ccrslegislation.info/CCR-State-Policy-Resources/common-core-status-map

National Research Council. (1989). *Everybody counts: A report to the nation on the future of mathematics education*. Washington, DC: National Academy Press.

Nicks, D. (2014). Indiana drops Common Core education standards. Time. Retrieved from http://time.com/36779/indiana-drops-common-core-education-standards/

Open the Floodgates? Indiana Becomes First State to Scrap Common Core. (2014, March 25). *Fox News*. Retrieved from http://www.foxnews.com/politics/2014/03/25/indiana-becomes-first-state-to-drop-common-core-standards/

Rehmeyer, J. (2010). Reasoning & sense-making in the math curriculum. *Critical Issues in Mathematics Education, 8*(7).

Standards in Your State. (2015). *Common Core State Standards Initiative.* Retrieved June 18, 2015, from http://www.corestandards.org/standards-in-your-state/

Strauss, V. (2014). Two more states pull out of Common Core. *The Washington Post*. Retrieved from https://www.washingtonpost.com/news/answer-sheet/wp/2014/06/05/two-more-states-pull-out-of-common-core/?utm_term=.ee8ed6ee8c62

Thames, M. H., & Ball, D. L. (2013). Making progress in U.S. mathematics education: Lessons learned - Past, present, and future. In K. R. Leatham (Ed.), *Vital Directions for Mathematics Education Research* (pp. 15-44). New York: Springer.

USING A NEXT-GEN “AGILE CURRICULUM” TOOL TO STRENGTHEN WEAK RATIO REASONING

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Math Mapper 6-8 (MM68) is a digital learning system (DLS) that scaffolds curricular coherence, supporting assembly and iterative improvement of diversely-resourced curricula. MM68 includes a learning map consisting of learning trajectories organized around the big ideas in middle grade mathematics and a diagnostic assessment and reporting system. We report on how a design-based implementation research (DBIR) study supported the development of a new framework for curriculum enactment, the “agile curriculum framework,” and explore how such a framework can productively blur lines among curriculum designer, user, and researcher.

Keywords: classroom assessment, learning trajectories, agile curriculum, digital learning system

Perspectives or Theoretical Framework for the Research

The research is grounded in constructivist learning theory (Confrey & Kazak, 2006; Thompson, 2002; von Glasersfeld,1982): student thinking around key ideas goes through predictable, albeit probabilistic, transformations from naïve to sophisticated thinking to reach a target concept. The sequence of the transformations can be described as *learning trajectories* (LTs) or *progressions* (Confrey, Maloney & Corley, 2014; Sarama & Clements, 2009; Simon, 1995). They begin from students’ prior knowledge, which is refined and revised based on their interactions with a carefully sequenced set of curricular tasks. Those tasks challenge students to solve new problems that represent more sophisticated reasoning. We emphasize that a learning trajectory draws fundamentally from genetic epistemology (Piaget,1970), enacting the idea that to “know” something, one has to consider what conditions create and render the need for the idea. However, LTs do not assume Piagetian stage theory (Lehrer & Schauble, 2015), and they do not manifest as biological development such as maturation. Movement along a trajectory occurs via building and refining schemes, through a process of assimilation and accommodation as students solve tasks (Simon & Tzur, 2004). It depends on instructional experiences and on students’ “opportunities to learn”: opportunities to undertake the carefully sequenced tasks, use appropriate tools, and engage in discussions and debates with teachers and peers. Unlike most meanings of instructional design, which involve logical deconstruction of how to reach targeted ideas (Gagne, 1965), LTs are grounded in empirical research on the patterns of responses shown by students in learning (sometimes called “emergent” (Gravemeijer, 1999)).

The Study

This study reports on the “agile curriculum framework” (Confrey, et al., 2018) and how it can be supported by Math Mapper 6-8 (MM68), a new type of digital learning tool. The agile curriculum framework (figure 1) describes how curricula are enhanced through iterative short-term (during instructional units) and long-term (across months and years) cycles, based on feedback.

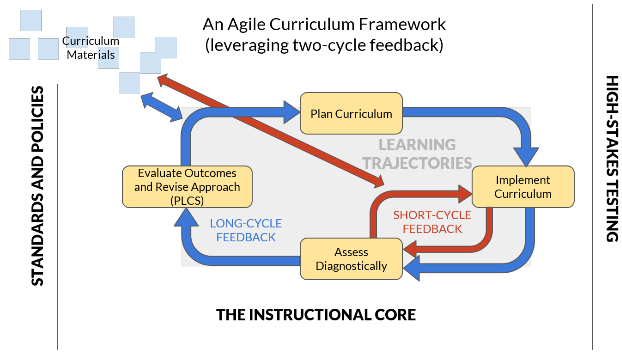

Figure 1. The agile curriculum framework (Confrey et al., 2018, p.7).

MM68 is a digital learning system (DLS) (Confrey, 2015) that is built on an explicit research-based foundation in learning trajectories (LTs), aligned to the Common Core State Standards in Mathematics (CCSS-M) (Confrey et al., 2014). MM68's front end is a learning map that organizes the content of middle grades math around nine "big ideas," each of which is subdivided into 2-5 "relational learning clusters." Each cluster comprises a set of constructs whose interconnected meaning supports the big idea. Each construct is embodied by an LT of 5-12 progress levels, which are used to build periodic "classroom assessments" (Pellegrino, DiBello, & Goldman, 2016). MM68 generates ongoing assessment reports—for teachers and students—that are used to refine instruction. In this paper, we present the results of a 4-week design study investigating if—and, if so, how—three elements interact to have positive impacts on students' learning: 1) teacher knowledge of LTs, 2) a curriculum designed to promote student progress along related LTs, and 3) the use of diagnostic assessments to monitor progress on those LTs. The goals of the study were to: a) improve sixth grade students' ratio reasoning, b) learn how teachers share, discuss, and act with students on assessment data, and c) use classroom observations and student data as one source of validation of the LT-based assessments.

Methods

To hold one critical factor constant in this complex system, a single set of curricular materials was created to be implemented across all sixth grade sections at a high needs middle school. The materials were aligned to the LTs in two ratio clusters: "key ratio relationships" (cluster 4) and "comparing ratios and finding missing values" (cluster 5). We conducted a design-based implementation research (DBIR) study (Penuel & Fishman, 2012). Our research conjecture (Confrey & Lachance, 2000) was that if teachers understand the LTs, they can assist students in progressing through the materials to build and demonstrate more sophisticated ratio reasoning. We further conjectured that *how* the teachers discussed the data from periodic assessments with students would provide an explanatory framework on how the LTs can most directly support improvement in practice. Therefore, we sought to observe how the teachers discussed and reviewed the data with students, how they adjusted their instruction, and whether and how students resolved to improve.

Data Sources and Analysis

Five assessments were administered during the study: One pretest and one posttest, scored digitally, both containing items from both clusters 4 and 5; two diagnostic assessments administered following instruction on each cluster (one containing only cluster 4 items and one containing only cluster 5 items), scored digitally; and an independent posttest from Student Achievement Partners administered on paper, scored manually by two members of the research team. The mean scores on the pretest, diagnostic assessments, and posttest were computed and the difference between the means was determined. Results are reported for students completing both the pretest and posttest (n=165 for cluster 4 and n=143 for cluster 5). Some students mistakenly took a cluster 4 assessment as a posttest, and others mistakenly took a cluster 5 assessment as a posttest, resulting in a difference between the number of students completing both the pretest and posttest for cluster 4 and the number of students completing both tests for cluster 5. Teachers were also observed implementing the curricular materials, administering assessments, and reviewing assessment feedback with their students. Teachers and students were interviewed to understand how they were interpreting and acting on assessment feedback.

Results and Discussion

For students who completed both the pretest and posttest, mean scores increased from 35.59% (SD=20.67%) to 50.19% (SD=22.85%) in cluster 4 (a gain of **+14.6**), and from 31.59% (SD=21.53) to 47.14% (SD=25.34%) in cluster 5 (a gain of **+15.55**). The effect sizes were 0.67 (cluster 4) and 0.66 (cluster 5). On the independent posttest, students achieved a mean score of 38.3% (SD=20.6%, n=262). Some difference between the independent posttest and the MM68 posttest is to be expected. The curriculum, diagnostic assessments, and MM68 posttests were built with the LTs as a common framework, providing a coherent classroom experience. The independent posttest was not built using this framework and the alignment between assessment and classroom experience was not as strong. However, as will be discussed below, the results of the independent posttest informed revisions to both the LT levels and long-cycle modifications to the curricular materials.

Examples of the type of short-cycle and long-cycle revisions described by the agile curriculum framework were evident in the ways researchers and teachers presented, discussed, and shared results and insights. These insights took the form of: a) recommendations about reporting and acting on diagnostic assessment data, b) general instructional suggestions, c) content-specific suggestions, and d) proposed changes to the curricular materials.

To encourage student discussion and reflection, and to help ensure that students perceived the diagnostics as constructive, teachers encouraged each other to focus on both strengths *and* weaknesses of students. Researchers worked with teachers to emphasize the LTs as a way of monitoring progress and guiding future learning. Teachers, in turn, focused on the interpretation of the scores as “getting started,” “showing some understanding,” or “showing proficiency” rather than the specific percent correct score. Students reported positive attitudes about revising and resubmitting answers and were observed “taking up the language” of the constructs and clusters to focus their learning. Researchers leveraged diagnostic assessment data to address gaps in student understanding by providing supplemental activities to be used during the unit.

Researchers also made long-cycle revisions to the curricular materials based on assessment data. Responding to weak performance in the construct “finding unit ratios,” researchers revised the curricular materials for future use by strengthening the treatment of unit ratio. Similarly, the independent posttest revealed insufficient treatment of the connection between part-to-part and part-to-whole ratios. The materials were revised to specifically address this gap.

Finally, researchers provided content-specific suggestions to inform future implementation of the curriculum. For example, researchers suggested teachers encourage students to consistently make, label, and use ratio tables and ratio boxes, and to use “broken arrow” diagrams to stress the multiplicative relations both as covariation and as correspondence (Vergnaud, 1988).

Conclusion

The DBIR work carried out in this partnership led to a variety of means of ongoing improvement, including professional development, curricular revisions, and tool redesign aimed at strengthening and supporting short-cycle and long-cycle instructional adaptations characteristic of the agile curriculum framework. Although this study reports on findings from one grade level team in one school, it nevertheless demonstrates how a school partnership can be formed to work together for ongoing improvements and the types of learning gains that can be achieved when students, teachers, and researchers all have a stake and a role in that improvement.

Acknowledgments

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References

Confrey, J. (2015). *SUDDS digital learning system and learning map.* Retrieved from https://sudds.ced.ncsu.edu/wp-content/uploads/2017/02/gates-report-07-2015-smaller_2.pdf

Confrey, J., & Kazak, S. (2006). A thirty-year reflection on constructivism in mathematics education in PME. A. Guttierez & P. Boero (Eds.). *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 305–345). Rotterdam, the Netherlands: Sense Publishers.

Confrey, J., & Lachance, A. (2000). Transformative teaching experiments through conjecture-driven research design. A. E. Kelly, & R.A. Lesh (Eds.). *Handbook of research design in mathematics and science education* (pp. 231-265). Mahwah, NJ: Lawrence Erlbaum Associates.

Confrey, J., Maloney, A., Belcher, M., McGowan, W., Hennessey, M., & Shah, M. (2018). The concept of an agile curriculum as applied to a middle school mathematics digital learning system. Manuscript submitted for publication.

Confrey, J., Maloney, A. P., & Corley, A. K. (2014). Learning trajectories: a framework for connecting standards with curriculum. *ZDM, 46*(5), 719-733.

Gagne, R. M. (1965). *The conditions of learning*. New York: Holt, Rinehart, and Winston.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning, 1*(2), 155–177.

Lehrer, R., & Schauble, L. (2015). Learning progressions: The whole world is NOT a stage. *Science Education, 99*(3), 432–437.

Pellegrino, J. W., DiBello, L. V., & Goldman, S. R. (2016). A framework for conceptualizing and evaluating the validity of instructionally relevant assessments. *Educational Psychologist*, *51*(1), 59-81.

Penuel, W. R., & Fishman, B. J. (2012). Large-scale science education intervention research we can use. *Journal of Research in Science Teaching*, 49(3), 281–304.

Piaget, J. (1970). *Genetic Epistemology.* New York: W.W. Norton.

Sarama, J. & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. New York, NY. Routledge.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 114–145.

Simon, M. A., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical thinking and learning*, 6(2), 91–104.

Thompson, P. W. (2002). Didactic objects and didactic models in radical constructivism. K. Gravemeijer, R. Lehrer, B. van Oers, & L. Verschaffel (Eds.). *Symbolizing and modeling in mathematics education* (pp. 197–220). Dordrecht, The Netherlands: Kluwer.

Vergnaud, G. (1988). Multiplicative structures. J. Hiebert & M. Behr (Eds.). *Number concepts and operations in the middle grades* (pp. 141–161). Reston, VA: National Council of Teachers of Mathematics.

von Glasersfeld, E. (1982). An interpretation of Piaget's constructivism. *Revue Internationale de Philosophie*, 36(142–143), 612–635.

RE-LEARNING CURRICULUM THROUGH FOCAL EXPERIENCES TO CREATE SPACE FOR DIALOGIC CURRICULUM

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Learning to teach mathematics is a complex endeavor. Particularly challenging is making sense of the various levels of curriculum (societal, technical, and enactment) and engaging in problem-solving, or dialogic, mathematics curriculum. This self-study examines how one early career teacher transformed her enactment of curriculum, through specific experiences during her first five years teaching science and mathematics. Narrative inquiry was used to examine past artifacts (i.e., lesson plans) and experiences. The findings highlight the importance of specific focal experiences that support teachers throughout their first five years of teaching to create space in the technical curriculum for enacting dialogic mathematics.

Keywords: Curriculum, Teacher Beliefs, Teacher Education-Inservice

Experience is the foundation for learning (Dewey, 1902), as such prospective teachers (PSTs) draw upon their many years of experience as mathematics student when they are learning to teach. Despite recent calls for dialogic instruction, which includes challenging problem-based curriculum, group work, and an emphasis on mathematical thinking over finding correct answers (NCTM, 2000, 2010), PSTs often draw on their own classroom experiences to define mathematics teaching. These formative experiences as a learner, often learning mathematics through direct mathematics instruction, limit PSTs' understanding of elementary mathematics (Ma, 1999). This self-study, drawing on levels of curriculum (Doyle, 1992), utilizes a narrative approach to examine how specific focal experiences can transform one's mathematics curriculum enactment. I begin by unpacking my learning to teach experience, using the lens of extant literature, and conclude with implications for teacher education.

Theoretical Framework

In this proposal, I utilize the levels of curriculum—societal, technical, and classroom enactment (Doyle, 1992)—to examine how an early career teacher's (ECT) understanding of curriculum levels changed as her experiences changed.

Societal Level of Understanding Curriculum

PSTs enter mathematics methods with varying beliefs about how school should be taught based on their own experiences (Ambrose, 2004). These ideas and understandings come from their experience of being students. This level of curricular understanding is termed the *societal level*. Such an understanding "may mislead prospective teachers into thinking that they know more about teaching than they actually do" making "it harder for them to form new ideas and new habits of thought and action" (Feiman-Nemser, 2001, p. 1016). Therefore, one of the first goals of many mathematics teacher educators (MTEs) is supporting PSTs in expanding their pre-existing beliefs of mathematics curriculum. Here the emphasis is often placed on support PSTs in gaining a better understanding of the realities and complexities of dialogic curriculum enactment (e.g., Ambrose 2004). Unfortunately, the societal level of curricular understanding is often in conflict with the ideas in mathematics methods.

Technical Level of Understanding Curriculum

The technical level of curriculum is often concrete, albeit frequently changing, and includes standards, textbooks, and curriculum documents (Doyle, 1992). While these standards can be straightforward they can also be challenging for prospective teachers to enact for a myriad of reasons, i.e., prospective elementary teachers may not have sufficient mathematical content knowledge (e.g., Ambrose, 2004). Additionally, as Westbury (2002) highlighted prospective teachers may view standards-based textbooks as "curriculum-as-a-document." In other words, novice teachers may view the textbook as *the curriculum* that they enact—not something that supports the curriculum. Due to this conflation between the technical curriculum and the enactment level of curriculum, ECTs too often rely on their understanding of mathematics curriculum at the societal level to support their enactment, rather than their new understandings from methods.

Enactment Level of Understanding Curriculum

As ECTs experience teaching they begin to understand curriculum at the level of enactment. At this level of understanding, teachers have more experiences and are more familiar with classroom variables. ECTs draw on these experiences and the resulting stories when deciding on their daily classroom curricula (or lesson plans). Ideally, by building on their understandings of dialogic mathematics from mathematic methods, ECTS understanding of curriculum will begin to expand. However, ECTs often face a tension between the technical curriculum and what works in classrooms, or at the enactment level of curriculum. This tension, can be viewed as recontextualization, or how teaching practices and ideas introduced in methods are enacted, or not, by ECTs (Ensor, 2001). In summary, the enactment level is the actual implementation (or teaching) of the curriculum, building on understandings that include their societal view, the technical curriculum, and understandings from teaching and mathematics methods.

Methods

In this self-study, I took my first five years of teaching as a case to offer a "deep and critical look at practices and structures of teacher education" (Zeichner, 1999, p. 11). I utilized a narrative approach (Carter, 1993) to identify both my curricular enactment and my understanding of curriculum levels. I open-coded (Strauss & Corbin, 1990) lessons plans, journal entries, and specific recalled experiences that comprised my narrative of learning-to-teach. Through this process I identified five specific focal experiences that impacted my curricular enactment and my understanding of curriculum levels.

Findings

The Starting Point—Societal View of Curriculum

Growing up particularly engaged in my classroom education I believed I understood curriculum. However, my view changed as I advanced in high school mathematics. We were learning material that was engaging on its own, yet I had no idea *why* we were learning it. I tried to trust the teacher, but while learning Conic Sections in Algebra II, I finally had to ask, "How do we use conic sections in the real-world?" In response, we (the class) were given a new project, to explore how Conic Sections are used in real life. As I was learning to teach, I regularly reflected on this early experience questioning how content, and curriculum were taught in schools. Specifically, I wondered why the real-world was often separate from the classroom.

Technical Curriculum and the Foundation for Creating Space

Several years later, I had earned a BS in Biology with a minor in environmental education, and I was an AmeriCorps member, responsible for running a Title 1 elementary school garden.

Through one school garden conference and a handful of education courses (for my minor and general interest), I knew I should connect the garden activities to standards. I thought this connection would facilitate classroom engagement. Having limited support in making connections I turned to the standards, and thus began my first real experience with the technical curriculum. I read (and re-read) the state standards, which formed the foundation for three garden activities I created. Each activity addressed several mathematics and literature state standards.

Through my narrative analysis, I also recognized that this year provided the foundation for developing my knowledge of students and their communities, or Community Funds of Knowledge (CFoK). I worked with families before and after school and I began to understand the wealth of knowledge, or CFoK, that *each* family had to offer. I found that connecting to and eliciting CFoK helped the garden flourish. During this year, I began to think of ways to connect this knowledge with the technical curriculum (a key component of my future curricular enactment). In this position, I experienced both the value of state standards and the importance of working with families and community.

Dialogic Curriculum Enactment in Three Steps

Phase 1: Technical curriculum is not enough. The following year, as I started my first teaching position, at a relatively new charter school, my developing understanding of curriculum was challenged. In preparing for the school, I studied the standards for middle school science and mapped these onto to the school calendar, ensuring I would "hit" each standard. During this process, my curricular understanding was limited to the technical level, despite having worked in the same neighborhood the year before, recognizing the value of CFoK. Yet, as my students filled the classroom, my view of curriculum complexified as I was met with the realities of being a new teacher in a school with few resources. Throughout this first year teaching my curricular focus included my students and my attempts of curricular enactment. I still recognized that standards had value, but I now understood that they were not sufficient to teach diverse middle school students. Rather there were *many* more details to attend to and I began to develop and implement approaches to connect to families and the community.

Phase 2: Engaging students in their education. The following year I taught at a well-established private school where my understanding of enacted curriculum changed again. This child-centered school was aligned with Dewey's (1902) emphasis on student experience as a critical component of learning. Therefore, with the support of a colleague and veteran math teacher, my students (and *not the standards*) became my key focus when developing lesson plans. My classes were much smaller, yet much more individualized and involved. Here I connected my understandings from my prior focal experiences (the importance of real-world mathematics, CFoK, and standards), and when I developed or enacted curriculum I drew on my knowledge of students, their engagement, and real-world mathematics. For example, through problem-solving lessons, I had students derive pi and identify real-life applications for quadratic functions. Overall, this teaching experience proved essential in my understanding of successful curriculum enactment, specifically my expanding view of what enactment could include—dialogic mathematics.

Phase 3: Creating space in the technical curriculum to engage students. When I began teaching at a large public high school, I was provided with structured mentoring, re-introduced to state standards and introduced to new technical aspects of the district. As such, my enactment level of curriculum began with the state standards and incorporated the district level expectations, but I also made space for student engagement, real-world applications, and student interest in their own education. For example, I began each year with the students writing a short essay entitled, "Why Mathematics?," in which students researched and explained why math was personally important. Building on my prior focal experiences, I now saw room in the technical curriculum to include enactment of curriculum that created a community of learners, supported each student in understanding *why* we were learning the content, focused on problem-solving, *and* sought to authentically addressed the standards.

Conclusion

This narrative approach to self-study offers insights into the complexities ECTs face when learning to teach, specifically understanding the varying levels of curriculum. For me, it took five years of teaching and key focal experiences to understand the complexities of curricular enactment and to recognize that, contrary to my pre-existing beliefs, there is *space* in the technical curriculum for a dialogic approach to teaching mathematics. While moving schools provided me with specific focal experiences that pushed my understanding of curricular levels, it is not a sustainable model for teacher education. Yet, my experience and narrative, highlights the challenges MTEs face (often in one-year or less) in supporting PSTs to move beyond a societal understanding of curriculum to one that includes enactment of research-based practices. This study suggests implementing explicit focal experiences (i.e., focus on child-centered learning) to support PSTs and ECTs in developing a comprehensive understanding of mathematics curriculum.

This and other self-studies, presented as case studies, may offer MTEs opportunities to begin the conversation around curriculum levels and identify additional focal experiences (Zeichner, 1999). Introducing case studies and focal experiences addressing curricular levels may support ECTs successful transition from a societal understanding of curriculum to an understanding that recognizes there is space in the technical curriculum for enacting dialogic mathematics. Future research examining the impact of case studies and curriculum level work in methods classes and during professional development would support the refinement of such an approach.

References

Ambrose, R. (2004). Initiating change in prospective elementary school teachers' orientations to mathematics teaching by building on beliefs. *Journal of Mathematics Teacher Education*, 7, 91-119.

Carter, K. (1993). The place of story in the study of teaching and teacher education. *Educational Researcher, 22*(1), 5-12.

Dewey, J. (1902). The Child and The Curriculum. Chicago, Illinois: The University of Chicago.

Doyle (1992). Curriculum and Pedagogy. In P. W. Jackson (ED.), *Handbook of research on curriculum* (pp. 486-516). New York: Macmillan

Ensor, P. (2001). From preservice mathematics teacher education to beginning teaching: A study in recontextualizing. *Journal for Research in Mathematics Education,* 296 – 320.

Feiman-Nemser, S. (2001). From preparation to practice: Designing a continuum to strengthen and sustain teaching. *Teachers College Record,* 103 (6), 1013-1055.

Ma, L. (1999) *Knowing and Teaching Elementary Mathematics.* Mahwah, NJ: Lawrence Erlbaum Associates.

NCTM (2000, 2010) Principles and standards for school mathematics. Reston, VA: NCTM.

Strauss, A. L., & Corbin, J. (1990). *Basics of qualitative research* (Vol. 15): Sage Newbury Park, CA.

Westbury, I. (2002). Toward an Understanding of the "Aims" of Music Education. The Handbook of research on music teaching and learning. 105-111.

Zeichner, K.M. (1999). The new scholarship in teacher education. *Educational Researcher, 9,* 4-15.

THIRD GRADE TEXTBOOKS' MODELS FOR MULTIPLICATION & DIVISION

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Multiplication and division are important topics that are formally introduced in third grade. Visual models and representations are key scaffolds for facilitating the teaching and learning of these topics, but there is little research on the prevalence of these models in classrooms. The present study examined six U.S. textbooks and found that three quarters of independent practice and homework tasks did not incorporate visual models. Additionally, textbooks vary significantly regarding which specific models are incorporated.

Keywords: Curriculum; Elementary School Education; Number Concepts and Operations.

Overview & Purpose

Multiplicative reasoning is an important topic that is typically introduced formally in third grade in U.S. schools. Analyses of textbooks suggest that many different visual representations are used in various textbooks in different nations (Davydov, 1991; Harries & Sutherland, 2000; Watanabe, 2003). The variety of representations available for multiplication and division is extensive, with Lay (1963) noting the benefits of set models, length models, area models, tree diagrams, and tables/figures. Although some research has focused on the affordances and constraints of particular models (e.g., Barmby et al., 2009; Huang & Witz, 2013), there has been relatively little research examining the prevalence of particular models in U.S. textbooks or their use by elementary teachers.

Various researchers have found that textbook content influences teachers' instructional decisions and content selection (Grouws et al., 2004; Porter, 2002). International comparisons of U.S. textbooks to those of other nations suggests that U.S. textbooks do not sufficiently promote the relationship between a visual representation and the symbolic notation associated, and there is less focus within such representations on the structure of multiplication (Harries & Sutherland, 2000; Watanabe, 2003). In his implementation of visual representations in a textbook, Davydov (1991) suggests that not only the use but the means of using representations is essential in promoting multiplicative meaning for children. Specifically, Davydov (1991) notes that while visual representations are important, they are not "equivalent to demonstrating the object conditions of multiplication…that create a basis for the subsequent transfer to operation with signs" (p. 30). Thus, the types of representations that are used in textbooks matter and may affect how teachers scaffold students' learning of multiplication and division. Although the prior literature suggests a comparison of children's engagement is both necessary and useful, the current study takes a different approach. In addition to the need for understanding which models (and the pedagogy associated with their use) promote multiplicative reasoning, there is a need to understand which models are currently promoted in U.S. textbooks. Thus, the purpose of the present study is to examine various U.S. third grade textbooks for the visual representations provided or solicited for students in tasks for independent work.

Sample & Methods

Six third grade textbooks were selected for analysis, based on their prevalence of use in area school districts. The textbooks include: enVision Math (2016); Eureka Math (2016); Everyday Mathematics (2016); GoMath! (2012); Investigations in Mathematics (2017); and My Math

(2018). Analysis focused on tasks most likely to be assigned to students for individual practice in class, or for individual homework. Additionally, tasks were analyzed only from sections of the book explicitly designated as focusing on multiplication or division by the publisher. For example, analysis of enVision Math included tasks within *independent practice* and *homework & practice* for sections the publisher aligned with multiplication and division mathematics standards. The current sample includes 4,937 tasks across all six textbooks.

Tasks were coded as either providing or soliciting specific visual representations. Although we did code for variations of representation, the present study focuses on eight specific nominal codes: *no visual representation*, *set model*, *length model*, *area model*, *number line*, *tree diagram*, *table/figure*, *other visual representation*. Each task was coded for types of provided and solicited representations it included. Provided representations are those illustrated within tasks whereas solicited representations required students to construct them in some manner. Both authors independently coded 100 tasks from one textbook. Interrater reliability was computed using the Kappa statistic and indicated near perfect agreement for provided representations (K=.96) and solicited representations (K=.89) between both authors (Landis & Koch, 1977). This supported the use of independent coding of remaining tasks across textbooks.

Analysis & Results

Figure 1 illustrates the distribution of provided and solicited models used to represent multiplication or division across all textbooks. Given the coding process, there are some tasks (n=31) that included both provided and solicited models. However, the descriptive statistics still provide a clear indication of how prevalent particular models are across textbook tasks designed to be individually assigned to students. Evident in Figure 1 is the relative dominance of tasks with no visual model for multiplication/division either provided or explicitly solicited. When such tasks did provide or solicit a visual model, the most common model across textbooks was the set model, followed by tables or figures, and then length models (excluding number lines).

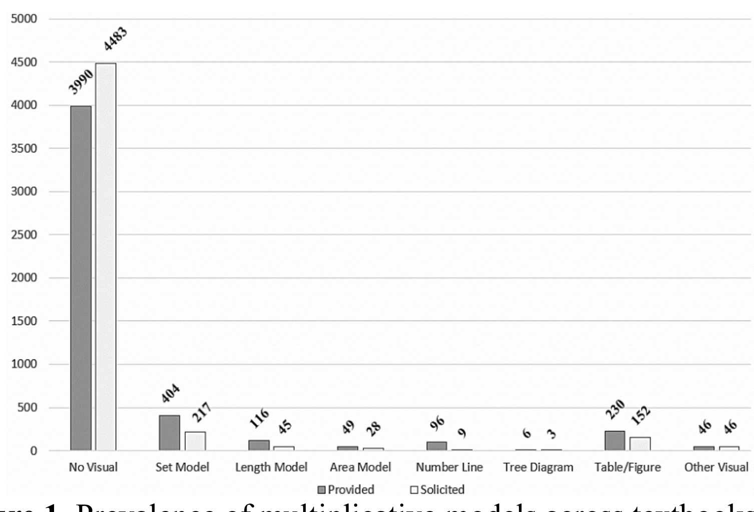

Figure 1. Prevalence of multiplicative models across textbooks.

Next, we sought to examine whether textbooks differed comparatively in regards to the visual models provided or solicited. The distribution of observed and expected counts for both provided (P) and solicited (S) visual models is presented in Table 1. A Chi-square statistic for provided models suggests that the observed distribution was independent from chance ($\chi^2(df=35)=554.78$, $p<.001$), and a similar result was found regarding solicited models

($\chi^2(df=35)=350.88$, $p<.001$). An examination of differences in both observed and expected counts in Table 1 suggests that while Eureka Math had fewer observed tasks with no visual model provided than was expected by chance, all other textbooks had more such tasks than were expected. However, it should be noted that 71.9% of Eureka Math tasks provided no visual representation. Other trends of note reveal that Eureka Math had comparatively higher frequencies of providing and soliciting set models, while enVision and Investigations had comparatively lower frequencies in both regards. Some textbooks differed in their consistency with either providing or soliciting the same model. For example, Everyday Math provided set models for only one task, but solicited a much higher frequency. Numerous other trends emerged across models, suggesting that, in general, certain textbooks emphasized the use of particular visual models over others, and these differences were statistically significant.

Table 1: Observed and Expected Counts for Visualizations of Multiplication and Division

	Set Model		Length Model		Area Model		Number Line		Tree Diagram		Table/ Figure		Other Visual		None	
	P	S	P	S	P	S	P	S	P	S	P	S	P	S	P	S
enVision	39	8	23	5	1	0	28	3	0	0	28	1	0	21	725	806
	69	*37*	*20*	*8*	*8*	*5*	*16*	*2*	*1*	*0*	*39*	*1*	*8*	*26*	*682*	*766*
Eureka Math	146	93	48	32	0	0	0	0	0	0	85	0	38	30	810	972
	92	*50*	*27*	*10*	*11*	*6*	*22*	*2*	*1*	*0*	*53*	*1*	*11*	*35*	*911*	*1023*
Everyday Math	1	36	0	0	10	6	0	0	0	0	6	0	4	8	167	138
	15	*8*	*4*	*2*	*2*	*1*	*4*	*0*	*0*	*0*	*9*	*0*	*2*	*6*	*152*	*171*
GoMath!	96	25	9	0	17	12	31	4	0	0	70	2	3	30	869	1022
	90	*48*	*26*	*10*	*11*	*6*	*21*	*2*	*1*	*0*	*51*	*1*	*10*	*34*	*885*	*994*
Investigations	13	2	36	8	21	10	0	2	0	0	11	0	0	19	419	459
	41	*22*	*12*	*5*	*5*	*3*	*10*	*1*	*1*	*0*	*23*	*0*	*5*	*15*	*404*	*454*
My Math	109	53	0	0	0	0	37	0	6	0	30	0	1	44	1000	1086
	97	*52*	*28*	*11*	*12*	*7*	*23*	*2*	*1*	*0*	*55*	*1*	*11*	*36*	*956*	*1074*

Note: Italicized numbers are expected by chance, and non-italicized numbers are observed. Provided models are designated in black text in column P, and solicited models are designated in gray text in column S.

Discussion

The present study describes the preliminary findings of an ongoing analysis of visual models in third grade textbooks. Findings suggest that current third grade textbooks' tasks for independent student work (both within and outside of class) engage students with few visual models of multiplication or division. Specifically, when accounting for both provided and solicited visual representations, approximately a quarter (27.7%) of analyzed tasks were designed to engage students with a visual model of multiplication and/or division. Additionally, there appeared to be little consistency across textbooks in regards to which visual models were more prevalent.

The results presented in this brief report reflect our preliminary analysis of models used in U.S. textbooks. Further analysis of the present data will focus on when representations are incorporated in each textbook, and with what aspects of multiplication and/or division. For example, a surface level analysis does suggest that textbooks differ in regards to the prevalence of visual representations in later chapters on multiplication/division, but analysis is currently ongoing.

There are several potential implications from the preliminary analysis provided here. One implication is that the relatively infrequent use of visual models appears at odds with research literature promoting specific use of such models (Sherin & Fuson, 2005). Another potential implication is that the inconsistency between textbooks on the use of particular models may promote confusion in professional discourse on such models for teachers (and students) who have used and are familiar with different textbooks. Although research suggests that textbooks do influence teachers' pedagogical decisions (Grouws et al., 2000; Porter, 2002), future research is needed to understand whether, and to what degree, the use of specific visual models in textbooks affects teachers' use of such representations in their lessons.

The present study focused on independent practice and homework assignment tasks, in regards to provided and solicited visual representations for multiplication and division. Results indicate that textbooks provide relatively few such tasks that incorporate visual models (27.7%). However, it is unknown how teachers use these tasks. It is possible that some teachers select primarily tasks with visual models, while other teachers may select mostly tasks without visual models. There are likely many factors that influence such task selection including pedagogical content knowledge, teaching philosophy, institutional obligations, etc. This preliminary analysis provides a report on only what visual models are present in tasks designed for independent work. Future work is needed to unpack how such distributions influence teachers' decision making, and affect students' mathematical learning.

References

Barmby, P., Harries, T., Higgins, S., & Suggate, J. (2009). The array representation and primary children's understanding and reasoning in multiplication. *Educational Studies in Mathematics, 70*(3), 217-241.

Davydov, V. V. (1991). A psychological analysis of the operation of multiplication. In V.V. Davydov (Ed.), *Psychological abilities of primary school children in learning mathematics* (pp. 9-85). Reston, VA: National Council of Teachers of Mathematics.

Grouws, D., Smith, M., & Sztajn, P. (2000). The preparation and teaching practices of United States mathematics teachers: Grades 4 and 8. In P. Kloosterman & F. Lester (Eds.), *Results and interpretations of the 1990 through 2000 mathematics assessments of the National Assessment of Educational Progress* (221-267). Reston, VA: NCTM.

Harries, T., & Sutherland, R. (2000). The representation of mathematical concepts in primary mathematics textbooks: a focus on multiplication. *Mathematics for living: The mathematics education into the 21st century project. Amman, Jordan*, 142-146.

Huang, H-M. E., & Witz, K. G. (2013). Children's conceptions of area measurement and their strategies for solving area measurement problems. Journal of Curriculum and Teaching, 2(1), 10-26.

Landis, J. R., & Koch, G. G. (1977). The measurement of observer agreement for categorical data. *Biometrics*, *33*(1), 159-174.

Lay, L. C. (1963). Times of the times. *The Arithmetic Teacher, 10*(6), 334-338.

Porter, A. C. (2002). Measuring the content of instruction: Uses in research and practice. *Educational Researcher*, *31*(7), 3-14.

Sherin, B., & Fuson, K. (2005). Multiplication strategies and the appropriation of computational resources. *Journal for Research in Mathematics Education*, *36*(4), 347-395.

Watanabe, T. (2003). Teaching multiplication: An analysis of elementary school mathematics teachers' manuals from Japan and the United States. *The Elementary School Journal, 104*(2), 111-125.

THE ROLE OF THE TWO-COLUMN PROOF IN THE GEOMETRY CLASSROOM

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Our research team surveyed members of the mathematics education community to gain insight into the community's perceptions of the two-column proof. We asked participants to describe the value of the two-column proof and discuss whether they would be in favor of eliminating it from the high school curriculum. There was a wide-range of diversity in the responses and we present several themes that we observed. We found that about 36% of the respondents were definitely or probably in favor of eliminating the two-column proof, 42% were definitely or probably not in favor of eliminating it, while about 22% were unsure.

Keywords: Reasoning and Proof, Geometry and Geometric and Spatial Thinking, Curriculum

Reform documents (e.g., Common Core State Standards Initiative, 2010; National Council of Teachers of Mathematics, 2000) have longed called for mathematical proof to be a part of K-16 students mathematical education. However, despite these efforts, proof still has a marginal role in schools (Stylianides, Stylianides, & Weber, 2017). Historically in the U.S., most students' experience with proof is limited to geometry; and a common way of engaging students with proof is through the two-column format (Herbst, 2002b). According to Stylianides et al. (2017),

> At least in the United States, proofs in geometry are sometimes written in a two-column format where a geometry statement appears in the left column and a reason for why that statement is logically permissible appears in the right column (Herbst, 2002b). Herbst (2002a) observed that this format places implicit (and sometimes conflicting) demands on teachers and students that constrain what is possible in a secondary school classroom. For instance, the two-column format implicitly requires giving students a statement (in the form of premises) and a conclusion, and testing students' abilities to reason logically from the statement to the conclusion. This practice discourages students from generating key ideas from a proof, constructing their own diagrams, or making conjectures by choosing their own premises and conclusions (Herbst & Brach, 2006). (p. 248)

Herbst (2002b) described how the two-column format developed as a way to meet a Committee of Ten mandate, introduced in 1893, that students learn to prove in geometry. The intention of this mandate, decreed by college professors, was to prepare students for college and create opportunities for students to develop mental discipline. The two-column proof emerged as a way to make it "possible for teachers to claim that they were teaching students how to prove and for students to demonstrate that their work involved proving" (Herbst, 2002b, p. 283). Conceptions of mathematics, and proof specifically, have shifted significantly since 1893, yet the ways in which proof is taught, especially in geometry, have not undergone much change.

How do members of the mathematics education community currently view the role of the two-column proof in high school geometry? Through an e-mail survey, we sought to gain insight into the perceptions held amongst members of the mathematics education community regarding the perceived value of the two-column proof and opinions regarding its place in the curriculum.

The Authors' Perspective on Proving as a Disciplinary Practice

Proving is an important aspect of mathematical practice (Hemmi, 2010), and students of all ages should have the opportunity to learn mathematics through proof (Stylianou, Blanton, & Knuth, 2010). Proof has a variety of meaningful functions and can be useful for engaging students in verification, explanation, discovery, and communication (Bleiler-Baxter & Pair, 2017; de Villiers, 1990). But the dominant two-column proof approach for teaching and practicing proof in high school geometry is counterproductive to students' productive engagement in mathematics. The two-column proof has evolved into a one-sided conceptualization of proof as a means of logical verification that deemphasizes the communicative aspects (e.g., negotiating the norms for acceptable evidence and explanations) of problem solving (De Villiers, 1998).

Herbst (2002b) argued that the two-column proof separates the practices of proving from the practices of knowing. We agree with Herbst (2002b) that "proving should be as natural [as other disciplinary practices such] as defining, modeling, representing, or problem solving" (p. 283). In order to reconceptualize proof as a disciplinary practice and to change the way we teach proof, we need to deemphasize the two-column proof because it is antiquated and counterproductive to the kind of thinking mathematics educators aim to foster in today's classrooms. Through our survey, we sought not only to determine if our views of the two-column proof were shared in the mathematics education community at large, but also to learn more about perceptions of the two-column proof that differ from our own.

Methodology

What are the perceptions of the two-column proof held by members of the mathematics education community? To answer this question, we designed a five-question survey which we sent to about 900 e-mail addresses gathered from public lists of attendees of mathematics education conferences. Each question received 152-172 responses. Question 1 asked the participants to describe whether they were a university professor, in-service teacher K-12 teacher, graduate student, or other. Question 2 was "Is there value in having students write two-column proofs?" Participants were asked to respond "yes" or "no." Question 3 requested an open response: "Why or why not?" Question 4 was "Do you support eliminating the two-column proof from the high school geometry curriculum?" The participants responded either definitely yes, probably yes, might or might not, probably no, or definitely no. Question 5 requested an open response: "Why or why not?" We present a summary of the quantitative results and a preliminary summary of qualitative themes that we see emerging from our data at this time.

Results

Quantitative Results

Participants included 133 university professors, 22 graduate students, 5 in-service K-12 teachers, and 12 participants who fell into the category "other." Regarding Q2, "Is there value in having students write two-column proofs?" we found 132/169 or 78% of participants responded "yes" and 37/169 or 22% of participants responded "no." In response to Q4, "Do you support eliminating the two-column proof from the high school geometry curriculum?", 28/171 or 16% responded definitely not, 44/171 or 26% responded probably not, 37/171 or 22% responded might or might not, 34/171 or 20% responded probably yes, and 28/171 or 16% responded definitely yes. Note that about 36% were probably or definitely in favor of eliminating the two-column proof, 22% might or might not, and 42% probably or definitely were not in favor of eliminating the two-column proof.

Qualitative Results

We now present fives themes that are emerging from our data during the preliminary stages of our analysis of the open responses to questions 3 and 5.

Theme 1: Proof is needed in the curriculum. Many participants expressed worry that if the two-column proof was eliminated, then proof would disappear entirely from the K-12 curriculum. Such participants wrote that they see value in students' reasoning, argumentation, and justification, and if the two-column proof was eliminated, they would like to see it replaced by another form of proof.

Theme 2: The two-column proof should be regarded as one, but not the only way to write proofs. Participants expressed a desire for students to be exposed to multiple formats, perhaps with two-column proof as a scaffold to other formats. Furthermore, students should choose the type of format with which they are most comfortable.

Theme 3: The two-column proof format helps students think logically and organize their thinking. Many participants described the value of the two-column proof as helping students to think logically or organize their thinking. Some participants wrote that the two-column proof format might help students understand the way mathematical theorems are logically dependent upon prior results.

Theme 4: The two-column proof format does not meet our goals for students' engagement with mathematics. Many participants wrote about how the two-column proof format runs contrary to other goals we have for our students (e.g., view proof as communication). Some participants noted that mathematicians do not write two-column proofs, and such proofs are inauthentic and contrary to the goal of having students think like mathematicians. Others claimed that the two-column proof emphasizes form rather than reasoning. Some participants claimed that high school students are not ready for the formal reasoning required by the two-column proof and would be better served by other justification and reasoning activities.

Theme 5: We would need professional development and detailed planning if we were to eliminate the two-column proof from the high school curriculum. Many participants noted that if there was a decision made to eliminate the two-column proof from the curriculum, then a lot of planning and professional development would be needed. Other participants had a desire to become more familiar with the research literature before making a decision about keeping the two-column proof in the curriculum.

Discussion

Our analysis reveals that members of the field of mathematics education have diverse opinions about the place of the two-column proof in the geometry curriculum. While most participants stated that the two-column proof does have pedagogical value, many of those participants' comments were reserved, noting that the two-column proof may do more harm than good. However, participants were also hesitant to say that they were in favor of eliminating the two-column proof from the high school geometry curriculum, citing concerns that this may lead to the total removal of proof from the curriculum.

Beyond expressing the need for more alternative types of proof formats in the schools, some participants expressed serious concerns regarding the negative aspects of the two-column proof. We do not believe it is wise to ignore these concerns. Herbst (2002b) noted,

> The two-column proving custom was an accomplishment of geometry instruction in the sense that it helped comply with a mandate. But that accomplishment did not come for free. It

> brought to the fore the logical aspects of a proof at the expense of the substantive role of proof in knowledge construction. (p. 307)

The mandate that students learn to prove in geometry still exists today in the Common Core (2010). The requirement that students prove in geometry is part of the state standards for all fifty states, including those not using the Common Core standards. The two-column format developed as a way to meet this mandate, and it is questionable whether the two-column format's dominant place in the curriculum could be diminished unless we remove entirely the standard that students learn to prove in geometry. Perhaps there are other creative options for making proof more meaningful for students in school. Moving forward, we believe that the field could benefit from research that explores students' understandings, proficiency in constructing, and comprehension of a variety of types of proof formats. The research community needs to be informed about the benefits and constraints of having students write proofs both in the two-column format and other formats. Additionally, we agree with many of our participants that teachers need professional development with proof if it is to gain more than a marginal place in K-12 mathematics classrooms.

References

Bleiler-Baxter, S. K., & Pair, J. D. (2017). Engaging students in roles of proof. *The Journal of Mathematical Behavior, 47,* 16-34.

Common Core State Standards Initiative. (2010). *Common core state standards for mathematics.* Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

De Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras*, *24*, 17-24.

De Villiers, M. (1998). An alternative approach to proof in dynamic geometry. *Designing learning environments for developing understanding of geometry and space*, 369-393.

Herbst, P.G. (2002a). Engaging students in proving: A double bind on the teacher. *Journal for Research in Mathematics Education,* 33(3), 176-203.

Herbst, P. G. (2002b). Establishing a custom of proving in American school geometry: Evolution of the two-column proof in the early twentieth century. *Educational Studies in Mathematics*, *49*(3), 283-312.

Herbst, P. G., & Brach, C. (2006). Proving and doing proofs in high school geometry classes: What is it that is going on for students? *Cognition and Instruction,* 24(1), 73-122.

Hemmi, K. (2010). Three styles characterising mathematicians' pedagogical perspectives on proof. *Educational studies in mathematics*, *75*(3), 271-291.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics.* Reston, VA: Author

Stylianides, Stylianides, & Weber. (2017). Research on the teaching and learning of proof: Taking stock and moving forward. In J. Cai (Ed.), *Compendium for Research in Mathematics Education.* Reston, VA: NCTM.

Stylianou, D. A., Blanton, M. L., & Knuth, E. J. (Eds.). (2010). *Teaching and learning proof across the grades: A K-16 perspective*. Routledge.

THE DIFFERENCES IN HOW CALCULUS TEXTBOOKS INTRODUCE THE CONCEPT OF DERIVATIVE COULD IMPACT CONCEPTUAL UNDERSTANDING

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Keywords: Curriculum Analysis, High School Education

Textbooks remain a common source of content, pacing, and instruction for mathematics educators. With this in mind, it is critical to examine how the textbooks that educators use as a primary resource introduce essential concepts. The purpose of this study is to explore how calculus textbooks introduce and display the definition of derivative, along with the specific examples and definitions they provide.

The following textbooks were selected for analysis from the College Board's suggested AP Calculus textbooks list: Anton & Biven (2015); Finney et al. (1999); Foerster (2005); Larson & Edwards (2018); and Stewart (2016); see poster for references. Each textbook was analyzed for the first instance of the definition of derivative, then recorded along with the context, and the examples immediately following the definition. Analysis suggests three distinct methods of introducing the concept and definition of derivative. Anton and Biven's Calculus textbook (2015) is denoted as an AP Edition, introducing the definition of derivative as follows: "The function f' defined by the formula f'(x) = is called the ***derivative of f with respect to x***" (p. 90). The example immediately following the definition uses the function $f(x) = x^2$ and asks the reader to find the derivative at x = 2. In this particular case f(2) = f'(2), which could be misleading as a primary example for students to record in their notes. The structure of the Foerster (2005) textbook lends pedagogically to a more problem-based approach to teaching calculus. The definition provided in the text box of section 3.4 states, "Derivative as a function (Δx or h form) f'(x) = " (p. 86). This textbook emphasizes Δx, possibly because of the focus on application problems. The current version of James Stewart's calculus textbook, (2016), begins with the definition of derivative at a point, also stated in a text box, says "The **derivative of a function f at a number a**, denoted by f'(a), is f'(a) = if this limit exists" (p. 151).

The choice of Foerster, (2005), and Larson and Edwards, (2018) to use Δx instead of h has several implications. While this is a robust definition and should be included when teaching the concept of derivatives, it may be intimidating and discouraging for students who do not come to calculus with a strong sense of context in math and/or science coursework to be exposed to as a primary definition (Sofronas et al., 2011). The pedagogical implications in this analysis range from introducing a critical concept in a variety of ways, to making the conscious decision not to begin with an example where f(2) = f'(2). While the latter seems like a minor detail, not attending to a pedagogical decision such as this can have an impact on students' conceptual understanding of the concept of derivative.

References

Sofronas, K. S., DeFranco, T. C., Vinsonhaler, C., Gorgievski, N., Schroeder, L., & Hamelin, C. (2011). What does it mean for a student to understand the first-year calculus? Perspectives of 24 experts. *The Journal of Mathematical Behavior*, *30*(2), 131-148.

USING TWITTER FROM AN ORGANIZATIONAL SENSEMAKING PERSPECTIVE TO SUPPORT STATEWIDE IMPLEMENTATION EFFORTS

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Keywords: Policy Matters, Standards, Technology

Local implementation of changes in state educational policy initiatives has proven to be a broad challenge for the field. One challenge to these efforts are the mixed messages and multiple interpretations that arise as policy change efforts are enacted locally and the infrastructuring work necessary to overcome these challenges (Spillane, et al., 2002). In other system-wide change efforts, social media tools have been used to mitigate this challenge (Terantino, 2012), however, there is a gap in the literature in mathematics education on the ways in which social media tools can be used to address these challenges.

After the recent adoption of new state mathematics standards, a partnership between our state education agency and researchers from four universities in our state was formed to assist in implementing new state mathematics standards. As a part of these efforts, we use organizational sensemaking (Weick, 1995) to guide the use of social media tools aimed at promoting common messages and interpretations of the new state mathematics standards and the ways in which these standards can be embodied in reform-based instruction with students. Theoretically, organizational sensemaking is the process that individuals within organizations go through when they encounter moments of uncertainty and ambiguity as they make sense of change (Weick, 1995). During sensemaking, researchers highlight three cyclic processes: *cues* that trigger sensemaking; *intersubjective meaning making* among members of organizations, and the *action* needed to continue to promote organizational sensemaking (Maitlis & Christianson, 2014).

For this proposed poster, we share how we have leveraged Twitter to support the statewide implementation of new mathematics standards. Using organizational sensemaking, we share an emerging framework for using social media to promote systemic coherence during implementation that goes beyond dissemination and communication to promote individual and collective learning. This poster will provide examples of how the framework guided our use of Twitter to provide opportunities to engage in the sensemaking processes.

References

Maitlis, S., and Christianson, M. "Sensemaking in organizations: Taking stock and moving forward." *The Academy of Management Annals* 8.1 (2014): 57-125.

Spillane, J. P., Reiser, B. J., & Reimer, T. (2002). Policy implementation and cognition: Reframing and refocusing implementation research. *Review of Educational Research*, 72(3), 387-431.

Terantino, J. M. (2012). An activity theoretical approach to examining virtual professional development and informal learning via social networks. *Virtual professional development and informal learning via social networks*, 60-74.

Weick, K. E. (1995). *Sensemaking in Organizations* (Vol. 3). Sage.

PERIODIC CONFUSION: EXAMINING TWO CONTEMPORARY TEXTBOOK UNITS ON TRIGONOMETRIC FUNCTIONS

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The purpose of this study was to examine two textbook units on trigonometric functions to understand what opportunities to learn they provide to students. Despite its reputed content difficulty, trigonometry is under-researched in the field. Obstacles to understanding trigonometry may stem from a lack of opportunities to develop deep conceptual knowledge, which the Common Core State Standards for Mathematics (CCSS-M) were designed to address. Two features associated with tasks that give students such opportunities are grappling with mathematics and explicit connections to concepts (Hiebert & Grouws, 2007). These features define high cognitive demand tasks, as operationalized by Stein and Lane (1996). Thus, to fully understand how each text gave students opportunities to learn trigonometry meant examining both the content and the intended cognitive demand of tasks. My research questions were:

1. What specific mathematics content do students have opportunities to learn by studying trigonometric functions using each of the two selected texts?
2. What are the intended levels of cognitive demand of tasks in the two selected texts?

The textbooks selected for this study were Pearson's *Algebra 2: Common Core* and the second (2011) edition of Key Curriculum Press's *Interactive Mathematics Program: Year 3*. I conjectured there would be important differences between the two texts; for example, IMP, an NSF-funded "reform" textbook, might have a higher percentage of problems at high levels of cognitive demand than Pearson, a "traditional" textbook. Over 950 student tasks were coded for trigonometry concepts implicated, using author-developed codes from a collapsing coding methodology, and level of cognitive demand, using the framework of Stein and Lane (1996).

Analysis revealed Pearson's text covered all trigonometry topics required by the CCSS-M, but it emphasized procedural fluency, with relatively few opportunities for students to engage with high cognitive demand tasks to build conceptual understanding. IMP, in contrast, provided ample opportunities for students to first develop conceptual understandings by using high cognitive demand tasks, followed by opportunities to build procedural fluency. However, IMP omitted some trigonometry content crucial to prepare students for Calculus from the CCSS-M. As a result, both texts may have shortcomings for students if enacted as intended. This work invites discussion of ways curricula can balance development of deep conceptual understandings of trigonometric functions with procedural fluency skills given limits on instructional time.

References

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (371-404). Reston, VA: NCTM.

Stein, M. K., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. *Educational Research and Evaluation, 2*(1), 50-80.

HOW TWO PROJECT-BASED MATHEMATICS TEXTBOOKS POSITION STUDENTS TOWARDS MATHEMATICS

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Keywords: textbook analysis, positioning theory, cognitive demand, voice of textbook

Mathematics is often seen as neutral and free of bias and subjectivity because it is comprised of numbers and calculations. However, the way that we use words and language to teach and communicate about mathematics sends a powerful message to students about who they are and what they can do in relation to mathematics (Langer-Osuna, 2017). Therefore, we need to attend more carefully to the language we use in mathematics curriculum materials.

Inspired by prior analyses of tasks and curriculum materials, I created a framework to analyze two problem-based textbooks: Core Plus (CP) 1 and the Mathematics Visions Project (MVP), Year 1. Herbel-Eisenmann (2007) analyzed the inclusive and exclusive imperatives used in textbooks and found that textbooks either position (Harre & van Langenhove, 1999) students as *thinkers* (asked to draw a conclusion or make a connection) or *scribblers* (asked to simply make a calculation). Additionally, I drew upon Stein and Lane's (1996) work with cognitive demand of mathematical tasks because the task itself sends messages to students about whether they are expected to engage as *high thinkers, low thinkers, high scribblers,* or *low scribblers*.

On in-class tasks, students were positioned as *high thinkers* in CP 50% of the time while students were positioned as *high thinkers* in MVP only 18% of the time. Students in the MVP curriculum were positioned most frequently as *low thinkers* on in-class tasks, 74% of the time. For CP, the percentage of times students were positioned as *high scribblers* increased from 13% (on the in-class tasks) to 36% (on homework). For MVP, the number of problems with high cognitive demand decreased (from the in-class tasks to homework) while the number of problems with low cognitive demand increased which resulted in students being positioned as *scribblers* 88% of the time on the homework! The CP materials collectively (in-class tasks and homework) almost equally position students as *high thinkers*, *low thinkers*, and *high scribblers*. The MVP materials collectively position students as *high scribblers* the majority of the time.

Because these curriculum materials both claim to align with the NCTM standards and the CCSS-M, I expected to find more similarities than differences. However, I found that the CP tends to position students more as *thinkers* than MVP does and that the CP homework problems tend to be more cognitively demanding than the MVP homework. I also found that there may be more opportunities for students to move out of the scribbler position in the CP materials.

References

Harre, R., & van Langenhove, L. (Eds.). (1999). *Positioning Theory: Moral Contexts and Intentional Action.* Malden: Blackwell Publishers, Inc.

Herbel-Eisenmann, B. (2007). From intended curriculum to written curriculum: Examining the "voice" of a mathematics textbook. *Journal for Research in Mathematics Education*, *38*(4), 344–369. https://doi.org/10.2307/30034878

Langer-Osuna, J. M. (2017). Authority, Identity, and Collaborative Mathematics. *Journal for Research in Mathematics Education*, *48*(3), 237–247.

Stein, M. K., & Lane, S. (1996). Instructional Tasks and the Development of Student Capacity to Think and Reason: An Analysis of the Relationship between Teaching and Learning in a Reform Mathematics Project. *Educational Research and Evaluation*, *2*(1), 50–80. https://doi.org/10.1080/1380361960020103

EXAMINING CRITICAL CONVERSATIONS DURING CO-DESIGN OF INSTRUCTIONAL GUIDANCE DOCUMENTS

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Keywords: Pacing Guide, Research-Practice Partnership, Design Based Research, Standards

In many districts, pacing guides strongly filter the sequencing, planning, and timing of instruction. Furthermore, they often vary from district to district and are created by local individuals or teams who may or may not have particular expertise in mathematics education. Research suggests that while pacing guides can support teachers in learning of curriculum (Bauml, 2015), pacing guides and the pressure to comply with them can also undermine the professionalization of teaching, prevent teachers from meeting the individual needs of their students, and decrease teachers' use of cognitively demanding tasks for the sake of "covering" the content in the pacing guide (Bauml, 2015; David, 2008). We report on a research-practice partnership (Penuel, Allen, Coburn, & Farrell, 2015) in which multiple partners across our state collaborated to mediate these challenges by designing pacing guides, which we renamed as instructional frameworks, to support districts and teachers in their implementation of new statewide mathematics standards adopted in 2017.

In this poster session, we provide an overview of activity theory (Engeström, Miettinen, & Punamäki, 1999) as our theoretical perspective and design-based implementation research (Fisherman, Penuel, Allen, Cheng, & Sabelli, 2013) as our approach to share how a co-design team of 70 diverse stakeholders (state agency consultants, administrators, district curriculum directors, teachers, and higher education faculty) brought different lenses to inform decision-making about the clustering and sequencing of content standards and the duration needed to teach them. We report on the tensions in critical conversations about clustering and sequencing of standards during the initial phases of our design process, highlighting the ways in which actors drew on previous experiences, assessment concerns, curriculum materials, and mathematical learning progressions/trajectories to inform decision-making during the development of the instructional frameworks.

References

Bauml, M. (2015). Beginning primary teachers' experiences with curriculum guides and pacing calendars for math and science instruction. *Journal of Research in Childhood Education*, *29*, 390–409.

David, J. L. (2008) What research says about pacing guides. *Educational Leadership, 66*(2), 87-88.

Engeström, Y., Miettinen, R., & Punamäki, R. L. (1999). Perspectives on activity theory. Cambridge University Press.

Fishman, B. J., Penuel, W. R., Allen, A. R., Cheng, B. H., & Sabelli, N. O. R. A. (2013). Design-based implementation research: An emerging model for transforming the relationship of research and practice. *National society for the study of education*, *112*(2), 136-156.

Penuel, W. R., Allen, A. R., Coburn, C. E., & Farrell, C. (2015). Conceptualizing research–practice partnerships as joint work at boundaries. *Journal of Education for Students Placed at Risk (JESPAR)*, *20*(1-2), 182-19

INFLUENCE OF INTEGRATED STEM CURRICULA ON INSTRUCTION

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Recent research has shown the importance of including STEM education programs in elementary schools (Dringenberg, Wiener, Groh, & Purzer, 2012). Teachers may integrate the STEM disciplines at varying degrees (Nathan, Tran, Atwood, Prevost, & Phelps, 2010), focusing more on one discipline than another. Given more and more teachers are being asked to integrate STEM into their instruction, we examined two fourth grade teachers, Ms. Khanna and Ms. Devries, implementation of DESCARTES, an integrated STEM curricula. We analyzed how DESCARTES influenced the teachers' conceptions of teaching and their instructional practices.

In the DESCARTES curriculum, students construct their own knowledge by engaging in an authentic context (designing and building boats) to solve a real-world problem (shipping cargo efficiently across a body of water). The students are presented with a challenge of designing a boat that could carry the most load across a body of water. The students engage in multiple hands-on activities to examine and test buoyancy, pressure, cross-sections, water displacement, speed, and volume. The students test similar factors in the DESCARTES software environment, which is a gamified design, simulation, prototyping, and collaborative environment.

Both teachers participated in a 45–minute initial interview prior to implementing DESCARTES. Field notes were recorded each day of the 4½ week unit. At the end of the unit, the teachers participated in a 45–minute final interview. Data were analyzed using an inductive approach.

Although the teachers were "veteran" teachers, they experienced some trepidation prior to implementing the DESCARTES curriculum. Ms. Khanna expressed, "I was unsure first [about the DESCARTES curriculum]. I was like, oh gosh! What did I get myself into? You know, thinking this is going to be hard. And it was." (Final Interview, 2016). Both Ms. Khanna and Ms. DeVries became aware how their STEM instruction showed different outcomes compared to their siloed-discipline instruction– learning gaps amongst students were almost non-existent and the teachers observed students' "willingness to not give up. They don't ever feel like they are trying to achieve the right answer. They are just constantly making it better" (Final Interview, 2016). After implementing the DESCARTES curriculum, the teachers noticed the positive effects it had on their instruction, student learning, and student engagement.

References

Dringenberg, E., Wiener, J., Groh, J., & Purzer, S. (2012). Measuring the impact of engineering outreach on middle school students' perceptions. *Proceedings of the American Society for Engineering Education (ASEE) IL-IN Regional Conference.*

Nathan, M. J., Tran, N. A., Atwood, A. K., Prevost, A., & Phelps, L. A. (2010). Beliefs and expectations about engineering preparation exhibited by high school STEM teachers. *Journal of Engineering Education*, *99*(4), 409-426.

Stohlmann, M., Moore, T. J., McClelland, J., & Roehrig, G. H. (2011). Impressions of a middle grades STEM integration program: Educators share lessons learned from the implementation of a middle grades STEM curriculum model. *Middle School Journal*, *43*(1), 32-40.

PROJECT VERSUS PROBLEM BASED LEARNING: IS THERE A DIFFERENCE?

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Keywords: Curriculum Analysis

Project and problem based learning are commonly used interchangeably. There are, however, important differences between the two. The Buck Institute for Education (2017) defined project based learning as a method focused on students learning by working for a period of time on an investigation in response to a problem or task. While problem based learning is a student-centered approach where learning is done through group work to explore open-ended problems (Center for Teaching Innovation at Cornell University). It appears problem based learning is driven by the problem, while project based is driven by the project, but both have the overarching goal of letting students lead the investigation.

The purpose of this study was to conduct a literature review to compare how researchers have used project and problem based learning to enhance mathematical learning over the last two years (2015 to 2017). The articles reviewed provided insight into contemporary ways researchers have enacted problem and project based learning. The goal of the literature review was to provide insight on how the different conceptualizations of project and problem based learning may influence the choices made by the researchers.

To collect the most recent articles about project and problem based learning, I conducted an EBSCO and ERIC search within scholarly peer-reviewed journal and keywords problem based learning in the title and mathematics and education within the entire paper. We repeated these searches in EBSCO and ERIC for project based learning. Initial search results found 65 articles total. Out of the 65 articles (37 problem based learning and 28 project based learning), only 36 matched the criteria of being empirical studies in English language peer-reviewed scholarly journals. Twenty articles were classified as focused on problem based learning and 16 articles were classified as focused on project based learning.

Conducting a literature review of both project and problem based learning has afforded the opportunity to research how the two methods are enacted in classrooms. Initial findings demonstrate a lack of consistency in both defining problem and project based learning and how they are used. Preliminary results also yield limited empirical studies within the last two years of project based and problem based learning in the discipline of mathematics. Mathematics was included or part of the project but the studies did not focus on mathematical concepts. Studies are also limited within the United States. The results across both studies show project and problem learning do enhance the instruction of mathematics.

References

Buck Institute for Education. (2017). Why Project Based Learning (PBL)? Retrieved from https://www.bie.org
Center for Teaching Innovation at Cornell University. (2018). Problem Based Learning. Retrieved from https://www.cte.cornell.edu/teaching-ideas/engaging-students/problem-based-learning.html

AN INTERNATIONAL COMPARISON OF MKT AND EDUCATIVE CURRICULUM MATERIALS FOUND IN TEACHERS' MANUALS

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Keywords: Curriculum, Curriculum Analysis, Mathematical Knowledge for Teaching

This study compares teachers' manuals from the U.S. and Japan to determine the extent and the ways in which mathematical knowledge for teaching (MKT) is developed in teacher manuals from each country. Ball, Thames, & Phelps (2008) discuss the importance of MKT because, in part, it plays a critical role in planning and enacting the intended curriculum (p. 399).

In addition to providing opportunities to develop MKT, there are other ways in which curriculum materials can support students' learning. According to Davis and Krajcik (2005), curriculum materials could be educative if they provide teachers with opportunities to learn *why* developing pedagogical content knowledge (PCK) is important. Therefore, this study is not only an analysis of the comparison of the textbooks that are educative, and the level at which MKT is developed. It is also an analysis between Japan and the US textbooks, which can provide insights into the differences in the materials provided to the teachers in each country. This study was guided by the following research questions:

1. In what ways do teachers' manuals provide opportunities for Japanese and US teachers to develop their mathematical knowledge for teaching (MKT) about triangle congruency statements? What are the similarities and differences?
2. In what ways do the teachers' manuals in Japan and the US provide opportunities for teachers to learn about the mathematics in the lesson and students' thinking about the mathematics? What are the similarities and differences?

For the purposes of this exploratory study, to draw a fair comparison between the teachers' manuals in Japan, *Core-Plus Mathematics* (Hirsch, 2008), a reform-based curriculum, was selected to represent the U.S. manual. The U.S. and Japanese curriculum materials had similar numbers of instances to develop MKT, although the Japanese curriculum instances were spread more widely among the different categories of MKT. However, there were more differences than similarities in the other opportunities teachers had to learn from these materials. The Japanese curriculum devoted more space and time to helping the teachers understand the *why* behind certain decisions, which could help teachers develop a deeper understanding of the lesson and also can provide insight into possible student thinking. In contrast, the US textbook focused mostly on areas where the students may struggle. Therefore, the curriculum materials were similar in providing opportunities to develop MKT, but they varied otherwise in the level at which they were educative for teachers.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content Knowledge for Teaching. *Journal of Teacher Education*, *59*(5), 389–407

Davis, E. A, & Krajcik, J. S. (2005). Designing Educative Curriculum Materials to Promote Teacher Learning. *Educational Researcher*, *34*(3), 3–14.

Hirsch, C. R. (2008). *Core-plus Mathematics: Contemporary Mathematics in Context. Course 2*. Glencoe/McGraw-Hill.

DYNAMIC GEOMETRY SOFTWARE (DGS) TASKS IN SECONDARY CURRICULA

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As Dynamic Geometry Software (DGS) becomes more ubiquitous in secondary mathematics classrooms, it is important that the development of tasks in mathematics curricula mirrors research findings concerning its use. Textbooks have a strong influence on teachers' selection of the content that they include and the instructional strategies that they utilize (e.g. Banilower et al., 2013). While research shows that the strategic use of technology has the potential enhance students' mathematical thinking and understanding (e.g., Hollebrands & Dove, 2011), the question remains as to whether the use of DGS as promoted in current secondary curricula leverages that potential. Thus, we pose the following research questions: (a) to what extent is DGS integrated into current secondary curricula, (b) how is DGS used in secondary curricula?

A nationally representative sample of 20 textbooks was chosen, the unit of analysis for coding each textbook being a mathematical task. Tasks were coded for whether or not they used technology, and what type (e.g., DGS, calculator). To answer the second research question, tasks were coded as using technology as amplifier or reorganizer (Pea, 1985), and whether the use of DGS involved one of three types of reasoning: exploration, justification, or verification (Oner, 2009). Technology is used as an *amplifier* if it is *only* used to perform computations with increased speed or precision, and is considered to be used as a *reorganizer* if the purpose of using technology is to support a shift in students' focus or thinking. *Exploration* involves observing relationships or patterns, *justification* involves providing reasoning for a conjecture, and *verification* involves testing a result; these codes are not mutually exclusive as a single task could ask students to do more than one.

A total of 1316 of 10,100 tasks were identified as making use technology, with 110 of those using DGS. Results indicate that 39 tasks used DGS as an amplifier, while 71 used DGS as a reorganizer. With regard to DGS use, 90 tasks included exploration, 22 included justification, and 48 included verification. Results will highlight associations (and implications) between these distinctions and whether the curriculum was (a) investigate vs. conventional, and (b) integrated vs. subject-specific.

References

Banilower, E. R., Weis, A. M., Campbell, K. M., Malzahn, K. A., Weiss, I. R., & Smith, P. S. (2013). *Report of the 2012 National Survey of Science and Mathematics Education*. Horizon Research.

Hollebrands, K. F. and Dove, A. (2011). Technology as a tool for creating and reasoning about geometry tasks. In T. P. Dick & K. F. Hollebrands (Eds.), *Focus in high school mathematics: Technology to support reasoning and sense making* (pp. 33-52). NCTM: Reston, VA.

Oner, D. (2009). The Role of Dynamic Geometry Software in High School Geometry Curricula: An Analysis of Proof Tasks. *International Journal for Technology in Mathematics Education, 16*(3).

Pea, R. D. (1985). Beyond Amplification: Using the computer to Reorganize Mental Functioning. *Educational Psychologist, 20*(4), 167-182.

CO-CONSTRUCTING STATEWIDE CURRICULUM FRAMEWORKS: PURPOSE AND PROCESSES

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Keywords: Curriculum Guides, Design Experiments, Statewide Collaboration

Addressing problems of practice and implementation at scale is an issue that continues to challenge education researchers (Cobb & Jackson, 2011). As part of a statewide research-practice partnership, we are currently using a design implementation research (Fishman et al., 2013) approach to organize the collaborative design of a multifaceted intervention for new state mathematics standards and the promotion of more equitable classroom learning opportunities for teachers. We convened multiple education partners in an ongoing project to create a statewide curriculum framework to support the implementation of new state mathematics standards.

Considerable research suggests that pacing guides and the pressure to comply with them can undermine the professionalization of teaching (Bauml, 2015). Novice teachers fear reprimand from administrators if they do not strictly adhere to the sequence and timing of the guides.

Teachers may sacrifice cognitively demanding, real world tasks to "cover" the content in the guide (David & Greene, 2007). Some schools with pre-packaged instructional programs require teachers to teach the same lessons on the same day (Achinstein & Ogawa, 2006). Despite the negative consequences, Kauffman, Johnson, Kardos, Liu, & Peske (2002) found that curriculum guides can be a significant resource for new teachers who are still building their professional knowledge. Our research-practice partnership convened K-5 and 6-8 Framework Design Teams to create two statewide collaboratively-designed Instructional Frameworks that could be adapted by all districts to implement the new state mathematics standards.

In this poster presentation, we describe our purposes for creating K-5 and 6-8 statewide instructional frameworks (i.e., pacing guide and instructional resources) and how findings from research shaped our work in the Design Teams. We then describe the processes we used to ensure that all stakeholders were invited to design (e.g., teachers, administrators, higher educators, etc.), and enough resources were included so that all districts would be able to adapt the frameworks to their specific community. We also present preliminary findings from "framework rollout" meetings across the State that illuminate the ways in which instructional leaders anticipate challenges to plan professional development in their personal districts.

References

Achinstein, B., & Ogawa, R. T. (2006). (In)fidelity: What the resistance of new teachers reveals about professional principles and prescriptive educational policies. *Harvard Educational Review*, *76*(1), 30–63.

Bauml, M. (2015). Beginning primary teachers' experiences with curriculum guides and pacing calendars for math and science instruction. *Journal of Research in Childhood Education*, *29*, 390–409.

Cobb, P., & Jackson, K. (2011). Towards an empirically grounded theory of action for improving the quality of mathematics teaching at scale. *Mathematics Teacher Education and Development*, *13*(1), 6-33.

David, J. L., & Greene, D. (2007). *Improving mathematics instruction in Los Angeles high schools: An evaluation of the PRISMA pilot program.* Palo Alto, CA: Bay Area Research Group.

Fishman, B. J., Penuel, W. R., Allen, A. R., Cheng, B. H., & Sabelli, N. (2013). Design-based implementation research. *National Society for the Study of Education*, *112*(2), 136-15.

TRENDS IN CROSS-NATIONAL COMPARATIVE EDUCATIONAL RESEARCH

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Keywords: Curriculum

The increase of worldwide social relations promoted by a global economy links the local to the distant in unprecedented ways. This interconnectedness plays a unique role in education highlighting common problems for education by providing opportunities for generalization and creating a system of lending and borrowing of educational ideas (Arnove, 2013). Contrasting different systems of education provides opportunities to illuminate factors that would otherwise go unnoticed (Cai, 2002). This research space has the potential to provide insights into successes and failures of educational policies to promote productive change. Mathematics education provides a unique space for cross-national comparisons. The purpose of this study is to uncover what is considered valued research in the realms of cross-national comparative educational research. This research was guided by the following research question: what are the trends in cross-national research in mathematics education top-tier journals in the last ten years?

Empirical research articles from five of the top peer reviewed journals ranked by Williams and Leathams (2017) were searched for studies focused on international comparisons: *Journal of Research in Mathematics Education, Educational Studies in Mathematics, Journal of Mathematical Behavior, Mathematical Thinking and Learning,* and *Journal of Mathematics Teacher Education*. To collect the articles, I did a keyword search for each journal using "cross-cultural comparison," "international comparison," and "cross-national comparison". I also set a range for publication between 2007 and 2017. Initially 39 articles matched the search criteria. After closer examination, 26 articles matched the criteria of empirical international comparisons. Examining the studies matching the criteria provided a picture of the current landscape in cross-national comparative education in mathematics education in the last 10 years. A spreadsheet was used to organize and analyze the data extracted from each article including mathematical/pedagogical topic discussed, databases or test used (e.g. PISA), and countries compared.

Initial findings indicate an under representation of articles that include a South American or African country in their study. While comparisons between the United States and China accounted for 30% of cross-national comparisons in top tier mathematics education journals. Additionally, seven of the 26 articles were textbook comparisons and fractions were the most common mathematical topic discussed.

References

Arnove, R. F., & Torres, C. A. (2007). *Comparative education: The dialectic of the global and the local. third edition*. Lanham, Maryland: Rowman & Littlefield Publishers, Inc.

Cai, J., & Hwang, S. (2002). Generalized and generative thinking in US and Chinese students' mathematical problem solving and problem posing. *The Journal of Mathematical Behavior, 21*(4), 401–421. doi:10.1016/S0732-3123(02)00142-6

Williams, S. R., & Leatham, K. R. (2017). Journal quality in mathematics education. *Journal for Research in Mathematics Education, 48*(4), 369-396. doi:10.5951/jresematheduc.48.4.0369

Chapter 3

Early Algebra, Algebra, and Number Concepts

Research Reports

Brief Research Reports

Posters

GRADE 5 STUDENTS' NEGATIVE INTEGER MULTIPLICATION STRATEGIES

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Twenty-four Grade 5 students participated in clinical interviews where they solved integer multiplication number sentences. Drawing on the theoretical perspective of strategies that students use with whole number multiplication and integer addition and subtraction, we describe the strategies that students employ when negative integers are incorporated with multiplication. The students, although drawing on similar strategies for whole number multiplication (e.g., repeated addition, direct modeling), used these strategies differently (e.g., using Unifix cubes to represent -1). The students also used unconventional strategies for solving integer multiplication, such as analogies and invented procedures. The results highlight the important constructions of students prior to formal instruction on integer multiplication, where prior research has been mainly situated in thinking about integer addition and subtraction.

Keywords: Number Concepts and Operations, Elementary School Education, Cognition

Investigations of strategies that students invent, and even struggle with, for integer multiplication number sentences, will provide teachers and researchers with insight into students' thinking about integers. With this understanding, we can begin to develop instructional strategies that support building on students' thinking about integer multiplication, a neglected topic in our field. In order to improve instructional approaches, we must first investigate students' constructions and reasoning.

Children invent sophisticated and robust ways of reasoning about integers and integer addition and subtraction (e.g., Bofferding, 2014; Bishop et al., 2014). As children approach addition and subtraction of integers for the first time, they use different strategies (Bofferding, 2010), ways of reasoning (e.g., Bishop et al., 2014; Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2016), and conceptualizations (e.g., Aqazade, Bofferding, & Farmer, 2017; Bofferding & Wessman-Enzinger, 2017; Wessman-Enzinger, 2015). Although there has been an increased focus on children's reasoning about integers (e.g., Aquazade et al., 2017; Bofferding, Aqazade, & Farmer, 2017; Bishop et al., 2016), investigations into integer multiplication remain overlooked.

The goal of this research report is to present an inaugural framework of strategies students created as they engaged with integer multiplication number sentences for the first time. Our research question focuses on students' invented strategies for integer multiplication number sentences (e.g., $-2 \times 3 = \square$): What strategies do Grade 5 students use as they solve integer multiplication number sentences?

Theoretical Perspective

Because children often build on their whole number knowledge and extend this to integer reasoning (Bofferding, 2014), looking towards strategies that children employ with whole number multiplication may provide insight into how children may begin to reason about integer multiplication. Multiplication and division problems are often approached by children through a variety of invented strategies, such as repeated addition or direct modeling with grouping collections of countable objects (Carpenter, Fennema, Franke, Levi, & Empson, 2015).

Carpenter et al. (2015) and Baek (1998) demonstrate that children are able to understand multiplication when they can invent their own strategies. Some of the strategies for single-digit (Carpenter et al., 2015) and multi-digit (Baek, 1998) multiplication with whole number include: direct modeling strategies, counting strategies, repeated addition, and derived fact strategies. The extent to which students will use similar strategies with negative integer multiplication is an open question.

With *direct modeling*, students model groups using manipulatives (e.g., Unifix cubes) or drawings. When students use *counting* strategies they may skip count accounting for groups, sometimes using fingers or choral counting. Students draw on *repeated addition* or *doubling* (e.g., $4 \times 3 = 3 + 3 + 3 + 3$). *Derived facts* strategies include drawing on factual knowledge and creating a new algorithm based on previously known facts (e.g., 2×3 may be solved by know that $2 \times 2 = 4$ and then 2 more added to that product is 6).

From the integer addition and subtraction literature, we know that students use a variety of strategies different from the CGI frameworks. These include using *computations* or *procedures* (Bishop et al., 2014), drawing on *recalled facts* (Bofferding & Wessman-Enzinger, in press), and making *comparisons* or *analogies* (Bishop et al., 2016; Bofferding, 2011; Wessman-Enzinger, 2017; Whitacre et al., 2017).

As we began our study, we drew on both single-digit and double-digit strategies for multiplication with whole numbers and strategies for integer addition and subtraction. We thought these strategies would provide insight into the ways that students may solve multiplication problems involving negative integers.

Methods: Participants, Interviews, and Analysis

We conducted clinical interviews (Clement, 2000) with 24 Grade 5 students from the rural Pacific Northwest. We selected Grade 5 students that did not have formal school experiences with integers; *Common Core State Standards* recommendations include integer operations in Grade 7 (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). We interviewed each student once, using the following integer multiplication number sentences (see Figure 1). Students solved the integer multiplication number sentences, with each number sentence provided on a singular piece of paper. Manipulatives and tools provided during this interview included: Unifix cubes, two-colored chips, empty number lines, and markers. We asked prompting questions throughout the interviews, without giving the students the answer or additional information. These types of questions included: “How did you come up with that?”; “Can you explain your thinking?”

$$3 \times 5 = \square$$

$$-2 \times 3 = \square$$

$$3 \times -4 = \square$$

$$-4 \times -2 = \square$$

Figure 1. Integer multiplication number sentences provided to students.

We videotaped and transcribed each interview. Our unit of data included the video clip, drawings, and transcripts associated with each integer multiplication number sentence. We began coding with the framework delineated in the analytical framework (Baek, 1998; Carpenter et al., 2015). For instance, we looked for the use of manipulatives and drawings for direct modeling strategies. We looked for choral counting, skip counting, and use of fingers for counting strategies. Using the definitions established for these various strategies, we employed constant comparative methods (Merriam, 1998). We modified the strategies to include the ways that students used negative integers, not previously captured with only positive integer multiplication strategies or integer addition and subtraction strategies. We met to compare codes and negotiated any disagreements. In the results section, we highlight the integer multiplication strategies that the students in our study used.

Results: Strategies for Integer Multiplication

We highlight the descriptions of the strategies, rather than focusing on correctness or incorrectness. Because the students have powerful strategies paired with some correctness, this is provides a space to understand children's thinking as a vehicle for leveraging discourse in the classroom in the future.

Direct Modeling

Example of direct modeling. Edie solved -2 × 3 = □, using a direct modeling strategy, that resulted in a solution of -6 (see Figure 2). Edie assigned the value of -1 to each Unifix cube. She constructed three groups of two blocks (see white blocks in Figure 2). Because she attributed the value of -1 to each of the white Unifix cubes, she modeled -2 × 3 = □, instead of 2 × 3 = □.

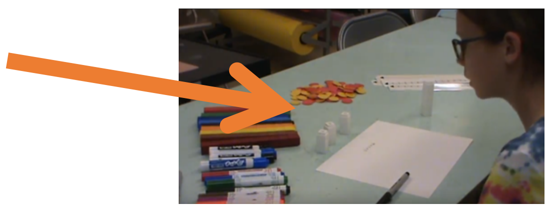

Figure 2. Example of direct modeling strategy for -2 × 3 = □.

The following transcript excerpt illustrates how Edie shared her strategy:

> (Reaches for Unifix cubes) I'm going to pretend this is negative… okay this is negative 2 (pulls off 2 white Unifix cubes) negative plus a negative would be a negative… so if these are negatives then that would 3 times the 2 negatives which would equal 6 negative (writes "-6" on paper).

Description of direct modeling. Students used a direct modeling strategy when they illustrated integer multiplication with physical tools (e.g., Unifix cubes, two-colored chips, pictures)—modeling (number of groups) × (number of things in each group) = total. The students who used direct modeling strategies determined the solutions to integer multiplication through physically manipulating and modeling with these objects.

The students used two-colored chips (one yellow side, one red side) to be a physical representation of the difference between a negative number and a positive number. Notably, the students flexibly used the colors. Sometimes, red chips represented negative integers and yellow chips represented positive integers; other times, red chips represented positive integers and yellow negative integers.

Using Unifix cubes, the students used the cubes to model multiplication as groups of the same amount of quantities. The students who used the cubes mapped values of -1 to each of the cubes. The cubes represented a way to account for groups of negative quantities and provided a physical way to add the groups together in order to determine their solutions.

Use of direct modeling. The students used direct modeling strategies seven times for 3 × 5 = ☐. For the number sentences with negative integers, the students used direct modeling strategies five times for -2 × 3 = ☐ and four times for 3 × -4 = ☐. Direct modeling was used only once for -4 × -2 = ☐, which is not surprising given the physical limitations of negative amounts of groups.

Repeated Addition and Subtraction

Example of repeated addition. Eliza solved 3 × -4 = ◻ using repeated addition (see Figure 3) and obtained the solution, -12. Eliza demonstrated repeated addition as she repeatedly added -3 four times in order to get her product of -12. Notably, she added -3 four times, instead of adding -4 three times; her strategy actually aligns to 4 × -3 = ◻ instead of 3 × -4 = ◻. Essentially, Eliza implicitly recognized the equality of 4 × -3 and 3 × -4, without commenting on it. In Figure 3, the black writing illustrates her final computed product. However, the red writing illustrates her repeated addition, which she wrote first.

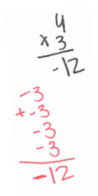

Figure 3. Example of repeated addition strategy.

Description of repeated addition and subtraction. Repeated addition, as a strategy, describes multiplication with adding positive integers repeatedly (Baek, 1998; Carpenter et al., 2015). The students in our study drew on repeated addition with negative integers. However, they also used *repeated subtraction* of positive integers.

Use of repeated addition and subtraction. The students used repeated addition strategies ten times for 3 × 5 = ☐. For the number sentences with negative integers, the students used repeated addition and subtraction four times for -2 × 3 = ☐ and three times for 3 × -4 = ☐. A student used repeated addition and subtraction only once for -4 × -2 = ☐, which is also not surprising given the challenges of adding -4 “negative two” times.

Recalled Fact

Example of recalled fact. Zoe first solved -2 × 3 = ◻ and obtained -6 as a recalled fact, even though it was her first time engaging with integer multiplication. She quickly stated the answer, -6, before the interviewers even completely finished reading the multiplication number sentence, -2 × 3 = ◻. Zoe relied on her factual knowledge of the product of 2 × 3 = 6, when questioned. With probing she justified her solution with a procedure “you just do 2 times 3 and then you make it a negative,” which will be discussed later.

Description of recalled fact. Within the CGI strategy framework, students often draw on facts to make derived facts (e.g., Carpenter et al., 2015). In our study with integer multiplication, students did not seem to use derived facts, but did use their factual knowledge about whole number multiplication quickly for integer multiplication without verbal explanation. Students used recalled facts when they stated their solutions to integer multiplication as a fact, likely memorized from whole numbers. Or, they drew on their memory so much that it did not require any form of deliberation. Students stated their solution quickly with an often “just is” explanation.

Use of recalled fact. The students used recalled fact four times for -2 × 3 = □, six times for 3 × -4 = □, and five times for -4 × -2 = □. The students demonstrated confidence with single digit whole number multiplication (e.g., fifteen stated the answer of 3 × 5 = □ as a recalled fact).

Procedure

Example of procedure. Lia solved 3 × -2 = □ with a solution of 4, using a procedure as a strategy. In this example, Lia solved the integer multiplication sentence by using a "negative integer as a singular subtrahend" procedure (see Figure 4). She first computed 3 × 2 by solving 3 + 3. Then, Lia incorporated the singular integer in the number sentence, -2, by subtracting 2 from the product of 3 × 2. This procedure is one of various types used in this study by the students.

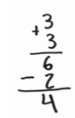

Figure 4. Example of procedure strategy.

Description of procedure. When students used an algorithm or created an invented procedure to find the solution they used the procedure strategy. Although this represents an addition to existing CGI framework for multiplication strategies (e.g., Baek, 1998), many integer researchers have stated that students use computational reasoning (Bishop et al., 2016) or procedures (Wessman-Enzinger, 2015; Bofferding & Wessman-Enzinger, in press) as they solve integer addition and subtraction problems. Thus, it seems to be a natural extension that students would also use computational and procedural strategies with integer multiplication.

The students in this study used different types of procedures (e.g., appending a negative sign to the solution, negative numbers as equivalent to zero, exclusive negativity). Describing the extensive use of procedures is beyond the realm of this research report. But, Zoe used the "appending a negative sign" procedure in her justification of derived fact strategy -2 × 3 = -6 when she stated that the negative sign is just "added on." Other students said that number sentences, such as -4 × -2 = □, needed to be "all negative," concluding that -4 × -2 = -8 based on a procedure of "exclusive negativity."

Use of procedure. Students used or invented various procedures for dealing with integer multiplication throughout the study (e.g., eleven times for -2 × 3 = ☐, sixteen times for 3 × -4 = ☐, and fifteen times for -4 × -2 = ☐). The students did not use a procedure for 3 × 5 = ☐ and used procedures only for multiplication number sentences with negative integers (e.g., -2 × 3 = ☐).

Counting

Example of counting. Cittie used counting on a number line to solve -2 × 3 = □, obtaining a solution of 7. Figure 5 illustrates Cittie's number line. She reasoned that she could start at -2 and counted in sequential order on the number line, moving right, to her destination, 7; she skip counted by 3, three times. Although this does not represent a correct solution, Cittie ordered the negative and positive numbers correctly and started her counting at -2, which represents beginning, ordered integer reasoning necessary for integer multiplication.

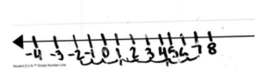

Figure 5. Example of counting strategy.

Description of counting. Students, sometimes with the help of a number line, counted in sequential order when solving multiplication number sentences. Students often described their strategies as "skip counting." The students sometimes drew on the number line, as a tool for facilitating their skip counting. In addition to number lines, students also used fingers and even cubes as number paths (Bofferding, 2010). Challenges with negative integer multiplication for students included deciding which direction to count in, as directions are not fixed like they are with whole number numbers, and determining the quantities of the skip counts (see, e.g., Cittie counting amounts of 3 three times).

Use of counting. The students used counting strategies three times for $3 \times 5 = \square$, four times for $-2 \times 3 = \square$, three times for $3 \times -4 = \square$, and eight times for $-4 \times -2 = \square$. When the students used counting strategies with integer multiplication, they sometimes did so with a number line. It is likely that students used counting strategies the most for $-4 \times -2 = \square$ because of challenges in physically representing this number sentence with strategies like direct modeling.

Analogy

Example of analogy. Jaxon first solved $-2 \times -4 = \square$, with analogy, determining that $-2 \times -4 = -8$. The following transcript highlights Jaxon's reasoning:

> Well, because it wouldn't really make as much sense for a negative multiplied by a negative to equal a positive. It's like, um, I'm not sure how to … it just wouldn't make as much sense. Because if a positive multiplied by a positive would equal a positive, then I would assume that it would be the same for a negative. And, it would be a negative times a negative would equal a negative.

In this excerpt, Jaxon compared $-2 \times -4 = \square$ to $2 \times 4 = \square$. Reasonably, he concluded that because a positive number times a positive number is positive (e.g., $2 \times 4 = 8$), then a negative number times a negative number is another negative number (e.g., $-2 \times -4 = -8$). Again, like the previous example where we highlighted a strategy with an incorrect solution, there is still powerful reasoning embedded in Jaxon's strategy. Jaxon connected his reasoning about whole numbers in a logical way (albeit not a culturally/mathematically correct way).

Description of analogy. Students used analogy when they connected previous knowledge about whole numbers to integers and compared it to a whole number multiplication number sentence for constructing or justifying new claims when solving the integer multiplication number sentences. Although this is an addition to the CGI framework for multiplication strategies, there is evidence that students use analogies with integer addition and subtraction (Bishop et al., 2016; Bofferding, 2011; Wessman-Enzinger, 2017; Whitacre et al., 2017). We distinguish this from recalled facts or procedures in that the students made explicit comparisons, with reasoning focused on these comparisons.

Use of analogy. The students did not use an analogy strategy for $3 \times 5 = \square$, $-2 \times 3 = \square$, or $3 \times -4 = \square$. But, students used analogy twice for $-4 \times -2 = \square$. Although not used often in this study, students use analogies frequently with integer addition and subtraction (e.g., Whitacre et al., 2017). We conjecture that if we gave more "negative number multiplied by negative number" number sentences we would have seen analogy strategies employed more—analogies seem like potentially productive strategies for these types of integer multiplication number sentences.

Counter Movement

Example of counter movement. Warren solved $-2 \times -4 = \square$, determining a solution of 8. The following transcript excerpt highlights Warren's solution of 8.

> Um. So. I… this subtraction, this negative symbol. Scratch out this one and this one … counters this one too. So it takes both these out and then it's just 4 times 2. … I just thought that since they're both negative numbers and that one's whole, it's basically, kinda like dividing except you're multiplying. And… you just kinda just… I just thought that you counter them. … It's kind of like dividing because instead of making this uh… -8, you just make it normal 8, which means that the number … if these numbers were whole numbers … well no since their negative numbers… if like cause they're not equal to ... they're not equal to 0 they're this way (left of 0) and the whole numbers are this way (right of 0) it's kind of like since these numbers were divided …these ones were dividing they would get smaller or go the opposite way. Like if these ones they would go this way.

At first, Warren's strategy sounds procedural because he talks about "scratching" out symbols, referencing the negative symbols in front of -2 and -4. Then, Warren references a "countering" of movements in the negative and positive direction (e.g., -1 × -1 = 1 or -1 $\boxed{v?}$ -1 = 1), making -8 "a normal 8." He discusses how we can treat this multiplication problem as "whole numbers" since multiplying the negative integers "counter" the directions of each other. Multiplying by -1 moves a number "this way (left of 0)" and multiplying it by -1 changes the direction.

Description of counter movement. Students use the counter movement strategy when they employ continuous movement or motion that "counters" each other. Use of this strategy includes a reference to changing directions, where the movement is countered or balanced. Consider this equation: -2 × -4 = (-1 × 2) × (-1 × 4) = (-1 × -1) × (2 × 4). Multiplying by negative one refers to a movement or translation in one direction and multiplying by the other negative one is a movement or translation in the other direction—consequently *countering* the overall movement.

Use of counter movement. Of the twenty-four students we interviewed, only one student constructed this strategy. Although this may not warrant the creation of a new category, the strategy uniquely helped Warren construct a correct solution to -2 × -4 = □, a notoriously challenging problem type. Honoring the student's use of the word "counter" and the use of continuous movement for constructing meaning, we called this strategy "counter movement."

Discussion and Final Remarks

The results of this work are significant in that we provide an inaugural framework for integer multiplication strategies that students use prior to school instruction, modified from CGI multiplication strategies frameworks and integer addition and subtraction literature. Previous research has focused on thinking and strategies of integer addition and subtraction (e.g., Bofferding, 2010; Bishop et al., 2014) and we extend the scholarly discussion on students' thinking about integers by describing their invented strategies for integer multiplication.

Our focus is on the powerful ways that students, prior to formal school instruction, solved integer multiplication number sentences, whether correct or incorrect. If we wish to support student inventions and discourse in the mathematics classroom, we must first understand their sophisticated reasoning (Carpenter et al., 2015). Jaxon, for example, obtained -2 × -4 = -8. Although we know this to be an incorrect solution, it is rooted in a logical analogy (e.g., if 2 × 4 = 8, then -2 × -4 = -8). As teachers and researchers, how do we promote conceptual change when students invent strategies that are logical, but not mathematically correct? We might consider pairing number sentences like 2 × -4 = □, where students had success in correct answers, with number sentences like -2 × -4 = □, where students had more difficulty, to leverage growth or change in the students' sophisticated reasoning (Bofferding et al., 2017).

Empowering students in the classroom requires building on their thinking. Consequently, we must first learn about the ways students enter the mathematics classrooms and the invented strategies they construct; then, we can draw on their reasoning to build future instructional interventions.

Acknowledgments

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References

Aqazade, M., Bofferding, L., & Farmer, S. (2017). Learning integer addition: Is latter better? In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th annual meeting of the North America Chapter of the International Group for the Psychology of Mathematics Education* (pp. 219– 226). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Baek, J. (1998). Children's invented algorithms for multidigit multiplication problems. In L. Morrow & M. Kenney (Eds.), *Teaching and Learning of Algorithms in School Mathematics* (pp. 151–160). Reston, VA: NCTM.

Bofferding, L. (2010). Addition and subtraction with negatives: Acknowledging the multiple meanings of the minus sign. In P. Brosnan, D. Erchick, & L. Flevares (Eds.), *Proceedings of the 32nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 703–710). Columbus, OH.

Bofferding, L. (2011). -5 – -5 is like 5 – 5: Analogical reasoning with integers. Paper presented at *American Educational Research Association*. Vancouver, BC: AERA.

Bofferding, L. (2014). Negative integer understanding: Characterizing first graders' mental models. *Journal for Research in Mathematics Education, 45*(2), 194–245. doi:10.5951/jresematheduc.45.2.0194

Bofferding, L., Aqazade, M., & Farmer, S. (2017). Second graders' integer addition understanding: Leveraging contrasting cases. In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th annual meeting of the North America Chapter of the International Group for the Psychology of Mathematics Education* (pp. 243– 250). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators

Bofferding, L., & Wessman-Enzinger, N. M. (in press). Nuances of prospective teachers' interpretations of integer word problems. In L. Bofferding & N. M. Wessman-Enzinger (Eds.), *Exploring the Integer Addition and Subtraction Landscape: Perspectives on Integer Thinking*. Springer.

Bishop, J. P., Lamb, L. L., Philipp, R. A., Whitacre, I., Schappelle, B. P., & Lewis, M. L. (2014). Obstacles and affordances for integer reasoning: An analysis of children's thinking and the history of mathematics. *Journal for Research in Mathematics Education, 45*(1), 19–61. doi:10.5951/jresematheduc.45.1.0019

Bishop, J. P., Lamb, L. L., Philipp, R. A., Whitacre, I., & Schappelle, B. P. (2016). Leveraging structure: Logical necessity in context of integer arithmetic. *Mathematical Thinking and Learning, 18*(3), 209–232. Carpenter, Fennema, Franke, Levi, & Empson, (2015).

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly, & R. Lesh (Eds.), *Handbook of Research Data Design in Mathematics and Science Education* (pp. 547–589). Mahwah, NJ: Lawrence Erlbaum Associates.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Washington, DC: Author. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Merriam, S. B. (1998). *Qualitative research and case study applications in education*. San Francisco, CA: Jossy-Bass.

Wessman-Enzinger, N. M. (2015). *Developing and describing the use and learning of conceptual models for integer addition and subtraction of grade 5 students*. Normal, IL: Proquest.

Wessman-Enzinger, N. M. (2017). Whole number and integer analogies. In E. Galindo & J. Newton (Eds.), Proceedings of the 39th annual meeting of the North America Chapter of the International Group for the Psychology of Mathematics Education (pp. 319– 322). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Whitacre, I., Azuz, B., Lamb, L. L. C., Bishop, J. P., Schappelle, B. P., & Philipp, R. A. (2017). Integer comparisons across the grades: Students' justifications and ways of reasoning. *Journal of Mathematical Behavior, 45,* 47–62.

SCALING CONTINUOUS COVARIATION: SUPPORTING MIDDLE SCHOOL STUDENTS' ALGEBRAIC REASONING

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Middle school is a critical time when students begin formal study of functional relationships in algebra. However, many students struggle in understanding functions as relationships between quantities that change according to a dependency relationship. We report on the influence of Scaling Continuous Covariation in fostering productive ideas about graphical relationships and rates of change. Scaling Continuous Covariation entails the ability to imagine a re-scaling to any increment for x and coordinate that scaling with associated values for y. We present findings from two students, one who reasoned with Scaling Continuous Covariation and one who did not, and report on how Scaling Continuous Covariation supported students' reasoning in three ways: (a) sense making about graphs, (b) forming constant rates of change, and (c) understanding constantly-changing rates of change.

Keywords: Algebra and Algebraic Thinking, Cognition, Middle School Education

Introduction: Supporting Function Understanding in Middle School

Functions and relations comprise a critical aspect of secondary mathematics, with recommendations for supporting students' algebraic reasoning emphasizing an early introduction to functional relationships in late elementary and middle school (Stephens, Ellis, Blanton, & Brizuela, 2017). Middle school in particular represents a key time when students enter a formal investigation of function and begin to develop the algebraic tools to express and represent different functional relationships. However, students' difficulty in acquiring the function concept is well documented (e.g., Stephens et al., 2017; Thompson & Carlson, 2017). In particular, students struggle to use functions to model real-world contexts that require a conceptualization of quantities and how they change together (Carlson et al., 2002; Monk & Nemirovsky, 1994).

One potentially fruitful approach to better support students' understanding of function and rates of change is an instructional emphasis on variational and covariational reasoning (Carlson, Smith, & Peterson, 2003; Kaput, 1994; Thompson & Carlson, 2017). Early research suggests that providing students with opportunities to reason covariationally can position them to make meaningful sense of functions (e.g., Ellis, 2007, 2011; Johnson, 2012; Moore, 2014), as well as the ideas in calculus (Thompson & Carlson, 2017). We report on a study investigating the reasoning of two middle school students who explored linear and quadratic growth within the context of continuously co-varying quantities. We found that a particular form of covariational reasoning, scaling continuous covariation, supported a robust understanding of graphical relationships and rates of change.

Background and Theoretical Framework

One contribution to the challenges in building and supporting a robust understanding of function is the lack of attention to variation in algebra curricula. Thompson and Carlson (2017) reviewed seventeen U.S. secondary textbooks, ranging from Algebra I through Precalculus, and found that all relied on a correspondence definition of function. Under this definition, y is a function of x if each value of x has a unique value of y associated with it (Farenga & Ness, 2005). This static view underlies much of school mathematics, with the associated set theoretic meaning of variable becoming the foundation for school definitions of function (Cooney & Wilson, 1996). Students' function concepts are consequently dominated by static images of arithmetic computations used to evaluate outcomes at individual values (Carlson & Moore, 2015). This results in students viewing functions through the lens of symbolic manipulations rather than as a mapping (Carlson, 1998).

Integrating the mathematics of change into students' investigation of functional relationships provides opportunities to interpret how a function's output values can change in relation to its input values, which is an essential component of making sense of dynamic situations (Carlson & Moore, 2015). These opportunities are typically reserved for introductory calculus courses, thus effectively restricting access to these forms of reasoning to the minority of students who will reach the highest level of high-school mathematics (Roschelle, Kaput, & Stroup, 2000). Thinking about functions covariationally, however, can support students' abilities to make sense of linear (Ellis, 2007; Johnson, 2012), quadratic (Ellis, 2011), exponential (Ellis et al., 2015), and trigonometric (Moore, 2014) functions.

Covariational Reasoning

Researchers have addressed covariational reasoning in a number of ways, but for the purposes of this paper we draw on work that considers the possible imagistic foundations that can support students' abilities to think covariationally (e.g., Castillo-Garsow, Johnson, & Moore, 2013; Thompson & Carlson, 2017). These researchers describe covariational thinking as the act of holding in mind a sustained image of two quantities' values varying simultaneously. One can imagine how one quantity's value changes while imagining changes in the other. A person thinking covariationally can couple two quantities in order to form a multiplicative object (Thompson & Carlson, 2017); once such an object is formed, one can then track either quantity's value with the immediate understanding that the other quantity also has a value at every moment.

Castillo-Garsow (2013) distinguished between two types of continuous variation, chunky and smooth. Chunky continuous variation entails thinking about values varying discretely, except that one has a tacit image of a continuum between successive values. One imagines that a change in values occurs in completed chunks, without imagining that variation occurs within the chunk. In contrast, smooth continuous variation entails an image of a quantity changing in the present tense; one can map from one's own experiential time to a time period within the mathematical context, thinking about a value varying as its magnitude increases in bits while simultaneously anticipating smooth variation within each bit (Thompson & Carlson, 2017). Building on these distinctions, Thompson and Carlson (2017) created a covariational reasoning framework that attends to students' images of quantities' values varying. They stressed that smooth continuous variational and covariational reasoning necessarily involves thinking about motion.

Ely and Ellis (in press) subsequently introduced a related but distinct form of reasoning they call *scaling continuous covariation*, which entails imagining that at any scale, the continuum is still a continuum and a variable takes on all values on the continuum. One can conceive of the continuum as arbitrarily or even infinitely "zoomable", in which the process of zooming will never reveal any holes or atoms. Thus, one can imagine a re-scale to any arbitrarily small increment for x and coordinate that scaling with associated values for y. Importantly, unlike smooth continuous covariation, this way of thinking does not fundamentally rely on an image of motion.

Just like chunky continuous covariation, scaling continuous covariation relies on partitioning a domain and then reasoning about the corresponding chunks of the covarying quantity. A chunky reasoner can re-chunk the domain to a different sized chunk, but then must re-run his or her chunking scheme from scratch. A person using scaling continuous reasoning, on the other hand, can abstract the chunking process and imagine the result of this chunking scheme for chunks at all scales. This enables one to generalize properties of the covarying quantity's chunks at all of these scales as well, enabling him or her to coordinate the covariation of the two quantities on increments of any scale. With this in mind, we can distinguish between students' use of chunky continuous and scaling continuous covariation and study how these reasoning types support their understanding of rates of change.

Methods

We conducted a 10-day, 15-hour videotaped teaching experiment (Steffe & Thompson, 2000) with two 7th-grade students in general general mathematics (neither had yet taken algebra). The first author was the teacher-researcher. We assigned gender-preserving pseudonyms to each student. The aim of the teaching experiment was to support the students' emerging understanding of linear, piece-wise linear, quadratic, and higher-order polynomial functions from a rate-of-change perspective.

Building on the literature emphasizing the importance of continuously-covarying quantities, we developed tasks to support a conception of linear growth as a representation of a constant rate of change, and quadratic growth as a representation of a constantly-changing rate of change. The tasks emphasized these ideas within the contexts of speed and area. The area tasks presented "growing rectangles", "growing stair steps", and "growing triangles" via dynamic geometry software, in which the students could manipulate the figure by extending the length and observing the associated growth in area (Figure 1).

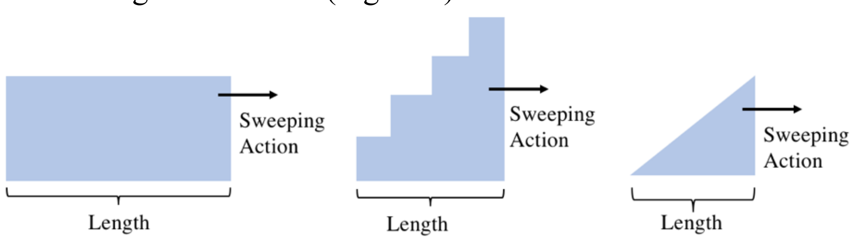


Figure 1. Growing rectangle, stair step, and triangle tasks.

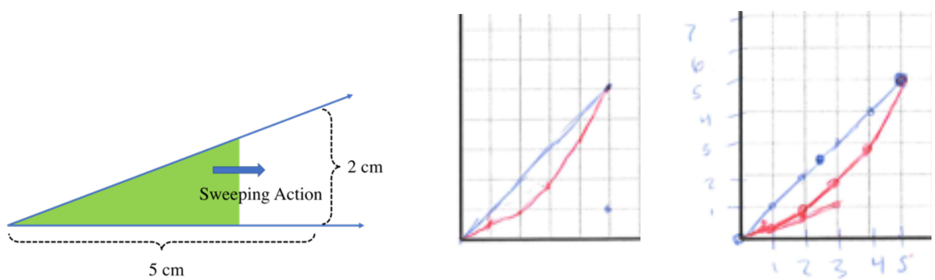
Relying on the videos and transcripts of each teaching session and copies of the students' work, we used the constant-comparative method (Strauss & Corbin, 1990) to analyze the teaching-experiment data in order to identify (a) students' forms of covariational reasoning, and (b) the students' conceptions of constant and changing rates of change. For the first round of analysis we drew on Thompson and Carlson's (2017) framework of variational and covariational reasoning. We used open coding to infer categories reasoning based on students' talk, drawings and graphs, gestures, and task responses. The first round led to an initial set of codes, which then guided subsequent rounds of analysis in which the project team met regularly to refine and adjust the codes in relation to one another. This iterative process continued until no new codes emerged. The final round of analysis was descriptive and supported the development of an emergent set of relationships between students' covariational reasoning and their conceptions of constant, changing, and instantaneous rates of change.

Results

One student, Wesley, exhibited evidence of scaling continuous covariation, while the other student, Olivia, exhibited evidence of chunky continuous covariation. We report three ways in which the form of covariational reasoning influenced students' sense-making about functional relationships: (a) reasoning about graphs, (b) constant rates of change, and (c) constantly changing rates of change. We address each in turn.

Reasoning about Graphs

Wesley and Olivia interpreted and discussed their graphs of quadratic phenomena differently. Wesley conceived of his graphs as smooth continuous curves that, for any given increment, regardless of its size, did not reduce to a straight line within the increment. In contrast, Olivia conceived of curved graphs as being composed of line segments. For instance, the students graphed the relationship between the total accumulated area and the length swept for a shaded cm by 5 cm triangle in which the area and the length swept out together (Figure 2).



(a) (b) (c)

Figure 2. Wesley (b) and Olivia's (c) graphs for the growing triangle (a).

Both students plotted five points in order to draw their graphs, and also drew the "average journey" graph that would be represented by a rectangle sweeping out the same total area for a length of 5 cm. The teacher-researcher (TR) asked the students whether their triangle graphs were curved everywhere, or whether they were piecewise linear. Wesley's response was that the graph was curved everywhere, explaining, "I think in between these points [indicates two points on his graph], if you added a bunch of little points in between, it would make a curve." Wesley understood that the total accumulated area increased with respect to each additional centimeter swept, but he also understood that this relationship would hold even for a smaller increment. He did not need an increment to be any particular size, such as 1 cm, in order to claim that the graph would be curved in between any two points; he already had an image of growth occurring with each increment, no matter how small. In contrast, Olivia could imagine re-sizing an increment, but within each increment, the graph would be a straight line:

> I think, because, if you made the increments even smaller like into 0.1 as your first point then I think it'd be, all the little lines together, I think they'd make a very subtle curve but relatively straight. So, when I did it with the increments as 1, I see them as straight, but if they were smaller they might *look* as if they were curved to make one big curve.

In order to better probe the students' thinking, the teacher-researcher asked the students to consider what the graph would look like if a perfectly precise robot could construct the graph with almost imperceptibly tiny increments:

TR: Would it be curved in between the points or straight in between the points?
Wesley: I believe it would be curved.

TR: What do you [Olivia] think?

Olivia: I think [long pause]. Well I mean, I think it'd be small enough to the point because the way I think of curves is a whole bunch of straight lines together to make a curve. So, I think if it was to the smallest possible thing even if it could go to infinity, but if it had to be down to the smallest possible things I think it'd be straight lines.

Olivia's remarks suggest that she still saw the graph as composed of straight segments for infinitesimal increments. She could explain the change in area with respect to each additional centimeter, but she viewed the rate of change in area with respect to length swept within a given increment to be constant. This imagery is consistent with chunky continuous covariation, in which one does not imagine change within a given increment. Wesley, however, remained consistent in his belief that such a graph would be everywhere curved, and provided supporting remarks such as: "There's tiny points *in between* those tiny points", "It goes on infinitely, kind of, the points", and "In between those, there's still more points, and it goes on forever."

Wesley appealed to a scaling image in his explanations. He described zooming to smaller and smaller scales, a process that could go on forever and never ground out at an atomic level. In this iterative process, he treated each re-scaled increment as being similar to the bigger increments. This image of scaling continuous *variation* supported his scaling continuous *covariation*, because he generalized across scales a property he noticed about the covariation of area and length: Namely, because the area grows at a changing rate over a large increment, it must also grow at a changing rate over increments at each smaller scale, and thus be curved everywhere. This generalization could extend to an infinitesimal scale just as his image of scaling continuous variation appeared to.

Constant Rates of Change

Wesley and Olivia could both discuss length and area growing together. However, their conceptions of the ratio of area to length differed. Olivia conceived of this relationship as a static ratio, whereas for Wesley, it was a rate of change. For instance, at one point the students investigated the way the area changed as a rectangle with a constant height of 4 cm grew in length (Figure 1). Both students produced a number of equivalent ratios to represent the rate at which the area accumulated relative to length. Olivia could not produce a ratio for a length less than 1 cm, whereas Wesley generated equivalent ratios such as 2 cm^2:0.5 cm, 0.4 cm^2:0.1 cm, and $4x cm^2$:x cm. Wesley explicitly referenced both quantities, but Olivia's descriptions appealed to an image of breaking the area into parts. We believe that Wesley's ratio of 4 cm^2 for 1 cm of length represented a rate. Thompson and Thompson (1992) described a rate as a reflectively abstracted constant ratio. A ratio is a multiplicative comparison of two taken-as-unchanging quantities, whereas a rate is a conception of a constant ratio variation as being a single quantity. It symbolizes the ratio structure as a whole while giving prominence to the constancy of the result of the multiplicative comparison. In order to understand the ratio as a rate, Wesley needed to have an image of change such that 4 cm^2:1 cm represented an equivalence class. Thus, he would need to understand that the unit ratio was simply a convenient measure of expressing the growth in area for a standard unit of length, and was just one of infinitely many equivalent ratios. A scaling continuous covariation image could enable this understanding, as Wesley would be able to mentally zoom in and out for different length increments, generalizing that any arbitrarily-sized increment of length would imply an associated amount of area adhering to the 4:1 ratio.

Both students also produced drawings of a 4 cm-high rectangle, but only Wesley believed that the shape of the rectangle would not change if the provided rate of 4 cm^2:1 cm were represented as 8 cm^2:2 cm. In justifying his belief, Wesley explained, "The height doesn't – like, it's not a different shape, it's the same. So, it [the rectangle] would be the same, I think." Wesley recognized that all of the ratios were instantiated in the same rectangle *height*; he appeared to understand the height as a representation of the rate of change of area, which does not depend on an amount swept. In contrast, Olivia could only justify the sameness of the picture for any equivalent ratio by converting the new ratio to the original 4 cm^2:1 cm ratio and comparing.

Constantly Changing Rates of Change

The students were also asked a series of questions about situations in which area was constantly changing, such as when the growing area was bounded by a line slanting upward at 45º-angle, or a slope of 1 (Figure 1). Wesley expressed his answer using the quantities length and area: "Every time you increase by 1 in. in length, the area for that will grow by [an additional] 1 in^2." Wesley's care in expressing the change in the growth in area for a specific length increase is additional evidence that he understood the unit ratio as a convenient representation of the constantly-increasing rate that depended on a particular increment. In contrast, Olivia had to draw pictures to visually determine the amount of increase from one increment to the next (Figure 3). She came to the same conclusion, but conceived of the 1-inch increments as "columns". When asked whether the size of the length increment mattered in terms of determining that the rate of change was constantly increasing, Olivia was uncertain and had to check with a new column size of 2 cm. She found that the new constantly-increasing rate was 4 cm^2:2cm for each of the columns. Wesley knew without having to check that the rate of the rate of change would remain invariant for any given increment, even though that value was dependent on the increment size. Scaling-continuous reasoning enabled him to generalize this observed property across all possible scales.

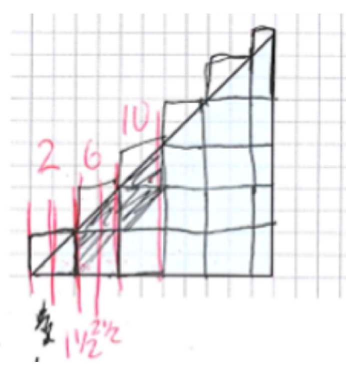


Figure 3. Olivia's drawing to find the amount of area accumulated for each 1-cm increment.

In a second example, when investigating the constantly-increasing rate of change of the area for a 3 cm:2 cm triangle, Olivia again relied on a visual strategy, explaining, "I counted it out." Wesley did not have to draw increments or make any calculations. He instead wrote "1.5", and then explained, "It [the rate of change] increased by the slope." Olivia's drawing, partitioning, and counting strategy was sensible given her image of growth across completed chunks. It enabled her to make calculations for each column and then compare the increases from one column to the next. Wesley's reliance on the slope of the triangle indicates a different conceptual foundation. He could conceive of the slope as a convenient way to express a unit ratio while also understanding that it was dependent on a particular chosen increment of 1 cm. For Wesley, the slope was a multiplicative object (Thompson & Carlson, 2017), which was produced by mentally uniting accumulated area and accumulated length simultaneously. Scaling continuous covariation could potentially support the development of such a multiplicative object because an image of re-scaling the continuum to any increment for x, including infinitesimal increments, while simultaneously coordinating that scaling with associated values for y provides a set of operations conducive to creating a new conceptual object "that is, simultaneously, one *and* the other" (Thompson & Carlson, 2017, p. 433, emphasis original). Scaling-continuous covariation enables the generalization of the ratio over any elapsed increment.

Discussion

Our findings indicate that scaling continuous covariational reasoning has the potential to support a meaningful understanding of constant and constantly-changing rates of change. In particular, it affords productive generalization of covariational properties across arbitrarily small, even infinitesimal, scales. Thompson and Carlson (2017) noted that the idea of a non-constant rate of change "is actually constituted by thinking of the function having constant rates of change over small (infinitesimal) intervals of its argument, but different constant rates of change over different infinitesimal intervals of the argument" (p. 452). As evidenced by Wesley's language, this image is compatible with scaling continuous covariation, in which one can imagine zooming to any scale, even an infinitesimal one, to visualize a tiny interval over which the function's rate of change is constant. Wesley recognized that the changes in the changes of area under a sloping line were uniform no matter the scale, which is precisely the constant second-differences characteristic that is unique to quadratic growth. Because he could imagine this at arbitrarily small scales, he could also connect this idea to the curvature of the area graphs he made.

In addition, scaling continuous covariational reasoning has the potential to support an image of instantaneous rate of change and other foundational ideas in calculus. A student who reasons with chunky continuous covariation may struggle to think about a rate of change that is not dependent on an elapsed amount of swept length. This was the case for Olivia, who needed to imagine a completed increment and an associated amount of area in order to create comparisons across same-size increments (Ely & Ellis, 2018). Wesley, however, was able to construct the height of a figure at any given instant as a multiplicative object representing the rate of change of the area compared to the length swept. This enabled him to conceive of the height as a potentiality; once it would sweep out, it would turn the potential rate into an amount of area depending on how much length has been swept. Alternatively, one could imagine the rate of change at a point to be an average rate of change over an infinitesimal interval, which offers a natural motivation for the limit definition of the derivative.

The case of Olivia and Wesley offers evidence that middle school students can develop powerful ideas about constant and changing rates, and that scaling continuous covariation could potentially offer a foundation for building sophisticated ideas about instantaneous rates of change. Situating students' exploration of functional relationships within contexts that foster images of covariation and address ideas of infinitesimal increments is critical for providing this foundation. Given the potential of scaling continuous covariation for supporting important ideas about function, we advocate for additional research to better understand the nature of this form of reasoning and its affordances for algebraic thinking.

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References

Carlson, M.P. (1998). A cross-sectional investigation of the development of the function concept. In E. Dubinsky, A.H. Shoenfeld, & J.J. Kaput (Eds.), *Research in collegiate mathematics education, III. Issues in Mathematics Education, 7*, 114 – 162.

Carlson, M.P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*, 352 – 378.

Carlson, M.P., Smith, N., & Persson, J. (2003). Developing and connecting calculus students' notions of rate of change and accumulation: The fundamental theorem of calculus. In N.A. Pateman, B.J. Dougherty, & J.T. Zilliox (Eds.), *Proceedings of the Joint Meeting of PME and PMENA* (Vol. 2, pp. 165 – 172). Honolulu, HI: CRDG, College of Education, University of Hawai'i.

Castillo-Garsow, C. W. (2013). The role of multiple modeling perspectives in students' learning of exponential growth. *Mathematical Biosciences and Engineering, 10*(5/6), 1437-1453.

Castillo-Garsow C. W., Johnson H. L., & Moore K. C. (2013). Chunky and smooth images of change. *For the Learning of Mathematics, 33*(3), 31–37.

Cooney, T. J., & Wilson, M. R. (1996). Teachers' thinking about functions: Historical and research perspectives. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp. 131–158). Hillsdale, NJ: Erlbaum.

Ellis, A. B. (2011). Generalizing-promoting actions: How classroom collaborations can support students' mathematical generalizations. *Journal for Research in Mathematics Education, 42*(4), 308–345.

Ellis, A.B. (2007). The influence of reasoning with emergent quantities on students' generalizations. *Cognition and Instruction*, *25*(4), 439 – 478.

Ellis, A.B., Ozgur, Z., Kulow, T., Williams, C.C., & Amidon, J. (2015). Quantifying exponential growth: Three conceptual shifts in creating multiplicative rates of change. *The Journal of Mathematical Behavior, 39,* 135 – 155.

Ely, R., & Ellis, A.B. (In Press). Scaling-continuous variation: A productive foundation for calculus reasoning. Scaling-continuous variation: A productive foundation for calculus reasoning. To appear in the *Proceedings of the twenty-first Annual Conference on Research in Undergraduate Mathematics Education*. San Diego, CA: San Diego State University.

Farenga, S. J., & Ness, D. (2005). Algebraic thinking part II: The use of functions in scientific inquiry. *Science Scope, 29*(1), 62–64.

Johnson, H. (2012). Reasoning about variation in the intensity of change in covarying quantities involved in rate of change. *The Journal of Mathematical Behavior, 31*(3), 313-330.

Kaput, J.J. (1994). Democratizing access to calculus: New routes to old roots. In A. H. Schoenfeld (Ed.), *Mathematical thinking and problem solving* (pp. 77–156). Hillsdale, NJ: Erlbaum.

Monk, S., & Nemirovsky, R. (1994). The case of Dan: Student construction of a functional situation through visual attributes. *CBMS Issues in Mathematics Education, 4*, 139 – 168.

Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education, 45*, 102–138.

Roschelle, J., Kaput, J., & Stroup, W. (2000). SimCalc: Accelerating students' engagement with the mathematics of change. In M. Jacobson & R. Kozma (Eds.), *Innovations in science and mathematics education: Advanced designs for technologies of learning* (pp. 47–75). Mahwah, NJ: Erlbaum.

Steffe, L., & Thompson, P. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. Kelly & R. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Stephens, A., Ellis, A.B., Blanton, M., & Brizuela, B. (2017). Algebraic thinking in the elementary and middle grades. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 386 – 420). Reston, VA: National Council of Teachers of Mathematics.

Strauss, A., & Corbin, C. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Newbury Park, CA: Sage Publications.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.

RELATIONSHIPS BETWEEN UNITS COORDINATION AND SUBITIZING

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This proposal explores relationships between young children's unit development/coordination and young children's subitizing. In particular, this theoretical commentary considers students' degrees of abstraction, students' development of actions on units, and students' operations with units when subitizing. As a result of this commentary, this author offers questions regarding how subitizing may elicit actions on units and perceptual/figurative material in a different order. These questions indicate possible alternative means in which comprehensive operations with natural numbers may develop. Possible educational implications and future research around these questions are also discussed.

Keywords: Early Childhood Education, Number Concepts and Operations

Steffe's (2017) plenary for PME-NA outlined comprehensive means in which radical constructivist learning theory has explained children's construction of mathematical concepts that have children's construction and coordination of units as a foundation. Steffe also explained that on average about 40% of first graders do not yet use figurative material to stand in for perceptual material when counting and unitizing. By third grade, this figure remains at about 5-8% of students. This finding suggests differences in children's number abstractions in early elementary grade levels that may also indicate distinct differences in their unitizing activity later. This theoretical commentary explores how young children's unitizing may relate to *subitizing* activity (individuals' quick apprehension towards the numerosity of a small set of items), which may explain differences in children's number abstractions in early grade levels.

Freeman (1912) first proposed that subitizing requires an individual's attention towards units of units when encoding number. MacDonald and Wilkins (under review) found that one child's subitizing activity may relate to her composite unit development. However, these findings are still exploratory and provide more questions than answers. In particular, if young children are developing units and/or acting on these units through their subitizing activity, how might this affect the development of their actions and operations? For instance, Steffe and Cobb (1988) found that young children develop a singular unit through their counting activity, which they iterate before engaging with partitioning. However, subitizing activity requires students partition patterned sets of items to associate with number before they iterate singular units.

If children order their actions on material differently, how might this affect their ability to coordinate actions and form operations? If both counting and subitizing promote children's coordination of actions earlier, which promotes early operation development, how would this affect children's number development? Therefore, when considering subitizing activity in relation to students' composite unit development, there seems to be new perspectives when investigating learning trajectories in number. Through this theoretical commentary, I will discuss how actions and operations may develop differently when young children subitize versus count. Resulting from this commentary are future research directions and educational implications.

The purpose of this commentary is to consider alternative learning trajectories that take on theoretical aspects in the Neo-Piagetian literature and may explain how subitizing relates to young children's development and their coordination of units. To discuss the intersect between subitizing and children's development of units, I will (a) provide a theoretical framework, (b) define unit construction, (c) describe how subitizing activity may relate to unit construction, and (d) propose future research that should consider how the two types of activity may relate.

Theoretical Framework

This theoretical commentary is grounded in the radical constructivist paradigm and more specifically, a Neo-Piagetian perspective. Essentially, in adopting a radical constructivist paradigm, I acknowledge that children learn through active engagement and reflection of their perceived reality. By adopting this paradigm, I also acknowledge that each individual constructs a unique mathematical reality that can be partially understood by others developing a second-order model of his or her mathematics.

Abstractions

With this paradigm, Piaget explains changes in mathematical realities in varying means through degrees of abstraction children rely on when engaging in mathematics. Piaget (1977/2001) described abstractions students rely as beginning as a reliance on *empirical abstractions* (abstractions of actions on perceived objects) towards *reflective abstractions* (abstractions on projected operations). Glasersfeld (1995) explains that individuals rely on two types of empirical abstractions (empirical abstractions and pseudo-empirical abstractions). *Empirical abstractions* explain children's attention towards rules and patterns when acting on perceived objects (e.g., counting manipulatives and knowing the last number word signifies the total). *Pseudo-empirical abstractions* are defined as children's ability to coordinate figurative material, which is indicative when students represent perceived objects with figurative material (e.g., fingers, tapping, verbal utterances), when solving a task. Students transitioning from empirical abstractions towards pseudo-empirical transitions internalize mathematical patterns and rules while coordinating their unitizing, regardless of material presented to them. As children internalize their actions and further step away from perceived material presented to them their degree of abstraction transitions from empirical abstractions towards reflective abstractions.

Piaget (1968/1970) first defined reflective abstractions as simply "coordinated actions" (p. 18). Glasersfeld (1995) further interpreted Piaget's (1977/2001) reflective abstraction delineations by describing two types of reflective abstractions (reflective abstraction and reflected abstraction) that children rely on when interiorizing mathematical patterns to form logical structures. *Reflective abstraction* (first subset of reflective abstraction) explains an individual's projection and reorganization of his or her coordinated actions or operations at another conceptual level (1995). *Reflected abstraction* (second subset of reflective abstraction) explains this same activity, but also explains that an individual is also aware of his or her projection and reorganization (1995).

Actions Versus Operations

As children rely on different degrees of abstractions, they develop operational fluency with number operations, as structures for number become interiorized. Boyce (2014) explained how these structures develop in weak forms versus relatively stronger forms. Essentially, Boyce distinguishes between children's development and coordination of actions versus their development and coordination of operations to explain different forms of reflective abstractions. For instance, if a child is capable of coordinating his or her <u>actions</u> to create a goal and develop a means in which to take a unit and iterate it, then the child has created goal-activity and an iterable unit (an abstract unit capable of iteration). The unit has become abstracted, but the operational structure has not been developed to allow student anticipation of his or her actions on the unit and in coordination with his or her other actions. This is an example of a lower form of reflective abstraction because the unit is acted upon in activity. An operation to anticipate this activity is not created. This would explain why a child may be transitioning away from a reliance on perceptual material towards figurative material, but still struggles to interiorize structures for natural numbers. Comparatively, if a child is capable of developing an iterating operation, then the operation is one that can be acted upon, not the unit. This allows for anticipated activity (c.f., Tzur & Simon, 2004; Steffe, 1992; Ulrich & Wilkins, 2017), which allows a child's natural number schema to become interiorized.

In this commentary, I will not focus on the operations, which are beyond the capability of a young child in the early childhood years, but on how early perceptual actions may explain later conceptual operations. For instance, Piaget (1968/1970) posits, "in developmental psychology … there is never an absolute beginning" (p. 19). What Piaget seems to be referring to as the "absolute beginning" in this argument is the beginning of logical structures. When the coordination of actions begin framing our discussion, Piaget posits that the coordination of actions can go back to biological or organic coordination of actions.

I posit that many of these early roots of biological coordination do not directly relate to a child's development of his or her mathematical structures for number, but may explain the root of his or her early perceptions and coordination of activity when explaining latter operation development and unitizing (Glasersfeld, 1981). Therefore, I want to focus in on the roots of early development and coordination of actions to determine how early forms of operations may explain unitizing that young children engage in. Coupling this focus with subitizing development may explain differences in young children's unit development. Thus, this proposal will further discuss how particular actions with subitizing may be important for young children and how the coordination of these actions with their counting actions may relate to earlier forms of operations.

Unit Construction and Unit Coordination

The term *unit* has become polysemous in the Neo-Piagetian field, as a unit symbolizes a variety of means in which unitizing develops relative to the context the proposed unit is set in. Ulrich (2015) cited Glasersfeld (1981) when defining unitizing as the "generalized and generative process of abstracting out the 'one'-ness from some aspect of experience" (as cited by Ulrich, 2015, p. 3). Frege (1884/1974) and Husserl (1887/1970) found in their work that children engaged in conceptual activity when required to cut "discrete items out of the flow of experience," which were found to promote the construction of "unitary wholes and ultimately of countable units" (as cited by Steffe & Cobb, 1988, p. 3). Steffe and Cobb posited that children younger than two years of age are capable of this activity, yet it is rare to find studies investigating or even theorizing this development or its nature in the early childhood years. For instance, Clements (1999) first proposed that subitizing relates to young children's number development and MacDonald and Wilkins (Under Review) found that one preschool student's subitizing activity related to her conceptual forms of units that she used when solving a counting task. However, it has not been determined how subitizing activity may relate to unitizing and how this development may relate to latter counting development.

A Unit

Boyce (2014) defines a unit as "something that has been unitized or set apart for further action" (p. 3) and characterizes a unit as "an object that can be transformed" (p. 4) and something that can be iterated (p. 24). A very different definition comes from Ulrich and Wilkins's (2017) study where they define a unit as "interiorized counting acts, so they can be used to enumerate the size of the sets of visible or invisible items and can themselves be counted" (p. 2). Finally, a third (very different) definition from Ulrich (2015) where a unit is defined as that, which "allows [students] to measure the number of items in a collection" (p. 3). These definitions are distinctly different because each researcher investigated different aspects of students' mathematics learning. Thus, the term unit becomes polysemous because this one word has multiple meanings that are related, but not alike. As described earlier, when the context becomes more sophisticated, the word, unit represents that which is the basis for measuring what the children are developing and coordinating within their respective mathematical structures. To provide insight into how a unit and unitizing action might relate or not relate to early childhood subitizing and number development, I will adopt Boyce's definition in hopes to explain early forms of empirical and pseudo-empirical abstraction development.

Actions and Operations on Units

The coordination of actions upon a unit explain how students develop operations. Boyce (2014) explains that a coordination of operations allow anticipatory frameworks (see Tzur & Simon, 2004) to develop, as operations are acted upon and allow for anticipation. This iterative cycle explains the nature of development and learning in mathematics. For instance, Norton and Boyce (2015) explain that children produce and coordinate four actions (unitizing, iterating, partitioning, and disembedding) when developing operations (e.g., distributing operation) that promote unit coordination. When students are able to operate flexibly and anticipate appropriate actions and results within a particular mathematics domain, Norton and Boyce posit they are able to do so because they have produced and coordinated levels of units. Coordination of these units results from students' development and coordination of the aforementioned actions.

Steffe and Cobb (1988) also found that young children evoked counting actions to develop early forms of units described as prenumerical singular units. These prenumercial singular units were described as evidence of young children's reliance on actions with perceptual material, figural patterns, motor patterns, verbal utterances, and abstract numbers (1988). As children distance themselves from perceptual material towards more abstract material, Steffe and Cobb found that children unitized the perceptual and figurative material before developing actions upon singular abstract units. Once actions upon abstract singular units are developed, children are able to iterate these units to construct numerical sequences that they segment. Through their segmenting actions, children develop composite abstract units. Once children iterate and partition composite units, they are capable of disembedding parts from whole sets. These actions promoted more operational understandings so that they are able to be aware of and work within mathematical structures for number. Steffe and Cobb found that when children could (1) count-on, (2) double count (i.e., 1, 2, 3, … one three; 4, 5, 6, … two threes), and (3) count by multiples (i.e., 3, 6, 9, 12) to solve problems, they were using all four actions to develop a nesting of sums for multiplicative and fractional reasoning.

With this learning theory in place, mathematics educational researchers have been able to explain nuanced development of children's counting, fractions, and multiplication. However, it is still not clear how subitizing activity may relate or not relate to children's unit development and coordination. Thus, these unit development and coordination learning theories will be discussed as related to different types of perceptual and conceptual subitizing.

Perceptual and Conceptual Subitizing Related to Composite Unit Development

Subitizing, initially defined in the psychology field (Kaufman, Lord, Reese, & Volkmann, 1949), describes individuals' quick attention to the numerosity of a small set of items. Unitizing and unit coordination have not been described in this research. Historically, subitizing has been described in the psychology field as a quantification encoding process and visual information processor where the numerosity of a small sets of items (ranging 1 to 5) are identified (Klahr, 1973). More recently in the mathematics education field, Sarama and Clements (2009) delineated a hypothetical trajectory of subitizing activity and explained subitizing as relying on either perceptual or conceptual activity when developing early number understanding. More specifically, Sarama and Clements explain that individuals *perceptual subitizing* use an encoding process for small numerosities, but also draw from attentional resources when associating a number word with the numerosity of the set. Individuals engaging with *conceptual subitizing* use relatively more advanced number understandings to make sense of larger sets of items (≥ 5) (Sarama & Clements, 2009). However, unit development and coordination may explain, in more nuanced ways, how conceptual subitizing develops in relation to number understanding.

Perceptual Subitizing and Unit Development

Freeman (1912) initially proposed that subitizing may introduce children to the perception of "units of units" when encoding number. To consider early forms of unit development with subitizing activity, I consider actions students use to determine how "units of units" may be produced through subitizing activity. Clements (1999) first described *perceptual subitizing* and *conceptual subitizing* activity as relying on actions that promote early unit development. Perceptual subitizing was defined as students' ability to intentionally quantify a set of items through their subitizing activity yet be unable to be aware of any mathematical processes. To be capable of engaging in perceptual subitizing, Sarama and Clements (2009) suggest that students would need to be capable of cutting away a set of items to determine these as a unit. Sarama and Clements defines conceptual subitizing as children's ability to be aware of units of units when quickly associating sets of items with number words. However, it is not yet clear what type of units students may be developing and coordinating when subitizing.

MacDonald and Wilkins (2016) found in an exploratory study that preschool children engage in unitizing that may relate to early forms of composite unit development. For instance, children were found to engage in *Initial Perceptual Subitizing* where simple associations were made between shapes or motion when intentionally naming a number word. I argue this is very similar to Sarama and Clements (2009) description for perceptual subitizing and explains early forms of figural or motor singular unit item development. This form of unit development would explain how students rely on patterns shown to them visually with figurative patterns and how they may even represent them rhythmically with motor patterns. Through this unit development, young children may begin acting upon the rules of the patterns instead of simply acting on the actions related to the patterns. This would be indicative of a child pointing to his or her paper to show the pattern or shape when justifying why he or she knows she saw "three."

MacDonald and Wilkins (2016) also found that young children could subitizing two or more subgroups of items before they were capable of composing these subgroups. This type of perceptual subitizing activity was described as *Perceptual Subgroup Subitizing*. Quite often, the orientations shown to the students were clustered and regular. For instance, an orientation shown to a child may have two dots in a column on the right-hand side of the mat and three dots in a triangular orientation on the left-hand side of the mat. These orientations afforded students the opportunity to unitize more than one subgroup without requiring them to partition or iterate.

MacDonald, Boyce, Xu, and Wilkins (2015) also found that when students were shown orientations with regular patterns that were symmetrical (i.e., four dots in a rectangular orientation) in nature, they were capable of unitizing and iterating subgroups. For instance, when Frank, a four year-old student, was shown four dots in a rectangular orientation he initially said he saw, "T … four" (p. x). This suggests that he iterated two to build up towards four. This also suggests that he was capable of using a unit of two and partitioning a unit of four. Through these actions, MacDonald et al. posited that Frank was capable of coordinating actions to produce a unit for four. The symmetrical aspects of the orientation may have also afforded Frank the opportunity to partition because the line of reflection fell upon these same partitioning lines. Empson and Turner (2006) found that early forms of partitioning that began with paper folding were foundational for students' construction of functional relationships. Thus, this activity may provide foundations for more sophisticated coordination of units later.

Perceptual Subitizing and Actions on Units

If students are capable of (1) subitizing and then composing units after subitizing – *Perceptual Ascending Subitizing*, or (2) composing and subitizing and then decomposing units – *Perceptual Descending Subitizing*, then MacDonald and Wilkins (2016) found students were building necessary activity for conceptual subitizing. The distinction between this activity and conceptual subitizing is that students are shown items that are clustered and patterned, which does not require partitioning. Thus, this Perceptual Ascending Subitizing and Perceptual Descending Subitizing can allow students to unitize and act on the units developed. It is not clear how these operations develop and what type of composite unit (perceptual or figurative) students are developing or coordinating. These early operations may develop through a coordination of counting actions, through a segmenting of numerical sequences, or through students' development of patterned or figurative patterns. Regardless, this transition from students' development of actions to their development of operations is key, as students are now capable of being aware of number structures that afford them units of units perspectives.

Furthermore, when children engage in Perceptual Descending Subitizing, MacDonald and Wilkins (2016) argue that children are engaging in bi-directional activity. For instance, students who subitize two clustered subgroups may state that they saw "two and three." Once asked, "how many is that altogether?" they may then compose these units and state "five." MacDonald and Wilkins explain that this is Perceptual Ascending Subitizing because the student is ascending from the groups to the total set. However, students who subitize and compose two clustered subgroups may state that they saw "five." When asked, "how do you know there are five there?" they would then need to reflect on their actions and decompose the total set. Steffe, Glasersfeld, Richards, and Cobb (1983) described this bi-directional activity when children reversed their counting actions and found that this activity provided foundation actions for reversible counting later. Steffe explained that the distinction between bi-directional counting and reversible counting was that in bi-directional counting children would rely on prenumerical units and in reversible counting, children would rely on abstract units. This distinction leaves a lot to educators to determine the nuances of the development between bi-directional counting and reversible counting. Thus, in Perceptual Descending Subitizing, I argue that children are not relying on abstract composite units and therefore require the activity and other means to represent these units when reversing their activity. Also, questions arise from some this development in subitizing. For instance, how could different types of perceptual subitizing activity describe different types of prenumercial composite units?

Conceptual Subitizing and Unit Action Coordination

Once children are capable of partitioning and unitizing subgroups from a total set of items, MacDonald and Wilkins (2016) found that children were capable of conceptual subitizing. When children conceptually subitize, it seems that are capable of partitioning and unitizing before they iterate units. Children inverse these actions when they are counting. For instance, when counting, Steffe and Cobb found that children iterated units before they were capable of partitioning or segmenting numerical sequences. As students develop operations, they begin coordinating actions, which I posit occurs when students reverse the order of these actions. For instance, if children only engage in counting, then they will develop schemes that promote a particular order of actions. Children will first unitize to develop units and then iterate their actions in accordance with these units in a 1:1 correspondence. Here the actions are still in the material presented to the children and number is also present in their actions on these objects. However, through a distancing of these, children's reliance upon particular items shown to them (e.g., through use of fingers, through use of motor actions) children's units become more abstract until they are capable of operating on a series of abstract units. These early operations involve children segmenting numerical sequences that they built through their unitizing and iterating of abstract units. Thus, when counting children unitize, iterate, and then partition.

Comparatively, if children only engage in subitizing, then they may develop schemes that promote a different order of actions. Children will first unitize to develop units (perceptual subitizing) and then partition these units to develop a unit of units understanding (conceptual subitizing). Only then, will they begin to iterate units (e.g., five and five make ten) to (de)compose multiplicative units. Thus, when subitizing children unitize, partition, and then iterate. I posit that when children engage in counting and subitizing, their actions become operations because they are now asked to change the order of their actions and coordinate them in operational structures that are more comprehensive. This allows for more sophisticated reflective abstractions where children can use when anticipating actions and solutions in mathematics (Boyce, 2014).

In closing, by investigating relationships between subitizing and composite unit development, mathematics educational researchers may be able explain differences in children's composite unit development and provide alternative trajectories in learning mathematics.

Future Research and Possible Educational Implications

This theoretical commentary provided insight into how subitizing may or may not relate to unit development and coordination. From this discussion, it seems evident that when students engaging in counting activity they unitize and iterate before partitioning their developed number sequences. This allows students opportunities to develop actions and abstract these actions on their developed units. However, students engaging in subitizing activity may be unitizing and partitioning before iterating their developed spatial patterns. This allows students opportunities to develop these same actions on their developed units, but in a different order. However, if students develop more than one order of actions, would this provide a more comprehensive set of operations and allow students operation development earlier? By leveraging this development, I posit that gaps that Steffe (2017) and others (e.g., Clements & Sarama, 2011; Siegler & Ramani, 2008) have found in early elementary school may shrink. Thus, future research should focus on how different forms of activity in early elementary years may provide students alternative means to develop operation and units in which to operate on. Findings from studies like this might serve educators alternative actions that would serve children's disembedding actions and distributive operations. Also, when engaging in trajectories with different ordered actions, would children be capable of simultaneously coordinating these actions to develop operations similar to a splitting operation (see Wilkins & Norton, 2011)? Finally, how might alternative trajectories be used to leverage development for students who require different curricula or educational support (e.g., students identified as having a learning disability)? Thus, future research in how subitizing and counting relate to unit development and coordination could serve theoretical frameworks and educational curricula and should be explored further.

References

Boyce, S. J. (2014). *Modeling Students' Units Coordinating Activity* (Doctoral dissertation, Virginia Polytechnic Institute and State University).

Clements, D. H. (1999). Subitizing: What is it? Why teach it?. *Teaching children mathematics, 5*(7), 400-405.

Clements, D. H., & Sarama, J. (2011). Early childhood mathematics intervention. *Science, 333*(6045), 968-970.

Empson, S. B., & Turner, E. (2006). The emergence of multiplicative thinking in children's solutions to paper folding tasks. *The Journal of Mathematical Behavior, 25*(1), 46-56.

Freeman, F. N. (1912). Grouped objects as a concrete basis for the number idea. *The Elementary School Teacher, 12*(7), 306-314.

Glasersfeld, E. von (1981). An attentional model for the conceptual construction of units and number. *Journal for Research in Mathematics Education*, 83-94.

Glasersfeld, E. von (1995). *Radical constructivism: A way of knowing and learning*. NY: Taylor & Francis group.

Kaufman, E. L., Lord, M. W., Reese, T. W., & Volkmann, J. (1949). The discrimination of number. *The American Journal of Psychology, 62*(4), 498-525.

Klahr, D. (1973a). Quantification processes. In W. G. Chase (Ed.), *Visual information processing* (pp. 3-31). NY: Academic Press.

MacDonald, B. L., Boyce, S., *Xu, C. Z. & Wilkins, J. L. M. (2015). Frank's perceptual subitizing activity relative to number understanding and orientation: A teaching experiment. In T. G. Bartell, K. N. Bieda, R. T. Putnam, K. Bradfield, & H. Dominguez (Eds.), Proceedings of the 37th Annual Psychology of Mathematics Education Conference, North American Chapter. (pp. 149-156). Lansing, MI.

MacDonald, B. L. & Wilkins, J. L. M. (Under Review). Amy's subitizing activity relative to number understanding and item orientation.

MacDonald, B. L. & Wilkins, J. L. M. (2016). Seven types of subitizing activity characterizing young children's mental activity. In S. Marx (Ed.), Qualitative research in STEM. New York: Routledge.

Piaget, J. (1970). *Genetic epistemology*. (E. Duckworth, Trans.). New York, NY: Columbia University Press. (Original work published in 1968).

Piaget, J. (2001). *Studies in reflecting abstraction.* (R. L. Campbell, Trans.). Philadelphia, PA: Psychology Press. (Original work published in 1977).

Sarama, J. & Clements, D. H. (2009).*Early childhood mathematics education research: Learning trajectories for young children.* New York, NY: Routledge.

Siegler, R. S., & Ramani, G. B. (2008). Playing linear numerical board games promotes low-income children's numerical development. *Developmental science, 11*(5), 655-661.

Steffe, L. P. (1992). Schemes of action and operation involving composite units*. Learning and individual differences, 4*(3), 259-309.

Steffe, L. P. (2017). Psychology in mathematics education: Past, present, and future. In E. Galindo & J. Newton (Eds). *Proceedings of the 39th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 27-56). Indianapolis, IN.

Steffe L. P., Glasersfeld E. von, Richards J. & Cobb P. (1983) Children's counting types: Philosophy, theory, and application. Praeger, New York.

Steffe, L. P. & Cobb, P. (1988). *Construction of arithmetical meanings and strategies*. New York, NY: Springer Verlag, Inc.

Tzur, R., & Simon, M. (2004). Distinguishing two stages of mathematics conceptual learning. *International Journal of Science and Mathematics Education, 2*(2), 287-304.

Ulrich, C., & Wilkins, J. L. (2017). Using written work to investigate stages in sixth-grade students' construction and coordination of units. *International Journal of STEM Education, 4*(1), 23.

Wilkins, J. L., & Norton, A. (2011). The splitting loope. *Journal for Research in Mathematics Education, 42*(4), 386-416.

STRATEGIES USED BY MEXICAN STUDENTS IN SEEKING STRUCTURE ON EQUIVALENCE TASKS

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This paper presents strategies identified in Mexican rural school students in seeking structure on equivalence tasks that involve the equal sign and tasks that do not. The results arise from the pilot study of a research project on the structure of numbers and numerical operations – a key aspect of early algebraic thinking.

Keywords: Algebra and Algebraic Thinking, Elementary School Education.

Research on algebraic thinking in young students is a trending topic worldwide (Kieran, 2018; Singh & Kosko, 2017). In this and other works, different approaches to early algebra that recognize in arithmetic a strong algebraic feature can be identified (Carraher & Schliemann, 2007). Studies on equivalence in numerical sentences have centered on relational thinking (Carpenter, Franke, & Levi, 2003; Molina & Ambrose, 2008), as well as on generalization, as main aspects of algebraic thinking. However, another important aspect is that of structure in arithmetic (Kieran, 2018).

Recent studies show some of the relationships between expressing the structural and the operational on equivalence tasks. For instance, Asghari and Khosroshahi (2016), with tasks that do not involve the equal sign, propose the existence of an operational approach in developing algebraic thinking in the context of the associative property. According to these researchers, mathematical thinking in elementary school may involve both an operational and a structural conception. Hence, the authors identify the development of algebraic thinking as operationally experienced in the ability to transform a numerical structure.

Schifter (2018) states that an important feature of early algebra includes observation, development, and justification of structural properties in numerical operations expressed in students' computational strategies. This is seen, for instance, in students' verbalizations of the property that in an addition, adding and subtracting the same amount does not affect the value of the expression. Thus, it is important to engage students in discussions on their strategies to determine the veracity or not of numerical sentences such as $57 + 89 = 56 + 90$. If students indicate a relational mode of thinking, it suggests that they are focused on the structure of such equalities. Schifter also analyzed students' thinking when they explored structural properties in related expressions in a sequence of expressions not involving the equal sign (14+1, 13+2, 12+3, 11+4).

In another work, Pang and Kim (2018) using sentences such as $67 + 86 = 68 + 85$ reported that participants tended to use computational strategies; however, they also showed their ability to use a structural approach. According to Pang and Kim, one structural strategy consists in observing that an addend increases by one and the other decreases also by one. In Schwarzkopf, Nührenbörger, and Mayer (2018), however, it is considered that describing patterns in a sequence of expressions such as $30 + 20 =$, $31 + 19 =$, $32 + 18 =$, … is not actually structural reasoning, even if they point out the importance of such thinking in patterns or regularities. These researchers agree with Mason, Stephens, and Watson (2009) in the sense that structural thinking is much more than only observing patterns.

It is clear, then, that there are different perspectives regarding the structural in arithmetic as a component of algebraic thinking, as well as the importance of its development. From here, the purpose of our work is to research the reasoning of Mexican students from rural schools in seeking structure on equivalence tasks that involve the equal sign and tasks that do not. The research question was: What are the strategies used by Mexican students from rural schools regarding structure on equivalence tasks?

Theoretical Framework

Structure in Numbers and Numerical Operations

One of the key aspects in developing algebraic thinking is the notion of structure; however, there are different perspectives on this notion. As mentioned in Kieran (2018), several researchers have worked in this area in the teaching and learning of algebra. Linchevski and Livneh (1999), for instance, recognize that students experience difficulties with structure in algebra and that these difficulties are due to lack of understanding of structure in arithmetic. Mason et al. (2009) have suggested that working with tasks that focus on relations rather than on procedures strengthens students' attention to the structural aspect of arithmetic. They refer to this as *structural thinking* and propose that it allows students to move away from the particular in a situation.

According to Kieran (2018) generalization-oriented activities encompass a structural aspect, but more attention is needed to the process that is complementary to generalizing, that is, the process of *seeing through mathematical objects,* decomposing and recomposing them in several structural ways. Kieran (2018, pp. 80-81) argues that to observe the structure of mathematical objects is to see through them. This means being aware of possible and different ways to structure number and numerical operations, for example, observing that 989 may be decomposed as $9 \times 109 + 8$, as $9 \times 110 – 1$, or as $9 \times 10^2 + 8 \times 10^1 + 9 \times 10^0$. According to this researcher, the generalization of mathematical ideas in arithmetic is linked to the idea of expressing structure. So generalization involves identifying the structural, and the structural involves identifying the general. Kieran (2018, p. 82) states that structure in numbers and numerical operations may be explained, firstly, by drawing on Freudenthal (1983, 1991, quoted in Kieran, 2018). That is, that the system of whole numbers constitutes an order structure, where addition is based on the order of this structure: in the addition structure, to each pair of whole numbers a third number (its sum) can be assigned. It is emphasized that, in Freudenthal's discussions of structure, there is not just one all-encompassing structure. He refers, for example, to order structure, additive structure, multiplicative structure, structure according to divisors, structure according to multiples, etc.

Based on the literature regarding perspectives on mathematical structure, specifically arithmetical structure, Kieran (2018) suggests promoting student experiences with equivalence through decomposition, recomposition, and substitution. Following Freudenthal, she points out that the structure in numbers and operations involves different *means of structuring*, according to factors, multiples, powers of 10, evens and odds, decomposition of primes, etc. Such structures expressed through decomposition, in other words, uncalculated forms, have properties. This perspective on structure constitutes a wider conceptualization of the fundamental aspect of structure in number and numerical operations as a means to develop early algebraic thinking. Taking into account the points made by Kieran (2018), as well as the suggestion of Schifter (2018) that structural properties can be implicit in students' procedures, this work will explore Mexican students' structure sense in equivalence tasks as evidenced through their strategies.

Methodological Considerations

Included in this report we present the preliminary results from an ongoing qualitative research aimed at investigating the strategies that students use in equivalence tasks.

Initial Task Design

Three tasks were designed in order to explore students' strategies; two of these did not include the equal sign. Task 1 aims at identifying the way in which students relate two numbers *a* and *b* with a third one *c* (i.e., its sum) and the rationale they use. It is a 4-item task with a main question: Can number *c* be written from numbers *a* and *b*? (*a* and *b* being specific numbers). Also, the task includes a generalization question: Can any number be written from other numbers? For all the questions, students were asked to provide an explanation.

Task 2 was based on the sequence proposed by Schifter (2018) and includes seven items. The aim is to observe the regularities students find in the proposed sequence based on this first item:

14+1
13+2
12+3
11+4
10+5

The rest of the items focus on two particular expressions from the sequence (e.g., 14+1 and 13+2). Students are asked to explain how to write an expression from the other. Another set of items focuses on discussing the equivalence of expressions without computation. The task ends with a question where a sequence of the same type is proposed, but involves subtracting; here we want to observe if students extrapolate from the discussion involving the case of adding.

Task 3 involves the use of the equal sign to show the equivalence of expressions, for instance, 4 + 5 = 4 + 3 + 2. The aim is to explore if students indicate relational thinking based on the structure of such equalities. The main goal in the task is to determine if the numeric sentences are true, as well as the possibility of rewriting them in an equivalent form. Task 3 also included numerical sentences with "big numbers".

Participants

Six sixth graders, ages 10 and 11, from a public Mexican school participated. This grade level was chosen because such students are finishing primary school and have been exposed to the official Mexican public education curricula. In the curriculum for the elementary school (SEP, 2016) the equivalence of numerical expressions is not mentioned. However, several tasks from the official textbooks have the potential to promote students' early algebraic thinking (Cabañas, Salazar, & Nolasco, 2017).

Data Collection

Prior to the unfolding of the designed activity, the teacher in charge of the group reviewed it. In her opinion, the students had never solved similar tasks; they had only worked with the use of the equal sign in an operational sense. The data collection technique was that of the Group Interview, so that students could verbalize their rationales. Data were obtained during three sessions, one session per task, with sessions lasting 30-40 minutes each. All six students participated in each of the three sessions.

Results and Discussion

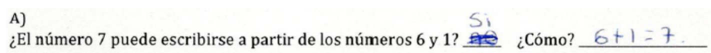
The preliminary results of an ongoing study are herein reported. The analysis focuses on the work of three students (S1, S2, and S3), those who participated most fully in the group interviews. Data for these results come from students' worksheets, videotaped footage of the sessions, and researcher's field notes.

Results from Task 1

Task 1 does not include the equal sign so as to see whether students use it spontaneously and, if so, in which way. Three of the items were the following:

1. *May number 7 be written from numbers 6 and 1? If so, how?*
2. *May number 19 be written from numbers 14 and 5? If so, how?*
3. *Is it correct to write number 7 as 3 + 4? As 8 + 2? Explain.*

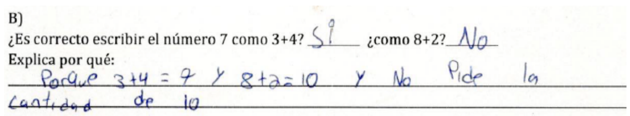
The students answered affirmatively items 1 and 2, their explanation being based on what in the literature is known as an operational use of the equal sign. For example, see S1's work in Fig. 1.


A)
¿El número 7 puede escribirse a partir de los números 6 y 1? Si ¿Cómo? 6+1=7.

May number 7 be written from numbers 6 and 1? Yes If so, how? 6 + 1 = 7

Figure 1. S1's operational form of justification.

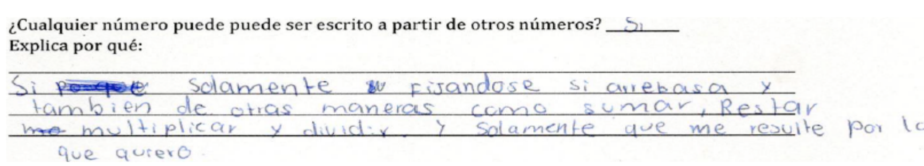
In his explanation, S1 relates 7 with 6 and 1 in an *operational sense*: 6 + 1 = 7, through a computational strategy. None of the students write, for example, 7 = 6 + 1, which would be recognized as a not strictly operational response. From a structural point of view, 6 and 1 can be interpreted as a decomposition of the number 7, which can then be recomposed from these numbers. In item 3, students answer in the same sense (Fig. 2) based on *the result they must obtain*.


B)
¿Es correcto escribir el número 7 como 3+4? Sí ¿como 8+2? No
Explica por qué:
Porque 3+4 = 7 y 8+2=10 y No Pide la cantidad de 10

B)
Is it correct to write number 7 as 3+4? Yes As 8+2? . No
Explain
Because 3+ 4 = 7 and 8 +2 = 10 and [3+4] does not ask for a 10

Figure 2. S3's justification in terms of the result.

The same idea is present in the answers involving a generalization. *Can any number be written from other numbers?* Students identify the generality in terms of the response that they must arrive at. This is observed in S1's final explanation (Fig. 3) where he states *"...only if I get what I want"*.


¿Cualquier número puede puede ser escrito a partir de otros números? Si
Explica por qué:
Si Solamente fijandose si quebasa y también de otras maneras como sumar, Restar multiplicar y dividir y Solamente que me resulte por lo que quiero.

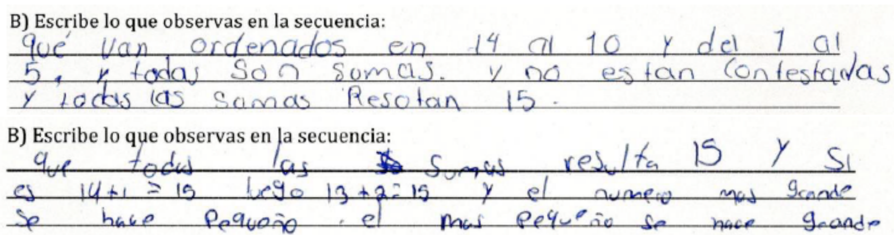
Can any number be written from other numbers?
Explain why
Yes, "*...only if I get what I want*

Figure 3. S1's general statement.

Students' answers show a lack of relational thinking by their use of the equal sign as a symbol that indicates the result. In other words, their strategy doesn't match with a notion of number decomposition, but with the idea of *operating with numbers* in order *to get a result*. The way in which they justify their answers – according to their teacher – shows how they have been systematically exposed to this way of thinking. In order to test whether the way in which Task 1 was designed led to the strategy that students used, Tasks 2 and 3 were designed differently.

Results from Task 2

The analysis of data from Task 2 (involving the sequence from Schifter, 2018) focuses on the features of the $a + b$ form of expressions that were observed by the students, as well as on the possibility of transforming one expression of the sequence into another expression of the same sequence. Regarding the first aspect, students' answers show that they identify the regularity in the sequence. For instance, they write *about the involved operation (addition)*, *the sum (the result)* and the existence of an *order* in the sequence (*increase* and *decrease* of the addends). See S2's and S3's answers to the first item (Fig. 4).

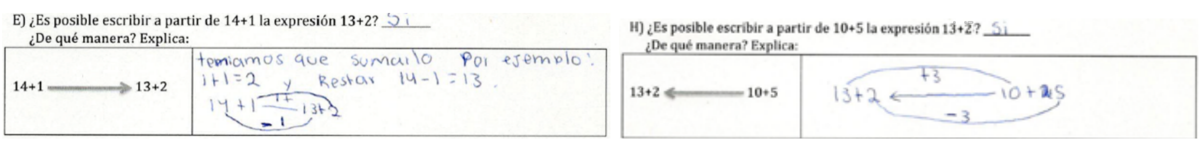


B) Write what you observe on the sequence
There is an order, in the 14 to 10 and in the 1 to 5, all of them are additions, they are not answered and all the additions result in 15.

In all the sums the result is 15, you have 14+1=15 then 13+2=15 and the biggest number becomes small, the smaller becomes big.

Figure 4. Features of the sequence as indicated by S2 (above) and S3 (below).

The kind of answers students produce regarding the presented sequence relates, on the one hand, to the kind of thinking they show throughout Task 1. That is, they identify the expression as an operation that must be carried out in order *to obtain a result*. This feature is clear in S2's response when he writes: *"…all of them are additions and they are not answered and all the additions result in 15"*. This suggests that these students do not see the expression as a mathematical object in itself, reflecting what is described in the literature as the *lack of closure* dilemma. On the other hand, there is some evidence of a train of thought that could be associated with *the structural*. According to Pang and Kim (2018), to identify patterns such as "increases by one and decreases by one" is a part of structural sense. This can be seen in S3's work (the lower half of Fig. 4) when he states: *"…and the biggest number becomes small, the smaller becomes big"*. However, he does not relate the feature he observes to the equivalence of the expressions. In the second part of Task 2, the students were asked how to obtain one expression in the sequence (e.g., 13 + 2) from another (e.g., 14 + 1). In these cases, all the students use an *additive compensation strategy*, as observed in Fig. 5.



E) Is it possible to write from 14+1 the expression 13+2? Yes
In which way? Explain
We have to add. For instance 1+1=2 and subtract 14 -1=13

H) Is it possible to write from 10+5 the expression 13+2? Yes
In which way? Explain
[Student illustrates an additive compensation strategy]

Figure 5. S1's additive compensation strategy.

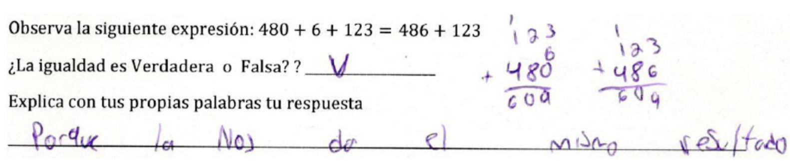
The aim of our research was to study how students move from one expression to another, if they decompose and recompose the involved numbers. It was noticed that they identify the parts of the expressions, but not as a mathematical object that can be decomposed and recomposed to transform one expression into another. It was also seen that students use a *compensation strategy*: adding and subtracting the same amount to and from the involved addends in order to obtain the second expression. The following is an extract from an interview when, in the course of presenting the task, the interviewer asked for a generalization of the student's strategy:

> *Researcher*: …Can you do it [referring to his strategy] in any case? Is there a rule for it?
> [After other students offer suggestions, S2 answers]
> *S2*: It's only a matter of adding and subtracting, depending on the required numbers.

Despite such structurally-related responses as S2 produced, there is not enough evidence, however, to determine if students identify an equivalence relationship among the expressions (e.g., 14+1 and 13+2). Nor is there enough evidence, with respect to the additive compensation strategy, to determine if they are aware that their strategy is generalizable to all additions (e.g., that 27 + 15 can be converted to, say, 30 + 12 or that 44 + 19 can be converted to 43 + 20) or simply applicable to the set of additive expressions provided in Task 2. If the latter, then — as suggested in Schifter (2018) — this would be an *ad hoc* strategy aimed at getting the numbers needed for the second expression from the first one, and viceversa.

Results from Task 3

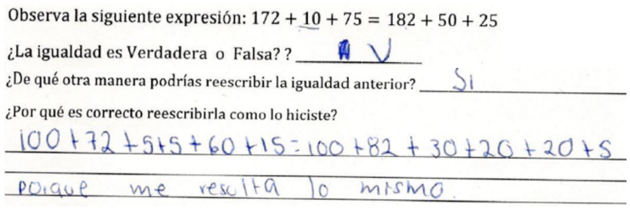
This task includes the equal sign – in expressions such as $a + b = c + d$. As mentioned, Task 3 involves "big" numbers to see if this deters the use of computational strategies.
On the one hand, students accept expressions such as $a + b = c + d$; however, they justify the equality of both sides by calculating the result on each side. Again, this computational strategy demonstrates that students are not relying on relational thinking. Their computational strategy is called upon in both cases, whether with "small" or "big" numbers (Fig. 6).



Observe the following expression: 480 + 6 + 123 = 486 + 123
Is the equality True or False? *T*
Explain with your own words
Because we get the same result

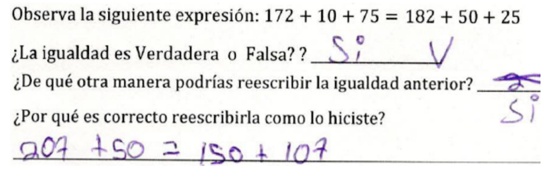
Figure 6. S3's computational strategy.

On the other hand, students rewrite the equalities in the form of other equivalent equalities according to two strategies. In the first of these strategies, they decompose each of the addends, but not in a way that shows a clear relationship between one side and the other of the equality (see S1's work in Fig. 7). In the second, which is based on calculating the *total (the result) for each side* without first decomposing the involved addends, students then look for two or more numbers for which they could obtain the same total (see S3's work in Fig. 8).


Observa la siguiente expresión: 172 + 10 + 75 = 182 + 50 + 25
¿La igualdad es Verdadera o Falsa? ? V
¿De qué otra manera podrías reescribir la igualdad anterior? Si
¿Por qué es correcto reescribirla como lo hiciste?
100 + 72 + 5 + 5 + 60 + 15 = 100 + 82 + 30 + 20 + 20 + 5
porque me resulta lo mismo

Observe the following expression: 172 + 10 + 75 = 182 + 50 +25
Is the equality True or False? *T*
In which other way could you re-write the previous equality? *Yes.*
Why it is correct re-write the expression in such a way?
100 + 72 + 5 + 5 + 60 + 15 = 100 + 82 + 30 + 20 +20 + 5
Because I get the same result

Figure 7. S1's equality rewriting.


Observa la siguiente expresión: 172 + 10 + 75 = 182 + 50 + 25
¿La igualdad es Verdadera o Falsa? ? Si V
¿De qué otra manera podrías reescribir la igualdad anterior? Si
¿Por qué es correcto reescribirla como lo hiciste?
207 + 50 = 150 + 107

Observe the following expression: 172 + 10 + 75 = 182 + 50 +25
Is the equality True or False? *Yes*
In which other way could you re-write the previous equality? *Yes.*
Why it is correct re-write the expression in such a way?
207 + 50 = 150 + 107

Figure 8. S3's equality rewriting.

This task shows that students accept equalities in the form of $a + b = c + d$. However, they transform them without relating the right and left sides except according to their totals. Their strategy is to maintain the equivalence through the two above-mentioned strategies. Hence, as observed in Figs. 7 and 8, there is not a natural inclination in students to re-express the equalities in such a way that both sides of the equalities look alike; for instance, 172 + 10 + 50 + 25 = 172 + 10 + 50 + 25, or 170 + 2 + 10 + 50 + 25 = 170 + 2 + 10 + 50 + 25, or even as 182 + 75 = 182 + 75. However, S1's work shows some structural sense according to the reviewed literature. Even when S1 and S3 write correct equalities, each side is considered on an individual basis. The left side is decomposed in one fashion and the right side in a different way, without showing explicitly the equality of both sides. S1 (Fig. 7) does not explain that both sides look more or less the same, he only mentions that the result (on both sides) is the same.

Conclusions

From the strategies students used, only one can be considered to illustrate a structural approach (S1 in Task 3, as shown in Fig. 7), even though the accompanying explanation refers to the result of both sides of the equality. The rest of the students' strategies are clearly computational, referring to the expected result, whether it involves operating with the numbers of an expression so as to calculate the result on both sides of an equality (the strategy observed in Tasks 1 and 3), or operating with the addends of one expression to obtain the addends of the other expression (the strategy observed in Task 2). In this sense, the presence or absence of the equals sign in the tasks seems not to influence the students in their chosen strategy.

Our results coincide with those reported by Pang and Kim (2018), in the sense that students tend to use computational strategies. This means that they show a strong operational sense, even when they accept equalities in the form of $a + b = c + d$. Nevertheless, this acceptance could be used as a base to promote the development of structural sense within algebraic thinking by designing tasks in such a way that students are explicitly requested not to pass through the intermediate step of computing the total for each expression in their work on judging the equivalence of the component expressions. Accepting expressions as bona fide numerical objects, and operating with and on these objects, is essential to seeking and expressing structure within the domain of arithmetic and thereby fostering the development of algebraic thinking.

Acknowledgements

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References

Asghari, A., & Khosroshahi, L. (2016). Making associativity operational. *International Journal of Science and Mathematics Education*, *15*(8), 1559-1577.

Cabañas, G., Salazar, V., & Nolasco, H. (2017). Tareas que potencian el desarrollo del pensamiento algebraico temprano en los libros de texto de matemáticas de primaria. In L. Aké y J. Cuevas (Coords.), Pensamiento algebraico en México desde diferentes enfoques. México: CENEJUS-UASLP.

Carpenter, T., Franke, M., & Levi, L. (2003). *Thinking mathematically. Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.

Carraher, D. W., & Schliemann, A. D. (2007). Early algebra and algebraic reasoning. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 669-706). Greenwich, CT: Information Age Publishing.

Kieran, C. (2018). Seeking, using, and expressing structure in numbers and numerical operations: A fundamental path to developing early algebraic thinking. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds. The global evolution of an emerging field of research and practice* (pp. 79-105). New York: Springer.

Linchevski, L., & Livneh, D. (1999). Structure sense: The relationship between algebraic and numerical contexts. *Educational Studies in Mathematics*, *40*(2), 173–196.

Mason, J., Stephens, M., & Watson, A. (2009). Appreciating mathematical structure for all. *Mathematics Education Research Journal*, *21*(2), 10-32.

Molina, M., & Ambrose, R. (2008). From an operational to a relational conception of the equal sign. Thirds graders' developing algebraic thinking. *Focus on Learning Problems in Mathematics*, *30*(1), 61–80. Retrieved from http://funes.uniandes.edu.co/550/1/MolinaM08-2885.PDF

Pang, J., & Kim, J. (2018). Characteristics of Korean students' early algebraic thinking: A generalized arithmetic perspective. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds*. *The global evolution of an emerging field of research and practice* (pp. 141-166). New York: Springer.

Schifter, D. (2018). Early algebra as analysis of structure: A focus on operations. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds. The global evolution of an emerging field of research and practice* (pp. 309-328). New York: Springer.

Schwarzkopf, R,, Nührenbörger, M., & Mayer, C. (2018). Algebraic understanding of equalities in primary classes. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds*. *The global evolution of an emerging field of research and practice* (pp. 195-212). New York: Springer.

Secretaría de Educación Pública (SEP). (2016). *Propuesta Curricular para la Educación Obligatoria 2016*. México, DF: SEP.

Singh, R., & Kosko, K. (2017). Exploring the structure of equivalence items in an assessment of elementary grades. In E. Galindo & J. Newton (Eds.), Proceedings of the 39th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Indianapolis: IN: PME-NA.

LEARNING FROM NAEP RELEASED ITEMS: U.S. ELEMENTARY STUDENTS' GRASP OF MULTIPLICATIVE RELATIONSHIPS

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Curricular analysis indicates that the U.S. students are introduced to multiplication in additive terms (as the replication of equal groups and repeated addition). But the virtue of this introduction for supporting students' understanding of the full range of multiplicative relationships is unclear. This paper reports an analysis of all grade 4 released NAEP items that expressed a multiplicative relationship, focusing on the range of relationship types, related quantities, and item difficulty. Results show that multiplicative items (1) frequently presented discrete quantities and equal group situations, (2) were easier when they involved discrete versus continuous quantities, and (3) within discrete items, equal groups and array items were easier than other types. These results provide measured support for the conjecture the additive introduction to multiplication may limit the development of elementary students' understandings.

Keywords: Number Concepts and Operations, Elementary School Education, Assessment and Evaluation

Objectives of the Study

Can analyses of released items from the National Assessment of Educational Progress (NAEP)—the "nation's report card"—and performance on those items shed light on questions of interest to mathematics educators, beyond what has been reported in summary volumes (e.g., Klosterman & Lester, 2007)? This analysis addressed that question for the broad content area of multiplicative relationships. "Multiplicative relationships" designates a set of tasks and situations, numerical and quantitative, that engage students in multiplicative reasoning and in carrying out numerical operations of multiplication or division. Given the introduction of multiplication and multiplicative relationships in grades 2 and 3 in U.S. classrooms, the analysis examined released items from grade 4.

Multiplicative relationships are significantly more diverse and challenging for students to master than additive relationships (Nunes & Bryant, 1996; Vergnaud, 1983, 1988). Students work on multiplicative relationships for many years, and this work is intensive in the upper elementary through middle school years. They are introduced to the operations of multiplication and division and some "applied" situations in grades 2 and 3. In the U.S., this introduction is essentially additive in nature. Quantitatively, multiplication is presented as involving the replication of equal-sized groups of discrete objects; numerically, it is presented as repeated addition (Smith, 2017). Research on students' understanding of different multiplicative relationships raises questions about whether this instructional foundation effectively supports students' access to and understanding of the full range of multiplicative relationships. This analysis of released NAEP items is one small step in a larger effort to address that critical question. The NAEP analysis complements the results of many prior studies that have assessed students' understanding of multiplicative relationships (as summarized in Greer [1992] and Harel & Confrey [1994]).

Two main questions focused the analysis: (1) what multiplicative situations (type and frequency) appear in released Grade 4 items and (2) how well do the performance results align with the additive introduction to multiplication? For example, are items presenting the replication of equal groups easier than other types of situations that are less amenable to replication and repeated addition?

Theoretical Perspective

Broadly, the larger inquiry that motivated this analysis was framed in constructivist terms. If learning is a social and psychological process of adapting prior understandings to cope with new and problematic mathematical situations (e.g., Wood, Cobb, Yackel, & Dillon, 1993), then how we introduce students to multiplicative relationships matters greatly for their subsequent work to engage the full multiplicative conceptual field (Vergnaud, 1983). Understanding multiplication and division means knowing where and why situations encountered in the world are "multiplicative," not additive. Mastery of the numerical aspects multiplication and division (i.e., basic facts, algorithms, and properties of operations) may contribute to understanding multiplicative relationships but is neither sufficient nor central.

A framework of types of multiplicative reasoning and quantitative situations that typically elicit such reasoning framed the analysis of the released items. *Quantities* are countable or measureable attributes of objects or collections of objects that are constituents of situations that student encounter and reason about in school and the everyday world (Smith & Thompson, 2008). To use their mathematical knowledge effectively in reasoning about and resolving these situations, students must consider the quantities involved and how they are related. Table 1 presents (and relates) different types of multiplicative reasoning and types of situations. But situations do not determine students' reasoning about them. Rather, the correspondence below reflects how prior research has characterized situations in relation to multiplicative reasoning. "Replication," an additive form of reasoning, has been included for "coverage" of the released items and because U.S. curricula treat replication as multiplicative.

Table 1: Types of Multiplicative Reasoning and Situations

Types of Multiplicative Reasoning	Types of Quantitative Situations
Replication	Equal groups; rectangular arrays; some area and volume situations
One-to-many	Unit conversion
Scaling	Comparison; price (discrete); rate/cost (continuous)
Successive partitioning	Folding; splitting
Composition	Cartesian product (discrete); area; volume (continuous)

Central to the analysis of situation types is the distinction between *discrete* and *continuous* quantities. Discrete quantities are sets of objects; their numerical value can be determined by counting. Continuous quantities are initially attributes of unsegmented objects (e.g., distances or lengths, areas, time periods between two events). Their measurement requires the selection and iteration of a unit (a smaller piece of the target quantity). Their numerical value is the number of such units that collectively fill up or "exhaust" the initial quantity.

Rectangular arrays and some area and volume situations are listed in Table 1 along with equal groups because (1) array situations support replication reasoning (when a row or column of objects is interpreted as a group) and (2) area and volume situations are often presented as arrays of squares or stacks of cubes that similarly support replication. Price indicates situations where some discrete number of items has been purchased. It is related to, but a special case of the more general set of rate/cost situations that accept measured quantities (e.g., “9.45 gallons of gas”). Cartesian product and area/volume are both multiplicative compositions, where the product differs from both factor quantities. In the former, “pants” and “shirts” are different quantities than “outfits.” Similarly, both rectangular area and prism volume are different quantities than the lengths from which they are composed.

The entries in Table 1 suggest that understanding multiplication involves grasping fundamentally different forms of multiplicative relationship that many students may see as conceptually distinct, especially early in their mathematical experience.

Methods

Items from nine Mathematics assessments (administered in 1990, 1992, 1996, 2003, 2005, 2007, 2009, 2011, and 2013) have been released for public examination (https://nces.ed.gov/nationsreportcard/nqt/). NAEP characterizes items by content area—*Number properties and operations* (NPO), *measurement* (M), *geometry* (G), *data, statistics and probability* (DSP), and *algebra* (A) and by format—multiple-choice, short constructed response, and extended constructed response. Released NAEP items are characterized by difficulty, as “easy” (performance ≥ 60% correct), “medium” (performance is between 40% and 59% correct), or “hard” (performance < 40% correct).

All 388 released grade 4 items were examined and coded by the author as either additive, multiplicative, or other. Multiplicative items presented one of the situation types listed above in Table 1 (including replication). *Additive* items presented one of three types of additive relationship—combine, separate, or compare. Additive items included area and volume/capacity items that presented collections of squares and cubes that supported counting and (additive) comparison. *Other* items presented content topics such as place value, ordering, estimating, fractions, graphing, and stating probability, where neither an additive nor multiplicative relationship between numbers or quantities was expressed.

Multiplicative items were found in all five content areas but were most common in NPO and M domains. All multiplicative items were first coded as “numerical” or “quantitative.” Numerical items presented written numerals and operations with minimal prose. “Quantitative” items were primarily expressed in words, where numerals were associated with quantities. Some items were presented entirely in prose (as “word problems”); others presented tables or figures with the written text, and the tabular or figural information was necessary for solving the item. Quantitative items were then coded for the type of quantities involved, discrete or continuous. Discrete items were further distinguished according to the type of relationship presented. Eight types proved sufficient for coding all discrete items: *Equal groups*, *equal shares*, *rectangular arrays*, *money*, *price*, *multiplicative comparison*, *unit conversion*, and *Cartesian product*. Continuous items presented length, area, or volume/capacity measurement situations where the quantity could not be evaluated with additive reasoning. Four types were sufficient for coding continuous items: *Unit conversion, equal shares, multiplicative comparison*, and *computation*.

Results

For a broad overview, Table 2 presents an overview of all 388 released grade 4 items by year. Column 3 lists the total number of multiplicative items; column 4 lists the total number of additive items; and columns 5–8 characterize the multiplicative items, first as numerical or quantitative items and then for quantitative items, as discrete and continuous.

Table 2: General Character of Grade 4 Items (1990-2013)

Year	All	Mult	Add	M; Number	M; Quan	M, Quan; Discrete	M, Quan; Continuous
2013	46	14	9	6	8	6	2
2011	49	20	14	7	13	10	3
2009	31	6	8	1	5	3	2
2007	54	14	12	3	11	9	2
2005	32	5	11	2	3	3	0
2003	59	23	9	3	20	16	4
1996	25	9	6	1	8	6	2
1992	59	20	13	6	14	12	2
1990	33	10	7	6	4	3	1
Total	388	121	89	35	86	68	18

The number of released items and the number of multiplicative items varied substantially across the nine assessments, but multiplicative items were generally more frequent than additive items. In most years, multiplicative quantitative items outnumbered multiplicative numerical items, often dramatically. Among quantitative items, those presenting discrete quantities were at least twice as frequent as those presenting continuous quantities.

More substantively, the distribution of discrete items across the eight types listed above was not uniform (Table 3).

Table 3: Frequency of Grade 4 Discrete Multiplicative Items by Type (1990-2013)

Year	Disc	Grps	Share	Array	Comp	Unit	Money	Price	C.P.
2013	6	2	0	0	2	1	0	0	1
2011	10	3	2	0	2	3	0	0	0
2009	3	1	0	0	0	1	1	0	0
2007	9	3	0	0	1	3	1	1	0
2005	3	2	0	0	1	0	0	0	0
2003	16	5	2	1	1	1	0	5	1
1996	6	2	0	2	0	1	0	1	0
1992	12	5	0	0	0	0	1	5	1
1990	3	0	0	1	0	0	0	2	0
Total	68	23	4	4	7	10	3	14	3

Note: “Grps” = equal groups, “Comp” = comparison, “Unit” = unit conversion, “C.P.” = Cartesian product

Equal groups was by far the most frequent type of situation presenting discrete quantities in multiplicative relationship, followed by price and unit conversion. Equal groups and unit conversion items always involved quantities with whole number values, where price items involved some whole number of items at a cost represented as decimal (e.g., \$0.87 or \$2.79).

Many authors have argued that multiplicative relationships are intrinsically more difficult for students to master than additive relationships (e.g., Vergnaud, 1983, 1988). Is this claim reflected in the NEAP results? Overall, the entries in Table 4 indicate an affirmative answer.

Table 4: Relative Difficulty of Grade 4 Additive and Multiplicative Items (1990-2013)

Year	All	Mult.	Add.	Avg % Corr; A	Avg % Corr; M	%A – %M
2013	46	14	9	46.8	43.6	3.2
2011	49	20	14	58.2	48.8	9.4
2009	31	6	8	57.4	52.8	4.6
2007	54	14	12	61.3	46.0	15.3
2005	32	5	11	53.3	58.2	-4.9
2003	59	23	9	57.8	50.8	7.0
1996	25	9	6	54.2	42.0	12.2
1992	59	20	13	45.2	40.4	4.8
1990	33	10	7	44.6	44.2	0.4
Total	388	121	89			

On seven of nine assessments, multiplicative items were more difficult than additive items, and in three (1996, 2007, and 2011) significantly so. The difference was negligible in 1990 and had the opposite sign in 2005.

But item difficulty across types of multiplicative relationships was the central focus of this analysis. Table 5 below reports percent correct for items presenting each type of discrete multiplicative relationship. The first value in right-most column lists the average percent correct for all items of that type. The values in parentheses are the average percent correct for a meaningful subset of those items, as explained below. The other columns list the number of items rated "easy," "medium," and "hard" and the percent correct for each item in those three categories.

Table 5: Difficulty of Grade 4 Discrete Multiplicative Items by Type (1990-2013)

Discrete sub-type	N	N Easy	%	N Med	%	N Hard	%	Avg %
Equal groups	23	4	75, 61, 80, 70	11	50, 53, 59, 53, 56, 46, 57, 50, 55, 47, 48	8	38, 35, 23, 21, 36, 39, 37, 37	49.0 (56.4)
Price	14	2	70, 62	4	58, 58, 53, 48	8	4, 35, 39, 31, 17, 8, 9, 21	39.3 (53.2)
Unit Conversion	10	6	66, 75, 65, 85, 78, 61	2	53, 44	2	17; 39	58.3 (62.9)
Compare	7	1	72	2	47, 47	4	32, 34, 24, 34	43.0 (36.3)
Array	4	1	79	2	50, 48	1	35	53.0
Money	3	1	60	1	58	1	20	46.0
Equal shares	4			2	47, 51	2	23; 38	39.8
Cartesian product	3			1	48	2	24; 28	33.3
Total	68	15		25		28		45.9

Items in three discrete types were somewhat easier overall (Unit Conversion, Array, and Equal Groups, in descending order), with average correct at or slightly above 50%. Six of the 10

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.

Unit Conversion items provided the conversion ratio explicitly in the item (e.g., 1 qt. = 2 cups). The two "hard" conversion items (39% and 17% correct) both presented 3:1 ratios, where the other eight items presented conversion ratios of 2:1, 5:1, 10:1, or 100:1. Arrays were either represented directly in diagram (n = 1) or described in words (n = 3). The "easy" item (79% correct) described a particularly familiar array—two rows of six cookies on a cookie sheet.

As shown, Equal Groups and Price items were often difficult; eight items of both types were "hard." But within both types, hard items often involved two or more steps, where a multiplicative relationship was involved in at least one step. For example, some two-step Price items asked for the change received for the purchase of a set of items at a given price when a specific bill was given for payment—requiring both multiplicative and additive reasoning. Some two-step Equal Groups items introduced more than one group (e.g., students in a class and buses with maximum capacity for students). Most multi-step items were more difficult than single step items of the same type. The average percent correct for the thirteen single-step Equal Groups items was 56.4% (as shown), where the corresponding average for the ten multi-step items was 39.2%. Similarly, the six single-step Price items were considerably easier (average 53.2% correct) than the eight multi-step items (average 28.9% correct).

By contrast, Compare, Equal Shares, and Cartesian Product items were more challenging, at 43%, 39.8%, and 33.3% average correct, respectively. The majority of Compare and Cartesian Product items were "hard," even when six of the seven Compare items involved a 2:1 ratio. Only one, presenting ten stars and five triangles in a 3 by 5 array and four possible ratios, was "easy" (72% correct). Without that item, average correct fell to 36.3%. Of the four Equal Shares items, none called for simply distributing some discrete quantity equally to a given number of recipients. One "hard" item (38% correct) asked students to distribute 24 wheels to bikes and wagons in two different ways; one "medium" item (47% correct) required interpreting the remainder after equal sharing. Cartesian Product items were difficult even though support was provided for solving two of the three (i.e., items provided the solution for smaller factors).

Finally, Table 6 presents the performance on the N = 18 multiplicative items that presented length, area, or volume/capacity quantities and within each quantity, the type of multiplicative relationship involved.

Table 6: Difficulty of Grade 4 Continuous Multiplicative Items by Type (1990-2013)

Continuous Sub-type	N	N Easy	%	N Med	%	N Hard	%	Avg %
Length	7			1		6		30.6
compare	2						27, 33	30.0
equal shares	4				47		27, 26, 23	30.8
unit convert	1						31	31
Area	6	1		1		4		36.5
compute rect	4						23, 24, 24, 19	22.5
compare	2		78		51			64.5
Volume/capacity	5	2		1		2		47.4
unit convert	3		67, 61				32	53.3
compute stack	1				56			56
compare	1						21	21
Total	18	3		3		12		37.2

Overall, the results show that multiplicative items presenting continuous quantities were generally more difficult than those with discrete quantities. Six of seven length items were "hard," as were four of six area items—even when the area items involved a single-step. However, the two items comparing areas— both presenting sectors of partitioned circles—were markedly easier (average 64.5% correct) than the corresponding discrete comparison items (n = 7; average 43.0% correct). The three volume/capacity unit conversion items were somewhat more difficult (average 53.3% correct) than the discrete unit conversion items (n = 10; average 58.3% correct). As was true for the discrete items, all three continuous items stated the conversion ratio in the item text.

Discussion

The analysis was revealing in two principal ways. First and generally, the careful examination of released NAEP items supported a finer-grained analysis of U.S. grade 4 students' successes and challenges—as a proxy measure of national understanding—than the published volumes have thus far (Klosterman & Lester [2007] and prior volumes in that series). The released item set provided greater access to the items as presented to students, their difficulty, and the details of item performance. Second and more specific to this inquiry, the analysis provided a measure of empirical support for the concern that the additive introduction of multiplication may present challenges for the growth of students' understanding beyond equal groups of discrete objects and repeated addition. However, that support was mixed and complicated by many factors outside the frame of the analysis. Since many factors other than quantity and relationship type likely contribute to item difficulty, the analysis shows the difficulty inherent in establishing what makes an item easy (or challenging) for students.

Some of the results are consistent with (a) the curricular introduction of multiplication as the replication of equal groups of discrete quantities and repeated addition and (b) the concern that that an additive foundation likely makes extension to a wider set of situations problematic. Additive items were on average easier than multiplicative items. Second, items posing multiplicative relationships among discrete quantities were much more frequent and generally easier than item involving continuous quantities. Third, situations involving Equal Groups were the most frequent of type of discrete multiplicative relationship and were easier than other discrete item types that are less amenable to interpretation as equal groups—specifically, Comparison and Cartesian Product. But this was not always the case; Unit Conversion items were easier on average than both Equal Groups and Array items. Overall, the results are consistent with and do not remove the concern that the additive introduction to multiplication and multiplicative relationships may support initial access to multiplicative relationships, but at the cost of later conceptual challenges as application extends both quantitatively and numerically (e.g., to fractions, decimals, and negative numbers).

There are numerous limitations to this analysis and more broadly to using NAEP released items to address questions about the effects of curricular approaches on student learning. Perhaps the most important is that the released item set stands in uncertain relationship to the larger corpus of items where NAEP has performance data. The conditions under which a NAEP item is released to the public are unknown. Also, where the type of multiplicative item and the type of related quantities may well influence the item difficulty, other factors do so as well, including the numerical values of the quantities (even within the set of whole numbers), the length and clarity of item prose, and the nature of support provided (e.g., stating conversion ratios or not). Third, multi-step problems present challenges for characterizing items as additive or multiplicative and for judging the sources of difficulty. Fourth, though the capacity to explain one's reasoning may be a better indicator of understanding than producing the right numerical answer, the number of released NAEP items requiring explanation (that is, both short and extended constructed-response items) have been small in number, in this target content area and likely in most others. The smaller the item set, the more perilous any conclusion drawn from such data becomes. Last, the analysis has been completed by a single person and the coding scheme must be shown to be reliable.

Despite these limitations, released NAEP items are an underused resource for mathematics education researchers who wish to address questions of learning at a national level. Where it is unlikely that any similar analysis of items selected for a given topic or response type (e.g. constructed-response) will resolve questions about student understanding, they may contribute to inquiries that draw on multiple sources of evidence. NAEP released item data, reflecting such a large and nationally-representative sample of students, is a unique source of evidence. Its analysis can provide either general support (as in this case), no support, or contradictory evidence for a given hypothesis.

References

Davis, B. & Simmt, E. (2006). Mathematics-for-teaching: An ongoing investigation of the mathematics that teachers (need to) know. *Educational Studies in Mathematics, 61*, 293–319.

Greer, B. (1992). Multiplication and division as models of situations. In D. A. Grouws (Ed.), *Handbook of Research in Mathematics Teaching and Learning* (pp. 276–295). New York, NY: MacMillan Publishing.

Harel, G. & Confrey, J. (Eds.). (1994). *The development of multiplicative reasoning in the learning of mathematics*. Albany, NY: SUNY Press.

Klosterman, P. & Lester, F. K. (Eds.). (2007). *Results and interpretations of the 2003 Mathematics Assessment of the National Assessment of Educational Progress*. Reston, VA: National Council of Teachers of Mathematics.

Nunes, T. & Bryant, P. (1996). *Children doing mathematics*. Oxford, UK: Blackwell Publishers.

Schwartz, J. L. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. Behr (Eds.), Number concepts and operations in the middle grades (pp. 41-52). Reston, VA: NCTM.

Smith, J. P. (2017). Conceptual deficits in curricular introductions to multiplication. In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 156–159). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Smith, J. P. & Thompson, P. W. (2008). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 95–132). Mahwah, NJ: Lawrence Erlbaum Associates.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127–174). New York, NY: Academic Press.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 141–161). Reston, VA: National Council of Teachers of Mathematics.

Vergnaud, G. (1994). Multiplicative conceptual field: What and why? In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 41–59). Albany, NY: SUNY Press.

Wood, T., Cobb, P., Yackel, E., & Dillon, D. (Eds.) (1993). Rethinking elementary school mathematics: Insights and issues (selections). *Journal for Research in Mathematics Education Monograph*, no. 6. Reston, VA: National Council of Teachers of Mathematics.

A LONGITUDINAL STUDY: THE EFFECTS OF TIME AND EARLY INSTRUCTION ON STUDENTS' INTEGER LEARNING

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Twenty-nine second graders received integer instruction (instruction-only) and three years later participated in our study again as fifth graders (instruction + time). In addition, we analyzed data from an additional 73 fifth graders to investigate the effect of time without having had the second-grade instruction (time-only). The findings indicate no significant difference between instruction-only and time-only groups, a significant improvement from the instruction-only to instruction + time group on integer addition and subtraction problems, and a significantly better performance for the instruction + time group on integer subtraction problems.

Keywords: Number Concepts and Operations, Cognition, Elementary School Education

The transition from a whole to negative number understanding requires conceptual change and sufficient time to process (Vosniadou, Vamvakoussi, & Skopeliti, 2008). Children as early as first grade can reason with negative numbers (e.g., Bofferding, 2014); however, they often do not learn negative number operations until seventh grade (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010). Therefore, upper elementary students may generate a conception of negative numbers based on their whole number understanding (e.g., Bofferding, 2014) without getting formal feedback. For example, students may always subtract a smaller number from a larger number (e.g., Murray 1985). In fact, Aqazade (2017) found fifth graders' preconceptions limited their ability to refine their conceptions of negative numbers and achieve higher scores compared to second graders. Through the conceptual change lens, we explored the role of time and early instruction on students' performance on integer addition and subtraction problems.

1. After being exposed to integer operations in second grade (instruction-only group, Year 1 of the study), how does students' performance on integer addition and subtraction problems differ three years later as fifth graders (instruction + time group, Year 4 of the study)?
2. In Year 4, how do the fifth graders who had integer instruction in second grade (instruction + time group) perform on integer addition and subtraction problems compared to fifth graders who did not have the instruction (time-only group)?

Framework Theory Approach to Conceptual Change

The framework theory for conceptual change refers to enrichment or restructuring of children's existing knowledge to accommodate new knowledge. Consistent with this, students initially interpret integer arithmetic through a whole number lens (Bofferding, 2014; Vosniadou, Vamvakoussi, & Skopeliti, 2008).

Interpretation of the Minus Sign, Addition, and Subtraction

The minus sign holds three different meanings: binary, symmetric, and unary (Vlassis, 2004). With the binary meaning or subtraction sign, students treat the minus sign as subtraction (e.g., solving -1 + 8 as 8 – 1 = 7; Bofferding, 2010). The symmetric meaning designates taking the opposite (Vlassis, 2004); with this interpretation, students may treat negative numbers as positive and add the negative sign to their answer (e.g., solving -7 – -7 = -0; Bofferding 2010). The unary meaning indicates negative numbers, as seen when students start with the negative number and count towards negative or positive directions (Bofferding, 2010). With positive numbers, subtraction involves a decrease in magnitude (Vosniadou, Vamvakoussi, & Skopeliti, 2008), but with integers, subtraction can result in a decrease or increase (Bofferding, 2014). Students' ideas of only subtracting the smaller number from a larger one, subtraction as moving downward, and addition as moving upward are challenged with integers (e.g., Bofferding, 2014; Murray, 1985).

Methods

Participants and Setting

We have collected data for four years as part of a larger study. Twenty-nine students participated as second graders in Year 1 of the study (*instruction-only group*) and three years later as fifth graders in Year 4 of the study (*instruction + time group*). Seventy-three additional fifth graders were in Year 4 who did not have integer instruction in their second-grade year (*time-only group*). All students were recruited from two rural, elementary schools in the Midwest (about 30% were English Language Learners, and 80% qualified for reduced-price lunch).

Design and Analysis

In Year 1, second graders (instruction-only group) completed a pretest on integer addition and subtraction. Then over three sessions, groups of two to four students explored integer order and symbols and played a movement game on a number path. Next, students individually compared integers and received immediate feedback. After the sessions, students participated in a whole-class lesson on integer addition and subtraction. Then, they completed a posttest. After both tests, we interviewed 20% of the students about their strategies. In Year 4, fifth graders (time-only and instruction + time groups) took an interviewed pretest involving integer addition and subtraction problems. We focus on the data from the Year 1 posttest and Year 4 pretest.

For analysis, we used a repeated measure ANOVA to compare the instruction-only and instruction + time groups' performance on the common test items (i.e., 1 – 4, -2 – -6, -5 – -5, -1 + 8, -9 + 2, and 7 + -3), which provided information about the role of time. Afterwards, we explored students' most frequent answers and strategies on the items. We used a median test to compare students' performance in the instruction-only group from Year 1 to students' performance in the time-only group from Year 4 on the common items to investigate the role of time only versus early instruction only. Finally, we used a median test to compare time-only and instruction + time groups on all 29 integer problems from the Year 4 pretest.

Findings

Research Question One: Role of Time

Students who participated in Year 1 as second graders performed significantly better on the addition and subtraction problems when solving the same problems three years later as fifth graders (addition, $F(1, 27) = 5.32$, $p = .029$ and subtraction, $F(1, 27) = 12.385$, $p = .002$).

Instruction-only versus instruction + time, solving 1 – 4. A majority of students responded 3, incorrectly commuting the problem; the average percentage of students answering 3 remained stable over time. Moreover, 11 (37%) students answered 3 in both years, suggesting time influenced little to change this conception of subtraction. A larger percentage of students answered -3 in Year 4 than Year 1, suggesting some students changed their thinking. In fact, 6 (21%) students correctly changed their answer from -5, 3, or -2 to -3. Those who changed from -5 to -3 indicated attention to the operation, those who changed from -2 to -3 indicated more accurate counting, and those who changed from 3 to -3 indicated advances in integer subtraction.

Table 1: Most Frequent Answers on Common Integer Subtraction Problems

Groups	1 – 4			-2 – -6			-5 – -5			
	0	3	-3	4	-4	-8	0	-0	10	-10
Instruction + Time	3%	55%	38%	34%	48%	14%	86%	10%	0%	3%
Instruction-Only	3%	52%	24%	28%	17%	34%	59%	3%	10%	28%
Time-Only	0%	70%	22%	21%	47%	23%	58%	25%	0%	14%

Note. Grey cells indicate the most frequent response in each group for each problem.

Instruction-only versus instruction + time, solving -2 – -6. For students in the instruction-only group, -8 was their most common response. Students answering -8 often added the absolute values of the integers and made the answer negative, a symmetric meaning of the minus sign. Likewise, they could get -4 by solving 6 – 2 = 4 and making the answer negative. Another strategy to get -8 was to start at one number and move in the negative instead of positive direction, a unary meaning of the minus sign. Students could get the correct response using their whole number understanding and solving 6 – 2 = 4; however a correct strategy would be to start at -2 and move in a less negative direction. Among students who were in both instruction-only and instruction + time groups, 5 (17%) students changed their answer from -8 (Year 1) to -4 (Year 4), and 4 (14%) changed their answer from 4 to -4. In addition, 7 (24%) students correctly changed their answers from -4, -8, and 8 to 4. Those who changed from -4 or -8 to 4 demonstrated an understanding of the binary and unary minus signs and directed subtraction; those who changed from 8 to 4 realized the problem involved subtraction, not addition.

Instruction-only versus instruction + time, solving -5 – -5. Students in the instruction-only group more often responded -10 compared to students in the instruction + time group. Students could answer -10 based on two solution strategies: using the unary meaning of the minus sign without correct directional movement or using the symmetric meaning with an incorrect use of operation. Compared to 1 – 4 and -2 – -6, students performed higher on this problem. Among students in both instruction-only and instruction + time groups, 15 (52%) students continued to answer 0, and 10 (34%) changed their answers from -10, 10, or -0 to 0. Those who changed from 10 to 0 noticed that the problem involved subtraction. Those who changed from -10 or -0 no longer relied on the symmetric meaning of the minus sign. Even though few students responded -0 in both groups, 2 (7%) students in the instruction-only group changed their correct answer to -0 three years later. This often occurred for students who initially ignored the negatives.

Research Question Two: Role of Early Instruction

Comparing the time-only versus instruction-only group's scores on the common items with the median test indicated no significant difference between their performances. These findings suggest the effects of early instruction could play a similar role as the effects of time without early instruction. Students' performance in the instruction + time group compared to the time-only group on integer addition and subtraction problems revealed no significant difference on the integer addition problems and a significant difference favoring the instruction + time group on the subtraction problems ($X^2 = 4.254$, $p = .039$) (see Table 1).

Instruction + time versus time-only, solving 1 – 4. Students in the time-only group answered 3 more frequently than students in the instruction + time group. Further, 7% of students in the time-only group provided negative answers close to -3 (i.e., -2, -1, -4).

Instruction + time versus time-only, solving -2 – -6. Interestingly, -4 was the most regular answer among both groups. The slightly more frequent responses of -8 among students in the time-only group compared to the instruction + time group is related to their lower percentage of correct answers. However, students in the time-only group answered with -0 in two instances, indicating a symmetric meaning of the minus sign. Actually, students in the time-only group responded with negative answers more often than the instruction + time group (73% versus 55%). Students in this group also provided positive answers including 7, 6, and 2.

Instruction + time versus time-only, solving -5 – -5. Even though both groups did not have 10 as their answers, the time-only group had positive answers such as 5 and 2. An answer of 5 could result if considering one of the -5's as worth as 0 and interpreting the other one as a positive five. Similar to -2 - -6, students in the time-only group had more negative numbers compared to students in the instruction + time group. For example, students in the time-only group more frequently answered -0, representing a symmetric treatment of the minus sign.

Implications and Discussion

We investigated the effects of time and early instruction on negative number operations on students' integer addition and subtraction performance. Consistent with conceptual change theory, time played an important role (Vosniadou, Vamvakoussi, & Skopeliti, 2008). The instruction-only group shifted toward a unary conception of the minus sign and interpreting the subtraction operation as corresponding to a directional movement three years later in the instruction + time group. The significant difference between the instruction + time and time-only groups on subtraction problems suggests early integer instruction can support understanding. The time-only group provided more negative answers, which could correspond to overusing the symmetric meaning of the minus sign. Overall, an early exposure to negative numbers was beneficial for students' integer learning, which could conceptually develop through time.

Acknowledgement

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References

Aqazade, M. (2017). *Time for learning integers: Changes in second and fifth graders' integer understanding.* (Unpublished master's thesis). Purdue University, West Lafayette, IN.

Bishop, J. P., Lamb, L. L., Philipp, R. A., Schapelle, B. P., & Whitacre, I. (2011). First graders outwit a famous mathematician. *Teaching Children Mathematics, 17*(6), 350-358.

Bofferding, L. (2010). Addition and subtraction with negatives: Acknowledging the multiple meanings of the minus sign. *Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 703-710). Columbus, OH: The Ohio State University.

Bofferding, L. (2014). Negative integer understanding: Characterizing first graders' mental models. *Journal for Research in Mathematics Education, 45*(2), 194-245.Murray, J. C. (1985). Children's informal conceptions of integer arithmetic. *Proceedings of the Ninth Annual International Conference for the Psychology of Mathematics Education*, (pp. 147-153). Utrecht, The Netherland.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics.* Washington, D.C. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Vlassis, J. (2004). Making sense of the minus sign or becoming flexible in 'negativity'. *Learning and Instruction, 14*(5), 469-484.

Vosniadou, S., Vamvakoussi, X., & Skopeliti, I. (2008). The framework theory approach to the problem of conceptual change. In S. Vosniadou (Ed.), *International handbook of research on conceptual change* (pp. 3-34). New York, NY: Routledge.

SECOND AND FIFTH GRADERS' INTEGER SUBTRACTION PERFORMANCE: LEARNING FROM CONTRASTING WORKED EXAMPLES

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We explored 105 second and fifth graders' performance on integer subtraction problems before and after analyzing different contrasting worked examples involving integers. The students, as part of a larger study, completed a pretest, were randomly assigned to intervention groups -- which differed in the problems they compared -- and participated in two small-group sessions, one whole-class lesson on integer subtraction, and a posttest. The students made progress in solving integer problems from pretest to posttest. In this paper, we focus on students who provided only positive or zero answers on their pretest. The trends in their posttest answers show important differences among the intervention groups concerning their use of number order and interpretations of operations in integer arithmetic.

Keywords: Number Concepts and Operations, Cognition, Elementary School Education

Theoretical Framework

Based on a framework theory view of conceptual change, children initially interpret mathematics numerical problems through a whole number lens, due to their experiences interacting with objects (Vosniadou, Vamvakoussi, & Skopeliti, 2008). Learning about negative numbers can challenge their framework theory of numbers in several ways, including their interpretation of minus signs and the meaning they ascribe to operations. Older students may refer to rules about the use of signs when solving problems with negative numbers rather than articulating conceptual justifications for their solutions (Bishop, Lamb, Philipp, Whitacre, & Schappelle, in press), possibly because they are less willing to change their prior conceptions (Aqazade, Bofferding, & Farmer, 2017). However, some research has shown that young children are willing to play with and make sense of negative numbers (Aze, 1989). We investigate how different problem contrasts might promote conceptual change with second and fifth graders.

Interpretations of Minus Signs and Operations

Using their framework theory for numbers, children's initial interpretations of the minus sign correspond to the subtraction operation, *the binary meaning* (Vlassis, 2004). With this interpretation, they often treat negatives as subtraction signs (e.g., solving 9 – -2 as 9 – 2 – 2 = 5; Bofferding, 2010) or ignore them completely. However, there are two additional meanings of the minus sign. The *symmetric* meaning of the minus sign corresponds to taking opposite of a number (Vlassis, 2004), as seen when students operate with integers as positive and make the answer negative or draw comparisons between positive and negative forms (e.g., solving -7 – - 7 = -0; Bofferding, 2010). The *unary* meaning of the minus sign designates a negative number (Vlassis, 2004), as seen when students start at a negative number and count in a correct or an incorrect direction (Bofferding, 2010).

Students learn positive number subtraction as counting down, which corresponds to a decrease in numerical magnitude (Vosniadou et al., 2008). However, with negative numbers, students need to learn that subtracting a negative number corresponds to counting up (Bofferding, 2014) and that subtraction could result in a decrease or increase in magnitude from the initial number. One way to help students distinguish the roles of the minus sign in operations involving negative numbers is through analyzing contrasting cases with worked examples.

Worked Examples and Contrasting Cases

Worked examples can help reduce the cognitive load of managing many new concepts or steps at a time so that students can focus on understanding the problems and using them to solve related problems (e.g., Hilbert, Renkl, Kessler, & Reiss, 2008). Likewise, contrasting cases can help students to notice important differences between problems to discern an underlying structure (Schwartz, Tsang, & Blair, 2016), learn new solution methods (e.g., Rittle-Johnson & Star, 2011), refine their prior understanding, and promote conceptual understanding (e.g., Namy & Gentner, 2002). With subtraction, helpful contrasts could include comparing positive minus positive problems with a) negative minus negative, b) negative minus positive, or c) positive minus negative problems. This research investigated the following questions: When learning integer subtraction, how do these contrasts and a short lesson benefit second and fifth grade students? What contrasts best help them make sense of integer subtraction problems?

Methods

Participants and Design

Participants included 95 second graders (from the larger study with 133 second graders) and 10 fifth graders (from a larger group of 74 fifth graders) from two rural, elementary schools in the Midwest (Free and reduced lunch: 30% and English Language Learners: 79%). This subset consistently answered with positive numbers or zero on the pretest (which is why the number of fifth graders in the subsample is so small). After a pretest, students were randomly assigned to a control or one of three experimental groups, participated in two small-group sessions, engaged in a whole-class lesson on integer subtraction, and took a posttest.

Data Sources and Procedure

Pretest and posttest. We focused our analysis on four, single-digit integer addition and 17 subtraction problems involving negative integers that were identical on both tests. We interviewed at least 20% of the larger sample after each test to learn more about their reasoning.

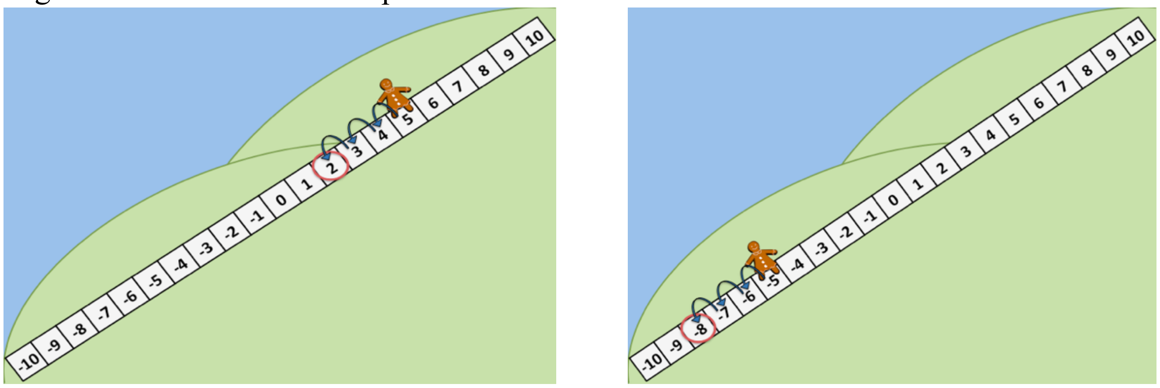
Small-group sessions. In two small-group, 20-minute sessions, groups of 2-3 students analyzed sets of contrasting integer addition problems (control group: Add) or subtraction problems (experimental groups: Negative Minus Negative, Negative Minus Positive, and Positive Minus Negative, named based on the problem type they initially saw contrasted with 5 – 3 in their first session). See Table 1 for the groups' contrasting examples seen each session.

Table 1: Examples of problems control and experimental groups compared each session

	Groups	First Session	Second Session
2nd	**Add (control)** (*n* = 25)	P + P vs. P + N	N + P vs. N + N
	Negative Minus Negative (*n* = 21)	P – P vs. N – N	N – P vs. P – N
	Negative Minus Positive (*n* = 26)	P – P vs. N – P	P – N vs. N – N
	Positive Minus Negative: (*n* = 23)	P – P vs. P – N	N – N vs. N – P
5th	**Add (control)** (*n* = 4)	P + P vs. P + N	N + P vs. N + N
	Negative Minus Positive: (*n* = 6)	P – P vs. N – P	P – N vs. N – N

Note: N = Negative number, P = Positive number

In their small-group sessions, students discussed the similarities and differences between the problems and pictures in contrasting, worked-examples (see Figure 1). Students solved related integer addition or subtraction problems at the end of each session.



Correct: 5 – 3 = 2 Correct: -5 – 3 = -8

Figure 1. Comparison of worked examples and corresponding problems that students analyzed.

Whole-class instruction. During the 30-minute lesson, students helped solve problems on a number path, where subtracting a negative number corresponded to moving in a less negative direction (or up) and subtracting a positive number corresponded to moving in a less positive direction (or down). We ordered the problems to emphasize differences (e.g., 3 – 3 then 3 – 4).

Analysis

First, we coded students' strategies based on the possible ways they could have obtained their answers. Then, we looked at their pattern of responses (e.g., Widjaja, Stacey, & Steinle, 2011) and only gave credit for correct answers if they *followed a pattern of using knowledge of negatives.* For example, a student who solved 1 – 4 = 5, -5 – 9 = 14, and 9 – -2 = 11 was classified as adding the absolute values of two integers. Therefore, we did not give credit for 9 – -2 = 11. When possible, we used our interview data to check the identified pattern of responses.

Results, Discussion, and Implications

On the pretest, based on students' overall strategies and verbal reports, the only problems students got correct happened without knowledge of negatives, so we treated all pretest scores as zero. On the posttest, based on the same analysis, students improved from the pretest. Table 2 shows students' average number correct for the posttest subtraction items.

Table 2: Overall subtraction average scores on the posttest across groups

	2nd Add (n = 25)	**2nd NMN** (n = 21)	**2nd NMP** (n = 26)	**2nd PMN** (n = 23)	**5th Add** (n = 4)	**5th NMP** (n = 6)
Overall Subtraction	0.88	0.62	**1.23**	0.43	**0.75**	0.17

Note. Add = Control group, NMN = Negative Minus Negative group, NMP = Negative Minus Positive group, PMN = Positive Minus Negative group,

Overall, second graders in the Negative Minus Positive experimental group had the highest performance across the majority of items. The performance of the groups followed an interesting pattern based on their answers. Second graders in the Positive Minus Negative group frequently reversed the order of the numbers–49% on average for the P – P problems and 35% on average for N – P problems–and subtracted the smaller absolute value from the larger, indicating a strong framework theory conception. The Add and Negative Minus Negative groups used this strategy frequently on two of the P – P problems (solving 1 – 4 with 3 and 0 – 9 with 9) and one of the four N – P problems (solving -7 – 4 with 3); otherwise, the majority of second graders answered 0, suggesting they ignored the negative sign without switching the order of the numbers. The second graders in the Negative Minus Positive group primarily answered 0 for for 1 – 4 (27%) and 4 – 5 (38%) and answered 9 for 0 – 9 (62%); overall, their performance on these items was double that of their second grade peers in other groups. Although they were most likely to answer 4 for -5 – 9 (27%), on -3 – 3, 46% answered 0, followed by 19% who correctly answered -6, and on -2 – 3, their top answers were evenly split amongst -5, 0, and 1 (each 19%). This group's initial comparisons better preserved their understanding of subtraction in terms of which direction to count when subtracting a positive number and helped them develop a better sense of the unary meaning of the minus sign. The fifth graders in the Negative Minus Positive group were most likely to reverse the numbers for 1 – 4 and 4 – 5 (42% on average) but answered 0 for 0 – 9 (50%). On -5 – 9, they were equally likely to answer 4 or -4 (33% each), and 50% answered -2 for -7 - -9, suggesting they were more likely to use the symmetric meaning of the minus sign. Even with our conservative analysis, students did make important changes from pretest to posttest in just over an hour of engagement; however, it may be worthwhile to have students participate in additional sessions, so that all problem types can be better contrasted.

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References

Aqazade, M., Bofferding, L., & Farmer, S. (2017). Learning integers addition: Is later better? In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 219-226). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Aze, I. (1989). Negatives for little ones? *Mathematics in School, 18*(2), 16-17.

Bishop, J. P., Lamb, L., Philipp, R. A., Whitacre, I., & Schapelle, B. P. (in press). Students' thinking about integer open number sentences. In. L Bofferding & N. Wessman-Enzinger (Eds.), *Exploring the Integer Addition and Subtraction Landscape: Perspectives on Integer Thinking.* Springer.

Bofferding, L. (2010). Addition and subtraction with negatives: Acknowledging the multiple meanings of the minus sign. *Proceedings of the 32nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Columbus, OH, October 28-31.

Bofferding, L. (2014). Negative Integer Understanding: Characterizing First Graders' Mental Models. *Journal for Research in Mathematics Education*, 194-245.

Hilbert, T. S., Renkl, A. Kessler, S., & Reiss, K. (2008). Learning to prove in geometry: Learning from heuristic examples and how it can be supported. *Learning and Instruction, 18*, 54-65.

Namy, L. L., & Gentner, D. (2002). Making a silk purse out of two sow's ears: Young children's use of comparison in category learning. (D. S. Lindsay, Ed.) *Journal of Experimental Psychology: General, 131*(1), 5-15.

Rittle-Johnson, B., & Star, J. R. (2011). The power of comparison in learning and instruction: Learning outcomes supported by different types of comparisons. In J. P. Mestre, & B. H. Ross (Eds.), *Psychology of Learning and Motivation: Cognition in Education* (Vol. 55). Elsevier.

Schwartz, D. L., Tsang, J. M., & Blair, K. P. (2016). *The ABCs of how we learn: 26 scientifically proven approaches, how they work, and when to use them.* W. W. Norton & Company.

Vlassis, J. (2004). Making sense of the minus sign or becoming flexible in 'negativity'. *Learning and Instruction. Special Issue*: The Conceptual Change Approach to Mathematics Learning and Teaching, 14(5), 469-484.

Vosniadou, S., Vamvakoussi, X., & Skopeliti, I. (2008). The framework theory approach to the problem of conceptual change. In S. Vosniadou (Ed.), *International handbook of research on conceptual change* (pp. 3–34). New York, NY: Routledge.

Widjaja, W., Stacey, K., & Steinle, V. (2011). Locating negative decimals on the number line: Insights into the thinking of pre-service primary teachers. *Journal of Mathematical Behavior, 30*, 80-91.

COMPARISONS WITH CLOSEST AND MOST: SECOND AND FIFTH GRADERS' CONCEPTIONS OF INTEGER VALUE

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Elementary students have difficulty learning integer values as the new knowledge conflicts with their whole number understanding. We administered a pretest and a posttest to 204 second and fifth graders and collected their answers to three types of integer comparison problems: which number is closest to 10, is most positive, and is most negative among three integers. Results showed that second and fifth graders had difficulty determining which of three negative numbers was most positive. They found it relatively easy to determine which of three positive numbers was most positive and which of three negative numbers was most negative. Students had some improvement in their integer value mental models, although their mental model shifts varied by the question phrasing.

Keywords: Number Concepts and Operations, Cognition, Elementary School Education

Traditional numerical comparison questions involving negative numbers largely test students' knowledge of convention as opposed to their conceptions of the numbers. These questions typically ask students to identify which of two integers is greater, larger, or most (e.g., Bofferding, 2014; Whitacre et al., 2017). Determining the larger of two negatives requires that students know that this question prioritizes order and not magnitude (i.e., even though -4 has a greater magnitude than -2, -2 is considered larger because it is further to the right on a number line). However, comparison questions rarely ask specifically about order (i.e., which number is closest to another number). In order to capture students' understanding of integer comparisons without forcing them to rely on convention, we explored the following research questions.

1. How do second and fifth graders' integer comparison performance differ based on the language of the comparisons (question phrasing: closest to 10, most positive, and most negative) and the numbers involved (all positive, all negative, positive or zero and negative)?
    a. In terms of number of students to correctly compare the integers?
    b. In terms of their mental models for integer values?
2. For second and fifth graders with initial value mental models on the pretest, how do their mental models change, depending on the question phrasings?

Theoretical Framework

Bofferding (2014) categorized students' mental model models of integer order and values into initial, synthetic, and formal, along with two transitional levels. Students exhibiting initial integer mental models ignore negative signs by operating with negative numbers as if they are positive. Students exhibiting a transition I mental model can differentiate between negative and positive numbers but sometimes treat the negative numbers as zero or positives. Students exhibiting synthetic integer mental models perceive the negative numbers as numbers below zero; though, they consider negative numbers with larger absolute values as greater than negative numbers with smaller absolute values (e.g., -6 > -1). In between the synthetic and formal integer mental models are transition II mental models, when students sometimes interpret negative numbers at the synthetic level but sometimes interpret them correctly. Students holding formal mental models know that positive and negative numbers are symmetric around zero and that negative numbers with bigger magnitudes have smaller values.

Students need to shift from a categorical to a continuum-based understanding of integers. With the introduction of negative numbers, the language use of more and less associated with numbers becomes challenging. Students with a categorical understanding might think that *more* corresponds to a greater absolute value (e.g., -5 > 3) and *less* corresponds to a smaller absolute value. They need to develop an understanding of directed magnitude language on a continuum where *more* has two meanings: *more positive* or *more negative* (Bofferding, 2014).

Methods

Participants and Setting

Participants came from three rural elementary schools in the midwestern United States where 74% to 87% of the students in each school received free or reduced price meals and 25% to 38% of the students were English Language Learners. Overall, 102 second graders and 102 fifth graders participated in this study to completion.

Design and Materials

The study involved a pretest, four small group sessions, a whole-class lesson, and a posttest. On the pretest, we conducted a whole-class written test including comparisons with the three question phrasings, and students chose answers from three positive numbers, three negative numbers, and a mixed set of positive or zero and negative numbers. Then students, who were randomly assigned to small groups, analyzed pairs of worked examples about integer operations. All groups saw similar problems but in different orders, and all students heard the language of *more positive* and *more negative* as part of the sessions. Next, students received a 30-minute, whole-class lesson that emphasized language use in integer operations. Finally, we conducted the posttest, which contained the same comparison problems as in the pretest.

Analysis

First, we calculated the number of students at each grade level who correctly answered each comparison problem. Next, based on each students' pattern of responses, we determined students' integer and value mental models for each question phrasing by using Bofferding's (2014) mental model framework. For example, we classified students who determined which integer was closest to 10, most positive, or most negative by selecting the largest absolute value for each question as exhibiting initial mental models. We aimed to find patterns in the shifts of students' mental models between pretest and posttest.

Results

Fifth and Second Graders' Gains from Pretest to Posttest

Both on the pretest and the posttest, second graders had difficulty determining which of three negative numbers was most positive; whereas, they had a much easier time determining which of three negative numbers was most negative. This is likely because they only had to consider the numbers' absolute values. Interestingly, they did better than chance in identifying which of three positive numbers was most negative. This required them to avoid focusing on the word "most" in the term. Their higher performance on identifying the most negative of positive numbers as opposed to the most positive of negative numbers may reflect their overall greater familiarity with positive numbers.

Second graders experienced an overall improvement from the pretest to the posttest, with the greatest improvement occurring on mixed comparison problems for the most positive question phrasing. This result suggests that students were attuned to the word *positive;* perhaps that reminder helped them avoid focusing solely on the largest absolute value, which may have been more likely if they interpreted closer to ten as closer to *a* ten.

Unsurprisingly, fifth graders had better overall performance than second graders on all integer comparisons. A similar, but slightly different, pattern was visible for fifth graders as second graders. Fifth graders did better on all positive comparisons than on all negative problems, even for the most negative question phrasing unlike second graders. They scored relatively low (71.0 on pretest and 82.6 on posttest) when choosing the most positive numbers from all-negative integers, as found with the second graders.

Mental Models on Pretest and Posttest

On the pretest, 61.8% of fifth graders, on average, exhibited formal integer value mental models while an average of 23.5% of fifth graders exhibited initial mental models. On the other hand, at the pretest, a majority of second graders did not know about negative numbers yet, so 49.3% of second graders exhibited initial integer value mental models while an average of 7.5% exhibited formal mental models.

On the pretest closest to 10 problems, 90.7% of second graders on average chose numbers with the biggest absolute values when all numbers were positive. The likelihood that students responded this way decreased for the *most positive* problems and was lowest for the *most negative* problems. These decreases indicate that those who exhibited initial mental models did not solely choose numbers with the biggest absolute values. Therefore, second graders tried to make sense of the compound meanings of *most positive* and *most negative*.

Second graders seemed to be more sensitive to *most* than *positive* or *negative* when they decided the meaning of *most positive* and *most negative*. This is consistent with Bofferding and Farmer's (2018) findings that students chose least of the cold (i.e., smallest negative) instead of least cold (i.e., warmest or biggest positive). For *most positive*, students would get similar answers if they focused on either *most* or *positive*. This is not the case for *most negative*. This mismatch could be a reason why the choices between biggest and smallest absolute values became more varied even for these comparisons. A second grader F08, for example, thought of *most negative* as most and *least negative* as less.

Mental Model Changes

Overall, we categorized both fifth and second graders as exhibiting higher integer value mental models on the posttest than on the pretest. The number of students classified as exhibiting Random, Initial, and T1 mental models decreased on the posttest while those classified as exhibiting Magnitude, T2, and Formal mental models increased. When focusing on students who started with initial mental models, the improvement took on different patterns between fifth and second graders (see Table 1).

On the posttest, fifth graders' mental models mostly shifted to Formal (with some at the Synthetic and Transition II levels). By contrast, most second graders either remained at the initial level or shifted their mental models across T1, Synthetic, and Formal, a much broader range. Interestingly, second graders who started with initial mental models had the largest percent of students exhibiting formal mental models on the posttest for the most negative question phrasing; whereas, fifth graders had the most formal mental models for the closest to 10 question phrasing. The numbers show that students did not always have the same mental model classifications for each question phrasing. Take second grader B04 as an example. This student "saw the negatives down and the rest that didn't have negatives were up" and exhibited a formal mental model for the closest to 10 phrasing. For *most positive*, though, the student chose the largest absolute values, and for *most negative*, the student chose the smallest absolute values.

Table 1: Students' Mental Model Shifts in terms of Percent for Those with Initial Mental Models on the Pretest for each Question Phrasing

Mental Model Changes	Second Graders			Fifth Graders		
	Closest to 10 (n=69)	Most Positive (n=49)	Most Negative (n=33)	Closest to 10 (n=22)	Most Positive (n=26)	Most Negative (n=29)
Initial to Random	8.7%	4.1%	0%	0%	3.8%	0%
Initial to Initial	53.6%	36.7%	51.5%	27.3%	11.5%	31.0%
Initial to Transition I	8.7%	24.5%	15.2%	4.5%	3.8%	3.4%
Initial to Synthetic	13.0%	22.4%	3.0%	4.5%	11.5%	3.4%
Initial to Transition II	5.8%	2.0%	0%	4.5%	0%	24.1%
Initial to Formal	10.1%	10.2%	30.3%	59.0%	50.0%	37.9%

Conclusions

Second graders' higher performance on the mixed questions for most positive and most negative compared to closest to 10 suggests that students do not necessarily coordinate order and value when they learn about negative integers. The implication of these results is that early integer activities need to explicitly highlight both elements of order and value to help students attend to these concepts. Moreover, regardless of their different grade levels, most students found it difficult to determine which integer was *most positive* among three negatives and which was *most negative* among three positives. This result might suggest that students need additional language support to shift from categorical to a continuum-based understanding of integers. Considering that those who started with initial mental models had different areas of improved progress, elementary teachers might consider multiple instructional ways to address students' various needs.

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References

Bofferding, L. (2014). Negative integer understanding: Characterizing first graders' mental models. *Journal for Research in Mathematics Education, 45*(2), 194-245.

Bofferding, L. & Farmer, S. (2018). Most and least: Differences in integer comparisons based on temperature comparison language. Advance online publication. *International Journal of Science and Mathematics Education.*

Whitacre, I., Azuz, B., Lamb, L. L.C., Bishop, J., Schappelle, B. P., & Philipp, R. A. (2017). Integer comparisons across the grades: Students' justifications and ways of reasoning. *Journal of Mathematical Behavior, 45*, 47-62. https://doi.org/10.1016/j.jmathb.2016.11.001

WHAT DOES THE LOW-ACHIEVING LABEL TELL US ABOUT STUDENTS' CONCEPTUAL UNDERSTANDING OF VARIABLES?

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This study investigated the differences in conceptions of variable among the groups of students labeled as having mathematics difficulties (MD) and typical mathematics achievement (TMA). Analysis students' responses on items involving the comparison of expressions involving generalized quantities provided data. The theoretical framework for this study was based upon the learning trajectory (LT) of the levels of sophistication of students' conceptions of variable. Data analysis included descriptive and inferential statistics. Results revealed little evidence of differences between MD and LMA students' conceptual understandings. That students labeled as MD and TMA can have similar levels of sophistication of conceptions of variables suggests that current measures of achievement provide an incomplete picture of students' understandings of algebra, and also disproportionately disadvantage those labeled as low-achieving.

Keywords: Algebra and Algebraic Thinking, Equity and Diversity, Instructional activities and practice

Algebra is often identified as an exemplar of mathematical abilities. Moses and Cobb (2001) described algebra as necessary for all students in to understand and succeed in a world with ever-increasing technological integration. Opportunities to be successful in algebra have been called equity and civil rights issues because of the limitations that low algebra achievement places upon individuals' career opportunities (Kaput, 1998; Moses & Cobb, 2001). Kaput and Blanton (2000) describe students' disappointing algebra understandings, public dislike of mathematics, and the inequity of academic tracking as problematic. These aspects can be linked to U.S. teaching practices and curricula that are focused on procedural skills instead of conceptual understanding. Kieran (2013) described the separation of procedural and conceptual understanding as a "false dichotomy" that has especially impacted the field of school algebra. She states that despite traditional instructional practices, even the symbolic aspects of algebra, have conceptual foundations that students need to develop. Since variable use is a major component of algebra and students' understanding of algebra has been identified as problematic, it is reasonable to hypothesize that students' understanding and use of variable may also be problematic. Students who see a variable as representing a single unknown value are more limited algebraic thinking, problem solving, equation solving, and making generalizations than those with a more sophisticated conception of variable. (Cai, Moyer, Wang, & Nie, 2011).

While the work done in this area is substantial, there is a gap in the research around examining specifically how the uses and conceptions of variables differ between successful algebra students and those who have difficulties in algebra. Research has suggested that students who have difficulties in mathematics differ from their typically-achieving peers in both mathematics-specific and more general characteristics. These differences have been attributed to environmental and cognitive factors. Mathematics-specific characteristics of students who have been labeled as low-achieving in mathematics are reported to include difficultly both storing and recalling basic arithmetic facts, which impact students' ability to achieve fluency in computation (Geary et al., 2012). They also experience delays in achieving accuracy and fluency in arithmetic operations and procedures and typically have a limited conceptual understanding of how and why the procedures work leading to a reliance upon less advanced solution strategies than their typically-achieving peers. I describe these students as having *mathematics difficulties* (MD).

There is not consensus about how to identify MDs from their typically-achieving peers (TMA) (Mazzocco, 2007). The "single-cutoff method", where students with MD are identified by creating one low-achieving and one typically achieving group of students based on their score on a standardize assessment is most commonly used. But this method has several draw backs including only providing a snapshot of a student's performance and typically focusing on procedural proficiency. Lewis (2014) warned the limited focus of research on MD students has resulted in low procedural fluency as a de facto defining characteristic of students with MDs. This neglects the more complex and conceptually-based aspects of mathematics. The cut-off score of 25% is most commonly used to identify students as MDs (Geary et al., 2012).

Statement of the Problem

Students are labeled as low-achieving and/or as having a MD based on the results of a single standardized and typically primarily procedural assessment. Such an assessment only provides insight into a portion of the mathematics that we want our students to know and understand. This leaves a significant gap in our understanding of specific differences between conceptions and use of variables amongst students with MD and TMA students. The purpose of this study is to investigate the differences between the responses of low- and typically-achieving students in an Algebra 1 course on items designed to assess students' ability to compare expressions involving variables.

Research Question

The general question under investigation is: Do the responses of students who are at or below the 25th percentile on the Iowa End of Course assessment (IEOC), and those of students who are above the 25th percentile differ on eight items designed to assess students' ability to compare expressions involving generalized quantities on a conceptual progress-monitoring tool at the beginning of an algebra course?

Theoretical Framework

The theoretical framework guiding this study builds on the notion that different uses of variables suggest corresponding levels of sophistication of conceptions of variable. A student's conception of variable is the student's idea of what a variable is, how it acts, and what it can represent. A student's use of variables refers to the actions that the student does in connection with this conception of variable. While we cannot know exactly what a student's conception of variable is, we can make inferences about that conception based on how the student uses variables. Blanton, Brizuela, Gardiner, Sawrey, and Newman-Owens (2015) described a progression of these conceptions and uses of variable in their learning trajectory (LT) that characterized increasingly sophisticated levels in students' thinking about variable and variable including: *Pre-Variable, Letter as Representing Variable with Deterministic Value, Letter as Representing Variable with Fixed but Arbitrarily Chosen Value, Letter as Representing Variable as Varying Unknown, and Letter as Representing Variable as Mathematical Object*.

In addition to this perspective about the connections between students' uses, misconceptions and conceptions of variables, the theoretical perspective taken with respect to students with MD is influential in the structure of my study. In this study, I view all learners as part of a normal distribution of mathematics achievement and growth as described by Geary et al. (2012). While groups of learners are not independent of each other, this does not preclude different, but overlapping, sets of characteristics. The perspective that all students are part of this distribution does not require that all students have the same factors in their mathematical achievement and growth. Specifically, I take the perspective that these MDs that students experience are the result of cognitive differences in comparison with their TMA peers (Lewis, 2014; Mazzocco, 2007). These differences are not deficits in the sense that students with MLDs are deficient in the ability, or are unable, to learn the content with which they are experiencing difficulty. Rather, students with MLDs have different connections and conceptions than their TMA peers within the foundational aspects of the content which inhibit the typical connections from these foundational topics to next level of content. Thus, these few who exhibit abnormal connections can be thought of as the left tail of the normal distribution described by Gearyet al. (2012).

Methodology

Using data collected as part of the *Algebra Screening and Progress Monitoring* project (Foegen & Dougherty, 2010), I analyzed students' responses to a subset of items on a conceptual algebra progress monitoring measure to identify and describe differences among the MD and TMA groups of students. The Project (Funding information) provided the context for my study. From the literature on students with MD (e.g., Geary et al., 2012), students were separated into two groups: TMA and MD. Students were assigned to one of these groups based on percentile rank of their Iowa End of Course (IEOC) assessment score. Students with scores at or below the 25th percentile MD. Students with scores higher than the 25th percentile were considered TMA.

Using the levels of sophistication of conceptions of variable from Blanton et al.'s (2015) LT, I coded student responses and multiple-choice options as consistent with of a specific TL level or misconception. In my study, these different descriptions of how students think of and use variables were used to classify students' responses on items designed to assess students' ability to compare expressions involving generalized quantities. These classifications were then compared across the MD and TMA groups using descriptive and inferential statistics and coded for the concepts of variable. It is important to note that letters as variables can have different roles in different situations. In all of the items that were considered for this study, the variables represented generalized quantities that were not dependent on other quantities. For example, one of the items asked students to compare the expressions "$5 + t$" and "$t + 3$".

Summary

Both the similarities and the differences identified in this study between MD and TMA student responses to items designed to assess their ability to compare expressions involving variables support the claim that most students in an algebra 1 course do not have and are not developing a conception of variable that allows for the variable to represent more than one value at a time. The few differences identified suggest that items with more complex operational, procedural, or property-based contexts are less accessible to MD students. The similarities identified support the claim that procedurally focused assessments provide incomplete descriptions of students. In both cases, MD students are disproportionately disadvantaged and these disadvantages impact not only their school experiences, but can have long-lasting impacts on career opportunities as well. These findings have implications for the classroom.

Closing Statement

In closing, this investigation revealed that the differences between MD and TMA students are limited and seem to be related to the level of complexity of the operations, procedures, and properties that are incorporated into an item. More than these differences, the similarities that were illuminated by this study impact how we perceive what successful and struggling students may look like, and what they need. The overall low proportion of students who have a sophisticated conception of variable suggests that all students need access to better conceptual instruction and experiences in algebra. The context and content of this instruction need to be appropriate for all students, which may mean that MD students required additional support than their TMA peers in developing these sophisticated conceptions of variables.

Despite the similarities that were identified in this study, students who are identified as MLD or MD may be disproportionally disadvantaged by traditional instructional practices and data-based decision making based on procedurally focused assessments and instruction these disadvantages need to be further investigated. These are the students most at risk to be "innumerate" (Geary et al., 2012, p. 206) because of the long-lasting impact that being labeled as low-achieving can have on school mathematics opportunities and on future career options. These are the students who need our help. Through experiences with and exposure to variables that represent more than just one value from a younger age, these students can begin to develop a solid foundation upon which to build their future successes, both mathematically and as a contributing member of society.

The identification of these limitations of the current ways in which we label students as provides an opportunity to reexamine how students are labeled and what we call a "successful student". This examination must lead us to consider how can we change our current instructional practices to provide more equitable access to all students, regardless of how they perform on a standardized test.

References

Blanton, M. L., Brizuela, B. M., Gardiner, A. M., Sawrey, K., & Newman-Owens, A. (2015). A Progression in First-Grade Children's Thinking About Variable and Variable Notation in Functional Relationships. Manuscript submitted for publication.

Cai, J., Moyer, J. C., Wang, N., & Nie, B. (2011). Examining students' algebraic thinking in a curricular context: A longitudinal study. In J. Cai & E. Knuth (Eds.), *Early algebraization* (pp. 161-185): Springer.

Foegen, A., & Dougherty, B. J. (2010). Algebra screening and progress monitoring. Measurement. Measurement (Goal 5) award from the Institute for Education Sciences, U.S. Department of Education. Award Number: R324A100087.

Geary, D. C., Hoard, M. K., Nugent, L., & Bailey, D. H. (2012). Mathematical cognition deficits in children with learning disabilities and persistent low achievement: A five-year prospective study. *Journal of Educational Psychology, 104*(1), 206-223.

Kaput, J. J. (1998). *Transforming algebra from an engine of inequity to an engine of mathematical power by "algebrafying" the K-12 curriculum.* Paper presented at the The nature and role of algebra in the K-14 curriculum: Proceedings of a national symposium.

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning. In J. J. Kaput, D. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 5-17). New York, NY: Routledge.

Kaput, J. J., & Blanton, M. L. (2000). *Algebraic reasoning in the context of elementary mathematics: Making it implementable on a massive scale*. Retrieved from Dartmouth, MA:

Kieran, C. (2013). The False Dichotomy in Mathematics Education Between Conceptual Understanding and Procedural Skills: An Example from Algebra. In R. K. Leatham (Ed.), *Vital Directions for Mathematics Education Research* (pp. 153-171). New York, NY: Springer New York.

Lewis, K. E. (2014). Difference Not Deficit: Reconceptualizing Mathematical Learning Disabilities. *Journal for Research in Mathematics Education, 45*(3), 351-396.

Mazzocco, M. M. (2007). Defining and differentiating mathematical learning disabilities and difficulties. In D. B. Berch (Ed.), *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities* (pp. 29-47). Baltimore, MD Paul H Brookes Publishing.

Moses, R., & Cobb, E. C., Jr. (2001). Radical equations: Organizing math literacy in America's schools: Beacon Press.

HOW QUANTITATIVE REASONING CAN SUPPORT GRAPH UNDERSTANDING IN ALGEBRA

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The construction and interpretation of graphs is a key mathematical activity, particularly at the middle school level, when students' experiences form the foundation for their reasoning about functions and relations. However, research demonstrates that students experience challenges in interpreting and understanding graphs. One promising avenue is an emphasis on graphs as representations of quantities varying in tandem. We present a case of two middle-school students, one who emphasized quantities and their relationships and one who did not. We found that attention to quantities fostered ratio concepts and supported appropriate slope conceptions.

Keywords: Algebra and Algebraic Thinking, Cognition, Middle School Education

Graphing is a key aspect of mathematical understanding and represents a "critical moment" in middle school mathematics for its opportunity to foster powerful learning (Leinhardt, Zaslavsky, & Stein, 1990). However, students experience a number of challenges in constructing, interpreting and making sense of graphs (e.g., Moore & Thompson, 2015). In addressing these challenges, researchers have offered several characterizations of students' understanding of graphs. For instance, Lobato, Rhodenhamel, and Hohensee (2012) differentiated between understanding slope as a mathematical object (a relationship between quantities' values) versus a physical object (a property of visual steepness), which is similar to Zaslavsky, Sela, and Leron's (2002) two conceptions of slope, analytic and visual. Moore and Thompson (2015) distinguished between static and emergent shape thinking, in which the former involves conceiving of a graph as a shape qua shape, and the latter entails envisioning a graph as a trace of covariation. In their work, Moore and Thompson (2015) point to the need to support students' abilities to make sense of graphs emergently. This is particularly true at the middle-school level, which is when students are typically introduced to function graphs and develop graph-related conceptions that will influence their future mathematics experiences. One potentially promising way to support productive conceptions is to emphasize covariation, in which students conceive of graphs as a representation of quantities varying in tandem (Moore & Thompson, 2015). Thus, we investigate the following question: How does attending to covarying quantities affect middle-school students' construction and interpretation of graphs? In order address this question, we present a case study of two students, one who regularly referenced quantities in graph construction and interpretation and one who did not, and discuss these students' resulting conceptions and sense making.

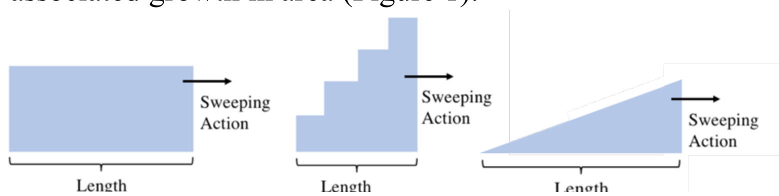
Theoretical Framework: Quantitative and Covariational Reasoning

Thompson (1994) defines a quantity as an individual's conception of the measurement of an attribute of an object. It is composed of a conception of an object, an attribute, an appropriate unit, and a process for assigning a value to the attribute. Speed, area, and length are all attributes that can be conceived as quantities. When students coordinate the variation in the values of quantities that change together, this is termed covariational reasoning (Thompson & Carlson, 2017). Covariational reasoning entails tracking either quantity's value with the explicit understanding that at every instance, the other quantity also has a corresponding value.

Researchers have characterized students' reasoning about quantities that change in tandem with respect to their graphing activities (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). This body of work indicates that students and pre-service teachers benefit from opportunities to use a covariation perspective when making sense of graph features such as slope, intercept, and root (e.g., Ayalon, Watson, & Lerman, 2015). We propose that an approach emphasizing covarying quantities can also be effective in supporting middle-school students' emerging conceptions of graphs.

Methods

We conducted a 10-day, 15-hour videotaped teaching experiment (Steffe & Thompson, 2000) with two 7th-grade pre-algebra students, Wesley and Olivia. The first author was the teacher-researcher. We developed tasks to support a conception of linear growth as a phenomenon of a constant rate of change, and quadratic growth as a constantly-changing rate of change. The tasks emphasized these ideas within the contexts of speed and area. The area tasks presented "growing rectangles", "growing stair steps", and "growing triangles" via dynamic geometry software, in which the students could manipulate the figure by extending the length and observing the associated growth in area (Figure 1).



(a) (b) (c)

Figure 1. (a) Growing rectangle, (b) stair step, and (c) triangle tasks.

Data sources included video and transcripts of each teaching session and copies of the students' work. Our analysis relied on the constant comparative method (Glaser & Strauss, 1967), and was guided by an attempt to account for the commonalities and differences in Wesley's and Olivia's graphical thinking. We developed explanatory accounts of these differences based on evidence from their written work, descriptions of their ideas, their drawings, and their gestures. We then compared and discussed these explanatory themes as a research team until we reached consensus.

Results: Quantitative Reasoning Influences Students' Ratio and Slope Conceptions

Wesley regularly referenced two quantities (area and length) when constructing and discussing his graphs, while Olivia did not. As an example, the students graphed the relationship between area with respect to the length of a growing triangle with a length to height ratio of 5 cm to 2 cm (see Figure 1.c), as well as the area of a rectangle that would sweep out the same total amount of area after a length of 5 cm (Figure 2). Both students labeled the *y*-axis "area" and the *x*-axis "length", although this is not shown in the cropped graphs in Figure 2.

Oliva discussed her graph's shape while Wesley discussed his graph's constituent quantities. When describing her graph, Olivia said, "It starts kind of low and then it gradually gets more curved and then steeper." Olivia described the visual features of the graph itself and did not reference the associated quantities of area and length in the growing triangle context. In contrast, when Wesley explained why the graph of the growing rectangle was straight, he said, "So every 1 cm in length, it's always going to be 1 cm in area." For Wesley, the graph represented a trace of the associated growth between area and length. Olivia did also at times reference the associated attributes area and length, but in a non-quantified manner, which we discuss below.

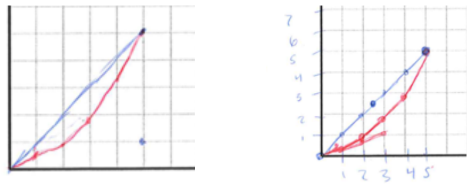


Figure 2. Wesley's (left) and Olivia's (right) graphs of the growing triangle (in red) and rectangle (in blue).

We found two major implications of an explicit attention to both quantities: (a) the development of ratio and rate, and (b) a conception of slope as a ratio. We illustrate these implications through Olivia and Wesley's approach to the "two growing rectangles" task, in which the rectangles grew in length while maintaining the same height (see Figure 1.a). The students observed one growing rectangle with an unspecified height, and then compared it to a second growing rectangle with a larger height (Figure 3).

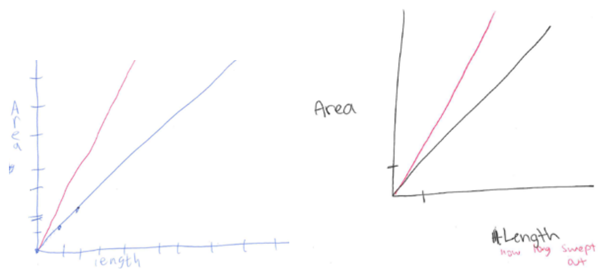


Figure 3. Wesley's (left) and Olivia's (right) graphs of the growing rectangles.

Olivia referenced the attributes length and area when explaining why her graphs were linear: "I sort of pictured it in my head…I knew it would line up because for every length that you've pulled it should be the same amount of area." Olivia mentally coordinated the two attributes, but did not quantify them. In contrast, Wesley decided to think of the height as 1 meter: "If you drag it out 1 meter, so that's the length is 1 meter and the height is 1 meter, and then to find the area you actually times the length by the height which is 1 times 1 is 1 square meter actually." When explaining why the graph of the second rectangle was steeper than the first, Wesley said, "The height is bigger than 1 meter now, then for every length that you pull it 1 meter, it gets more area [than before]." Olivia, in contrast, explained, "The steeper it is, the longer height of the wall it is", where "wall" meant the height of the rectangle.

For Olivia, the slope of her graphs represented one attribute, area, rather than a ratio of area to length. This attention to one attribute also encouraged Olivia to rely on thematic associations (Moore & Thompson, 2015) when considering a graph's slope. Namely, a feature of the graph, such as constant slope, represented a quality of the motion she observed when a rectangle grew, which she described as "consistent". For instance, Olivia justified a constant slope for a growing rectangle graph by stating, "It went consistently as like a straight line", and "The rectangle, it grew at a consistent rate." The association between "consistent" growth and a constant slope also resulted in Olivia initially graphing the area versus length of a growing triangle as a straight line rather than a curve. Her justification was similar in this case: "Mine is going up consistently." Wesley, in contrast, described slope as the ratio of two quantities: "[Slope] means the area covered in a certain length."

Both students frequently created a standard unit for length so that they could compare amounts of area across uniform intervals of length. For Olivia, this process reduced her attention to one attribute, area. In contrast, Wesley attended to both quantities, and created unit ratios. This was evident in his decision to consider the area of the rectangles from Figure 2 in relation to 1 meter of length. In another case, Wesley explained his linear graph of a growing rectangle by remarking, "The height is 1 cm. So, every time it's pulled out 1 cm, the area gets greater by 1 cm squared." Wesley could also conceive of this as a multiplicative comparison; for instance, for a 4-cm high rectangle, he explained, "To get how much the area accumulates by, you do *x* [an unknown length] times 4." When considering growing triangles, Wesley also compared amounts of area accumulated per a standard unit of length, and the unit remained explicit: "Every inch it goes it, like, it goes, it covers more area for that inch so it keeps getting steeper and steeper."

Discussion

Explicit attention to both quantities and a coordinated change in quantities offered meaningful affordances for the creation of unit ratios, the understanding of slope as a ratio, and the ability to conceive of a graph as a representation of coordinated change. Given students' difficulties in conceiving slope as a ratio-of-change (e.g., Lobato et al., 2012), the case of Wesley and Olivia suggests that an emphasis on quantitative reasoning can be a productive route towards meaningful sense-making with graphs. However, simply relying on the use of quantitatively-rich contexts is not sufficient; it does not guarantee that students will attend to both quantities or develop images of coordinated change. Teachers should therefore encourage students to attend to both quantities represented in graphs, to make that attention explicit in their language, and to ask questions that require students to coordinate variation in quantities.

References

Ayalon, M., Watson, A., & Lerman, S. (2015). Functions represented as linear sequential data: Relationships between presentation and student responses. *Educational Studies in Mathematics, 90*(3), 321-339.

Carlson, M.P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66-86.

Glaser, B. G. & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research.* Hawthorne, NY: Aldine Publishing Company.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning and teaching. *Review of Educational Research, 60*(1), 1-64.

Lobato, J., Rhodehamel, B., & Hohensee, C. (2012). 'Noticing' as an alternative transfer of learning process. *Journal of the Learning Sciences 21*(3): 433–82.

Moore, K.C., & Thompson, P.W. (2015). Shape thinking and students graphing activity. In T. Fukawa-Connelly, N. Infante, K. Keene, & M. Zandieh (Eds.), *Proceedings of the Eighteenth Annual Conference on RUME* (pp. 782 – 789). Pittsburgh, PA: WVU.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. Kelly, & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–306). Mahwah, NJ: Lawrence Erlbaum Associates.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel, & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179–234). Albany, NY: SUNY Press.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.

Zaslavsky, O., Sela, H., & Leron, U. (2002). Being sloppy about slope: The effect of changing the scale. *Educational Studies in Mathematics, 49*(1), 119 – 140.

FUTURE MIDDLE GRADES TEACHERS' INCREMENTAL ALIGNMENT OF KNOWLEDGE WITHIN THE MULTIPLICATIVE CONCEPTUAL FIELD

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We used interviews to examine future middle grades teachers' capacities to coordinate reasoning with quantities and with multiplication equations. Results include a mathematical analysis of multiplication as coordinated measurement and a (still emerging) psychological framework. In particular, we characterize future teachers' expanding coordination as incremental refinement and alignment of diverse knowledge resources.

Keywords: Teacher Knowledge, Teacher Education, Multiplication

Despite decades of research on topics in the multiplicative conceptual field (MCF; Vergnaud, 1983), multiplication and division with whole numbers and fractions, proportional relationships, and linear functions of the form $y = mx$ continue to pose significant challenges for many students and teachers. In contrast to research that has emphasized teachers' deficits with respect to topics in the MCF, we are investigating ways that a measurement approach to multiplication can help future teachers construct sound explanations for mathematics they will teach.

Theoretical Frame

Our framework combines mathematical and psychological perspectives. Figure 1 shows the quantitative definition of multiplication upon which we have converged. It applies to situations in which there is a quantity (the product amount) that is simultaneously measured with two different measurement units (a "base unit" and a "group"). The most important aspects of this definition are (a) writing the multiplicand and multiplier in a consistent order to highlight a common underlying structure for multiplication, division, and proportional relationships (e.g., Beckmann & Izsák, 2015) and (b) interpreting N, M, and P in Figure 1 as numbers that result from measuring quantities in terms of some designated unit (a base unit or a group).

N	$\bullet$	M	$=$	P
How many base units make one group exactly?		How many groups make the product amount exactly?		How many base units make the product amount exactly?

Figure 1. A quantitative definition for multiplication based in measurement.

The definition in Figure 1 can be used to coordinate an important swathe of the MCF—for instance, by viewing division as multiplication with an unknown factor and proportional relationships as instances where values for two of N, M, and P co-vary while the value for the third remains fixed (Beckmann & Izsák, 2015). Furthermore, the definition in Figure 1 can be used equally well with both whole numbers and with fractions. Figures like that shown in Figure 2a can support the measurement perspective on unit fractions if one asks how many of the long strip make the short strip exactly (1/3). We have found that future teachers have little problem answering such questions and can extend this measurement perspective from unit fractions to non-unit fractions, including improper fractions (Figure 2b). This appears to be a reliable

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.

foothold for future teachers when extending the measurement definition of multiplication shown in Figure 1 from whole numbers to fractions.

Figure 2. Interpreting fractions from a measurement perspective. (a) 1/3 of the long strip makes the short strip exactly. (b) 4/3 of the short strip (4 copies of 1/3) makes the long strip exactly.

Our psychological perspective is informed by diSessa's (2006) knowledge-in-pieces epistemology. Knowledge-in-pieces is a constructivist perspective in which learners come to know by using and refining knowledge as they construct interpretations of their interactions with the physical and social environment. The perspective characterizes the evolution from novice to expert knowledge as piecemeal construction, refinement, and reorganization of diverse fine-grained knowledge resources that are connected to varying degrees and whose use is often sensitive to context. Examples of cognitive mechanisms include refining the contexts in which resources are applied, forming new connections among resources, and loosening connections among others. In the present study, we examined the ecology of resources that future middle grades teachers used as they coordinated the definition of multiplication shown in Figure 1 with diverse problem situations that are contained in Vergnaud's (1983) MCF.

Methods

In Fall 2016, we recruited six future middle grades teachers who were enrolled in a 2-semester sequence of mathematics content courses (Number and Operations followed by Algebra). The second author taught both courses; the first author conducted semi-structured interviews. The interviews were spaced a few weeks apart and were coordinated with whole-class instruction, most often so that the interviews provided information about the future teachers' reasoning before specific topics were treated in the course. We video recorded the interviews, collected all written work generated during the interviews, and transcribed the interviews verbatim. The present report is based on analysis of talk, gesture, and inscription as captured in the interview videos, transcripts, and written work. We wrote analytic notes to capture our interpretations of how future teachers were reasoning moment-to-moment. In some cases, we took future teachers' statements as direct, reliable reports of their thinking. In other cases, we made inferences about aspects of future teachers' reasoning that they would not likely be able to report directly.

Results

All of the future teachers demonstrated some alignment between reasoning with quantities and with arithmetic computations—for instance, all connected iterating groups of units with multiplication and connected partitioning quantities into equal-sized pieces with division by a whole number. At the same time, *not* all future teachers connected partitioning quantities into equal-sized pieces with multiplication by a unit fraction. More generally, to different degrees, the future teachers experienced challenges expanding the range of situations in which they successfully aligned reasoning with quantities and with arithmetic computations (from their point of view and ours). Consistent with the knowledge-in-pieces epistemological perspective, such expanding alignment appeared to involve refinement and reorganization of a complex ecology of knowledge resources.

We organized knowledge resources we identified into four groups having to do with (a) meanings for multiplication (e.g., repeated addition, "of"), (b) meanings for the equal sign (e.g., correspondence, balance), (c) meanings for numbers (e.g., counts, measures), and (d) computation (e.g., algorithms, factor-product combinations). We make three points, first, from a knowledge resources point of view, reasoning at any given moment is supported by a set of activated resources. In some cases, those resources can be well-connected and support one another. The discussion above about connections among partitioning, division by a whole number, and multiplication by a unit fraction is one example. Second, differences in sets of resources activated by an individual helps explain variation in how that individual reasoned both within and across tasks. An individual might interpret the equal sign at one point as indicating a correspondence and, a few moments later, as stating that the numbers of units on the left and right hand sides are the same (balance). Such variation in reasoning, which was not uncommon in the interviews, can both support and constrain incremental alignment. Third, coming to perceive a common underlying structure, like the meaning of multiplication shown in Figure 1, across situations can be characterized as a gradual expansion in which one aligns thinking about quantities in those situations and about multiplication equations. Such expansion can occur through the gradual accumulation of fine-grained adjustments to the organization of knowledge resources.

We draw examples from Nina's reasoning to illustrate gradually expanding coordination. During her first interview, she demonstrated facility modeling problem situations with whole-number multiplication and connecting partitioning, division by a whole number, and multiplying by a unit fraction. At the same time, she reported that she had not thought before about a consistent interpretation of multiplication that worked both with whole numbers and with fractions. During the interview, Nina wrote equations to model word problems that described 4 cans with 3 tennis balls in each, 2 bags with 5 soccer balls in each, 1/5 of 4 ounces of tomato paste, and 1/5 of 1/3 cup of oatmeal. For the whole-number examples, Nina drew appropriate pictures to generate and explain her equations, "4 x 3 = 12" and "2 x 5 = 10." For the examples involving fractions, she again drew appropriate pictures of the situations but struggled at first to use those pictures to explain equations. She was able, however, to use her connection that "of means multiply" to calculate the correct answers of 4/5 and 1/15.

Nina was able to expand her coordination of reasoning with quantities and arithmetic computation by asking "what times 5 makes 4" and "what 5 things of equal amount will add up to 1/3." Thus, she reinterpreted situations presented with a fractional multiplier (e.g., 4 • 1/5 = ?) as one using a whole-number multiplier (e.g., ? • 5 = 4). The particular form in which Nina aligned her drawing and her computation appeared influenced by a further expectation she articulated that multiplication should make numbers larger. This incremental alignment appeared supported by her initial connection between partitioning and multiplying by a unit fraction. During her second interview, Nina continued to expand connections she made between reasoning with quantities and with arithmetic as she worked on the following problem:

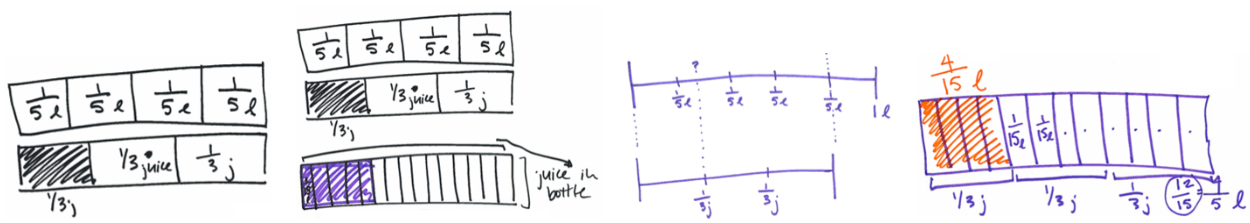
> A full bottle contained 4/5 of a liter of juice. Then you drank 1/3 of the juice in the bottle. What fraction of liter [sic] of juice did you drink?

Nina began by drawing a strip diagram that showed appropriate relationships between the bottle partitioned into 4 parts (each of which was also 1/5 liter) and partitioned into 3 parts (see Figure 3a). She knew she needed further partitions to answer the question but had two competing ideas: "I don't know if I should find a common denominator between 3 and 4, because there's only 4 pieces, or if I need to find it between 3 and 5, because I want to know how much of a liter."

Faced with competing, reasonable ideas and no apparent way to prioritize one approach to partitioning over the other, Nina chose the second option.

Nina continued her work by drawing a new strip and shading 5 of 15 mini-pieces (see Figure 3b). She interpreted the shaded region: “So this is 1/3 of the juice. Also 5/15 a liter, I think.” When the first author asked if she had other ways of thinking about the task, Nina explained “1/3 of 4/5” meant to multiply and computed “1/3 x 4/5 = 4/15.” She recognized that she had two different answers but did not understand how to reconcile them. After Nina spent approximately 3 minutes trying to diagnose her inconsistent answers, the interviewer moved on to other tasks.

Nina took a fresh pass at the Juice problem 25 minutes later at the end of the second interview and this time accomplished greater coordination between her reasoning with drawn quantities and with computations. She considered the equivalent fractions 4/5 and 12/15, saw that she should partition the bottle into 12 parts, and created a new drawing that coordinated the liter, the juice bottle, partitioning, and her calculated answer of 4/15. Nina’s final comment suggested that she perceived new alignment that had eluded her during her first attempt: “That makes me feel better.” Although Nina did not make any specific comment, she might have learned that partitions in drawings (4 parts and 3 parts in this case) can be a reliable guide for partitioning. Such an adjustment is a further example of refinement in her ecology of resources that supported alignment of reasoning with quantities and with arithmetic computation.



(a) (b) (c) (d)

Figure 3. (a) Nina’s initial drawing for the bottle situation. (b) Nina interprets her drawing to show 5/15 as the solution. (c) Nina makes a new drawing at the end of the interview. (b) Nina partitions the bottle into 12 pieces.

Conclusion

Future teachers entered our courses with initial capacities to coordinate reasoning with quantities and with arithmetic; and, we examined the knowledge resources that they used and transformed to achieve alignment across a wider range of situations in the MCF.

Acknowledgments

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References

Beckmann, S., & Izsák, A. (2015). Two perspectives on proportional relationships: Extending complementary origins of multiplication in terms of quantities. *Journal for Research in Mathematics Education*, *46*(1), pp. 17–38. doi: 10.5951/jresematheduc.46.1.0017

diSessa, A. A. (2006). A history of conceptual change research: Treads and fault lines. In R. K. Sawyer (Ed.), *The cambridge handbook of the learning sciences* (pp. 265–282). New York, NY: Cambridge University. doi: http://dx.doi.org/10.1017/CBO9780511816833.017

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of Mathematics Concepts and Processes* (pp. 127-174). New York: Academic Press.

A COVARIATIONAL UNDERSTANDING OF FUNCTION: PUTTING A HORSE BEFORE THE CART

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Supporting students developing understandings of function has been a notoriously elusive task. In this report, we present Thompson and Carlson's (2017) description of a covariational meaning of function and provide an empirical example of a student who maintains meanings compatible with this description. We use this students' activity to illustrate nuances in Thompson and Carlson's description and to highlight how such meanings can be powerful.

Keywords: Function, Cognition, Teaching Experiments

Several researchers have examined teachers' and students' understandings of univalence and arbitrariness, key mathematical properties typical to a formal function definition. Univalence is the property that for each element in the domain there is a unique element in the range. Arbitrariness refers to a function needing not be defined by a known correspondence rule. Researchers have shown that students and teachers do not perceive a need for the univalence and arbitrariness properties of function (e.g., Breidenbach, Dubinsky, Hawks, & Nichols, 1992). Addressing the property of univalence, Even (1990) noted, "some serious questions are raised by the fact that, without prompting, none of the subjects could come up with a reasonable explanation for the need for the property of univalence" (p. 531). In an effort to re-conceptualize the notion of function in school mathematics, Thompson and Carlson (2017) presented a covariational meaning of function, which we elaborate on in the next section. We include an example of student activity to highlight nuances in Thompson and Carlson's (2017) description of a covariational meaning of function and illustrate how such a meaning can be productive for a student. Our goal is to show how a student who has developed such a meaning has the horse (i.e. foundational understandings) needed to pull a cart (i.e. the formal definition of function).

Theoretical Perspective: A Covariational Understanding of Function

Drawing on their body of work and the growing body of literature highlighting the importance of students reasoning about quantities that change in tandem, Thompson and Carlson (2017) proposed a definition of function rooted in covariational reasoning. They described, "A function, covariationally, is a conception of two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that…every value of one quantity determines exactly one value of the other" (p. 444). Rather than foregrounding univalance, Thompson and Carlson (2017) foregrounded an individual *constructing* an invariant relationship; once an individual has conceived of such a relationship, she can begin to investigate properties of that relationship. Hence univalence becomes a particular property of this invariant relationship. Thompson and Carlson (2017) avoid including dependent and independent variables in their definition. They explained, "What is independent and what is dependent will depend entirely on the person's conception of the situation and which way they envision dependence, if they envision dependence at all" (p. 444). Thompson and Carlson highlighted, however, a conceived function entails some cognitive sense of dependency of one quantity to another, as an individual must think of one quantity before the other. Thompson and Carlson (2017) added, "it is through covariation that the dependency becomes crystalized in her thinking as being invariant across quantities' values" (p. 444). However, the extent to which students absolutely maintain a dependency of one quantity in relation to another when conceiving of an invariant relationship between two quantities remains an open question.

Data Sources and Results

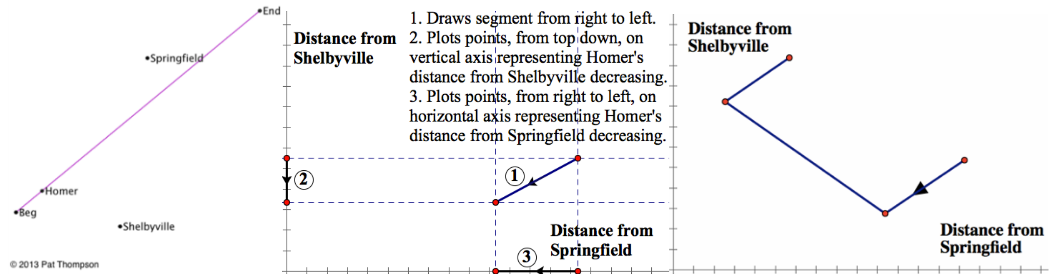
To provide an example of a student who maintained meanings consistent with those described by Thompson and Carlson (2017), we draw from data collected during a semester-long teaching experiment (Steffe & Thompson, 2000) with an undergraduate student, Arya. Throughout the semester, Arya repeatedly conceived of and constructed relationships between covarying quantities in dynamic situations and represented these covariational relationships graphically. During these activities we intentionally gave little to no focus on univalence and arbitrariness. In what follows, we focus on Arya's activity towards the end of the teaching experiment that is particular to Thompson and Carlson's (2017) description of a covariational meaning of function and refer the reader to Paoletti (2015) for detail on her specific activities.

Conceiving of an invariant relationship in the *Car Problem*

Arya addressed an adaptation of the *Car Problem* designed by Saldanha and Thompson (1998). Consistent with Saldanha and Thompson's use of the task, we asked Arya to represent the relationship between an individual's (Homer's) distances from two cities (Shelbyville and Springfield) as he travels back-and-forth on a road (Figure 1a). We adapted the task by asking about "function" after Arya constructed her graph. Because the relationship is such that neither represented distance is a function of the other distance, we conjectured Arya may spontaneously consider other quantities in the situation that were not directly represented in the graph.

Arya's activity suggests she conceived of "two quantities varying simultaneously such that there is an invariant relationship between their values" (Thompson & Carlson, 2017, p. 444). Arya consistently focused on the relationship between Homer's distances from the two cities through constructing the relationship in the situation and then representing her conceived relationship graphically. To illustrate, she first described the directional covariation of Homer's distance from each city (e.g., as Homer moves from the beginning of his trip, the distance from each city decreases), and then drew a segment from right to left corresponding to decreasing ordinate and abscissa magnitudes (indicated by (1) in Figure 1b). Arya pointed to the applet and described, "We start off... far from Springfield and pretty close to Shelbyville [*pointing to Beg. on computer screen then traces along road*]. Then... you're getting closer to Shelbyville for a little ways and closer to Springfield as we're moving along the road". With respect to her graph, Arya marked horizontal and vertical dashed lines from each graphed point to the vertical and horizontal axes, respectively, to verify that she represented distances from Shelbyville and Springfield each decreasing (indicated by (2) and (3) in Figure 1b). Arya continued such actions to construct and justify the other two segments in her graph (Figure 1c).

Consistent with claims by Piaget et al. (1977) and Thompson and Carlson (2017), Arya did conceive of one quantity first when constructing a relationship in the situation and graph; however, Arya's actions did not imply she conceived an explicit dependent-independent relationship of either distance to the other. Arya first described how Homer's *distance from Springfield* varied. Before drawing her second segment (seen in Figure 1c), Arya first described how Homer's *distance from Shelbyville* varied . Finally, before drawing the third segment, Arya again first described how Homer's *distance from Springfield* varied. Arya maintained a focus on two quantities simultaneously covarying whilst alternating which quantity she considered first in order to accurately construct and represent the relationship she conceived.



(a) (b) (c)

Figure 1. (a) The Car Problem applet, (b) a recreation of Arya's initial work, and (c) a recreation of Arya's final graph.

Addressing questions about "function" in the *Car Problem*

We next asked Arya if she could "talk about anything in this situation in terms of things being functions?" Arya first determined that neither graphed quantity was a function of the other graphed quantity, stating, "If you take either like the distance from Springfield or the distance from Shelby[ville] as your input you're going to have more than one output in some places." A researcher then asked, "Is there anything else we could have asked you about that may or may not have represented a function?" Arya considered either distance from a city "and how it travels over total distance." Specifically, she considered using either Homer's distance from a city or total distance as the input quantity to make conclusions regarding the 'function-ness' of each possiblw input-output pair (e.g., she concluded that Homer's distance from Springfield is a function of his total distance traveled, see Table 1). Notably, Arya referred only to the dynamic image of the situation as she considered if each distance from a city corresponded to exactly one total distance traveled and vice versa; Arya did not use a graph, equation, or table.

Because Arya added an arrow to her completed graph (Figure 1c), we conjectured she may reason about the trace of her graph as being defined parametrically. A researcher asked, "What if your input was total distance traveled and your output was… like a pair of values. Where that pair is your distance from Springfield and your distance from Shelbyville... What do you think about that case?" Arya discussed two possible total distances: (1) Homer making one trip from Beg. to End and (2) Homer traveling back and forth along the road accumulating total distance. Arya stated that in either case the relationship represented a function if she considered total distance as the input but the inverse relationship only represented a function in the first case.

We use Arya's activity to highlight how a student who conceives of an invariant relationship between quantities consistent with Thompson and Carlson's (2017) description can use this understanding to determine if there was a "relationship between their values that has the property that… every value of one quantity determines exactly one value of the other" (Thompson & Carlson, 2017 p. 444). In total, Arya's image of the situation enabled her to consider 10 different possible functional relationships (see Table 1), regardless if the relationship was explicitly represented by a graph, equation, or table.

Table 1: The relationships Arya considered as possibly representing functions

Input	Output	Function?
Distance from Springfield	Distance from Shelbyville	No
Distance from Shelbyville	Distance from Springfield	No
Total distance traveled	Distance from Springfield	Yes
Distance from Springfield	Total distance traveled	No

Total distance traveled	Distance from Shelbyville	Yes
Distance from Shelbyville	Total distance traveled	No
Total distance traveled (one trip)	(Distance from Shelbyville, Distance from Springfield)	Yes
Total distance traveled (accumulated)	(Distance from Shelbyville, Distance from Springfield)	Yes
(D. from Shelbyville, D. from Springfield)	Total distance traveled (one trip)	Yes
(D. from Shelbyville, D. from Springfield)	Total distance traveled (accumulated)	No

Discussion

Arya's activity highlights how a student who understands "two quantities varying simultaneously such that there is an invariant relationship between their values" can leverage this understanding to determine if "every value of one quantity determines exactly one value of the other" (Thompson & Carlson, 2017 p. 444). Further, Arya's activity highlights how a conceived relationship between covarying quantities can serve as the *something* that students are reasoning about when discussing 'function' in various representations (e.g., Thompson, 1994).

Arya's activity addressing the *Car Problem* allowed us to clarify Thompson and Carlson's (2017) elaboration of Piaget et al.'s (1977) notion that an individual must conceive of one quantity varying first, but this does not imply necessary dependency. We note that the quantity Arya considered first switched throughout her activity, illustrating that a student can move flexibly between considering either of two quantities first as she conceives of and represents a relationship between these quantities. Returning to the opening analogy, we posit that supporting students in constructing invariant relationships between quantities has the potential to provide them with the horse to pull the cart that is the definition of function in school mathematics.

Acknowledgments

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References

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*(3), 247-285.

Even, R. (1990). Subject Matter Knowledge for Teaching and the Case of Functions. *Educational Studies in Mathematics, 21*(6), 521-544.

Paoletti, T., 2015. Pre-service teachers' development of bidirectional reasoning. (Ph. D. Dissertation). University of Georgia, Athens, GA, Retrieved from https://getd.libs.uga.edu/pdfs/paoletti_teo_j_201508_phd.pdf.

Piaget, J., Grize, J.-B., Szeminska, A., & Bang, V. (1977). Epistemology and psychology of functions (pp. 84-97). Dordrecht, The Netherlands: Reidel.

Saldanha, L. A., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensen, K. R. Dawkings, M. Blanton, W. N. Coulombe, J. Kolb, K. Norwood, & L. Stiff (Eds.), *Proceedings of the 20th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 298-303). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-307). Hillside, NJ: Erlbaum.

Thompson, P. W. (1994). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in Collegiate Mathematics Education: Issues in Mathematics Education* (Vol. 4, pp. 21-44). Providence, RI: American Mathematical Society.

Thompson, P. W., & Carlson, M. P. (2017). Variation, Covariation and Functions: Foundational Ways of Thinking Mathematically. In J. Cai (Ed.), *Handbook of Research in Mathematics Education*. Reston, VA: National Council of Teachers of Mathematics.

OPERATIONALIZING FUNCTIONS BY PROGRAMMING ROBOTS

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Because functions cut across many areas within mathematics, they are especially important for mathematical understanding. Yet research shows that students struggle to develop the functional representation fluency needed for more sophisticated mathematics. This pilot study examines how students use multiple representations of functions in a unit that incorporates functions to program the movement of a robot. Preliminary results indicate that students conceptualize a bidirectional relationship between the algebraic and robotic representations of a function but that the relationships between the other representations remain unidirectional or disconnected. We hypothesize that students' strong connection between the algebraic and robotic representations can be leveraged to facilitate increased understanding of the relationship between the other representations.

Keywords: Algebra and Algebraic Thinking, High School Education, Middle School Education

The concept of a function is essential to the study of mathematics. While students might be most comfortable with one representation of a function, they need the competency to compare, relate, select, use, and flexibly convert between the multiple representations (National Council of Teachers of Mathematics, 2000). The functional representation fluency expected by the NCTM Standards and needed for more sophisticated mathematics has been shown to be difficult for students (Knuth, 2000; Leinhardt, Zaslavsky, & Stein, 1990; Pérez, 2014; Van Dyke & Craine, 1997; Van Dyke & White, 2004). Students have trouble making the connections between different representations even when they contain the same information (Knuth, 2000). In studying 12th grade students' performance on function questions in the 2009 National Assessment of Educational Progress (NAEP) mathematics assessment, Pérez (2014) found that students had more difficulty with function items that involved identifying the appropriate graphs and equations in real-world contexts. This pilot study looks at learner engagement with multiple representations of functions in the context offered by using functions to program robots.

Background

This study in progress is part of a larger National Science Foundation study on integrating computational thinking practices and dispositions in the mathematics curriculum. Following an initial engagement with modeling and programming, several participating teachers have expanded on the project's original conception to develop a unit on functions, programming, and robots. The teachers created a unit where students would apply their knowledge of a velocity-time function of the form $\boxed{y = mx + b}$ within a programming script to control the velocity of a small robot's physical movement. In the equation, the y-value represents the velocity at which the robot travels, the m-value represents the acceleration, the x-value is the time in seconds, and the b-value represents the initial velocity of the robot.

The robots were built using Arduino microcontrollers and students were exposed to programming through the Arduino language. Within the programming script, students manipulated a linear equation by changing the slope and y-intercept (Figure 1, Line 33) to control their robot's velocity. The domain, incrementing from 0 to 5, was manipulated in the for-loop (Figure 1, Line 30).

```
29
30      for(float x=0; x<=5; x++) {
31
32
33       float y=(0*x+0);
34
```

Figure 2. Linear Equation and Domain Conveyed in the Arduino Code

Data Collection

The unit was implemented in 8th grade mathematics classes at a public middle school and 11th grade mathematics classes at a career-technical public high school. One lead teacher taught all 8th grade classes and another lead teacher taught all 11th grade classes. The middle school unit lasted 7 days, while the high school unit lasted 10 days. Each class was video and audio recorded during the unit. There was also a research assistant present to observe, take field notes, and interact with the students and teachers during the sessions.

Representations of Functions: Challenges and Opportunities for Students

The three common representations of functions engaged in classroom settings are algebraic equations, tables, and graphs. The relationships are represented in Figure 2 (see also Leinhardt, Zaslavsky, & Stein, 1990; Van Dyke & Craine, 1997). The double arrows indicate bidirectional fluency between those representations for information and the ability to create those representations in either direction.

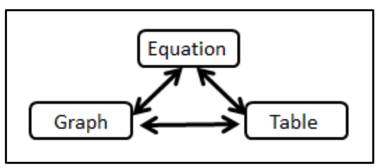

Figure 2. Ideal Configuration of Three Representation Fluency

In the context of the programming robots unit, where the robot's velocity is controlled by a linear function, a fourth physical representation of the robot's movement is conceptualized. The relationships between the four representations are represented in Figure 3.

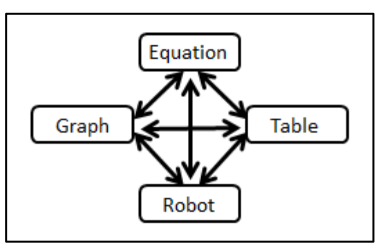

Figure 3. Ideal Configuration of Four Representation Fluency

During initial activities, students discovered how the slope and y-intercept in the linear equation affected the robot's velocity and movement over a fixed interval of time. This created an equation-to-robot relational connection. The students were then confronted with a series of challenges to further develop these connections and their reasoning with functions. For example, one challenge was to drive the robot forward past a target, stop, reverse, and then finally stop on the target. To achieve these goals, students programmed the robot by modifying a linear equation (Figure 1, Line 33) and its domain (Figure 1, Line 30).

One goal of the unit was for students to develop and reinforce fluency in the different functional representations. The meaningfulness of tables and graphs as functional representations was enhanced by their role as tools for inferring the robot's behaviors and the appropriateness of the equation for a given task. For example, the tables and graphs offered salient representations for the behavior of the robot in moving forward, stopping, and then reversing back to its original position. In this task, the robot's behavior represented in tabular form would translate to an equal number of positive and negative velocities that sum to zero over the domain. Understanding this representation can guide students' selection of appropriate slope and y-intercept values in the equation to satisfy the required outcome.

Expected Findings

Preliminary data show that students conceptualized a bidirectional relationship between the equation and robot. However, the relationship between the equation, table, and graph were unidirectional. In other words, most students only completed the table and drew the graph after they determined an appropriate equation for the robot to complete the challenge. In this scenario, tables and graphs served as documents for record of occurrence. They were not used to make predictions. This supports the idea that students did not yet recognize that the tables and graphs offered salient information to help them identify an appropriate equation for the task. This also suggests that, for many learners, there was not a connection between the robot's movement representation and the tabular or graphical representations since those representations were considered last and created from the equation. Thus, based on whole class observations, a general preliminary conception of the students' understanding of the relationships between the four functional representations is expressed in Figure 4.

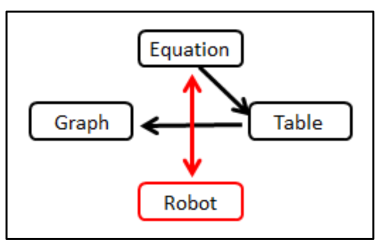

Figure 4. Observed Configuration of Students' Four Representation Fluency

Discussion

Students' strong understanding of the behavior of the robot in relation to the equation is an opportunity to leverage more fluency between all the representations. This bidirectional equation-to-robot connection can be used to build fluency between the robot's movement representation and the tabular and graphical representations. When tables and graphs are readily displayed at the time the robot moves, then there is opportunity for the robot's behavior to be related to these representations. The representation of the robot's movement constrains and grounds the more abstract functional representations (Ainsworth, 2008). The robot's movement is a familiar action to the students that can be used to support their understanding and reconcile any misconceptions regarding the information displayed through algebraic, tabular, and graphical representations (Figure 5a → Figure 5b). Then those developed connections with the robot could be used to mediate the bidirectional relationships between the equation, table and, graph.

A possible trajectory in using a fourth representation – which is contextualized here with the robot's movement – is displayed in Figure 5. A focused look at this trajectory hypothesis will happen during future iterations of the design experiment. Our goal is developing students' proficiency with mathematics to operationalize functions in all their representations and in real world situations. This is an important area for our nation's students to show growth (Pérez, 2014).

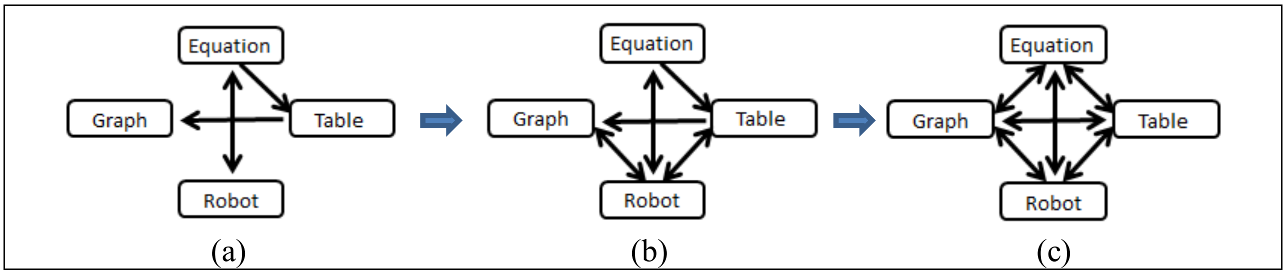

Figure 5. A Hypothesized Trajectory to the Ideal Configuration of Four Representation Fluency

Presentation Overview

The presentation will provide an overview and background of the pilot study. This will be followed by a discussion of the opportunities for student growth in functional representation fluency. The presentation will conclude with how the findings will inform future iterations of the design experiment.

Acknowledgement

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References

Ainsworth, S. (2008). The educational value of multiple-representations when learning complex scientific concepts. In John, G.K., Reiner, M., & Nakhleh, M. (Eds.), *Visualization: Theory and practice in science education* (pp. 191–208). New York, NY: Springer.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of educational research, 60*(1), 1–64.

Knuth, E. J. (2000). Understanding connections between equations and graphs. *The Mathematics Teacher, 93*(1), 48–53.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Pérez, A. (2014). *Functions Matter: Using Performance on NAEP to Examine Factors Influencing Students' Function Achievement* (Doctoral dissertation). Retrieved from ProQuest Dissertations and Theses database. (Order No. 3637172)

Van Dyke, F., & Craine, T. V. (1997). Equivalent representations in the learning of algebra. *The Mathematics Teacher, 90*(8), 616-619.

Van Dyke, F., & White, A. (2004). Making graphs count. *Mathematics Teaching,* 188, 42-45.

EXPLORING FACTORS RELATED TO HIGH SCHOOLERS' ALGEBRA ACHIEVEMENT: A REVIEW OF DISSERTATIONS USING HSLS:09 DATA

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The difficulty that students demonstrate when it comes to learning algebra in secondary school has been long documented and researched. The authors conducted a research synthesis of the literature that uses data from the High School Longitudinal Study of 2009 with a focus on algebra achievement. We've summarized the results of 13 dissertations across 2 outcome variables (9th and 11th grade algebra achievement). We employed optimal resource theory (ORT) as a research framework to inform best-practices for improving student outcomes. Our findings reveal that several malleable factors at the student-, teacher-, school-, and parent-level were found to be related to algebra achievement.

Keywords: HSLS:09 Data, Algebra Achievement, High School, and Dissertation Review

Given the importance of high school students' performance in algebra and the role of algebra as a gatekeeper to higher level mathematics and the pursuit of a career in Science, Technology, Engineering, and Mathematics (STEM) fields, it is imperative we have a clear understanding of the factors that are related to high schoolers' algebra achievement. The purpose of this paper is to summarize the results of dissertations using the nationally representative High School Longitudinal Study of 2009 (HSLS:09) to investigate the malleable student-, teacher-, school-, and parent-level characteristics that are associated with algebra achievement.

Theoretical Framework

This study employs optimal resource theory (ORT) introduced by Anderson (2015) as a research framework to inform best-practices for improving student outcomes. Positive student achievement is among the main interests of ORT. Five principles governing this growth advise to: (1) account for the multiplicity and complexity of factors that influence student development, (2) account for externally controlled factors where practicable, (3) examine manipulable factors that may be internally controlled, (4) focus on incremental progress or reasonable outcomes rather than on comprehensive change, and (5) focus on maximizing progress with available resources, despite external circumstances. Therefore, to advise schools on improving student achievement, ORT would target influential factors that are within school control (such as teaching practices, course offerings, etc.) while taking into account available resources and external factors beyond school-based control that also affect student achievement (such as race, gender, socioeconomic status, etc.). ORT was chosen for this study because the nature of HSLS:09 allowed the dissertations we reviewed to examine a multitude of factors associated with student achievement in algebra while controlling for various external circumstances. Furthermore, our research synthesizes the findings of these studies with a focus on malleable factors. Guided by ORT framework, this synthesis should help to inform teachers, schools, and parents' decisions regarding how best to improve their students' mathematics achievement. Our research question was: What are the student, teacher, school, and parent level factors that are related to high school students' algebra achievement in the U.S.?

Method

In this study, we conducted a synthesis of research literature that used data from the High School Longitudinal Study of 2009 (HSLS:09) and focused on mathematics achievement as the outcome variable. The HSLS:09 is an ongoing nationally representative study consisting of about 24,000 9th-graders from 944 schools in fall 2009. The first follow-up was of 11th graders in spring 2012. The surveys were completed by students, parents, math and science teachers, administrators, and counselors. The students also took a cognitive mathematical assessment, which was designed to provide a measure of student achievement in algebraic reasoning at two points in time (9th and 11th grade). The test framework was designed to assess a cross-section of understandings representative of the major domains of algebra and the key processes of algebra. The test and item specifications describe six domains of algebraic content (the language of algebra; proportional relationships and change; linear equations, inequalities, and functions; nonlinear equations, inequalities, and functions; systems of equations; sequences and recursive relationships) and four algebraic processes (demonstrating algebraic skills; using representations of algebraic ideas; performing algebraic reasoning; solving algebraic problems) (Ingels, et al, 2011, p. v).

Data Sources

We performed a search of the literature through June 2017 that used data from the HSLS:09 and related to mathematics. We excluded all reports and executive summaries from our search, and focused on journals, dissertations, research briefs, conference proceedings, and books. Data collection occurred in 4 stages: (1) Identification: identify records through database searching using variations of the search terms HSLS:09 and math; (2) Screening: screen the records' abstracts for HSLS:09 and Mathematics; (3) Eligibility: screen the methods section for algebra achievement as an outcome variable; and (4) Inclusion: determine the number of studies included in the qualitative synthesis. We searched the following databases: Google Scholar, ERIC, PsycINFO, Education Source, Academic Search Complete, Mental Measurements Yearbook with Tests in Print, JSTOR, Project MUSE, Sociological Abstract, and ProQuest Dissertation and Thesis A&I. We used *RefWorks* to eliminate duplicate records. The number of records remaining after duplicates were removed was 70. Out of the 70 records, we found 28 articles, one research brief, 35 dissertations, 5 conference proceedings, and one book. Preliminary results from a focus on the articles were presented at the 2018 National Council of Teachers of Mathematics (NCTM) Research conference. This study will focus on the dissertations. The methods and results section of each dissertation was reviewed to screen for the use of either 9th or 11th grade algebra achievement as the outcome or dependent variable. After this screening, we were left with 13 dissertations to review. Data was extracted from each dissertation that met our inclusion criteria. The studies were categorized based on the two primary outcome variables; 9th grade algebra achievement (Amar, 2016; Briggs, 2014; Cope, 2013; He, 2014; Kim-Choi, 2015; Larrain, 2015; Onsongo, 2015; & Rochmes, 2014) and 11th grade algebra achievement (Alexander, 2015; Howard, 2015, John, 2017; Maldonado, 2016; & Saw, 2016). Across the 13 dissertations, we classified the level of the factors/predictors/independent variables (IV) as either parent, school, teacher, or student. The factors were then cross tabulated and classified within the four levels and two outcomes with the reported significance level.

Results

The results of the systematic review included both non-malleable and malleable variables at the student, teacher, parent, and school levels that were related to high school students' algebra achievement. Since socio-economic status (SES) was defined as a composite of parent/guardian education, occupation, and family income status, we classified SES as a parent level variable. The significant ($p < .05$) non-malleable factors found were related to SES, region/location of school, race/ethnicity, gender, prior achievement or previous coursework, disability status, first language, and nationality. Next, we will focus on the significant ($p < .05$) malleable IV variables, since these are variables that can be influenced directly.

Parent Level

Parent involvement (as reported by the parents) was found to be positively associated with 11th grade algebra achievement (Howard, 2015).

School Level

Percentage of Advanced Placement (AP) enrollment was found to be positively associated with both 9th and 11th grade algebra achievement (Larrain, 2015; Saw, 2016).

Teacher Level

Class achievement (math teacher perception of the average level of achievement of students in their class) was found to be positively associated with both 9th and 11th grade algebra achievement (Cope, 2013; John, 2017). Also, math literacy (conceptual teaching emphasized by the teacher) was found to be positively associated with 11th grade algebra achievement (Maldonado, 2016). Cope (2013) found that heavy emphasis on teaching math concepts, effectively explaining math ideas, and performing computations with speed and accuracy were positively associated with 9th grade algebra achievement, while heavy emphasis on developing computational skills, the nature and history of math, and reasoning mathematical were negatively associated with 9th grade algebra achievement. In addition, teaching experience (the number of years taught) was positively associated with 9th grade algebra achievement (Cope, 2013).

Student Level

Math identity (students see themselves as a “math person” and believe others do too), math self-efficacy (students’ confidence in their ability to do math), and school engagement (arriving on time and bringing proper materials for class) were found to be positively associated with both 9th and 11th grade algebra achievement (Alexander, 2015; Briggs, 2014: Cope, 2013; Howard, 2015; John, 2017; Kim-Choi, 2015; Larrain, 2015; Onsongo, 2015). Math interest (math is the student’s favorite course and they enjoy it) was found to be positively associated with 9th grade algebra achievement, but negatively associated with 11th grade algebra achievement (Howard, 2015; Kim-Choi, 2015). The number of hours spent per day on homework and math utility (perception of the usefulness of math) were found to be negatively associated with 9th grade algebra achievement (Kim-Choi, 2015; Larrain, 2015). However, student-teacher relationship (students’ perceptions of whether they were treated fairly and with respect) and educational aspiration (highest level of education students expects to attain) were positively associated with 9th grade algebra achievement (Kim-Choi, 2015; Onsongo, 2015). Math effort (how often students pay attention in class, turn in assignments, or keep trying in class) and Peer network (peer academic orientation and peer communication) were found to be positively associated with 11th grade algebra achievement (Alexander, 2015; John, 2017).

Discussion

Based on the ORT theoretical framework and summary of the 13 dissertations that used a nationally representative dataset, we found that one-third of the significant factors affecting algebra achievement were external (non-malleable) factors beyond school-based control (such as race/ethnicity, gender, socioeconomic status, etc.). The remaining factors (that are within school control) were represented at the parent-, school-, teacher-, and student-level, noting that most of these factors explored by researchers were at the student level (approximately 68%). In accordance with ORT principle (1), the dissertations included several factors that may influence algebra achievement. Principle (2) recognizes that these factors likely operate differently depending on external circumstances, thus, race/ethnicity and socioeconomic status—factors out of the student or high school's control—were included as controls. Principle (3) stipulates a focus on malleable factors, so parent involvement, advanced placement course enrollment, and teaching emphasis—something that schools, teachers, or parents could look to change—were included as the main factors of interest. Based on these findings, researchers may provide practical recommendations for schools by considering ORT principles (4) and (5) which respect that progress may be gradual and that schools have finite resources.

References

Anderson, K. A. (2015). An Introduction to optimal resource theory: A framework for enhancing student achievement. *The Journal of Negro Education, 84*(1), 25-39.

Alexander, N. N. (2015). *Statistical models of identity and self-efficacy in mathematics on a national sample of black adolescents from HSLS:09*. (2016-26515-042). Retrieved from EBSCOhost.

Amar, A. (2016). *Does family income moderate the relationship between mental health and academic achievement?* Retrieved from ProQuest Dissertations & Theses A&I.

Briggs, C. (2014). *Mathematics: Self-efficacy, identity, and achievement among African American males from the high school longitudinal study*. (2015-99070-501). Retrieved from EBSCOhost.

Cope, L. M. (2013). *The impact of teachers' characteristics and self-reported practices on students' algebra achievement*. (2014-99031-237). Retrieved from EBSCOhost.

He, Z. (2014). *Examining the academic resilience in mathematics performance for the underprivileged ninth graders using the national data from the high school longitudinal study (HSLS: 09)* Retrieved from https://ttu-ir.tdl.org/ttu-ir/bitstream/handle/2346/58948/HE-DISSERTATION-2014.pdf?sequence=1.

Howard, N. R. (2015). *The influences of mathematics self-efficacy, identity, interest, and parental involvement on STEM achievement in algebra for female high school students* (Ph.D.). Retrieved from ProQuest Dissertations & Theses A&I. (1648432042).

Ingels, S. J., Pratt, D. J., Herget, D. R., Burns, L. J., Dever, J. A., Ottem, R., Rogers, J.E., Jin, Y. & Leinwand, S. (2011). *High school longitudinal study of 2009 (HSLS:09): Base-year data file documentation* (NCES 2011–328). U.S. Department of Education. Washington, DC: National Center for Education Statistics. Retrieved February 21, 2018 from http://nces.ed.gov/pubsearch.

John, T. A. (2017). *The impact of teacher perception of professional learning communities on student algebraic achievement* (Ph.D.). Retrieved from ProQuest Dissertations & Theses A&I. (1899859079).

Kim-Choi, S. (2015). Math-related, school contextual, and demographic variables associated with mathematics achievement of low-performing high school students. Retrieved from ProQuest Information & Learning.

Larrain, R. F. (2015). High school students' mathematics self-efficacy and achievement: Considering ethnicity and school belonging. (Ph.D., Fordham University). Retrieved from ProQuest Dissertations and Theses.

Maldonado, S. I. (2016). *The impact of mathematical literacy practices on Latina/os' achievement* (Ph.D.). Retrieved from ProQuest Dissertations & Theses A&I. (1781592011).

Onsongo, E. N. (2015). *Self-efficacy, academic engagement, and student-teacher relationships for ninth-grade African American male students' algebra I achievement: A structural equation model* (Ph.D.). Retrieved from ProQuest Dissertations & Theses A&I. (1778866558).

Rochmes, J. E. (2014). *Teachers' beliefs about students' social disadvantage: Exploring high school contexts and teachers' influence on the achievement gap* Retrieved from https://deepblue.lib.umich.edu/bitstream/handle/2027.42/108967/jrochmes_1.pdf?sequence=1.

Saw, G. (2016). *Three essays on estimating the effects of school and student improvement interventions* (Ph.D.). Retrieved from ProQuest Dissertations & Theses A&I. (1804423019).

CONCEPTUAL ANALYSIS OF STUDENTS' SOLVING EQUAL SHARING PROBLEMS

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In the present study, we conduct conceptual analysis based on schemes and operations to explain how students with different mathematical levels would solve equal sharing problems. We suggest two distinctive schemes (distributive sharing scheme and splitting scheme for composite units) as the key schemes each of which independently supports the students' solving equal sharing problems.

Keywords: Number Concepts and Operations, Rational Numbers

Equal sharing problems provide students with an opportunity to conceive fraction as a complex entity of multiplication, division, and ratio (Empson, 1999; Empson, Junk, Dominguez, & Turner, 2006). As Kieren (1988) notes, to support students' development of more advanced fraction knowledge, students are to reason with partitive quotient construct as well as part-whole construct. Abstraction of the partitive quotient notion of fraction could be established when students engage in equal sharing activities (Charles & Nason, 2000). The purpose of this study is to present our conceptual analysis of students' schemes and operations to solve equal sharing problems. In particular, we suggest the two distinctive schemes – distributive sharing scheme and splitting scheme for composite units – as the schemes, each of which supports students solving equal sharing problems.

Distributive Sharing Scheme with Partitioning

Young students construct what it means to share fairly based on their prior experiences of dealing out collections to establish fair shares (Wilson, Edgington, Nguyen, Pescosolido, & Confrey, 2011). Thus, students can spontaneously develop, what we named, *distributive sharing schemes* (DS schemes) from their very early age. The situation of a DS scheme is to share a whole number, p units of quantity among a whole number, q persons and the goal is to equally share the quantity among the people. The activities of the scheme consist of *distributing* operations. The student with the DS scheme distributes a unit or multiple units of the to-be-shared quantity to each of the sharing people in turn and the result is one person's share accumulated by repeating the distributing activities. The scheme is closed when the quantity is exhausted by the distribution and each person has an equal share. Quantifying the result of the sharing activity (i.e., measuring the accumulated amount gaining from the distributing activities) is not part of the DS scheme. For instance, for sharing 20 cookies among five people, which is normatively called a partitive division problem, a student whose DS scheme activates for the situation distributes 20 cookies one by one to each person until all the cookies are allotted to the people. She might not realize that one person gets to have four cookies until reflecting on her activities and results upon a teacher's request. One reason for implementing a DS scheme in partitive whole number division is spontaneity of its development from young students' daily life experiences, and another reason would be their lack of operating with a three-levels-of-units structure. In this case, they are to construct a composite unit, 20, as a unit of five units each containing four units so that they can take one part (four cookies) from the equally partitioned five-part whole (20 cookies). As students' multiplicative concepts deepen enough to construct such a three-levels-of-units structure, they are unlikely to use a DS scheme for the partitive whole number division due to its inefficiency.

However, when those students encounter a situation of equally sharing some number of same-sized objects among some number of people, where the result is a fractional quantity, because it requires more than their whole number knowledge, they may feel challenged in figuring out the quantity for one person's share. When establishing a goal of sharing, say, three pizzas among five people, they might attempt to partition the three pizzas into five equal parts at one go. As they realize there is no easy or practical way of cutting the three pizzas into five pieces, the students may go back to use their DS scheme, which would let them attend to sharing subsets of the whole rather than partitioning the whole at once. On the other hand, as students start to learn fractions and thus diverse partitioning operations are ready at hand for their mathematical activities, they should be able to use them as assimilating operations of their DS schemes for solving equal sharing problems. For the problem situation stated above, sharing three pizzas among five people, they can give one-half pizza to each person by halving each of the three pizzas and then partition the remaining one-half pizza into five equal shares. Especially, when students understand sharing multiple units as "multiple instances of sharing a single unit" (Wilson et al., 2011, p. 233), we judge that the students engage in *distributive partitioning operations* (Steffe & Olive, 2010). As reported in previous literature (e.g., Charles & Nason, 2000; Empson et al., 2006; Lamon, 1996; Hackenberg & Lee, 2016), the levels of sophistication in students' use of partitioning strategies increase as their understandings of fraction become more mature. However, one common characteristic of the partitioning strategies derived from students' use of DS schemes is to partition subsets (one or more items) of the whole at a time at their convenience, rather than the whole multiple items at once. In terms of three hierarchical multiplicative concepts based on the ways by which students generate and coordinate composite units (Hackenberg & Tillema, 2009; Steffe, 1992), we conjecture students with the *second multiplicative concept* (MC2) can construct distributive partitioning operation for sharing p items among q people, where q does not evenly divide p. However, a deficiency in the distributive partitioning operation by a DS scheme at the MC2 level is its inability to switch a referent unit in naming the resulting share. For example, even if an MC2 student succeeds to find one person's share by partitioning each of three pizzas into five, taking one part from each pizza, and combining them, she might not know the sharing result is both $\frac{3}{5}$ of a pizza and $\frac{1}{5}$ of the whole three pizzas. To view the result in relation to the two referent units requires "being able to switch between three-levels-of-units structures, which is outside of MC2 students' ways of operating." (Hackenberg & Lee, 2016, p. 260)

Splitting Scheme for Composite Units

A splitting operation for composite units "simultaneously splits each unit in a composite unit containing the units into an equal but unknown number of subunits" (Steffe & Olive, 2010, p. 320). The assimilating operations of the splitting scheme for composite units (SCU scheme) are the operations that produce the *generalized number sequence* whose operations "entail coordinating the basic units of two number sequences, such as the unit of three of a sequence of such units and a unit of four of a sequence of such units, prior to engaging in activity" (Steffe & Olive, 2010, p. 320). Such coordinating ability, considered as a functional accommodation of the splitting operation for composite units (Hackenberg, 2010), plays a critical role in transforming equal sharing problems into the situations for an SCU scheme. For example, for the sharing problem of four bars among 10 people, a student may want to *split* four units of a bar into 10 parts at once. Then the student would attempt to partition each of the four units into some number of parts to generate enough number of parts in total with a goal that the generated number of parts could be evenly divided by the number of people. To find an appropriate number that serves the goal prior to partitioning activities, the student should be able to coordinate two three-levels-of-units structures. In other words, she needs to implement the following steps simultaneously at the level of mental representation: 1) to distribute some number of parts, say, 10 parts to each of the four bars and form a composite unit, 40 as a unit of four units each containing 10 units, 2) to view the composite unit, 40 as a unit of 10 units each of which contains four units. Such coordinating operation enables her to transform an equal sharing problem into a situation for her SCU scheme so that the student can take a fractional part of the same-sized multiple items, which turns into the situation of whole number partitive division. Thus, from the perspective of the student with SCU scheme, to solve an equal sharing problem is equivalent to taking a fractional part of the whole*,* of which the student is explicitly aware throughout the problem solving process. Moreover, given that construction of the SCU scheme is predicated on the construction of the *third multiplicative concept* (MC3), we hypothesize that the student potentially construct flexibility in viewing the sharing result in relation to two referent units (a bar and the whole four bars) without much difficulty. Behavioral distinction between the splitting operation in an SCU scheme from the distributive partitioning operation in a DS scheme is the way to take one person's share. To make one person's share with four bars each of which is partitioned into 10 parts, a student using distributive partitioning operation gathers one part from each of the four bars because each part is the result of sharing each bar. In contrast, a student whose SCU scheme activates for the problem takes a chunk of four parts (maybe the leftmost four parts) of 40 parts because four is $\frac{1}{10}$ of 40.

Our conceptual analysis of schemes and operations in solving equal sharing problems informs that *distributive partitioning scheme* (Steffe & Olive, 2010) develops as the most sophisticated form of DS scheme when splitting operations for composite units are available for the DS scheme as assimilating operations. Despite that, a student who can solve the above problem using her SCU scheme might not feel the necessity to activate her DS scheme with distributive partitioning operation. The SCU scheme with the coordinating ability of two three-levels-of-units structures is enough to find one person's share by splitting the to-be-shared quantity by the number of people at once.

Conclusions and Implications

In this paper, we suggest DS scheme and SCU scheme as the schemes, each of which have its own developmental path and independently supports students' problem solving in equal sharing situations. We separated the two schemes for their distinctive characteristics in nature. DS scheme, itself, is a non-measuring scheme constructed from a very early age of young students, and its main activity (operation) is *distribution* which is directly linked to kinetic motion of sharing even though partitioning operations can be embedded in the assimilatory structure. SCU scheme is a quantitative scheme, and its main operations are various partitioning operations including a splitting operation for composite units. Thus, the activities implemented by the scheme entail quantification process of the sharing result. The results of our conceptual analysis are aligned with previous research findings that even lower grade students could successfully find one person's share (using DS scheme from our perspective) but were challenged when they were to name the amount of share in terms of referent units due to the lack of other mathematical resources such as multiplicative reasoning and fractional knowledge (Empson et al., 2006;

Hackenberg & Lee, 2016; Lamon, 1996, Steffe & Olive, 2010). Our results also complement the study of Hackenberg & Lee (2016) by explaining why four out of the six MC2 students, in their clinical interview, showed distributive partitioning operations, but not distributive partitioning schemes, and only one of the MC3 students used the distributive partitioning operation. We conjecture that the four MC2 students might have implemented distributive partitioning operations as parts of their DS schemes. Five other MC3 students, ironically due to their matured multiplicative reasoning, might not have felt logical necessity to activate their DS schemes with distributive partitioning operations. In this study, we argue that the required partitioning schemes and operations for solving equal sharing problems (and naming one person's share) are on a par with MC3, which is a highly advanced multiplicative concept at the elementary school level. Thus, mathematics teachers are advised to be sensitive to students' level of multiplicative concepts and partitioning operations. Hasty introduction of various partitioning strategies to students at the level of 2nd multiplicative concept or lower might produce an unexpected result that they adopt the partitioning solutions only as efficient, algorithmic strategies for finding one person's share, ignoring quantification process of the share in relation to relevant referent units.

References

Charles, K., & Nason, R. (2000). Young children's partitioning strategies. *Educational Studies in Mathematics, 43*(2), 191-221.

Empson, S. B. (1999). Equal sharing and shared meaning: The development of fraction concepts in a first-grade classroom. *Cognition and Instruction, 17*(3), 283-342.

Empson, S. B., Junk, D., Dominguez, H., & Turner, E. (2006). Fractions as the coordination of multiplicatively related quantities: A cross-sectional study of children's thinking. *Educational Studies in Mathematics, 63*(1), 1-28.

Hackenberg, A. J. (2010). Students' reasoning with reversible multiplicative relationships. *Cognition and Instruction, 28*(4), 383-432.

Hackenberg, A. J., & Lee, M. Y. (2016). Students' distributive reasoning with fractions and unknowns. *Educational Studies in Mathematics, 93*(2), 245-263.

Hackenberg, A. J., & Tillema, E. S. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fraction composition scheme. *Journal of Mathematical Behavior. 28*, 1-18.

Kieren, (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 162-181). Hillsdale, NJ: Lawrence Erlbaum.

Lamon, S. J. (1996). The development of unitizing: Its role in children's partitioning strategies. *Journal for Research in Mathematics Education, 27*(2), 170-193.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences, 4*(3), 259-309.

Steffe, L. P. & Olive, J. (2010). *Children's fractional knowledge*. New York, NY: Springer.

Wilson, P. H., Edgington, C. P., Nguyen, K. H., Pescosolido,R. C., & Confrey, J. (2011). Fractions: How to share fair. *Mathematics Teaching in the Middle School, 17*(4), 230-236.

USING VISUAL MODELS IN FRACTION DIVISION: NUMBER LINES SUPPORT CHILDREN'S ACCURACY AND CONCEPTUAL UNDERSTANDING

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Reasoning about fraction division is difficult for children and adults. We examined the relative effectiveness of three types of diagrams, number line, rectangular area, and circular area diagrams, for increasing children's accuracy and conceptual understanding of fraction division as compared to no diagram at all. Children who used number line and circular area diagrams to solve fraction division problems had the highest rates of accuracy. Children rated problems presented with circular area diagrams as the least difficult. Children who completed fraction division problems with number lines were most likely to consistently reason with conceptually sound, quotitive division models.

Keywords: Number Concepts and Operations, Problem Solving, Rational Numbers, Cognition

Reasoning about fraction operations is a critical aspect of the development of children's mathematics understanding (e.g., National Mathematics Advisory Panel, 2008, p. xviii). Despite its importance, fraction division is difficult for children (e.g., Mack, 2001; Siegler, Thompson, & Schneider, 2011) and teachers (e.g., Luo, Lo, & Leu, 2011). The Common Core State Standards for Mathematics (CCSSM) places an emphasis on using visual models (especially, the number line) to learn the effects of fraction operations and to represent them in a variety of situations, with visual models for fraction magnitudes introduced in first and second grade and visual models for fraction multiplication and division in fourth and fifth grade (CCSSM Writing Team, 2013). In practice, empirical studies demonstrate that area models are particularly common during classroom instruction (e.g., Webel & DeLeeuw, 2016). However, there has been no systematic investigation of the relative benefits and limitations of area models and number lines.

Across the fields of mathematics education, psychology, and mathematics, as well as among practitioners, there is considerable disagreement over which models best support students' fraction reasoning. For example, U.S. preservice teachers appear to be most accurate when reasoning in area contexts (Luo et al., 2011). However, reasoning with number lines may prevent common misconceptions that stem from the whole number bias (e.g., Ni & Zhou, 2005). *Looking back*, there is a wealth of research on visual models and models for division, but little is known about how differences in visual models affect children's fraction division understanding. *Looking ahead*, we hope our collaboration across Psychology and Mathematics Education fosters new perspectives, increased collaboration, and *crosstalk between these disciplines*.

We propose that number lines are more effective at supporting children's deep understanding of fraction operations than commonly-used area models. Our work is guided by a prominent psychological theory of numerical development, the *integrated theory of whole number and fractions development* (Siegler et al., 2011), which suggests the centrality of number lines as a tool for understanding the magnitude of all rational numbers. Number lines may better support children's fraction operation understanding because they better support children's fraction magnitude understanding (e.g., Hamdan & Gunderson, 2017; Siegler et al., 2011). Also, given that number lines support children's understanding of both fraction and whole number magnitudes, they are theorized to facilitate transfer and integration across whole number and fraction concepts (e.g., Siegler at al., 2011). Additionally, a number line allows students to represent more than one numerical magnitude on a single, common scale, which may better serve to highlight the multiplicative structure of fractions. Among area models, Wu (2011) notes the inflexible and limited nature of circular over rectangular models.

We investigated which types of diagrams best support children's accuracy, and elicit sound conceptual models of division, when solving fraction division problems. We hypothesized that any diagram would be beneficial in comparison to no diagram. However, we expected that children who completed problems with number line diagrams would have higher *accuracy rates* and show greater *conceptual understanding* than those who completed problems with circular area diagrams. We expected that rectangular area diagrams would elicit similar performance to number line diagrams, due to their linear nature. However, it is possible that children are more familiar with area models, and therefore more accurate on them.

Method

Participants

Participants were 62 children in late Spring of 5th grade or Fall of 6th grade (*M* age = 11.6*y*, *SD* = 1.4*y*; 45.6% girls; 75.0% White) from one public school in Northeast Ohio. Ohio state standards for mathematics education are aligned with the CCSSM, with fraction division first introduced in 5th grade. At this school, 18.30% of children qualify for the free and reduced price lunch program. Data collection occurred in two phases. Phase 2 included an additional 70 children.

Tasks

Children were randomly assigned to one of four between-subjects diagram conditions (see Figure 1) as they solved 18 fraction division problems: (a) *circular* area diagrams (n = 17), (b) *rectangular* area diagrams (n = 14), (c) *number line* diagrams (n = 16), and (d) *no diagrams* provided (n = 15). Problems varied by divisor and dividend type, including unit fractions, proper fractions, mixed numbers, and whole numbers (e.g., 1/3 ÷ 1/9 = ?, 2 1/4 ÷ 1/4 = ?, 2/5 ÷ 4 = ?). Each provided diagram represented six whole units (e.g., six circles). Whole units were partitioned into the denominator units of the smaller operand. Each problem was presented on a separate page in a random order. Children received no feedback. After solving each problem, children rated their confidence (0% to 100%) and perceived difficulty (*not difficult at all* [1] to *very difficult* [4]).

Procedure

Participants were tested individually in their school by the first author or an undergraduate researcher. We introduced the task by telling children that we were interested in the strategies they used to solve "*new kinds of math problems*", and asking children to show their work for each problem. Then, the researcher demonstrated "*how you can show your work*" using a whole number division example (6 ÷ 2). The researcher spoke about making "*a group of six*" and showing or thinking about "*how big six is*", making "*a group of two*" and showing or thinking about "*how big two is*", and finding "*how many times a group of two goes into a group of six*". In the diagram conditions, the researcher drew a diagram, matching that child's randomly-assigned diagram condition, to demonstrate a quotitive relationship between six, two, and three. In the *no diagram* condition, the experimenter simply wrote the numerals '6' and '2'. Instructions were scripted, and apart from instructions to "*show*" (diagram conditions) or "*think about*" (no diagram condition) the numbers, the script was identical in all conditions. We chose to model whole number division given previous research suggesting that children are more likely to successfully model fraction division immediately after modeling whole number division (Sidney & Alibali, 2017). We chose to model quotitive division given that children favor quotitive models for fraction division (e.g., Fischbein, Deri, Nello, & Marino, 1985).

Results & Discussion

Accuracy

We coded accuracy as whether or not the children wrote the correct answer to each problem somewhere on its page. For each child, we calculated the percentage of correctly answered problems out of the total number of problems attempted. All but six children attempted all 18 problems; data from all children were included in the analyses. Children were more accurate with fraction divisors (43% accuracy) than whole number divisors (25%), $t(56) = 3.30$, $p < .01$. This may be unsurprising given the nature of the quotitive example we provided; division of a fraction by a whole number may be better understood through a partitive model of division. Note that we did not observe this striking difference in accuracy by divisor type in the *no diagram* condition, $t(56) = 0.15$, $p = 0.88$. This may be indirect evidence that, in contrast to the children in the diagram conditions, children given no diagrams were not thinking about the conceptual structure of fraction division (either partitive or quotitive); reasoning with a diagram may invite children to reason conceptually. Furthermore, children in the *number line* condition had the highest overall rate of accuracy (48%), followed by children in the *circular* condition (43%), *no diagram* condition (33%), and *rectangular* condition (31%). However, in an ANCOVA on children's average accuracy, with diagram condition as the independent variable and problem order as a covariate, the overall effect of condition did not differ from zero, $F(3, 57) = 0.90$, $p = .45$. The pattern of accuracy suggests that there may be advantages to both number lines and circular area diagrams.

Conceptual Models

We examined children's conceptual models in each condition by coding children's written work on each problem based on a coding scheme (see Sidney & Alibali, 2017) aimed at categorizing children's overt strategies for fraction division. Critically, children's work on each problem was categorized as reflecting quotitive division, partitive division, or neither. In the number line condition, 12 out of 16 children demonstrated a quotitive or partitive divison model on at least 50% of problems. In contrast, children in either area diagram condition, *circular* (35%) or *rectangular* (43%), demonstrated quotitive or partitive division less often: 12 out of 16 children *vs*. *area* diagrams: 12 out of 31 children, $p = .01$, Fisher's exact probability test. In the *no diagram* condition, students' work rarely reflected a conceptual model. Thus, number line diagrams were most likely to elicit sound conceptual models of fraction division.

Difficulty

Finally, we examined children's difficulty ratings. Children in the *circular* condition rated problems as being less difficult ($M = 1.52$) than children in the *rectangular* ($M = 2.01$), *number line* ($M = 1.92$), and *no diagram* conditions ($M = 1.90$), $t(57) = -2.73$, $p < .01$. Looking across our analyses of accuracy and difficulty, children in the *number line* and *circular* conditions had similar levels of accuracy, and yet children in the *circular* condition reported less difficulty. If these findings are replicated in our full sample, it may be important for teachers and students to know that number lines may be challenging to reason with, but the benefits in conceptual understanding are "worth" the difficulty.

Conclusion

Overall, our findings suggest that diagrams support children's thinking about the *conceptual structure* of division, and that number lines in particular may elicit sound conceptual models of division. Both circular area models and number line models were beneficial for problem solving *accuracy*. Finally, because we reminded children of quotitive whole number division, our findings may provide evidence that children who reasoned with number lines were more likely to *transfer* across whole number and fraction division problems. In the full study, we hope to more closely examine this idea.

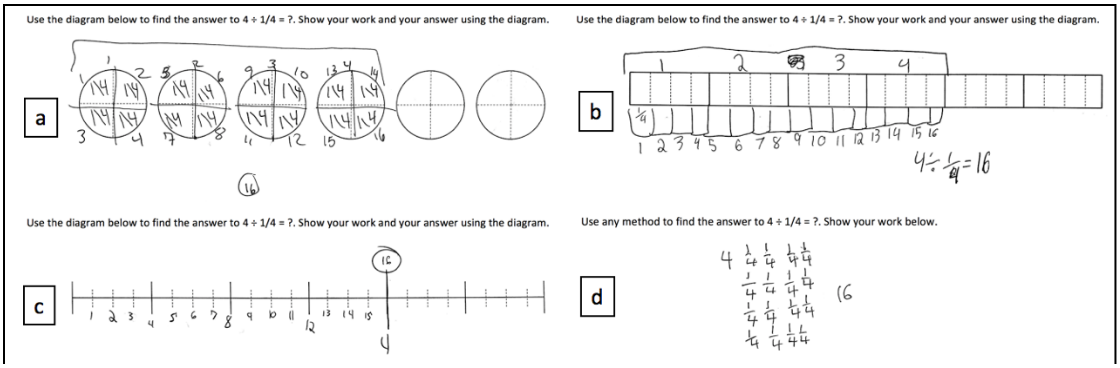

Figure 1. Solved fraction division problems, by condition. An example of correct student work from each condition is shown: circular (a), rectangular (b), number line (c), and no diagram (d).

Acknowledgments

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References

Common Core Standards Writing Team. (2013). *Progressions for the Common Core State Standards in Mathematics: Fractions*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

Fischbein, E., Deri, M., Nello, M. S., & Marino, M. S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16*(1), 3-17.

Hamdan, N., & Gunderson, E. A. (2017). The number line is a critical spatial-numerical representation: Evidence from a fraction intervention. *Developmental Psychology, 53*(3), 587-596.

Luo, F., Lo, J.-J., & Leu, Y.-C. (2011). Fundamental fraction knowledge of preservice elementary teachers: A cross-national study in the United States and Taiwan. *School Science and Mathematics, 111*(4), 164-177.

Mack, N. K. (2001). Building on informal knowledge through instruction in a complex content domain: Partitioning, units, and understanding multiplication of fractions. *Journal for Research in Mathematics Education, 32*, 267–295.

Ni, Y. & Zhou, Y. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychology, 40*(1), 27-52.

NMAP (2008). Foundations for success: The final report of the National Mathematics Advisory Panel. Washington, DC: U.S. Department of Education.

Sidney, P. G., & Alibali, M. W. (2017). Creating a context for learning: Activating children's whole number knowledge prepares them to understand fraction division. *Journal of Numerical Cognition, 3*(1), 31-57.

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology*, *62*(4), 273-296.

Webel, C., & DeLeeuw, W. W. (2016). Meaning for fraction multiplication: Thematic analysis of mathematical talk in three fifth grade classes. *The Journal of Mathematical Behavior, 41*, 123-140.

Wu, H. H. (2011). *Understanding elementary school mathematics*. Washington, DC: Mathematical Association of America.

CONNECTIONS AMONG CURRICULUM, TASKS, AND LINGUISTICALLY DIVERSE SECONDARY STUDENTS' UNDERSTANDINGS OF RATES OF CHANGE

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We examine how two 9th grade integrated mathematics teachers from a linguistically diverse school introduced linear and exponential rates of change, and we describe how their students demonstrated learning on a written assessment and a set of clinical interviews. The teachers collaborated on some aspects of their lesson planning, but used different curriculum materials. We found evidence that students in these two classes learned different concepts. Our preliminary analysis, using Lobato et al.'s (2013) focusing framework, indicates that students' performance can be traced back to the teachers' use of particular curricularly-influenced tools in the instructional environment such as "ratio tables."

Keywords: Algebra, Rate of Change, Curriculum

How does a teacher's use of curriculum materials shape student learning in linguistically diverse classrooms? This paper grows from a five-year design-based research study investigating how secondary mathematics teachers can design learning environments in which English Learners (ELs) develop robust understandings of critical concepts. Our study is conducted in a linguistically diverse school and is focused on the content area of linear and exponential rates of change. The first phase of the design research included an analysis of student assessments and interview tasks. In this analysis, we noticed that the students in two different teachers' classes answered both interview and assessment questions in consistent, but different ways. For example, students in one class used the general formulas for linear and exponential functions more frequently than students in the other class, yet they struggled more so than the students in the other class to use the formulas in a contextualized problem. The teachers had used different curriculum materials to teach the exponential rate of change unit. These observations lead to the following research question: What connections can be found among the curriculum, tasks, and student understandings of concepts related to exponential rates of change? More specifically, what were the different foci of the classroom activities that may have supported the students' interpretations of the tasks involving exponential rates of change? Given the use of two different curricula, what different concepts were in evidence in the students' learning?

Theoretical Framework

This study is rooted in a sociocultural approach to researching the mathematics learning of linguistically diverse students (Moschkovich, 2015), connecting students' opportunity to learn to broader dimensions of the learning environment such as curriculum and assessment (Zahner, 2015). Our analytical approach is informed by Lobato, Hohensee, and Rhodehamel's (2013) focusing framework for students' mathematical noticing, based on the notion that "what students notice mathematically has consequences for their subsequent reasoning" (p. 809). Lobato et al. described that what students notice is shaped by social interactions in the classroom, their interactions with mathematical tasks or curricular materials, and the nature of the mathematical activity in which they participate. The framework has four components. First, *centers of focus* are the mathematical features of a task that individual students notice that can be inferred through their written work, what they say, or through their gestures. The next three components, focusing interactions, mathematical tasks, and the nature of mathematical activity, all contribute to the emergence of the centers of focus. To illustrate the use of this focusing framework, Lobato et al. discussed the mathematical activities of two classrooms both focused on learning about slope as a rate of change, but with different types of activities. As they hypothesized, the students in each class developed different ways of reasoning about slope, influenced by the nature of the mathematical activities, tasks, and social interactions in their respective classes.

Although we adopt a focus on the appropriation of mathematical discourse (Moschkovich, 2015), we noticed a phenomenon similar to Lobato et al.'s (2013) in our study. Specifically, we noted that students in one teacher's class tended to use the equation, both recursive and explicit, as a resource for reasoning about exponential functions. At the same time, students in the other class were fairly adept at using the ratio / difference table to identify and build exponential / linear functions. We hypothesized that both of these trends could be traced back to the instructional environment and the centers of focus that emerged in each class (Lobato et al., 2013).

Methods

Setting

Our study was conducted at City High School (a pseudonym), a linguistically diverse comprehensive high school in California. We observed two Integrated Mathematics 1 (IM1) classes with two different teachers. The IM1 students at City High are primarily in the ninth grade. In the 2016-2017 school year when we collected this data, about 30% of the 9th grade students at City High were classified as English Learners (ELs) and about 56% of the 9th graders were formerly classified as ELs. The classes we observed for our study had similar proportions of ELs and former ELs as the overall ninth grade population.

Participants

We worked with two IM1 teachers, whom we call Mr. S and Ms. G. We observed their classes as they taught both the linear and exponential rates of change units in the fall and spring, respectively, but we focus on the results of the spring classroom observations, student interviews, and student assessments that reflected the introduction of exponential rates of change and a comparison of linear and exponential rates. Mr. S, who had taught at City High for eight years, was in his third consecutive year of teaching IM1. He has a mathematics degree and speaks both English and Spanish. Ms. G had 12 years of experience teaching at City High School and was teaching IM1 for the second year. She has a chemistry degree and holds a credential in mathematics. She is a monolingual English speaker. The IM1 students in their classes also participated in our study. All students were asked to take a pre- and post-assessment. Forty-nine students completed the assessments and 20 students volunteered to participate in problem-based interviews.

Data Sources

The data for this report came from three sources: classroom observations, pre- and post-assessments, and student interviews. The classroom observations coincided with the main instructional days during one unit on linear and exponential functions. These observations were video recorded, and the observer took structured field notes. We spent eight days observing in Mr. S's class and 10 days in Ms. G's class. The assessments consisted of four questions consisting of three problems from the IM1 textbook adopted at the school and one problem from map.mathshell.org. A total of 49 students completed both the pre- and the post-assessment. The student interviews were 45-minute long problem-based interviews using tasks from the IM1 textbook. The interviews were conducted in English and Spanish depending on the students' language preference. A total of 20 students participated in the 17 interviews we conducted. Three interviews were conducted with pairs of students.

Data Analysis

After observations, each researcher wrote reflections about the observation noting interesting mathematical and linguistic moments in the class as well as highlighting notes most pertinent to our study and questions that arose that we may wish to investigate at a later time. A high-level overview of each teacher's unit was created, consisting of the number of days in the unit, a basic outline of topics covered, and notable moments that highlighted the relationship between mathematics and language. Next, a detailed summary of each day of the unit was written, noting what mathematical content was covered, the types of contexts and activities used, unexpected or salient student contributions, and the ways the teachers supported language access and development.

The student assessments were scored using a researcher-created rubric, and all student responses were recorded in a database for further analysis. Statistical analyses showed that there were no significant differences between Mr. S's and Ms. G's classes before or after instruction, but the pre-post gain for the classes was statistically significant ($t(48)=5.7509$, $p<0.001$, using a paired t-test). Overall, about 78% of the students earned a higher score on the post-assessment, with an average gain of 20.5%. We then analyzed the database for patterns in the data. While the overall class performances were not statistically different from each other, we found interesting patterns in the data revealing a difference in foci.

Each of the student interviews were transcribed and coded. Each researcher created a summary of each interview by question and recorded overall impressions of the interview as well as any questions that arose for future interviews or analysis. We created content-analytic summary tables (Miles, Huberman, & Saldaña, 2014) for each question to summarize responses across all student interviews in order to find any patterns that emerged in the data.

Results

As we observed the teachers, we noticed differences in their presentations of the exponential rates of change units and we linked these to the different curriculum resources they were using. Table 1 below summarizes some of these differences. Note the use of multiple representations and multiple contexts in Ms. G's presentation compared to the primary use of tables and bank accounts in Mr. S's presentation of the material.

Table 1: Comparison of Features of the Exponential Rates of Change Units

Teacher	Days	Curriculum	Representations Emphasized	Task Contexts	Common Difference & Common Ratio
Mr. S	10	CME	Tables	Bank accounts	Delta column in table (1st day) Ratio column in table (4th day)
Ms. G	8	MVP	Multiple representations (tables, graphs, equations)	Allowance Filling/draining a pool Book shipments Repeatedly cutting paper into equal pieces Growing dot patterns	Arithmetic & Geometric Sequences (Delta and ratio columns were not introduced until the 8th day.)

Our analysis of the assessments led to several observations, and, in the interest of brevity, we will share two. First, of the students who could correctly identify an exponential function written in recursive function notation (36.7%) and explain why it was exponential (18.4%), all of these students were in Ms. G's class. We traced this difference in student answers back to the more frequent use of function notation and the focus on arithmetic and geometric sequences in Ms. G's class. Ms. G's emphasis on using phrases such as "multiply by three to get to the next term" instead of "multiply by three each time" was reflected in the student's explanations. Second, the majority of the students who could correctly fill in the output and "ratio" columns of a table for a given exponential equation (68.4%) were in Mr. S's class. As shown in the table above, this difference in student performance can be traced back to the earlier introduction of and continued emphasis on using a ratio column in a function table in Mr. S's class.

During the student interviews, the students were asked to describe, in their own words, what it means to be a linear function and what it means to be an exponential function. The students gave a variety of descriptions including graphical descriptions, recursive descriptions, and general equations in explicit or recursive forms. Of the students who gave the explicit forms of the equations (20%), all of these students were in Ms. G's class. Of those who referenced delta and ratio columns (20%), all were in Mr. S's class. Ms. G introduced the explicit forms on the 4th day of her unit and emphasized their use throughout the rest of the unit (7 days), whereas Mr. S emphasized the use of delta and ratio columns in tables throughout his unit (7 days). Additionally, on a question involving a comparison of banks offering simple interest versus compound interest, of the students who could correctly identify which offer would be the best in the long run and explain why, 83.3% were in Mr. S's class. As shown in the table above, Mr. S focused on a similar context (bank accounts) during his unit, while Ms. G's unit consisted of multiple contexts.

Discussion

Although we found differences in students' understanding of concepts related to exponential rates of change, it is important to recall that there were no significant differences in the post-assessment results of the two classes and that both classes had significant gains on the post-assessment. However, it is clear that what happens in the classroom, what emerges as centers of focus, impacts student learning. Our results don't clearly indicate that one approach is better than the other, only that the differing approaches yielded different profiles of student understandings. These results led us to consider the following question. What are the affordances of having an equation-centric view versus a table-centric view? An equation-centric view more easily suggests the existence of continuous covarying quantities, whereas the table-centric view points to the structure of the invariant ratio, which is a nice parallel to the collinear points conception (i.e., invariant slope).

References

Lobato, J., Hohensee, C., & Rhodehamel, B. (2013). Students' mathematical noticing. *Journal for Research in Mathematics Education*, *44*(5), 809–850. https://doi.org/10.5951/jresematheduc.44.5.0809

Miles, M. B., Huberman, A. M., & Saldaña, J. (2014). *Qualitative data analysis: A methods sourcebook*. Thousand Oaks, CA: SAGE Publications, Inc.

Moschkovich, J. (2015). Academic literacy in mathematics for English Learners. *Journal of Mathematical Behavior*, *40*, 43–62. https://doi.org/10.1016/j.jmathb.2015.01.005

Zahner, W. (2015). The rise and run of a computational understanding of slope in a conceptually focused bilingual algebra class. *Educational Studies in Mathematics*, *88*(1), 19–41. https://doi.org/10.1007/s10649-014-9575-x

RELATIONSHIPS BETWEEN UNITS COORDINATION AND UNDERSTANDING CALCULUS CONCEPTS

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Keywords: Post-Secondary Education, Rational Numbers, Number Concepts and Operations

We report on data from a 14-session paired-student constructivist teaching experiment investigating relationships between calculus students' *levels of units* and their understandings of differential and integral calculus concepts. We describe our initial assessments of students' units coordination (Norton, Boyce, Phillips, Anwyll, Ulrich, & Wilkins, 2015), students' reasoning during subsequent video-recorded teaching experiment activities, and conjectures of connections between students' levels of units and reasoning about graphs, including their covariational reasoning (Thompson & Carlson, 2017).

Methods, Results, Discussion

In Summer 2017 we conducted clinical interviews with students enrolled in Calculus I at a public university to assess their levels of units. Debbie was assessed as assimilating with two levels of units, and Trevor was assessed as assimilating with three levels of units. Our learning goals for students in a follow-up paired-student teaching experiment were for them to construct (a) more powerful ways of reasoning about functions, (b) connections between different contexts of the derivative, concavity of its graph, and second differences of average rates of change over intervals of uniform width, and (c) the accumulation function and fundamental theorem of calculus (Thompson & Silverman, 2008). Our goal was understanding affordances and constraints in students' constructions. We noticed persistent differences in students' reasoning about rates of change of linear functions, particularly through the ways the students reasoned with tables and graphs. Debbie reasoned more often with coordination of values; Trevor had a greater propensity for engaging in continuous covariational reasoning. Assimilating with three levels of units does not explain the plethora of differences noticed in their schemes, however, as Trevor's lack of imagery for trigonometric functions constrained his understanding of the integral $\boxed{\int_0^{\pi} \sin t \, dt}$. Future work may explore both how to support students' construction of covariational reasoning before they get to calculus and to continue to support conceptual understandings of calculus for students who assimilate with two levels of units.

References

Norton, A., Boyce, S., Phillips, N., Anwyll, T., Ulrich, C., & Wilkins, J. L. M. (2015). A written instrument for assessing students' units coordination structures. *Mathematics Education, 10*(2), 111-136.

Thompson, P., & Carlson, M. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.) *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: NCTM.

Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. In M. Carlson & C. Rasmussen (Eds.), Making the connection: Research and teaching in undergraduate mathematics, 117-131. Washington, DC: Mathematical Association of America.

Ulrich, C. (2015). Stages in constructing and coordinating units additively and multiplicatively (part 1). *For the Learning of Mathematics, 35*(3), 2-7.

EVOLUTION OF DEVELOPMENTAL STUDENTS' MATHEMATICS BACKGROUND KNOWLEDGE

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Far too many students began their postsecondary mathematics education in remedial mathematics (Bailey, 2009, Schwartz, 2007). In the California State University (CSU) system, approximately a third of their incoming freshmen are considered unprepared for college level mathematics courses (CSU, 2012). For some students in the CSU, a year will elapse before they can enroll in a college level mathematics course. Across the CSU, unless exempted, every admitted student is required to take the Entry Level Mathematics (ELM) test, which aims to measure proficiency in basic skills need to succeed in a college level mathematics course. 50 on the ELM is a cutoff score that determines whether a student needs mathematics remediation or not.

It is important to note that the ELM is not a diagnostic test; as such, it does not shed light on specific contents that students are struggling with. To that end, in this study, we used Second Year Algebra Readiness Test (SYART) to understand the mathematical background knowledge of 1100 students who received a score below 50 on their ELM test. These students were enrolled in a two, four-week courses designed to prepare them for a college math course: beginning algebra and intermediate algebra. In both classes, students met their instructor five times a week, and every class, except exam days, they would spend approximately 30 minutes in cooperative learning that utilizes active learning strategies such as *think-pair share, peer lesson,* and *wait time*. A pre/posttest analysis of SYART showed that students' overall score improved significantly. On average, beginning algebra students' SYART score improved by approximately 39.5%. Using a two-sample t-test, session one witnessed a statistically significant growth with a *p-value of* $\boxed{3.3 * 10^{-63}}$.

To summarize, students improved their performance in several topics of the test. However, the biggest growth were observed in the following topics: Exponents and square roots; Scientific notation, Linear equations and inequalities, Polynomial and quadratic equations. However, several students were still below a critical level in some topics. Specifically, students continue to struggle in graphical representation of solution of equations and inequalities. Still, there is a strong evidence to conclude that the four-week intervention in math remediation has a considerable impact.

References

Bailey, T. (2009). Challenge and opportunity: Rethinking the role and function of developmental education in community college. New Directions for Community Colleges, 145, 11-30. Community college research center publications.

California State University. (2012). Fall 2011 final regularly admitted first-time freshmen remediation system-wide.

Schwartz, A. E. (2007). New standards for improving two-year mathematics instruction. Education Digest, 73(2), 39-42

SEEING THE MATH IN PATTERNS: CHILDREN'S ATTENTION TO NUMERICAL INFORMATION IN A REPEATING PATTERN TASK

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Introduction

A growing body of research has established associations between children's early patterning skills and their formal mathematics knowledge (e.g., Rittle-Johnson, Fyfe, Hofer, & Farran, 2017). To better understand these associations, we sought to assess whether children "see the mathematics" in repeating patterns – that is, whether they spontaneously attend to mathematical information in patterns, such as the number of items in the part that repeats (e.g., the pattern is two circles followed by one square, then it repeats). Research in cognitive development suggests there is high variability in how often children attend to this type of precise quantitative information—referred to as children's "spontaneous focusing on numerosity" or SFON—and that individual differences in SFON predict later mathematics performance (Hannula, Lepola, & Lehtinen, 2010). In this study, we assess children's spontaneous attention to numerical information in a repeating pattern task and examine how this relates to their mathematics skills.

Method and Results

Participants were 36 children ranging from 5 to 13 years old. The same sample was studied in Fyfe, Evans, Matz, Hunt, and Alibali (2017), but for a different set of research questions. Children completed 24 pattern extension problems (e.g., ● ● ■ ● ● ■ ● __) by predicting the next item in the sequence and explaining their selection. We categorized these explanations based on whether they focused explicitly on numerical information in the pattern (i.e., the quantity of specific elements in the unit of repeat, such as "the pattern goes two-one, then two-one") or focused solely on the feature information (i.e., the concrete characteristics of the elements in the unit of repeat including size and shape, such as "it goes circle circle square then a circle"). Children also solved 9 arithmetic problems (e.g., $2 + 4 + 5 + 2 =$ __, $4 + 7 – 7 =$ __).

Children provided number explanations on close to 20% of pattern items, with 58% of children providing a number explanation at least once. The frequency of providing number explanations was not correlated with total pattern scores, $r(34) = .08$, $p = .65$; thus, attention to numerical information was not related to success on the pattern task. However, attention to numerical information was predictive of children's calculation skills. The frequency of providing number explanations was a significant, positive predictor of children's total score on the arithmetic problems, $\beta = .28$, $p = .04$, even after controlling for total pattern scores, nonverbal IQ, and verbal working memory capacity. In contrast, the frequency of providing feature explanations was not a predictor of children's scores on the arithmetic problems, $\beta = .06$, $p = .68$. These results provide a window into how children "see the mathematics" in patterns.

References

Fyfe, E. R., Evans, J. L., Matz, L. E., Hunt, K. M., & Alibali, M. W. (2017). Relations between patterning skill and differing aspects of early mathematics knowledge. *Cognitive Development, 44*, 1-11.

Hannula, M. M., Lepola, J. Lehtinen, E. (2010). Spontaneous focusing on numerosity as a domain-specific predictor of arithmetical skills. *Journal for Experimental Child Psychology. 107*, 394-406..

Rittle-Johnson, B., Fyfe, E. R., Hofer, K. G., & Farran, D. C. (2017). Early math trajectories: Low-income children's mathematics knowledge from ages 4 to 11. *Child Development, 88*, 1727-1742.

RELATING UNITS COORDINATING AND READINESS FOR CALCULUS

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Introduction/Theoretical Framing

Students sometimes experience success in school mathematics if they learn to reason with three levels of units *in activity*, which means they "build" an ephemeral third level of units as part of their way of reasoning rather than assimilating situations with a units (of units (of units)) structure—this difference has implications for the development of multiplicative and algebraic reasoning in the middle grades (Ulrich, 2015). Some students pursue STEM majors in college assimilating with two levels of units, and research suggests connections between students' ways of coordinating units and their ways of understanding rates of change in Calculus (Byerley, 2016). What connections exist, if any, between students' units coordination and their readiness for Calculus? One measure of student readiness for Calculus is The Precalculus Concept Assessment [PCA] (Carlson, Oehrtman, & Engelke, 2010). Past results suggest that a 50% score on the 25-item assessment differentiates student success in calculus: 77% of students scoring more than 12 passed their introductory calculus course, while 60% of students scoring less than 13 received a grade of D or F, or withdrew from the course.

Methods/Results

We assessed the units coordination and calculus readiness of 32 students enrolled in first-term Calculus at a university in United States. Each student completed the PCA and participated in a 15-minute clinical interview, conducted by the second author using the methods and descriptors described in Norton, Boyce, Phillips, Anwyll, Ulrich, & Wilkins (2015). Each student was assessed as either assimilating with three levels of units [S3] (14 students) or as assimilating with two or fewer levels of units [S2] (18 students). A Wilcoxon Signed-Ranks Test indicated that mean PCA scores for students in group S3 ($M = 15.71$) and students in group S2 ($M = 9.78$) were significantly different ($Z = 4.94$, $p < 0.001$). Additionally, 71% of students in group S3 scored more than 12 on the PCA compared to 33% of students in group S2. These results suggest a connection between students' units coordination and Calculus readiness. Implications for attending to students' units coordination and teaching and learning Calculus are discussed.

References

Byerley, C. (2016). *Secondary teachers and calculus students' meanings for fraction, measure, and rate of change*. (Unpublished doctoral dissertation). Arizona State University, Tempe, AZ.

Carlson, M., Oehrtman, M., & Engelke, N. (2010). The precalculus concept assessment: a tool for assessing students' reasoning abilities and understandings. *Cognition and Instruction, 28*(2), 113-145.

Norton, A., Boyce, S., Phillips, N., Anwyll, T., Ulrich, C., & Wilkins, J. (2015). A written instrument for assessing students' units coordination structures. *Mathematics Education, 10*(2), 111-136.

Ulrich, C. (2015). Stages in constructing and coordinating units additively and multiplicatively (part 1). *For the Learning of Mathematics, 35*(3), 2-7.

MODELING COGNITIVE DEMAND ACROSS MATHEMATICAL DOMAINS

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M-capacity is an age-dependent construct (ranging from 2 to 5 among middle school students) describing the number of schemes or actions one can simultaneously hold in working memory at a time (Pascual-Leone, Johnson, & Agostino, 2010). We present a model that uses M-capacity to explain and predict students' performance on tasks across different mathematical domains (multiplicative reasoning, fractions, algebraic reasoning), integrating schemes and actions from units coordination research. Units coordination (Steffe, 1992) describes students' engagement with various levels of units in mathematical tasks. Students can assimilate up to three levels of units at a time and account for additional levels of units through activity. Students are able to solve tasks with different levels of cognitive demand depending on their stage of units coordination and their M-capacity.

In the model, circles, boxes, and triangles represent schemes for assimilating one, two, and three levels of units, respectively; arrows represent mental actions. A uni-directional arrow represents a mental action carried out in activity; a bi-directional arrow represents a reversible action interiorized as part of a scheme. Figures 1(a)-(c) illustrate theoretical capacities for students at each of three stages of units coordination to reason through a multiplicative task: 8x3.

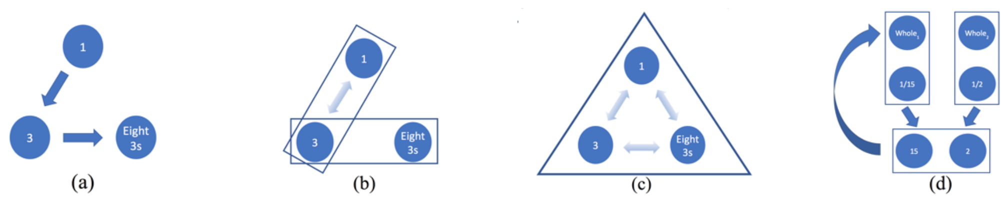

Figure 1: (a)-(c) Stage 1-3 unit coordination; (d) Stage 2 student solving a fractions task.

Figure 1(d) is our model applied to a students' work found in the literature (Hackenberg & Tillema, 2009). Sara was solving the task: "You decide to share that piece (one-fifteenth) of cake between two people. How much of the cake would one person get?" (page 7). The task involved six levels of units for the student to coordinate (circles). As a Stage 2 student, Sara could assimilate the units into three two-level schemes (boxes). Assuming a middle school student M-capacity of 5, she could carry out mental actions for coordinating the first two schemes with the third, through activity (short arrows). However, limited to five schemes/actions, she would not be able to coordinate results from the third scheme (30) back to the whole (long arrow) without relying on figurative material, which fits Hackenberg and Tillema's (2009) descriptions.

References

Hackenberg, A. J., & Tillema, E. S. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fraction composition schemes. *The Journal of Mathematical Behavior*, *28*(1), 1-18.

Pascual-Leone, J., Johnson, J., & Agostino, A. (2010). Mental attention, multiplicative structures, and the causal problems of cognitive development. In *The Developmental Relations among Mind, Brain and Education* (pp. 49-82). Springer, Dordrecht.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and individual differences*, *4*(3), 259-309.

PRELIMINARY GENETIC DECOMPOSITION FOR QUADRATIC RELATIONSHIPS IN REAL-WORLD CONTEXTS

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Keywords: algebra & algebraic thinking, learning trajectories (progressions), cognition

Review of Literature & Conceptual Framework

Students struggle to solve quadratic functions in authentic contexts (Didis & Erbas, 2015), due in part to reading comprehension (Pape, 2004). This paper therefore outlines a preliminary genetic decomposition (Dubinsky & McDonald, 2001) of the mental constructions that support student's conceptions of introductory-level authentic quadratic relationships including function, covariation, and reading comprehension.

Methods & Results

APOS Theory suggests students construct increasingly sophisticated mental structures (Action, Process, Object, and Schema) to understand mathematical concepts. The function strand is from Arnon et al. (2014), and the covariation strand is based on Carlson, Jacobs, Coe, Larsen, and Hsu (2002). For covariation, an Action conception constitutes students identifying how changes in one variable relate to changes in the other, and a Process conception involves students explaining the directionality of related values. An Object conception describes students recognizing that different coefficients cause the relationship between the x- and y-values to differ in measurable ways, and a Schema conception shows students can differentiate between quadratic relationships based on the relationships between the x- and y-values.

Interpreting Pape's (2004) work to coincide with the stages of APOS finds students have a Direct Translation Approach (DTA) approach to reading comprehension at the Action and Process levels, meaning they cannot translate the text, context, units, or mathematical relationships to a problem solution. Meaning-Based Approach (MBA) full-context reading comprehension occurs at the Object level when students can translate the text to mathematical symbols and justify their choices based on context. An MBA-justification approach occurs at the Schema level when students can comprehend at the MBA full-context level while justifying their choices mathematically.

References

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Fuentes, S. R., Trigueros, M., & Weller, K. (2014). *APOS theory: A framework for research and curriculum development in mathematics education.* New York, NY: Springer.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352–378.

Didis, M. G., & Erbas, A. K. (2015). Performance and difficulties of students in formulating and solving quadratic equations with one unknown. *Educational Sciences: Theory & Practice, 15*(4), 1137–1150. doi: 10.12738/estp.2015.4.2743

Dubinsky, E., & McDonald, M. A. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In D. Holton (Ed.), *The teaching and learning of mathematics at university level: An ICMI study* (pp. 275–282). Dordrecht: Kluwer Academic Publishers.

Pape, S. J. (2004). Middle school children's problem–solving behavior: A cognitive analysis from a reading comprehension perspective. *Journal for Research in Mathematics Education, 35*(3), 187–219.

COVARIATION GRAPHING PRACTICES: THE CHANGE TRIANGLE

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Keywords: Algebra and Algebraic Thinking

Creating and interpreting graphs in algebra remains challenging for many students. Past research has identified important ideas and common misconceptions related to graphing (Leinhardt, Zaslavsky, & Stein, 1990). Roth and Bowen (2001) challenge this work by noting that experts in science sometimes demonstrate misconceptions in their interpretation of graphs common in science but not directly related to their research. Roth and Bowen suggest that competent interpretation of graphs requires knowing graphing practices that are associated with specific types of graphs and the social contexts in which they are situated. In this study, we identified a family of practices associated with the use of the *change triangle* (see Figure 1) in a function-based algebra class (Carlson, 2016). We recorded instances of an experienced function-based algebra teacher using the change triangle while teaching, and then analyzed those instances to create descriptions of the ways the instructor attended to and reasoned about different elements of the change triangle.

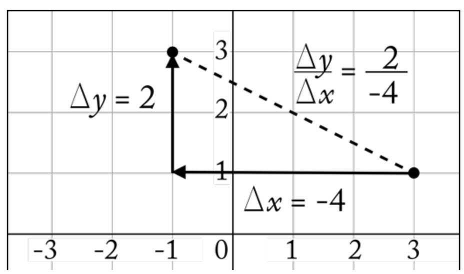

Figure 1. Change Triangle. Elements include a starting point, horizontal and vertical vectors, an ending point, and a dotted or solid line or line segment connecting the two points.

The change triangle is superimposed on graphs to invoke and support covariational reasoning (Thompson & Carlson, 2017) while solving a variety of problems (e.g., finding the vertical intercept of a line given a constant rate of change and an ordered pair). Different elements and multiple copies of the change triangle are attended to and coordinated in different ways to support a variety of quantitative comparisons and achieve multiple purposes. By making these practices more explicit to students, we not only support students in creating and using change triangles to identify and reason about covariation, but also allow for discussions about the norms and strategies for creating and using diagrams to support mathematical thinking.

References

Carlson, M. (2016). *Pathways college algebra: Student workbook* (3rd ed.). Phoenix, AZ: Rational Reasoning.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research*, *60(1),* 1-64.

Roth, W. M., & Bowen, G. M. (2001). Professionals read graphs: A semiotic analysis. *Journal for research in mathematics education*, *32(2),* 159-194.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.

BORROW, TRADE, REGROUP, OR UNPACK? INSTRUCTIONAL METAPHORS CURRICULAR RESOURCES INSTILL AS FOUNDATIONS FOR BASE-TEN NUMBER

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Given that analogical reasoning is a cognitive mechanism that humans use to understand new ideas (Gentner & Bowdle, 2008), instructional metaphors have their place in mathematics education and specifically base-ten number instruction (English, 2013; Pimm, 1981). Metaphors used while learning base-ten number require still more nuanced investigation (Nurnberger-Haag, 2018). The term *trade*, for example, has been treated in practice and research as though it is the target mathematical idea (e.g., Fuson & Briars, 1990; Saxton & Cakir, 2006); however, this is an instructional metaphor that arose due to Dienes blocks (Nurnberger-Haag, 2018). Thus, this study asked: What instructional metaphors do elementary textbooks use for base-ten number?

A preliminary analysis of base-ten number in the second through fourth-grade teachers' editions of 12 U.S. textbook series were coded for metaphors expressed in words (verbal metaphors; Nurnberger-Haag, 2018; see poster for list of textbooks). Ten series expressed more than one metaphor. Metaphors such as *grouping*, *composing/decomposing,* and *bundling* reflect crucial early number concepts (CCSSI, 2010), yet these insufficiently represent base-ten number because these terms imply these processes occur within same unit levels, rather than building the crucial base-ten number concept of composite units (Nurnberger-Haag, 2018; Steffe & Cobb, 1988). Nevertheless, these were the most common metaphors, occurring in 92% of the textbook series. Despite trade metaphors violating base-ten number concepts by failing to communicate the idea of units within units, modeling unintended operations, and opening what should be a closed problem system (Nurnberger-Haag, 2018), 75% used a trade metaphor. Four series used *carry/borrow* along with other metaphors. The packing/unpacking metaphors theoretically predicted to best represent base-ten number operations and concepts (Nurnberger-Haag, 2018) were found in the teacher, but not student pages, of a single series. This analysis combined with future research on how students learn with specific metaphors could inform revisions of textbooks for improved base-ten number understanding for all learners in elementary classrooms.

References

Common Core State Standards Initiative (CCSSI). (2010). *Common core state standards for mathematics*. Retrieved from http://www.corestandards.org/the-standards/mathematics

English, L. D. (Ed.). (2013). *Mathematical reasoning: Analogies, metaphors, and images*. New York: Routledge.

Fuson, K.C., & Briars, D.J. (1990). Using a base-ten blocks learning/teaching approach for first-and second-grade place-value and multidigit addition and subtraction. *Journal for Research in Mathematics Education, 21*(3), 180-206.

Gentner, D., & Bowdle, B. (2008). Metaphor as structure-mapping. In R. Gibbs (Ed.), *The Cambridge Handbook of Metaphor and Thought* (pp. 109-128). New York, NY: Cambridge University Press.

Nurnberger-Haag, J. (2018). Borrow, trade, regroup, or unpack? Revealing how instructional metaphors portray base-ten number. L. Jao and N. Radakovic (Eds.), *Transdisciplinarity in Mathematics Education: Blurring Disciplinary Boundaries* (pp.215-237). Cham, Switzerland: Springer International Publishing AG.

Pimm, David. (1981). Metaphor and analogy in mathematics. *For the learning of mathematics, 1*(3), 47-50.

Saxton, M, & Cakir, K. (2006). Counting-on, trading and partitioning: Effects of training and prior knowledge on performance on base-10 tasks. *Child Development, 77*, 767-785.

Steffe, L.P. & Cobb, P. (1988). *Construction of arithmetical meanings and strategies*. New York: Springer.

HOW BENCHMARKS AFFECT THE NATURAL NUMBER BIAS AND STRATEGY USE IN FRACTION COMPARISON

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Keywords: natural number bias, strategy use, fraction comparison

Fractions are difficult for many people. One source of difficulty is people's tendency to overextend natural number reasoning to fractions. For example, when people are asked to choose the larger of two fractions, natural number components can interfere with reasoning about magnitudes, yielding a "natural number bias" (Ni & Zhou, 2005). However, not all studies reveal the bias, and some studies have revealed a reverse bias (e.g., DeWolf & Vosniadou, 2015). In this study, we investigated whether encouraging people to use benchmarks (reference numbers, e.g., ½) in fraction comparisons would help them to activate fraction magnitudes and overcome a potential bias. We also examined patterns of strategy use.

Adults solved complex fraction comparison problems and reported their strategies on a trial-by-trial basis. All fractions were smaller than 1, and none of the pairs had common numerators or denominators. Half of the pairs were congruent (i.e., the larger fraction had the larger components) and half were incongruent (i.e., the larger fraction had the smaller components). The congruent and incongruent sets were balanced in terms of the fractions' magnitudes relative to common "benchmarks" (i.e., reference points, specifically, ¼, ½, or ¾). In "straddling" problems, one fraction was smaller and the other larger than one of these benchmarks. In "in-between" problems, both fractions were in between two adjacent benchmarks. In a special subcategory of "in-between" problems, both fractions were either smaller than ¼ or larger than ¾; in these problems, one fraction was close to 0 or 1, which may be especially salient benchmarks. Some participants also received a tip that benchmarks could be useful.

Overall, we found a reverse "smaller components—larger fraction" bias. Participants varied in their strategy use across problem types, indicating that they used strategies adaptively. On problems in which one fraction was close to 0 or 1, they used generally incorrect, component-based strategies much more often than on other problems. For the other two problem types, participants used component-based strategies less often, and used benchmark strategies somewhat more often. The tip about using benchmarks had little effect.

Participants used strategies adaptively in ways that made good use of the affordances of different problems (Alibali & Sidney, 2015), including the fractions' relative positions to benchmarks. Thus, patterns of strategy use may at least partially explain the occurrence and the direction of the natural number bias in fraction comparison. To better understand the natural number bias and why it varies across studies and across samples, it will be critical to understand the strategies people use in making specific fraction comparisons.

References

Alibali, M. W., & Sidney, P. G. (2015). Variability in the natural number bias: Who, when, how, and why. *Learning and Instruction, 37*, 56-61.

DeWolf, M., & Vosniadou, S. (2015). The representation of fraction magnitudes and the whole number bias reconsidered. *Learning and Instruction, 37*, 39-49.

Ni, Y., & Zhou, Y. D. (2005). Teaching and learning fraction and rational numbers: the origins and implications of whole number bias. *Educational Psychologist, 40*, 27–52.

ASSESSMENT OF K-2 RELATIONAL REASONING SKILLS: STRENGTHS AND LIMITATIONS OF ITEM TYPES AND FORMATS

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Keywords: Assessment and Evaluation, Number Concepts and Operations, Reasoning and Proof, Elementary School Education

Numeric relational reasoning is often defined as the ability to recognize and analyze relationships between numbers or expressions (Baroody, Purpura, Eiland, Reid, & Paliwal, 2016; Jacobs, Franke, Carpenter, Levi, & Battey, 2007). Students using numeric relational reasoning can use known facts to derive new facts (e.g., using 5 + 5 to solve 6 + 5), solve complex equations by transforming expressions using composition and properties of operations (6 + 5 = ☐ + 4), and recognize when calculations aren't necessary (e.g., 5 + 8 = ☐ + 5). Due to the predictive relationship between numeric relational reasoning and mathematics achievement (Aunio & Niemvirta, 2010; Nunes et al., 2007), it is important to understand how numeric relational reasoning can be assessed so that instruction can be modified to improve students' competence with this construct.

The purpose of this literature synthesis was to (1) identify what instruments currently exist that assess K-2 students' numeric relational reasoning competence, (2) determine different item types and formats for assessing this construct, and (3) determine the depth of reasoning needed by students for each item type.

After an extensive search, seven assessments were found that include items to assess K-2 students' numeric relational reasoning skills. These assessments include differing levels of content representation for numeric relational reasoning and its main components. Three main item types were utilized on these assessments: items with concrete or visual representations, word problems, and items with abstract notation only. The depth of reasoning elicited by these items varied, in part because the item formats did not make the students' reasoning visible. It is recommended that items focus not only on the correct answer but also evaluate students' reasoning strategies to better capture whether students are reasoning relationally.

Acknowledgments

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References

Aunio, P., & Niemivirta, M. (2010). Predicting children's mathematical performance in grade one by early numeracy. *Learning and Individual Differences*, *20*(5), 427-435.

Baroody, A. J., Purpura, D. J., Eiland, M. D., Reid, E. E., & Paliwal, V. (2016). Does fostering reasoning strategies for relatively difficult basic combinations promote transfer by K-3 students? *Journal of Educational Psychology*, *108*(4), 576-591.

Jacobs, V. R., Franke, M. L., Carpenter, T. P., Levi, L., & Battey, D. (2007). Professional development focused on children's algebraic reasoning in elementary school. *Journal for Research in Mathematics Education*, *38*(3), 258-288.

Nunes, T., Bryant, P., Evans, D., Bell, D., Gardner, S., Gardner, A., & Carraher, J. (2007). The contribution of logical reasoning to the learning of mathematics in primary school. *British Journal of Developmental Psychology*, *25*, 147-166.

Chapter 4

Geometry and Measurement

Research Reports

Brief Research Reports

Posters

WHAT INFLUENCES DO INSTRUCTORS OF THE GEOMETRY FOR TEACHERS COURSE NEED TO CONTEND WITH?

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This paper reports on a project aimed at developing a system of professional support for the improvement of the Geometry for Teachers course that mathematics departments teach to preservice secondary teachers. We share data from interviews with 20 instructors to report on how they perceive their position of geometry instructors and the work they do in the course. To inspect this set of interviews, we use the framework of professional obligations to the discipline, to individual students, to the institution, and to the classroom community. We share how references to these professional obligations emerged in the interview data.

Keywords: Geometry and Geometrical and Spatial Thinking, Post-Secondary Education, Teacher Education-Preservice

Introduction

This paper reports on a study of instruction at the college level, specifically focused on the geometry course that many universities offer and is taken by prospective secondary teachers (Geometry for Teachers, or GeT hereafter). We report how GeT instructors perceive their position and the work they do in the GeT course in relation to institutional stakeholders. The literature on instruction at the college level is emerging (Mesa, Wladis, & Watkins, 2014) and in order to frame our focus, we can profit from considering as background the literature on K-12.

The study of mathematics instruction at the K-12 level has often considered the classroom as a container within which interactions among teacher, students, and content unfold. The influence of institutional context on instruction has not always been part of that consideration. Cohen et al.'s (2003) instructional triangle calls attention to the environments in which instruction is situated but most studies of instruction pay little attention to how those environments influence instruction. Some of that is justified on the received wisdom that instruction is "loosely coupled" with administration (Weick, 1976). Awareness in our field of the importance to look at the relationships between institutional and instructional issues has been brewing, particularly from research focused on systemic reform (e.g., Cobb, Jackson, Smith, Sorum, & Henrick, 2013) and on equity (e.g., Lubienski, 2002; Walker, 2007). The realization, particularly from the latter research, is that some of the phenomena that happen at the classroom level (e.g., little access to good mathematics) owes to issues that are structural (e.g., tracking, teacher placement, school climate). As the era of accountability starts to affect also higher education (Levine, 2017), there is reason to consider how environments affect instruction also at the college level. On this matter the K-12 literature can provides some theoretical resources.

Chazan, Herbst, and Clark (2016) describe the position of the teacher as one that connects the institution and its stakeholders with the roles and relationships at play in instruction. Attempts to improve instruction have often relied on the agency of teachers. Chazan et al. (2016) contend that such attention to agency needs to be complemented with attention to structural issues at play in educational institutions and in society at large if we are to understand how instruction can be improved. This paper argues that the GeT course is one that could benefit from instructional improvement and asks the question of how instructors of this course experience the influence of institutional issues that might support or hinder that improvement.

We inscribe the paper within an improvement effort that tries to follow the approach described Bryk, Gómez, Grunow, and Le Mahieu (2015), which challenges the usual improvement paradigm of diffusion of innovation. Rather than conceiving a solution to a problem and seeking to implement it with fidelity and evaluate its results, the approach described by Bryk et al. (2015) starts from creating a *networked improvement community* (NIC). Using an organizational learning perspective, this approach engages the whole network in designing, monitoring, and continuously developing solutions that might improve the system, particularly by attending to variation in performance. This methodology for improvement starts from understanding the system that produces the outcomes that need to be improved. Our attention to how instructors perceive their institutional positions, at the hinge between the institution and classroom instruction, is key in understanding the system in question.

The Need for Improving the Geometry Course for Teachers

A geometry course (HSG, hereafter) has been part of the US high school curriculum for more than 100 years (Sinclair, 2008) and it has traditionally been key in inducting students into mathematical practices such as conjecturing and proving. The changes proposed by the Common Core (Common Core State Standards Initiative, 2010), and the corresponding state assessments have substantial impacts on the HSG curriculum (Wu, 2014). These changes have been accompanied by an increase in the use of students' achievement for individual teachers' accountability (Roth McDuffie et al., 2015). Likewise, the description of "highly qualified teachers" ushered in by the No Child Left Behind [NCLB] legislation (Bush, 2002) suggests the need for teachers to have substantial content preparation in the disciplines they will teach. This suggests an institutional pressure on those who prepare teachers, to align what they teach preservice teachers with what the latter will need to teach their own students. This pressure can be seen in the CAEP standards adopted by 33 states through partnership with NCATE which state that teacher education provider programs must "ensure that candidates demonstrate skills and commitment that afford all P-12 students access to rigorous college- and career-ready standards (e.g., […] Common Core State Standards)." The pressure can also be understood by examining the MET II documents that call secondary teacher preparation programs to offer courses specifically designed to focus on mathematics at the high school level from an advanced perspective, including "address[ing] the CCSS approach to Euclidean geometry based upon translations, rotations, reflections and dilations" (CBMS, 2012, p. 7). But, are GeT courses providing teacher candidates with the knowledge they need in order to teach HSG? Grover and Connor (2000) surveyed the content and instructional practices of geometry courses at 108 randomly selected U.S. colleges and universities and found that GeT course content varies greatly: From a review of middle and high school topics to the development of elementary axioms or a study of non-Euclidean and projective geometries using alternative transformational and analytic approaches. A comparison of the textbooks used for GeT courses produced similar differences. Grover and Connor (2000) concluded that there is no typical curriculum for GeT. This variability in GeT courses prompts questioning their usefulness for teachers (Wu, 2011).

It seems that institutions should have reason to be interested in improving the GeT course by better aligning what is taught to future teachers with the knowledge they need to teach HSG.

However, because Euclidean geometry is no longer an area of active mathematical research, the geometry content high school students study has few stewards in university mathematics departments (Steen, 1988; see also Atiyah, 2001). With the GeT course being a service course for fewer students than other service courses (such as calculus or linear algebra), it is hard for mathematics departments to create local communities to steward the GeT course. Improvement is needed but it may need more than local attention; it could use a network approach.

We are interested in improving the GeT course using the approach described by Bryk et al. (2015), which requires us to start from understanding the system in need of improvement. The GeT course is one where environmental influences (e.g., instructors need to prepare PSTs to teach geometry) could connect with the outcomes of the course (viz., better mathematical knowledge for teaching geometry of preservice teachers). As some research has shown connections between MKT and the mathematical quality of instruction as well as K-12 student outcomes (Hill, Rowan, & Ball, 2005; Hill et al., 2008), increasing MKT would be desirable. Based on performance data, Clements (2003) suggested that students' knowledge of geometry could use improvement. Increasing MKT in geometry might be one lever. Given that the need for improvement in HS geometry instruction points to the possibility to improve the GeT course, a question that can be asked is whether a process of continuous improvement based on principles of organizational learning can be used productively to improve geometry for teachers.

Our project has started developing a networked improvement community by bringing individuals together. We began by locating institutions with large teacher preparation programs—as we are interested in the undergraduate geometry course serving future teachers, rather than geometry courses in general. Within those institutions, we looked at mathematics departments for geometry courses serving secondary mathematics pre-service teachers and identified instructors of those courses as the natural candidates to be members of this community. We are also incorporating other stakeholders, including high school geometry teachers and mathematics supervisors who influence certification policies at state levels. Our first step has been to do a set of initial interviews of instructors of the GeT course. The interviews help us describe members of this group in terms of their professional position as instructors of college students.

Theoretical Framework: Practical Rationality of GeT Instructors

As we consider the effort involved in improving GeT, we are keen to note that like the case in K-12 instructional improvement, the improvement of curriculum and instruction for GeT is likely to need more than resources and networks: It needs know-how, anchored in an understanding of what instructional practice is like in its institutional context (Halverson, 2003). In a review of the research literature on collegiate mathematics, Speer, Smith, and Horvath (2010) reported that there exists "very little research [that] has focused directly on teaching practice—what teachers do and think daily, in class and out, as they perform their teaching work" (p. 99). Further, they argue that "the community's efforts to support instructors as they learn to teach college mathematics is often not informed by data and research on what is involved in teaching college mathematics" and recommend conducting research in this area to guide the professional development efforts designed to improve collegiate mathematics (p. 111). Specifically, in considering what is involved in the improvement of GeT, it would be helpful to understand how GeT instructors negotiate the multiple demands of their role preparing HSG teachers. How do faculty relate to the dual expectation that GeT be a university mathematics class and prepare students to teach high school geometry?

In their account of the practical rationality of mathematics teaching, Herbst and Chazan (2012; see also Chazan, Herbst, & Clark, 2016) identify sources of justification that instructors might use to make their actions reasonable or sensible. Many actions in teaching mathematics go without saying, they don't call for justification but rather are habitual or normative. But quite often instructors deviate from the norm—e.g., instead of correcting a mistaken response, an instructor might ask his or her class to consider whether the response makes sense. Herbst and Chazan (2012) propose that such departures from the norm might be perceived as justifiable by a professional if they help meet one or more of four professional obligations: An obligation to the discipline of mathematics, an obligation to students as individuals, an obligation to the class as a community, and an obligation to the institutions that make room for instruction.

The obligations framework has important implications for the ways we think about supporting GeT instructors in improving their instructional practice. In particular, we can use this framework to unpack the tensions that they perceive as undergirding GeT instruction. For example, the institutional obligation (to the Teacher Education program and State certification agency calling for mathematics departments to offer GeT courses) might compel the GeT instructor to cover the content that his or her students would need to teach in schools, while the instructor's obligation to the discipline of mathematics might compel them to bring in considerations that are more general (e.g., that Euclidean geometry is only one geometry, but different choices of postulates might give rise to different geometries) in order to better represent the discipline. Because time is limited, instructors can't just avoid these tensions, they need to manage them. We see the many tensions that might exist among the professional obligations of undergraduate instructors as fertile places where to start an inquiry toward the systemic improvement of the GeT course. In this paper, we begin that work by sharing what we've learned from GeT instructors through a set of initial interviews. Our research questions are: (1) How does the framework of professional obligations help us understand the professional position of geometry instructors? and (2) How does such understanding help us move toward alignment between the GeT course and the HSG course?

Methods

We recruited 20 participants (8 men and 12 women) from 17 universities across the U.S., all of whom were identified as mathematics department faculty, hold doctoral degrees in mathematics or mathematics education, and have recently taught a geometry course aimed at serving pre-service teachers in large teacher preparation programs. We developed a semi-structured interview protocol to learn about the instructor, their context, and their dispositions toward improving the GeT course. The protocol had three sections: 16 questions about the GeT course and the challenges that come with teaching it, 3 questions (with follow-ups) about the various forms of content covered in the course, and 3 questions about the background of the participant. In this paper we focus responses to the first section, and questions such as: "What experiences do you aim for your students to have in the geometry course for teachers?" and "What are the expectations that shape the geometry course for teachers that you teach?"

We piloted the interview with two non-participants who are experienced instructors of the GeT course before administering it to the rest of the participants. We conducted the interviews using online video-conferencing software that allowed us to capture audio and video records of the interaction. The analysis of the interview data was a multi-step process. We began by taking field notes during the interview and then improving those notes with one or two reviews of the video after the fact. Using the professional obligations framework, we individually coded participants' contributions across the first section's interview questions, meeting together to compare, discuss, and reconcile our understanding of each category as well as identify emerging themes.

Results

In this section, we use the professional obligations of mathematics teaching to organize initial gleanings from the interviews. We share the various ways that each of the professional obligations emerged in the data, illustrating each theme with examples from the data. Lastly, because part of what we aim to do is to understand how well the obligations framework helps us account for instructors' perceptions of their position as instructors of the GeT course, we also share data that did not fall neatly within any of the four professional obligations framework.

Instructors Dispositions toward the Disciplinary Obligation

We had hypothesized that that GeT instructors would relate to their professional position by recognizing an obligation to the discipline of mathematics. Unsurprisingly, we observed a plethora of responses that illustrated various dispositions toward the disciplinary obligation. Three themes emerged from that analysis; a disposition to attend to (1) geometry as a body of confirmed and correct knowledge that GeT students need to know; (2) mathematics as a practice of inquiry, discovery, invention, or knowledge development that people need to engage in; (3) geometry as a set of models which are useful for solving problems. In Figure 1 we illustrate these themes with participants' responses:

Mathematics as ...	Participant Response
A body of confirmed and correct knowledge	"I do think it's important for mathematicians to teach this course… it's a surprisingly mathematically sophisticated course. I think that it's mathematically sophisticated enough that mathematicians should be teaching it" (MV).
A practice of inquiry, discovery, invention, or knowledge development	"High school teachers were definitely as good as I was in recognizing patterns or solving problems. But when they found an answer, they were happy to move on. They didn't seem to me to need to have a rigorous argument about why this pattern worked or why things were the way they were...that's something that I want to convey to my students" (RR).
Providing models useful for solving problems	"[Community members such as businesses] want their people to be problem solvers and have the ability to deal with an unfamiliar situation. ... I think that that's the main value of the class to the society as a whole" (SA).

Figure 1. Examples of data coded as evidence of GeT instructors' disciplinary obligation. Note: Two letters are used to refer to each of the different participants.

Instructors Dispositions toward the Individual Obligation

We hypothesized that GeT instructors would relate to their professional position through a recognition of their obligation to the individual student. While not as prevalent as the evidence for instructors' disciplinary obligation, we observed many participant responses that we agreed would be categorized under the individual obligation. We share two themes that emerged from that analysis: the obligation to attend to the individual as: (1) a cognitive being (who can think, process mentally, etc.); (2) an emotional being (who can feel anger, joy, fear, disgust, sadness, etc.). Figure 2 illustrates these themes with participants' responses:

Individual as	Participant Response
A cognitive being	"[The mathematical experience is] a good thing for a person's spirit…to be challenged and succeed" (IL).
An emotional being	"So I've definitely have gotten some good feedback about people who were scared about, or nervous about teaching geometry at the high school level and after they took the college geometry course at our university they felt like they were ready, or maybe even excited about teaching geometry" (RL).

Figure 2. Examples of data coded as evidence of GeT instructors' individual obligation.

Instructors Dispositions toward the Institutional Obligation

We hypothesized that GeT instructors would relate to their professional position through a recognition of their obligation to the institution. Three themes emerged from that analysis: the obligation to attend to the institution as: (1) a place that provides service to young members of society; (2) a place that has external accountability (teacher preparation programs and certification agencies); (3) an organization that has rules, policies, etc. We illustrate these themes (see Figure 3) with participants' responses to various interview questions:

Institution as ...	Participant Response
A place that provides service to young members of society	"[T]he content in the course and the student work in the course is related to how well they do on things like Praxis tests as well as how it relates to how they actually teach in the classroom two years later" (IF).
A place that has external accountability	"We redesigned it based on the MET II document, actually based on the MET and then revised again with the MET II, and also focusing on the um Common Core Stand—State Standards as to what geometry the teachers are going to have to teach. And so we have totally revamped the course so the focus is on those aspects of geometry" (IF).
An organization that has rules, budgets, etc	"The catalogue affects me - the description talks about axiomatics and finite geometries. While finite geometry is a nice playground, because they can build the models and see everything, I wouldn't have to do that. That course description constrains me" (SL).

Figure 3. Examples of data coded as evidence of GeT instructors' institutional obligation.

Instructors' Dispositions toward the Interpersonal Obligation

We hypothesized that GeT instructors would relate to their professional position through a recognition of their obligation to the interpersonal collective of the classroom. Unlike the previous three obligations, we found few of the instructors' responses that could be filed under the interpersonal obligation. Thus, here we have just one theme that emerged from our analysis:

the obligation to attend to the group of students as a discourse community, that needs to partake of communicative exchanges (Figure 4).

Group of students as ...	Participant Response
A discourse community, that needs to partake of communicative exchanges	"So I want them to collaborate mathematically in and outside of class, I want them to get experience communicating mathematics both in person, we do a lot of presentations at the board, a lot" (MV).

Figure 4. Examples of data coded as evidence of GeT instructors' institutional obligation.

Those Responses Falling Outside the Four Professional Obligations

Prior to closing this section, we take a moment to review some of the instructor responses that felt important to describe how GeT instructors relate to their professional position but were not captured by the professional obligations. These exceptions are rare in the data, as we have only found two instances. These instances can be attributed to instructor's personal resources, including knowledge, experience, beliefs, and identity. One of the responses came in the context of discussing whether or not GeT instructors held any responsibility for teaching students how to engage in work that was specific to the work of teaching, like creating questions for a geometry exam or understanding students' difficulties with geometric proof. RR's response serves as an example of how an individual's knowledge or experiences can elevate the tensions that one experiences in teaching the GeT course:

> I was trained as a mathematician, I was not given any formal training on student teaching/learning - anything I know about student teaching/learning is something that I've read or picked up from [a mathematics education colleague] but there are other [mathematicians] who don't have [mathematics education colleagues] who come at these courses - geometry, or upper level algebra/analysis course who don't have any of that experience. (RR)

Conclusion

The professional obligations framework is useful to understand the professional position of GeT instructors, positions toward the GeT course, though the data speaks also about the importance to attend to personal resources. The interviews suggest that efforts to improve the course by developing a networked improvement community may need collective awareness of the variety of ways in which individuals relate to their positions as instructors.

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References

Atiyah, M. (2001). Mathematics in the 20th Century. *The American Mathematical Monthly, 108*(7), 654-666.

Bryk, A. S., Gomez, L. M., Grunow, A., & LeMahieu, P. G. (2015). *Learning to Improve: How America's Schools Can Get Better at Getting Better.* Cambridge, MA: Harvard Ed. Press.

Bush, G. W. (2002). No Child Left Behind, Act of 2001, Pub. L. No. 107-110, § 115. Stat, 1425, 107-110.

Chazan, D., Herbst, P., & Clark, L. (2016). Research on the teaching of mathematics: A call to theorize the role of society and schooling in mathematics instruction. In D. Gitomer and C. Bell (Eds.), *Handbook of research on teaching (5th edition, pp.* 1039-1097). Washington, DC: AERA.

Clements, D. H. (2003). Teaching and learning geometry. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 151–178). Reston, VA: NCTM.
Common Core State Standards Initiative. (2010). *Common Core State Standards for Mathematics.* Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.
Conference Board of the Mathematical Sciences (CBMS). (2012). *The mathematical education of teachers II*. Providence, RI: American Mathematical Society.
Cobb, P., Jackson, K., Smith, T., Sorum, M., & Henrick, E. (2013). Design research with educational systems: Investigating and supporting improvements in the quality of mathematics teaching and learning at scale. *National Society for the Study of Education Yearbook*, *112*(2), 320-349.
Cohen, D. K., Raudenbush, S. W., & Ball, D. L. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis, 25*(2), 119-142.
Grover, B. W., & Connor, J. (2000). Characteristics of the college geometry course for preservice secondary teachers. *Journal of Mathematics Teacher Education, 3*(1), 47-67.
Halverson, R. (2003). Systems of practice: How leaders use artifacts to create professional community in schools. *Education Policy Analysis Archives, 11*(37).
Herbst, P., & Chazan, D. (2012). On the instructional triangle and sources of justification for actions in mathematics teaching. *ZDM, 44*(5), 601-612.
Hill, H. et al. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction. *Cognition and Instruction, 26*(4), 430-511.
Hill, H., Rowan, B., & Ball, D. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *Amer. Ed. Research Journal, 42*, 371-406.
Levine, A. (2017). How the academic profession is changing. In S. Steinberg (Ed.), *The American academic profession* (pp. 42-53). New York: Routledge.
Lubienski, S. T. (2002). Research, reform, and equity in U.S. mathematics education. *Mathematical Thinking and Learning, 4*, 103-125.
Mesa, V., Wladis, C., & Watkins, L. (2014). Research problems in community college mathematics education: Testing the boundaries of K—12 research. *Journal for Research in Mathematics Education, 45*(2), 173-192.
Roth McDuffie, A., Drake, C., Choppin, J., Davis, J. D., Magaña, M. V., & Carson, C. (2017). Middle school mathematics teachers' perceptions of the Common Core State Standards for Mathematics and related assessment and teacher evaluation systems. *Educational Policy*, *31*(2), 139-179.
Sinclair, N. (2008). *The history of the geometry curriculum in the United States*. Charlotte, NC: Information Age.
Speer, N. M., Smith III, J. P., & Horvath, A. (2010). Collegiate mathematics teaching: An unexamined practice. *The Journal of Mathematical Behavior, 29*(2), 99-114.
Steen, L. A. (1988). The science of patterns. *Science, 240*(4852), 611-616.
Walker, E. N. (2007). Why aren't more minorities taking advanced math? *Educational Leadership, 65*(3), 48.
Weick, K. E. (1976). Educational organizations as loosely coupled systems. *Administrative Science Quarterly, 21*(1), 1-19.
Wu, H. H. (2011). The mis-education of mathematics teachers. *Notices of the AMS*, *58*(3), 372-384.
Wu, H. H. (2014). Potential impact of the common core mathematics standards on the American curriculum. In Y. Li & G. Lappan (Eds.), *Mathematics curriculum in school education* (pp. 119-142). Dordrecht, The Netherlands: Springer.

DEVELOPING AND USING DEFINITIONS FOR PRISMS AND PYRAMIDS

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Teaching geometry courses for preservice elementary teachers, we observed that difficulty with classifying shapes (and, in particular, composite 3-D shapes) persists even after work with simple shapes to support the writing of accurate and unambiguous definitions. We conducted a self-study of our teaching of 3-D concepts to uncover the concept images of pyramid and prism that emerge. We sought to understand the nature of those observed difficulties. We found that using both simple and composite shapes in classification activity exposed more nuanced and complex concept imagery than simple shapes alone. Opportunities to articulate assumptions create a space for all learners to make the language more precise and to create concept definitions that are more resilient.

Keywords: Geometry and Geometrical and Spatial Thinking, Instructional Activities and Practices, Teacher Education-Preservice

Background

Human perceptions of the physical world are primarily made up of 3-D shapes. Many curricula around the world provide opportunities to identify and name three-dimensional solids in early grades (Sinclair, Cirillo, & de Villiers, 2017). However, only a few studies on learners' definitions and conceptions of 3-D solids were discussed in the review conducted by Sinclair, Cirillo, and de Villiers (2017). Bozkurt and Koc (2012) reported that many of the first-year Turkish pre-service elementary teachers (PSETs) in their study found it difficult to define prism; 60% of them either could not provide a definition for prism or could not go beyond stating the fact that it was a term for 3-D shape. Another study identified a variety of concept images that Turkish PSETs hold about the base of 3-D shapes. Many were limited and/or contradictory in nature (Horzum & Ertekin, 2017). Tanguay and Grenier (2010) found that preservice secondary teachers had difficulty defining and describing regular polyhedra, which hindered their later attempts to develop a proof for the existence of only five possible regular polyhedra.

In our geometry courses for PSETs, we have observed the difficulty that our students have in classifying shapes, and, in particular, composite 3-D shapes. This difficulty persists even after significant work with simple pyramids and prisms to support the writing of accurate and unambiguous definitions. Initially, we speculated that this difficulty was related to our PSETs' ability to write and apply formal definitions, but wondered if it was also related to unarticulated concept imagery. The challenge was to create opportunities to articulate problematic concept images and expose the hidden contradictions that make classification difficult. Our analyses have uncovered layers of complexity in PSETs' conceptions of prism and pyramid.

We will provide findings related to two research questions:

1. What concept images of pyramid and prism emerged from in-class activities that focused on defining and classifying 3-D solids?
2. What is the nature of PSETs' difficulty in using established concept definitions and images to classify composite 3-D shapes as pyramids, prisms, or neither?

Theoretical Framework

In our work to help PSETs understand prisms, pyramids, and related concepts, we strive to create opportunities for PSETs to experience cognitive disequilibrium (Piaget, 1985). This is the moment when there is an imbalance between prior knowledge (schema) and experiences that cannot (yet) be explained by it. The process of engaging students in the act of defining is one of iteration and revision; we move back and forth between examining concrete shapes built from wooden models or other commercially made building materials, and writing and revising emerging concept definitions. The activity described later in this paper, *Prism, Pyramid or Neither?* is one of our attempts to perturb the equilibrium of our students in the hopes that they are able to articulate deeply held concept imagery about these shapes and to demonstrate how resilient their conceptions have become.

We used Tall and Vinner's (1981) framework of concept definition and concept image to frame our PSETs' experiences with classification activity. Tall and Vinner describe a *concept image* as "the total cognitive structure that is associated with the concept which includes all the mental pictures and associated properties and processes" (p. 152). These authors distinguish this from a *concept definition* or "a form of words used to specify that concept" (p. 152). For example, one of our students described a pyramid as having "a tippy or pointed top, a base opposite to the top, and triangles around the top point." Individual concept definitions may be different from the "formal concept definition," which is a definition accepted by the mathematical community.

Data Collection and Analysis

This study followed the *Self-study of Teacher Education Practices* as we undertook action research to systematically study our own practices and to make our knowledge and beliefs, along with the dilemmas, decisions, and reflections, explicit (Vanassche & Kelchtermans, 2015). Self-study research makes it possible to share what we have learned from our practices so that it can be examined and transformed by other teacher educators (Bullough & Pinnegar 2001). We adopted an "inquiry as a stance" approach and acknowledged that self-study is an ongoing and complex process (Cochran-Smith, 2003).

We collected data in a geometry course required for all PSETs in a Midwest university including lesson plans and observation notes of about 100 minutes of lessons, lesson stories written by assigned students spanning the first two lessons of the semester, as well as written work from 58 students from three different sections taught by the same instructor on classifying composite polyhedrons.

We will present three stories, built from our data. First, we will use collected data to describe two episodes of classroom instruction related to classifying polyhedra and creating concept definitions for prisms and pyramids. Data collected at the classroom level were analyzed in order to examine existing and emergent concept imagery as the class worked to construct concept definitions for prism and pyramid. Once these definitions had been constructed, we used the quantitative results of the *Prism, Pyramid or Neither?* assignment to determine areas of both success and struggle for individuals involving classifying polyhedra using those classroom-constructed concept definitions. In this activity, we showed PSETs composite polyhedra built by composing pyramids and prisms in different ways. We asked them to identify each as a prism, pyramid, or neither. We were able to interpret their written justifications and identify specific concept images that interfered with classification activity.

Findings and Discussion

Emergent Images and Definitions

In this section, we discuss two classroom episodes. These episodes are amalgams of data from three separate teachings of the same content. They are written to represent the depth and breadth of conversations that occurred, even if each varied in minor ways from the others.

Episode 1. Working in small groups of 3 to 4, preservice elementary teachers were asked to come up with different ways to categorize a set of 15 wooden 3-D shapes that included both polyhedra (e.g., triangular prism, rectangular pyramid) and non-polyhedra (e.g., cylinder, sphere) as seen in Figure 1a. They were asked to record their thoughts on the question, "How are items in a category like one another and how are they different from other shapes?" A variety of categories was proposed and discussed during the follow-up whole-class discussion.

Many issues emerged during this part of the lesson that gave rise to the need for more precise definitions. For example, students had different meanings for the word *face*. Some considered *flatness* as part of their definitions of *face*; thus, a hemisphere would have only one face. Others argued that a hemisphere had two faces—a flat one and a curved one. The word *side* was also problematic. Some used the word *side* to refer to the *faces* of a prism, while others used that word to refer to the *edges* of a prism.

These discussions led to a classification scheme that separated polyhedra from non-polyhedra, with polyhedra being 3-D shapes that had only straight edges and flat faces. The instructor then assigned students to learn more about polyhedra, prisms, and pyramids by visiting the interactive 3-D shapes by Annenberg Learner (https://www.learner.org/interactives/geometry/3d.html).

Episode 2. To begin the second lesson, PSETs shared what they had learned from the website about the definitions as well as their current thoughts on the similarities and differences between pyramids and prisms. Initially, many students had a limited conception of *base*, similar to their Turkish counterparts (Horzum & Ertekin, 2017). The instructor helped students to transcend orientation-dependent conceptions of *base* by drawing attention to identical triangular and hexagonal prisms placed in different orientations (Figure 1b, 1c).

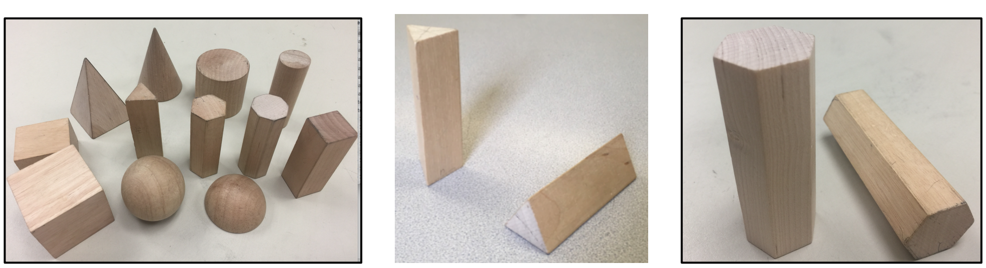

a b c

Figure 1. Classifying wooden models.

Our definition for the base of a prism as a special type of paired, opposite, congruent, and parallel faces required more clarification. When exposed to the hexagonal prisms in Figure 1c, PSETs encountered multiple pairings that fit this description. Introducing the idea of congruent *cross-sections* encouraged the distinction between base and lateral faces (non-base face). Similar discussions explored the idea of *apex* in pyramids as a special kind of vertex where all the edges from the base connected. Finally, the PSETs arrived at the class definitions: prisms are polyhedra with two congruent, parallel bases and all lateral faces are rectangles; and pyramids are polyhedra with a base, an apex opposite to the base, and all lateral faces triangles.

To further solidify the concept definitions and images, the instructors passed out to each group two composite shapes that were made up of two prisms, two pyramids, or one of each and asked the students to name the two shapes that made up the composite shape, describe the way they were connected, and decide whether the composite shape was a prism, pyramid, or neither.

This proved to be a challenging task for many students. For example, one group of students was not sure if the composite shape in Figure 2 was a pyramid. They could identify that this composite shape was made by connecting the base of a square pyramid with a lateral face from a hexagonal prism. They thought it could be named a pyramid because there were an apex and a base, but they admitted it didn't quite look like a typical pyramid. The instructor reminded students to justify their decision using the definition of pyramid that the class had agreed upon. Finally, students examined the shape further and noticed that some of the lateral faces weren't triangles, thus confirming that this composite shape was not a pyramid.

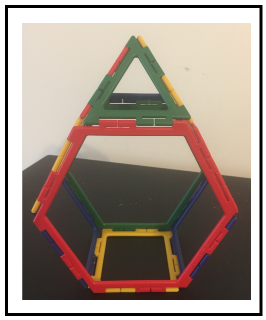


Figure 2. Is this composite shape a pyramid?

Classification Difficulty

We were also able to document our PSETs' difficulties in using established concept definitions and images to classify composite 3-D shapes. After the class activities described above, PSETs were asked to complete an online quiz. The online quiz had two components. The first component asked them to determine if each of 12 composite shapes was a prism, pyramid, or neither. The second part asked them to "write a careful justification to explain whether a composite shape shown is a prism, pyramid, or neither" for three composite shapes selected at random from a subset of 4 of the original 12 shapes.

Because of space limitations, we included here results from only six items: two from the items with the highest percentage of correct responses and four from the items of lowest percentage to provide insights into PSETs' overall performance. We use bold print to indicate correct responses. While the majority of PSETs were able to correctly identify that Figure 3a was a prism and 3b a pyramid, the other shapes were not so easily classified. The quantitative data alone are evidence of the difficulty our students faced in applying definitions to classify 3-D shapes. However, the justifications for these classifications (provided in the next section) illustrate strongly held concept images that might help explain the nature of these difficulties. Below, we describe several concept definitions and images strongly held by some PSETs when classifying composite shapes. We drew support from both the analysis of the quantitative data summarized above and the analysis of the written work.

Difficulties associated with pyramid. Despite the class discussion tending to the classification of the composite shape in Figure 2, about a quarter of PSETs still classified the shape in Figure 3c as a pyramid because it had an apex, a base, and triangular lateral faces. For example, Dakota (mistaking the hexagon for an octagon) wrote, “The composite shape is a pyramid because it has an octagonal prism with another octagon pyramid on top. The definition of a pyramid includes a polygonal base—an octagon, and lateral faces that are triangles that run into one vertex.” She and many PSETs still ignored the fact that some lateral faces on Figure 3c were not triangles; thus, it could not be a pyramid.

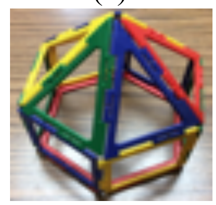
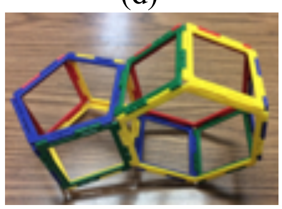
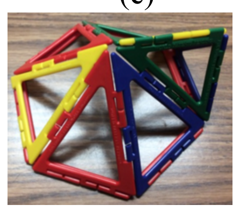
	(a) 	(b) 	(c) 
Prism	**94.83%**	5.17%	6.90%
Pyramid	0.00%	**84.48%**	25.86%
Neither	5.17%	10.34%	**67.26%**
	(d) 	(e) 	(f) 
Prism	32.76%	3.45%	**41.38%**
Pyramid	1.72%	48.48%	6.90%
Neither	**65.52%**	**48.48%**	51.72%

Figure 3. Classification of composite shapes by PSETs (N = 58).

Shape 3e was another difficult one to classify. It was made up of a hexagonal pyramid and a triangular prism connecting through their lateral triangular faces. Half of the PSETs judged it to be a pyramid and half of them judged it to be neither. Our analyses of the written justifications revealed that the class definition of pyramid as “polyhedra with a base, an apex opposite to the base, and all lateral faces are triangles” might not be explicit enough to help PSETs to correctly classify Figure 3e. For example, Ryann wrote, “Pyramid. Because it has one face that is a polygon and all the other lateral faces are triangles they also come to a point.” Ryann’s concept definition of pyramid was previously sufficient to classify all of the wood blocks in Figure 1a, but it was not sufficient to tackle the complexity of this composite figure.

Some PSETs did recognize that there seemed to be odd faces disconnected from the apex that were not bases. This was a moment of disequilibrium (Piaget, 1985) that proved important. Some decided to modify their concept definition to accommodate the new type of face. Katelyn wrote,

> The composite shape is a pyramid because even though it has an extension on one of the lateral faces, all lateral faces are triangles and both bases are polygons. It is a combination of a triangular pyramid and a hexagonal pyramid. It fits the definition of a pyramid, which is a polyhedron where the base is a polygon and all lateral faces are triangles. It breaks no rules and meets all the requirements to be considered a pyramid.

Our analysis uncovered additional concept images and definitions of pyramid that were useful in helping the PSETs to make the correct classification. For example, some PSETs decided that Figure 3e was not a pyramid because "it had a side stick out." Alex stated that for a shape to be classified as a prism or pyramid, "The shape would have to be able to lay on any one of its sides." Jasmin said this is not a pyramid because it is "capable of rocking back and forth." The concept image of a pyramid that could lay stable on each of its many faces was a strong one. Others had included in their concept definitions of pyramid the requirement of having only one base. As Anjou reasons, Figure 3e isn't a pyramid because it has two bases:

> Our definition for pyramid is—a polyhedron where the lateral faces are triangles, has one base which is a polygon. Now. Although all the lateral faces meet at a common point, this shape has more than one base so it has more than one base so it can't be a pyramid either.

Difficulties associated with prism. About 51.72% of the students declared that Figure 3f was neither a prism nor a pyramid. Most justified eliminating pyramid as a choice due to the lack of an apex. However, the fixed orientation impacted PSET's ability to see it as a prism. As Jessica says, "This shape is not a prism because all of the lateral faces are not rectangles, some are triangles and also this shape does not have two congruent bases."

Just like Jessica, many PSETs had the concept image of a base as a face on which the whole 3-D shape sits. So if a student assumes the square as the base, the octagons become lateral faces and it is impossible to find another congruent square parallel to that square base. The idea that shapes retain their form as they are rotating in space (or, for 2-D, on a plane) is a critical conceptual understanding that students need to develop in making sense of both 2-D and 3-D shapes. Some were able to overlook the orientation and identified this shape as a prism, as Jamal wrote, "This composite shape is a prism because it has two congruent bases when I flip the shapes sideways (yellow shapes at the bottom) and the rest of the lateral faces are rectangles which fits in the definition of a prism."

What is it about the shown orientation that renders it unrecognizable? The answer is to return to concept imagery around the term *base*. When classifying simple polyhedra in class, students always oriented their prisms so that they were resting on one of the bases. By doing so, it became a habit to define *base* as a face on which the whole 3-D shape "sits." The singular case is also described as "the bottom" (as in a basement) and the pair of bases as "the bottom and top" of a prism. Jessica explains: "This shape is not a prism because all of the lateral faces are not rectangles, some are triangles and also this shape does not have two congruent bases."

The case of overgeneralization. As seen in Figure 3a and 3b, the majority of the PSETs were able to recognize the composite shape in 3a as a 9-gonal prism made by connecting a hexagonal prism and a pentagonal prism at a congruent lateral face. Also, they recognized the shape in 3b as a pyramid made by connecting two triangular pyramids at a congruent lateral face. Unfortunately, that led some students to believe that the overall shape of a 3-D composite is a prism if it is made up of two prisms, and is a pyramid if it is made up of two pyramids, and is neither if it is made up of a prism and a pyramid.

The statement is obviously false but the challenge is to recognize what parts of this statement contain some truth. It is true that composing a prism and a pyramid will always result in a shape that is neither a prism nor a pyramid. Figure 3c is one example. In the composition process, the figure cannot retain the property of having two congruent parallel bases (thus, not a prism) nor can it cannot retain the property that all lateral faces are triangles (thus, not a pyramid).

The first two parts of the generalization, however, are not always true. In Figure 3e we see a counterexample of a composite shape, made up of two pyramids, that is not a pyramid. The two shapes were connected via a lateral face such that their bases did not create a composed pyramid; their bases are not in the same plane as in Figure 3b. Another counterexample is shown in Figure 3d, which was composed of the same two prisms in Figure 3a by matching the same lateral faces, but twisted so that their original bases would not lie on the same plane.

The composite polyhedra were selected for this activity because they had the potential to bring a variety of concept imagery to the surface related to the terminology used to define prisms and pyramids. However, this last discussion indicates that this activity has the potential to generate even more false generalizations if we do not recognize the complexity of this topic.

Conclusions and Implications

We have chosen to closely examine our instruction related to prisms and pyramids partly because of the difficulties we observed PSETs having with classification activity related to both 3-D and 2-D shapes. However, we were surprised at the depth to which we were able to take our analysis, indicating that composing, decomposing, and classifying 3-D shapes is far more complex than we previously thought. The power of self-study is to uncover assumptions, and we feel that the methodology was successful in that regard.

From this experience, we find that it's not enough to use simple ready-made solids such as wooden blocks when exploring 3-D concepts with PSETs. While they are sufficient to sort polyhedra from non-polyhedra, they lack the complexity necessary to lead to a deep discussion about related concepts (e.g., base, lateral face, edge, side, and apex) to make the properties of prisms and pyramids clear. Using simple polyhedra allows for ambiguity and assumption.

Composite shapes and complex polyhedra (including platonic solids) have the power to cause disequilibrium (Piaget, 1985) and perturb the concept imagery that PSETs take for granted. The concept images that PSETs have of prisms and pyramids as well as related concepts are myriad and rich, but often go unarticulated. Opportunities to articulate assumptions create a space to make the language more precise and to create concept definitions that are more resilient.

Supporting students in their examination and classification of polyhedrons has long-term implications at all levels. The act of composing and decomposing are central to the development of measurement concepts (Feikes, Schwingendorf, & Gregg, 2008). A robust understanding of prism and pyramid is important to the future study of measurement concepts such as volume and surface area. Many concept images are formed throughout the teaching episodes presented here. Specifically, using cross sections to make distinctions between bases and lateral faces on prisms has great promise when it comes time to develop formulas for volume.

Furthermore, teaching the act of defining rather than a memorized definition (de Villiers, 1998) creates a space to challenge and refine concept imagery that conflicts with more formal concept definitions. This, in turn, supports a more robust understanding of the concepts we are trying to define. Providing a myriad of activities that help students assimilate increasingly complex shapes into their schema for polyhedra, prisms, and pyramids challenges assumptions and opens up opportunities for nuance and precision in the way we are all able to collectively negotiate meaning and shared understanding.

Both the theoretical frameworks by Piaget and Tall and Vinner have supported the design and analyses of numerous mathematics education research projects in the last 40 years. In this study, we found them also to be helpful in illuminating our quest for understanding the nature of difficulties behind 3-D classification. One area for future research is to continue to explore the use of their constructs to understand the nature of student difficulty with other challenging topics related to 3-D solids, such as surface area and volume, for both PSETs and K-12 learners. Another study could focus more on the general conceptions of definition held by PSETs and the impact of curricular moves on those conceptions. Leikin and Zazkis (2010), looking across multiple research studies, argued that teachers' concept images and their understanding of the notion of definition influence the ways in which teachers introduce mathematical content to their students. Working in conjunction with methods instruction, it would be important to extend the study into a field experience where PSETs were tasked with designing and/or implementing lessons on 3-D shapes with elementary students.

References

Bozkurt, A., & Koc, Y. (2012). Investigating first year elementary mathematics teacher education students' knowledge of prism. *Educational Sciences: Theory and Practice*, *12*(4), 2949–2952.

Bullough, R. V., Jr., & Pinnegar, S. (2001). Guidelines for quality in autobiographical forms of self-study research. *Educational Researcher, 30*(3), 13–21.

Cochran-Smith, M. (2003). Learning and unlearning: The education of teacher educators. *Teaching and Teacher Education, 19*, 5–28.

de Villiers, M. (1998). To teach definitions in geometry or to teach to define? In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 248–255). Stellenbosch, South Africa: University of Stellenbosch.

Feikes, D., Schwingendorf, K., & Gregg, J. (2008). *Connecting mathematics for elementary teachers*. Boston: Addison-Wesley Longman.

Horzum, T., & Ertekin, E. (2017). Prospective mathematics teachers' understanding of the base concept. *International Journal of Mathematical Education in Science and Technology*, *49*(2), 176–199.

Leikin, R., & Zazkis, R. (2010). On the content-dependence of prospective teachers' knowledge: A case of exemplifying definitions. *International Journal of Mathematical Education in Science and Technology, 41*(4), 451–466.

Piaget, J. (1985). *The equilibration of cognitive structures: The central problem of intellectual development*. Chicago, IL: University of Chicago Press.

Sinclair, N., Cirillo, M., & de Villiers, M. (2017). The learning and teaching of geometry. In J. Cai (Ed.), *Compendium in research in mathematics education* (pp. 457–489). Reston, VA: National Council of Teachers of Mathematics.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151–169.

Tanguay, D., & Grenier, D. (2010). Experimentation and proof in a solid geometry teaching situation. *For the learning of mathematics*, *30*(3), 36-42.

Vanassche, E., & Kelchtermans, G. (2015). The state of the art in self-study of teacher education practices: A systematic literature review. *Journal of Curriculum Studies, 47*(4), 1–21.

EVALUATING GEOMETRIC DEFINITIONS

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Definitions are fundamental to the work of geometry, but many students struggle with understanding terms, and fail to learn the role of definitions in mathematics. We explore how ten college-level students created and evaluated definitions for common geometric terms. Using the idea of concept image and definition, we hoped to learn about the links between how students think about geometric terms, how they define them, and how they evaluate definitions. We found that despite demonstrating strengths in their understanding, none of them evaluated definitions by constructing a biconditional argument. We suggest ways of addressing the issue.

Keywords: Geometry and Geometrical and Spatial Thinking, Reasoning and Proof

Definitions play an essential role in teaching and learning mathematics (Vinner, 2002; Zazkis & Leikin, 2008). They shape the relationship between a concept image and a concept definition and form an essential part of one's knowledge structure that affects their thinking processes (Tall & Vinner, 1981). However, research has demonstrated that students experience difficulties in constructing and evaluating definitions (e.g., Zazkis & Leikin, 2008). The definitions that learners generate mirror their understanding of particular mathematical concepts (Zazkis & Leikin, 2008), and serve as a lens for examining their understanding of what a mathematical definition entails. Examples are illustrations of concepts and principles, and we view the act of defining and evaluating definitions similarly, which allows us to draw inferences about the participants' knowledge of the concepts involved in the definitions. The purpose of this study is threefold: (1) to investigate the process of constructing and evaluating definitions for basic geometrical concepts, (2) to investigate concept images of college students for these concepts, and (3) to analyze the relationship between college students' concept images with how they construct and evaluate definitions.

Theoretical Framework

Three tools framed our research; concept image, concept definition, and proof schemes. Hershkowitz (1990) described types of behaviors that explain the quality of the concept images an individual has. We used two in this study, limited and complete concept image. Individuals with a limited concept image make judgments based on a few prototypical examples plus some properties drawn from those examples. Whereas, individuals who can provide a wide variety of examples and all the important properties of these examples are considered to have a complete concept image. Images can either be static or dynamic. To have a dynamic image of rectangle, for example, would mean to have an image of all the possible variations in rectangle that still maintain the defining properties. Whereas having a static image includes only few examples of the figure and being unaware of how changes to one image might lead to another. We consider this type of image to be related to Harel and Sowder's (1998) transformational reasoning.

Only definitions by property are discussed because they were the only types of definitions the participants created. Zaskis and Leikin (2008) evaluated participants' definitions using a framework, with three criteria: accessibility and correctness, richness, and generality. We adopted these criteria except accessibility. The correctness of a definition means that the definition includes a set of necessary and sufficient conditions. Richness addresses whether the definitions include properties other than the well-known properties (i.e. sides and angles), and targets whether the definition shows signs of a robust understanding. Generality refers to whether the definition relates to general objects of the concepts being defined rather than specific ones, and is similar to de Villiers's and colleagues (2009) meaning for economical definitions.

Definitions are arbitrary, and it is valuable to have a diversity of equivalent definitions because one can chose which definition best suits their problem solving interests (Borsai, 1992; Sinclair, et al., 2012). Having a rigid understanding of definitions might imply that the individual sees definitions as given a priori and fails to see the role that humans play in creating them. By equivalent definitions, we mean logically equivalent where one must show that a biconditional relationship exists. However, we examined the types of reasons the participants gave using Harel and Sowder's (1998) taxonomy of proof schemes, where a proof scheme is a way of thinking about how to ascertain or persuade for an individual or a community.

Methodology

In this section we describe the participants, data sources, and the analysis of the data. Ten college students participated in the study. Nine were secondary mathematics education or mathematics majors, one was an accounting major with a minor in mathematics. They were given between four and five geometry terms and asked the following questions in sequence. Can you draw an example of the term? They were continually asked for a new, different example until they described aspects of the image they were changing. We hoped by asking these questions we would get a sense of the images that students had for the terms, how diverse those images were, and whether or not their images were dynamic. Next, we asked them to define the term in as many ways as they could. This was followed by asking them if are there any properties they felt were important for someone to know about that were not included in their definition(s). With these questions we hoped to get a sense of the types of definitions they had for the terms in regards to the definitions correctness, richness, and generality. Finally, they were presented definitions and asked to decide if the definitions were valid or not and to explain their reasoning. The interviews were recorded with one camera capturing the participant's work and another camera capturing the interaction. Any work that was performed was scanned, and thick descriptions were created that included, transcripts, work, drawings, gestures, and tone.

To analyze the data, we independently watched each participant's video, collecting all the images they created for each term and their names. Each definition the participant gave was coded for correctness, richness, and generality. This gave us a sense of their concept image. Finally, we coded the participants' reasons for accepting or rejecting the alternative definitions using proof schemes. We met to discuss the summaries of our work and resolved disagreements collaboratively. After doing this for all of the participants' videos, we met to discuss key themes. After elaborating our meaning for these themes we went back through and again watched one participant's interview at a time to categorize their work by these themes.

Results

In this section, we describe the participants' concept images and definitions as well as how they evaluated definitions in the sections that follow.

Participants' Concept Images

All of the participants had a complete concept image for the majority of the terms. However, we noted four relevant features. First, all the participants were consistently able to create an example of each figure in the class with one exception explained below. Second, almost all demonstrated evidence of thinking hierarchically and transformational reasoning (Harel &

Sowder, 1998). For example, Sally drew the black parallelogram in Figure 1(a), marked the angle and said she could continually draw new, different examples by changing the angle, but keeping the sides the same length. When asked to draw an example of what she meant, she drew the red parallelogram. The subtle, but important, distinction between what Sally claimed to be doing and the image she drew is captured in Figure 1(b). Similarly, many participants had difficulties creating the images they described, and when they did create the images in their minds, the images did not behave the way they drew them. So, although they demonstrated transformational reasoning, their images could misrepresent their thinking.

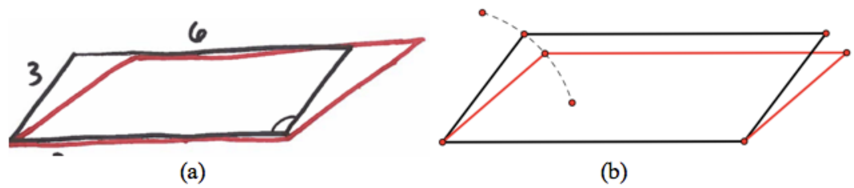

Figure 1. (a) Sally's drawing to demonstrate how to dynamically change angle measure while maintaining side lengths in a parallelogram. (b) Researchers' construction of Sally's description.

The exception described above came when the participants were asked to draw a generic term like quadrilateral. Almost all participants consistently drew a special case as their first image, and continued to draw special cases. As Sally put it, "when I'm asked to draw a triangle, I always just draw a right triangle." Because the participants regularly reasoned empirically, their specialized images left them open to verifying false claims, and making false claims. For example, it was not uncommon for them to claim a median of a triangle is also perpendicular to the opposite side because the shapes they reasoned with were specialized.

Participants' Concept Definitions

We asked the participants for several definitions, but only three participants were able to create multiple definitions. And in these cases, the definitions were not different. For example, James originally defined a rectangle as a quadrilateral with four right angles. When asked for another, different definition, he simply swapped quadrilateral for parallelogram. In total, 50 definitions were created and only four were invalid. Another fifteen definitions failed to meet necessary conditions because the participants used terms like shape or figure instead of polygon or quadrilateral. We considered these omissions to be minor. While evaluating the following definition for triangle – a triangle has 3 sides and 3 angles – Cameron said, "It would help if there was some other information … like it's a polygon … but that's probably understood." We felt that the participants were quite skilled at creating at least one definition. Eighteen of them not only meet necessary and sufficient conditions, but were also economical.

Evaluating Researcher-Generated Definitions

Given their rich concept images and ability to create definitions, we felt the participants would do well at evaluating definitions. They evaluated 38 definitions in total, seven incorrectly, and three correct justifications used invalid reasons. Thus, the participants were inaccurate when evaluating definitions over 25% of the time. Given their success in the other areas, we wondered what the cause was.

The participants evaluated definitions three ways, using non-examples, examples, and arguments. To use non-examples, the participants tried to imagine a figure that was a non-example of the term that fit the properties used in the researcher-generated definition. There were two different ways that they evaluated definitions using examples. For the first way, they imagined examples and checked whether the term had the properties in the researcher-generated definition. James, for instance, was evaluating the definition, a parallelogram is a quadrilateral with diagonals that bisect each other, when he said, “Yeah, I think so. I’m just thinking by looking at these [pointing to the pictures of parallelograms he had drawn]. Yeah, they bisect each other”. For the second way, they imagined what figure(s) could be created if they drew an example with the properties in the researcher-generated definition and checked to see if those examples matched their concept image. For example, Eve drew several examples that fit the following definition: a parallelogram is a quadrilateral that has a pair of opposite sides that are parallel and equal in length. Then said, “Anything you could give would fit the description of a parallelogram.” Interestingly, none of the participants used both methods when evaluating the definitions. Had they done so, they would have, at least empirically, proved the biconditional nature of the definition. The participants’ arguments additionally only proved the conditions in the researcher-generated definition produced the conditions in their definition. Again, not proving the biconditional nature of the definition. We explore the potential reasons for this next.

Discussion

Students need to take a greater role in the process of creating and evaluating definitions (Borsai, 1992; Vinner, 2002). Kobiela and Lehrer (2015) showed that when 6th-grade students were given these opportunities they learned about the concepts and about the process of creating a definition. Evaluating definitions, valid and not, presents an opportunity for valuable and difficult mathematical work. By examining non-definitions, students get the opportunity to learn about how definitions are created. Examining alternative definitions provides students with an opportunity to work in an authentic mathematical environment, and see that some, but not all, properties meet necessary and sufficient conditions. Moreover, it provides students with an authentic opportunity to use proof; something the participants did spontaneously and regularly. Finally, students will have the opportunity to sharpen their own understanding of the terms as alternative definitions help students make connections between terms they see as distinct, and to see distinctions between terms that are similar.

References

Borsai, R. (1992). *Learning mathematics through inquiry*. Portsmouth, NH: Heinemann

de Villiers, M., Govender, R., & Patterson, N. (2009). Defining in geometry. In T. Craine (Ed.). *Understanding geometry for a changing world.* Reston, VA: NCTM.

Hershkowitz, R. (1990). Psychological aspects of learning geometry. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and cognition* (pp. 70–95). Cambridge, UK: University Press.

Harel, G., & Sowder, L. (1998). Students’ proof schemes: Results from exploratory studies. In A. H. Schoenfeld, J. Kaput, E. Dubinsky (Eds.), Research in Collegiate Mathematics Education vol. 3, Providence, RI: American Mathematical Society.

Kobiela, M., & Lehrer, R. (2015). The codevelopment of mathematical concepts and the practice of defining. *Journal for Research in Mathematics Education, 46*(4), pp. 423-454

Sinclair, N., Pimm, D., & Skelen, M. (2012). *Developing essential understanding of geometry grades 9-12*. Reston, VA: NCTM.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169.

Vinner S. (2002) The role of definitions in the teaching and learning of mathematics. In: Tall D. (ed.) *Advanced Mathematical Thinking* (pp.65-81). Mathematics Education Library, vol. 11. Springer, Dordrecht.

Zazkis, R., & Leikin, R. (2008). Exemplifying definitions: A case of a square. *Educational Studies in Mathematics, 69*, 131-148.

EXPLORING TRIGONOMETRIC RELATIONSHIPS: IS IT A FUNCTION?

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We examined conceptual understandings of preservice secondary mathematics teachers as they reasoned about chord length and arc length in a directed-length representation related to the sine function. We characterized the ways in which our participants understood the functional relationship between the geometric objects by describing various aspects of their concept images, and the progression of the images over time. Concept image components progressed from less useful to more useful, eventually aligning with components of a standard definition of function and key features of the sine function.

Keywords: Reasoning and Proof, Geometry and Geometrical and Spatial Thinking, Teacher Education-Preservice

Understanding trigonometric functions is important as that understanding is "a pre-requisite for understanding topics in Newtonian physics, architecture, surveying, and many branches of engineering" (Weber, 2005, p. 91). However, practices for teaching trigonometry have led to difficulties including "underdeveloped angle measure understandings" (Moore, 2014, p. 103) as well as poor connections to the unit circle (Moore, LaForest, & Kim, 2016). Even with a knowledge of right-triangle trigonometry, students may not be able to define function, a foundational concept in mathematics, nor know how to describe sine as a function (Weber, 2005). Clement (2001) described common perceptions of function including: (a) a relation that passes the vertical line test; (b) a machine that gives an output for an input; and (c) a correspondence following a clear pattern, rather than arbitrarily matched values. Many believe functions should be: (a) given by a rule, (b) continuous, and (c) one-to-one (Clement, 2001).

Weber (2005) and Moore (2014) concluded that using quantitative reasoning was a promising avenue for learners to make sense of and articulate properties of the sine function. However, in Weber's study, students used measurements, rather than focusing on quantities. Hertel and Cullen (2011) found that preservice teachers (PSTs) were able to make sense of trigonometric relationships when using a directed-length representation of the basic functions. In a directed-length representation, a vector (i.e., with direction and length), related to a circle arc, represents a trigonometric function. We build on the work of Hertel and Cullen by considering PSTs' sense making using directed lengths as objects or quantities about which to reason.

This study focused on the first in a sequence of learning activities in which PSTs reasoned quantitatively about two dynamically changing objects in a circle, and whether those objects could be considered inputs and outputs of a function. Our work was guided by the research question: Which aspects of PSTs' conceptual understanding are activated when exploring directed-length representations related to the sine function?

Theoretical Perspectives

The design of our instruction was informed by quantitative reasoning. Thompson (1990) defined a quantity as "a quality of something that one has conceived as admitting some measurement process. Part of conceiving a quality as a quantity is to explicitly or implicitly conceive of an appropriate unit" (p. 5). Quantities in our study were related in a single diagram, implying the presence of a common unit. In our analysis, we distinguished between PSTs' concept images and the concept definition (Moore, 1994; Tall & Vinner, 1981; Vinner & Hershkowitz, 1980). One's *concept image* "refers to the set of all mental pictures that one associates with the concept, together with all the properties characterizing them" (Moore, 1994, p. 252). *Concept definition* "refers to a formal verbal definition that accurately explains the concept in a non-circular way, as might be found in a mathematics textbook" (Moore, 1994, p. 252).

Methods

The participants in this study were 23 PSTs, working in groups, who had completed at least 60 hours toward a degree in mathematics with a focus on secondary education. The context of the study was a semester-long course focused on technology in mathematics education. In this report, we discuss data from the first week of a 6-week instructional sequence on trigonometry.

The instructor displayed an arc and a chord, in *Geogebra* (see Figure 1). Both objects changed as point C rotated counterclockwise around the circle. The instructor gestured to the animated model and asked "Is this a function?"

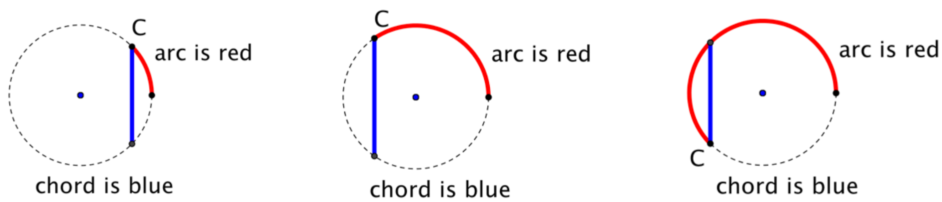

Figure 1: Covarying quantities as point C varies.

Results

All groups traversed a similar path in which the following aspects of their concept images were activated: (a) the vertical line test, (b) function matching, (c) independent and dependent variables, (d) univalence, and, finally (e) function matching revisited: the sine function. PSTs are identified by their group. For example, StudentA-1 is person number 1 in group A. Text in parentheses, describes gestures or tone; text in brackets offers contextual clarification.

The Vertical Line Test

PSTs in groups A and C initially attempted to use the vertical line test to determine whether the relationship constituted a function, indicating that this test was part of their concept image of "function." However, in this situation, it was not reasonable to use the vertical line test, which would require interpreting the dynamically changing components of the circle as "graphs" on a coordinate plane. StudentC-3 claimed the chord isn't a function because "the vertical line goes through the whole thing." Eventually, StudentC-2 said the vertical line test cannot be used as a criterion for determining whether the quantities are related by a function "because we're not dealing with a coordinate plane. You can only use vertical lines for a coordinate plane."

A Collection of Known Functions

Students in groups B and C seemed to reason that if they could map the situation they were investigating to a known function, then they could conclude that the situation they were discussing must represent a function. For example, StudentB-1 described the behavior of the dynamic diagram components as a parabola, noting "…as the length of that's increasing it gets bigger and then smaller and bigger again…"

The next concept image components, discussed by all three groups, were independent aspects of the concept definition: the identification of independent and dependent variables and the univalence requirement, that is, the requirement of one unique output for any input (Even, 1993).

Independent and Dependent Variables: Quantifying Length and Angle

To identify the independent and dependent variables, PSTs considered the length of the blue chord and length or angle of the red arc as quantities. StudentC-1 asked, "What is your output when you're talking about the chord?" To which StudentC-2 stated "It could be the length of the chord. Because the length of the chord depends on the position." StudentC-1 noted a connection between the position of point C and the length of the arc, "So given the length of the arc is the length of the chord?" and later clarified "Well, that place (pointing to animated diagram) is also in reference to the length of the arc and then it repeats as soon as you hit 360 (rotated his finger in a circle). StudentC-2 also made reference to an angle of 360˚, but it was not clear whether the PST was referring to the arc angle or the measure of the associated central angle.

Univalence

While clarifying independent and dependent variables, PSTs explored ramifications of repeated value situations. In so doing, PSTs focused on univalence and struggled to distinguish a function from a one-to-one function. StudentC-2 wanted to "restrict it to an interval" from 0 to π/2 to ensure the relationship was a function. StudentC-1 challenged the limitation, implying repeated outputs are allowed, by asking "Why isn't it a function all the way to π?" StudentC-2 responded "Because once we go all the way to π [past π/2] the chord starts shrinking again and we go back to the lengths that we've already had." StudentC-1 pointed out that "it's a different position on the circle" and StudentC-4 then explained "it would still be a function. It won't be one-to-one, but it will be a function."

After deeming the relationship a function on [0,π], the group considered whether more than two of the same outputs was acceptable, as when point C was more than halfway around the circle. To avoid dealing with additional repeated outputs, StudentC-1 proposed the idea of directed length, "Oh yeah, you're technically doing negatives. Like an opposite. You could interpret it [the blue chord's repeated length] as the opposite 'cause that length would be like flipped over."

Function Matching Revisited: The Sine Function

StudentB-1 claimed the relationship between objects constituted a sine function. This conclusion followed mention of oscillation, unit circle, and key ordered pairs. StudentB-1 said each key input value (0, π/2, π, 3π/2, and 2π) and stated or nodded for the corresponding output values. Likewise, StudentC-1 followed up on his earlier conjecture that the chord could be signed (i.e., positive when the arc was between 0 and π and negative when the arc was between π and 2π). StudentC-1 concluded "This is like (moved his finger counterclockwise around an imaginary circle)… Right?...Wait a second.... Hold up, this is like the sine function? You start at zero and the length gets longer up to π/2… and then it goes back down. This is like sine!"

PSTs concluded that the relationship between the arc and chord could be conceived of as a function by considering quantities (i.e., the length of the arc and chord) and labeling them as independent and dependent variables. Their work sorting independent and dependent variables also resulted in articulating differences between the univalence requirement for functions and the criteria for one-to-one functions. However, most PSTs had not yet recognized that unless they only considered directed length, they were dealing with an absolute value. And no PST had realized that the function based on the full chord was actually twice the sine function.

Discussion

As PSTs explored quantitative relationships we noted two conclusions. First, PSTs engaged in reasoning and sense making. Like PSTs in Even's study (1993), our PSTs had a preference for the vertical line test. But our PSTs identified limitations of this test, the importance of dependent and independent variables, as well as differences between one-to-one and univalence. Second, as we tracked concept image components activated as PSTs determined whether or not a situation could be a function, we noted a path that lead them away from a less helpful aspect of a concept image (i.e., vertical line test) toward more effective components linked to a concept definition.

Throughout this exploration, PSTs determined the relationship between the arc and the chord could be conceived of as a function by considering each as quantities and noting that length was the quality of the object (Thompson, 1990) that was relevant. We see this move toward precision as important in the PSTs' development of more robust conceptual understanding of trigonometric functions and functions in general.

Although we hoped that representing the angle as an arc of a circle would help PSTs negotiate a geometric definition of sine, it was clear PSTs still struggled with representations of angle (Akkoc, 2008), using radian and degree measure interchangeably. Likewise, they appear to have conflated central angle, arc angle, and arc length, without noting the need to consider the magnitude of the circle's radius.

After the PSTs reached consensus that the relationship between the arc and the chord was indeed a function, some PSTs conjectured it was the sine function. For those PSTs, the recognition of familiar aspects of their concept images of sine (e.g., embedded in circle, oscillating behavior, known ordered pairs), may have led to the conjecture. In that moment, those PSTs may have added to their concept image of sine as they encountered a function defined by a relationship between a circle's arc and related chord. Having reorganized their concept image of the sine functions in this way, perhaps the PSTs were more ready to consider a directed-length definition of sine, build connections among this definition, the right triangle definition, and the unit circle definition, and use that connected understanding as a potent tool for making sense of all trigonometric functions.

References

Akkoc, H. (2008). Pre-service mathematics teachers' concept images of radian. *International Journal of Mathematical Education in Science and Technology*, *39*(7), 857–878. doi:10.1080/00207390802054458

Clement, L. L. (2001). What Do Students Really Know about Functions? *Mathematics Teacher, 94*(9), 745–748.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education, 24*(2), pp. 94–116. doi:10.2307/749215

Hertel, J., & Cullen, C. (2011). Teaching trigonometry: A directed length approach. In L. R. Wiest & T. Lamberg (Eds.), *Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1400–1407). Reno, NV: University of Nevada, Reno.

Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education*, *45*(1), 102–138. doi:10.5951/jresematheduc.45.1.0102

Moore, K. C., LaForest, K. R., & Kim, H. J. (2016). Putting the unit in pre-service secondary teachers' unit circle. *Educational Studies in Mathematics*, *92*, 221–241. doi:10.1007/s10649-015-9671-6

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics*, *27*, 249–266. doi:10.1007/BF01273731

Tall, D., & Vinner, S. (1981). Concept images and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*, 151–169. doi:10.1007/BF00305619

Vinner, S., & Hershkowitz, R. (1980). Concept images and common cognitive paths in the development of some simple geometrical concepts. In R. Karplus (Ed.), *Proceedings of the 4th International Conference for the Psychology of Mathematics Education* (pp. 177–184). Berkeley, CA: University of California.

Thompson, P. W. (1990). A theoretical model of quantity-based reasoning in arithmetic and algebra. San Diego: Center for Research in Mathematics & Science Education, San Diego State University.

Weber, K. (2005). Students' understanding of trigonometric functions. *Mathematics Education Research Journal*, *17*(3), 91–112. doi:10.1007/BF03217423

EXPLORING GENDER DIFFERENCES IN A SYMMETRY SOFTWARE INTERVENTION FOR YOUNG CHILDREN

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Symmetry is a foundational geometric concept that receives minimal attention in early childhood mathematics. Differing informal play experiences involving symmetry exploration may contribute to gender differences in symmetry understanding. This study sought to explore whether boys' and girls' performance on symmetry tasks differs after a symmetry software intervention. A significant gender effect benefiting boys was found on post-test rotation tasks but not on reflection or translation tasks, controlling for pre-test scores. A gender effect was also not significant for identifying or explaining symmetric transformations at post-test. The findings have implications for learning opportunities and modes of assessment for all children.

Keywords: Gender, Geometry and Spatial Thinking, Early Childhood Education, Technology

Symmetry is present in everyday life and is a theme in children's play and creative activities. Despite children's natural interest in symmetry, learning standards addressing symmetry do not appear in the Common Core State Standards until grade four (NGACBP, 2010). A software program was developed to expand young children's understanding of three symmetric transformations—reflection, translation, and rotation. The purpose of this study is to explore whether there were differences between boys' and girls' symmetry understanding following the symmetry software intervention. This study sought to answer the following questions: (1) Do boys and girls differ in their ability to accurately create symmetric transformations after the intervention when controlling for pre-intervention symmetry aptitude (measured by the Symmetry Graphical Assessment) and treatment status? (2) Are there differences between boys and girls in their identification and explanations of symmetric transformations, as measured by the Video Transformation Task, after the intervention when controlling for treatment status?

Theoretical Perspectives

This research is based on the theoretical perspectives of constructivism, socio-cultural theory, and, to a lesser extent, the idea of intellectual honesty. The software and study design were informed by constructivism—the theory of learning that posits that learners do not simply absorb information but instead actively construct knowledge from their experiences (e.g., Piaget, 1970). The software functions as a "phenomenaria" (Perkins, 1991), intentionally designed to allow children to manipulate and explore symmetric transformations. Vygotsky's (1978) socio-cultural theory, especially the idea that social interaction is key to learning, was embedded into the study design with a research assistant (RA) facilitating exploration and prompting the child to verbalize observations. The software served as the "more knowledgeable other" (Vygotsky, 1978) by providing definitions and examples and offering feedback and solution strategies. The introduction of the mathematically complex topic of symmetric transformations to young children was influenced by Bruner's (1960) idea that "any subject can be taught effectively in some intellectually honest form to any child at any stage of development" (p. 33).

Because geometry is often taught in a cursory manner in early childhood (Clements, 2004), children's symmetry experiences often occur in informal contexts. Gender differences benefiting boys in symmetry-related spatial tasks such as mental rotation have been documented (e.g., Maeda & Yoon, 2013), but gender differences may be attributable to socio-cultural or experiential factors (e.g., Fennema & Sherman, 1977). For example, symmetry is a recurring characteristic of children's block building (Seo and Ginsburg, 2004), but boys engage in block play more frequently than girls (Kersh, Casey, & Young, 2008). Though certain play experiences may help children develop symmetry understanding, gender differences in engagement in these types of play may contribute to gender differences in symmetry understanding.

Methods

Materials: Symmetry Software

A computer program was designed to teach reflection, translation, and rotation to young children. Cognitive principles for the design of mathematics software for young children guided software development (Ginsburg, Jamalian, and Creighan, 2013). Visual and audio feedback identify mistakes and provide solution strategies to the users.

Research Design and Procedure

The study was conducted using a pre- and post-test between-subjects randomized experimental design with two conditions: the treatment condition, which consisted of nine symmetry software sessions (three each for reflection, translation, and rotation), and the control condition, which consisted of nine sessions using a non-symmetry-focused mathematics software. Experimental group software activities included guided explorations (during which an RA prompted the child to verbalize observations), viewing instructional videos with real world examples, and completing tasks that involved placing shapes on the screen to create the specified symmetric transformation and receiving feedback from the software.

Setting and Participants

The study was conducted in an urban public elementary school in the Northeastern US. The participants included 86 children from the school's first and second grade classrooms—43 were randomly assigned to the experimental group (24 girls and 19 boys) and 43 were assigned to the control group (21 girls and 22 boys). Participants' ages ranged from 5.8 to 7.8 years.

Measures

Symmetry Graphical Assessment. The Symmetry Graphical Assessment, a paper-and-pencil instrument designed by the primary investigator (PI) to measure students' ability to generate reflections, translations, and rotations, was administered at pre- and post-test. The instrument included explanations and examples for the symmetries to ensure that it assessed symmetry concept understanding rather than familiarity with relevant vocabulary. Two RAs were trained to implement the pre-/post-test scoring scheme. Inter-rater reliability, estimated using Cohen's kappa, was equal to 0.878.

Video Transformation Task. The Video Transformation Task was designed by the PI to measure participants' ability to identify and explain reflection, translation, and rotation and was implemented at post-test. Participants watched six short videos of symmetric transformations and were asked to identify the symmetry and explain their reasoning. Identification of the symmetric transformation in each video was scored for accuracy (0 or 1). Explanations were coded for the presence of words or phrases indicating conceptual understanding of the symmetries. Explanations were assigned a score of 1 for indicating conceptual understanding or 0 for not indicating conceptual understanding of the symmetry. One RA coded students' explanations for indicating conceptual understanding of the symmetric transformation. Inter-rater reliability between the coding by the RA and master coding by the PI, estimated using Cohen's kappa, was equal to .90.

Results

Post-Test Outcomes

Treatment group girls achieved, on average, higher reflection task scores than boys in the same group; however, treatment group boys achieved higher average scores on translation and rotation tasks than girls in the same group. Similar gender patterns on post-test reflection, translation, and rotation tasks were observed in the control group (see Table 1).

Table 1: Descriptive Statistics on the Post-Test Outcomes by Group

Post-test outcome	Treatment group		Control group	
	Boys (N = 19) Mean (*SD*)	Girls (N = 24) Mean (*SD*)	Boys (N = 22) Mean (*SD*)	Girls (N = 21) Mean (*SD*)
Reflection	53.57 (*16.99*)	56.93 (*18.68*)	49.19 (*19.40*)	51.43 (*16.82*)
Translation	60.28 (*20.37*)	55.49 (*25.36*)	47.10 (*19.00*)	45.02 (*21.31*)
Rotation	58.70 (*12.87*)	51.60 (*16.11*)	61.53 (*13.84*)	53.11 (*19.59*)

A MANCOVA model was estimated to test for the joint effect of treatment condition and gender on the three outcomes, controlling for pre-existing abilities in the outcomes as measured at pre-test. MANCOVA model assumptions for multivariate normality of the outcomes and homogeneity of variance covariance of the outcomes among groups were met. A statistically significant effect for treatment condition (Pillai=0.17, F=5.39, p=.002) and gender (Pillai=0.09, F=2.73, p=0.049) on the three outcomes was observed, controlling for pre-test score. On average, boys' scores were 1.24% higher on reflection tasks, 5.38% higher on translation tasks, and 10.19% higher on rotation tasks than girls. Separate ANCOVA models on each outcome showed a significant gender effect on rotation tasks [F(1, 82)=4.56, p=0.035], but not reflection [F(1, 82)=2.64, p=0.108] or translation tasks [F(1, 82)=0.09, p=0.76].

Video Transformation Task Outcomes

Descriptive statistics showed that girls in the treatment group achieved, on average, higher scores in the identification and explanation of symmetries than boys in the same group on the Video Transformation Task (see Table 5). For control group participants, boys achieved higher average scores on accurate symmetry identification than girls, but girls achieved higher average scores on explanations of symmetries than boys.

A MANOVA model was estimated to test the joint effect of treatment condition and gender on accuracy and explanations in the video transformation task. MANOVA model assumptions for multivariate normality of the outcomes and homogeneity of the variance covariance matrix among groups were met. There was a statistically significant treatment effect on accuracy and explanations [Pillai=0.27, F(1, 81)=14.90, p<0.001,], but a gender effect was not significant [Pillai=0.04, F(1, 81)=1.56, p=.22]. Separate ANOVA models were estimated for accuracy and explanations. The results show a significant effect on accuracy scores for treatment [F(1, 82)=17.86, p <0.001] but not for gender [F(1, 82)=0.03, p =.86.]. Similarly, the results show a significant treatment effect on explanation scores [F(1, 82)=27.71, p<0.001] but not a gender effect [F(1, 82)=1.88, p=.17].

Table 2: Descriptive Statistics on the Video Transformation Task Outcomes by Group

	Treatment group		Control group	
	Boys (N = 19) Mean (*SD*)	Girls (N = 24) Mean (*SD*)	Boys (N = 21) Mean (*SD*)	Girls (N = 21) Mean (*SD*)
Accuracy	4.63 (*1.26*)	4.75 (*1.54*)	3.41 (*1.74*)	3.19 (*1.44*)
Explanations	3.95 (*0.97*)	4.42 (*1.21*)	2.62 (*1.32*)	2.90 (*1.51*)

Discussion and Scholarly Significance

After a symmetry software intervention, a significant gender effect benefiting boys was found on post-test rotation tasks but not on reflection or translation tasks, controlling for pre-test scores. A gender effect was not significant for identifying or explaining symmetries. Boys' more accurate performance on rotation tasks at post-test is in line with existing literature (Maeda & Yoon, 2013). However, even though boys have more play experiences that lend themselves to symmetry exploration than girls, boys and girls were similar in their identification and explanation of reflection, translation, and rotation and in their performance on reflection and translation tasks. Boys' and girls' similar performance on certain symmetry tasks indicates the importance of testing for aptitude in different types of symmetry content.

The presence of a significant gender effect on post-test rotation items, and the absence of a significant gender effect on accuracy and explanations on the video transformation tasks, points to the importance of multiple means of assessment for young children. While some children may feel confident taking a paper-and-pencil task, others may feel less comfortable with the written format. Including verbal tasks in assessments for children provides an important opportunity to reveal understanding not observed in traditional written assessments.

The absence of significant gender differences on almost every measure in the study indicate that teaching symmetry concepts to young children has the opportunity to benefit both girls and boys—building on their natural interest in symmetry and preparing them for success in higher level mathematics and career opportunities both in and out of mathematics.

References

Clements, D.H. (2004). Geometric and spatial thinking in early childhood education. In D. H. Clements, J. Sarama, & A.-M. DiBiase (Eds.), *Engaging young children in mathematics: Standards for early childhood mathematics education* (pp. 267-297). Mahwah, NJ: Lawrence Erlbaum Associates.

Fennema, E., & Sherman, J. (1977). Sex-related differences in mathematics achievement, spatial visualization and affective factors. *American educational research journal, 14*(1), 51-71.

Ginsburg, H. P., Jamalian, A., & Creighan, S. (2013). Cognitive guidelines for the design and evaluation of early mathematics software: The example of MathemAntics. In L. T. English & J. T. Mulligan (Eds.), *Reconceptualizing Early Mathematics Learning* (pp. 83-120). Dordrecht, Netherlands: Springer.

Kersh, J., Casey, B.M., & Young, J. M. (2008). Research on spatial skills and block building in girls and boys. In O. N. Saracho and B. Spodek (Eds*.), Contemporary perspectives on mathematics in early childhood education*, 233-251. Charlotte, NC: Information Age Publishing, Inc.

Maeda, Y., & Yoon, S.Y. (2013). A meta-analysis on gender differences in mental rotation ability measured by the Purdue spatial visualization tests: Visualization of rotations. *Educational Psychology Review*, *25*(1), 69-94.

National Governors Association Center for Best Practices, Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Retrieved from http://corestandards.org/assets/CCSSI_Math%20Standards.pdf

Perkins, D.N. (1991). Technology meets constructivism: Do they make a marriage? *Educational Technology, 31*(5), 18-23.

Piaget, J. (1970). *Genetic epistemology.* New York City: Columbia University Press.

Seo, K. H., & Ginsburg, H. P. (2004). What is developmentally appropriate in early childhood mathematics education? Lessons from new research. In D. H. Clements, J. Sarama, & A.-M. DiBiase (Eds.), *Engaging young children in mathematics: Standards for early childhood mathematics education* (pp. 91-104).

Vygotsky, L. S. (1978). *Mind in society.* Cambridge, MA: Harvard University Press.

MEASUREMENT AND CONSERVATION OF LENGTH: A TEACHING EXPERIMENT WITH TONY AND SAM

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The way measurement is taught in elementary school differs from the Piagetian concept of operational measurement and conservation of length. The purpose of this study is to investigate how two first grade students conceptualize measurement and conservation of length. Results show that although both students began at the same Piagetian stage, one student progressed further and achieved operational measurement and conservation of length. Differences in each student's thinking are analyzed and insights to future research are offered.

Keywords: Elementary School Education, Measurement, Learning Trajectories, Standards

The concept of *measurement* and the process of *measuring* are connected, but different. When students construct operational *measurement*, they internalize how to coordinate sub-division and change of position (Piaget, Inhelder, & Szeminska, 1960/1981). *Measuring* however occurs when students compare objects to see which one is longer or taller; examples of *measuring* include directly comparing objects, relating object lengths with nonstandard items, or reporting standardized units from a ruler (VDOE, 2016). This examination mirrors Lamon's (2007) findings that demonstrate that some students are not being taught the concept of *measurement*, but rather the process of *measuring*. This inhibits students from learning that some objects cannot be measured, which stunts their conceptual understanding of *measurement* and rational numbers. National standards, however, have kindergarteners measure by comparing objects before first-graders iterate a tool the length of an object (CCSSI, 2018; NCTM, 2000). This suggests a focus on sub-division and change of position which helps students develop *measurement*. The purpose of this paper is to investigate how two first-grade students conceptualize measurement and conservation of length.

Conceptual Framework

When a student coordinates sub-division and change of position, Piaget et al. (1960/1981) believe conservation of length is attained, resulting in operational measurement. Three stages precede operational measurement. In Stage 1, four to five-year olds compare objects with visual and perceptual estimates. Students five to six years old manually transfer objects side by side for comparisons in Stage II. More sophisticated students at this stage use their body (e.g., hand) to compare objects. Between Stage II and III, six to seven-year olds compare lengths intuitively with tools, but can only compare the tool to each object, not between both objects. Finally, in Stage III students seven and older develop conservation through operational measurement and can use tools to compare between objects. These stages suggest students in first grade may develop operational measurement and conservation of length. National standards (e.g., CCSSI, 2018; NCTM, 2000) support this argument, but some state standards (e.g., VDOE, 2016) fail to address the development of conservation of length through operational measurement, implying students may learn *measuring* without learning *measurement*.

Methods

Sam and Tony (pseudonyms) were two first grade boys who attended a public elementary school and had similar demographics. Three, fifteen-minute clinical interviews (Clement, 2000) using tasks from Piaget et al. (1960/1981) were conducted to determine how Sam and Tony each conceptualized measurement, conservation of length, and their zones of potential construction (ZPC; Norton & D'Ambrosio, 2008). From this data, individual constructivist teaching experiments (Steffe & Thompson, 2000) were designed consisting of six, fifteen-minute teaching episodes during which hypotheses were made and later tested. Tasks from *Everyday Mathematics, Grade 1* (McGraw Hill, 2014) were chosen based on their ZPC and research hypotheses. Sam's and Tony's responses were compared against Piaget et al.'s (1960/1981) stages to build a second-order model of conceptualization of measurement and conservation of length. Each teaching episode was recorded, transcribed, and coded for observations according to Piaget et al.'s stages. Conceptual analysis was used to determine what mental structures Sam and Tony used while performing each task (Thompson, 2008), which afforded a refined second-order model of how Sam and Tony conceptualized measurement and conservation of length.

Results

Sam

The preliminary second-order model proposed during Sam's clinical interviews was consistent with Stage IIA for conservation (Piaget et al., 1960/1981). He could compare endpoints of objects visually or in activity with his fingers, but never realigned the moved objects to show length was conserved, and could not explain curves have length. He did manually move a tower he built next to another tower to compare their heights. He could not use a common measure to compare the lengths of objects. Sam also showed no desire to sub-divide a whole or change a tool's position along the whole, counting the moves between endpoints.

Sam began the first teaching episode using manual transfer to measure a table side with a string of Unifix cubes. He could not however measure his toys with tools that were shorter or longer than the toy because the tools were "too long" or "too short." If the tools did not exactly align with the toy's length, Sam rounded his measure up. For example, when a pencil was longer than his transformer, Sam reverted to visual transfer and guessed the length of the transformer was one pencil long. During another episode, Sam measured the perimeter of a table with a pencil, but repeatedly lost count and ignored when the pencil was longer or shorter than the table's edge. Subsequent episodes were used to practice counting and the motor skills involved in change of position. Sam still struggled to count by ones while pointing to each count, and consistently lost his count on tasks. Because he regularly did not align his countable unit with the endpoint of the measured object and put his finger between units without noticing it took up space, episodes three and four focused on sub-division. After much practice, he accurately measured the height of a table and sofa with a marker, and did not lose count. Since Sam improved in counting, motor skills, and sub-division, episode five entailed comparative length tasks. Sam again reverted to visual transfer to relate the lengths of objects, and only used a paperclip to measure when prompted. Sam ended the teaching episodes being intuitively able to state that a pen moved to a different location would not change length.

Sam's re-assessment of Piaget et al.'s (1960/1981) tasks from the clinical interviews, showed his behaviors were slightly more sophisticated than Stage IIA. He inconsistently accounted for the lengths of curves between a straight and curvy pipe cleaner using visual transfer. He claimed the lengths were the same even though "one is bent and curled around so it's shorter." Similarly, he knew two parallel rows of toothpicks had the same length because he counted them, but when one row was zig-zagged he said the straight row was longer without counting. When assessing the lengths of two identical parallel pipe cleaners with one moved forward, Sam tried measuring the lengths with a paperclip, but lost count and concluded one was longer. He then ignored this finding and noticed the moved pipe cleaner was “longer” on one side than the unmoved pipe cleaner. Finally, when comparing the heights of a tower he built to one prebuilt on a higher surface, he used manual transfer to compare the towers until satisfied their heights were equal. He knew the uneven surfaces made a difference in the tower heights, but when his tower was moved back to the lower surface, he added blocks to make the heights “equal.”

Tony

The preliminary second-order model proposed during Tony’s clinical interviews was consistent with Stage IIA for conservation (Piaget et al., 1960/1981). He based measurements on visual estimates, focused on the endpoints of objects and used his fingers to compare lengths. He also used manual transfer to relate objects, but his explanations suggested he was beginning to see measurement as the intervals between two endpoints.

During the first episode, Tony measured his shoe with his hand, reporting it as seven hands long, but recognized this was incorrect and used eight Unifix cubes instead. This suggested he was transitioning to Stage IIB. Tony consequently practiced iterating a tool to determine lengths and compared objects using paperclips. He struggled at first using a shorter or longer tool but improved. When asked to measure a paper strip, Tony resisted using a longer pencil as his unit, but then decided the paper strip was half a pencil. Tony needed support with fractions since he independently noticed tools either did not exhaust the whole or surpassed it. It was hypothesized that part-whole relationships (Norton & McCloskey, 2008; Steffe, 2010) were in his ZPC.

Tony had no prior knowledge of fractions at this point in the year. Using Cuisenaire rods, he developed an understanding of halves, thirds, and fourths, as exemplified by his explanation: “1/3 is like if there’s one whole piece and there’s 3 pieces and these are the same, that’s 1/3.” In another task, he showed that four individual units made up the whole, demonstrating a part-whole fraction scheme (Norton & McCloskey, 2008; Steffe, 2010). Tony was also able to partition a whole into units, and understood unit pieces needed to be equal in size. For example, Tony made two approximate “cuts” to a pencil, but then realized the pieces were not equal and adjusted his cuts. Tony verified each piece was identical to the others, indicating he was able to equipartition a whole to determine an iterable unit (Steffe, 2010), and knew he could iterate any one of them to recreate the whole (Norton & McCloskey, 2008). Tony further demonstrated his fraction scheme by drawing lines to segment a paper strip into thirds on one side and fourths on the other. This indicated partitioning, disembedding, and iterating, giving way to an equipartitioning fractional scheme in activity (Norton & McCloskey, 2008; Steffe, 2010). Tony applied this knowledge to iterate a tool, and then partition the tool in the last iteration to determine what fractional piece of the tool was left over. When measuring a pencil with paperclips, he estimated six, then determined it was five and one third paperclips. This suggested Tony had progressed to Stage IIB. Through explorations, Tony learned curves have length, making curved objects longer than straight ones. For example, he measured a straight path of 6.5 paperclips and a curved path of 9.5, and knew, “9.5 is bigger than 6.5 so it [the curved path] is longer.” He also learned that once objects have a measurement, length is conserved.

Tony’s re-assessment of Piaget et al.’s (1960/1981) tasks, showed he progressed beyond Stage IIA. He knew a curved pipe cleaner would be longer than a straight one due to the curves, and justified his answer with a paperclip. He understood the length of the pipe cleaner would not change despite its shape, and that two equivalent pipe cleaners would be the same length, no matter how they were arranged. Tony then counted the number of toothpicks to determine the two rows were equal and knew zig-zagging made no difference. He was initially confused when one toothpick was broken in half, but then demonstrated that two halves completed a whole, making the rows the same. In the final task, he used a paperclip to measure the heights of two towers, and knew the difference in table heights affected that of the towers. Tony's behaviors therefore supported operational conservation of length indicative of Stage III.

Conclusions

This study shows that although students may begin in the same developmental stage, each student needs different supports to cultivate more sophisticated ways of thinking. Both students, however, were able to progress in their conceptualization of measurement. Sam progressed from Stage IIA to transitioning between Stage IIA to IIB, whereas Tony progressed from Stage IIA to Stage III, reaching operational measurement and conservation of length (Piaget et al., 1960/1981). In the teaching experiment, Tony needed support to use fractions, a concept Sam never entertained. This is supported by Lamon's (2007) findings that measurement is directly related to fraction concepts. However, it is unclear why Tony was able to make conceptual strides while Sam was unsuccessful. Addressing this is beyond the scope of this paper, so future research is needed to explore the factor that may have caused a difference in each student's development of operational measurement and conservation of length. Overall, this study also shows that the way measurement is being taught in some schools hinders the development of operational measurement and conservation of length, at least in the case of Sam.

References

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Research design in mathematics and science education* (pp. 547–589). Mahwah: Lawrence Erlbaum Associates, Inc.

Common Core State Standards Initiative. (2018). *Measurement and data.* Retrieved from http://www.corestandards.org/Math/Content/MD/

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629–668). Charlotte, NC: Information Age.

McGraw Hill Education. (2014). *Everyday mathematics: Student math journal* (Vols. 1–2). Columbus, OH: McGraw–Hill Education.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Norton, A. & D'Ambrosio, B. S. (2008). ZPC and ZPD: Zones of teaching and learning. *Journal for Research in Mathematics Education, 39*(3), 220–246.

Norton, A., & McCloskey, A. V. (2008). Modeling students' mathematics using Steffe's fraction schemes. *Teaching Children Mathematics, 15*(1), 48–54.

Piaget, J., Inhelder, B., & Szeminska, A. (1981). *The child's conception of geometry*. (E. A. Lunzer, Trans.). New York, NY: Norton. (Original work published 1960).

Steffe, L. P. (2010). The partitive and the part-whole schemes. In L. P. Steffe & J. Olive (Eds.), *Children's fractional knowledge* (pp. 75–122). New York; NY: Springer.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267–307). Hillsdale, NJ: Erlbaum.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sépulveda (Eds.), *Plenary Paper presented at the Annual Meeting of the International Group for the Psychology of Mathematics Education,* (Vol 1, pp. 31–49). Morélia, Mexico: PME.

Virginia Department of Education. (2016). *Mathematics standards of learning for Virginia public schools.* Retrieved from http://www.doe.virginia.gov/testing/sol/standards_docs/mathematics/2016/stds/k-12-math-sol.pdf

INVESTIGATING VOLUME AS BASE TIMES HEIGHT THROUGH DYNAMIC TASK DESIGN

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This study aims to engage students in dynamic tasks of extruding surfaces on a certain heights and reasoning about volume as a continuous quantity that depends on the size of the base and the height of extrusion, what we call as Dynamic Measurement for Volume (DYME-V). This paper describes two of our DYME-V tasks and presents data from a design experiment with one pair of fifth grade students showing DYME-V's potential for developing students' conceptual understanding of volume as a multiplicative relationship of base times height.

Keywords: volume, capacity, technology, dynamic measurement

The different 'faces' of Volume

In school mathematics, volume measurement focuses on packing the space within a three-dimensional object with a two-dimensional array of cubic units iterated in the third dimension (Curry & Outhred, 2005) and quantifying that packing in terms of the total number of cubic units used. In other words, students need to develop a mental picture of a unit structure, find the number of units required to cover the base of the 3-D shape and multiply that number by the number of layers (Curry, Mitchelmore, & Outhred, 2006). This approach of volume measurement is called *volume as packing* (Clements & Sarama, 2009, Curry & Outhred, 2005).

The literature presents several difficulties that students experience while packing a 3-D space by unit cubes. Examples include students just counting the number of cubes that are visible to them ignoring the other cubes, or counting the number of squares on the cubic units shown on the visible two faces and double that count (Figure 1), ignoring in that way the three dimensions of the object (Ben-Haim, Lappan &Houang, 1985). Similarly, Battista and Clements (1996) argued that students struggle in coordinating the separate views of arrays and integrate them to form a coherent mental model. This struggle is also present even when students are given a real 3D objects and are asked to use real cubes to measure it. Curry and Outhred (2005) found that students have difficulty in packing a bigger cuboid by enumerating smaller cubes, because students cannot always determine the successive positions of the cubes while iterating. As a result, students leave gaps and overlaps in the empty space.

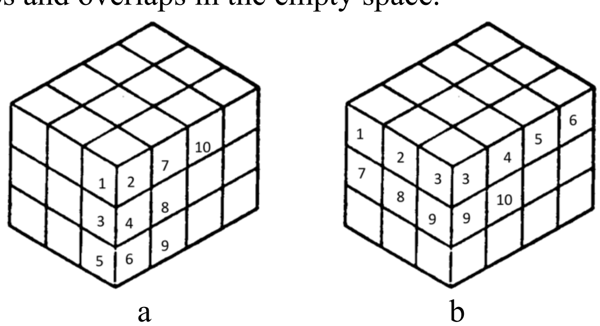

Figure 1. Students count (a) the faces of the cubic units and (b) the visible cubic units

The literature on volume measurement in math education distinguishes between *Volume as packing*) versus *Volume as filling.* Volume as filling is about filling a 3-dimensional space with iterations of a fluid unit that takes up the space of the container (Clements & Sarama, 2009, Curry & Outhred, 2005), an approach that seems to be a more continuous approach towards measuring the volume of an object over *volume as packing*. According to Piaget (1960), in order for students to understand the multiplicative relationship of volume, they need to conceptualize it as a continuous quantity, therefore the volume as filling approach seems promising for developing a conceptual understanding of the volume formula.

While exploring volume as filling, Curry and Outhred (2005) asked students to find the number of cups of rice that would fit into a jug after one cup was poured in the jug. They found that students treated the height of the rice in the cups as a unit length, which they iterated to fill the jug, ignoring the other dimensions of the jug. Similarly, while studying students' conceptions of volume, Piaget and his colleagues (Piaget, 1968; Piaget, Inhelder & Szeminska, 1960) found that elementary school students thought that the volume had been reduced when the liquid was poured into a wider glass. This predominant use of a single dimension to make three-dimensional judgments was termed by Piaget as 'centration hypothesis' and it is found to occur even when adults perceive volume in boxes at the grocery store (Raghubir, 1999).

Exploring volume dynamically

The literature above comes in contrast with research stating that students already have a dynamic sense of volume that we may use to develop students' conceptual understanding of this concept. While exploring similarity, Lehrer, Strom and Confrey (2002) discussed how students visualized volume "like pulling" the area through the height of the cylinder, in other words looking at three-dimensional objects as 2-D unfoldings (Lehrer, Strom & Confrey, 2002). The same study reported that students were able to find the volumes of cylinders by estimating the surface area of the base and then calculating the volume as the product of area and height.

The idea of "pulling" an area through a height is conceptually different than filling a jar with cups of liquid. In exploring students' thinking of dimensions in geometry, Panorkou & Pratt (2016) discussed the dual nature of capacity stating that "one can see the space as incorporating objects; in this sense, the space contains the objects. At the same time, the space can be thought of as generated by the objects" (p. 213). The generation component of volume is also described by Lehrer, Slovin, Dougherty and Zbiek (2014) who discussed how we can generate attributes through motion and gave the example of a volume being generated by sweeping an area through a length. To illustrate this generation, imagine extruding a two-dimensional rectangular surface of area '*ab*' for a height or depth of '*c*' to generate a space of '*abc*' (Figure 2).

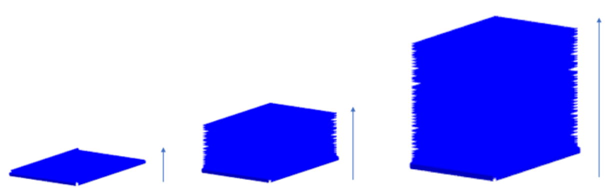

Figure 2. Volume as a continuous structure

The difference between this approach and the filling/packing approaches lies in the distinction between a matrix which is made up of a limited number of elements [a certain number of blocks/cups] and "one which is thought of as a continuous structure with an infinite number of elements" (Piaget et al. 1960, p. 350). We call this approach to measurement *Dynamic Measurement for Volume (DYME-V)*, and it involves extruding surfaces on a certain height and reasoning about volume as a continuous quantity that depends on the size of the base and the height of extrusion. DYME-V seems promising not only for developing a meaning of volume for rectangular cuboids but also for other 3-D objects, such as triangular prisms and cylinders.

Consequently, our goal was to test the conjecture that it is possible for children to visualize volume as a dynamic continuous structure and as a product of the area of base and height through careful task design. More specifically we explored: a) What type of tasks and tools may be used for developing students' DYME-V? b) What forms of DYME-V reasoning are made visible and can be seen to develop as a result of students' systemic engagement in these tasks?

We used a design-based research methodology to engineer particular forms of learning and study how those forms of learning develop with the particular context of volume measurement (Cobb et al., 2003). For our task design, we used the dynamic feature of extrusion and tracing of Geogebra to enable students to generate 3-D objects by extruding 2-D surfaces and reason about volume. This paper presents two tasks that we implemented with one pair of fifth grade students, Ashley and Maggie, and discuss the generalizations they made about volume during their interaction with the specific tasks. We met with Ashley and Maggie for 8 one-period sessions (45-50 min). The students represented various abilities according to their teacher and the prior knowledge they had on measuring volume was by water displacement from their science class.

The Case of Ashley and Maggie

In the first task, we asked students to extrude surfaces in different planes and reason about what they observe (Figure 3). As they dragged the surfaces, students reasoned that, *'it spreads out and looks like it is layering itself.'* When we asked them if they could count the number of layers they stated, '*there are millions of layers and together they make a 3D cube,'* claiming that the size of the shape *'depends on how far you stretch them out.'*

Figure 3. Extruding rectangular surfaces to create 3D shapes

Building on the students' notion of *layers* and *stretching*, the next task aimed to shift their attention to the value of base in order to make judgements about volume. We presented two surfaces of different area to students and asked them to stretch them for 1-inch each (Figure 4).

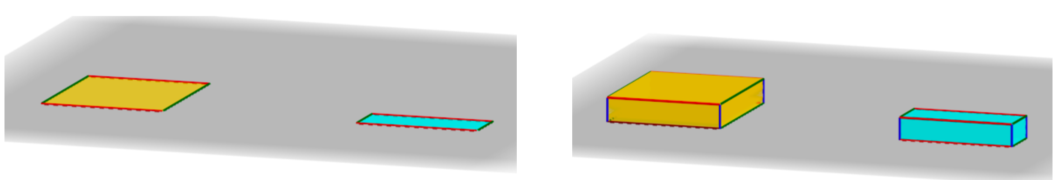

Figure 4. Stretching a 5 x 6 inches^2 and a 5 x 2 inches^2 surfaces by 1-inch height.

Researcher:	What is the size of each box?
Ashley:	This one is 10 [right] and this one is 30 [left].
Researcher:	How much space will each one fit? How much space is the first one and how much is the second one?
Maggie:	The same amount?

Ashley: I don't think so. It cannot be, because if this one [yellow on the left] is bigger, then it has to have more space.

Researcher: How much more though?

Ashley: This one is 5 times 6 which is 30 and times 1 is 30. That's for the yellow one. And that one is 5 times 2 which is 10 and 10 times 1 is 10. So 30 minus 10 is 20.

As we prompted the students to say more about why they multiplied, Maggie responded, *'because it just added on to another one. So like a second one, which means multiplying by 2'* [describing how two stretches of 1-inch are equal to one 2-inch stretch]. It appeared that they considered composites of 1-inch stretches and could realize volume as filling and continuous. In subsequent tasks, when we asked the students to state how they can find the space covered they responded, *'by multiplying the height and area, well, area is typically the base.'*

Concluding remarks

DYME-V is an exploratory study which examines how students reason about volume as a continuous quantity when they are exposed to technological tasks developed in Geogebra. Our results suggest that the approach of dynamic measurement has the potential to engage students in dynamic tasks that foster the development of volume as a continuous construct to ultimately reasoning about volume as base *times* height. Moving forward we plan to conduct more design experiments for exploring the progression of students' DYME-V reasoning as they interact with the tasks, and examine patterns and difference between different pairs of students.

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References

Battista, M. T., & Clements, D. H. (1996). Students' understanding of three-dimensional rectangular arrays of cubes. Journal for Research in Mathematics Education, 258-292.

Ben-Haim, D., Lappan, G., & Houang, R. T. (1985). Visualizing rectangular solids made of small cubes: Analyzing and effecting students' performance. Educational Studies in Mathematics, 16(4), 389-409.

Clements, D. H. & Sarama, J. (2009). Learning and teaching early math: The learning trajectories approach. Routledge: New York.

Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. Educational researcher, 32(1), 9-13.

Curry, M., Mitchelmore, M., & Outhred, L. (2006, July). Development of children's understanding of length, area, and volume measurement principles. Paper presented at the Thirtieth Annual Meeting of the International Group for the Psychology of Mathematics Education, Prague, Czech Republic.

Curry, M., & Outhred, L. (2005). Conceptual understanding of spatial measurement. Building connections: Theory, research and practice, 265-272.

Lehrer, R., Slovin, H., Dougherty, BJ, and Zbiek RM (2014) Developing essential understanding of geometry and measurement for teaching mathematics in grades 3-5. Reston: National Council of Teachers of Mathematics.

Lehrer, R., Strom, D. & Confrey, J. (2002). Grounding metaphors and Inscriptional resonance: Children's emerging understanding of Mathematical Similarity. Cognition and Instruction, 20(3), 359-398.

Panorkou, N. and Pratt, D. (2016). Using Google SketchUp to research students' experiences of dimension in geometry. Digital Experiences in Mathematics Education, 2 (3), pp. 199-227.

Piaget, J. (1968). Quantification, conservation, and nativism. Science, 162(3857), 976–979.

Piaget, J., Inhelder, B., & Szeminska, A. (1960). The child's conception of geometry. London: Routledge and Kegan Paul.

Raghubir, P., & Krishna, A. (1999). Vital dimensions in volume perception: Can the eye fool the stomach? Journal of Marketing Research, 36(3), 313–326.

INVESTIGATING STUDENTS' PROOF REASONING AS THEY TRANSITION FROM VERBAL PLANNING TO WRITTEN PROOF

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The importance of doing formal axiomatic proofs and students' difficulties with proof have been well documented in research. This paper focuses on formal proofs that use triangle congruence postulates, which students create in high school geometry. Examining student work with proofs in an clinical interview setting, we analyzed students' transition from planning their proofs to writing their formal proofs using a two-column format. We found that much of the reasoning students described in their planning did not match the reasoning they used in their written proofs. Using a lens of spatial/logical structuring, we illustrate this finding by providing an illustrative example in which a student's planning does not match the deductions that she wrote in her formal proof.

Keywords: Geometry and Geometrical and Spatial Thinking, Reasoning and Proof, Cognition

It has been well documented in research that the majority of students struggle to construct valid proofs in high school geometry (Chazan, 1993; McCrone & Martin, 2004; Senk, 1985). This paper focuses on formal proofs that use triangle congruence postulates, which students create in high school geometry. It takes a psychological constructivist perspective (Battista, 2001) to investigate how students transition from orally planning their proofs to writing their proofs formally using the two-column format. It illustrates how the reasoning students convey during the planning of their proofs many times does not match the reasoning conveyed in their written formal proofs, and how this mismatch provides researchers insights into students' proof reasoning.

Conceptual Framework

The reasoning involved in constructing geometric proofs is quite complex and involves using four types of structuring: spatial, geometric, logical, and axiomatic (Battista, 2008). *Spatial structuring* is the cognitive operation of constructing a spatial organization or form for an object or set of objects (Battista, 2008). Battista (2008) defined *geometric structuring* as using formal geometric concepts and properties (e.g., congruence, parallelism, slope, length etc.) to describe a geometric shape's spatial structure. For a geometric structuring of a shape to make sense to a person, it must evoke an appropriate interiorized spatial structuring of the shape as well as interiorized formal geometric concepts used to describe the shape (Battista, 2008). Both spatial and geometric structuring are needed to begin constructing a geometric proof as students view/draw a diagram in terms of both its visual-spatial properties and the geometric conceptualizations that are given to describe it.

However, to construct a geometric proof, not only does an individual need a linked spatial and geometric structuring, but he/she must also be able to integrate this linked spatial and geometric structuring to the third and fourth types of structuring—logical and axiomatic structuring. *Logical structuring* is the process of making a series of deductions assumed to be consistent with the rules of logic in an attempt to prove the desired conclusion from the given premises. In the two-column proof format, logical structuring is the set of conclusions that one deduces and writes in the left-hand column. One major error that can occur in a proof's logical structure is when the argument has a logical gap in it. A logical *gap* occurs between two deductions when students deduce a conclusion by applying an axiom or theorem whose premises have not been established by the given conditions or prior deduced conclusions in their proof. *Axiomatic structuring* is the process of situating and justifying deductions and logical structuring within a given axiomatic system. In the two-column proof format, axiomatic structuring can be thought of as the sequence of deductions in the left-hand column coupled with the required justifications in the right-hand column. A two-column proof exhibits a correct logical structure if the sequence of conclusions in the left-hand column is correct in that each conclusion can be justified with one axiom or previously proved theorem in the axiomatic system. A two-column proof exhibits a correct axiomatic structure if every conclusion exhibits a logically valid deduction (correct logical structuring) and each conclusion is correctly justified in the right-hand column by a relevant axiom or theorem.

Method

The data was collected by conducting a series of one-on-one semi-structured task-based interviews to seven high school geometry students who were asked to complete a series of proof problems. All participants were volunteers who were currently enrolled in a proof-based geometry course in which they had already completed a unit on triangle congruence proofs. Students were interviewed for five one-hour sessions in which they "thought aloud" as they worked on twelve proof problems. For each proof problem, students were first asked to verbally (orally) plan their proof and only after they had described their proof plan verbally were they asked to write out their formal proof. All interviews were video recorded, transcribed, and later analyzed using the constant comparative method and retrospective analysis.

Sample Results and Discussion

In this paper, we discuss how students transitioned from verbally planning their proofs to writing their proofs formally using the two-column format, along with difficulties they faced, and differences between their plans and written proofs. When evaluating a written proof with a gap, the proof itself typically provides little to no insight as to the reason(s) why the students might have skipped/missed deductions. In contrast, in this interview setting, many students demonstrated important proof reasoning in their verbal explanations that was not explicitly conveyed in their written deductions for proof. Therefore, the present study found that students' verbal explanations many times provided critical insight into the reasons why some students missed or skipped deductions in their written proofs. Due to space limitations, we only present one example of student work to illustrate that what many students said during their verbal proof planning did not always match what they wrote later in their formal proofs and how this mismatch affected their proofs' logical and axiomatic structuring.

Rose's Verbal Plan for Problem M {See Figure 1 below for Problem M}

Rose: So, DE is perpendicular to EG is down here [marks $\angle DEF$ as a right angle]. And then the same thing with BF [marks $\angle BFG$ as a right angle]. …

IN: And how did you know they were right angles?

Rose: I was marking for perpendicular. *If it is perpendicular it has to be a right angle.* So, F is the midpoint of EG, which means that since it is the midpoint these [points to $\overline{EF}$ and $\overline{FG}$] have to be the same length. [Rose marks $\overline{EF}$ and $\overline{FG}$ congruent]. And then DF is parallel to BG. So [marks $\overline{DF}$ and $\overline{BG}$ parallel] I drew parallel signs. And [Triangles] DEF and BFG; so, proving that those two triangles [are congruent]. So, since this [points at $\overline{EG}$] is a straight line and these [outlines $\overline{DF}$ and $\overline{BG}$] are parallel they intersect *at the same angle* [make X gesture with her arms] which makes these [marks $\angle DFE$ and $\angle BGF$ congruent] Corresponding Angles. And I just proved that the triangles because Angle-Side-Angle [*points to* $\angle BGF$, $\overline{FG}$, *and* $\angle BFG$]. **[Later in the episode, after Rose has completed her written proof for Problem M**].

IN: These ones right here [points to $\angle BFG$ and $\angle DEF$] you said were 90-degree angles. Is that right?
Rose: Yeah. I feel like it is a self-explanatory given that they are perpendicular they would have to be right angles.

Rose's Formal Written Proof for Problem M

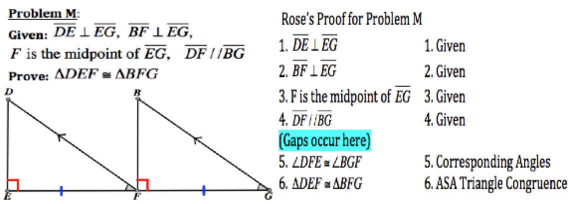

Figure 1

Correct Formal Proof for Problem M

1. $\overline{DE} \perp \overline{EG}$	1. Given
2. $\overline{BF} \perp \overline{EG}$	2. Given
3. F is the midpoint of $\overline{EG}$	3. Given
4. $\overline{DF} // \overline{BG}$	4. Given
4a. $\angle E$ and $\angle BFG$ are right angles	4a. Def. of $\perp$ lines
4b. $\angle E \cong \angle BFG$	4b. Right Angles are Congruent Theorem
4c. $\overline{EF} \cong \overline{FG}$	4c. Def. of midpoint
5. $\angle DFE \cong \angle BGF$	5. Corresponding Angles
6. $\Delta DEF \cong \Delta BFG$	6. ASA Postulate

Figure 2

Rose had three errors in her formal proof's logical structure due to missing deductions that were relevant and critical to the validity of her formal proof. First, Rose never explicitly wrote a deduction in her proof that stated that Angles E and BFG are right angles by the definition of perpendicular segments. From her oral description (in particular her statement about "self-explanatory"), we infer that Rose did not write this deduction because she assumed that the argument was intuitively obvious, therefore there was no need to write it out. Although Rose had clearly demarcated these right angles (in red) on the diagram (see Figure 1), since she did not write it as a deduction there is no explicit evidence in her written proof on how this conclusion was justified within the axiomatic system. If Rose's proof was evaluated solely from what she had written, one would have to make presumptions about whether Rose understood that these angles are right angles by definition of perpendicular segments, which might or might not be reflective of Rose's reasoning. Since it is not clear from only reading Rose's written proof that she understood this issue, the missing deduction would need to be classified as a gap in her written proof's logical structure. However, when considering her oral explanations with her written proof, it is clear that Rose had the correct intuitive reasoning about the missing deduction, but failed to formalize it in her written proof.

Second, Rose did not explicitly state a second deduction about the two right angles being congruent in either her verbal explanations or her written proof. This missing deduction was critical to the proof's logical structure because it was used later to draw the last deduction that Triangles DEF and BFG congruent by the ASA Postulate. This is evidenced in her verbal planning when Rose explicitly pointed to Angle BGF, Segment FG, and Angle BFG on the diagram to represent the congruent parts from Triangle BFG when explaining why the two triangles, BFG and DEF, were congruent. From her gesture to Angle BFG, which she had demarcated as a right angle on the diagram (see Figure 1), it can be inferred that Rose used the notion that the two right angles were congruent to satisfy the premise for a second pair of congruent angles needed to apply the ASA Postulate. However, Rose had not explicitly established that these two angles are right angles nor that they are congruent in her written proof. From reading her written proof, it is not clear how Rose was applying the ASA Postulate when she only had one pair of congruent angles (see line 5 in Figure 1) explicitly established in her proof. As a result, this is a gap in her formal proof's logical structure.

The third error that Rose made in her written proof was that she needed to explicitly write a deduction that stated that Segments EF and FG are congruent by the definition of midpoint. This missing deduction was also critical to her proof's logical structure because she used it as the corresponding congruent sides premise to apply the ASA Postulate in line 6 of her written proof (see Figure 1). This was exhibited when Rose pointed to Segment FG as her congruent side in

Triangle BFG as she was explaining why she could deduce the triangles congruent by the ASA Postulate. Importantly, Rose verbally stated that Segments FG and EF have the same length by definition of midpoint during her verbal planning of the proof, indicating that Rose had the conceptual understanding of the premises of this deduction, but she did not explicitly convey this understanding in her written proof. This illustrates again that students can have gaps in their written proofs even when they actually have correct reasoning about missing deductions. So, it is not so much a gap in logical structuring but one of axiomatic structuring, misconceiving the level of detail needed for a written formal proof.

Concluding Remarks and Implications

Findings from all the clinical interviews suggest that most of the proofs that students wrote were not formally correct, but that many students wrote proofs that were not reflective of the sound proof reasoning evidenced in their oral plans. In many instances, students had developed sound intuitive ideas for proofs, but they did not write proofs that were rigorous enough to stand up to scrutiny due to gaps/omissions in their written proofs' logical and/or axiomatic structures. Most of the time, teachers and researchers assess students' proof reasoning only by evaluating their written proofs. The present study shows that in many instances there is more going on with students' proof reasoning than what is reflected in their writing. A two-column proof provides only so much information about what the student was thinking when he or she drew a deduction, especially in flawed proofs. Many times, when students make errors or omissions in their written proofs, it is not clear or evident from simply reading their proof what student reasoning was behind these errors. For example, when students are missing deductions in their written proofs, the teacher or researcher usually cannot distinguish between students who did so intentionally because they thought the deductions were obvious from students who did it because of faulty reasoning.

In contrast, as the present study illustrates, evaluating students' proofs using both their verbal explanations and their written proofs can provide teachers and researchers with deeper insight into students' reasoning. This insight not only helps teachers and researchers better understand students' proof reasoning difficulties, but also helps them diagnose students' errors and misconceptions so they can develop more effective curriculum and instructional interventions to remedy the difficulties. Additionally, this paper illustrates that having students verbally plan their proofs before writing them reveals that some students have correct intuitive reasoning about some deductions absent from their written proofs. Due to the deeper insight gained into students' reasoning, it is recommended here that when possible, teachers and researchers should evaluate students' proofs using both their verbal explanations and their written proofs.

References

Battista, M. T. (2008). Development of the shape makers geometry microworld. Research on technology and the teaching and learning of mathematics: Volume 2 cases and perspectives (pp. 131-156)

Battista, M. T. (2001). How do children learn mathematics? research and reform in mathematics education. In T. Loveless (Ed.), *The great curriculum debate: How should we teach reading and math?* (pp. 42-84). Harvard: Brookings Press.

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics, 24*(4), 359-387.

McCrone, S. M., & Martin, T. S. (2004). Assessing high school students' understanding of geometric proof. *Canadian Journal of Math, Science & Technology Education, 4*(2), 223-242.

Senk, S. L. (1985). How well do students write geometry proofs? *The Mathematics Teacher, 78*(6), 448-456.

THE EFFICACY OF AN ALTERNATIVE HIGH SCHOOL GEOMETRY CURRICULUM ON STUDENT ACHIEVEMENT WITH PROOFS

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Keywords: Geometry, Reasoning and Proof, Curriculum, High School Education

Over the last 30 years, professional organizations (e.g., NCTM, 1989; NCTM, 2000) and policy makers (CCSSI, 2010) have continued to make recommendations that high school geometry students should learn proof, but students have continued to struggle with it (McCrone & Martin, 2009; Senk 1985). Furthermore, some geometry teachers have claimed that they do not have any instructional strategies to scaffold the learning of proof and that students must either "sink or swim" when learning it (Cai & Cirillo, 2014).

In response to the continued struggles, the present study established baseline data on achievement on proof for students who learned geometry using a recently developed curriculum that emphasizes a semester of making conjectures through investigations followed by a five-part proof progression. For a detailed description of the proof progression, see Nirode (2018).

Out of 78 students in four classes at a single high school, 56 students enrolled in the study. Near the end of the third quarter, students took an 8-question test. Questions 1 and 2 asked students to put a proof in order given all the statements and reasons. Questions 3 and 4 had students filling in blanks for a partially completed proof. For questions 5 and 6, students wrote a proof after being provided with a diagram, the given, and what to prove. For questions 7 and 8, students wrote a complete proof from a conditional statement.

The researcher scored the tests. The proof puzzles both had 8 statements and 8 reasons for students to put in order for a total of 16 points. The next two questions both had 6 blanks in the proof for students to complete for a total of 6 points. Then, the researcher scored questions 5–8 using Senk's (1985) four-point rubric. Also, like Senk's study, for questions 5–8, a proof with a score of 3 or 4 was classified as correct.

Students did exceptionally well with both proof puzzles ($M = 15.25$, $SD = 1.08$ and $M = 15.39$, $SD = 1.89$). Next, students did well on filling in the blanks ($M = 3.91$, $SD = 1.40$ and $M = 4.34$, $SD = 1.40$). For the two proofs where students were provided with a diagram, the given, and what to prove, the means for both proofs were essentially correct scores ($M = 3.25$, $SD = 1.22$ and $M = 2.93$, $SD = 1.42$). Finally, students struggled when writing a proof from a given conditional statement ($M = 1.29$, $SD = 1.29$ and $M = 1.29$, $SD = 1.29$).

References

Cai, J., & Cirillo, M. (2014). What do we know about reasoning and proving? Opportunities and missing opportunities from curriculum analyses. *International Journal of Educational Research*, 64, 132–140.
CCSSI. (2010). *Common core state standards for mathematics*. Washington, DC: Author. Retrieved from http://www.corestandards.org/wp-content/uploads/Math_Standards1.pdf.
McCrone, S. M., & Martin, T. S. (2009). Formal proof in high school geometry: Student perceptions of structure, validly, and purpose. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proofs across the grades: A K–16 perspective* (pp. 204–221). New York, NY: Routledge.
NCTM (1989). *Curriculum and evaluation standards for mathematics*. Reston, VA: Author.
NCTM (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
Nirode, W. (2018). A proof progression for geometry. *Mathematics Teacher*, *111*(7), 512–519.
Senk, S. L. (1985). How well do students write geometry proofs? *Mathematics Teacher*, *78*(6), 448–456.

YOUNG CHILDREN'S RESOURCES FOR DEFINING ASPECTS OF 3D SHAPE

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Keywords: early childhood education; geometry and geometrical and spatial thinking

Studies of young children highlight relations between their cognitive development and experiences moving in space (Oudgenoeg-Paz et. al., 2015). Yet, current early mathematics instruction often ignores the resources children have to make sense of foundational properties of space. Recent programs highlight the accessibility of geometric concept for young children (Hawe et al., 2017), but less is known about how early experiences of space can also support their engagement in mathematical practices. Expanding attention to young children's development of mathematical practice aligns with sociocultural views of mathematics as situated in the everyday activity and evolving practices of local communities (Lave et al.,1984). I present a design study conducted in a rural 1st grade classroom aimed at co-developing children's concepts of 3D shape and defining practices to answer these research question: What resources from children's everyday experiences with 3D shape and space help them define and conceptualize properties of 3D shape? How do children's resources support the development of a new classroom practice of defining?

Using grounded theory, I analyzed video and field notes from eight days of instruction. I first categorized ways students described, built, and interacted with 3D structures. Then, I traced how these different categories supported episodes of collective defining practices using Kobiela and Lehrer's (2015) framework of aspects of defining. I found four themes in students' engagement with 3D structures. These included judgements about a structure's smoothness (i.e., sliding a hand around a structure; ability to roll; stability in relation to properties of closure; and height versus width. These four themes also proved consequential to the classroom's development of a practice of defining. In particular, the teacher's press on students to articulate what properties of structures they used to make judgement corresponding to these themes helped establish definitions for properties and classes of 3D shapes. For example, when comparing examples of right prisms, students noted each could roll around their rectangular faces but some rolled better. By exploring what impacted the structures ability to roll, students coordinated rolling to the number of sides on the bases. Students then generated a generalization that these bases needed to match and that they played an important structural role of holding all the rectangles or squares together. This final generalization was later positioned as a definition for prisms.

References

Hawes, Z., Moss, J., Caswell, B., Naqvi, S., & MacKinnon, S. (2017). Enhancing Children's Spatial and Numerical Skills through a Dynamic Spatial Approach to Early Geometry Instruction: Effects of a 32-Week Intervention. Cognition and Instruction, 35(3), 236-264.

Kobiela, M., & Lehrer, R. (2015). The codevelopment of mathematical concepts and the practice of defining. Journal for Research in Mathematics Education, 46(4), 423-454.

Lave, J., Murtaugh, M., & de la Rocha, O. (1984). The dialectic of arithmetic in grocery shopping. In B. Rogoff & J. Lave (Eds.), Everyday cognition: Its development in social context. Cambridge, MA: Harvard University Press.

Oudgenoeg-Paz, O., Leseman, P., & Volman, M. (2015). Exploration as a mediator of the relation between the attainment of motor milestones and the development of spatial cognition and spatial language. Developmental Psychology, 51(9), 1241.

Chapter 5

Inservice Teacher Education/Professional Development

Research Reports

Brief Research Reports

Posters

OPENING SPACE FOR CHANGE AND EMPOWERMENT THROUGH PHILOSOPHICAL AND STRUCTURAL CONTEMPLATION IN TEACHER PROFESSIONAL DEVELOPMENT

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In this theoretical article, I examine the norms of teacher professional development with an eye towards modifying them to allow for genuine systemic change. I first argue that current norms restrict professional development practice to "pseudo-activity" which necessarily operates within rather than against larger societal structures. I then propose philosophical and structural contemplation of mathematics as an example of a currently non-normed practice that has the potential to effect substantive change.

Keywords: Equity and Diversity, Teacher Education-Inservice/Professional Development

> In our frenetic academic life, it is not easy to find the time and the will to contemplate. Partly because one of the necessary requisites for contemplation is the absence of a concern with the applicability of our thoughts, in these days in which time is money, some will ask: why lose money with all this philosophical/contemplative waste of time? The ethos of scientific research today makes plain that empty words are not enough; we must set to work, do it instead of just talking about it. What we need, some say, is engagement in action, quick solutions ready to be implemented, evaluated and, eventually, discarded, so that the entire process can start again. (Pais, 2012, p. 82)

> The threat today is not passivity but pseudo-activity, the urge to 'be active', to 'participate', to mask the Nothingness of what goes on. (Žižek, 2006, p. 334)

The 40th annual North American Conference of Psychology in Mathematics Education invites us to consider the enduring challenges of the last 40 years as well as ways to enact change moving forward. With that in mind, my goal in this article is to lay out an argument that current teacher professional development norms preclude the opportunity for genuine systemic change, and to suggest a path forward that would allow for such change. Specifically, I argue for a widening of the norms of acceptable mathematics teacher professional development in the United States to allow for philosophical and structural contemplation. To that end, the purpose of this article is twofold: (1) to provide a critical analysis of the ways professional development opportunities systematically privilege forms of professional development that focus on "pseudo-activity," activity that has here-and-now implications for the classroom but preserves the hegemony of school mathematics, and (2) to illustrate a discouraged form of professional development and the power it could have to enable critique of the hegemony of school mathematics and open space for empowerment. I will first argue that current professional development practice focuses on activities that might have immediate or directly visible impacts on teaching practice, and that such practices preserve the hegemony of mathematics and narrow space for empowering the students and teachers who interact with mathematics (Pais, 2012). I will then outline how structural and philosophical consideration of mathematics could have powerful implications for teachers and students, since such consideration exposes the discipline to critical interaction while opening space for the empowerment of the people who interact with it. In essence, the former argument is meant to establish that the norms of teacher professional practice are narrow in some way, while the latter is an existence proof of a potentially valuable practice that currently lies outside the realm of normed practice.

Throughout this paper, I will use the phrase "structural and philosophical contemplation of mathematics" to refer to thinking about the epistemological and ontological assumptions on which the discipline is built as well as to considering the ways the discipline acts as a framing or scaffolding device that shapes how we think of other aspects of our world. The former contemplation I would call philosophical and the latter structural. The lines between these two types of contemplation are fuzzy, but since I use them in tandem as a single construct throughout this article, this fuzziness should not be problematic.

Critical Analysis of Constraints on Teacher Professional Development

> I often experience it firsthand when I am asked to speak with teachers and administrators regarding mathematics achievement and persistence among African American students. Despite insisting on the complexity of these issues, some version of the following is inevitably asked: 'What you have said is fine, but tell me, specifically, what I should do today when I go back to my school or classroom to work more successfully with African American students?' In most cases, this is a sincere request. (Martin, 2009, p. 304)

Professional development is commonly taken to refer to the learning opportunities that teachers engage in to improve their professional practice (Feiman-Nemser, 2001). Professional development thus includes, but is not limited to, mandated staff development offered by districts, reading professional journals, and attending professional conferences.

In order to argue that the norms of acceptable mathematics teacher professional development should be widened, I first contend that the current norms of practice presently exclude certain types of professional practice. In brief, I contend that teachers have limited time to devote to professional development activities, and that professional structures restrict the sorts of professional development that might be considered acceptable. None of this should be taken as a critique of teachers; instead, it is a critique of larger forces that shape the decisions that teachers might reasonably make.

Teachers work long hours during the schoolyear. The OECD's large-scale international study found that teachers in the United States work an average of roughly 1,900 hours per year, more than 30 of the 32 other countries included in the dataset (OECD, 2017, p. 388). Assuming the majority of this workload occurs during the school year rather than over breaks, it is not out of the question that many teachers in the United States might average 50 or more hours per workweek during the school year, 25% more than the prototypical 40 hour workweek. This finding is echoed by the 2011-2012 Schools and Staffing Survey which reported that public school teachers in the United States average 52.2 hours of work per workweek (SASS, 2012) as well as by numerous smaller-scale non-scientific surveys (e.g. Banning-Lover, 2016).

Given the long hours teachers work, they must use their time strategically. Teachers have many obligations, including but not limited to: (1) teaching, (2) planning and preparation of lessons, (3) review and marking of student work, (4) communication with parents and guardians, (5) supervision of students outside of teaching time, (6) teamwork with colleagues, (7) participation in mentoring or support groups, and (8) other managerial or extracurricular obligations such as serving as department head or running academic clubs (OECD, 2017, pp. 390-391). Along with all of these obligations, teachers are expected to find time for professional development. Given the limited time remaining for such activities, teachers are pressured to make the most efficient use of professional development time possible.

What strategic choice of how to use limited professional development time might a teacher reasonably make? As is suggested by Martin's (2009) quote at the beginning of this section, one enticing choice is to focus on activities with immediate or directly visible implications for the classroom. In many cases, this choice is made for teachers by districts, in the form of full- or half-day sessions which focus on the dissemination of teaching techniques or strategies coupled with "inspirational" lectures (Feiman-Nemser, 2001). Such mandatory professional development informs norms of teacher professional development, suggesting that it should center on activities with immediate or directly visible implications for the classroom. This norm is further reified by the professional structures that provide additional professional development opportunities for teachers.

The National Council of Teachers of Mathematics (NCTM) is perhaps the largest professional institution devoted to mathematics education and teaching, boasting more than 60,000 members and 230 affiliates (NCTM, 2017a). NCTM offers a variety of journals and meetings devoted to the professional development of practitioners as well as the practical dissemination of research (or implications thereof) to practitioners. Given the size and pervasiveness of this institution, looking at its journals and meeting schedules can offer some insight into the norms of mathematics teacher professional development practice in the United States.

In reviewing the feature articles from the most recent (as of the time of this writing) issues of every NCTM journal dedicated to K-12 teachers (see *table 1*), I found that *every* feature article

***Table 1*: NCTM practitioner journal feature articles reviewed by the author**

Journal	Teaching Children Mathematics	Mathematics Teaching in the Middle School	The Mathematics Teacher
Articles	Bleiler-Baxter, Stephens, Baxter, & Barlow (2017) Bray & Blais (2017) Earnest, Radtke, & Scott (2017) Firmbender, Casa, & Colennese (2017) Lambert, Imm, & Williams (2017) Lomax, Alfonzo, Dietz, Kleyman, & Kaxemi (2017) McCormick, & Essex (2017) Perry, & Lewis (2017) Shumway, & Pace (2017) Turrou, Franke & Johnson (2017)	Banes, López, Skubal, & Perfecto (2017) Groth, Jones, & Knaub (2017) Matsuura, Sword, & Finkelstein (2017) McNamara (2017) Oslund, & Barton (2017) Roy, Bush, Hodges, & Safi (2017)	Beigie (2017) Borkovitz, & Haferd, (2017) Dougherty, Bush, & Karp (2017) Gerver, Santucci, & Leventhal (2017) Goldenberg, & Carter (2017) Jaqua (2017) Lommatsch (2017) McCaffrey, & Matthews (2017) Samuels (2017) Taff (2017)

focused on immediate or directly visible implications for classroom practice. This review covered 10 articles from *Teaching Children Mathematics*, 6 from *Mathematics Teaching in the Middle School*, and 10 from *The Mathematics Teacher*. All of these articles either: (1) described an exemplar lesson or course (e.g. Earnest, Radtke, & Scott, 2017), (2) describe a pedagogical technique (Oslund & Barton, 2017), or (3) offer some guiding principles for a certain style of teaching (e.g. Roy, Bush, Hodges, & Safi, 2017). Provided the reader is teaching the relevant course, every article in here contains ideas that could be immediately implemented in class and that could be pointed out to observers as teacher practice responding to professional development. Cursory analysis of additional articles spanning the years 2013-2017 echoes this finding.

Reviewing the proceedings of the annual meeting and exposition of NCTM (2017b) produces analogous results. Reviewing all 708 scheduled talks suggests that all, save for the administrative meetings, focus on topics with immediate or directly visible implications for the classroom which could be classified under the same three broad headings mentioned above.

Taken together, I claim these results suggest that a strong, though sometimes implicit, message is being sent to teachers: professional development time should be spent on activities with immediate or directly visible implications for practice. One might propose to counter this claim by arguing that teachers simply prefer professional development activities of this sort, and school districts and NCTM are acting to meet their needs. The truth or falsity of this counter-claim, however, is irrelevant to my claim. Regardless of whether or not the counter-claim is true, the current state of district mandated professional development and NCTM's professional offerings nonetheless send a message about what type of professional development is acceptable, thus informing the norms of teacher professional practice.

Focusing on professional development with immediate implications can be very valuable. However, maintaining such a focus to the point of excluding other sorts of professional development is potentially problematic, as it restricts the ways that teachers might think about solving problems (Putnam & Borko, 2000) and risks locking us into larger structures that are themselves destructive (Pais, 2012). In particular, I note that mathematics learning is itself a destructive force when not exposed to critique (Ernest, 2016), and that "here-and-now" professional development works within the discipline rather than subjecting it to critique. Thus, if there are valuable types of professional development currently excluded from the norms of professional development that might respond to this weakness, then we should make efforts to widen those norms.

Structural and Philosophical Consideration of Mathematics as Productive Professional Development

> If mathematics is objective, it makes no sense to be concerned with learners' cultures and lived experiences. If mathematical achievement can be accurately and fairly measured with standardized tests of routinized items, it makes no sense to develop more "subjective" assessments of mathematical understanding. And if mathematics is inherently too difficult for many to master, it makes no sense to try to teach all students rigorous aspects of the discipline. (Ellis & Berry III, 2005)

> Often these processes operate at a level below consciousness; they remain unexamined or even unnoticed, in which case the task at hand is to render them visible and expose them to critique. (Greer & Mukhopadhyay, 2012)

In order to argue that the norms of acceptable teacher professional development should be widened, I now contend that there exist potentially powerful forms of professional development that are currently excluded from the norms of acceptable professional development evidenced previously. In particular, this existence proof focuses on the potential power of structural and philosophical consideration of mathematics as a means to disempower the discipline while opening space for the empowerment of the people who interact with it, such as students and teachers. Noting at the outset that such contemplation falls outside of the normed space of professional development with immediate or directly visible implications for practice, my argument is structured as follows: (1) mathematics plays a large and widespread role in our lives, (2) perception of mathematics as objective and value-free empowers the discipline in potentially destructive ways, (3) substantial evidence conflicts with this perception, and (4) adopting a view of mathematics as fallible and value-full better reflects this evidence and opens space to empower people while disempowering the discipline.

Mathematics plays a powerful and pervasive role in our lives. Mathematics coursework is expected of us for a minimum of twelve or thirteen years of our lives, and many more years are required for many of jobs popularly labeled as the "best" jobs available to us (Ward, 2017). Mathematics underlies much of modern technology which has become so deeply integrated into our everyday lives. The tools of mathematics are utilized by statisticians whose work, in turn, informs policy at all levels of the government. Mathematics is everywhere, not just in the sense that anything can potentially be mathematized, but in the sense that mathematics has played some role in almost every facet of our lives.

Perhaps even more so than science (Gould, 1981/1996), mathematics is often perceived as objective and value-free (Ernest, 1991). This hegemonic perception gives power to the discipline while taking power away from those who interact with it (Ernest, 2016; Greer & Mukhopadhyay, 2012). For example, if one holds the belief that mathematics is objective and value-free, then one might take the persistent "racial gap" in academic achievement as evidence that people of color are intellectually inferior to whites (e.g. Herrnstein & Murray, 1994; Jensen, 1969), reinforcing hegemonic positioning and reifying white supremacy through appeal to the assumed nature of mathematics (Greer & Mukhopadhyay, 2012).

Mathematics is neither objective nor value-free. Focusing first on the former, Ernest (1991) identified and refuted several assumptions required for an absolutist view of mathematics, the view that mathematics is certain and unchallengeable. Without recreating the entirety of Ernest's argument, I will highlight several key take-aways in support of the opposing fallibilist view of mathematics: (1) The informal proofs that mathematicians publish are commonly flawed, (2) there now exist proofs that cannot be checked by humans for correctness, and (3) we cannot know that any but the most trivial of axiomatic mathematical systems are secure.

I focus now on the latter, that mathematics is not value-free. To that end, I make the following non-exhaustive list of some ways in which mathematics is value-full:

- Aesthetics drives all aspects of mathematics, from what mathematical questions people ask to the ways they construct argument and proof for inspection by others (Burton, 1999; Sinclair, 2009; Wells, 1990).
- Mathematics and mathematical meaning-making vary from culture to culture, indicating that mathematics itself is a cultural product (D'Ambrosio, 1985; Lipka, Wong, Andrew-Ihrke, & Yanez, 2012; Meaney, Trinick, & Fairhall, 2013; Thomas, 1996).
- Mathematics is socially-mediated and mathematical proofs are discursively constructed (Burton, 1999; Lakatos, 1976).
- The version of mathematics taught in schools and practiced by mathematicians is Eurocentric (Joseph, 1987).

- People of marginalized backgrounds have different experiences with mathematics than white males for reasons that cannot be explained by effort or ability (Martin, 2009; Stinson, 2013)

Note that these arguments that mathematics is not value-free can be taken as further support of the fallible nature of mathematics, since these values will influence what sorts of mathematical questions are asked, what can be taken as evidence in favor of mathematical claims, and generally what counts as mathematics or mathematical knowledge (e.g. Thomas, 1996).

Adopting a view of mathematics as fallible and value-full more accurately reflects this evidence and also serves to open space for the empowerment of people who interact directly or indirectly with the discipline. Recalling the earlier example of the "racial gap" in academic and mathematics achievement, this view of mathematics allows one to ask questions such as: (1) Whose brand of mathematics is being named "mathematics," (2) what value judgements and assumptions underlie the metrics used to measure achievement in this brand of mathematics, and (3) how does the culture of the test-takers interact with the cultures and values of this brand of mathematics and the measures of achievement? Thus, rather than mathematics being empowered to marginalize people, people are empowered to critically interact with mathematics (Greer & Mukhopadhyay, 2012).

Adopting such a view of mathematics requires opportunities for structural and philosophical contemplation of the discipline itself, with no concern for immediate or directly visible actions to be taken in the classroom. Opportunities for such critical analysis of and interaction with mathematics as a structure in our lives and society can be empowering (Greer & Mukhopadhyay, 2012) and might even be necessary for the construction of a more just society (Pais, 2012), but are currently excluded from the norms of acceptable professional practice for teachers. Consequently, I conclude that the norms of teacher professional development need to be widened to allow for such contemplation.

Conclusion

In this article, I have argued for a widening of the norms of acceptable teacher professional development in the United States to allow for structural and philosophical contemplation of mathematics. I began by providing evidence that the norms of professional development for mathematics teachers currently exclude professional development activities that lack immediate or directly visible implications for practice. I then argued that such "pseudo-activity" reinforces the hegemony of mathematics in destructive ways, and illustrated how structural and philosophical contemplation of mathematics could enable productive critique of the discipline if (re)introduced into the realm of acceptable professional development practice. Taken together, I conclude that the norms of teacher professional practice should be widened to allow for such structural and philosophical contemplation.

The machinery through which these norms could be widened is nontrivial, and I make no specific recommendations for how it should be accomplished; my goal here has been to invite conversation, not dictate prescription. It could be that some individual or group in a position of appropriate influence might push school districts, NCTM, or other professional institutions to make room for structural and philosophical contemplation. It could be that teacher-educators, as individuals or as a group, might take it upon themselves to adopt discourse patterns that validate such practice. It could be the culmination of a million small acts of agency, conspiring to change our professional world.

Given the immensity of the challenge, it would be tempting to take no action at all. In trying to do what is right, it seems rational to start with the set of things that are possible, and look within that for the things that are right; if the task of normalizing structural and philosophical contemplation is impossible, then it is discarded at the outset. However, when we are ourselves linked tightly into the structures that influence our lives, it can be hard to distinguish what is genuinely impossible and what is impossible only within the current system (Pais, 2012; Putnam & Borko, 2000). Thus, I suggest instead that we start with the set of things that are right, and find ways to make those things possible.

References

Banes, L. C., López, G., Skubal, M., & Perfecto, L. (2017). Co-constructing written explanations. *Mathematics Teaching in the Middle School, 23*(1), 30-38.

Banning-Lover, R. (2016). 60-hour weeks and unrealistic targets: Teachers' working lives uncovered. *The Guardian*. https://www.theguardian.com/teacher-network/datablog/2016/mar/22/60-hour-weeks-and-unrealistic-targets-teachers-working-lives-uncovered

Beigie, D. (2017). Solving optimization problems with spreadsheets. *The Mathematics Teacher, 111*(1), 26-33.

Bleiler-Baxter, S. K., Stephens, D. C., Baxter, W. A., & Barlow, A. T. (2017). Modeling as a decision-making process. *Teaching Children Mathematics, 24*(1), 20-28.

Borkovitz, D. K., & Haferd, T. (2017). Clock buddies: An accessible, engaging problem-solving activity with rich mathematical content. *The Mathematics Teacher, 111*(1), 16-24.

Bray, W. S., & Blais, T. V. (2017). Stimulating base-ten reasoning with context. *Teaching Children Mathematics, 24*(2), 120-127.

Burton, L. (1998-1999). The practices of mathematicians: What do they tell us about coming to know mathematics? *Educational Studies in Mathematics, 37* (2), 121-143.

D'Ambrosio, U. (1985). Ethnomathematics and its place in the history and pedagogy of mathematics, *For the Learning of Mathematics, 5*, 44-48.

Dougherty, B. J., Bush, S. B., & Karp, K. S. (2017). Circumventing high school rules that expire. *The Mathematics Teacher, 111*(2), 134-139.

Earnest, D. Radtke, S. & Scott, S. (2017). Hands together! An analogue clock problem. *Teaching Children Mathematics, 24*(2), 94-100.

Ellis, M. W. & Berry III, R. Q. (2005). The paradigm shift in mathematics education: Explanations and implications of reforming conceptions of teaching and learning. *The Mathematics Educator, 15*(1), 7-17.

Ernest, P. (1991). *The philosophy of mathematics education*. London: Falmer.

Ernest, P. (2016). The collateral damage of learning mathematics. *Philosophy of Mathematics Education Journal*, (31), 1-24.

Feiman-Nemser, S. (2001). From preparation to practice: Designing a continuum to strengthen and sustain teaching. *Teachers College Record, 103*, 1013-1055.

Firmbender, J. M., Casa, T. M., & Colennese, M. W. (2017). Write on. *Teaching Children Mathematics, 24*(2), 84-92.

Gerver, R., Santucci, L., & Leventhal, H. (2017). Building a high school math research curriculum. *The Mathematics Teacher, 111*(1), 34-39.

Goldenberg, E. P., & Carter, C. J. (2017). A student asks about (-5)! *The Mathematics Teacher, 111*(2), 104-110.

Gould, S. J. (1996). *The mismeasure of man*. W. W. Norton & Company: New York. 1981.

Greer, B. & Mukhopadhyay, S. (2012). The hegemony of mathematics. In Skovmose & Greer (Eds.) *Opening the cage: Critique and politics of mathematics education* (pp. 229-248). The Netherlands: Sense.

Groth, R. E., Jones, M., & Knaub, M. (2017). Working with noise in bivariate data. *Mathematics Teaching in the Middle School, 23*(2), 82-89.

Herrnstein, R. J., & Murray, C. A. (1994). *The bell curve: Intelligence and class structure in American life*. New York: Simon & Schuster.

Jaqua, K. M. C. (2017). Mathematical selfies: Students' real-world mathematics. *The Mathematics Teacher, 111*(1), 54-59.

Jensen, A. R. (1969). How much can we boost IQ and scholastic achievement? *Harvard Educational Review, 39*, 1-123.

Joseph, G. G. (1987). Foundations of Eurocentrism in mathematics. *Race & Class, 28*(13), 13- 28.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery.* Cambridge: University of Cambridge Press.
Lambert, R., Imm, K., & Williams, D. A. (2017). Number strings: Daily computational fluency. *Teaching Children Mathematics, 24*(1), 48-55.
Lipka, J., Wong, M., Andrew-Ihrke, D. & Yanez, E. (2012). Developing an alternative learning trajectory for rational number reasoning, geometry and measuring based on Indigenous knowledge. In Mukhopadhyay & Roth (Eds.) *Alternative forms of knowing (in) mathematics* (pp. 159-181). The Netherlands: Sense.
Lomax, K., Alfonzo, K., Dietz, S., Kleyman, E., & Kaxemi, E. (2017). *Teaching Children Mathematics, 24*(2), 112-119.
Lommatsch, C. W. (2017). Calculus in your career: Putting the “relate” back in related rates. *The Mathematics Teacher, 111*(2), 112-118.
Martin, D. B. (2009). Researching race in mathematics education. *Teachers College Record*, *111*(2), 295-338.
Matsuura, R., Sword, S., & Finkelstein, T. (2017). The search for hidden structure. *Mathematics Teaching in the Middle School, 23*(2), 90-97.
McCaffrey, T. & Matthews, P. G. (2017). An emoji is worth a thousand variables. *The Mathematics Teacher, 111*(2), 96-102.
McCormick, K. K., & Essex, N. K. (2017). Capturing children’s multiplication and division stories. *Teaching Children Mathematics, 24*(1), 40-47.
McNamara, J. (2017). Return of the tug-of-war. *Mathematics Teaching in the Middle School, 23*(1), 40-47.
Meaney, T., Trinick, T. & Fairhall, E. (2013). On size does NOT fit all: Achieving equity in Maori mathematics classrooms. *Journal for Research in Mathematics Education, 44*, 235-263.
NCTM (2017a). About NCTM. http://www.nctm.org/About/
NCTM (2017b). NCTM annual meeting and exposition: Program book. https://www.nctm.org/uploadedFiles/Conferences_and_Professional_Development/Annual_Meeting_and_Exposition/NCTM2017AnnualProgramBook_opt.pdf
OECD (2017). *Education at a glance 2017: OECD indicators*. OECD Publishing, Paris.
Oslund, J. A., & Barton, J. (2017). Creating zines: Supporting powerful math identities. *Mathematics Teaching in the Middle School, 23*(1), 20-28.
Pais, A. (2012). A critical approach to equity. In O. Skovmose & B. Greer (Eds.) *Opening the cage: Critique and politics of mathematics education* (pp. 49-92). Boston, MA: Sense.
Perry, R. R., & Lewis, C. C. (2017). How can students learn fraction (de)composition? *Teaching Children Mathematics, 24*(1), 30-39.
Putnam, R. T. & Borko, H. (2000). What do new views of knowledge and thinking have to say about research on teacher learning? *Educational Researcher, 29*(1), 4-15.
Roy, G. T., Bush, S. B., Hodges, T. E., & Safi, F. (2017). Mathematics discussion: Expectations matter. *Mathematics Teaching in the Middle School, 23*(2), 98-105.
Samuels, J. (2017). A graphical introduction to the derivative. *The Mathematics Teacher, 111*(1), 48-53.
SASS (2012). *Number and percentage of public school teachers who are regular full-time teachers and average number of hours per week that regular full-time teachers spent on selected activities during a typical full week, by state: 2011–12*. https://nces.ed.gov/surveys/sass/tables/sass1112_2013314_t1s_005.asp
Shumway, J. F., & Pace, L. (2017). Preschool problem solvers: CGI promotes mathematical reasoning. *Teaching Children Mathematics, 24*(2), 102-110.
Sinclair, N. (2009). Aesthetics as a liberating force in mathematics education? *ZDM Mathematics Education, 41*, 45 – 60.
Stinson, D. (2013). Negotiating the “White male math myth”: African American mal students and success in school mathematics. *Journal of Research in Mathematics Education, 44*(1), 69 – 99.
Taff, J. (2017). Rethinking the order of operations (or what is the matter with dear aunt sally?). *The Mathematics Teacher, 111*(2), 126-132.
Thomas, R. (1996). Proto-mathematics and/or real mathematics. *For the Learning of Mathematics*, *16*(2), 11-18
Turrou, A. C., Franke, M. L., & Johnson, N. (2017). Choral counting. *Teaching Children Mathematics, 24*(2), 128-135.
Ward, M. (2017). These are the 10 best jobs of 2017. *CNBC*. https://www.cnbc.com/2017/04/26/these-are-the-10-best-jobs-of-2017.html
Wells, D. (1990). Are these the most beautiful? *The Mathematical Intelligencer, 12*(3), 37-41.
Žižek, S. (2006). *The parallax view.* Cambridge, MA: MIT Press.

SUPPORTING A MATHEMATICIANS' INSTRUCTIONAL CHANGE IN UNDERGRADUATE MATHEMATICS THROUGH FACULTY COLLABORATION

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To reform instruction by moving towards student-centered approaches, research has shown that faculty benefit from support and collaboration (Henderson, Beach, & Finkelstein, 2011; Speer & Wagner, 2009). In this study, we examined the ways in which a mathematician's instruction unfolded during his participation in a faculty collaboration geared towards reforming instruction and aligning it with inquiry oriented instruction (IOI) (Kuster, Johnson, Andrews-Larson, & Keene, 2017). Results indicate the participant's mathematics background and research interests influenced how he used student thinking in his instruction. Further, there existed a tension between IOI and anticipating student thinking. Lastly, results highlight the importance of active participation in faculty collaboration to support instructional change.

Keywords: Post-Secondary Education, Teacher Education-Inservice/Professional Development

Over the last decade there have been numerous calls for reform in undergraduate mathematics education (e.g., President's Council of Advisors on Science and Technology [PCAST], 2012). These calls for reform draw on research that has shown the benefits of student-centered instruction (e.g., Freeman et al., 2014). To address these calls, change is needed in the instruction of undergraduate mathematics. For example, *A Common Vision* gave a general call that instruction should move away from traditional lecture as the sole instructional method in undergraduate mathematics (Mathematics Association of America [MAA], 2015).

Given these calls for instructional reform, faculty want to make changes to their instruction. However, research has shown that even when working with research-supported curricular materials, mathematics faculty are often unprepared to undertake the challenge of changing their instruction (Henderson et al., 2011; Wagner, Speer, & Rosa, 2007). Current research is providing mathematics faculty with support needed to change their instruction.

There are also calls for departments and faculty members to collaborate specifically on the pedagogy (MAA, 2011). One research-based method of support is faculty collaborations geared towards collectively improving instruction (e.g., Nadelson, Shadle, & Hettinger, 2013). In particular, researchers are studying how mathematics faulty come to use research-based instructional strategies in their classrooms in the context of faculty collaboration. This study explored the experiences of a mathematician who participated in one such faculty collaboration that addresses the numerous calls for reform in undergraduate mathematics education and instruction. The study addressed the following overarching research question: *1) In what ways does one mathematician's experiences in an online faculty collaboration on inquiry oriented differential equations relate to his instructional practice?* And the following sub research questions: *a) How does his instructional practice unfold over his first implementation of inquiry oriented differential equations and in what ways does it align with inquiry oriented instruction? b) How does his participation unfold in the online faculty collaboration?*

Literature Review

In this section, we briefly describe the instruction that the faculty collaboration sought to support. Following this we briefly discuss relevant research on instructional change.

Inquiry Oriented Mathematics

The faculty collaboration focused on inquiry oriented mathematics and instruction. Rasmussen and Kwon (2007) defined inquiry oriented (IO) environments as teaching where students are inquiring into the mathematics, while the teachers are inquiring into the students' mathematical thinking. In this study, we focused on inquiry oriented differential equations (IODE) which has been shown effective for student understanding of differential equations (Kwon, Rasmussen, & Allen, 2005; Rasmussen, Kwon, Allen, Marrongelle, & Butch, 2006).

Inquiry oriented instruction. In inquiry oriented mathematics, it is clear that the role the teacher plays is important for advancing the mathematical agenda. Kuster et al. (2017) recently defined four focal components of inquiry oriented instruction (IOI): generating student ways of reasoning, building on student contributions, developing a shared understanding, and connecting to standard mathematical language and notation. The focal components of instruction are guiding principles of IOI. It is important to note that the four focal components very rarely occur independently; oftentimes, these components overlap and occur in the complexities of an IO classroom. Further, there are local practices of IOI. The local practices of IOI (see Table 1) are an elaboration on the four focal components of IOI. While the focal components are guiding principles of IOI (i.e., ways of composing and discussing IOI), the local practices are specific actions that instructors do in an IO classroom.

Table 1: Inquiry oriented instructional local practices (Kuster et al., 2017)

Local Practice	**Description**
1	Teachers facilitate student engagement in meaningful tasks and mathematical activity related to an important mathematical point.
2	Teachers elicit student reasoning and contributions.
3	Teachers actively inquire into student thinking.
4	Teachers are responsive to student contributions, using student contributions to inform the lesson.
5	Students are engaged in one another's thinking or reasoning.
6	Teachers guide and manage the development of the mathematical agenda.
7	Teachers introduce language and notation when appropriate and support formalizing of student ideas/contributions.

Overview of Instructional Change

Here we first describe barriers to instructional change and then what the research community knows about facilitating and sustaining instructional change.

Barriers to instructional change. One barrier to instructional change is faculty's knowledge for teaching with student-centered instructional strategies. Research has shown that some faculty lack the necessary skills to enact student-center instruction (Hayward, Kogan, & Laursen, 2015), sometimes because they lack specialized content knowledge relating to instruction and being prepared to respond to student questions productively (Wagner et al., 2007). Further, faculty have stated that student resistance, lack of student buy-in, and student attitudes of school are reasons why they do not use student-centered instruction (DeLong & Winter, 1998). The most often cited environmental reason by faculty to not use student-centered instruction is how much more time it takes than teacher-centered instruction (Henderson & Dancy, 2017). Likewise, faculty say they stray away from student-centered instruction because they have a certain amount of material that needs to be covered over the course of one semester (Hayward et al., 2015).

Facilitating and sustaining instructional change. Henderson et al. (2011) outlined four categories of instructional change strategies that are elaborated on in this section: disseminating curricula and pedagogy, developing reflective faculty, enacting policy, and developing a shared vision. Borrego and Henderson (2014) elaborated on these four categories of change by defining eight change strategies that fit within the framework. This study considered two of these change categories that we discuss here: scholarly teaching and faculty learning communities. Scholarly teaching is when "individual faculty reflect critically on their teaching in an effort to improve" and faculty learning communities are when a group of faculty come together and "support each other in improving teaching" (Borrego & Henderson, 2014, p. 227). These two strategies can work together to improve undergraduate mathematics instruction.

Methods

This study focused on one participant from an IODE online faculty collaboration (OFC). This qualitative instrumental case study (Stake, 1995) was bounded by the participant's participation in the OFC and his classroom teaching. This work comes from the TIMES project, which supported university mathematics faculty in shifting their practice towards an IO practice. TIMES offered three supports: the IO materials (in this case IODE), a summer workshop, and the weekly OFC. Here we first highlight pertinent details on the OFC.

Online Faculty Collaborations

The IODE OFC met weekly during the semester they are teaching IODE, virtually via Google Hangouts to conduct lesson studies that were modified Japanese lesson studies (Demir, Czerniak, & Hart, 2013) led by a facilitator. The main goals of the OFC were to: 1) aid teachers in making sense of the instructional IODE materials, 2) thinking through the sequences of tasks, how students might approach the tasks, how to structure instruction around the tasks to support student learning, and 3) assist teachers in developing and enhancing their instructional practice.

Participant

The focus of this study is one participant from the IODE OFC, Dr. DM. The OFC consisted of the facilitator (Dr. KK), two graduate research assistants (GRA1 and GRA2), five faculty teaching the materials for the first time (Drs. DM, AB, PR, CD, ST). The sampling of Dr. DM was purposeful in nature (Yin, 2013) and there were several reasons for that choice. First, he was and is passionate about his participation in TIMES and to this day continues with IOI in his IODE classroom. Second, he became a facilitator for the project in future semesters following his participant experience. Furthermore, Dr. DM filmed every class of the semester, which was more than was expected of the other TIMES participants, affording a plethora of possible data sources and a semester-long look at instruction.

Data Collection and Analysis

Classroom data. Video data from Dr. DM's classroom were collected. Classroom video was chosen to match the units covered in the OFC lesson studies. The OFC discussed Unit 6 and Unit 9. In addition to those units, Unit 1-2 as an introductory unit and Unit 12 were analyzed. All units lasted a different amount of time. The IOI framework discussed above (Kuster et al., 2017) was designed to capture IOI in action. Consequently, we used the framework as an a priori analytical framework for coding Dr. DM's classroom instructional practice to answer research question 1a. In particular, we used the local practices (LP) of IOI. The IOI framework also contained "evidences," not shown above, of each LP; these evidences served as codes that were collapsed to each LP. LP1 was not coded for unique observable instances in the data. After the first round of coding, we went back again and revisited analysis logs and made adjustments to the coding as necessary. In this step, we looked for emergent themes from the data.

OFC data. Each OFC was screencast recorded. All weeks of the OFC were analyzed except week 6 because the data was corrupted and week 8 because Dr. DM was unable to attend that week (in total 9 OFCs were analyzed). Weeks 1 and 2 were introductory weeks. Lesson study 1 took place over weeks 3-5 and lesson study 2 took place over weeks 6-10. Lastly, a debrief OFC occurred during week 11. All videos were transcribed. To analyze Dr. DM's participation in the OFC we coded the transcripts with specific a priori codes and frameworks: the role of the speaker (production design from Krummheuer, 2007), the role of the listener (reception design from Krummheuer, 2011), and conversation categories (Keene, Fortune, & Hall, under review). These frameworks were adapted to fit the context of this study and are discussed in the results. In a broad sense, we considered Dr. DM's active versus passive participation.

Interview data. The interview data served as a third data source to relate Dr. DM's experiences in the faculty collaboration to his instructional practice. Furthermore, this data offered Dr. DM's personal perspective on being part of a faculty collaboration. Entrance and exit semi-structured interviews were conducted. All interviews were audio recorded and transcribed. Transcripts of both interviews were open coded (Yin, 2013).

Results

Instructional Practice

Central to IOI is the facilitation of mathematics where students are actively inquiring into the mathematics while the teacher is actively inquiring into the students' mathematical thinking (Rasmussen & Kwon, 2007). Dr. DM's instruction focused predominantly on LP2, eliciting student ways of reasoning and contributions (see Table 2). Dr. DM less often actively inquired into why his students were making such contributions (LP3), less often used those contributions to push the agenda forward (LP4), and less often had students engage in one another's thinking (LP5; although this happened frequently in Unit 1-2). Note that frequencies were scaled and rounded to represent the same amount of class time; each unit lasted a different number of days.

Table 2: Frequencies of Dr. DM's Local Practices of Inquiry Oriented Instruction

Practice	Unit 1-2	Unit 6	Unit 9	Unit 12
2	58	52	66	26
3	17	24	16	4
4	17	16	15	8
5	42	26	14	2
6	14	16	6	4
7	3	14	8	2

Table 2 is very telling of Dr. DM's instruction. He was very interested in generating student contributions. While some of the questions asked were ones from the IODE tasks themselves, he often would ask his own questions in his own way as a means to address something that he wanted to focus on or have his students think about. While students had opportunities to engage in others' contributions as they were written on the board, they less often had opportunities to engage in others' thinking, as Dr. DM did not tend to follow up with questions to have students elaborate on their thinking. Essentially, after students made contributions, Dr. DM would more often move on. We cannot know for sure if Dr. DM was so in tune with the students in his class and the mathematics itself, that he did actually know why his students were thinking along certain lines. However, LP3 and LP4 are about making explicit to the rest of the class such thinking and thus Dr. DM's LP frequencies were reflective of the fact that he didn't often make public his inquiring into student thinking.

Dr. DM's instruction did not necessarily change from the beginning of the semester to the end of the semester. As discussed across the totality of Dr. DM's instruction his most frequent LP was LP2, eliciting student ways of reasoning and contributions. However, when comparing the four units of analyzed instruction there were contrasts between the unit instructional portraits. Namely, the way Dr. DM's instruction unfolded was tied to 1) how and when he used student thinking in his class, 2) the mathematical task itself, and 3) his mathematical beliefs, rooted in his mathematical research arena.

First, in Unit 1-2, Dr. DM frequently (more often than any other unit when comparing across scaled time) engaged students in one another's thinking. In particular, this unit was the unit where his students' thinking was most at the forefront of the class and he oftentimes used that thinking to advance the mathematical agenda. When student thinking was made prevalent to the rest of the class, Dr. DM's instructional portrait reflected that.

Second, it was observed that Dr. DM's instruction was influenced by the mathematical task. Specifically, if a unit was a more scaffolded unit with limited options for student exploration (e.g., Unit 12), the questions that Dr. DM would ask were limited in scope and thus he used less IOI LPs in those units. In this particular case, Unit 12 was a very algebraic unit where students, being led by the teacher, develop an understanding of how to find the eigenvalues of a system of differential equation and use that information to find the associated eigenvectors and in turn the solution to the system of differential equations. Consequently, the questions Dr. DM could ask and the probing he could do was significantly impacted and all IOI LPs occurred less often.

Third, when the mathematics of the unit was associated with Dr. DM's mathematical research interests he would focus on getting students to get to "the way [he] view[s] the mathematics" rather than having his students' work or ideas at the center of the development of the mathematical agenda. Unit 9 dealt with the development of the phase plane which was a crucial tool in Dr. DM's research. The instructional portrait of that unit had the highest amount of eliciting student ways of reasoning and contributions (LP2) and in comparison, a very low frequency of LP3-5 (the other practices associated with student thinking). Many of the questions that Dr. DM asked were of his own accord and not generated from the whole class discussion. Because he knew the mathematics so intimately, he was most interested in getting students to see the mathematics the way he does, rather than letting the mathematics emerge from the students.

Participation in OFC

Recall the goal of the OFC was to support cohorts of mathematicians as they came to learn about IOI and IODE. Table 3 highlights the participation frequencies based on role and conversation. For the purposes of space, we only discuss active and passive participation here rather than all the more specific roles adapted from Krummheuer (2007, 2011). Additionally, we adapted frameworks from our previous work (Keene et al., under review) but here only include four broad conversation categories rather than each individual conversation topic.

Rather than growth throughout the semester, Dr. DM immediately jumped into the active role in the OFC and that active role was consistent throughout the semester. Similar to his classroom instruction there was not a change but rather how his role looked depended on the content of each OFC. For example, if the week focused on doing mathematics, he rarely authored topics because he simply was partaking in the conversation, however, he was very active in those weeks as he has a real passion for mathematics. Additionally, when the OFC focused on sharing of his videos, he authored frequently those weeks and the conversation focused on pedagogy as he sought advice on, for example, how to speed up his class because he was running out of time at the end. Table 3 highlights Dr. DM's most active role related to pedagogical issues.

Table 3: Frequencies of Speaker / Listener Codes by Participation / Conversation Category

Conversation Category	Speaker		Listener	
	Active	Passive	Active	Passive
Pedagogical Issues	137	16	82	55
Mathematical Issues	70	6	72	40
Student Issues	63	2	20	23
OFC Issues	97	24	91	156

Discussion

In this section, we provide an answer to the overall research question.

Mathematics Background

Dr. DM's mathematics background played a role in how his instruction panned out throughout the semester and how he participated in the OFC. In both cases his mathematical content knowledge (rooted in his background and research interests) was placed on top of his interest in enhancing his pedagogical practice. By that we mean, in his teaching his view of mathematics sometimes was the view of mathematics that he was guiding his students towards. Likewise, in his participation in the OFC, his mathematical understanding was one of the driving factors for his interest in enhancing his pedagogical practice. Namely, he sought support on how he can get his students to that same level of awe and understanding.

This conclusion supports previous work from Speer, Wagner, and colleagues (Speer & Wagner, 2009; Wagner et al., 2007). However, there are important distinctions that shed light on this topic and provides discussion for faculty collaborations going forward. Most importantly, that is the subtle notions of what a mathematician's mathematical content knowledge is. In their work, Speer and Wagner noted that their participant had a strong understanding of the mathematical content but that did not help in terms of his analytic scaffolding (i.e., meaning facilitation of discussion). Similarly, Dr. DM also had a strong understanding of the mathematical content across all units. However, the difference lies in the fact that in some units he was able to provide analytic scaffolding, namely, he was able to use his students' ideas in the class (LP3: actively inquiring into student thinking, LP4: being responsive to student contributions, LP5: engaging students' in one another's thinking, LP6: guide the mathematical agenda). Yet, he was more likely to do that when the mathematical content wasn't his specific research interest. Consequently, we concur with Speer and Wagner and posit that one's mathematics background is not sufficient to successfully use student thinking in one's class, however, the level to which one understands that content makes a difference in their instruction.

Tension Between Agenda and Inquiry

This first relationship translates into a second one, as there is a tension between what a mathematician wants to do in his/her class and IOI. In the case of Dr. DM, his focus, for some of the content from the course, was to get his students to his view of the mathematics. This ultimately leads to a tension between one's teaching agenda and inquiry. If in inquiry, student thoughts are central to the development of the mathematical agenda (Kuster et al., 2017), then imposing one's own view of mathematics does not align with an inquiry perspective. The reason this causes a tension is because being passionate about your research inherently is not a bad thing, nor trying to get your students to see the beauty of mathematics. However, in so doing, one privileges their understanding over that of their students.

Anticipating Student Thinking

A third relationship that relates Dr. DM's instruction to his participation in the OFC considers how one anticipates student thinking and how student thinking is used in instruction. We know from extant literature that mathematicians often struggle to implement novel teaching (if it is new to them) and in particular struggle with how to respond to and deal with student contributions in a productive and successful way (Wagner et al., 2007). However, this was not an issue for Dr. DM as he was in an OFC supporting his instruction. He never noted that he was unsure what his students were going to do. Yet, he seldom actively inquired into his students thinking. This indicates he either knew what his students were thinking or simply did not probe into their thinking; we cannot know which one.

We also know from extant literature on the possible successes of anticipating student thinking in professional development settings (e.g., Demir et al., 2013). In the OFC, participants spent 1-2 weeks for each lesson study doing the mathematics of the units of focus and then anticipating how their students may approach the tasks. This model was based off of the Japanese lesson study (Demir et al., 2013). Ultimately, we conjecture there is a tension between anticipating student thinking and inquiry oriented instruction. By that we mean that because a critical component of inquiry oriented instruction is inquiring into student thinking and engaging with unexpected contributions, if student contributions are overwhelmingly "anticipated," mathematics faculty may struggle to engage with those unexpected contributions if they spend a large amount of time anticipating what their students will do.

Active Participation in Faculty Collaborations

The fourth relationship centers around active participation in the OFC positively impacting instruction. As discussed, Dr. DM was an active participant in the OFC. Additionally, his goals were clear in that he was there to enhance his pedagogical practice. Dr. DM's passion for IODE and IOI bled into both his instruction and participation in the OFC. This OFC was an example of what Borrego and Henderson (2014) defined as a faculty learning community. "STEM undergraduate instruction will be changed by groups of instructors who support and sustain each other's interest, learning, and reflection on their teaching" (Borrego & Henderson, 2014, p. 233). Dr. DM was supported by and supported his fellow colleagues in learning about and reflecting on inquiry oriented instruction. This indicates that for successful instructional change, faculty learning communities or faculty collaborations needs to be designed with ensuring active participation from all involved.

Conclusion

This area of research is ripe for future research directions. The instruction of undergraduate mathematics courses is a hot button item in undergraduate mathematics education research today. More importantly, the research community still needs to know more about how we can support endeavors to reform instruction, how can they be scaled up, and how do we measure success? In this qualitative instrumental case study, while not generalizable, we can conclude that the OFC supported Dr. DM's desire to reform his instruction. This work has highlighted how those faculty collaborations can be improved moving forward and most importantly highlights that instructional change is possible if the time and effort are put into it.

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References

Borrego, M., & Henderson, C. (2014). Increasing the use of evidence-based teaching in STEM higher education: A comparison of eight change strategies. *Journal of Engineering Education*, *103*(2), 220–252.

DeLong, M., & Winter, D. (1998). Addressing difficulties with student-centered instruction. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies*, *8*(4), 340–364.

Demir, K., Czerniak, C. M., & Hart, L. C. (2013). Implementing Japanese lesson study in a higher education context. *Journal of College Science Teaching*, *42*(4), 22–27.

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences*, *111*(23), 8410–8415.

Hayward, C. N., Kogan, M., & Laursen, S. L. (2015). Facilitating instructor adoption of inquiry-based learning in college mathematics. *International Journal of Research in Undergraduate Mathematics Education*, *2*(1), 59–82.

Henderson, C., Beach, A., & Finkelstein, N. (2011). Facilitating change in undergraduate STEM instructional practices: An analytic review of the literature. *Journal of Research in Science Teaching*, *48*(8), 952–984.

Henderson, C., & Dancy, M. H. (2007). Barriers to the use of research-based instructional strategies: The influence of both individual and situational characteristics. *Physical Review Special Topics - Physics Education Research*, *3*(2), 1–14.

Keene, K. A., Fortune, N., & Hall, W. (under review). Supporting mathematics faculty's instructional change: Using class videos in an online working group. Under review at the *International Journal of Research in Undergraduate Mathematics Education.*

Krummheuer, G. (2007). Argumentation and participation in the primary mathematics classroom: Two episodes and related theoretical abductions. *Journal of Mathematical Behavior*, *26*(1), 60–82.

Krummheuer, G. (2011). Representation of the notion "learning-as-participation" in everyday situations of mathematics classes. *ZDM - International Journal on Mathematics Education*, *43*(1), 81–90.

Kuster, G., Johnson, E., Keene, K. A., & Andrews-Larson, C. (2017). Inquiry-oriented instruction: A conceptualization of the instructional principles. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies*, 1–18. http://doi.org/10.1080/10511970.2017.1338807

Kwon, O. N., Rasmussen, C., & Allen, K. (2005). Students' retention of mathematical knowledge and skills in differential equations. *School Science and Mathematics*, *105*(5), 227–240. http://doi.org/10.1111/j.1949-8594.2005.tb18163.x

Mathematical Association of America. (2011). *Partner discipline recommendations for introductory college mathematics and the implications for college algebra*. (S. L. Ganter & W. E. Haver, Eds.). Washington, DC: Author. Retrieved from http://www.maa.org/sites/default/files/pdf/CUPM/crafty/introreport.pdf

Mathematical Association of America. (2015). *A common vision for undergraduate mathematical sciences programs in 2025*. (K. Saxe & L. Braddy, Eds.). Washington, DC: Author. Retrieved from http://www.maa.org/sites/default/files/pdf/CommonVisionFinal.pdf

Nadelson, L. S., Shadle, S. E., & Hettinger, J. K. (2013). A journey toward mastery teaching: STEM faculty engagement in a year-long faculty learning community. *Learning Communities Journal*, *5*, 97–122.

President's Council of Advisors on Science and Technology. (2012). *Engage to excel: Producing one million additional college graduates with degrees in science, technology, engineering, and mathematics*. Washington, DC: White House Office of Science and Technology Policy. Retrieved from www.whitehouse.gov/administration/eop/ostp/pcast/docsreports

Rasmussen, C., & Kwon, O. N. (2007). An inquiry-oriented approach to undergraduate mathematics. *Journal of Mathematical Behavior*, *26*(3), 189–194.

Rasmussen, C., Kwon, O. N., Allen, K., Marrongelle, K., & Burtch, M. (2006). Capitalizing on advances in mathematics and K-12 mathematics education in undergraduate mathematics: An inquiry-oriented approach to differential equations. *Asia Pacific Education Review*, *7*(1), 85–93.

Speer, N. M., & Wagner, J. F. (2009). Knowledge needed by a teacher to provide analytic scaffolding during undergraduate mathematics classroom discussions. *Journal for Research in Mathematics Education*, *40*(5), 530–562.

Stake, R. E. (1995). *The art of case study research*. Thousand Oaks, CA: Sage.

Wagner, J. F., Speer, N. M., & Rossa, B. (2007). Beyond mathematical content knowledge: A mathematician's knowledge needed for teaching an inquiry-oriented differential equations course. *Journal of Mathematical Behavior*, *26*(3), 247–266.

Yin, R. (2013). *Case study research: Design and methods* (5th ed.). Thousand Oaks, CA: Sage.

BEGINNING TEACHER FEEDBACK IN THE CONTEXT OF A CO-TAUGHT INCLUSION MATH COURSE

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As schools turn towards co-teaching models to implement inclusion practices (US Department of Education, 2010), teachers are increasingly being asked to operate with a co-teacher (Scruggs et al., 2007). However, most research on co-teaching remains at the survey level; it does not examine factors that may influence co-teaching relationships over time (Scruggs et al., 2007). This paper seeks to address that gap by examining the feedback a first and second year co-teacher give one another as they teach a high school mathematics course. The question under consideration is: what kinds of feedback may beginning co-teachers offer one another and how may it influence their classroom practice over time? The results of this work contribute to the broader body of co-teaching research by offering insights into a specific case of teacher collaboration.

Keywords: Teacher Education- Inservice/Professional Development; Equity and Diversity; Affect, Emotion, Beliefs, and Attitudes

Inclusion, the practice of blending students with and without documented special needs in one classroom, has become increasingly supported in schools across United States by adopting a co-teaching model where one teacher has content knowledge and the other teacher, knowledge of special education practices (US Department of Education, 2010). When implemented well, co-teaching has been shown to be an effective way to teach all students (Cook & Friend, 1995). However, research indicates that teachers often do not have adequate supports, such as common planning time, resources, and training, and that without these advantages, these co-teaching relationships are not nearly as successful (Bouck, 2010; Dieker & Murawski, 2003; Scruggs, Masteropieri, & McDuffie, 2007). Because under-resourced, beginning co-teacher situations are common, they should be explored in greater depth to better understand how to support educators in this position.

The current body of co-teaching research is primarily limited to teacher perception studies that provide snapshot insights into co-teacher relationships (Scruggs et al., 2007). A 2012 study by Rytivaara and Kershner is an exception; their case study of two social studies co-teachers suggests that co-teachers may shift towards more student-centered practices when they collaborate effectively. This is an exciting prospect because not all teachers have access to more traditional professional development (PD) programs to improve their practices; therefore, if co-teachers can offer productive feedback towards one another, co-teaching might offer an alternative form of PD to foster productive changes in teacher pedagogy over time. Indeed, Rytivaara and Kershner (2012) emphasize the importance of "social and collaborative" (p. 1001) reflection as key to successful professional learning and note that in theory, co-teaching gives participant teachers the opportunity to engage in a "peer-learning relationship" (p. 1001).

The purpose of this paper is to consider the role of feedback given by two co-teachers, myself and my co-teacher, "Peter," in our "peer-learning relationship" (Rytivaara & Kershner, 2012, p. 1001) and examine whether and how our classroom practice changed over time. The study was conducted using action-research methodology (e.g. Chazen, 2000; Lampert, 1990; Townsend, 2014). The feedback Peter and I exchanged as we worked together for the first time to teach an Algebra II and Trigonometry course are analyzed to address the following research question: what feedback may beginning co-teachers offer one another and in what ways may it influence their classroom practice over time?

Theoretical Framework

This work was conducted based on the belief that learning is situated (Greeno, 1991; Lave & Wegner, 1991) and that teachers and students are "elements or aspects of an encompassing system of social practices" (Cobb & Bowers, 1999, p. 5). Therefore, this study's intention is not to strive for generalization to other contexts, but instead to seek initial patterns from the data themselves and produce context-dependent insight into a specific case, as opposed to producing "high level theory" (Flyvbjerg, 2006, p. 223). As Flyvbjerg (2006) argues, it is essential to consider a breadth of cases before developing more general theory, and that the cases themselves are useful for producing a more "nuanced view of reality" (p. 223). This study will contribute to the literature by illustrating the case of two beginning co-teachers, Peter and myself, as we collaborate together for the first time.

Research Methods

Study Context and Participants

I started teaching high school mathematics in an urban district in the Northeastern U.S. during the 2014-2015 school year. Prior to teaching, I spent two years working on a PD research project that sought to increase teachers' ability to attend and respond to their students' mathematical ideas. Based on this background, I hoped that Peter and I could work together towards developing a more responsive classroom environment for all of our students (Hammer & Shifter, 2001), where we would build our lessons on students' ideas and craft an environment where students shared and discussed one another's mathematical thinking.

Peter had taught for one year in a resource room in another district before moving to the research site, which is a relatively high-performing urban high school with approximately 1,700 students. He often reported that he loved math in high school, but found it difficult in college and changed his major to Spanish. Although he did not take many college-level mathematics courses, from our experience together it was clear that he was quite proficient in the topics we taught.

Peter and I co-taught one course together, from January to June 2015, but he taught two other classes with two other teachers that semester as well. His planning time did not align with any of the teachers he was assigned to work with, so most planning occurred via email or after school. Hence, Peter and I had little common planning time and had received no co-teacher training before teaching together. This situation is representative of many other co-teachers across the U.S. (Bouck, 2010; Dieker & Murawski, 2003; Scruggs et al., 2007).

As stated previously, Peter and I taught an Algebra II and Trigonometry course. The curriculum covered a review of polynomials, linear and quadratic functions and their transformations, and then introduced exponential and logarithmic functions, rational functions, limits, the unit circle, solving trigonometric equations, proving identities, and finally, graphing trig functions and trig function application problems.

Study Design

This study was designed as action research (Adelman, 1975; Lampert, 1990; Townsend, 2014), as I occupied the dual role of teacher and researcher whose goal was to foster a more responsive environment with my co-teacher. While some researchers have argued that the primary purpose of action research is to change practice, rather than to produce research (Elliott, 1991; Kemmis & McTaggart, 1982), I sought to design a study where the results could be similar to other situations with broader research implications. Hence, I strove for it to be as authentic to a traditional co-teaching experience as possible, only intending to give feedback and argue for responsive teaching practices when the moment arose organically in conversation, rather than overtly pushing my agenda. Therefore, while I hoped that our practices would become more responsive to our students' needs, I did not explicitly communicate that to Peter.

Data Collection and Analysis

The data were collected from late January to late June 2015. I recorded 26 of the approximate 40 conversations we had we planned together. These conversations were of varying length (5 minutes to 1.5 hours) and occurred almost entirely after school. I also made daily field notes that summarized the day's class and my impressions from that day, and collected my 123 email correspondences and the assignments Peter and I developed for our students.

The data were first coded line-by-line using grounded theory methods (Charmaz, 2014). For instance, in the transcript below (Figure 1), Peter and I discussed how to address low student test scores. The lines on the right indicate the presence of some of the codes given to each piece of transcript. Peter says "part of me is like, we spent two weeks on that, but it was very choppy because of all of the other stuff…" This was coded as "blaming students for poor performance," as he says "we spent two weeks on that" and the second part, "But it was very choppy because of all of the other stuff," was coded as "taking responsibility for student performance." This entire segment was coded as "exploring why students did not do well on an assessment," because Peter discussed how time, student motivation, and instruction may have influenced the test scores. Over 200 codes were developed after analyzing all transcripts.

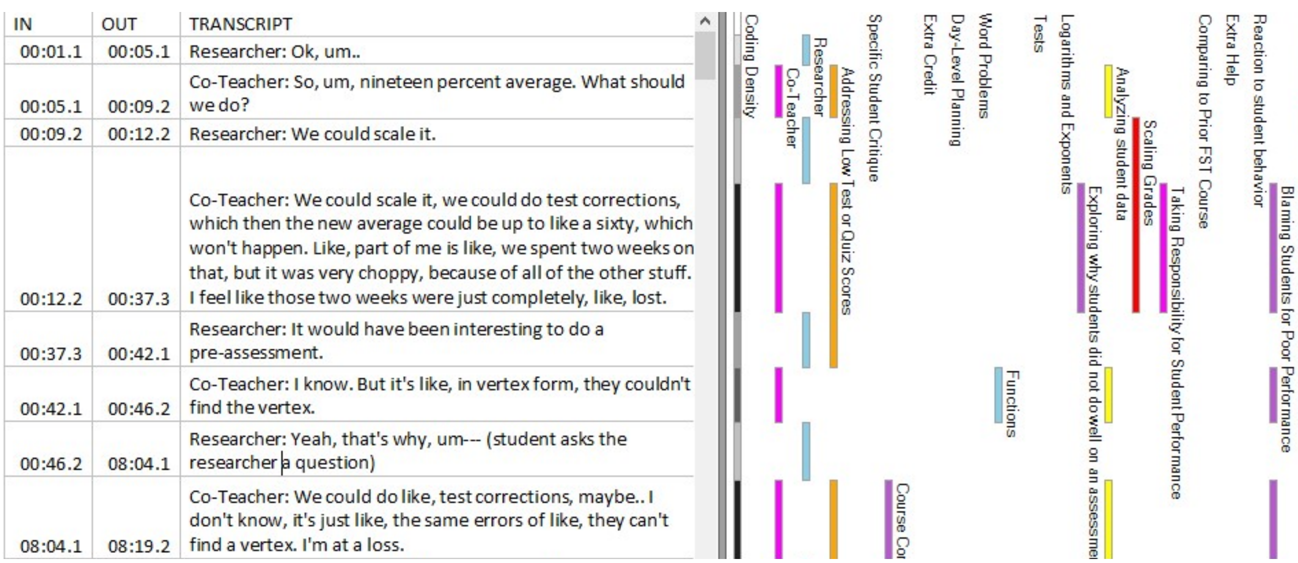

Figure 1. Example of the initial coding process using Nvivo software.

Next, the 200 codes were sorted into 43 broader categories (Charmaz, 2014) to identify larger themes in the data. After this initial analysis was completed, I returned to the research question to examine the feedback more directly. All moments of feedback were classified based on their degree of specificity, as specificity is considered a hallmark of quality feedback (Gan & Hattie, 2014; Hattie & Timperley, 2007). Three codes were developed based on this premise:

Vague: Feedback that is merely positive or negative and does not appear to add any advice or insight or explain our reasoning. For instance, "That packet is awesome" and "That looks great!" are examples of vague feedback because we do not explain why the subject at hand is "awesome" or "great."

Moderately Specific: Feedback that provides some level of detail as to the strengths or weaknesses of the subject. For example, "I think it's a little long but the questions are great!" In this quote, I validate Peter's hard work ("the questions are great!") but provide the feedback that I believe the assessment he developed is too long. However, I does not explain why I believe it is too long or provide suggestions to address the issue.

Specific*:* Feedback that is detailed and explains the feedback-giver's reasoning. This feedback should also include propositions for changes. Following Peter's development of a quiz, I replied with the following:

> Overall, I really like the structure of the quiz and I think the questions are great, but I think that the graphs are <u>really</u> hard—I think that in order for us to determine what they understand and what they don't, some of them should just have a phase shift, some should just have amplitude, some should just have a vertical shift, and some could have a combination to see what they're capable of. I think it's a little much to have all three shifts in most problems.

In this example, I give the feedback that I think the quiz is "really hard," but then grow more specific: I delve into why I believe this is the case and offers suggestions to remedy the situation.

Results

The purpose of this paper is to describe the types of feedback we gave one another and how it may have changed our classroom practice. Therefore, the results are organized by first giving an overview of the feedback we gave one another, and then examining the ways in which our class changed over time.

Feedback Exchanged

There were two instances of feedback in the 26 recorded conversations between Peter and me and both occurred near the end of the semester in June. In the first instance, I followed up on a discussion Peter had with one of our students, Jalen. I recorded the events that transpired the day Peter talked with Jalen in my field notes:

> Peter stopped by my class after school and helped Jalen, one of our students who passed first quarter and then started to slack. He worked on revisions from our Limits test, but had essentially forgotten what limits were so Peter was reteaching him. Early on, Peter asked Jalen what he planned to study after graduating. Jalen said he was interested in music and in the medical field, and so Peter told him that if he was interested in studying medicine he might need to take calculus. Because limits are a 'calculus-y' concept, understanding them is very important and kind of a preview to what he'd be learning in calculus. I thought it was a good approach for Peter to relate the content to Jalen's interests, but the tone kind of (unintentionally, I think) implied that if he didn't master limits, calculus would be really hard. I don't think that Peter meant to give Jalen that message, but I'm afraid that Jalen left with a fear that he couldn't succeed in what he hoped to pursue. Also, Peter was trying to encourage Jalen, 'See, you're getting it, this stuff is easy.' But I don't think that Jalen was understanding the material to the degree that Peter thought he was, and Jalen seemed a little upset by his assertion-- he didn't want to protest, I don't think, [or] contradict what Peter was saying by pointing out that no, he didn't understand, but it was a little uncomfortable. I didn't really know how to help, either, but I probably should have said something (June 4th, 2015).

This quote contains an important reflection. I wrote, “I didn’t really know how to help.” Not knowing how to approach areas of disagreement with Peter was common in the field notes; there are a few instances where I wrote notes such as, “I don’t really know how to talk about that with him,” especially early and mid-semester. In this case however, I was so bothered that I decided to discuss it with Peter the next time we were both after school:

Me: Yeah, I think that... I wasn't sure... like, the other day, when you were saying, um, like, ‘You got it! You got it!’ to him, I was like, ‘I don't-- I dunno if he has it yet.’

Peter: No, but like, maybe he doesn't have it, but at least if he's taking a step in the right direction? Sometimes positive reinforcement. I think sometimes I might do that a little to much. Like a sine graph, I sometimes may go way overboard with praise, sometimes I may be like, ‘screw you.’

Me: Yeah, I guess it's just making sure it's the right kind of praise too. I just remember some of my math professors, when they were trying to make us feel better about what we were learning, they would be like, ‘This is so easy! You shouldn't feel bad about this. This is so easy!’ And then I would be like, ‘No, it's not easy! Now I just feel dumb.’ And until I was that student. I was like, ‘Oh, I should make sure I don't call something ‘easy.” Sometimes I use the euphemism of ‘straightforward,’ which doesn't really…

Peter: I definitely called limits easy, because it's like, we're looking at two things. They either come together or they go in different directions (June 6th, 2015).

I tried to provide feedback on the type of praise that Peter offered, that it may have been premature to assert that Jalen understood the material. Peter countered, saying that he gave “positive reinforcement” to Jalen’s work. I then attempted to argue by connecting to my own personal experience, citing that when I was told by a well-intended professor that the content was “easy,” it made me feel worse. Peter disagreed, saying that the material was “easy.” In this example, Peter did not appear to accept my feedback to adjust the way he discussed content with students. He continued to argue that the content was “easy” and that he did provide appropriate praise.

The second instance of feedback occurred later that same day. In class, Peter had tried to “cold-call” students to hold them accountable for paying attention. As the co-teachers reflected on how class went that day, the following discussion took place:

Peter: I'm happy. I think today some people made sense of things. I feel like today was somewhat of a success.

Me: Well it was funny how many people you called upon to ask for a definition and how many people still just like, weren't listening.

Peter: Yeah, weren't even listening to the group. And like, it's sad but I think it's something that we have to keep doing.

Me: Yeah, I think it's good to do that. You probably could have kept it up a bit longer too.

Peter: Yeah.

Me: But it's hard when there are so many people who are sitting and you can only call on them some of the time (June 6th, 2015).

This small piece of my feedback, that “you probably could have kept it up a bit longer too,” is the one piece of pedagogical feedback given over the course of the semester. However, I wrote in my field notes that “I think that if we were friends it might be easier for us to give each other feedback on curriculum materials, etc.” It appears that I felt uncomfortable giving Peter feedback because we were not friends. Yet, my feedback that Peter could have “kept it up a bit longer too,” along with my critique of his conversation with Jalen, both may indicate that by the end of the semester I felt that our relationship was strong enough to offer small pieces of pedagogically-based feedback face-to-face with Peter.

While there was little feedback given during face-to-face conversations, there were 19 instances of feedback given via email between Peter and me, significantly more than the 2 instances given in-person.

Table 1: Number of specific, moderately specific, and vague pieces of feedback per month given via email.

Month	Specific	Moderately Specific	Vague
January	0	1	4
February	0	1	0
March	2	0	0
April	1	1	0
May	3	2	1
June	3	0	0

As Table 1 shows, as the semester progressed, there was an increase in the frequency of specific feedback and a decrease in vague feedback. Again, this may indicate that as we grew more comfortable with one another, we felt that we could give more specific critiques. All feedback related to developing course materials and assessments. For example, after grading a test on the unit circle, I wrote in an email:

> The kids bombed the test. The highest grade was in the 40s… I like the idea of having a revisions/make-up day at some point before the end of the semester, but I was thinking that we give them advanced warning and give them a group test on Tuesday or Wednesday, then average the scores… I wonder whether it makes sense to just forget about the sum/difference formulas and focus on graphing and mastering what they do know... what do you think? (May 31st, 2015).

Peter replied:

> I agree about tomorrow. I feel like the multiple representations are important. Perhaps we could use desmos or have them extend the graph to show the two graphs, or we could [ask] them [to] compare a sine graph shifted to the cosine graph (y = sin(x + pi/2) vs. y = cos(x)) so they can see the similarity between the graphs (June 1st, 2015).

This quote conversation is representative of the types of conversations Peter and I had. Here, I proposed omitting the sum and difference formulas moving forward, although it would be on the final, and instead continue work on graphing. Peter gave the feedback that he agreed with my suggestion, and then built on my comments by proposing multiple representations using technology. We often acknowledged one another’s ideas about content, discussing order of topics, material development, and assessing student understanding, and agreed or pushed back accordingly.

Changes in Classroom Practice

When Peter and I began our class in January 2015, we often opened with a "Do Now" warm-up problem for students to try, followed by direct instruction and then individual or group work. We often alternated between providing direct instruction and circulating the room to ensure that the students were on-task. A similar structure prevailed in the end of the semester. Hence, our overall classroom structure did not appear to change from the beginning to end of the course and we did not shift towards using the responsive teaching practices to the extent I had hoped for. This is perhaps representative of the lack of feedback and conversation around making our classroom more student-centered; there were only two pieces of feedback about student interactions and they occurred in June.

Peter and I did change how we delivered material to students. Initially we developed large packets of problems and notes per unit, but as the semester progressed we transitioned to daily assignments based on our perception of our students' understandings from the day before. Peter and I often alternated between making the next day's assignment and would share our lesson through email to get feedback. This shift could perhaps reflect our focus on students' understanding of content through the assignments and assessments we created, as opposed to the mathematical ideas students shared in class. This focus was especially prevalent in the feedback we gave one-another via email. While our daily structure remained the same, that our assignments became more attentive to students' mathematical knowledge raises the possibility that given more time and experience, Peter and I may have learned to practice more responsive teaching methods as well, not just more responsive assignments.

Discussion and Implications

The results show that we gave each other pedagogical feedback face-to-face only once, at the end of the semester, and otherwise gave no feedback in-person. However, we gave each other content-based feedback much more frequently over email, and this feedback increased in specificity as the semester progressed. This is an important indication of quality feedback (Hattie & Timperley, 2007; Shute, 2008). The late increase in feedback may indicate that we grew more comfortable with one another as the semester progressed and suggest that for co-teachers to give one-another feedback, they need to know one-another first. This builds upon prior research, which has found that for a co-teaching relationship to be successful, it needs to be like a marriage: "requiring effort, flexibility, and compromise for success" (Scruggs et al., 2007, p. 14). Therefore, future research should examine structures that could be put in place to help teachers develop stronger relationships, therefore fostering an environment more conducive to giving and receiving feedback.

Peter and my feedback was quite substantive as well at times, suggesting that even beginning co-teachers can give feedback to improve the content presented to a course. However, our lack of feedback towards other aspects of teaching highlights that we were not good sources of feedback for improving pedagogy, and could explain why there were no shifts in classroom practice over the semester while there were changes in how we created and delivered course material. This builds upon current feedback research, which primarily focuses on feedback given by teachers to students or from administrators to teachers (Khachatryan, 2015), by illustrating that even beginning teachers can offer suggestions for improvement.

While more co-teaching relationships should be examined to see the greater variety of the ways in which co-teachers may provide feedback and shift their practices, schools might consider using all, not simply veteran, teachers when implementing feedback protocols. This is because even beginning teachers may have important contributions to make to improving school experiences for students.

References

Adelman, C. (1975). The function of classroom observation within the methodology of the ford teaching project. *British Educational Research Journal, 1*(2), 42–43.
Bouck, E. (2010). Co-Teaching… Not Just a Textbook Term: Implications for Practice. Preventing School Failure: *Alternative Education for Children and Youth, 51*(2), 46:51.
Chazen, D. (2000). *Beyond Formulas in Mathematics in Teaching: Dynamics of the High School Algebra Classroom*. Teachers College Press: New York.
Charmaz, K. (2014). *Constructing Grounded Theory* (2nd Ed.). SAGE Publications: London.
Cobb, P. & Bowers, J. (1999). Cognitive and situated learning perspectives in theory and practice. *Educational Researcher, 28*(2), 4-15.
Cook, L. & Friend, M. (1995). Co-Teaching: Guidelines for creating effective practices. *Focus on Exceptional Children, 28*(3), 1-16.
Dieker, L., & Murawski, W. (2003). Co-Teaching at the Secondary Level: Unique Issues, Current Trends, and Suggestions for Success. *The High School Journal, 86*(4), 1-13.
Elliott, J. (1991). Action research for educational change. Buckingham, England: Open University Press.
Ellison, N., Heino, R., & Gibbs, J. (2006). Managing impressions online: Self-presentation processes in the online dating environment. *Journal of Computer-Mediated Communication, 11*(2).
Flyvbjerg, B. (2006). Five misunderstandings about case-study research. *Qualitative Inquiry, 12*(2), 219-245.
Gan, M., & Hattie, J. (2014). Prompting secondary students' use of criteria, feedback specificity, and feedback levels during an investigative task. *Instructional Science, 42*(6), 861-878.
Greeno, J. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education, 22*, 170-218.
Hammer, D. & Schifter, D. (2001). Practice of inquiry in teaching and research. *Cognition and Instruction, 19*(4), 441-478.
Hattie, J., & Timperley, H. (2007). The Power of Feedback. *Review of Educational Research, 77*(1), 81-112.
Khachatryan, E. (2015). Feedback on Teaching from Observations of Teaching: What Do Administrators Say and What Do Teachers Think About It? *NASSP Bulletin, 99*(2), 164-188.
Kemmis, S., & McTaggart, R. (1982). *The action research planner*. Geelong, Victoria: Deakin University Press.
Lampert, M. (1990). When the Problem is not the Question and the Solution is not the Answer: Mathematical Knowing and Teaching. *American Educational Research Journal, 27*(1), 29-63.
Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge University Press.
Rytivaara, A., & Kershner, R. (2012). Co-Teaching as a Context for Teachers' Professional Learning and Joint Knowledge Construction. *Teaching and Teacher Education,* 28, 999-1008.
Scruggs, T. E., Masteropieri, M. A., & McDuffie, K. A. (2007). Co-Teaching in Inclusive Classrooms: A Metasynthesis of Qualitative Research. *Exceptional Children, 73*(4), 392-416.
Shute, V. (2008). Focus on Formative Feedback. *Review of Educational Research, 78*(1), 153-189.
Townsend, A. (2014). Weaving the Threads of Practice and Research: Reflections on Fundamental Features of Action Research. In F. Rauch, A. Scuster, T. Stern, M. Pribila, & A. Townsend (Eds.), *Promoting Change through Action Research* (pp. 7-22). Boston: Sense Publishers.
U.S. Department of Education. (2010). Inclusion. Retrieved from http://eric.ed.gov/?ti=Inclusion

FRAMING ELEMENTARY SCHOOL TEACHERS' CONSIDERATIONS WHEN ENGAGING WITH THEIR OWN STUDENT WRITTEN WORK

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Professional development (PD) is widely used to aid teacher development and improve student learning outcomes. In PD, skilled facilitators use research-based frameworks to support and provide a lens for teachers' engagement with artifacts of practices (e.g., student written work). However, often what is unaccounted for in PD design is the use of teachers' insider knowledge of their students. This study explored considerations that arose when teachers engaged with written work from their classes outside PD. Results identify the range and prevalence of considerations used in teachers' conversations. Also discussed are implications for professional developers and those who support teacher learning through the analysis of written work.

Keywords: professional development, frames, student written work, elementary school teachers

Professional development (PD) is widely used to support teacher development and improve learning outcomes. In PD, the complex work of teaching is decomposed into manageable parts, and teacher learning is often supported through engagement with artifacts of practices (e.g., student written work) led by experienced facilitators (Kazemi & Franke, 2004). In addition, the use of research-based frameworks (Carpenter, Fennema, Franke, Levi, & Empson, 2014) and protocols of instruction (Krebs, 2005; Little, 2003) provides a lens to support teachers' engagement with classroom artifacts in PD with the intent that the same lens is used in their classrooms. However, what is often unaccounted for in most PD is the potential role of teachers' insider knowledge of their children and the influence of their engagement with artifacts of practice.

Take for example the task of examining written work, which is a prevalent activity inside and outside PD. In PD, teachers are asked to bring in samples of written work from their own class as they utilize research-based frameworks to understand their children's thinking. Facilitators and collaborative groups play a role in co-constructing what to consider as teachers engage with their written work with the hope those considerations will remain the same once teachers leave the PD. Outside of PD, teachers engage with written work individually in their classrooms and sometimes collaboratively with their peers (e.g., within grade level meetings). When teachers are in their classrooms, engaging with written work is typically constructed by the individual teacher's insider knowledge of their children and is guided by the need to make decisions in the moment or in preparation for the next day. Considerations are influenced by goals set for each child, the class as a whole, or curricular expectations. When teachers work collaboratively with their peers, considerations for engaging in the written work are driven by school-level expectations, standards, accountability, and curriculum (Little, 2003). In collaborative settings, the goal of analyzing the written work is often not closely linked to the teachers' individual child but related to understanding classroom performance, more generally. The different settings in which teachers examine written work—in PD contexts, teachers' individual classrooms, or school-based teacher groups—position written work differently, and teachers draw upon different relationships in their analyses. These differences highlight the use of teachers' knowledge and beliefs but also the context of teaching should be considered when trying to understand teachers' engagement (Webel & Platt, 2015).

Conceptual Framework

In this study, I chose to think about teachers' considerations as "frames" to look beyond what teachers say as they work with written work, and instead focus on the nature of the engagement. Framing or frame analysis was theorized by sociologist Erving Goffman to explain how individuals structure information for sense-making. Goffman (1974) argued that we all actively classify, organize, and interpret our life experiences to make sense of them by filtering relevant information and discarding what is not needed depending on the situation. Individuals, whether aware or unaware, construct frames on a day to day basis to make sense of the world around them. I argue framing is a broad construct that includes knowledge, beliefs, dispositions, and experiences. In the case of engaging with written work, the frames used by the PD facilitator, the individual teacher, or grade level teams could shape individual teachers' engagement. For example, in schools and PD settings, the framing of written work could include making sense of children's mathematical thinking (Kazemi & Franke, 2004; Little, 2003), understanding misconceptions (Krebs, 2005), or checking for correct answers (Horn, 2007).

Researchers in mathematics education have applied framing in PD to understand teachers' noticing in video clubs (Sherin & Russ, 2014) and in schools to investigate teachers' conversations when categorizing students (Horn, 2007) and in teacher learning through collaborative groups (Bannister, 2015; Louie, 2016). Across these studies, framing served as an analytical approach to provide the meaning of teachers' conversations. For example, Bannister (2015) noted individual teachers expressed different conceptions of how to classify a "struggling student" based on the needs of their own students. However, the framing of "struggling students" was co-constructed by the group's collective interpretation which was not representative of every teacher's interpretation.

In PD settings, facilitators guide teachers through the use of research-based frameworks that frame ways to structure meaning of written work. For example, Kazemi and Franke (2004) used a research-based children's mathematical thinking framework (Carpenter et al., 2014) to promote elementary school teachers' analysis of written work in professional development. During the PD, teachers shared written work samples from their classrooms while the facilitator questioned teachers to attend to and make sense of strategies according to the children's mathematical thinking framework. Findings showed that, over time, teachers learned to frame how they attended to and made sense of children's mathematical thinking in ways that were reflective of the framework used in the PD. In essence, the design of the PD guided teachers' framing of the written work as intended.

The Kazemi and Franke (2004) study is representative of studies that have provided evidence that teachers can learn to use a *children's thinking* frame as they engage with written work during PD, which is a controlled setting. There is also some evidence that teachers who have adopted a children's thinking frame in PD, continue to use this frame in their own classrooms (Sherin & van Es, 2009). However, we do not know if there were other frames that arose as teachers engaged with written work because that was not the focus of these types of studies. Looking beyond the *children's thinking* frame used in PD to examine written work can provide insight into other frames that were not addressed in PD. Further, by looking at the frames that arise outside the context of PD when teachers engage with written work after having taught a lesson, we can better understand the issues that could emerge in a more naturalistic and applied setting.

In this study, I chose to explore frames outside of a *children's thinking* frame and outside of PD. I draw heavily upon Sherin and Russ's (2014) conception of *interpretative* frames, which were defined as structures that described the way teachers' attention grows out of and informs their reasoning. Sherin and Russ (2014) used interpretative frames to understand the relationship between teachers' reasoning about classroom events and the role of reasoning used in teachers' noticing of short video excerpts. Analysis of teachers' conversations in video clubs identified 13 interpretative frames that characterized the network of ideas teachers used to reason when analyzing video excerpts from their own classrooms during PD. My study is similar to the work of Sherin and Russ (2014) because I investigated the frames used when engaging with children's ideas, but there are some important differences. Sherin and Russ (2014) worked with middle and high school teachers who were analyzing video of classroom events. I chose to focus on elementary teachers' individual engagement with their own written work after a classroom lesson.

Research Focus and Questions

The focus of this study was to explore considerations elementary school teachers used when examining their own students' written work. Specifically, I addressed the following questions: What considerations arose in elementary school teachers' conversations when engaging with written work from their classrooms? More specifically, what frames were utilized (outside the *children's thinking* frame provided in the PD) when teachers engaged with written work from their classrooms?

Methods

Participants

This study involved 42 grades 3–5 teachers who were a subset of teachers who participated in the Responsive Teaching in Elementary Mathematics (RTEM) study — a professional development design study focused on research-based knowledge of children's fraction thinking and how teachers can use this knowledge to be responsive to children's thinking in their classrooms. The 42 participants were teachers whose classrooms were observed solving an equal sharing fraction problem (Empson & Levi, 2011), completed 1, 2, or 3 years of PD in the RTEM study, reflected a range in years of teaching experience and prior experiences with research on children's thinking. They were drawn from three school districts in the southern region of the United States, and these districts included not only a diversity of ethnicities and achievement levels but also purposeful variations in instructional supports and constraints, which could influence teachers' engagement with written work.

Professional Development

The 3-year PD consisted of 8.5 total days each calendar year, including the summer and academic school year. Teachers engaged in numerous activities focused on using research-based frameworks for children's thinking of whole-number concepts and fractions to support responsiveness to children's thinking. Additionally, teachers participated in practices (e.g., questioning and noticing) that built on children's mathematical thinking. Equal sharing problems were a type of problem widely used in the PD to support building children's foundational understanding of fractions and conceptual knowledge of working with fractions. An example of an equal sharing problem is: *Six children want to share ten brownies so that everyone gets the same amount. How much brownie can each child have?*

Data Sources and Analysis

Data Sources. Data included 42 audio-recorded, semi-structured teacher interviews that took place following an observation of each teacher's classroom instruction. Teachers were asked to pose an equal sharing problem during their instruction that they considered appropriate for their class. After the lesson, teachers were immediately interviewed. I am focused on the portion of the interview in which teachers were asked to identify a student's written work that was interesting to them and then discusses what stood out about that child's thinking. Additional questions asked teachers to describe the details of the child's strategy, their understandings, and next instructional steps.

Data Analysis. The analysis of teachers' conversations occurred in multiple steps and was guided by the question: "What frames beyond children's thinking do teachers typically draw upon when making sense of their own students' written work?" Again, the focus for looking beyond children's thinking was to look for what teachers considered beyond what was addressed in the PD. The first analysis stage focused on segmenting the interview transcripts into idea units when a single topic was discussed. In the second stage, I determined the focus of each idea unit. During the third stage, I used an iterative process looking across the idea units to generate five frames for conversations about the teachers' own written work. I was inspired by the approach of Sherin and Russ (2014) but did not begin with their 13 *interpretative* frames for teachers' engagement with video. Instead, I generated frames that emerged from the data. Finally, in stage four, I used the five frames identified in the third stage to code the entire data set. In this final phase of analysis, I identified themes across frames to create categories and noted how many teachers used each of the five frames.

Results

The main result of the analysis of teachers' conversations was the identification of five frames used in the teachers' conversations. These five frames fell within two broad categories: non-mathematical or mathematical performance comparisons. Non-mathematical frames highlighted affective aspects or personal aspects of the child and their sense making. Mathematical performance comparison frames highlighted the child's mathematical performance in comparison to the child's performance in previous problems, the performance of others in the class on this problem or previous problems, or curricular or testing goals for that grade level. Table 1 describes each of the frames that emerged and includes the number of teachers who used for each frame.

Non-Mathematical frames

Confidence, behavior. This first category consisted of two frames in which teachers focused on some non-mathematical or personal aspects of the child in a discussion of the written work and sense-making. Non-mathematical frames were used to highlight the progress the child has made despite certain issues or as a way of making excuses for what the child was unable to accomplish. In the *confidence* frame, the teachers highlighted a child's mathematical confidence concerning their mathematical thinking. For example, when asked to describe the details of the child's strategy, Teacher 1 noted, "At first they did nothing. I had to walk them through the problem. They have the knowledge to solve the problem, but does not have the confidence to do it on their own without assistance." In this excerpt, Teacher 1 did not attend to the details of the child's strategy but excused the child's inability to solve the problem independently due to a lack of confidence when additional support was not provided. The confidence frame was used by 24% of the teachers.

Behavior was another non-mathematical frame in which teachers identified specific behaviors of a child and related that to their written work. In these instances, teachers' statements suggested that a child's behavior had a causal relationship to what was represented in the written work. For example, in response to the question, "What stood out to you in the child's work?" Teacher 2 stated, "I could not understand why they chose to draw so many lines. They were on the right track, but struggles with attention and gets side-tracked easily and probably started to draw a lot of elaborate lines and arrows." A closer look at the child's strategy indicated the lines and arrows were used to purposefully denote passing out to the sharers. Teacher 2 attributed the details in the child's representation to his difficulty focusing on the problem. Only two teachers used the behavior frame, but this frame was included because it speaks to ways others have found that teachers categorize students (Horn, 2007).

Mathematical Performance Comparison Frames

Past performance, class performance, broader scope. This second category included three frames that highlighted the mathematical performance of the individual child in comparison to that individual child, the class, or curricular goals. Teachers discussed the child's work in a descriptive or evaluative manner regarding its consistency with the teacher's knowledge and experiences from previous involvement with that specific child, children in the classroom, research, curriculum, etc. In the *past performance* frame, teachers highlighted how the child's performance on the problem compared with prior work from that child and often mentioned typical strategies used by the child or his or her progress over time. For instance, in a description of a child who solved the problem using a valid strategy but had an incorrect answer, Teacher 3 said, "They drew two rectangles and split each into 6 pieces, but no answer was written. I was surprised because yesterday, they had an invalid strategy. They might know more than I thought." These statements indicated the teacher originally expected the child to solve the problem using an invalid strategy based on their work on the previous day. The teacher was surprised the child used a valid strategy and acknowledged a potential underestimation of the child's understandings. The *past performance* frame was widely used by 60% of the teachers.

Related to the *past performance* frame was the *class performance* frame in which teachers highlighted how this child's performance on the problem compared with the performance of the rest of the class on this problem or previous similar problems. For example, when probed about a child's understandings for a problem about 20 cookies being shared by 8 friends, Teacher 4 stated "They understand that if there are 20 [items], you have 2 groups of 8 with a remainder of 4. They did the division in their head, while everyone else first passed out wholes individually. In some ways, they are more advanced than the others." This statement shows the teacher's comparison of the child's strategy and level of understanding to the majority of the class. Teachers also used the *class performance* frame as a way to consider how the child might offer support to others in the class or for others to help the child in their thinking and was also widely used by 67% of the teachers.

The third frame in the mathematical performance comparison category, the *broader scope* frame, drew on teachers' expectations from goals that were beyond the performance of the specific children in the teachers' class. In this frame, teachers highlighted how a child's performance on this problem compared with the broader curricular or testing goals for the grade level. Teacher 5 provided an example of the *broader scope* frame when asked to share next steps for the selected child: "I want [the class] to start notating their thinking more since on their [state standardized test] they have to write out everything. They cannot draw to show their work." Here, the teacher framed her analysis to focus on the use of notation for the child and the entire class based on their knowledge of testing expectations. The *broader scope* frame was used by 29% of the teachers.

Table 1: Frames Used to Discuss Teachers' Own Student's Written Work

	Written Work Frames	Definition	No. (%) of teachers using frame *N = 42*
Non-Mathematical	Confidence	Teacher highlights a child's confidence related to his or her problem-solving performance	10 (24%)
	Behavior	Teacher highlights a child's behavior related to his or her problem-solving performance	2 (5%)
Mathematical Performance Comparison	Past Performance	Teacher highlights how a child's performance on this problem compares with prior work from this ***child***	25 (60%)
	Class Performance	Teacher highlights how a child's performance on this problem compares with the performance of the rest of the ***class***	28 (67%)
	Broader Scope	Teacher highlights how a child's performance on this problem compares with the ***broader curriculum goals*** or testing for the grade level	12 (29%)

Extended Example of Frames in Use

When teachers talked about their written work, they often used multiple frames or the same frames multiple times throughout the conversation. The following example shows the use of various frames in a conversation with Ms. Young and written work from Jordan (pseudonyms were used). Ms. Young posed the following problem to the class: *10 friends want to share 19 brownies equally. How much brownie will each friend get?* Jordan solved the problem by drawing 19 rectangles to represent the brownies. She numbered the first ten brownies to give each friend one brownie. Jordan then split the next five brownies into halves and gave 1/2 to each friend. Finally, Jordan split the remaining four brownies into fifths and gave each person 1/5 from every two brownies, totaling 2/5. Jordan combined 1, 1/2, and 2/5 incorrectly and wrote 1 7/10 as her final answer.

Interviewer: What stood out to you about Jordan's work?

Ms. Young: Jordan's answer did not match her work. It's okay, but she is just not confident in her work. That is why she added them altogether. However, the rest of the class did what I expected them to do. Most of the class broke up the brownies into tenths, like I expected. Jordan did not do what I expected her to do.

Ms. Young was surprised that Jordan responded to the problem incorrectly despite having a valid strategy. Ms. Young used the *confidence frame* as a rationale for Jordan's mistake of adding the fraction pieces incorrectly. According to Ms. Young, Jordan's overall lack of confidence in her work was the reason for adding incorrectly. Ms. Young compared Jordan's difference in partitioning to the class, using the *class performance* frame. Ms. Young expected most of the class including Jordan to solve the problem by partitioning based on the number of sharers (10) and Jordan did not partition as anticipated, which was a reason her work stood out.

Later in the conversation, Ms. Young used the *past performance* frame to discuss Jordan's understandings:

Interviewer: Based on Jordan's work, what do you think she understands?
*Ms. Young***:** I think she can be flexible with her fractions. It's just interesting that she broke it up into halves first and then fifths. Usually, she does division and writes out the answer (without drawing a picture), so I don't know why she ended up drawing a picture today.

Ms. Young expected Jordan to solve the problem with a more advanced strategy than using a picture because according to Ms. Young, Jordan typically partitioned mentally and used symbols. Additionally, Jordan typically solved the problem by partitioning based on the number of sharers, but for this problem used halves and fifths. Jordan's variation in partitioning demonstrated flexibility with fractions for Ms. Young. In sum, Ms. Young's interview reflected a use of multiple frames to position her thinking about written work produced by a single child.

Discussion

This study investigated considerations drawn upon by elementary school teachers who examined written work from their classrooms. Findings suggest that teachers used their insider knowledge to think about the specific moment, past experiences, and their vision for the individual child and the class. The broad category of frames, non-mathematical and mathematical performance comparisons highlights topics that are typically overlooked or not foregrounded in the design of PD.

It is important to note that all teachers in this study used a *children's thinking* frame in their engagement with written work, but also included additional frames. The five frames identified suggest teachers often acknowledged an individual child's learning while considering the learning of others in the classroom (Webel & Platt, 2015). The use of multiple frames provided insight into how teachers coordinated what was shared in PD along with many other ideas of how to support student learning.

Table 1 summarized each of the frames found in the teachers' conversations and indicated some frames were used more by teachers than others. Overwhelmingly, teachers used more frames in the mathematical performance comparison category. This finding was not surprising given the task occurred during teachers' math lessons and the design of the PD was focused on children's mathematical thinking. It was somewhat surprising that the *past performance* frame was used as frequently as the *class performance* frame given the task was centered on the discussion of an individual child's work. The *past performance* and *class performance* frames were used by 60% and 67% of the teachers, respectively.

Conclusion

A closer look into the use of *frames* highlighted ways a single piece of written work came in and out of focus in teachers' discussion. Sometimes teachers only attended to aspects within the child's written work. Other times, teachers demonstrated an ability to maintain focus on the individual child's written work while comparing that work to others in the class or the class as a whole. Additionally, teachers exhibited foresight by thinking beyond the work of the individual child or the entire class to consider other curricular expectations. The different ways teachers attended to the child's written work linked to certain frames but highlighted another important feature to consider. These findings suggest the need for PD to afford opportunities for teachers to think about the complexity of honoring the individual child's thinking while keeping in mind thinking of the entire class and the connection to broader curriculum goals.

In PD, teachers are provided with a *children's thinking* frame that shapes their interaction with artifacts of practice (e.g., written work). In this study, teachers used the *children's thinking* frame but also other frames when engaging with written work. As the analysis of written work continues as a widespread practice in PD, those who design PD should consider how the use of frames–beyond those shared in PD–shapes teachers' engagement. Teachers in this study grappled with considering goals related to individual child, the class, and curricular expectations. The identification of the five frames used in teachers' conversations of written work can provide professional developers with ways to address contextual aspects of teaching. Further research is needed to understand the interplay of multiple frames and the overall nature of teachers' engagement with written work in the context of PD and in the classroom.

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References

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (2014). *Children's Mathematics: Cognitively Guided Instruction.* Portsmouth, NH: Heinemann.

Empson, S. B. & Levi, L. (2011). *Extending Children's Mathematics: Fractions and Decimals.* Portsmouth, NH: Heinemann.

Goffman, E. (1974). *Frame analysis: An essay on the organization of experience*. Harvard University Press.

Horn, I. S. (2007). Fast kids, slow kids, lazy kids: Framing the mismatch problem in mathematics teachers' conversations. *The Journal of the Learning Sciences*, *16*(1), 37-79.

Kazemi, E., & Franke, M. L. (2004). Teacher learning in mathematics: Using child work to promote collective inquiry. *Journal of mathematics teacher education*, *7*(3), 203-235.

Krebs, A. S. (2005). Analyzing child work as a professional development activity. *School Science and Mathematics*, *105*(8), 402-411.

Little, J. W. (2004). Looking at child work in the United States: A case of competing impulses in professional development. *In C. Day & J. Sachs (Eds.), International handbook on the continuing professional development of teachers*, 94-118.

Louie, N. L. (2016). Tensions in equity- and reform-oriented learning in teachers' collaborative conversations. *Teaching and Teacher Education*, *53*, 10–19

Sherin, M. G. & Russ, R. (2014). Teacher noticing via video: The role of interpretive frames. In B. Calandra & P. Rich (Eds.) *Digital video for teacher education: Research and practice.* New York: Routledge.

Sherin, M. G., & van Es, E. A. (2009). Effects of video club participation on teachers' professional vision. *Journal of Teacher Education*, *60*(1), 20–37.

Webel, C., & Platt, D. (2015). The role of professional obligations in working to change one's teaching practices. *Teaching and Teacher Education*, *47*, 204-217.

USING INTERPRETIVE FRAMES TO INFORM SELECTIONS OF ARTIFACTS OF STUDENT THINKING

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An important decision that professional development (PD) facilitators must make when preparing for activities with teachers is to select an appropriate tool for the intended learning goals of the PD (Sztajn, Borko, & Smith, 2017). One important and prevalent tool is artifacts of student thinking (e.g. Jacobs & Philipp, 2004). In this paper we add to the literature on artifact selection for professional development by discussing the affordances and constraints of different written artifacts of student thinking. Through a professional noticing assessment, we examine the interpretive frames (Sherin & Russ, 2014) that were invoked by 72 secondary teachers regarding 6 students' written strategies to proportional reasoning tasks. We characterize different ways teachers might make sense of different artifacts of student thinking, and discuss for what purposes PD facilitators might select particular written solutions.

Keywords: Written Artifacts of Student Thinking, Secondary Teachers, Interpretive Frames, Professional Development

In their review of research about professional development (PD) of mathematics teachers, Sztajn, Borko, and Smith (2017) found two striking similarities across the programs illustrated in the research. First, there was a similar vision for effective mathematics teaching across the programs, namely that teaching should include facilitation of interactions among students as they engage in rich mathematical tasks, that teachers should elicit and use students' emerging ideas to reach a mathematical goal, and that an overarching goal should be advancing understanding for all students. Second, the PDs were structured in similar ways; leaders *actively engaged* teachers in activities (as opposed to lecturing) about mathematics content and the teaching and learning of said content, and programs spanned many months and included many hours of work. Additionally, many PD programs focus on students' mathematical ideas as a way to support teacher learning, through the use of video artifacts (e.g. Sherin & van Es, 2005) or written artifacts (e.g. Jacobs & Philipp, 2004; Kazemi & Franke, 2004). Herein, we focus on written student work, given its prominent use in PD and its accessibility for teachers. To support PD leaders, we investigate the affordances and constraints of different features of written student work and how they might influence what teachers notice.

There are many ways teachers might interact with written student work. For example, National School Reform Faculty (2014) has over 200 different protocols that teachers and PD leaders can use to engage with students' written work, each supporting different discussions about a variety of pedagogical topics. However, little is known about *which* strategies teachers might be interested in analyzing and discussing (for selection criteria for videos of student thinking, see Sherin, Linsenmeier, & van Es, 2009). Hence, we investigated teachers' perceptions of six different written solution strategies, with the aim of informing future PD leaders as they select particular strategies to engage their teachers.

Conceptual Framework

In order to understand teachers' perceptions of the different artifacts, we draw upon the construct of *interpretive frames* (Sherin & Russ, 2014). Sherin and Russ defined interpretive frames as "structures that describe the ways in which a teacher's selective attention both grows out of and informs a teacher's knowledge-based reasoning, and vice-versa" (p. 3). In particular, what teachers notice is both contextual and interdependent. That is, what teachers notice depends on what they noticed previously, and is influenced by their beliefs, values, and/or knowledge. Teachers actively (yet tacitly) create particular frames through which they view their surroundings in order to make sense of their surroundings.

In their chapter, Sherin and Russ (2014) identified 13 different types of interpretive frames that teachers' created while making sense of videos of classroom lessons. Consider the following examples. In a video of a student loudly tapping their pen on a desk, a teacher viewing the event through an *evaluative* frame might claim the student is off task, and must be a bad student. Alternatively, a teacher viewing the event through an *affective* frame might express an emotional reaction to the event, and talk about how pen tapping irritates him/her. As another example, a teacher viewing the event through a *principle* frame might see the pen tapping as evidence the student is off-task, and describe this instance by citing a principle such as "students tend to act disruptively when they don't have access to the task." Importantly, each frame supported a different *way* of noticing the event, and teachers' perceptions of the event provide evidence of the creation of particular frames. In this study we focus on investigating two types of frames teachers create: *narrative* frames and *personal* frames. Narrative frames refer to when teachers simply describe what they notice, which may or may not include identifying causal relations. Personal frames refer to when teachers experience a personal connection to what they notice, which may include emotional reactions to what they notice, or desires to interact with the student/strategy in some way. We focus on these two frames because we believe they have much potential to influence the depth at which teachers engage with students' ideas.

Methods

To understand how different pieces of written student work influence what teachers notice, 72 practicing and prospective secondary teachers completed a survey. In the survey teachers responded to prompts about six different strategies for solving two proportional reasoning tasks. We then analyzed teachers' responses by identifying *when* teachers created particular interpretive frames, and *for which strategies* teachers created these frames (Sherin & Russ, 2014). In the next four sections we describe the participants, survey, strategies, and our analysis.

Participants

The secondary teachers in this study come from two teacher populations in the southwestern United States. First, 30 prospective secondary mathematics teachers were recruited from a large urban university. All intended to become secondary teachers, and none had begun the post-baccalaureate credential program offered at that university, nor student teaching. Second, 42 experienced practicing teachers (grades 6 - 12) were recruited from the southwestern region of the United States. All teachers had at least 4 years of experience teaching, and an average of 13.1 years of teaching experience.

We recognize that there are important differences among the two teacher groups that may influence what the teachers notice; however, parsing out these differences is beyond the scope of this paper. Our purpose in this paper is to investigate and discuss some of the ways different artifacts of student thinking pique secondary teachers' curiosity. For that reason, in the rest of the paper the term "teacher" will refer to both prospective and practicing teachers.

Survey

The survey consisted of three parts. In part 1, teachers first solved a missing-value proportional reasoning task, and then considered three student's written strategies for solving that task (see table 1). Next, teachers responded to three professional noticing prompts about the

Table 1: Tasks, Students, Strategies, and Descriptions

Student	Strategy	Our Description of the Strategy
	Task: Each day, 6 mice eat 18 food pellets. How many food pellets do 24 mice eat?	
A	$6\overline{)18}$ 3 24 × 3 62 food pellets	Student divides 18 pellets by 6 to find a unit rate of 3 pellets per mouse. Student multiplies unit rate by 24 mice. Student makes a mistake when multiplying 24x3.
B	$\frac{18}{6} = \frac{x}{24}$ $\frac{432}{6} = \frac{6x}{6}$ $\boxed{72 = x}$	Student uses the traditional cross-multiplication strategy, setting up pellets on top and mice on bottom of the fraction. Student manipulates equation correctly.
C	24 ÷ 6 = 4 18 × 4 = 72	Student divides 24 mice by 6 mice to find the number of groups of 6 mice in 24. There are 4 groups of 6 mice, so there should be 4 groups of 18 pellets.
	Task: Each day, 8 caterpillars eat 12 leaves. How many leaves do 20 caterpillars eat?	
D	8÷12=1½ 8÷8=1 12−8=4=½ of 8 1½×20=X 30 leaves 1×20=20 20×1=20 20/2=10 20+10=30	Student divides 12 leaves by 8 caterpillars (mistakenly writes 8 divided by 12), to find each caterpillar eats 1.5 leaves. The student divides in a non-standard way. The student then multiplies 1.5 leaves x 20 caterpillars, using the distributive property to help with the multiplication.
E	8 caterpillars × 3 = 24 caterpillars 12 × 3 = 36 leaves 36 − 4 = 32 24 − 4 = 20 Caterpillars | leaves 20 — 32	Student multiplies both 8 caterpillars and 12 leaves by 3, apparently scaling up the ratio to 24 caterpillars and 36 leaves. The student subtracts 4 from both quantities to obtain the 20 caterpillars, and arrives at 32 leaves. This last part exhibits additive reasoning, and is not correct.
F	8×3= 24 24 − 4 = 20 12÷2=6 12×3= 36 −6 30 30 leves	Student multiplies 8 caterpillars by 3 to get 24 caterpillars, then subtracts 4 to get 20 caterpillars. Student likely recognized that 4 caterpillars is half of 8 caterpillars, because next the student finds half of 12 leaves (6 leaves). The student then multiplies 12 leaves by 3, and subtracts a "half group of leaves" from 36 to get 30 leaves

students' mathematical thinking (Jacobs, Lamb, & Philipp, 2010). Part 2 was similar to part 1, except with a different task and with different strategies (table 1). In this paper, we discuss responses to the first two prompts: (a) Describe in detail what these students did in response to the task, and (b) What did you learn about these students' mathematical understandings? Hence,

we collected data on both what the teachers attended to and interpreted in the students' strategies, and consequently the narrative frames teachers created with respect to the strategies.
In the third part of the survey, teachers revisited all six of the strategies they had previously considered, and responded to four prompts. In this paper, we discuss the analysis of the first two prompts: (a) Is there a student you would like to talk to further? If yes, which student would you like to talk to further, and why? (b) Would you be interested in discussing a particular solution with other teachers? If yes, which solution would you discuss with other teachers, and why? Hence, we collected data on whether teachers wanted to interact with a specific student or strategy, and consequently the personal frames teachers created with respect to the strategies.

Written Artifacts of Student Thinking

The mathematical tasks and written artifacts can be seen in Table 1. To differentiate among the strategies, we focus on 6 characteristics: (a) strategy type, (b) integer/non-integer ratios, (c) exhibits non-standard calculation strategies, (d) conceptually correct/incorrect, (e) correct/incorrect calculations, and (f) work includes all steps/work or is missing steps. For strategy type, we identified what type of strategy the student used according to the literature on proportional reasoning (Carney et al., 2015; Lobato & Ellis, 2010). Strategies A and D employ a unit rate strategy, in which the student finds how many pellets (or leaves) one mouse (or caterpillar) eats, and then multiplies this number by the new number of mice (or caterpillars). Strategy B employs a cross-multiplication strategy, which provides little evidence of the degree to which the student has a conceptual understanding of proportions (Cramer, Post, & Currier, 1993). Strategies C, E, and F employ different scaling strategies, where the student scales up the original ratio (strategy C), scales up and adds equal amounts (strategy E), or scales up and adds proportional amounts (strategy F).

For integer ratios, we identify whether the student used a scalar or unit rate multiplicative relationship (Carney et al., 2015), and whether that relationship was an integer ratio or non-integer ratio (Tourniaire & Pulos, 1985). For strategies A and C, the scalar and functional multiplicative relationships are integer ratios, and for strategies D and F they are non-integer ratios. We did not include strategies B or E, because strategy B does not use either the scalar or unit rate multiplicative relationship when solving, and strategy E includes additive reasoning. For the other four categories, we simply looked for whether there was evidence the strategy exhibited that characteristic or not. For example, strategy D exhibits non-standard calculations, while the others do not. Table 2 summarizes the characteristics we identified for each strategy.

Table 2: Characteristics of Strategies

Student	Strategy Type	Integer Ratios?	Correct Calc's?	Standard Calc's?	Correct Concept?	All Steps?
A	Unit Rate	Yes	No	Yes	Yes	Yes
B	Cross-Multiplication		Yes	Yes		Yes
C	Scale Up	Yes	Yes	Yes	Yes	Yes
D	Unit Rate	No	No	No	Yes	Yes
E	Scale Up w/ Adding Equal Parts		Yes	Yes	No	Yes
F	Scale Up w/ Adding Prop. Parts	No	Yes	Yes	Yes	No

Analysis

Analysis was conducted by the first author. In parts 1 and 2, teachers described the students' strategies and understandings. Hence, all teachers created narrative frames in some form as they noticed the student's work and described what the student did. Consequently, the first author focused solely on whether the teacher's description got the gist of the strategy, or the key underlying reasoning of the strategy. Examples include descriptions similar to our own, or descriptions of the strategy we identified (e.g. "Student A found the unit rate and solved for 24 mice"). Non-examples include identifying a strategy other than the one we identified, claiming the student was confused when evidence existed that the student was not, or admitting confusion about what the student did (e.g. "I have no idea what this student did"). More than 20% of the data was double coded by another researcher, and interrater reliability was 87%.

In part 3, teachers had a choice of talking or not talking to/about a student. However, it turned out that 100% of the teachers chose to talk to/about at least one student. The first author coded for *which* student a teacher selected, and *what* they wanted to talk about. Responses fell into four categories: (a) learn more about the strategy, (b) help the student, (c) share the strategy with the class or with other teachers, or (d) discuss other mathematical topics. More than 20% of the data was double coded by another researcher, and interrater reliability was 91%.

Finally, the first author also looked for instances when teachers spontaneously expressed excitement toward a strategy (e.g. "Love it," "This student is a genius," "Student F is my favorite"), which we took as evidence that the teacher created a personal connection with the strategy (i.e. a personal frame). Even though we did not actively seek to collect data on teachers' emotions, 24% of the teachers spontaneously expressed excitement for at least one strategy at some point in the survey. More than 20% of the data was double coded by another researcher, and interrater reliability was 93%.

Results

For narrative frames, we counted instances when the teacher's response captured the gist of the strategy. Percentages of teachers that that captured the gist of the strategy can be seen in Table 3. Almost every teacher captured the gist of strategy B, which was the cross-multiplication strategy. Comparing the correct unit rate strategies (i.e. A & D) with the correct scaling strategies (i.e. C & F), we see that the scaling strategies were more challenging for teachers to capture the gist of the strategy. Additionally, within these four conceptually correct strategies, it appears the non-integer ratio strategies (i.e. D & F) were more challenging than the integer ratio strategies (A & C), respectively, but only marginally more challenging than their counterparts. Strategy E, which included additive reasoning, was more challenging than the unit rate and cross multiplication strategies, but less challenging than the other two scalar strategies. For personal frames, we counted instances when a teacher wanted to talk to/about a particular student, and then categorized responses based on what they wanted to talk about. We also looked for evidence that a teacher enjoyed a particular strategy. Percentages can be seen in Table 3. When considering the conceptually correct scalar and unit rate strategies (i.e. A, C, D,

Table 3: Percentages of teachers capturing gist, wanting to talk to/about a student, and expressing emotion

N = 72	A	B	C	D	E	F
% of teachers that captured gist of the strategy	89%	97%	63%	83%	69%	57%
% of teachers who want to talk to/about a student	8%	15%	25%	33%	60%	75%
% of teachers expressing excitement	0%	3%	3%	11%	4%	18%

& F), scalar strategies elicited more teachers wanting to talk to/about the student than the unit rate strategies, and the strategies with non-integer ratios were more intriguing to the teachers than the integer-ratio strategies. Strategy F (which was the most challenging, included non-integer ratios, used a scaling up strategy, and appeared to not show all steps) elicited the largest percentage of teachers wanting to talk to/about the student, as well the largest percentage of teachers expressing excitement. Strategy E, which had the conceptual error, elicited the second highest percentage of teachers wanting to talk to/about the student. Strategy D, which had the non-standard algorithms, elicited the second highest percentage of teachers expressing excitement. Looking at what excited the teachers for strategy D, ¾ of the teachers who expressed excitement specifically mentioned the non-standard algorithms.

The percentages of which students teachers wanted to talk to/about and why can be seen in table 4[1]. Notice that almost half of the teachers wanted to learn more about strategy F, which supports the notion that strategy F was the most challenging and intriguing to teachers. Strategy F also elicited the highest percentage of teachers who wanted to share the strategy, either with their class or with other teachers.

Looking at strategy E, which had the conceptual error, we see that this strategy elicited the highest percentage of teachers (38%) wanting to help the student. In contrast, less than 10% of the teachers wanted to help students A and D (respectively), who exhibited calculational errors in their work. (The teachers who wanted to help students C and F assumed they were confused.)

When considering the conceptually correct scalar and unit rate strategies (i.e. A, C, D, & F), we see two more pieces of evidence that scalar strategies and strategies with non-integer ratios are more interesting than their counter-parts. First, more teachers wanted to learn about scalar strategies than unit-rate strategies, and more teachers wanted to learn more about the strategies with non-integer ratios than the strategies with integer ratios. Second, more teachers wanted to share the scalar strategies with others than the unit rate strategies, and more teachers wanted to share the strategies with non-integer ratios than the strategies with integer ratios. When sharing strategies, teachers often either expressed excitement for the strategy, or valued the different ways of thinking exhibited in the work.

Finally, teachers wanted to discuss a variety of other things with their peers. This included learning about some new mathematics, anticipating other solutions, discussing the importance of multiple solutions, discussing pedagogical strategies for teaching proportional reasoning, and assessment strategies. Notice that the cross multiplication strategy had a large number of teachers wanting to discuss other mathematical ideas. All of these teachers wanted to discuss whether the cross-multiplication strategy exhibited a high level of understanding, or not. This

Table 4: Percentages of teachers wanting to talk to/about a student and why

N = 72	A	B	C	D	E	F
Want to learn about strategy	3%	4%	14%	18%	22%	46%
Want to help student	4%	1%	4%	6%	38%	8%
Want to share strategy		4%	7%	4%	6%	18%
Want to discuss other topics	1%	6%	1%	7%	1%	8%
Total wanting to talk to/about student	8%	15%	25%	33%	60%	75%

means that not only did these teachers recognize that this student might have used a memorized procedure, but they were also interested in discussing such implications with other teachers.

[1] In table 4, the columns do not sum to the totals because some teachers created multiple different personal connections for the same student. Additionally, the totals do not sum to 100% because some teachers chose to talk to/about multiple students.

Discussion

To summarize, we found that some strategies elicited teachers' desire for discussions more often than others, albeit for different reasons and in different ways. In this section we discuss the strategies and the interpretive frames teachers created, and for what purposes PD leaders might select particular strategies.

Strategy F was clearly the most exciting, interesting, and challenging for teachers. We conjecture that this strategy was difficult and interesting for four reasons. First, the scalar strategy type appears to be less familiar to teachers than the unit rate or cross multiplication strategy. Second, the non-integer ratios afforded certain complexities that would not have been available in a task structure with integer ratios. In particular, this student has to both iterate and partition the ratio, and then subtract a partitioned ratio from the larger ratio (Lobato & Ellis, 2010). In strategy C, the student need only multiply the 18 pellets by a whole number; there is no need to partition the original ratio. Third, the missing step was a challenging step for many teachers to notice, but also an important part of the strategy. Finally, the computations are all quite simple at first glance. We wonder if the simple computations created a slight misdirection in teachers' expectations as they considered the student's strategy. Perhaps the excitement teachers expressed arose from prevailing in understanding a challenging strategy.

Student E was also often chosen by teachers, but for different reasons than for student F. Many of those who wanted to talk to/about student E wanted to help student E fix the error, or develop a stronger understanding. Contrasting this with the other two strategies that exhibited errors, strategy A and strategy D, we see that the conceptual error in strategy E was more interesting to teachers than the calculational errors in strategies A and D. Hence, it appears conceptual errors intrigue teachers more than calculational errors.

Strategies D and F elicited the most excitement from teachers. For strategy D, ¾ of the teachers who exhibited excitement specifically cited the non-standard division and/or multiplication algorithms as exciting. Strategy F appeared to be exciting in general. Considering the two strategies, we highlight four similarities that may have supported such excitement. In particular, strategy F and the non-standard algorithms of strategy D were correct, based on underlying concepts, complex, and unfamiliar to teachers. We conjecture that other strategies with these qualities may also elicit excitement from teachers.

Another idea that emerged from the data was that 6% of the teachers wanted to discuss the underlying mathematics of strategy B, and whether it counted as a deep conceptual understanding or not. It appears that these teachers recognized that the student might have been following a memorized procedure, and wanted to discuss with other teachers what this might mean with respect to learning and teaching. We believe that conversations like the one these teachers wanted to have could be productive for teachers. Perhaps other more traditional strategies, when surrounded by non-standard or conceptually-based strategies, could spark conversations among teachers about the underlying mathematics.

In their study of video artifacts of student thinking, Sherin, et al. (2009) rated videos based on three particular characteristics, and looked for which types of videos supported conversations among teachers. They concluded that there wasn't a particular characteristic that was more important than others, but rather certain combinations of characteristics supported better discussions. Our results seem to support this idea, that no single characteristic makes a solution interesting, but rather a combination of characteristics.

Earlier we mentioned that we recognized there were important differences between the two teacher groups from which we collected our data. In our analysis we noticed that there were indeed differences among the teacher groups. For example, the four teachers who wanted to discuss whether the cross multiplication strategy exhibited a deep conceptual understanding were all practicing teachers. In contrast, the two teachers who wanted to show off the cross multiplication strategy (because they valued the strategy) to other teachers were prospective teachers. However, due to the scope of this paper we did not aim to parse out these differences, and instead focused on investigating what interpretive frames our teachers invoked and how they were invoked across the different strategies. In future work we will disaggregate our data and investigate differences in interpretive frames among the teacher groups.

We end by emphasizing that different artifacts serve different purposes. Each strategy seemed to have its own unique set of challenges, and teachers wanted to respond to different ideas based on the different strategies. In our work we only looked at 6 different strategies and the qualities associated with these strategies, and aggregated data from a diverse group of secondary teachers. We wonder what other kinds of strategies researchers might consider, what other combinations of qualities might pique teachers' interest, and how teachers' interests differ across different populations of teachers.

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References

Carney, M. B., Hughes, G. R., Brendefur, J. L., Crawford, A. R., & Totorica, T. (2015). Conducting classroom assessment based in cognition: The case of proportional reasoning. In C. Suurtam (Ed.), *Assessment to enhance teaching and learning* (pp. 209-218). National Council for Teachers of Mathematics.

Cramer, K., Post, T., & Currier, S. (1993). Learning and teaching ratio and proportion: Research implications. In D. Owens (Ed.), *Research Ideas for the Classroom* (pp. 159-178). NY: Macmillan Publishing Company.

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169–202.

Jacobs, V. R., & Philipp, R. A. (2004). Mathematical Thinking: Helping Prospective and Practicing Teachers Focus. *Teaching Children Mathematics*, *11*(4), 194-201.

Kazemi, E., & Franke, M. L. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education*, *7*(3), 203-235.

Lobato, J., & Ellis, A. (2010). *Developing Essential Understanding of Ratios, Proportions, and Proportional Reasoning for Teaching Mathematics: Grades 6-8*. Reston, VA: National Council of Teachers of Mathematics.

National School Reform Faculty. (2014). *National School Reform Faculty Harmony Education Center*. Retrieved from https://www.nsrfharmony.org/.

Sherin, M. G., Linsenmeier, K. A., & van Es, E. A. (2009). Selecting video clips to promote mathematics teachers' discussion of student thinking. *Journal of Teacher Education, 60*(3), 213–230.

Sherin, M. G., & Russ, R. (2014). Teacher noticing via video: The role of interpretive frames. In B. Calandra & P. Rich (Eds.), *Digital video for teacher education: Research and practice* (pp. 3 - 20). New York: Routledge.

Sherin, M. G., & van Es, E. A. (2005). Using video to support teachers' ability to notice classroom interactions. *Journal of Technology and Teacher Education, 13*(3), 475–491.

Sztajn, P., Borko, H., & Smith, T. (2017). Research on mathematics professional development. In J. Cai (Ed.), *First Compendium for Research in Mathematics Education* (pp. 793-823). Reston, VA: National Council of Teachers of Mathematics.

Tourniaire, F., & Pulos, S. (1985). Proportional reasoning: A review of the literature. *Educational studies in mathematics*, *16*(2), 181-204.

PROFESSIONAL DEVELOPMENT AND KNOWLEDGE GAINS FOR HIGH SCHOOL MATHEMATICS TEACHERS

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This study examined the effects of the Local Systemic Change Through Teacher Enhancement Initiative (LSC) on 1,596 in-service high school mathematics teachers' knowledge. Because data are clustered by schools, a hierarchical linear model was calculated. It was found that five of the characteristics of the professional development, time to reflect, support to implement, hours attended, assistance to implement from LSC, and participation in message boards, had statistically significant effects on increasing teacher knowledge, but time to work with other teachers, race, educational background, and years taught did not have a statistically significant effect.

Keywords: High School Education, Teacher Knowledge

Introduction

There are many different ideas about what makes professional development effective, but there is some agreement on what key features are important (Borko, 2004; Darling-Hammond, Wei, Andree, Richardson, & Orphanos, 2009; Desimone, 2009; Garet, Porter, Desimone, Birman, & Yoon, 2001; Heck, Banilower, Weiss, & Rosenberg, 2008). Desimone (2009) argues for her Core Conceptual Framework with five key features of professional development that are content focus, active learning, coherence, duration, and collective participation, and that these key features lead to an increase in teacher knowledge. In the Eisenhower professional development programs, Garet et al. (2001) found that focus on content knowledge, active learning, and coherence were all statistically significant predictors of enhanced knowledge and skills when controlling for sponsor, traditional or reform, span, and contact hours.

Between 1997 and 2006, the National Science Foundation funded 86 professional development projects, 48 of which were for math teachers, across the country through the Local Systemic Change Through Teacher Enhancement Initiative (LSC) (Heck et al., 2008). According to Heck et al. (2008), the main goals of the LSC projects were to include all teachers in targeted schools with a minimum of 130 hours of professional development and to implement LSC-designed curriculum materials in the teachers' classrooms. The individual projects were designed to include key features of effective professional development, but the main focus was to increase teachers' content knowledge, pedagogical content knowledge, and the use of investigative pedagogical strategies (Heck et al., 2008). To assess these outcomes, teachers were expected to complete a questionnaire at the end of a year of participation in the LSC project (Heck et al., 2008).

Research Question

What factors of an LSC professional development for high school math teachers predict perceived increase in knowledge by the teacher?

Method

Data

The data was collected from the National Science Foundation's Local Systemic Change through Teacher Enhancement Initiative (LSC). This project offered professional development for K-8 and 6-12 teachers in mathematics and science over the school years 1996-97 to 2005-06 (Banilower, Heck, & Weiss, 2007; Heck et al., 2008). Each year of the project, principals and teachers were required to complete a Principal Questionnaire or Teacher Questionnaire, with a response rate of 80% expected for the teachers and 90% expected for the principals (Heck et al., 2008). Several of the questions in the questionnaires changed in the initial two years, but the questions stabilized by the third year (Heck et al., 2008). For questions that were changed, data from later questionnaires were both recoded to the original scale and included in the new scale with missing values for the original questionnaires. The data set is available by request from Horizon Research.

Participants

The full data set included over 80,000 participants' responses to the four teacher questionnaires, K-8 math, 6-12 math, K-8 science, and 6-12 science, but the data were filtered to focus just on the 9-12 math teachers which left 3,064 of the full data set. Removing any participants who were missing data for the variables of interest left 1,596 teachers in the sample data to be analyzed.

The participants are all 9-12 mathematics teachers who took part in LSC professional development. The participants are split almost evenly between males (45%) and females (55%). The participants are primarily white (80%), with other races under 10% (black). For educational background, the majority had an undergraduate major in mathematics or mathematics education (60%) and certification to teach (60%). Additionally, a large percentage held a graduate degree in mathematics or mathematics education (38%). On average, the participants had 12.30 years of teaching experience.

Table 1: Descriptive Statistics of Teacher Control Variables by Sample and Original Data

	Sample Data						Original Data				
	M	SD	Min	Max	N		M	SD	Min	Max	N
Male	.43	.50	0	1	1,596		.45	.50	0	1	3,023
Hispanic	.03	.17	0	1	1,596		.03	.18	0	1	3,064
Indian	.01	.11	0	1	1,596		.01	.11	0	1	3,064
Asian	.02	.15	0	1	1,596		.03	.17	0	1	3,064
Black	.09	.29	0	1	1,596		.10	.31	0	1	3,064
Pacific Islander	.00	.04	0	1	1,596		.00	.03	0	1	3,064
White	.83	.37	0	1	1,596		.80	.40	0	1	3,064
No Race Indicated	.02	.14	0	1	1,596		.03	.17	0	1	3,064
Undergraduate Major	.61	.49	0	1	1,596		.60	.49	0	1	3,020
Undergraduate Minor	.10	.30	0	1	1,596		.10	.30	0	1	3,020
Graduate Major or Minor	.38	.49	0	1	1,596		.38	.49	0	1	3,020
Teaching Certificate	.63	.48	0	1	1,596		.60	.49	0	1	3,020
None of the Above Education	.07	.26	0	1	1,596		.08	.27	0	1	3,020
Years Taught	11.86	7.67	1	21	1,596		12.30	7.79	1	21	3,038

Note. $N = 3{,}064$ for the original data and $N = 1{,}596$ for the sample data. Years taught is on a scale of 1 to 21+ years, with the maximum coded as 21. For all other variables, 0 indicates "no" and 1 indicates "yes."

For the original data set, the data are nested within 19 professional development projects with a range of 1 to 400 participants in each project. There mean number of teachers per project is 161.26 with a standard deviation of 117.94. For the sample data, the data are nested within 18 professional development projects with a range of 3 to 223 participants in each project. There are a mean number of 88.67 teachers per project with a standard deviation of 61.74.

Table 2: Descriptive Statistics by Project ID by Sample and Original Data

	Sample Data			Original Data	
Project ID	Count	Percent		Count	Percent
04	56	3.51		98	3.20
09	77	4.82		153	4.99
10	93	5.83		188	6.14
18	223	13.97		400	13.05
19	13	.81		16	.52
28	0	.00		1	.03
31	51	3.20		124	4.05
36	98	6.14		195	6.36
39	129	8.08		249	8.13
40	58	3.63		101	3.30
42	153	9.59		224	7.31
44	95	5.95		232	7.57
48	68	4.26		123	4.37
58	144	9.02		284	9.27
71	183	11.47		369	12.04
72	3	.19		8	.26
73	4	.25		11	.36
74	120	7.52		219	7.15
78	28	1.75		58	1.89

Note. $N = 3{,}064$ for the original data and $N = 1{,}596$ for the sample data.

Measures

The control teacher variables come primarily from the teacher questionnaire. The variables for gender and race are dummy variables created from participants choosing one of a list of choices, as seen in Table 1. Participants also chose from a list of educational backgrounds, but were allowed to choose all choices that applied to them, as seen in Table 1. For years taught, participants chose from a list with ranges of years taught (0-2, 3-5, 6-10, 11-15, 16-20, 21-25, 26 or more). These were initially coded as dummy variables, but were operationalized as the mean of each range to turn them into more meaningful continuous variables.

Most of the covariate measures also come directly from the teacher questionnaire. These include the amount of time to work with other teachers during the professional development, time to reflect during professional development, and support to implement what was learned in the professional development. All three of these variables came directly from the questionnaire, with a rank of 1 to 5, with 1 labeled as "Not at all" and 5 labeled as "To a great extent." The number of hours of professional development attended was listed in 11 categories on the questionnaire (0, 1-9, 10-19, 20-39, 40-59, 60-79, 80-99, 100-129, 130-159, 160-199, and 200 or greater). These were initially coded as dummy variables from 1 to 11, but were operationalized as the mean of each range to change them into continuous variables with more meaning in the context of the research question.

The final three professional development variables, assistance to implement the professional development from LSC, participation in the LSC message boards, and increase in knowledge from professional development, are each means of multiple questions from the questionnaire. All of the original questions for these measures are on the same scale as the number of times participants did each activity (0, 1-2, 3-4, 5-6, and 7 or more). Assistance to implement the professional development is the average of four sources of assistance, coaching by LSC staff based on observations, LSC teacher leaders, LSC district staff, or LSC mathematicians or mathematics educators, with $\alpha = .7247$ with all four measures included. Participation with message boards was the average of three sources of participation, reading messages, posting messages, and discussion groups, but the $\alpha = .5107$ with all three sources and $\alpha = .6440$ with just reading messages and posting messages. Thus, participation is only the average of the two sources that leave the highest α, reading and posting on message boards. Perceived increase in knowledge was initially the average of three sources of knowledge, mathematics content knowledge, understanding of how children think/learn about mathematics, and ability to implement high-quality mathematics instructional materials, with $\alpha = .8887$. However, $\alpha = .8907$ by removing mathematics content knowledge, so perceived increase in knowledge is just the average of the two measures of teaching knowledge.

Table 3: Descriptive Statistics of Professional Development Variables by Sample and Original Data

	Sample Data						Original Data				
	M	SD	Min	Max	N		M	SD	Min	Max	N
Amount of time to work with other teachers	2.84	1.23	1	5	1,596		2.81	1.22	1	5	1,962
Level of time to reflect	2.76	1.19	1	5	1,596		2.73	1.18	1	5	1,957
Level of support to implement what was learned	3.42	1.15	1	5	1,596		3.39	1.15	1	5	1,952
Hours attended	93.78	67.04	5	200	1,596		62.07	69.32	0	200	2,843
Level of assistance to implement from LSC	1.43	1.62	0	7	1,596		1.40	1.60	0	7	1,908
Amount of participation in message boards	1.25	1.40	1	6	1,596		1.25	1.40	0	7	1,919
Perceived increase in knowledge	3.32	1.09	1	5	1,596		3.32	1.10	1	5	1,941

Note. $N = 3{,}064$ for the original data and $N = 1{,}596$ for the sample data.

Analysis

Given that the original analysis was concerned with how the teachers were able to implement the LSC curriculum materials in their classrooms, the possible covariates and control variables were limited by the data collected in the teacher questionnaire. The covariates used were amount of time to work with other teachers during professional development, level of time to reflect during professional development, level of support to implement what was learned in professional development, hours of professional development attended, level of assistance to implement professional development from LSC, and amount of participation in LSC message boards. Research has shown collective participation (Desimone, 2009; Franke, Carpenter, Levi, & Fennema, 2001; Garet et al., 2001) and duration (Banilower, Heck, & Weiss, 2007; Garet et al., 2001; Heck et al., 2008) are both predictors of teachers' gains in knowledge from professional development. Other research has shown that aid to implement what is learned in professional development is related to teachers' long term use of what was learned in the professional development (Carpenter, Fennema, & Franke, 1996; Fennema et al., 1996; Franke et al., 2001). Participation in online forms of professional development has also been shown to be effective in helping teachers to gain knowledge (Boling & Martin, 2005; Dede, Breit, Ketelhut, McCloskey, & Whitehouse, 2005; Herrington, Herrington, Hoban, & Reid, 2009). The possible teacher control variables were gender, race, educational background, and number of years teaching. All were included, because it is likely that each will have an influence on how teachers view what they have learned from the professional development projects.

Results

The sample data are nested in 18 LSC professional development projects, so a hierarchical linear regression model was calculated. This model produced similar results to the multiple regression model, with the same professional development covariates being statistically significant predictors of perceived increase in knowledge from the professional development. In model 2, these variables include time to reflect on what was learned in the professional development, support to implement what was learned in the professional development, hours of professional development attended, assistance to implement what was learned in the professional development from LSC, and participation in LSC message boards; all were statistically significant at $p < .001$. Unlike the multiple regression however, the only statistically significant teacher control variable was gender ($t(1595) = -6.00$; $p < .001$).

Table 4: Hierarchal Linear Model of Increase in Knowledge on Professional Development Variables and Teacher Control Variables

	Model 1	Model 2
	Perceived increase in knowledge	Perceived increase in knowledge
Time to work with other teachers	.0147	.0101
Time to reflect	.0958***	.1081***
Support to implement	.2067***	.1877***
Hours attended	.0047***	.0045***
Assistance to implement from LSC	.1112***	.1077***
Participation in message boards	.0683***	.0791***
Male		-.2719***
Hispanic		.2645
Indian		-.1649
Asian		-.1416
Black		.0763
Pacific Islander		-.8439
No Race Indicated		-.0513
Undergraduate Major		.0191
Undergraduate Minor		.0754
Graduate Major or Minor		.0109
Certification to Teach		.1726
Years Taught		-.0074
Constant	1.6898***	1.8256***
N	1596	1596

Note. t statistics in parentheses, $*** \; p < .001$, $** \; p < .01$, $* \; p < .05$. Reference group is white for race and no education in mathematics or mathematics education.

Discussion

Both the multiple regression and the hierarchical linear model agree on the factors of the LSC professional development that related to the teachers' perceived increase in knowledge. The number of hours of professional development attended has the greatest standardized coefficient, so it has the greatest impact on the projected increase in knowledge. Specifically, for every one standard deviation increase in number of hours attended by a teacher, there is a projected .2889 standard deviation increase in knowledge. This agrees with the research findings that say that the duration of a professional development is one of its key features (Desimone, 2009; Garet et al., 2001; Heck et al., 2008). Another result that agrees with research is that the support and the assistance to implement the professional development is related to the increase in knowledge. Franke et al. (2001) found that the teachers felt the support of the professional development team was a critical factor in their ability to sustain what they learned in the professional development.

Finally, participation in the message board was also a statistically significant predictor of the teachers' increase in knowledge, which agrees with research that online professional development can help teachers increase their knowledge (Boling & Martin, 2005; Dede et al., 2005; Herrington et al., 2009).

One result that does not agree with research is about the time to work with other teachers. Research shows that collective participation is a key feature of professional development (Desimone, 2009) with a positive relationship to increased knowledge (Garet et al., 2001). However, the multiple regression found working with other teachers to have a negative relationship that was not a statistically significant predictor of increase in knowledge.

Limitations

These results are based on self-reported survey data from the participants about their views on the professional development project and what they learned from the professional development project. However, Desimone (2009) argues that surveys are a reliable data source for behavior-based constructs about frequencies of events, such as teachers' experiences in professional development and experiences implementing the professional development. This data is a mix of behavior-based questions and evaluative questions, so using a survey may not be the most reliable source of data collection. Another limitation is that the teachers are rating their own perceived increase in knowledge, rather than using a pretest and posttest of knowledge to quantify how much teachers learned. This makes it difficult to know how much the teachers actually learned from attending the professional development.

Next Steps

This analysis focused on teacher knowledge, but not on how teacher knowledge affects teaching practices or student achievement. Research has shown that teacher knowledge has an impact on both practice in the classroom and on students' achievement (Ball, Thames, & Phelps, 2008; Hill, Rowan, & Ball, 2005; Kennedy, 1999). Although data is not available about student achievement for the LSC projects, the teacher questionnaire included questions about teaching practice. Thus, a next step would be to analyze whether the change in knowledge predicts teaching practice, as was found in other research on professional development (Desimone, 2009; Garet et al., 2001; Heck et al., 2008).

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References

Ball, D. L. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407. http://doi.org/10.1177/0022487108324554

Banilower, E. R., Heck, D. J., & Weiss, I. R. (2007). Can professional development make the vision of the standards a reality? The impact of the National Science Foundation’s local systemic change through teacher enhancement initiative. *Journal of Research in Science Teaching*, *44*(3), 375–395. http://doi.org/10.1002/tea

Boling, C. J., & Martin, S. H. (2005). Supporting teacher change through online professional development. *Journal of Educators Online*, *2*(1), 1–15. http://doi.org/10.1.1.411.8268

Borko, H. (2004). Professional development and teacher learning: Mapping the terrain. *Educational Researcher*, *33*(8), 3–15. http://doi.org/10.3102/0013189X033008003

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal*, *97*(1), 3. http://doi.org/10.1086/461846

Darling-Hammond, L., Wei, R. C., Andree, A., Richardson, N., & Orphanos, S. (2009). *Professional learning in the learning profession: A status report on teacher development in the United States and abroad*. National Staff Development Council. Washington, DC. http://doi.org/10.1006/jfbi.2002.2063

Dede, C., Breit, L., Ketelhut, D. J., McCloskey, E., & Whitehouse, P. (2005, November). An overview of current findings from empirical research on online teacher professional development. In *Online Teacher Professional Development Conference at the Harvard Graduate School of Education, Cambridge, MA.* http://doi.org/10.1.1.117.1285

Desimone, L. M. (2009). Improving impact studies of teachers' professional development: Toward better conceptualizations and measures. *Educational Researcher*, *38*(3), 181–199. http://doi.org/10.3102/0013189X08331140

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education*, *27*(4), 403–434. http://doi.org/10.2307/749875

Franke, M. L., Carpenter, T. P., Levi, L., & Fennema, E. (2001). Capturing teachers' generative change: A follow-up study of professional development in mathematics. *American Education Research Journal*, *38*(3), 653–689. http://doi.org/10.3102/00028312038003653

Garet, M. S., Porter, A. C., Desimone, L., Birman, B. F., & Yoon, K. S. (2001). What makes professional development effective? Results from a national sample of teachers. *American Education Research Journal*, *38*(4), 915–945. http://doi.org/10.3102/00028312038004915

Heck, D. J., Banilower, E. R., Weiss, I. R., & Rosenberg, S. L. (2008). Studying the effects of professional development: The case of the NSF's local systemic change through teacher enhancement initiative. *Journal for Research in Mathematics Education*, *39*(2), 113–152. http://doi.org/10.2307/30034894

Herrington, A., Herrington, J., Hoban, G., & Reid, D. (2009). Transfer of online professional learning to teachers classroom practice. *Journal of Interactive Learning Research*, *20*(2), 189–213.

Hill, H., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, *42*(2), 371–406. http://doi.org/10.3102/00028312042002371

Horizon Research. (2006). National Science Foundation's (NSF) Local Systemic Change through Teacher Enhancement initiative (LSC) Public Release Data [Data file]. Retrieved from http://www.horizon-research.com/LSC/.

Kennedy, M. M. (1999). *Form and Substance in Mathematics and Science Professional Development. NISE brief.* Retrieved from http://www.eric.ed.gov/ERICWebPortal/recordDetail?accno=ED435552

WHERE'S THE MATH? A STUDY OF COACH-TEACHER TALK DURING MODELING AND CO-TEACHING

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This study explores how two instructional coaches enacted modeling and co-teaching cycles with five elementary teachers during mathematics instruction. A content analysis of the coach-teacher talk from 11 planning meetings and 23 lessons reveals that the coaches and teachers seldom engaged in mathematical conversations. Instead, they primarily had low-depth discussions about curriculum, other instructional materials, and assessment. Implications for school districts with instructional coaching models are discussed.

Keywords: Teacher Education-Inservice/Professional Development

Introduction

Recent reforms promote an ambitious vision of high-quality math instruction for all students (Martin & Herrera 2007; NCTM, 2014; Common Core State Standards Initiative, 2011). As this vision represents a significant shift from how many teachers learned and taught math (Hiebert, 1999), enacting this type of instruction requires much support (Ball & Cohen, 1999). To address this challenge, many schools are enlisting the help of instructional coaches, as their intensive, one-on-one support can embody many facets of effective professional development (Desimone & Pak, 2017). Given the significant financial investment coaching requires (Knight, 2012) it is critical that we understand its enactment, including emergent challenges and supports that can make it successful.

Literature and Research Questions

To impact classroom teachers' knowledge and instruction, both *individual* and *group* settings are important places of learning (Campbell & Griffin, 2017; Cobb & Jackson, 2015). However, research on *one-on-one* activities has received relatively little attention (Cobb & Jackson, 2011; Gibbons & Cobb, 2017).

In their conceptual analysis, Gibbons and Cobb (2017) identified two potentially productive coaching activities for individual teachers: *modeling* and *co-teaching*. According to the authors, these activities are potentially productive as they meet the standards of high-quality professional development, and have demonstrated a positive impact on teachers.

Despite the popularity of these two strategies, there is a surprising lack of research describing how modeling and co-teaching can be used effectively with experienced, practicing teachers. Most studies examining modeling and co-teaching have focused on pre-service teachers (Clarke et al., 2014; Scantlebury et al., 2008) or mentoring programs aimed to support novice in-service teachers (Feiman-Nemser, 2001). While a few studies have explored modeling or co-teaching with practicing teachers, they have primarily focused on literacy coaches (Bean et al., 2010; Vanderburg & Stephens, 2010). A small handful of math education studies with school-based coaches have touched on the practices of modeling and co-teaching, although that was not the main focus of their work (Campbell & Griffin, 2017; Ellington et al., 2017).

To address this gap, this study looks in-depth at the modeling and co-teaching cycles enacted by two coaches and five teachers during math instruction. Specifically, the following research question is addressed: What is the nature of coach-teacher talk during modeling and co-teaching cycles?

Framework

Prior literature suggests that two elements are important for teacher learning during professional development: (1) high-depth interactions; and a (2) focus on mathematics content.

High-Depth Interactions

In her 2003 paper, Coburn called upon education researchers to rethink how they conceptualize scale when talking about education reform. According to the author, scale has traditionally been operationalized in a quantitative sense, with the goal of increasing the number of schools and teachers involved in a reform. Coburn (2003), however, argues that this is a superficial way to measure scale-up, and that careful attention must be given to the four dimensions of depth, sustainability, spread and shift.

The dimension of depth is applicable to this study as it has been conceptualized as one way to demonstrate the opportunities teachers have to learn when engaged in social interactions (Coburn & Russell, 2008). Specifically, Coburn and Russell (2008) distinguish between low- and high-depth interactions, with low-depth focusing on "surface structures and procedures, such as sharing materials, classroom organization, pacing, and how to use the curriculum" (p. 212) and high-depth addressing "underlying pedagogical principles of the approach, the nature of the mathematics, and how students learn" (p. 212). Thus, it can be argued that teachers have limited opportunity to engage in meaningful learning if they are primarily exposed to low-depth interactions. We apply Coburn's (2003) concept of low- and high-depth interactions to understand teachers' opportunities to engage in meaningful learning experiences during modeling and co-teaching cycles.

Focus on Mathematics Content

Current research on effective professional development reflects a consensus that there must be a content focus (Desimone, 2009; Desimone & Pak, 2017). Furthermore, current literature on high-quality mathematics instruction states that teachers must possess a deep understanding of the math they teach (Martin, 2007; NCTM, 2014). Thus, in addition to engaging in high-depth interactions, teachers must also be provided with opportunities to deepen their understanding of the math content they teach during professional development.

Methods

Participants and Context

The participants included two elementary instructional coaches, Meg and Claire. Meg was in her second year as a coach and had spent 21 years prior as a teacher, and Claire was in her third year as a coach and had previously been a teacher for 10 years. During this study Meg modeled instruction for Teachers Michelle and Mackenzie (grades 3 and 4), while Claire co-taught with Teachers Cathy, Caroline and Cecilia (grades 5, 1, and 4). All teachers were rather experienced (range of 9-23 years of teaching), and all teachers and coaches were white females.

Data Collection

Primary data collection methods included observations and resulting field notes (Bogdan & Biklen, 2011), as well as transcripts generated from audio recordings of 11 planning meetings and 23 modeled or co-taught lessons (see Table 1).

Table 1: Observation Data for All Coach-Teacher Pairs

	Modeling		Co-Teaching		
Coach	Meg		Claire		
Teacher	Michelle	Mackenzie	Caroline	Cecilia	Cathy
Grade Level	3rd	4th	1st	4th	5th
Lessons Observed	7	6	4	3	3
Planning Meetings	1	1	2	4	3

Data Analysis

All audio recordings were first transcribed using InqScribe software. The primary author then carefully read each lesson transcript and searched for places where the coach and teacher *directly engaged* with one another, typically through conversations and singular comments.

To look at coach-teacher engagement across the planning meetings and lessons, it was helpful to have a set of common codes. The authors primarily engaged in a process of open coding (Creswell, 2013) to develop two levels of emergent codes: (1) Level-1 Parent Codes; and (2) Level-2 Codes. Table 2 presents the Level-1 and -2 codes and how they are clustered.

Table 2: Level-1 and Level-2 Codes

Level-1 Codes	Management	Pedagogy	Content	Planning and Logistics	Contextual Factors	Other
Level-2 Codes	1) Classroom Composition and Attendance 2) Classroom Management	1) Grouping 2) Assessment 3) General Pedagogy	1) Curriculum, Activities, and Materials 2) Mathematics	1) Technology 2) General Plans for Coaching Cycle 3) Facilitator's Role 4) Time and Schedule	1) Relationship Building 2) External Requirements	1) External Individuals 2) Other

During the coding process, all codes were mutually exclusive and assigned at either the sentence (planning meeting data) or exchange (lesson data) level. After coding, we noticed a lack of talk directly focused on math, but there were many instances in which math-related terms were at least mentioned. To better capture all math-related talk (beyond the math-focused talk that was coded as Mathematics), we used a "Mathematics Indicator" to flag instances when the coaches and teachers used math words and phrases while discussing other topics, such as the curriculum, without attending to the mathematical meaning of those words and phrases.

The primary author coded all data and engaged in a reliability process with two independent coders trained in mathematics education research. During the reliability process, random subsets of data were independently coded by the both the primary author and an independent coder. Then, the two individuals met to reconcile differences. After talking through all areas of disagreement, overall, the coders agreed on over 97% of all assigned codes.

Last, to better understand the depth of the coach-teacher talk, the authors used Coburn and Russell's (2008) definitions of low-, medium- and high- depth and mapped their Level-2 Codes onto these three categories. Some of the Level-2 Codes closely mapped onto the definitions. For example, the exchanges coded as Curriculum, Activities and Materials mapped onto the low-depth category, as Coburn and Russell considered talk about "materials" and "how to use the curriculum" as low-depth. Other Level-2 Codes, such as General Pedagogy, did not cleanly fit with a single depth level, and such codes were divided into sub-codes and placed in the appropriate category. After the coding process, the authors tabulated frequencies and percentages for all sets of codes to better understand the substance of the coach-teacher talk. For the planning conversations, percentages were tabulated at the character-level using NVivo software, while for the lesson-level data, percentages were calculated at the exchange-level.

Results

We begin by illustrating what was typical regarding the coach-teacher talk, including the most prevalent topics with examples. Given space constraints, we report data for the practices of modeling and co-teaching without details for each coach-teacher pair. We then explore the prevalence and examples of the Mathematics code, as well as the Mathematics Indicator. Last, we discuss the depth of all coach-teacher talk.

Modeling

Here, we describe typical coach-teacher talk for Coach Meg and Teachers Michelle and Mackenzie during their modeling cycles, which were focused on implementing Calendar Math. We were invited to observe one planning meeting for each pair, as well as 13 total modeled lessons. Transcripts from the planning conversations and modeled lessons were coded with the set of 15 Level-2 codes noted above.

Overall, the coach and teachers most frequently discussed: (1) Curriculum, Activities and Materials (25%); (2) Assessment (14%); and (3) Classroom Management (12%) (see Table 3). Each topic is discussed in more detail below.

Table 3: Coach-Teacher Talk for Modeling and Co-Teaching Cycles

	Modeling			**Co-Teaching**		
	Planning Meetings	Modeled Lessons	Overall	Planning Meetings	Co-Taught Lessons	Overall
	n=2	n=13		n=9	n=10	
Content						
Curriculum, Activities and Materials	26%	24%	25%	16%	23%	20%
Mathematics	4%	3%	4%	2%	3%	3%
Total	30%	26%	28%	18%	26%	22%
Pedagogy						
Grouping	0%	0%	0%	10%	11%	11%
Assessment	21%	7%	14%	18%	15%	17%
General Pedagogy	7%	1%	4%	18%	8%	13%
Total	28%	7%	18%	48%	34%	41%
Management						
Classroom Composition and Attendance	1%	13%	7%	0%	1%	1%
Classroom Management	2%	22%	12%	1%	2%	2%
Total	3%	35%	19%	1%	4%	3%
Planning and Logistics						
Facilitator's Role	2%	4%	3%	2%	10%	6%
General Plans for Coaching Cycle	7%	0%	4%	2%	0%	1%
Technology	0%	0%	0%	0%	8%	4%
Time and Schedule	16%	6%	11%	10%	4%	7%
Total	25%	10%	18%	14%	21%	18%
Contextual Factors						
Relationship Building	4%	15%	10%	4%	10%	7%
External Requirements	1%	2%	2%	4%	0%	2%
Total	4%	17%	11%	8%	10%	9%
Other	13%	4%	9%	12%	4%	8%

Curriculum, Activities and Materials (CAM*).* When engaged in talk about CAM, across both cycles, the coach and teachers primarily discussed the materials needed to enact Calendar Math. This included conversations about the calendar, calendar pieces, number line, money, markers, wipes for the dry erase boards, popsicle sticks that would be used to elicit student participation, notebooks, making copies, and laminating materials. Typical exchanges coded as CAM included statements such as “I understand you have notebooks,” “Do you have a sticky chart paper?” and “Do we have a dry erase marker?”

Assessment. Assessment was the second most prevalent theme during both modeling cycles. While engaged in assessment talk, the coach and teachers primarily conversed about assessment logistics, such as selecting which pre-made assessment to give students, as well as when students would take assessments and how long it would take. The following planning meeting excerpt is typical, illustrating how Coach Meg and Teacher Mackenzie decided when to give their pre-assessments:

Meg: ‘Cuz the addition and the multiplication can be done at any time.
Mackenzie: These two can be?
Meg: Well, I mean, yeah. Unless you want to do a beginning baseline and an end baseline. It's completely up to you.
Mackenzie: I mean, we switch for almost, they come in at 11:15 and we don't switch back until 12:35.
Meg: That should be enough time to get ‘em all in. (11/15/16)

Classroom Management. Classroom Management, the third most prevalent theme, most commonly surfaced during coach-teacher talk from the modeled lessons, rather than the planning meetings. The coach and teachers either praised students (“Such hard workers, Meg!”), or discussed classroom incentives (“I can honestly say that once everybody got focused and centered in here, they earned it today.”) and challenging students (“I’m going to have to take her to the office if she won’t do what she needs to do.”).

In summary, during both modeling cycles, the coach and teachers primarily discussed immediate concerns related to planning and implementing the lessons, including materials, assessment logistics, and issues related to classroom management. We now examine the coach-teacher talk during the co-teaching cycles.

Co-Teaching

We were invited to observe 2-4 planning meetings and 3-4 co-taught lessons for each of the three coach-teacher pairs. Overall, the coach and teachers most frequently discussed: (1) Curriculum, Activities and Materials (20%); (2) Assessment (17%); and (3) General Pedagogy (13%). Each topic is explored below.

Curriculum, Activities and Materials. As CAM-related talk during the modeling cycles tended to focus on the materials, a different trend emerged in the co-teaching cycles as most of the CAM talk centered on the curriculum and activities. In particular, the coach and teachers often talked about issues related to: timing as they sequenced activities (“Do we want to try and have them do the first two problems…?”); the difficulty of the curriculum (“This is a hard lesson.”); what students in groups should work on (“This would be good to do in a small group—this chart down here.”); and understanding and/or navigating the curriculum (“Everyday Mathematics I think can be a little confusing...When I go through and look at this, I always look at the game to see if it’s something that I can do whole group.”).

Assessment. Similar to the modeling cycles, coach-teacher talk about assessment frequently surfaced during the co-teaching cycles. However, unlike the modeling cycles where the coach and teachers commonly discussed assessment logistics, during the co-teaching cycles it was more common for the coach and teachers to monitor student learning ("They did really well with expanded form the other day.") or use data to inform their instructional plans ("Why don't we use their independent work and kind of break them up into smaller groups?").

General Pedagogy. Last, unlike the modeling cycles where General Pedagogy rarely surfaced, during the co-teaching cycles, it emerged as one of the most frequently discussed topics. In particular, when the coach and teachers engaged in pedagogical conversations, they typically planned and/or created original materials that went beyond the district-provided curriculum ("We could just have an empty spot where they could write the number sentence or they can actually write a story problem."), discussed differentiation ("Ok, so the on-level group, we decided that it's just going to be higher level factors."), or engaged in more theoretical pedagogical talk ("So…you're doing more like student-led work out here during your centers?").

Hence, in some ways, the coach-teacher talk during co-teaching was quite different than the modeling coach-teacher talk. Specifically, modeling talk more often focused on materials, assessment logistics, and classroom management, while co-teaching talk more often focused on the curriculum and activities, monitoring student learning, utilizing data to inform their teaching, and engaging in pedagogical talk. Despite these differences, however, much of their discussions were rather similar, especially in terms of their limited math focus and depth.

Where's the mathematics?

The lack of math-focused conversation is striking across the modeling and co-teaching cycles. Specifically, only 2-4% of all planning meeting exchanges, and 3% of all modeled and co-taught lesson talk was coded as "mathematics" (see Table 3 above). In the rare instances in which the Mathematics code was used, it primarily captured procedurally-driven conversations about simple computation problems ("1 + 6 is not 5. Right? 1 + 6 would be 7."), definitions ("A multiple are all of the answers to a multiplication problem… The multiples of 4 are 4, 8, 12, 16 because 4 x 1 is 4, 4 x 2 is 8, 4 x 3 is 12."), and mathematical rules or procedures for textbook activities ("The box is not correct 'cuz 2 x 1 is not 3."). Conceptually driven mathematical conversations were, by far, more rare.

Mathematics Indicator. Although math-focused discussions were rare, the coaches and teachers often used math words and phrases while talking about other topics, such as assessment, without attending to the mathematical meaning of those words and phrases. Such instances were tagged with a Mathematics Indicator to reflect that a math term or phrase was listed, however it was not coded as Mathematics because it did not reflect a conversation about the content. For example, the exchange below received a primary code of Curriculum, Activities and Materials.

Claire: So, Wednesday, are we going back to 2.2?
Cathy: <u>Yes, which is exponents.</u>
Claire: <u>I really like that exponents lesson.</u> (3rd planning meeting, 9/19/16)

In addition to the CAM code, the underlined sentences were assigned a Mathematics Indicator, as the coach and teacher used the math word "exponents" while discussing the topic for the textbook chapter.

The Mathematics Indicator most frequently surfaced as the coaches and teachers named math words and phrases while discussing the Curriculum, Activities and Materials ("Lessons 2 and 3 are *exponents.");* Assessment ("This is probably a group that I feel like has a pretty good handle on *place value* compared to last year."); and General Pedagogy ("I don't know how to respond to these whole big *arrays* in your head.").

Depth

As explained above, we used Coburn's (2003) concept of *depth* to explore teachers' opportunities to learn. During both the planning meetings and the modeled/co-taught lessons, the coaches and teachers engaged in primarily low- (63-92%) and medium- (7-33%) depth interactions, while high-depth interactions seldom occurred (1-6%) (see Table 5). This suggests that the coach-teacher talk heavily emphasized surface-level structures and procedures, rather than focusing on topics such as how students learn mathematics, for example. Still, more medium-depth interactions occurred during co-teaching (27%) than modeling (10%), and low-depth interactions were more prevalent during modeling (87%) than co-teaching (70%).

Table 5: Depth for All Modeling and Co-Teaching Cycles

	Low	Medium	High
Modeling Cycles			
Planning Meetings	82%	12%	6%
Modeled Lessons	92%	7%	1%
Overall	87%	10%	4%
Co-Teaching Cycles			
Planning Meetings	63%	33%	5%
Co-Taught Lessons	76%	20%	2%
Overall	70%	27%	4%

Discussion

In this study of two coaches and five elementary teachers, we found that coach-teacher interactions during modeling more often focused on materials, assessment logistics, and classroom management, while interactions during co-teaching more often focused on the curriculum, monitoring student learning, utilizing data to inform teaching, and engaging in pedagogical talk. These differences may be specific to the two particular coaches involved, or may be due to differences in the focus of the cycles (e.g., with modeled lessons centered around Calendar Math, which was materials-intensive and outside the regular curriculum). However, some differences seem likely due to the fact that during modeling, the teachers generally assisted the coach (consistent with the greater focus on materials and classroom management), while during co-teaching, the teacher and coach implemented lessons together. These differences in foci may have prompted the greater depth during the co-teaching discussions.

Still, across the modeling and co-teaching cycles, conceptual discussions about mathematics content were extremely rare. This is disappointing, considering decades of research recommending that effective professional development should have a content focus (Desimone, 2009; Desimone & Pak, 2017). The lack of mathematical depth of discussions may be due, in part, to the fact that both coaches were generalists who lacked specialized mathematics training and were expected to provide professional development in all content areas.

Although our findings require replication before definitive conclusions can be drawn, the results suggest two potential guidelines for school districts. First, professional development for instructional coaches should help coaches gain a deep understanding of math content and pedagogical content knowledge across the developmental spectrum, which will help facilitate deeper conversations about math teaching. Second, more consideration should be given to implementing a content coaching model at elementary schools. In this model, although coaches would likely be shared between schools, they would be content experts who are better able support teachers in their area of expertise.

References

Ball, D. L., & Cohen, D. K. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In G. Sykes and L. Darling-Hammond (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 3 – 32). San Francisco: Jossey-Bass.

Bean, R. M., Draper, J. A., Hall, V., Vandermolen, J., & Zigmond, N. (2010). Coaches and coaching in reading first schools. *The Elementary School Journal, 111*(1), 87-114.

Bogdan, R. C., & Biklen, S. K. (2011). Qualitative research for education: An introduction to theory and methods. Boston: Allyn & Bacon.

Campbell, P. F., & Griffin, M. J. (2017). Reflections on the promise and complexity of mathematics coaching. *The Journal of Mathematical Behavior*, *46*, 163-176.

Clarke, A., Triggs, V., & Nielsen, W. (2014). Cooperating teacher participation in teacher education: A review of the literature. *Review of Educational Research, 84*(2), 163 – 202.

Cobb, P., & Jackson, K. (2011). Towards an empirically grounded theory of action for improving the auality of mathematics teaching at scale. *Mathematics Teacher Education and Development*, *13*(1), 6-33.

Cobb, P., & Jackson, K. (2015). Supporting teachers' use of research-based instructional sequences. *ZDM*, *47*(6), 1027-1038.

Coburn, C. E. (2001). Rethinking scale: Moving beyond numbers to deep and lasting change. *Educational Researcher, 32*(6), 3 – 12.

Coburn, C. E., & Russell, J. L. (2008). District policy and teachers' social networks. *Educational Evaluation and Policy Analysis, 30*, 203 – 235.

Common Core State Standards Initiative. (2011). *Common core state standards for mathematics.*

Creswell, J. W. (2013). *Research design: Qualitative, quantitative, and mixed methods approaches*. Sage publications.

Desimone, L. M. (2009). Improving impact studies of teachers' professional development: Toward better conceptualizations and measures. *Educational researcher*, *38*(3), 181-199.

Desimone, L., & Pak, K. (2017). Instructional coaching as high-quality professional development. *Theory Into Practice, 56*(1), 3 – 12.

Ellington, A., Whitenack, J., & Edwards, D. (2017). Effectively coaching middle school teachers: A case for teacher and student learning. *The Journal of Mathematical Behavior*, *46*, 177-195.

Feiman-Nemser, S. (2011). Lessons from an exemplary support teacher. *Journal of Teacher Education, 52*(1), 17 – 30.

Gibbons, L. K., & Cobb, P. (2017). Focusing on teacher learning opportunities to identify potentially productive coaching activities. *Journal of Teacher Education*, *68*(4), 411-425.

Hiebert, J. (1999). Relationships between research and the NCTM standards. *Journal for Research in Mathematics Education, 30*(1), 3 – 19.

Knight, D. S. (2012). Assessing the cost of instructional coaching. *Journal of Education Finance*, *38*(1), 52-80.

Martin, T. S., & Herrera, T. (2007). *Mathematics teaching today: Improving practice, improving student learning*. National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all.*

Scantlebury, K., Gallo-Fox, J., & Wassel, B. (2008). Co-teaching as a model for preservice secondary science teacher education. *Teaching and Teacher Education, 24*(4), 967 – 981.

Vanderburg, M., & Stephens, D. (2010). The impact of literacy coaches: What teachers value and how teachers change. *The Elementary School Journal*, *111*(1), 141-163.

MATHEMATICS EDUCATION STAKEHOLDERS PROFESSIONAL NETWORKS AND USE OF RESEARCH EVIDENCE

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In this paper, we present findings from an exploratory study of mathematics education stakeholders to understand their professional networks, and acquisition and use of research on mathematics teaching and learning. Evidence suggests that mathematics leaders are key to promoting organizational sensemaking and are more likely to acquire and use research on mathematics teaching and learning which has important implications for improvement efforts at scale.

Keywords: Design Experiments, Policy Matters, Standards, Professional Development

Introduction

Historic approaches to bridging the research-practice divide have often focused on improving the quality of research dissemination efforts to move evidence from research to use in practice. These approaches privileged researchers' perspectives, and though they achieved some success, the field lacks empirical understanding of what forms of research evidence are being drawn upon and used in practice (Finnigan, Daly, & Che, 2012). To improve the use of research in practice, scholars have identified a number of resources and characteristics of schools and districts, such as the influence of mid-level decision makers in organizations because they often "straddle policy and practice and are well poised to put research to work" (Tseng, 2012, p.5). Others challenge the dissemination model and argue for other ways to relate the work of researchers and practitioners, such as research-practice partnerships (RPPs). RPPs have recently gained traction as a promising approach for educational improvement (Coburn & Penuel, 2016). In RPPs, researchers and practitioners identify and commit to addressing a shared problem through long-term, mutualistic collaborations that include research, development, or evaluation (Coburn, Penuel, & Geil, 2013).

As part of an RPP with our state education agency focused on improving the process of implementing new mathematics standards, we conducted an exploratory study of the professional networks and research uses of mathematics teachers, mathematics leaders (e.g. school-based coaches and curriculum facilitators, district-based math leaders), and school-based administrators (e.g. principals, assistant principals). This paper reports results from a questionnaire that was developed to inform the design of professional learning opportunities for mathematics teachers and leaders that centralize research evidence on mathematics teaching and learning. In doing so, we aim and address the following research questions: (1) *From whom and for what purposes do mathematics teachers, mathematics leaders, and administrators have significant conversations about mathematics teaching and learning?; (2) From what sources and to what extent do mathematics teachers, mathematics leaders, and administrators look for and use research on mathematics teaching and learning?;* and (3) *What do mathematics teachers, mathematics leaders, and administrators identify as primary barriers to using research in their role?*

Background

Our partnership began after our state adopted new mathematics standards. State agency leaders were interested in using an improvement science approach in build and refine a process of implementing state academic standards. During the first year of the partnership, we negotiated and specified research evidence and its use in practice among mathematics teachers and leaders as the focus of our work together and co-designed a variety of professional learning opportunities and materials for mathematics teachers and leaders that embody research on student learning, instructional practice, and teacher learning in professional development (Wilson, McCulloch, Webb, Stephan, Mawhinney, & Curtis, 2017).

To inform the design of these efforts, we developed an exploratory questionnaire to understand the ways mathematics education stakeholders acquire and use evidence from mathematics education research in their practice. In what follows, we briefly summarize the literature on professional networks and research use, outline our theoretical perspective, and describe the development and administration of the exploratory questionnaire. We then highlight key findings and conclude with implications for others interested in promoting both organizational learning and the use of research in practice with attention to scale.

Professional Networks and Research Use

Scholars studying the ways research evidence informs practice and policy define research use as the act of drawing on and interacting with research evidence in the course of decision making (Coburn & Turner, 2011; Honig et al., 2017; Tseng, 2012). Investigations of practitioner and policy maker uses of research have identified several broad ways that evidence is used, for example, instrumental use which results in changes in practice (Nutley et al., 2007), conceptual use which results in changes in knowledge (Weiss et al., 1977), or imposed use in which practitioners are pressured to use research by agencies or policy makers (Weiss et al., 2005), among others. In addition, Honig and colleagues (2017) reported that though practitioners claim to use research in their work, it is often interpreted through existing schema to reinforce previous decisions or rejected ideas that conflict with prior understandings.

Much of the scholarship on research use has focused on local or statewide data, such as student achievement scores or local measures. However, little attention has been paid to the ways mathematics education stakeholders use research evidence on mathematics teaching, students' mathematical learning, and/or mathematics teacher learning to inform their decisions. For the purpose of this exploratory study, we wanted to focus on similarities and differences in the ways teachers and leaders acquire and use research evidence on students' mathematical learning, mathematics teaching practices, and mathematics teacher learning given their direct relation to the work of teachers and coaches in the classroom.

Studies on research use in practice often focus on singular roles (e.g. administrators, central office leaders), downplaying the social ecology and complex nature of research use processes within school and district initiatives. Relationships, organizational structures and contexts, and policy all complicate and influence the ways practitioners use research (Tseng, 2012). In addition, acquiring and using research evidence for improvement is a "multilevel phenomena" and occurs both within formal organizational structures and in informal social interactions (Daly et al., 2014). Scholars working in this area have called for more attention to developing an understanding of the ways research is used in practice within district systems between leaders and teachers (Honig & Venkateswaran, 2012) and how research supports implementation and improvement efforts (Daly & Finnigan, 2010). Some have found that small changes at different levels of the system can add up to larger organizational improvements (Coburn & Turner, 2012).

Research-practice partnerships are an increasingly popular approach to improve the use of research in practice (Coburn & Penuel, 2016). Proponents suggest partnerships support both sensemaking and use of research evidence in practice by creating opportunities for mutual engagement with research, translating research into tools that may serve as vehicles for learning and improvements to practice, and serving as models of the use of research (Fishman et al., 2013). However, research has shown that intermediary organizations are insufficient absent leaders who continue to learn about research use while also teaching others (Honig, 2017). Moreover, research has highlighted the importance of "opinion leaders" positions in social networks (Palinkas et al., 2011) as conduits for both the acquisition and use of research in practice (Tseng, 2012). Thus, for this exploratory survey, we were interested not only in the types of mathematics education research used, but also in the reasons mathematics teachers, mathematics leaders, and administrators interact with one another and the ways these interactions may relate to the acquisition and use of research.

Theoretical Perspectives

Our partnership uses design-based implementation research (DBIR) (Fishman et al., 2013) to organize our development, implementation, and research efforts. DBIR focuses not only on developing and refining tools and environments for learning but also on creating structures and supports necessary to scale and sustain them (Fishman et al., 2013). Throughout implementation, partners seek to improve the design of implementation efforts, generate theories of learning and implementation, and create supporting infrastructures to develop capacity and sustainability. Specifically, we use Lave and Wenger's (1991) ideas of participation, practice, and boundaries as a frame for designing for teacher and leader learning. We use organizational sensemaking (Weick, 1995) to broadly frame standards implementation as an organizational learning problem, and draw upon two sets of constructs related to the processes and kinds of resources that individuals or collectives within larger systems use when experiencing ambiguity or violated expectations during periods of systemic change.

Our design process is guided by a set of design and implementation principles derived from our commitments to supporting mathematics teaching and learning, theories of learning and implementation, the research on teacher learning and professional development, and the diverse expertise of our partners (Wilson et al., 2017). In this paper, we attend to our principals of utilizing research on mathematics teaching, connecting teachers and leaders, and designing coordinated tools and resources to better understand mathematics teachers, leaders, and administrators professional networks and the ways they acquire and use research evidence on mathematics teaching and learning.

Methods

Questionnaire Design

In the spring of 2017, we developed a questionnaire to inform ongoing efforts to co-design statewide initiatives related to standards implementation and the promotion of equitable learning opportunities for mathematics education stakeholders. We began by asking a set of demographic questions related to respondents' school district, role, years of experience, and grade-band. We then focused on three constructs identified by the partnership as important to the design of learning opportunities for mathematics teachers and leaders: professional networks, use of research in practice, and instructional vision. In this paper, we focus our efforts on statewide responses from mathematics *teachers*, mathematics *leaders* (e.g. school- and district-based mathematics coaches, curriculum facilitators), and school *administrators* (e.g. principals, assistant principals) related to professional networks and research use to inform our design.

To answer our first research question focused on professional networks, we asked respondents two questions: with whom and how often respondents had "significant exchanges about mathematics teaching and learning in the past year"; and what the primary reason was for the majority of these exchanges. To answer our second research question focused on research use, we asked respondents questions about their acquisition and use of research specific to mathematics education (e.g. research on students' mathematical thinking, mathematics teaching practices, mathematics teacher learning). These questions asked respondents to indicate how often they looked for research on topics related to mathematics teaching, the social sources of the research they encountered, and how likely they were to use research. To answer our third research question focused on barriers in using research, we asked respondents to select two main barriers to using research from eight possible choices.

Several of the questions were adapted from a validated and reliable survey developed by The National Center for Research in Policy and Practice (ncrpp.org). The survey's purpose was to characterize how leaders perceive, acquire, and use research to inform their decision-making. To reduce fatigue effect bias, we selected a subset of the survey questions most directly related to our research questions and implementation efforts and modified portions of these questions to reflect our focus. In addition, there were a set of questions focused on respondent demographics and another set of questions to directly inform future co-design and implementation efforts. Prior to administering the survey, we field tested the items with members of the partnership. We modified and clarified individual items as needed to ensure each measured the constructs of interest.

Data and Analysis

The questionnaire was distributed through listservs at the state agency to mathematics teachers, administrators, and mathematics leaders. The agency estimated the total number of educators on the listservs to be approximately 20,000. The questionnaire was open for 17 days during late May and June, 2016. Potential respondents received one message to invite participation upon its opening and three follow-up reminders throughout the open period.

Responses from those who completed 80% of the items and gave consent for research were considered complete and as data. Our quantitative analysis proceeded in two stages. First, we used descriptive statistics and graphical displays to explore responses to individual questions and then groups of questions. We then used responses to demographic items to organize a search for relationships among professional networks and research use. Because the questionnaire was developed with a primary goal of informing our design efforts, findings are descriptive.

Findings

In total, 1,605 teachers, 197 leaders, and 104 administrators (N=1,906) responses were collected, with a response rate estimated to be approximately 20%. While low, the responses represented 100% of the eight state education regions and 114 of the 115 school districts in the state with a mean and median of 17 and 11 responses per school district respectively. To better understand the representation of respondents across our state, a ratio of students per respondent was calculated by taking the ratio of the total number of students in a district per number of respondents in a district. These scores were then averaged across all districts in each of the eight regions across the state to obtain an average ratio per region. The mean and median of these region averages were 948 and 916 students per respondent respectively (std. dev.=167) – indicating a reasonably acceptable distribution of respondents per region across the state. In what follows, we share findings organized by our research questions.

From whom and for what purposes do mathematics teachers, mathematics leaders, and administrators have significant conversations about mathematics teaching and learning?

Respondents were asked to self-report how often they had significant conversations about mathematics teaching and learning in the last year by choosing *none*, *1-5 times*, *6-10 times*, or *greater than 10 times*. Results were translated to a scale of 0-3 (0 [*none*] – 3 [>*10*]) and means for each role are shown in Table 1. Results indicate that all roles self-report that they regularly engage in conversations about mathematics teaching and learning with teachers in schools. Mathematics leaders self-report that they more regularly engage with teachers and administrators about mathematics teaching and learning than do teachers or administrators, suggesting that leaders span boundaries within and across schools within districts.

Table 1: Mean scores for likelihood of having significant exchanges about mathematics

	Teachers	Math Leaders	Administrators
Teachers within schools	2.45	2.58	2.25
Teachers across district	1.27	1.89	1.05
Math Leaders	1.31	1.97	1.59
Administrators	1.55	2.03	1.77
Higher Ed Faculty	0.81	0.89	0.80

Taken that mathematics leaders self-reported the regularity of their conversations with others, as well as teachers and administrators self-reports of their conversations with math leaders, Table 2 highlights the predominant focus of math leaders' conversations. Results indicate that math leaders have frequent conversations about mathematics teaching and learning across many domains of mathematics education. With teachers, these conversations focus on planning for instruction, instructional practices to support student learning that align with mathematics standards, and resources that can be used in instruction. With one another, these conversations focus on professional development and other activities to support teacher learning that focus on mathematics standards and instructional practices. With administrators, these conversations focus on curriculum, assessments, and instructional practices. Taken together, the frequency and focus of these exchanges may indicate that mathematics leaders are centrally connected to district-wide implementation efforts related to mathematics teaching and learning.

Table 2: Percentage of respondents indicating the predominant focus of their exchanges.

	Math Leaders conversations with...were focused on...
Teachers in schools	Planning (35%), Teaching Practices (17%), Resources (15%)
Teachers across district	Standards (22%), Planning/PD (17%), Teaching Practices (16%)
Math Leaders	Teaching (20%), Standards (17%), PD (15%)
Administrators	Curriculum (21%), Assessment (18%), Teaching Practices (16%),

From what sources and to what extent do mathematics teachers, mathematics leaders, and administrators look for and use research on mathematics teaching and learning?

Drawing from our design and implementation principle focused on utilizing tools based on research on mathematics teaching and learning, we sought to explore stakeholders' frequency and use of research in practice. Respondents were asked questions to indicate how frequently they (1) looked for research in the past year and (2) previously used research in their role on a set of mathematics teaching and learning related topics. Results were translated to a scale of 0-3 (0 [*never*], 1 [*rarely*], 2 [*sometimes*], 3 [*often*]) and means across respondents for teachers (n=677), mathematics leaders (n=81), administrators (n=36) are shown in Table 3. For the survey, each respondent was given the set of questions related to professional networks and then randomly assigned questions related to either research use or instructional vision, thus the number of respondents for research use and instructional vision are roughly half of the total number of respondents.

Table 3: Frequency of prior use of research and looking for research in the past year

How frequently have stakeholders looked for & used research on math teaching and learning?	**Teachers**		**Math Leaders**		**Administrators**	
	Looked For	Prior Use	Looked For	Prior Use	Looked For	Prior Use
Students' mathematical thinking	1.7	2.0	2.0	2.5	1.8	2.3
Mathematics teaching practices	1.8	2.1	2.3	2.5	2.0	2.5
Math professional development	1.4	1.8	2.1	2.2	1.6	2.2
Resources for instruction	2.1	2.2	2.4	2.5	2.0	2.4
Assessment practices	1.8	2.0	2.1	2.3	1.7	2.4

Results indicate that all roles self-report that they look for and using research on mathematics teaching and learning. In addition, mathematics leaders self-report data suggest that they both look for and use research more often than teachers and administrators across each domain of research on mathematics teaching and learning. Moreover, data indicate that respondents are more likely to use research than they are to look for it, which suggests that they may acquire research from a variety of sources. Respondents were then asked to indicate how likely they were to acquire research from a list of social resources. Results were translated to a scale of 0-3 (0 [*never*], 1 [*rarely*], 2 [*sometimes*], 3 [*often*]) and means for each role are shown in Table 4.

Table 4: Mean scores likelihood of acquiring research from the following sources

	Teachers	**Math Leaders**	**Administrators**
Teachers within schools	2.1	1.7	2.3
Teachers across district	1.5	1.7	1.7
Math Leaders	1.4	1.9	2.0
Administrators	1.3	1.2	2.1
Higher Ed Faculty	0.6	1.2	0.4
Professional Associations	1.1	1.9	1.4
Research or Practice Journals	0.8	1.9	1.2

Results indicate that all roles self-report that they rarely to sometimes acquire research from the list of sources. For teachers, they reported that they were most likely to acquire research from other teachers. For administrators, they reported that they were most likely to acquire research from other administrators or mathematics leaders. For mathematics leaders, they reported that they were most likely to acquire research from other math leaders, professional associations, or journals. In addition, if one assumes that higher education faculty have access to or are producing research on mathematics teaching and learning, mathematics leaders reported that they were more likely to acquire research from faculty than teachers or administrators.

What do mathematics teachers, mathematics leaders, and administrators identify as the primary barriers to using research in their role?

Drawing from our design and implementation principles and our commitment to build capacity for sustainability at scale, we sought to explore what stakeholders' identified as the primary barriers to using research in their role. Respondents were asked to choose two main barriers from a list of hypothesized barriers we gathered through our conversations with educators. Results were tabulated across respondents for each role as shown in Table 5.

Table 5: Percentages of primary identified barriers to using research in practice

	Teachers	**Leaders**	**Administrators**
Time constraints	37%	32%	33%
Lack of PD that shares research	21%	22%	20%
Cost/Lack of access to journals	9%	17%	20%
Lack of relevance or ease of use	19%	11%	10%
Timeliness of research (e.g. out of date)	8%	8%	13%
Not valued in my education community	3%	7%	2%
Lack of high-quality evidence	5%	3%	3%

Results indicate that the primary barrier to using research in practice across all roles was time constraints. Secondly, results indicate that teachers, leaders, and administrators all reported a lack of access to professional development focused on sharing research findings as a barrier.

Implications and Discussion

The purpose of this exploratory study was to inform our ongoing efforts to co-design professional learning opportunities for mathematics teachers and leaders. Results of our analysis suggest that mathematics leaders are key aspects of school and district communication infrastructures related to mathematics teaching and learning. An examination of responses about their use of research provides some evidence that this communication shares research findings with others, on occasion. Moreover, leaders are more likely to seek out, promote, and use research on mathematics teaching and learning.

Results from this questionnaire will inform our upcoming efforts to co-design learning opportunities for leaders and teachers focused on mathematics teaching and learning that embody both research and attention to scale. For researchers and practitioners engaged in large scale improvement initiatives, results from our descriptive questionnaire mirror emerging findings about the influences of mid-level decision makers in decisions and the role of professional networks in introducing new ideas to schools and districts. We also note that self-reports of practice that are deemed favorable are regularly overestimated. While it is promising that leaders reported these practices, it is also noteworthy that teachers, leaders, and administrators were unlikely to have significant conversations about research on mathematics teaching and learning with higher education faculty. If we, as mathematics education researchers purport to both engage in and share these forms of research, it is important for us to consider these results.

As we consider the conference theme, "Looking Back, Looking Ahead", we see attention to mathematics leaders as one way to meet enduring challenges of shaping mathematics teaching.

Leveraging their professional networks and supporting them in learning to use research in practice may promote coherence and systemic change as they mitigate shifting policies, changes in curriculum, and expectations of industry. We see this work as supporting research-based mathematics teaching that leads to more equitable learning opportunities for students.

References

Battista, M. T., Fey, J. T., King, K. D., Larson, M., Reed, J., Smith, M. S., ... & Sutton, J. T. (2007). Connecting research and practice at NCTM. *Journal for Research in Mathematics Education*, *38*(2), 108-114.

Cai, J., Morris, A., Hwang, S., Hohensee, C., Robinson, V., and Hiebert, J. (2017). Improving the impact of educational research. *Journal for Research in Mathematics Education, 48*(1), 2-6.

Coburn, C. E., & Penuel, W. R. (2016). Research–practice partnerships in education: Outcomes, dynamics, and open questions. *Educational Researcher*, *45*(1), 48-54.

Coburn, C. E., Penuel, W. R., & Geil, K. E. (2013). Research-practice partnerships. *A Strategy for Leveraging Research for Educational Improvement in School Districts*. William T. Grant Foundation, New York, NY.

Coburn, C. E., & Turner, E. O. (2011). The practice of data use: An introduction. *American Journal of Education*, *118*(2), 99-111.

Daly, A. J., & Finnigan, K. S. (2010). A bridge between worlds: Understanding network structure to understand change strategy. *Journal of Educational Change*, *11*(2), 111-138.

Daly, A. J., Finnigan, K. S., Jordan, S., Moolenaar, N. M., & Che, J. (2014). Misalignment and perverse incentives: Examining the politics of district leaders as brokers in the use of research evidence. *Educational Policy*, *28*(2), 145-174.

Fishman, B. J., Penuel, W. R., Allen, A. R., Cheng, B. H., & Sabelli, N. (2013). Design-based implementation research: An emerging model for transforming the relationship of research and practice. *National Society for the Study of Education*, *112*(2), 136-156.

Finnigan, K., Daly, A., & Che, J. (2012). The acquisition and use of evidence district-wide. In *Annual Meeting of the American Educational Research Association, Vancouver, Canada*. Retrieved from http://www.wtgrantfoundation.org/resources/studying-the-use-of-research-evidence.

Honig, M. I., Venkateswaran, N., & McNeil, P. (2017). Research use as learning: The case of fundamental change in school district central offices. *American Educational Research Journal*, *54*(5), 938-971.

Honig, M. I., & Venkateswaran, N. (2012). School–central office relationships in evidence use: Understanding evidence use as a systems problem. *American Journal of Education*, *118*(2), 199-222.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge University Press.

Nutley, S. M., Walter, I., & Davies, H. T. (2007). *Using evidence: How research can inform public services*. Bristol: Policy Press.

Palinkas, L. A., Holloway, I. W., Rice, E., Fuentes, D., Wu, Q., & Chamberlain, P. (2011). Social networks and implementation of evidence-based practices in public youth-serving systems: a mixed-methods study. *Implementation Science*, *6*(1), 113.

Tseng, V. (2012). The use of research and evidence in policy and practice, *Social Policy Report*, *26*(2), 1-16.

Weick, K. E. (1995). *Sensemaking in organizations*. Thousand Oaks, CA: SAGE.

Weiss, C. H. (1977). Research for policy's sake: The enlightenment function of social research. *Policy analysis*, 531-545.

Weiss, C. H., Murphy-Graham, E., & Birkeland, S. (2005). An alternate route to policy influence: How evaluations affect D.A.R.E. *American Journal of Evaluation*, *26*(1), 12-30.

Wilson. P.H., McCulloch, A., Webb, J., Stephan, M., Mawhinney, K., & Curtis, J. (2017). Partnering for professional development at scale. *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1431-1434). Indianapolis, IN. Purdue University.

PREPARING SECONDARY MATHEMATICS AND SCIENCE TEACHER LEADERS IN RURAL DISTRICTS

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This study examines secondary mathematics and science teacher perceptions of teacher leadership during the first year of a professional development program focused on preparing teacher leaders in rural schools. It also begins to offer details as to what content-focused teacher leadership looks like and how teachers in rural schools enact teacher leadership. Data collection includes interview and survey analysis. Findings indicate four areas of growth for participants and project staff: participants began to expand their thinking and influence beyond the classroom, advocate more for students, develop a richer understanding of what content specific teacher leadership looks like, and gain a deeper understanding that teacher leadership in rural districts may be easier given the context of smaller settings but may also be more challenging in terms of teacher burnout.

Keywords: High School Education, Middle School Education, Teacher Knowledge, Teacher Education – Inservice/ Professional

Introduction

Teacher leadership has been in the literature for over 40 years and more recently, calls have been made to focus on content-focused teacher leadership (Wenner & Campbell, 2016). In 2014, the National Research Council (NRC) held a day and a half convocation to discuss the current and possible ways to better develop and utilize STEM teacher leadership. One result of the convocation was a publication that summarized convocation outcomes such as what STEM teacher leaders can do to effect policy, current models for empowering teacher voices, professional development for STEM teacher leaders, and research in the field of STEM teacher leadership (NRC, 2014). One finding from the report was the impact professional development "can make toward creating a robust corps of STEM teacher leaders" (NRC, 2014, p. 44). Mohan, Galosy, Miller, & Bintz (2017) completed a study to review and synthesize existing research on science and mathematics teacher leadership development programs. In their initial review of 89 research abstracts and 70 programs, their final study included 18 research articles and 15 programs. The review of these articles and programs, in addition to discussions with science and mathematics teacher leadership development leaders, resulted in four "focal areas" or recommendations for science and mathematics teacher leadership programs: programs and the teacher leadership development landscape, purposes of teacher leadership development programs, attributes of teacher leadership development programs, and research on teacher leadership development programs. Two key recommendations within these focal areas were the need to ensure teachers from underserved and underrepresented areas have leadership development opportunities and the importance of sharing findings of science and mathematics leadership development programs (Mohan et al., 2017).

Rural districts represent an underserved and underrepresented area where a focus on teacher leadership development is needed (Anderson, 2008). Rural areas also demonstrate a need for strong teacher professional development to address challenges such as fewer resources for instruction, lower teacher salaries, and less opportunities for professional development for teachers (Bush, 2005; Goodpaster, Adedokun, & Weaver, 2012; Showalter, Klein, Johnson, Hartman, 2017). Specifically, professional development in mathematics and science is needed as already existing teacher shortages in STEM areas are even greater and student performance in and access to higher level mathematics and science coursework is low in rural communities (Irvin, Byun, Smiley, & Hutchins, 2017; Showalter, Klein, Johnson, Hartman, 2017).

In response to this need for content-focused rural teacher leadership development, the Rural Secondary Science and Mathematics Teacher Leader Project (RSSMTL), in collaboration with 10 schools in 15 districts, was developed to prepare secondary mathematics and science teachers in rural districts. This study analyzes data from the first year of the program to determine how science and mathematics teacher perceptions of teacher leadership change through a teacher leader professional development program and what it looks like to become a mathematics or science teacher leader in a rural school.

RSSMTL focuses on Wenner & Campbell's (2016) definition of teacher leadership as "teachers who maintain K-12 classroom-based teaching responsibilities, while also taking on leadership responsibilities outside of the classroom" (p. 7). Given that change takes long time support, we are in the midst of a five-year grant funded program to develop strong secondary mathematics and science teachers in rural areas.

RSSMTL Conceptual Framework

Based on research literature in rural content specific teacher development and leadership as well as the needs articulated by our school partners, the below conceptual framework guided the development of the RSSMTL.

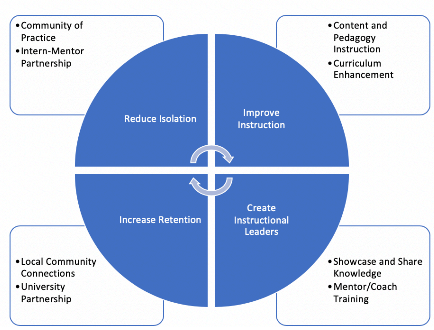

Figure 1. The Conceptual Framework for RSSMTL

Project Components

The RSSMTL conceptual framework four components are explained below and program elements that teachers complete to address each of the four components are described. Table 1 also presents the program components and elements.

Component 1: Reduce Isolation. Teachers in rural schools struggle with isolation given the sometimes smaller size of their schools. A science teacher, for example, may be the only person teaching physics at her school. Component 1 and the affiliated project elements are designed to reduce teacher isolation (Baird, Prather, Finson, & Oliver, 2006). The participating teachers and project team are creating a ***community of practice*** where we are able to learn together as teacher leaders in rural districts. As part of the project, teachers work with our preservice teachers who typically would not be placed in their schools due to the longer commutes but is now more possible with funding to supplement travel costs. Teachers speak to the value of having content partners with whom they can now talk about their teaching – both in terms of their student teachers (***intern-teacher relationships***) as well as fellow teachers in the program. For example, as the only science teacher teaching physics at her school, Brianna can now talk with her RSSMTL colleague who also is the only physics teacher at her rural school. As the RSSMTL participants increase their spheres of influence and connection as teacher leaders, they have been able to share those connections with teachers in their communities.

Component 2: Improve Instruction. Teachers in rural districts often lack access to high quality professional development or to content-specific specialists to aid in continuous improvement of their instruction (Hickey & Harris, 2005; Howley & Howley, 2005). If rural teachers do receive professional development, it is often not well aligned to their specific needs (Jimerson, 2004). Therefore, this component and aligned program elements are designed to improve teacher participant instruction. ***Content and pedagogical instruction*** emphasizing inquiry based teaching practices (Anderson, 2007; Silver, Kilpatrick, & Schlesinger, 1990) coupled with both Project Based Learning (Krajcik, Czerniak, & Berger, 2002) as well as Place Based Learning (Sobel, 2004) are foundational to the program. We meet for a two-week Summer Instructional Leadership Academy (ILA) each summer and spend time learning about these three pedagogical approaches. For example, in terms of ***curriculum enhancement***, each teacher has developed a Project Based Learning unit. Teachers implement and record at least four lessons during the academic year focused on these curricular and instructional changes. They also take three online content specific graduate courses to enhance their content knowledge. These courses are offered online so that teachers do not have to leave their communities to participate in the course yet are able to virtually connect with the instructor and their RSSMTL colleagues throughout the course. Teachers complete projects that are integrated into their daily work of teaching so they can immediately apply their learnings with their students. Teachers have been able to, in turn, as teacher leaders share the content and pedagogical content knowledge they are learning with teachers in their districts. Hence, the initial void of content specific curriculum specialists in some of our smaller, more rural districts that was part of the initial impetus for this work, is beginning to be filled by RSSMTL participants.

Component 3: Increase Retention. Teachers in rural districts tend to have higher rates of retention than in urban or suburban schools (Provasnik et al., 2007) though rural districts do report difficulties in recruiting and retaining high quality mathematics and science teachers (Friedrichsen, Chval, & Tuescher, 2007; Monk, 2007). Strong professional development, community involvement and local school support improves the retention of high quality teachers in rural districts (Goodpaster, Adedokun, & Weaver, 2012). Hence, Component 3 and the accompanying program elements are designed to increase retention among participants. As part of RSSMTL, teachers develop ***local community connections***. As part of the Project Based and Placed Based Learning, teachers connect with resources in their areas to both bring into their classrooms as well as take their students into the community. For example, Tammy connected with a local restaurant to have her geometry students re-design one of their take-out boxes. As part of the project, the restaurant management came to the school to view student presentations on their designs and then selected the best designs to visit their local restaurant. In addition, teachers plan local STEM community nights at their schools to engage parents and the local community in the work of their students. For example, several teachers focused their STEM nights on the eclipse that occurred on August 21, 2017. Furthermore, ***university partnerships*** continue to develop as student teaching interns and their supervisors connect with the teachers as they work in their classrooms. Teachers also co-teach methods courses alongside project team members. For example, as part of the content-focused methods course, teachers come to campus to share with student teachers as well as the methods courses travel to the teachers' schools to view lessons taught to their students.

Component 4. Create Instructional Leaders. Empowering teachers to develop their own instructional practices as well as supporting their ability to support the development of colleague's instructional practices offers teacher ways to become instructional leaders. It also addresses the lack of content-specific specialists in smaller rural districts by equipping classroom teachers with these skills (Hickey & Harris, 2005). Therefore, Component 4 and its accompanying program elements address this need. Teachers in the RSSMTL program ***showcase and share knowledge*** through multiple avenues. First, they complete the National Board Certification process for their specific content area. This process allows them to systematically and deeply reflect on their practice while simultaneously encouraging them to improve and strengthen that practice (Lustick & Sykes, 2006). As National Board Certified teachers, participants are more effective teachers (Cowen & Goldhaber, 2016) and better prepared to take on leadership roles (Sato, Hyler, & Monte-Sano, 2014). In addition, teachers write articles for professional journals as well as present their Project-Based Learning units at state and national conferences. Finally, at the close of the RSSMTL program, teachers will plan and host a professional development conference for their colleagues that focuses on lessons they have learned as part of the RSSMTL program. They also complete ***Mentor Teacher and Coach Training*** to enhance their work with student teachers and induction teachers as well as strengthen their skills to work alongside their colleagues to improve collective practices (Lotter, Yow, & Peters, 2014; Yow & Lotter, 2016).

Table 1. RSSMTL Program Components & Elements

Component	Focus	Aspects	Program Element
1	Reduce Isolation	Community of Practice	Relationships developed among teachers with RSSMTL colleagues and project staff
		Intern-Mentor Relationships	Teachers hosting student teacher interns
2	Improve Instruction	Content and Pedagogy Instruction	Inquiry Based teaching practices coupled with Project Based and Place Based
			Online pedagogical and content graduate coursework
		Curriculum Enhancement	Project Based Unit Development
3	Increase Retention	Local Community Connections	Project Based and Place Based Learning Units that involve local community resources
			STEM Community Nights
		University Partnerships	Teachers hosting student teaching interns
			Co-teach methods courses
4	Create Instructional Leaders	Showcase and Share Knowledge	National Board Certification
			Articles and Professional Presentations
			Professional Development Conference for Colleagues
		Mentor and Coach Training	Complete Mentor Teacher and Coach Training

Methods

Participants

Twenty teachers participate in the RSSMTL program. They were selected from a pool of applicants based on a principal reference, a written essay, transcript and licensure test review, and an interview. All 20 teachers have master's degrees and include ten mathematics teachers (six middle school and four high school) and ten science teachers (four middle school and six high school). Twelve are Caucasian and eight are African-American. Three are male and seventeen are female.

Data Collection

Data collection included a pre-and post-interview and a pre-and post-survey.

Interview. Prior to beginning the program, teachers were interviewed using a semi-structured interview protocol. The interviews lasted between 45 to 60 minutes and were audio recorded. The interview protocol asked about their beliefs about instruction and teacher leadership. In terms of instructional beliefs, teachers were asked questions such as how they thought students learned best, how they structured their lessons, and how they assessed student learning. In terms of teacher leadership beliefs teachers were asked, for example, how they defined teacher leadership and how they saw themselves or others as teacher leaders. A year later, the teachers were asked the same teacher leader questions with a sharper focus on being a mathematics or science teacher leader in a rural context.

Survey. Teachers also completed a teacher leadership survey (Triska, 2007). This survey contained 13 Likert items with a 1 (Never) to 4 (Frequently) scale and 11 items with a 1 (Disagree) to 4 (Agree) scale. Sample questions from the Never to Frequently section included how often they "Tried a strategy in your classroom that you had never tried before?" or "Voiced your personal thoughts about teaching or learning with other teachers?" Sample questions from the Disagree to Agree section included "You modeled reflection leading to improvement of practices in your classroom, which may have impacted other teachers" and "You played an important role in building the professional community here at school." A year later, teachers were administered this same survey though they were asked to do so both a written and verbal format. Teachers completed the survey by hand during the post-interview while also being asked to think aloud as they completed the survey to offer examples or additional context to their Likert scale responses.

Analysis

Data analysis included using SPSS to determine the descriptive statistics on the Triska (2007) teacher leadership survey. The teachers rated themselves highly on the pre-survey so there was no significant statistical change compared to the post-survey. Therefore, the main data analysis for this study was conducted through analysis of the teacher leadership survey think aloud alongside the pre- and post-interview responses. All interviews were audio recorded, transcribed, and inputted into Nvivo 11. Interviews were coded using a constant comparative method (Boglan & Biklen, 1989). The first two authors along with two graduate research assistants initially coded all interviews separately and a list of common consensus codes was determined through group discussion. Then, all researchers recoded the interview transcripts using the revised codes, met again to discuss any variations in the coding, and decided upon themes that represented participant perceptions and enactment of teacher leadership.

Results

Results indicated four findings with regard to the teacher perceptions and growth as teacher leaders. First, teachers are developing as teacher leaders whose focus is expanding to beyond their classrooms. Second, teachers are continuing, and more deliberately, serving as advocates for their students. Third, a more collaborative and comprehensive understanding of what content specific teacher leadership looks like is developing. Fourth, teacher leadership roles in rural districts may be more natural to obtain given the context of a sometimes smaller familiar setting, but may also be more challenging as these roles lead to other tensions including feeling overworked and stressed.

Finding 1: Expanding Focus to Beyond the Classroom

As we began the program, teacher conceptions about teacher leadership were more classroom-focused. Now, they are beginning to think about teacher leadership as also including their influences beyond the classroom. Jenny, for example, initially spoke about a teacher leader as "being receptive to her students – taking their needs and helping them grow." Now, teachers reflect on their current conceptions of teacher leadership as not only helping students grow, but also "being a role model to and sharing with my peers" as well as "presenting at conferences." They are beginning to feel responsibility to serve their students, colleagues, and districts.

Finding 2: Continuing to Advocate for Students

Teachers have always been champions for their students. Through the program, however, their voice as advocates for students has grown stronger. Their understanding that such a voice is part of their role as a teacher leader has also grown – part of being a teacher leader is having a "student-centered mindset" (Hunzicker, 2013). Kyana reflected, "but same is not equal in every school. Our children, being rural, need more outside experiences." She continued to speak specifically to advocating for the best teaching practices: "Math, we have to really make sure that we push for the best way for our kids to learn."

Finding 3: Understanding of Content Teacher Leadership

The teachers and project team are gaining a better understanding of what content-specific teacher leadership means. In mathematics, the teachers speak to the need to remain current in their knowledge of best practices for teaching mathematics. They also speak to the need to be involved with vertical teaming where they collaborate with teachers across grade bands to better understand, for example, what their algebra students learned in their previous pre-algebra course and what content they need to have a strong understanding as they progress into geometry. Science teachers speak to the need to "write grants" as teacher leaders to acquire the materials they need to complete engaging and safe laboratory activities. Mike reflected, "It takes much more to prepare for science." Particular foci and needs are specific to mathematics and science teacher leadership.

Finding 4: Recognizing Rural Teacher Leadership May Be More Natural But Challenging

Teacher leadership roles, both formal and informal, often times come more naturally to teachers in rural settings. For some teachers, they grew up in the community and are well respected so are often asked to take on leadership roles. For other teachers, their rural school is also a small school, so they are asked more often to serve. For example, Jessica noted "wearing lots of hats" as one of two mathematics teachers in her high school. For example, her school needed last-minute prom preparation help so she took her geometry class to the gym. She integrated their polygon lesson while arranging tables: "I turn our prom experience into a math experience." Tawanda noted, "I think it's easier to be [a teacher leader in a rural area] because you don't have to deal with so many. You're on a first name basis. They know me at the district office. They know who we are. They know what we do." However, with added roles come additional responsibility, time commitments and stress. Therefore, further reflection on how best to balance the natural yet challenging dichotomy of these roles needs exploring.

Discussion

In this initial year of the program, RSSMTL participant perceptions of teacher leadership as well as a developing picture of what it looks like to be a mathematics or science teachers leader in a rural school began to emerge. Through the program's focus on four areas (reducing isolation, improving instruction, increasing retention, and creating instructional leaders), four findings surfaced. First, teachers began to expand their focus to beyond the classroom. They began to think more deliberately and felt more empowered to take their expertise outside of their classroom walls to begin to impact and influence their colleagues and larger professional community (Beachum & Dentith, 2004). Second, though many had already served as advocates for their students, they learned of new ways and found an increased sense of agency to advocate for their students. In some cases, they learned of content-based extracurricular activities to offer to their students whose rural locale may have prevented their previous participation in such activities. They also began developing a new knowledge base about strong instructional practices and content expertise that enhanced their teaching and began to advocate for their subjects and these teaching practices (Hunzicker, 2013). Third, the collective knowledge of content-focused teacher leadership continues to evolve. Findings indicated that there are specific content area aspects that pertain to mathematics or science teacher leaders. For example, the concept of vertical teaming and planning within the mathematics community is a fertile ground for continued research in how best to prepare and position mathematics teacher leaders to serve in shepherding such efforts. Likewise, materials and supplies specific to successful lab and field experiences in the sciences lends itself to grant writing among science teacher leaders – and similar manipulative and resource materials in the mathematics classroom may also prove another content-specific area of mathematics teacher leadership. This finding adds to the complex nature of teacher leadership as it pertains to specific content areas (Wenner & Campbell, 2017). Finally, teacher leadership within rural schools can be more natural but at the same time challenging. The familiarity of being "known" in often smaller rural schools often makes it easier for teachers to assume teacher leader roles. With fewer administrators, there are often more opportunities for teachers to become teacher leaders. These same opportunities, however, sometimes lead to overburdening burnout (Little & Bartlett, 2002; Lieberman & Miller, 2004) so additional research into sustainable teacher leadership is warranted.

The complexity of content-based teacher leadership continues to evolve so additional research on how mathematics and science teachers perceive and enact teacher leadership in rural contexts is needed. The RSSMTL program adds to the limited literature around this field and as the program progresses and additional data is collected and analyzed, the authors hope to continue to add to this research literature so all students and teachers have access to high quality learning and leading.

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References

Anderson, K. D. (2008). Transformational teacher leadership in rural schools. *Rural Educator, 29*(3), 8-17.

Anderson, R. D. (2007). Inquiry as an organizing theme for science education. In S. K. Abell & N. G. Lederman (Eds.), *Handbook of research on science education* (pp. 807 – 830). Mahwah, NJ: Erlbaum.

Baird, W., Prather, J., Finson, K., & Oliver, S. (2006). Comparison of perceptions among rural versus nonrural secondary science teachers: A multistate survey. Science Education, 78(6), 555-576.

Beachum, F., & Dentith, A. M. (2004). Teacher leaders creating cultures of school renewal and transformation. *Educational Forum, 68*(3), 276-286.

Bush, W. S. (2005). Improving research on mathematics learning and teaching in rural contexts. *Journal of Research in Rural Education*, *20*(8), 1-11.

Cowan, J., & Goldhaber, D. (2016). National Board certification and teacher effectiveness: evidence from Washington state. *Journal of Research on Educational Effectiveness*, *9*(3), 233-258.

Friedrichsen, P., Chval, K. B., & Teuscher, D. (2007). Strategies and sources of support for beginning teachers of science and mathematics. *School Science and Mathematics, 107*(5), 169-181.

Goodpaster, K. P., Adedokun, O. A., & Weaver, G. C. (2012). Teachers' perceptions of rural STEM teaching: Implications for rural teacher retention. *Rural Educator*, *33*(3), 9-22.

Hickey, W. D., & Harris, S. (2005). Improved professional development through teacher leadership. *The Rural Educator, 26*(2), 12-16.

Howey, K. R. (1988). Why teacher leadership?. *Journal of Teacher Education*, *39*(1), 28-31.

Howley, A., & Howley, C. B. (2005a). High-quality teaching: Providing for rural teachers' professional development. *The Rural Educator, 26*(2), 1-5.

Hunzicker, J. (2013). Attitude has a lot to do with it: Dispositions of emerging teacher leadership. *Teacher Development, 17*(4), 538-561.

Irvin, M., Byun, S. Y., Smiley, W. S., & Hutchins, B. C. (2017). Relation of opportunity to learn advanced math to the educational attainment of rural youth. *American Journal of Education*, *123*(3), 475-510.

Jimerson, L. (2004). Teachers and teaching conditions in rural New Mexico. Retrieved from The Rural School and Community Trust website: http://www.ruraledu.org/user_uploads/file/teachers_newmexico.pdf

Krajcik, J., Czerniak, C., & Berger, C. (2002). *Teaching science in elementary and middle school classrooms: A project-based approach*. Boston: McGraw-Hill.

Little, W., & Bartlett, L. (2002).Career and commitment in the context of comprehensive school reform. *Theory and Practice, 8*(3): 345-54.

Lieberman, A. & Miller, L. (2004). *Teacher leadership*. Wiley & Sons: San Francisco: CA.

Lotter, C., Yow, J. A., & Peters, T. T. (2014). Building a community of practice around inquiry instruction through a professional development program. *International Journal of Science and Mathematics Education*, *12*(1), 1-23.

Loucks-Horsley, S., Hewson, P. W., Love, N., & Stiles, K. E. (2003). *Designing professional development for teachers of science and mathematics*. Thousand Oaks: Corwin Press.

Lustick, D., & Sykes, G. (2006). National Board Certification as professional development: What are teachers learning? *Education Policy Analysis Archives, 14*(5). http://dx.doi.org/10.14507/epaa.v14n5.2006

Mohan, L., Galosy, J., Miller, B., & Bintz, J. (2017). *A synthesis of math/science teacher leadership development programs: Consensus findings and recommendations* (Research Report No. 2017-02). Colorado Springs, CO: BSCS.

Monk, D. H. (2007). Recruiting and retaining high-quality teachers in rural areas. *The Future of Children, 17,* 155–174. doi:10.1353/foc.2007.0009

National Research Council. (2014). *Exploring Opportunities for STEM Teacher Leadership: Summary of a Convocation.* S. Olson and J. Labov, Rapporteurs. Planning Committee on Exploring Opportunities for STEM Teacher Leadership: Summary of a Convocation, Teacher Advisory Council, Division of Behavioral and Social Sciences and Education. Washington, DC: The National Academies Press.

Provasnik, S., KwealRamani, A., Coleman, M. M., Gilbertson, L., Herring, W., & Xie, Q. (2007). *Status of education in rural America* (NCES 2007-040). Washington, DC: National Center for Education Statistics, Institute of Education Sciences, U.S. Department of Education.

Rowley, J. (1988). The teacher as leader and teacher educator. *Journal of Teacher Education, 39*(3), 13-16.

Sato, M., Hyler, M. E., & Monte-Sano, C. B. (2014). Learning to lead with purpose: National Board Certification and teacher leadership development. *International Journal of Teacher Leadership*, *5*(1), n1.

Showalter, D., Johnson, J., Klein, R., & Hartman, S. L. (2017). *Why rural matters 2015-2016: Understanding the changing landscape*. Rural School and Community Trust.

Silver, E.A., Kilpatrick, J. & Schlesinger, B. (1990). *Thinking through mathematics: Fostering inquiry and communication in mathematics classrooms.* New York: The College Board

Sobel, D. (2004). *Place-based education: Connecting classrooms & communities* (3rd ed.). Great Barrington, MA: The Orion Society.

Strange, M., Johnson, J., Showalter, D., & Klein, R. (2012). *Why rural matters 2011-12. A report of the rural school and community trust policy*. Retrieved from The Rural School and Community Trust website: http://files.ruraledu.org/wrm2011-12/WRM2011-12.pdf

Triska, J.W. (2007). *Measuring teacher leadership* (Unpublished master's thesis). Humboldt State University: Arcata, CA.

Wenner, J. A., & Campbell, T. (2017). The theoretical and empirical basis of teacher leadership: A review of the literature. *Review of Educational Research, 87(*1), 134-171.

Yow, J. A., & Lotter, C. (2016). Teacher learning in a mathematics and science inquiry professional development program: first steps in emergent teacher leadership. *Professional Development in Education*, *42*(2), 325-351.

SITUATING DEFICIT DISCOURSE IN THE CONTEXT OF SOCIAL INTERACTION

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The persistence of deficit discourse has been one of the enduring challenges in mathematics education. This paper aims to contribute to the discussion by situating deficit discourse in the context of social interaction. Drawing on conversation analysis and discursive psychology, I offer an analysis of a conversation that happened in an equity-oriented professional development session. The findings show that deficit discourse was mobilized to do a particular context-specific action, and the action was achieved with multiple rhetorical devices. Thus, successfully countering deficit discourse involves micropolitics and rhetorical resources and skills.

Keywords: Equity and Diversity; Professional Development; Discourse Analysis

As Suh, Theakston-Musselman, Herbel-Eisenmann, and Steele (2013) noted, one of the enduring challenges in mathematics education is the persistent deficit discourse often targeted at marginalized students and families. Its ideological function is to blame the victims of social injustice and sustain the status quo (Valencia, 2010). Following Valencia, I define *deficit discourse* as spoken or written texts that legitimize the unjust social order based on the distorted deficit characteristics of the oppressed.

This paper aims to contribute to the discussion with an effort to situate deficit discourse in talk-in-interaction among mathematics teachers (MT) and mathematics teacher educators (MTE). By doing so, I argue that deficit discourses are mobilized as discursive resources for a participant to pursue a context-specific action; hence, I suggest it is important not only to identify what deficit discourses are but also to examine the practices of performing deficit discourses in a particular context of social interaction.

Linguistic Turn in Examining Deficit Discourse

Deficit discourse is a delicate and complex issue. As Tiezzi and Cross (1997) warned, focusing solely on an individual MT's belief as the root of deficit discourse may lead to MTE's own deficit perspective toward the MT. Deficit discourse is *a social* phenomenon which requires a view from a systemic perspective as well as an examination of the individuals. To do so, Parks (2010) and Suh and colleagues (2013) took a linguistic turn in which they treated language as performance or constructive of the social world. Language, in their views, is the object of study for its own sake to understand the social world rather than a medium to see a person's mind.

Their analyses revealed the discourses that are pervasive in mathematics education such as "the metaphor of a narrow path" (Parks, 2010), "the individual maturation storyline," and "the institutional tracking storyline" (Suh et al., 2013). These types of discursive resources, the authors argued, constrain MTs' ways of conceptualizing students' strength and assets and limit the MTs' own agency to utilize the diverse set of strengths in their classrooms. These discursive patterns often lead to a deficit-oriented statement such as "My students don't have …." In the individual maturation storyline, for instance, brain development is an important part, and the students are positioned as individuals lacking maturation to successfully engage in a particular learning task. Although Parks (2010) and Suh and colleagues (2013) revealed discursive limitations that are often overlooked by researchers, a few questions still remain in the linguistic turn: Why is this deficit discourse deployed here and now? How is it accomplished? To answer these questions, I examine how discursive resources are effectively mobilized in moment-to-moment interactions by the participants. By drawing on conversation analysis and discursive psychology as analytic tools, I examine a non-representative piece of data from a broader research project focused on equity-oriented professional development (PD).

Analytic Framework: Action Sequence and Rhetorical Redescription

Briefly speaking, conversation analysis (CA) concerns how people perform conversations in ordinary settings (e.g., dinner time, phone calls). CA grew out of ethnomethodology and speech act theory and has contributed to multiple fields by revealing the hidden practices in our daily socialization (e.g., turn-taking, sequencing action, repairing). In this paper, I focus on how actions are sequenced in social interactions (Schegloff, 2007), in particular, a storytelling sequence (Jefferson, 1978). Following the ethnomethodology tradition, CA validates claims based on participants' observable actions in talk-in-interaction, what the participants recognize, and how the participants treat those actions.

To examine the rhetorical effectiveness of the talk-in-interaction, I also draw on discursive psychology (DP). Influenced by ethnomethodology, social constructionism, and post-structuralism, DP approaches social psychology from a discursive perspective. I draw on Potter's (1996) work on the rhetorical aspect of discourses situated in social interactions. Thus, I ask how rhetorical resources are mobilized for participants to construct a discourse and how the discourse builds a version of fact.

Examining Deficit Discourse in the Context of Social Interaction

The conversation below happened in the early stage of the second round of an action research cycle, which focuses on incorporating students' active involvement and agency to guide teachers' action research projects. Prior to the excerpt below, one MT, named Renee, addressed her concern about having her first graders explore the suspension practice in the school using mathematics, and a tension emerged between Renee and the MTEs. On one hand, Renee mentioned that her first graders were not mature enough to think beyond themselves at their stage of development. On the other hand, the MTEs tried to persuade her to remain open-minded about her students' capacity and suggested her to pursue the idea of fairness instead of the institutional practice of suspension, and then the following conversation happened:

Excerpt: The Corpus Callosum

MTs: Carol, Danisha, Ramona, Renee, and Kathy; MTE: Susan (in pseudonyms)
Conventions: pause in 1/10 sec, (.*x*); overlapping talk, [ ]; vowel elongation, : ; emphasis, _ ; unrecoverable speech, ( ); rise and fall in intonation, ↑ ↓; low volume, ° °; comment, (( ))

```
01  Sus:   okay so there's suggestion of direction to go.
02         Kathy, you [have] something to add?=
03  Kat:              [Well]

04  Kat:   =I just had a thou:ght. you guys were at (auditorium)?
05                (0.4)
06  Kat:   [you know wel]coming back,
07  Dan:   [YE↑:::S]
08  Ren:   (                ) I wasn't th[ere::.]        Uh Heh Heh Heh heh
09  Kat:                                                   [↑I ↓kno:w.]
10  Kat:   but there was this wonderful speake:r and he talked about
11                the bra:::in.
12                (.)
13  Kat:   [(          )]  [                ]  [and don't   ] ↑QUOTE ME on this
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14  Car:  [he was good.]                          [he was good.]
15  Ram:                 [That's right.]
16  Kat:  but the two parts of the brai::n, there is that thing in the middle?
17               (0.3)
18  Kat:  anyone remembers what is that called?= ((puts two fists together))
19  Car:  =the [(0.2)] corpus=
20  Ram:    [the]
21  Sus:  =the corpus [callosum]
22  Kat:              [corpus] calla- Yeah. (.) ↑that is not even developed
23               about age ten, (sir/so). (0.5) so how do we expect our children to
 24              (0.2) know right and wrong and to
25               [problem solve, do this multiplication,] and division
26  Ram:  [we would have to teach them and have to help them develop it.]
27  Kat:  when they don't even (0.2) truly have the brain [capa]city.
28  Car:
 [right]
29               (0.2)
30  Kat:  so I am just wondering little six year olds.
31               (0.6)
32  Car:  °they are not there.°
33  Kat:  right. I mean they give you what they in↑terpret it is::.
34               (0.3)
35  Ren:  right it [um what I am thinking it is-] is (0.2) that=
36  Kat:                 [right and wrong but]
37  Ram:  =that's true. ((nods slowly))
```

The conversation above contains Kathy's storytelling sequence (line 10-23) about the brain development in relation to children's limited mental capacity, which Suh et al. (2013) named as "the individual maturation storyline." To further the discussion, I ask the following questions: how is the individual maturation storyline situated in this particular action sequence? What are the discursive resources upon which Kathy draws?

Based on where the storytelling is placed in the conversation, the storytelling is formulated as a dispreferred response (i.e., a refusal) to the suggestion of pursuing the concept of fairness. When a response to the suggestion was made relevant (line 1), Kathy opened a storytelling pre-sequence (line 4). In line 23, the refusal was clearly articulated when Kathy stated, "so how do we expect our children to know… ." Kathy's dispreferred response was also recognized by Renee, and after this excerpt, Renee further expands Kathy's refusal by offering her own hypothetical account. Note that the deficit image of young children was invoked not for its own sake but rather as an account to justify Kathy's refusal to the suggestion. In other words, the deficit discourse was mobilized by Kathy to do a context-specific action; that is, in this case, to refuse the MTE's suggestion.

A closer look at the sequence shows how pre-sequences were placed to elicit the solidarity among the teachers against the MTE's suggestion. In line 4, Kathy asked, "you guys were at (auditorium)?" The question was targeted to "you guys", the teachers; thus, it framed the story as 'our shared' story. Moreover, it assigned higher epistemic status on the teachers as co-witnesses since none of the MTEs witnessed the scene. The move toward the solidarity among *all* MTs became more apparent when Renee responded with "I wasn't there::." (line 8). Kathy gave her sympathetic response, "↑I ↓kno:w." (line 9), and it was followed by Renee's loud laughs indicating her affiliation to Kathy.

The sequence contains multiple rhetorical resources such as vivid detailed descriptions that can be used to build up the facticity of an account. For instance, Kathy aided her description of the brain with an embodied gesture when she put her two fists together to model the shape of the brain. Note that she also invoked the image of “little six year olds” (line 30) instead of other possible alternatives (e.g., first graders, children). This depicted the students as ‘underdeveloped’ children. Also, the sharp numerical comparison between the age of 10 (when children reach full development) and age of 6 (first graders) may have enhanced the rhetorical effect.

While mobilizing the rhetorical resources, Kathy also effectively established her neutrality to the description. Kathy indicated that her story was told by “this wonderful speaker”, a third-party male person. When she said, “don’t ↑QUOTE ME on this” (line 13), she reaffirmed her footing as an ‘unbiased messenger.’ To avoid accountability, she also expressed uncertainty when she offered her own idea by saying, “So I am just wondering …” (line 30).

The rhetorical account formulated by Kathy was, indeed, powerful. In contrast to Kathy’s deficit-oriented description, another teacher in the room, Ramona, initially interpreted the same event quite differently. She mentioned, “We would have to teach them and have to help them develop it.” (line 26). That is, the speaker advocated for the need of better support from the teachers to promote brain development. Later in line 37, however, Ramona conceded to Kathy’s argument by stating “That’s true.” and nodding slowly.

The above analysis offers a way to see how a particular deficit discourse is situated in the context of social interaction. I illustrated how a deficit discourse was mobilized to do a refusal, and it involved the delicate handling of social relationships and managing multiple rhetorical devices. To effectively challenge and reframe deficit discourse, then, we need to understand deficit discourse *within the context*. The task will involve disentangling the micropolitics in social interactions, and rhetorical resources and skills will be needed to construct and reinforce the asset-based reality in which diverse strengths of students are valued.

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References

Jefferson, G. (1978). Sequential aspects of storytelling in conversation. *Studies in the organization of conversational interaction* (pp. 219-248). New York, NY: Academic Press, Inc.

Parks, A. N. (2010). Metaphors of hierarchy in mathematics education discourse: the narrow path. *Journal of Curriculum Studies*, 42(1), 79-97.

Potter, J. (1996). *Representing reality: Discourse, rhetoric and social construction.* London: Sage.

Schegloff, E. A. (2007). *Sequence organization in interaction: A primer in conversation analysis* (Vol. 1). Cambridge, England: Cambridge University Press.

Suh, H., Theakston-Musselman, A., Herbel-Eisenmann, B., & Steele, M. D. (2013). *Teacher positioning and agency to act: Talking about “low-level” students.* Paper presented at the North American Chapter of the International Group for the Psychology of Mathematics Education, Chicago, IL.

Tiezzi, L. J., & Cross, B. E. (1997). Utilizing research on prospective teachers' beliefs to inform urban field experiences. *The Urban Review, 29*(2), 113-125.

Valencia, R. R. (2010). *Dismantling contemporary deficit thinking: Educational thought and practice.* New York, NY: Routledge.

EXPLORING TEACHERS' DECISIONS IN UNIT DESIGN AND IMPLEMENTATION

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This study investigates elementary mathematics teachers' decisions in designing and implementing a fraction unit aligned with reform-oriented learning and teaching practices and the factors that influence their decisions. Four 4th grade teachers from different schools designed a fraction unit and implemented it in their classrooms. The data sources include video-recordings of unit design sessions, classroom unit implementations and unit-debriefing sessions; pre and post semi-structured interviews including beliefs questions; and a Likert-scale survey on beliefs in addition to a fraction survey with items to determine teachers' levels of mathematical knowledge for teaching (MKT). The findings suggest that one of the teacher's relatively high level of MKT and her mathematical beliefs compatible with reform-oriented ideas might have affected her decisions and caused low fidelity in implementing the designed unit.

Keywords: Teacher Education-Inservice/Professional, Curriculum, Mathematical Knowledge for Teaching, Teacher Beliefs.

The agenda for reform efforts in K-6 mathematics education have aimed at increasing student math success by focusing on learning with understanding. In efforts to foster critical thinking and reasoning, problem solving and individual knowledge construction in classrooms, reform advocates have attempted to develop reform-oriented mathematics curricula to be used in schools. By acknowledging students as the constructers of math knowledge, their teachers were also expected to take on new roles, such as facilitators of learning as opposed to more traditional teacher roles that positioned them as knowledge holders and transmitters. Although reform advocates have set their goal and agenda clearly, their efforts have encountered pushback from teachers - they have struggled to implement the reform-oriented curricula. Researchers have explored this issue and asserted various reasons including teachers' lack of understanding of the pedagogy proposed within the curricula (Manouchehri & Goodman, 1998), teachers' beliefs and background in mathematics and in general (Remillard, 2005), and teachers' ideas about the appropriateness of the curricula for their students (Stein, Grover & Henningsen, 1996). Additionally, teachers are expected to use a curriculum for which they are not a part of the development process and that causes unfamiliarity with the ideas proposed in the curriculum (e.g. Kirk & MacDonald, 2001). As such, it begs the question, would teachers implement curricula with greater fidelity of they were actively engaged in the development process?

In this regard, I wanted to explore the role of agency in teachers' decisions about designing and implementing curriculum and the factors affecting their decisions through the whole process. Since developing a year-long curriculum would be difficult within the length of the study, the teachers were involved in designing a fraction unit focused on developing fourth grade students' understanding of a fraction concept. The research questions that guided the study are:

What factors influence fourth grade elementary math teachers' decisions in designing and implementing a fractions unit?

a. What are the design decisions made in designing the unit?
b. What are the implementation decisions made in instructing the unit?
c. To what extent does the unit designed by teachers align with the unit implemented?

i. In incidents of alignment, what factors contributed?
ii. In incidents of misalignment, what factors contributed?

Conceptual Framework

This study is designed as part of a larger professional development (PD) program that aimed to improve the quality of elementary teachers' mathematics instruction by developing their mathematical knowledge for teaching (MKT) (Ball, Thames &Phelps, 2008). Ball, Thames, and Phelps (2008) refined Shulman's (1986) earlier categories - content knowledge, curricular knowledge, and pedagogical content knowledge (PCK) - by encapsulating these knowledge under the umbrella of MKT and further subdividing content knowledge and PCK into components (for additional information on different types of knowledge required for teaching, see Ball, Thames & Phelps (2008) and Hill, Ball & Schilling (2008)). They suggest the components of teachers' MKT - specifically common content knowledge (CCK), specialized content knowledge (SCK), knowledge of content and students (KCS) and knowledge of content and teaching (KCT) - affect their instructional practices. Thus, this study considers teachers' MKT as a factor influencing their decisions in designing and implementing a fraction unit.

In addition to MKT, teachers' mathematical beliefs affect their practices (Ernest, 1989; Pajares, 1992). The relationship is sometimes positive; if a teacher holds reform-oriented beliefs about student learning, then she would use reform-oriented teaching methods in her classroom (e.g. Cross, 2009). On the other hand, there are other studies that provide evidence to cases where teachers' mathematical practices do not necessarily reflect their beliefs, or their beliefs do not necessarily match with their practices (e.g. Beswick, 2012). Leatham (2006) provided an explanation to that inconsistency in his *sensible systems framework*. He claims that the conflict between beliefs and practices is not due to lack of a direct relationship, but it is due to the effect of other beliefs that happen to affect practices more than the desired ones. For example, when we realize there is an inconsistency between a teacher's belief, which we believe to affect or bring about a certain type of instruction, and her practice, we should consider there may be other beliefs or factors that are possibly influential on the action. So, teachers' beliefs was another factor possibly affecting teachers' decisions in designing and implementing a fraction unit.

Methods

The study falls under the embedded mixed methods case study (Creswell & Clark, 2011) as the quantitative data were collected before qualitative data and were used to further explain some of the differences in teachers' decisions about designing and implementing the unit. Thus, the data were analyzed using both quantitative and qualitative data analysis approaches. The participants were four 4th grade elementary teachers from four schools across two school districts in the Midwest. All of the participants had at least one year of teaching experience.

The data was drawn from a larger project on teacher PD conducted by researchers from a neighboring university. Data sources included video recordings of unit design sessions where teachers collaboratively designed a fraction unit, classroom sessions where teachers implemented the unit, and unit debriefing sessions where teachers discussed video clip examples from each other's classrooms; audio-recordings of teacher interviews; a belief survey; an itemized MKT survey; and unit reflection forms developed for each teacher by the researcher.

The data analysis incorporated both quantitative and qualitative methods. First, the videos from the unit design sessions were analyzed to determine the details of the designed fraction unit. The final product - the unit to be implemented - included a sequence of tasks selected by teachers along with the corresponding pedagogical considerations (PC) to be employed during the enactment of tasks. Second, the unit implementation videos were analyzed with a focus on teachers' decisions about tasks, the sequence of tasks and the specific PCs used during instruction. In addition to that, the video recordings of the unit debriefing sessions in which teachers discussed their unit implementation by showing short video clips from their classroom were also used to frame the details of implemented unit for each teacher. Third, the level of alignment was determined by comparing i) the number of tasks, ii) the sequence of tasks and iii) the PCs from the designed unit and the implemented unit for each teacher. Finally, the results from beliefs and MKT surveys in addition to findings from the interviews and unit reflection forms were used to explain what factors affected teachers' decisions and how.

Results & Discussion

The designed unit consisted of 12 tasks that focused on identifying the whole, partitioning, equivalence, comparing, and ordering the fractions. In addition to that, a total of 19 PCs were included under the relevant tasks. An example of a PC was "using manipulatives in introducing improper fractions" included under the task about exploring improper fractions. The findings revealed that although teachers had agency in designing the unit, the fidelity of their implementation varied (Table 1).

Table 1. Alignment percentages of fidelity components across cases

Teacher	Number of tasks from designed unit	Sequence of tasks according to designed unit	Employed PCs for the tasks from designed unit
Candy	42%	100%	35%
Britney	58%	100%	56%
Ralph	58%	71%	22%
Tamara	33%	100%	37%

In this paper, I will focus on Candy's case because although she did not have the highest fidelity levels for all components, her case clearly shows the complexity underlying the unit implementation process. Candy used only 5 of the 12 tasks in the exact same sequence with the designed unit. Regarding PCs, she used only 35% of the relevant PCs for the tasks. The findings from the surveys and interviews revealed that she held reform-oriented mathematical beliefs- she believed mathematics is a human endeavor that is continuously changing. Regarding learning and teaching math, she believed "students are the constructers of knowledge" and math instruction should be centered around student thinking. Her score from the MKT survey revealed she had relatively high level of understanding about fractions (CCK, SCK), knew how to interpret student thinking (KCS) and how to respond to student productions meaningfully (KCT).

Regarding the fidelity level for the number of tasks, Candy explained that her students needed help with partitioning, so she decided to spend more time on the relevant tasks. Even though this made her unable to move forward with the additional tasks in a timely manner, the way she implemented the tasks made her instruction more valuable in terms of providing her students opportunities to work on partitioning more extensively by discussing, questioning and actively producing – a feature of classrooms that align with reform-oriented practices. Also, realizing that her students would not be able to move forward in building strong conceptions of more advanced fraction concepts mastering partitioning and deciding to spend more time with relevant tasks indicate that Candy has strong grasp of the knowledge at the intersection of content and students. In other words, Candy's MKT levels appeared to influence her decisions about unit implementation.

There is no evidence to suggest that Candy's students' understanding of fractions increased through the unit, but the interaction captured in her instructional video, her responses to interview questions and survey items and her reflection on her experience during unit debriefing session all indicate that she placed student thinking at the center and built her instruction around it. Even though her fidelity level for PCs appear to be low, among all the teachers, Candy was the one who incorporated the PCs related to questioning, listening student responses and encouraged whole group discussion more frequently. In her reflection she stated, "Having students explain and show is more helpful than paper-pencil." This might indicate that Candy's reform-oriented beliefs about students, learning and teaching mathematics influenced her decisions in implementing the unit. So, although it appears that Candy's participation in designing the unit did not lead to considerably high levels of fidelity in her implementation, our observations would suggest that her teaching of the unit aligned closely with instruction suggested in reform documents.

Teachers' understanding of the content, teaching and students appear to have an impact on their implementation of the curricula. Curriculum developers should consider this while working with teachers and confirm that teachers have a substantial understanding of the overall goals and instructional practices of the curriculum and students' learning trajectory within the curriculum in order to enable their adherence to main goals of the curricula. Further results and implications will be shared during presentation.

References

Ball, D. L, Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Beswick, K. (2012). Teachers' beliefs about school mathematics and mathematicians' mathematics and their relationship to practice. *Educational Studies in Mathematics*, *79*(1), 127-147.

Creswell, J. C., & Clark, V. P. (2011). *Designing and conducting mixed methods research*. Thousand Oaks, CA: SAGE publications, Inc.

Cross, D. I. (2009). Alignment, cohesion and change: Examining mathematics teachers' belief structure and its influence on instructional practice. *Journal of Mathematics Teacher Education, 12*(5), 325-346.

Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In P. Ernest (Ed.), *Mathematics teaching: The state of the art* (pp. 249-254). Philadelphia, PA: The Falmer Press.

Hill, H.C., Ball, D. L., &Schilling, S.G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic specific knowledge of students. *Journal for Research in Mathematics Education*, *39*(4), 372-400.

Kirk, D., & MacDonald, D. (2001). Teacher voice and ownership of curriculum change. *Journal of Curriculum Studies, 33*(5), 551-567.

Leatham, K. (2006). Viewing mathematics teachers' beliefs as sensible systems. *Journal of Mathematics Teacher Education, 9*, 91-102.

Manouchehri, A., & Goodman, T. (1998). Mathematics curriculum reform and teachers: Understanding the connections. *Journal of Educational Research, 92*(1), 27–41.

Pajares, F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research, 62*(3), 307-332.

Remillard, J. T., (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research, 75*(2), 211-246.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*(2), 455–488.

CHANGING TEACHING PRACTICE: EXAMINING PROFESSIONAL DEVELOPMENT IMPACT ON MATHEMATICS DISCUSSION LEADING PRACTICE

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Decades of research has shown that most professional development fails to result in changes to teachers' classroom practice. This project explores an innovative professional development focused squarely on mathematics teaching practice and seeks to understand the features of the professional development that effect classroom level change.

Keywords: Teacher Education – In-service/Professional Development, Classroom Discourse, Elementary School Education

Problem Statement

Supporting teachers to improve their practice is a fundamental challenge for professional development. Decades of research have demonstrated that many common approaches to professional development do not adequately support improvements in teachers' capabilities (e.g., Cohen & Hill, 2001). In response, there has been increased work to develop new forms of professional development. Careful studies are needed to understand features of professional development design that support the learning of practice and how key variations in the design impact teacher learning. Further, many studies of professional development do not examine changes in teaching practice and such studies are needed.

This study is grounded in our project's work over the last decade to address the challenge of supporting the learning of practice using an approach to professional development situated in a common "live" case of elementary mathematics instruction occurring as part of a summer program for fifth grade students. The class comprises primarily Black youth, along with a small number of Latinx and white children, mostly from low-income families. The teacher is experienced and comfortable with making her practice visible and open to others. The approach uses this classroom as a "common text" for working on practice, where participants are not only watching and discussing, but are engaged in developing and refining teaching practice. Participants' engagement approximates a form of "legitimate peripheral participation," (Lave & Wenger, 1991) through structured conversations about the lesson plans, close observation, analysis of student tasks, and examination of records of teaching and learning. In addition, participants receive professional development focused on leading mathematics discussions.

The research explores the impact of participation in these structured ways on teachers' practice, as well as on their knowledge and dispositions. Our contribution is to rigorously study the impact of our professional model in situ to determine whether and how the work transfers into classrooms. Specifically, our initial study seeks to answer the following questions: What do teachers learn from structured participation in the class? Does their participation impact their own teaching practice, and if so, in what ways? Does the addition of professional development focused on a particular teaching practice impact teachers' own practice, and if so, in what ways?

Theoretical Framework

Mathematics teaching is something that people do; it is not merely something to know. Teachers must use knowledge flexibly and fluently as they interact in specific contexts with students, with the aim of helping those students become proficient with mathematics. This interactive and dynamic view of instruction can be represented by the "instructional triangle" (Cohen, Raudenbush, & Ball, 2003), a conceptualization of teaching as interactions among teachers, students, and content, in an environment. This conceptualization has important implications for the design of professional development. It means that professional development must attend to the specialized ways that teachers must understand mathematics, how that knowledge of mathematics interacts with teaching practices that support the learning of their students, and how the context of teaching interacts with all of these factors.

Our professional development is designed to support this type of professional learning and draws on recent work supporting teachers' learning of mathematics content and teaching practices. The work of "video clubs" as a means to increase capabilities at noticing student thinking (van Es & Sherin, 2008) in conjunction with research on lesson study as a means for teachers to learn from teaching and to develop and share practitioner knowledge (Perry & Lewis, 2009) informed the design of our peripheral participation model. Our project builds on these efforts and seeks to learn about impact on teacher practice.

We grounded our professional development in a particular decomposition (Grossman et al., 2009) of the teaching practice of leading a mathematics discussion. We define discussion as "a period of relatively sustained dialogue among the teacher and multiple members of the class" in which students respond to and use one another's ideas to develop collective understanding (TeachingWorks, 2015). Our decomposition is informed by research on orchestrating productive discussions (Smith & Stein, 2011), the concept of talk moves (Chapin, O'Connor, & Anderson, 2013), and research on decomposing practices to support learning (Boerst, Sleep, Ball, and Bass, 2011). The table below shows our decomposition of leading discussions with practices organized into three areas of work: (1) framing, (2) orchestrating, and (3) recording/representing content.

Table 1: Areas of Work and Practices for Leading Mathematics Discussions

Areas of Work		
Framing	*Orchestrating*	*Recording/representing content*
Launching Concluding	Eliciting student thinking Probing student thinking Orienting students towards the thinking of others Making contributions	Keeping accurate public records Using representations to convey key ideas

Methods

The study seeks to determine the extent to which peripheral participation in elementary mathematics teaching and focused professional development impact teachers' actual practice and factors that support their enactment of practice, including their ability to notice the specific work of teaching, and children's mathematical strengths. The broader study focuses on elementary teachers (n = 24), all of whom engaged in peripheral participation in elementary mathematics teaching for five consecutive days (~25 hours) and half of whom (n = 12) participated in additional professional development across the five days (10 hours) focused on leading a mathematics discussion. We collected and are analyzing a set of pre- and post-measures including measures of teachers' language in talking about students and teaching, and videos of classroom mathematics discussions which are being analyzed using a tool focused on the work teachers do when leading mathematics discussions (Selling, Shaughnessy, Willis, Garcia, O'Neill, & Ball, 2015).

In order to increase the comparability of teachers' skill with discussion leading practice, our team designed a "common" discussion plan based on work developed in a prior study (Garcia, Selling, & Wilkes, 2015; Selling, Shaughnessy, Willis, Garcia, O'Neill, & Ball, 2015). This enabled us to control for task selection and discussion structure. The accompanying technique checklist tool, which captured techniques related to practices named in our decomposition, was expanded to include advanced techniques that could be expected from experienced teachers.

For this initial study, we focus in particular on the impact of the full professional learning experience (~35 hours) on teachers' skill with leading mathematics discussions. To focus in on this impact, we selected four case study teachers who represented a range of skill with leading mathematics discussions in the pre-intervention data set. We then examined pre- and post-intervention video of their common discussion lessons. Videos were double coded by members of the research team using the technique checklist tool. Discrepancies in coding were resolved by a consensus discussion and reference to code books developed by the research team.

Analysis

Analysis of pre- and post- common discussion lesson videos utilized the technique checklist tool to examine particular techniques used before and after the intervention. Techniques were coded as present, not present, and not applicable. In some cases where we would expect to see a technique used frequently, for example "asks questions that deepen student reasoning," additional codes of once, more than once, and frequently replaced the code of present. We then compared participants at the technique level to determine whether their use of the technique increased, stayed consistent, or decreased from pre- to post-intervention. Table 1 below shows the positive areas of change for each of our case study teachers in each area of work of leading a mathematics discussion. For example, T12 improved in all 3 techniques from pre-post in the area of launching a discussion.

Table 2: Number of Techniques in Each Area of Work Showing Positive Change

	Launching (3)	Concluding (3)	Eliciting (3)	Probing (5)	Orienting (4)	Making Connections (3)	Making Contributions (5)	Recording/ Representing Content (4)
T12	3	0	1	3	3	2	1	2
T14	0	2	0	1	0	2	1	0
T18	0	2	0	0	1	0	0	0
T22	1	0	2	1	1	3	2	1

The initial analysis shows that the three out of four teachers increased their use of techniques in four areas of work: probing, orienting, making connections, and making contributions. Additionally, half of the case study teachers increased their use of techniques in the remaining four areas of work.

In addition to the presence of techniques, the team also coded for problematic areas, including unequally distributed participation, ineffective probes, and incorrect/inaccurate mathematical contributions. All case study teachers had fewer issues in their post-intervention discussion than their pre-intervention discussion. Notably, T12 improved in 7 of 8 issue areas.

Discussion

Initial results of the study show promise for the potential of the current professional development design to impact teaching practice. Peripheral participation paired with a focused discussion leading professional development increased teachers' use of techniques core to the work of leading productive mathematics discussions. Next steps for this work are twofold. First, we will examine impacts of the professional development on other indicators including teachers' mathematical knowledge for teaching and their skill with noticing and naming student strengths. Second, we will examine and compare the impact of peripheral participation with full participation on teachers' discussion leading practice and associated factors.

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References

Boerst, T., Sleep, L., Ball, D. L., & Bass, H. (2011). Preparing teachers to lead mathematics discussions. *Teachers College Record*, *113*(12), 2844-2877.

Chapin, S., O'Connor, C., & Anderson, N. (2013) *Classroom discussions in math: A teacher's guide for using talk moves to support the Common Core and more, Grades K–6*, 3rd Edition. Math Solutions.

Cohen, D. K., & Hill, H. C. (2001). *Learning Policy : When State Education Reform Works*. New Haven, CT, USA: Yale University Press.

Cohen, D. K., Raudenbush, S., & Ball, D. L. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis, 25* (2), 119-142.

Garcia, N., Selling, S.K., & Wilkes, C. (2015). Standardized assessments of discussion leading practice: Are they valid measures? Proceedings of the Thirty-seventh Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. East Lansing, MI: Michigan State University

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. (2009). Teaching practice: A cross-professional perspective. *The Teachers College Record*, *111*(9), 2055-2100.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. New York: Cambridge University Press.

Perry, R. R., & Lewis, C. C. (2009). What is successful adaptation of lesson study in the US?. *Journal of Educational Change*, *10*(4), 365-391.

Selling, S.K., Shaughnessy, M., Willis, A.T., Garcia, N., O'Neill, M.K., & Ball, D.L. (2015). Standardized assessments of beginning teachers' discussion leading practice: Is it possible and what can we learn? Proceedings of the Thirty-seventh Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. East Lansing, MI: Michigan State University

Smith, M. & Stein, M. K.(2011). *5 practices for orchestrating productive mathematics discussions*. National Council of Teachers of Mathematics: Reston, VA.

TeachingWorks. (2015) *Leading a group discussion.* Unpublished technical report. University of Michigan: Ann Arbor, MI.

van Es, E. A., & Sherin, M. (2008). Mathematics teachers' "learning to notice" in the context of a video club. *Teaching and Teacher Education, 24*(2), 244-276.

TEACHER NOTICING OF STUDENTS' PRIOR KNOWLEDGE

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We investigate teacher noticing of students' prior knowledge in the context of a professional development intervention that integrated lesson study, animation discussions, and video clubs with in-service geometry teachers. We analyzed video club discussions throughout the program in relation to teachers' noticing of students' prior knowledge and found that the discussions in the middle of the two-year intervention had the highest level of noticing, coinciding the lesson revision process. The findings suggest that the lesson revision process spurred connections between teachers' observations of students' prior knowledge and pedagogical decisions.

Keywords: Teacher Education-Inservice/Professional Development, Geometry and Geometrical and Spatial Thinking

This study investigates the effects of professional development on teacher noticing of students' prior knowledge in geometry instruction. We consider how teachers attend to and interpret student thinking while discussing classroom videos. The intervention integrated three strategies, namely, animation discussions (Chazan & Herbst, 2012), video clubs (Sherin & Han, 2004), and lesson study (Fernandez, 2002), with the objective of supporting teachers' attention to student thinking. We consider the ways in which teachers can support students' development of mathematical understandings in secondary schools by noticing their prior knowledge.

Theoretical Underpinnings

Professional Development to Promote Teacher Noticing

Lesson study engages teachers in a cyclical process of conducting instructional material research, planning a research lesson, teaching and observing the implementation of the lesson, and reflecting upon student thinking observed during the implementation of the lesson (Fernandez, 2002). Teachers may engage in the process of revising and re-teaching the lesson to optimize student learning opportunities (Lewis & Hurd, 2011). Traditional live observations may be difficult in the U.S., which prompted us to replace live observations with video clubs. During video clubs, teachers discuss videos from their own classrooms. Sherin and colleagues demonstrated that initially teachers tend to focus on various aspects of instruction, but sustained video club meetings enable teachers to increase their attention to student thinking (Sherin & Han, 2004). Van Es's (2011) analysis of *how* and *what* teachers notice in video clubs demonstrated that the sophistication of teachers' analyses of students' thinking increased over time.

Students' Prior Knowledge

Our conceptualization of students' prior knowledge goes beyond the mathematical prior knowledge and includes students' multiple mathematical knowledge bases (Roth McDuffie, et al., 2014). This understanding of students' prior knowledge acknowledges that students draw upon their prior experiences, including their cultural and community knowledge, when approaching a math problem. Especially in problem-based math instruction, when students learn a target concept through solving a complex mathematical task, students' prior knowledge, both in relation to the mathematical task and the context it is situated within are of great importance.

Research Question

What students' prior knowledge do teachers notice and how do teachers consider students' prior knowledge in video clubs throughout a professional development intervention?

Methods

Five high school geometry teachers from high-needs schools participated in 20 three-hour sessions. All participants taught the lesson while members of the research team videotaped. In subsequent sessions, the teachers participated in video clubs where they discussed students' problem-solving strategies. The research team selected the video clips following criteria established by Sherin, Linsenmeier, and van Es (2009). In year two, the teachers revised and taught the lessons and engaged in video clubs.

We selected four sessions, 3, 4, 12, and 19, as representative video clubs at the start, middle, and end of the intervention. With an entire video club discussion as our unit of analysis, we independently coded the discussions using a rubric that considers teacher noticing of students' prior knowledge based upon van Es' (2011) framework (Table 1). The framework investigates *what* prior knowledge teachers notice and *how* they notice students' prior knowledge.

Table 1:Framework for Teacher Noticing of Students' Prior Knowledge

	Level 1	Level 2	Level 3	Level 4
What students' prior knowledge do teachers notice?	No mention of students' prior knowledge.	Mention students' mathematical prior knowledge in terms of what they have learned in class previously.	Mention students' prior knowledge in relation to the task, including mathematical, contextual, and/or practices.	Attend to relationship between students' prior knowledge and pedagogical issues.
How do teachers notice students' prior knowledge?	No mention of students' prior knowledge.	Describe students' mathematical prior knowledge.	Describe students' mathematical, contextual, and out-of-school prior knowledge.	Make connections between students' prior knowledge and principles of teaching and learning.
		Primarily evaluative comments about students' prior knowledge, with some interpretive comments.	Analyze and interpret noteworthy student actions in relation to their prior knowledge.	On the basis of interpretation, propose alternative pedagogical solutions to use students' prior knowledge.
		Refer to specific events and interactions as evidence.	Refer to specific events and interactions as evidence and elaborate on them in relation to students' prior knowledge and how it shapes their mathematical understanding.	All attributes of Level 3

Findings

The majority (67%) of the selected video club discussions were coded as Level 3 in regard to *what* students' prior knowledge teachers noticed (Table 2). Throughout the professional development intervention, the teachers consistently discussed students' prior knowledge in relation to their work on the task, and frequently considered sources of students' prior knowledge beyond mathematical content taught in school. However, in most discussions the teachers did not connect students' prior knowledge to pedagogical decisions. Thus, in terms of *how* teachers noticed students' prior knowledge during the video club discussions; we coded most of these video club discussions at Level 2 (56%). While it is possible that the teachers offered some interpretation of students' prior knowledge, the focus of the discussion was mostly on evaluation.

Table 2:Teachers' Noticing of Students' Prior Knowledge in Video Club Discussions

	Level 2	Level 3	Level 4
What Prior Knowledge Teachers Notice	11%	67%	22%
How Teachers Notice Students' Prior Knowledge	56%	33%	11%

Figure 1 illustrates the relationship between *what* and *how* teachers noticed students' prior knowledge for each video club over time. The majority of the video club discussions (5 out of 9) were coded at the same level for both what and how teachers notice students' prior knowledge. In the later discussions, the levels differed more frequently, but *what* students' prior knowledge teachers noticed was always rated at the same level or higher than the level of *how* teachers noticed students' prior knowledge. Additionally, teachers' discussions reached the highest levels of noticing during the middle of the professional development. At this point in year two, the teachers were in the process of revising the lesson for the second year.

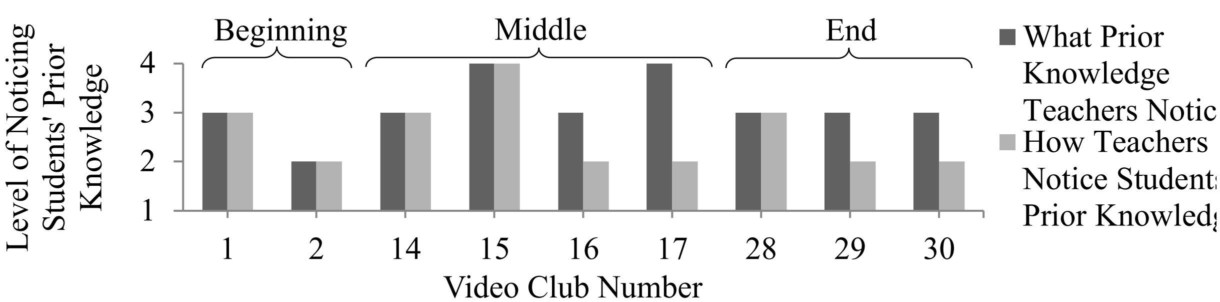

Figure 1. Levels of Noticing Students' Prior Knowledge Per Video Club

Discussion

The findings of our coding for *what* and *how* teachers notice students' prior knowledge show the video club discussions achieved the highest levels in the middle of the intervention. This finding suggests that teachers' motivation to revise the lesson and to re-teach it, as part of their engagement in lesson study, provoked them to examine pedagogical issues in combination with students' prior knowledge. Specifically, the highest levels of the theoretical framework require teachers to go beyond describing, evaluating, and interpreting students' prior knowledge by making connections with their instruction. It seems that the impending need to make changes to the lesson using their observations of students' work in the video club provoked discussion of pedagogical issues. This finding suggests that lesson study can be a motivation for increasing teacher noticing and supports the idea that video clubs can be valuable for lesson reflection.

Although prior research findings suggest that teacher noticing increases over time (van Es & Sherin, 2008), our findings suggest that teacher noticing is not an activity that necessarily increases in sophistication over time. Instead, teacher noticing is purposeful, and highly motivated by activities that promote connections between observations of student learning and teaching. In our case, the need for revising the lesson with the goal of optimizing student learning opportunities provided the conditions for teachers' highest levels of noticing. One implication for teacher education is that activities promoting connections between noticing student thinking and using student thinking in instruction can further teacher noticing.

Conclusion

Combining lesson study and video clubs can provoke teachers to achieve a higher level of noticing because they are motivated to study tasks for optimizing student learning opportunities. In particular, the process of revising the lesson prompts teachers to make connections between observations of student thinking and instructional improvement. Our findings support the importance of implementing the full lesson study cycle. Considering difficulties allocating resources for lesson study, it is important that the lesson revision step continues to be enacted to maximize teachers' learning opportunities.

Acknowledgments

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References

Chazan, D., & Herbst, P. (2012). Animations of classroom interaction: Expanding the boundaries of video records of practice. *Teachers College Record, 114*(3), 1-34.

Fernandez, C. (2002). Learning from Japanese approaches to PD: The case of lesson study. *Journal of Teacher Education, 53*(5), 393-405.

Lewis, C. C., & Hurd, J. (2011). *Lesson Study Step by Step: How Teacher Learning Communities Improve Instruction* (With DVD). Portsmouth, NH: Heinemann.

Roth McDuffie, A. R., Foote, M. Q., Bolson, C., Turner, E. E., Aguirre, J. M., Bartell, T. G., ... & Land, T. (2014). Using video analysis to support prospective K-8 teachers' noticing of students' multiple mathematical knowledge bases. *Journal of Mathematics Teacher Education*, *17*(3), 245-270.

Sherin, M. G., Jacobs, V. R., & Philipp, R. A. (Eds.). (2011). *Mathematics teacher noticing: Seeing through teachers' eyes*. New York: Routledge.

Sherin, M. G., & Han, S. Y. (2004). Teacher learning in the context of a video club. *Teaching and Teacher Education*, *20*(2), 163-183.

Sherin, M. G., Linsenmeier, K. A., & van Es, E. A. (2009). Selecting video clips to promote mathematics teachers' discussion of student thinking. *Journal of Teacher Education*, *60*(3), 213-230.

van Es, E. A., & Sherin, M. G. (2008). Mathematics teachers "learning to notice" in the context of a video club. *Teaching and Teacher Education*, *24*(2), 244-276.

van Es, E. A. (2011). A framework for learning to notice student thinking. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 134-151). New York: Routledge.

USING COACHING CYCLES TO TRANFER AND SUSTAIN EFFECTIVE INSTRUCTIONAL PRACTICES

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We illustrate a coaching cycle approach situated within a larger professional development design that focused on infusing high quality mathematics tasks and differentiation within inclusive elementary mathematics classrooms. Our design supported teams of general education and special education teachers with integrating tasks and differentiation strategies into co-taught mathematics lessons through reflective coaching sessions. This project yielded positive results in instructional practice and co-teaching collaborations.

Keywords: teacher education-inservice/professional development, instructional activities and practices, elementary school education, equity and diversity

Introduction

Professional development is necessary for teacher growth. In order to be effective, professional development must directly tie to the teachers' practice, focus on specific content, align with school improvement initiatives, strengthen teacher collaboration, and provide consistent follow-up (Wei, Darling-Hammond, Andree, Richardson, & Orphanos, 2009). Often, observations and coaching cycles are strategies used to provide the consistent follow-up and support the transfer of practices learned from professional development to teachers' classrooms. Specific to mathematics teaching and learning, coaching has been linked to increased student achievement, improved teaching practices, changes in teachers' beliefs, and sustained professional development concepts (e.g., Baldinger, 2014; Campbell & Malkus, 2011).

Overview of Professional Development Project

The overarching goals of the professional development project included increasing teachers' content and pedagogical content knowledge, creating communities of learners, particularly between general education and special education teachers, focusing on reducing the achievement discrepancy noted between students with and without disabilities, and shifting instructional practices in inclusive mathematics classrooms. Specifically, three areas were targeted to create the conceptual framework for the overall project: (a) increasing the use of high-quality mathematics tasks (Stein, Smith, Henningsen, & Silver, 2001), (b) incorporating meaningful differentiation strategies to meet the needs of all learners (e.g., Tomlinson et al., 2003), and (c) integrating effective co-teaching models and practices (Friend & Cook, 2010). To meet the goals of the professional development project and provide support to the teams of general education and special education teachers, we designed a longitudinal professional development plan that used multiple formats of professional development delivery (Guskey, 2000). Our design included both synchronous (face-to-face) and asynchronous (independent) opportunities for learning, with on-site coaching sessions to help sustain and transfer the learning from the whole-group sessions to teachers' actual classrooms.

Setting and Participants

The setting was one county district in a southern state located near a large research university. The school district identified a need for professional development in relation to co-teaching in mathematics to increase co-teaching and differentiation. Participants were from eight elementary schools and included 22 general education teachers and 13 special education teachers or paraprofessionals (n= 35 teachers). Co-teaching teams consisted of at least one general education teacher and one special education teacher or paraprofessional with varying numbers of teams per school (ranging from 1-6) for a total of 21 unique co-teaching teams.

Coaching Cycles

Theoretical Framework for Coaching

The theoretical framework for the coaching cycle centers on the cognitive apprenticeship model for learning (Collins, Brown, & Newman, 1987). Cognitive apprenticeship is similar to the traditional or trade apprenticeship ideals of an expert and a novice socially interacting in order to guide the learning of a task in a specific domain (Collins et al., 1987; Dennen, 2004). Unlike traditional apprenticeship models, "cognitive apprenticeship refers to the fact that the focus of the learning-through-guided-experience is on cognitive and metacognitive, rather than on physical, skills, and processes" (Collins et al., 1987, p. 5). The expert-novice relationship translates into a teacher-learner relationship (Atkinson, 1997), where learning occurs through a sequenced use of three teaching methods: modeling, coaching, and fading (Collins et al., 1987).

Coaching Cycle Development and Design

To ensure transfer and support the implementation of the lessons learned from our synchronous professional development sessions (e.g., Guskey, 2000), a coaching cycle was included in our larger professional development model to provide individualized support for each team of general and special education teachers (or paraprofessionals) in their inclusive mathematics classrooms. The participating teams of co-teachers were divided among the three lead facilitators (two elementary mathematics educators and one special education educator). Teams of co-teachers were asked to plan and deliver co- taught, high quality mathematics lessons that included differentiation when needed to meet the needs of all learners.

Similar to the design of coaching sessions used by Baldinger (2014), where there was a pre-session, observation, and post-session, we designed our coaching sessions to use a similar cyclical approach. Prior to implementing the coaching cycle, the facilitators developed a protocol to use during all coaching sessions to increase consistency in the expectations and coaching conversation among facilitators. The protocol established: (a) coaching sessions are to follow each synchronous professional development session, (b) the facilitator is to receive the co-teaching mathematics lesson plan at least 24 hours prior to the scheduled observation, (c) each lesson will be scored using the same instruments (discussed in more detail below), (d) a 30-minute coaching session/conversation will follow each lesson observation with discussions about the planning and delivery of the lesson (focused on co-teaching practices, differentiation strategies, and the quality of the mathematics task), and (e) the coach will record anecdotal notes about the lesson and coaching session.

Coaching Cycle Implementation

The Co-Teaching Checklist (Murawski & Lochner 2011), the Mathematics Classroom Observation Protocol for Practices (MCOP^2; Gleason, Livers, & Zelkowski, 2017), and a debriefing form (adapted from Villa, Thousand, & Nevin, 2013) focused on the three fundamental domains of our professional development model (i.e., high quality mathematics, differentiation, and co-teaching implementation) and were used for the data collection during the lesson observations and as a guide for the coaching conversations.

The Co-Teaching Checklist (Murawski & Lochner 2011) allowed us to measure specific co-teaching aspects, such as both teachers assist student with and without disabilities, the environment showcases a collaborative atmosphere, and the conversations and questions used support inclusive practices. The MCOP^2 (Gleason et al., 2017) allowed us to measure the extent instruction aligns with the Standards for Mathematical Practices and the shared responsibility and authority between teachers and students across 16 indicators. The debriefing form (adapted from Villa et al., 2013) allowed us to use the data collected from the Co-Teaching Checklist and the MCOP^2 coaching conversation and discussions surrounding co-teaching, differentiation, and the mathematics. The teachers first identified things that went well in the lesson, things they would do differently, and their co-teaching approach. Following the teachers' sharing, the facilitator then identified things that went well, as well as wonderings and suggestions. The coaching sessions ended with identifying next-steps and take-aways to focus on in future lessons.

Project Outcomes

A mixed methods design was used to determine the impact of coaching and professional development on teaching practices. Tentative, pre-post results from the MCOP^2 (Gleason et al., 2017) indicate a significant change in teacher facilitation and student engagement. Specifically, teachers increased their facilitation of high quality mathematics practices and meaningful learning experiences with students, and had more students engaged in opportunities to develop positive mathematics practices. These results indicated teachers were moving away from a teacher directed approach to a student-centered approach. Tentative results from the Co-Teaching Checklist (Murawski & Lochner, 2011) indicated more teachers were engaged in co-teaching and exhibiting effective co-teaching behaviors during the observations.

Our qualitative data across the course of the project also revealed positive benefits of the coaching cycles. Data from a sub-sample of culminating teacher reflections (n=9) showed positive results and benefits regarding the coaching sessions. Coding the responses revealed various themes, much of which centered on teacher growth and changes in teaching practices. Additionally, many teachers appreciated the coaching sessions allowed for immediate feedback and reflection time (see Table 1 for all themes and frequency of occurrences).

Table 1: Themes from Teacher Reflections about Coaching Sessions

Themes	Frequency
Growth as a teacher	7
Immediate feedback	4
Change of teaching practice	3
Most beneficial component of professional development	2
Reflection time	2
Energized	2
Accountable for co-teaching	1
Increase school communication	1
Ideas/strategies for extending lesson	1

Summary

Coaching sessions tied to intentional professional development yielded positive and visible results. Classroom observations shifted to a more student-centered practice, and an increase in co-teaching collaborations and behaviors. Teachers found the coaching cycles to be the most beneficial aspect of the professional development. Both our quantitative and qualitative results support the benefit of a coaching component to support and sustain professional development as echoed in prior research (e.g., Baldinger, 2014; Campbell & Malkus, 2011).

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References

Atkinson, D. (1997). A critical approach to critical thinking in TESOL. *TESOL Quarterly, 31*(1), 71-94.

Baldinger, E.M. (2014). *Learning together: Looking for learning in coach-teacher interactions*. Paper presented at the National Council of Teachers of Mathematics Research Conference, New Orleans, LA.

Campbell, P. F., & Malkus, N. N. (2011). The impact of elementary mathematics coaches on student achievement, *The Elementary School Journal, 111*, 430–454.

Collins, A., Brown, J. S., & Newman, S. E. (1987). *Cognitive apprenticeship: Teaching the crafts of reading, writing, and mathematics* (Technical Report No. 403). Washington, DC: National Institute of Education. Retrieved from http://files.eric.ed.gov/fulltext/ED284181.pdf.

Dennen, V. P. (2004). Cognitive apprenticeship in educational practice: Research on scaffolding, modeling, mentoring, and coaching as instructional strategies. In D. H. Jonassen (Ed.), *Handbook of research on educational communications and technology, 2nd edition* (p. 813–828). Mahwah, NJ: Lawrence Erlbaum Associates.

Friend, M., & Cook, L. (2010). *Interactions: Collaboration skills for school professionals* (6th ed.), Boston: Pearson.

Gleason, J., Livers, S., & Zelkowski, J. (2017). Mathematics Classroom Observation Protocol for Practices (MCOP^2): A validation study. *Investigations in Mathematics Learning, 9(3), 111-129.*

Guskey, T. R. (2000). *Evaluating professional development*. Thousand Oaks, CA: Corwin Press.

Murawski, W. W., & Lochner, W. W. (2011). Observing Co-Teaching: what to ask for, look for, and listen for. *Intervention in School and Clinic, 46*(3), 174-183.

Stein, M. K., Smith, M., Henningsen, M., & Silver, E. (2001). *Implementing standards-based mathematics instruction: A casebook for professional development.* New York, NY: Teachers College Press.

Tomlinson, C. A., Brighton, C. Hertberg, H. Callahan, C. Moon, T. R. Brimijoin, K. Conover, L., & Reynolds, T. (2003). Differentiating instruction in response to student readiness, interest, and learning profile in academically diverse classrooms: A review of literature. *Journal for the Education of the Gifted, 27*(2/3), 119-145.

Villa, R. A., Thousand, J. S., & Nevin, A. I. (2013). *A Guide to Co-Teaching: New Lessons and Strategies*. Corwin, Thousand Oaks: CA.

Wei, R. C., Darling-Hammond, L., Andree, A., Richardson, N., & Orphanos, S. (2009). Professional Learning in the Learning Profession: A Status Report on Teacher Development in the US and Abroad. Technical Report. *National Staff Development Council*.

CHARACTERIZING REHEARSALS WITH IN-SERVICE TEACHERS

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To address the chasm between learning about effective teaching practices and being able to implement those practices competently, teacher preparation programs have begun to adopt rehearsals as a pedagogy. Teaching rehearsals are opportunities to develop an understanding of instructional practices in an authentic, but scaffolded environment. These have been shown to help novice teachers gain competency in complex teaching practices. In this study, we explore this pedagogy with in-service secondary teachers. Results indicate that in these rehearsals the teacher educators were less directive in their suggestions and the participating teachers were more assertive in making suggestions as compared to rehearsals with pre-service teachers.

Keywords: Rehearsals, Pedagogies of enactment, Professional development

Introduction

The divide between theory and practice in teacher preparation has long been recognized (Shulman, 1998). To compound this issue, what teachers see and experience in schools often differs considerably from the current conception of good instruction (Shulman, 1998). This chasm creates a tension between reform methods and current practices. As a result, when teachers are faced with the realities of teaching, they tend to adopt teaching practices they have experienced as students or through classroom observations (Korthagen et al., 2001). Exploring solutions to this historic divide, Grossman and colleagues (2009) examined the pedagogy of professional education across a variety of professions. From their comparative study emerged the notion of an *approximation of practice*, a method of scaffolding preparation in which pre-professionals engage in smaller, more manageable components of practice. This structure embeds training in the actual work of the profession as well as makes the details of professional actions and reasoning more visible.

Building on Grossman's work, several researchers have developed practice-based models of professional preparation in which the instructional focus shifted from primarily developing students' knowledge bases to gaining competencies in enacting particular instructional practices. These models centered around a particular approximation of practice, referred to as rehearsal, in which pre-service teachers (PSTs) engage in authentic, yet controlled instructional activities (IAs). During a rehearsal peers play the role of students, which provides a context where the teacher educators can introduce particular problems of practice and interrupt the flow of the lesson to offer in-the-moment suggestions about the details of instructional moves and decisions. Furthermore, this context allows rehearsals take place within a community organized around a common vision of instruction and shared values.

Rehearsals were conceived as a support for pre-service teacher preparation programs and consequently have been used predominantly in this context. However, the issue of how to implement effective teaching practices in real classrooms continues to challenge even experienced teachers. As such, we felt in-service teachers might also benefit from engaging in rehearsals. In this study, we explore the nature of rehearsals with in-service teachers.

Literature Review

The overwhelming majority of research on rehearsals has been embedded in the context of pre-service elementary programs (see Pfaff, 2017 for an exception). Early studies provided thorough characterizations of rehearsals as well as a theoretical rationale behind their use (e.g. Kazemi, Franke, & Lampert, 2009; Lampert & Grazini, 2009). Later, McDonald, Kazemi, and Kavanagh (2013) built on this structure, outlining an instructional cycle organized around pedagogies of enactment, in which rehearsals are preceded with various practice-based activities in preparation and then followed with the live enactment in front of students. As a more consistent conceptualization of rehearsals has emerged in the field, recent studies have provided detailed descriptions of how PSTs grappled with the nuances of a particular teaching practice (e.g. Ballinger, Selling, & Virmani, 2016; Elliot, Aaron, & Maluangnont, 2015; Pfaff, 2017)

Another area of analysis has been with the rehearsals themselves and in particular the teacher educator's (TE's) leadership and organization. Reviewing a collection of over 90 rehearsals, Lampert et al. (2013) analyzed the various exchanges when a TE interrupted the flow of the lesson to document the substance of the interaction (i.e. what was being talked about) and the structure of the interaction (i.e. how it was being talked about). Similarly, Kazemi, Ghousseini, Cunard, and Turrou (2016) reflected on five years of working with PSTs to give insights into how teacher educators can structure rehearsals to make them productive for PSTs.

We build on this last category of work that gives insight into how TEs can structure the rehearsal experience to make it productive for its participants by extending this work with in-service teachers. In particular, we investigated the nature of the interruptions in the teaching during the rehearsal. Following Lampert et al. (2013), we investigated both the substance and structure of TE interruptions. We did so with the assumption that coding the substance and structure of these interruptions would reveal differences between rehearsals with PSTs and in-service teachers, bringing into greater relief the choices TEs make when leading rehearsals.

Methods

The authors of this paper offered a 20-hour professional development (PD) session for experienced middle school teachers that lasted one week. The PD focused on how to teach figural pattern tasks to help students develop an understanding of variables as measuring quantities. The PD made use of rehearsals and an IA targeting this content area. The overall lesson was decomposed into four similar iterations of the same instructional structure. Each iteration served as the basis of a new rehearsal and differed by the representation and consequently the level of abstraction that students were asked to use to engage in the figural pattern generalization. Over the last two days of the PD (4 hours each day) the teachers participated in six rehearsals. We had fourteen middle school teachers volunteer and participate in the PD. Each of the fourteen was involved in planning at least one of the rehearsals, but only six teachers had the opportunity to serve as the acting teacher (AT) in a rehearsal.

We began our analysis identifying each interruption during the 6 rehearsals. We counted an interruption as beginning when the flow of the lesson stopped. This happened when a TE, participating teacher, or the AT interjected a comment about the lesson. Generally, we counted interruptions lasting until the instruction resumed. However, there were times when several independent topics were discussed without the AT resuming teaching. We counted these discrete discussions as separate interruptions when the substance of the interaction changed notably.

We then coded who initiated each interruption as well as the structure and the substance of the interruption. To develop our coding scheme for structure and substance codes, we starting with the codes used by Lampert et al. (2013), but then added new codes where the Lampert et al.

(2013) codes did not accurately capture the nature of the interaction. We developed new codes by first identifying interruptions where a new code was necessary. We then made descriptive notes about what was discussed during these exchanges as well as the nature of these interactions. Finally, we used grounded theory (Strauss & Corbin, 1998) to create categories that captured the characteristics of these interruptions. This work resulted in the addition of eight new structure codes and one new substance codes. Space restrictions do not allow us to describe each of the codes in our coding scheme, but we describe codes on an as-needed basis. Finally, the percentages of each code were determined relative to the total number of interruptions and then compared to the percentages reported by Lampert et al. (2013).

Results

Below are the percentages of the structure and substance codes that appeared in our study as well as the corresponding codes and values from Lampert et al. (2013) (see Table 1). In all, we had 71 interruptions over the 6 different rehearsals. Several of the interruptions aligned with more than one substance or structure code. In all there were 83 different structure (initiated) codes, 93 different structure (taken up) codes, and 101 different substance codes. 23 of these interruptions (32.4%) were initiated by participating teachers.

Table 1: Coding of interruptions

Structure	Us (Initiated)	Us (Taken Up)	Lampert et al.	Substance	Us	Lampert et al.
Alternate Move	21.13	12.68		Student Thinking	33.80	13.95
Discussion	19.72	28.17	17.29	Elicit and Respond	32.39	35.74
Scaffold Enactment	15.49	15.49	21.09	Representation	23.94	23.64
Share Thinking	16.90	12.68		Select and Sequence	21.13	0
Interpreting the Situation	8.45	12.68		Content Goals	11.27	14.03
Directive	8.45	12.68	60.85	Managing Time	9.86	4.34
Rationale	7.04	12.68		Mathematics	4.23	11.94
Contextualizing Student Thinking	7.04	5.63		Orienting Students	1.41	7.05
Managing Rehearsal	5.63	5.63		Body/Voice	1.41	2.95
Highlighting	4.23	2.82		Process Goals	1.41	6.67
Negotiating Next Move	1.41	8.45		Student Engagement	0	21.55
Evaluative Feedback	1.41	1.41	28.14	Attending to IA	1.41	17.29

Discussion

Overall, our rehearsals with in-service teachers were characterized by a more active involvement by the teachers. Teachers were willing to interrupt the flow of the lesson to initiate discussions about teaching. Most often this was done to give more context to teaching decisions, reflect on various aspects of teaching, or to offer advice.

In comparing the relative frequency of our structure codes with those reported by Lampert et al. (2013) the differences are striking. The majority of Lampert et al.'s (2013) interruption were directive (when a TE tells the teacher what to do), 60.85%, while only 8.45% of our interruptions were initiated as directive. Similarly, there was much less evaluative feedback (when a TE makes a judgement about a move) in our rehearsals. While Lampert et al. reported that 28.14% of exchanges were TEs giving evaluative feedback, only 1.41% of our exchanges were. Reflecting on these results, it appears that when working with teachers who possess significant experience, we wanted to create a less directive and evaluative environment to draw from their expertise and position them as more equal partners. While we as teacher educators brought particular expertise to the rehearsal, we acknowledged that the teachers themselves had much understanding to share. Consequently, we structured the rehearsals in a way that provided teachers opportunities to learn from our expertise, but also to share their understanding as a way to reorganize their previously developed skills towards a new orientation of instruction.

There were also several notable differences in the substance of the interruptions. Two of these include our talk about student thinking and selecting and sequencing student ideas. The most frequently discussed topic in our rehearsals was student thinking (where participants discuss the details of students' ways of reasoning), with 33.80% of exchanges addressing this topic compared to only 13.95% of exchanges in Lampert et al.'s study. Another difference was the frequency of exchanges in which we talked about how to select and sequence student ideas. This code was in our top four, with about 21.13% of exchanges referencing this idea. This is significant because Lampert et al. (2013) did not feel the need to use this code to categorize their exchanges. These differences in substance may be in part attributable to the nature of our IA. In contrast to IAs reported in the literature, which tend to be shorter activities, our IA covered a multi-day unit. Such structure afforded teachers more opportunities to monitor student work and think about how to to leverage student thinking to reach conceptual goals (Smith & Stein, 2011).

Conclusion

This study provides an image of what rehearsals can look like with in-service teachers. As such, it helps to highlight decisions TEs need to make when leading a rehearsal. In particular, TEs will need to decide if the teachers they are working with need more directive feedback on and practice in implementing complex teaching practices or would benefit from reflective discussions about what to do next and why. TEs who wish to create a more reflective than directive environment could consider using some of the types of interruptions discussed here (e.g. suggesting alternative moves, interpreting the situation, or asking teachers to share their thinking). These decisions will likely depend on the experience level of the teachers with whom the TE is working and the goals the TE has in mind for the teachers.

References

Baldinger, E. E., Selling, S. K., & Virmani, R. (2016). Supporting novice teachers in leading discussions that reach a mathematical point: Defining and clarifying mathematical ideas. *Mathematics Teacher Educator*, 5(1), 8-28.

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. (2009). Teaching practice: A cross-professional perspective. *The Teachers College Record*, 111(9), 2055-2100.

Kazemi, E., Franke, M., & Lampert, M. (2009). Developing pedagogies in teacher education to support novice teachers' ability to enact ambitious instruction. In *Crossing divides: Proceedings of the 32nd annual conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 12-30). Adelaide, SA: MERGA.

Lampert, M., Franke, M. L., Kazemi, E., Ghousseini, H., Turrou, A. C., Beasley, H., & Crowe, K. (2013). Keeping it complex using rehearsals to support novice teacher learning of ambitious teaching. *Journal of Teacher Education*, 64(3), 226-243.

McDonald, M., Kazemi, E., & Kavanagh, S. S. (2013). Core practices and pedagogies of teacher education: A call for a common language and collective activity. *Journal of Teacher Education*, 64(5), 378-386.

Pfaff, E. (2017). The Role of Teacher Rehearsal in Classroom Mathematics Discourse. Vanderbilt University. Retrieved from http://etd.library.vanderbilt.edu/available/etd-04122017-151329/unrestricted/Pfaff.pdf

Shulman, L. S. (1998). Theory, practice, and the education of professionals. *Elementary School Journal*, 511-526.

Smith, M. S., & Stein, M. K. (2011). *5 practices for orchestrating productive mathematical discussions*. Reston, VA: National Council of Teachers of Mathematics.

STUDYING LESSON STUDY: CASES OF SECONDARY GEOMETRY TEACHERS

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When teachers anticipate student responses, they are likely to be prepared to respond to student thinking in the moment and facilitate rich mathematical discussions. In this study, three participants in lesson study, secondary geometry teachers, anticipated student responses and planned instructor responses for selected tasks. The researcher, then, observed the teachers enact the tasks and investigated how the anticipation of student responses influenced task enactment. The findings suggest that although participation in lesson study provides opportunities to anticipate student responses, this feature may not be sufficient in supporting teachers with enacting tasks in ways that facilitate open-ended discourse and maintain high levels of cognitive demand.

Keywords: lesson study, geometry, secondary

Some research suggests that when teachers attend to student thinking, they increase their content and pedagogical knowledge and implement student-centered instruction (Fennema, Carpenter, Franke, Levi, Jacobs & Empson, 1996). For instance, when teachers anticipate student responses, they can prepare to improvise instruction and facilitate higher levels of mathematical discourse and problem solving (Stein, Engle, Smith & Hughes, 2008). Participation in the lesson study process provides teachers with natural opportunities to focus on student thinking (Murata, Bofferding, Pothen, Taylor, & Wischnia, 2012). This study explored, in what ways do teachers anticipate student responses before teaching a lesson and respond to student thinking during a lesson?

Theoretical Framework

Lesson study has been found to improve teaching by increasing collaboration, improving content and pedagogical knowledge, and gaining insight to student thinking (Murata et al., 2012; Lewis, Perry & Hurd, 2009). The process of lesson study includes a small group of teachers using curriculum materials to plan a lesson, teach and observe the lesson, and reconvene to debrief and revise the lesson, and lesson study is often supported by knowledgeable others or experts in the field (e.g., school administrators or university professors).

During the planning phase of lesson study, teachers study curriculum to plan or revise a research lesson (Lewis et al., 2009). During this time, teachers focus on student thinking by anticipating possible student responses and instructor responses for tasks in the lesson. This is an important aspect in lesson study because the process of anticipating student responses during lesson study has supported teachers' in using generated student responses to encourage mathematical discussion and reasoning (Suh & Seshaiyer, 2014), and allowed teachers to think in advance about how to organize the sharing of student responses (Inoue, 2011).

Next, during the enactment phase of lesson study, a member of the lesson study team teaches the research lesson, and the rest of the team observes the lesson. During the lesson enactment, observers pay close attention to student thinking and collect data (e.g., student responses). The person teaching the lesson uses the anticipated student responses to influence how they will respond to student thinking during the lesson and strategically choose how to share student responses during a whole-class discussion (Lewis et al., 2009, Murata et al., 2012).

Methods

This study was carried out during the second year of a larger study *Proof in Secondary Classrooms* (PISC) (NSF; Award #1453493, PI: PI Michelle Cirillo) which focuses on developing an intervention for the introduction of proof in secondary classrooms. Three secondary teachers, Logan, Bruce, and Raven had experience teaching geometry for seven, six, and one year respectively. Logan and Bruce taught at the same private high school. Raven taught at a different private high school. For this sub-study, the teachers participated in lesson study as professional development for piloting 16 research-based lessons, focused on proof in geometry. Within the cycle of lesson study, teachers anticipated student and instructor responses for selected tasks in three different lessons. Then teachers enacted the lessons, and the researchers observed enactment.

Data Sources

Two representative tasks per lesson were selected as units of analysis (n = 17; 6 tasks for 3 teachers, excluding one skipped by Logan). The selected tasks addressed critical concepts that students must know to successfully engage with Euclidean geometry proof (i.e., investigating geometric concepts, drawing conclusions, and deductive structure). The teachers' anticipated student responses and planned instructor responses were audio-recorded and transcribed. The lesson enactments were video-recorded, and the selected task enactments were transcribed. The transcripts of the teachers' anticipated student responses and the task enactment transcripts were analyzed using open coding to look for patterns and themes.

Findings

Avoiding Misconceptions Before the Lesson

When the teachers engaged with the tasks and anticipated student responses, they often put forth how students would respond correctly. When teachers were asked how they would respond to correct answers, they said they would facilitate a discussion about correct responses. However, when asked about possible students' misconceptions, the teachers often indicated they would redirect students to a correct response. This report will focus on two tasks which exemplified the enactments of the tasks. For the first example (see Figure 1), students were asked to fold patty paper to explore angles formed by intersecting lines. Each teacher indicated that students would most likely form perpendicular lines and indicated they would facilitate a discussion about other possible responses as well as encourage students to explore other configurations to discover that vertical angles were congruent and adjacent angles were supplementary. In the second example (see Figure 2), the teachers indicated, they would expect the students to name vertical angle pairs or notice vertical angles congruency, and they would lead a discussion about those responses.

Student Task

- Fold the patty paper so that two intersecting lines are formed by the creases.
- Measure each of the angles with your protractor.
- Write a conjecture about what you notice.

Figure 1. Lesson 2, Task 1

Avoiding Misconceptions During the Lesson

When Logan and Bruce enacted the lessons, they increased the possibility that students would achieve anticipated correct responses by modifying tasks or asking closed or fill-in-the blank questions. For example, for *Lesson 2, Task 1*, Bruce first indicated he would want the students to avoid forming perpendicular lines. However, when Logan suggested leading an open-ended discussion about perpendicular lines as a special case, Bruce agreed with him. Yet when it came time to enact the task, Logan skipped the task due to time constraints, and Bruce modified the task by saying "*Fold the paper, so you can make two intersecting lines. Try not to make them perpendicular."* Bruce's modification of *Lesson 2, Task 1* limited the intended opportunities for students to explore the concept of vertical angles formed by intersecting lines and lowered the cognitive demand of the task which influenced limited classroom discussion and opportunities for Bruce to respond to student thinking in the moment.

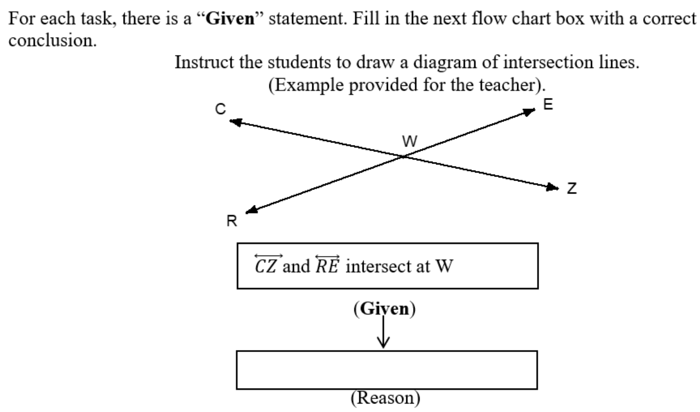

Figure 2. Lesson 7, Task 2

The following transcript from Logan's enactment of Lesson 7, Task 2 is representative of the closed and fill-in-the-blank questions that were observed.

Logan: So, guys, what could we conclude from this? John, what do you think?
John: Angle CWR and angle EWZ are corresponding angles
Logan: Are they called corresponding angles?
John: Uh, congruent.
Logan: They're congruent angles, right? … angle CWR is congruent to angle EWZ.

There were multiple possible responses for this task. However, instead of facilitating an open discussion, after one student responded correctly, Logan quickly moved to the next task.

Considering Multiple Student Responses During the Lesson

In contrast to Logan and Bruce**,** Raven was more likely to respond to student thinking as it occurred by considering multiple student responses and facilitating student discussion. In Raven's enactment of *Lesson 2, Task 1* she did not modify the task. As she predicted, most of the students folded the paper and formed perpendicular lines, so she facilitated an open-ended discussion about other possible ways to fold the lines as she had anticipated in her instructor responses. See the following transcript below for a sample of Raven's enactment of the task.

Raven: Just from observing them, what do you guys notice?
Students: Four right angles from several students. [Quietly]
Raven: Four right angles.
Christine: And they each have vertical angles.
Raven: There's vertical angles there. Ok…How many of you folded it, so you got for right angles, or it looks like four right angles. [About six visible students raise their hands] …Ok. Well, I want to talk about what Christine said for a second. Christine said there's two pairs of vertical angles formed… Do you think that happens every time you have intersecting lines? …
Mark: They don't have to be all congruent.

Jessica: Yeah, they don't have to all four be congruent.
Raven: Measure a pair of vertical angles on your paper and tell me if they're congruent…

Discussion

Previous literature (e.g., Fennema et al., 1996; Stein et al., 2008; Smith, 1996), has suggested that anticipating student responses might contribute to teachers' providing more student-centered instruction by facilitating mathematical discussions because they are prepared to respond to student thinking in the moment. However, previous studies (e.g., Smith, 1996; Wood, Cobb, & Yackel, 1991) have indicated that reform teaching has been difficult to implement because teachers are uncomfortable with releasing control, and mathematics teachers believe they must teach by telling because they believe the teacher's demonstrations are essential for student learning (Smith, 1996). While the teachers in this study indicated they would facilitate open-ended discourse, they often retreated to traditional instruction. It is also possible that anticipating student responses is not sufficient for changing teachers' instructional practices. Other features and conditions for teaching might influence teachers' decisions such as time for implementing a lesson, the cognitive demand required of the task, and teachers content and pedagogical knowledge. The design of this study created opportunities to look closely at cases of task enactment and how anticipating student responses during lesson study might influence teachers' attention to student thinking. Anticipating student responses is a key feature of lesson study, but future research might provide insight as to which features best support teachers.

Acknowledgments

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References

Fennema, E., Carpenter, T., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education*, *27*(4), 403–434. doi: 10.2307/749875

Inoue, N. (2011). Zen and the art of neriage: Facilitating consensus building in mathematics inquiry lessons through lesson study. *Journal of Mathematics Teacher Education*, *14*(1), 5–23. doi: 10.1007/s10857-010-9150-z

Lewis, C., Perry, R. R., & Hurd, J. (2009). Improving mathematics instruction through lesson study: A theoretical model and North American case. *Journal of Mathematics Teacher Education*, *12*(4), 285-304. doi: 10.1007/s10857-009-9102-7

Murata, A., Bofferding, L., Pothen, B., Taylor, M., & Wischnia, S. (2012). Making connections among student learning, content, and teaching: Teacher talk paths in elementary mathematics lesson study. *Journal for Research in Mathematics Education*, *43*(5), 616–650. doi: 10.5951/jresematheduc.43.5.0616

Smith III, J. P. (1996). Efficacy and teaching mathematics by telling: A challenge for reform. *Journal for Research in Mathematics Education*, *27*(4), 387–402. doi: 10.2307/749874

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, *10*, 313–340. doi: 10.1080/10986060802229675

Suh, J., & Seshaiyer, P. (2014). Examining teachers' understanding of the mathematical learning progression through vertical articulation during Lesson Study. *Journal of Mathematics Teacher Education*, *18*(3), 207–229. doi: 10.1007/s10857-014-9282-7

Wood, T., Cobb, P., Yackel, E., Wood, T., & Cobb, P. (1991). Change in teaching mathematics: A case study. *American Educational Research Journal*, *28*(3), 587–616. doi: 10.3102/00028312028003587

IMPLEMENTING MATHEMATICAL MODELING FOR EMERGENT BILNGUALS

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In this study, a researcher collaborated with a middle school mathematics teacher through co-planning, teaching and reflecting upon mathematical modeling lessons to support emergent bilinguals to make sense of mathematics. We investigate how this collaboration impacts the process of co-teaching and co-planning in terms of the interactions between the researcher and the teacher. Our analysis reveals how each role changed and maintained during the collaboration and what factors were discussed to develop and implement modeling-based lessons for emergent bilinguals.

Keywords: Emergent bilinguals, mathematical modeling, modeling based lesson, co-teaching

Purpose of the Study

The purpose of this research was to develop and implement mathematical modeling curriculum that effectively connects to Emergent Bilinguals (EBs; a.k.a. English language learners) through a teacher-researcher collaboration. The project originated from the misconception, yet another myth, that it is not appropriate to provide rigorous mathematics or word problems to EBs due to their English proficiency (Reeves, 2006). If EBs receive only easy tasks by this misconception, they would not have an opportunity to engage in problem-solving that cultivate a deeper understanding in high-level mathematics. Moreover, researchers (e.g., Chval & Chavez, 2011) have recommended making connections between mathematics and EBs' life experience to make sense of mathematics problems. In order to respond to these demands and recommendations, we designed this project between a teacher and a researcher in the co-development and co-teaching of mathematics lessons based on modeling problems that incorporate EBs' real-world contexts. The research question is: "How do teacher and researcher make decisions for developing and implementing a mathematics curriculum in a way of supporting EBs to make sense of mathematical modeling?"

Perspectives

Mathematical Modeling

Principles to Actions (National Council of Teachers of Mathematics, 2014) states an excellent mathematics program requires all students to have access to a high-quality mathematics curriculum and high expectations. Thus, we chose to use mathematical modeling tasks because they require students to do cognitively demanding activities and involve a real-world context that is related to students' cultural and life experiences. Mathematical modeling is also included in the standards of mathematical practice of the Common Core State Standards for Mathematics (NGA Center & CCSSO, 2010). Anhalt (2014) argued a well-designed mathematical modeling problem can engage EBs due to the real-world connection and the insufficiency of given information is necessary because a real-life problem does not provide all necessary information unlike a textbook word problem. This perception is echoed in how Dan Meyer created 3-act tasks that introduce a real-life story contacting conflict that does not include all necessary information and requires students to identify and search for the missing parameters. Meyer (2011) explained each step of the 3-act task: Act 1 is to identify the central conflict of a real-world story; Act 2 is to look for resources and develop new tools; and Act 3 is to resolve the conflict and set up an extension. We claim that it is necessary to add one more step when working with EBs because they may need further information to make sense of the story in Act 1. Therefore, the four actions are: (1) Make sense of the real-life story, (2) Identify the problem in the story, (3) Build a strategy and gather information, and (4) Resolve the problem and look for extension.

Design-based Research

The collaboration of a teacher and a researcher plays a crucial role in this project because their expertise complement each another's needs and build off the other's strengths. Hence, we employ multi-tier design-based research (DBR; Brown, 1992; Cobb, Confrey, Lehrer, & Schauble, 2003) in order to build an effective relationship among a researcher, a teacher, and also students. In this multi-tiered teaching experiment, a teacher acts as a researcher and teacher, and a researcher acts as a teacher, a learner and as an investigator, as they co-develop, co-teach, and analyze the lessons together (Jung & Brady, 2016).

Methods

Context and Participants

This project was conducted in one mathematics classroom in an urban middle school with 76.4% students of free and reduced lunch and 28.4% EBs, including many refugee students. There are more than 100 languages spoken in this district. There were 11 EBs in grades 7 and 8, consisting of 6 female and 5 male students in the EB-specialized mathematics class. The teacher had both mathematics (grades 5-12) and ELL (K-12) endorsement. She had 3 years of teaching experience when this project began and English monolingual. All EBs stayed in the U.S. within two years and their mathematics and English proficiency levels varied, but all scored 0 (lowest level) on the initial English language proficiency assessment conducted by the school district. Among 11 students, there were 3 Spanish speakers, 3 Swahili speakers, 2 Arabic speakers, 2 Burmese speakers, and 1 Karenni speaker.

Data Collection and Analysis

Applying the DBR collaborative research design, the researcher and the teacher met regularly for approximately one semester to co-develop modeling lessons and co-teach the lessons. All meetings were audiotaped, and the classroom teachings were videotaped. In every teaching session, the students were asked to write their work and reflect on their learning experience in a written form. The cycle for one lesson took two weeks: lesson planning in 1st week and teaching in 2nd week. After the 12 weeks, we interviewed five students about their experience and learning as well as an exit survey for the teacher. The data was analyzed qualitatively. Audio recordings and video recordings were transcribed when needed. The research team read all data multiple times and conduct open-coding (Strauss & Corbin, 1990) focusing on the research question. First, each coder identified the decision-making points in co-planning sessions and debriefings and examined who initiated each decision-making process because we believe the one who initiated a conversation with a new topic has a leading role. Based on the results of coding, codebook (Saldaña, 2013) was drafted, constructed by several generalizing and merging codes through several discussions. Then, a map of codes connecting them through phases such as planning, implementation, and debriefing was made to see how each decision-making impacts the following procedures of teaching. In addition, we investigated how the decision-making topics are repeated and addressed differently as time evolves. The multiple sources of data were triangulated.

Results

From the map with the topic of decision-making points during planning and debriefing sessions as well as teaching sessions, we identified the topics that appeared often and repeated in many sessions although the mathematical concepts and the tasks implemented were different. The topics we found in all lessons were how to group students and how to have students share their solutions. The topics we found in four or five weeks were task/context selection, making guiding questions for each Act, including the names students are familiar with in the story problem, assessing students' vocabulary understanding, using objects/manipulatives, timing for each Act, and integrating students' life experiences in the problems.

We found that most topics addressed in planning phases appeared in the following teaching phases but there was some variation. For example, assessing students' understanding of the words embedded in the modeling problems were often discussed and planned how to help students learn those words. These plans were usually executed during teaching phases. In contrast, the plans about writing did not go as planned, especially during the beginning lessons. In the first lesson, the EBs took a long time to write their reflection so we had to postpone the closing of the lesson with completing the writing. Based on reflecting on this factor during debrief, the student's reflection journal form was modified several times to make it more open-ended and provide more guidance. Grouping students was another topic that was discussed in all lessons. Same language speaker groups were used in the first lesson and mixed-level groups were used in the second lesson based on observation of student-student interactions during the first lesson result. After these two weeks, students chose their groups until the last lesson because this group choice seemed to make the EBs work more comfortably. Similar to grouping students, having students share their solution was found in each lesson. However, like writing a reflection journal, implementing this plan was challenging due to time constraints because EBs needed an additional time to solve the problems. After having this challenge a few times, a decision was made to have student's presentation in the beginning part of the lesson to share their process earlier opposed to waiting for EBs to find their solutions. We inferred the high frequency of some topics reveals its importance in terms of teaching mathematics for EBs.

Discussion

Our results demonstrate the teacher-researcher collaboration model allows both the researcher and the teacher to participate in and contribute to lesson planning. The repeated topic analysis showed what the researcher and the teacher consider important factors of developing effective lessons for EBs. We found the majority of repeated topics concentrated on Act 1: Make sense of the real-life story and Act 4: Resolve the problem and look for extension. Act 4 also includes students' sharing of their solutions and Act 1 is one of the reasons why modeling is an effective way to teach EBs mathematics (Anhalt, 2014). This result is well aligned with what the teacher explained in the exit survey.

> ELs [EBs] need digestible bites, especially on new concepts. I think the most important steps were Act 1 and Act 4. Without background knowledge, students do not have a platform to build off of. With ELs [EBs], I believe you can make no assumptions of what they already know. Act 4 ties all of the language and mathematical reasoning together to one final concept. In Act 4, students share out their reasoning and final answers as a group. They are also given an extension problem to go above and beyond the concept we had planned for the day. At the very end, students reflect on what they learned throughout the lesson through pictures or sentences in English or their native language. This ensures students are on the same page as they leave class that day.

We found one crucial factor from the discussion around timing. Timing was the one particularly discussed in the beginning lesson planning. The main topic about timing was how much we should invest for Act 1 because it seemed this step sometimes needed a large portion of the entire lesson period. The goal of Act 1 was to provide contextual clues (Cummins, 2000), connect with students' life experience (Aguirre et al., 2013), and support students' understanding of words embedded in the main problem before the problem is given. The researcher and the teacher discussed how many minutes they should plan for this step and decided to set sufficient time and agreed to extend the time if needed with the belief that it would take a longer time if the EBs do not fully understand the problem (Jackson, Garrison, Wilson, Gibbons, & Shahan, 2013). Once EBs understand the problem, the rest of the procedures can be accelerated. We found the researcher-teacher collaboration model sheds light on the important factors to design a lesson for EBs and this can be an effective form of teacher professional development.

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References

Aguirre, J. M., Turner, E. E., Bartell, T. G., Kalinec-Craig, C., Foote, M. Q., Roth McDuffie, A., & Drake, C. (2013). Making connections in practice: How prospective elementary teachers connect to children's mathematical thinking and community funds of knowledge in mathematics instruction. *Journal of Teacher Education*, *64*(2), 178–192. https://doi.org/10.1177/0022487112466900

Anhalt, C. O. (2014). Scaffolding in mathematical modeling for ELLs. In *Common Core State Standards in Mathematics for English language learners* (pp. 111–126). Alexandria, VA: TESOL International Association.

Brown, A. L. (1992). Design experiments: Theoretical and methodological challenges in creating complex interventions in classroom settings. *The Journal of the Learning Sciences*, *2*(2), 141–178.

Chval, K. B., & Chavez, O. (2011). Designing math lessons for English language learners. *Mathematics Teaching in the Middle School*, *17*(5), 261–265.

Cobb, P., Confrey, J., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, *32*(1), 9–13.

Cummins, J. (2000). *Language, power, and pedagogy bilingual children in the crossfire*. Tonawanda, NY: Multilingual Matters Ltd. Retrieved from http://search.ebscohost.com/login.aspx?direct=true&scope=site&db=nlebk&db=nlabk&AN=80547

Jackson, K., Garrison, A., Wilson, J., Gibbons, L., & Shahan, E. (2013). Exploring relationships between setting up complex tasks and opportunities to learn in concluding whole-class discussions in middle-grades mathematics instruction. *Journal for Research in Mathematics Education*, *44*(4), 646–682. https://doi.org/10.5951/jresematheduc.44.4.0646

Jung, H., & Brady, C. (2016). Roles of a teacher and researcher during in situ professional development around the implementation of mathematical modeling tasks. *Journal of Mathematics Teacher Education*, *19*(2–3), 277–295. https://doi.org/10.1007/s10857-015-9335-6

Meyer, D. (2011, May 11). The three acts of a mathematical story. Retrieved from http://blog.mrmeyer.com/2011/the-three-acts-of-a-mathematical-story/

National Council of Teachers of Mathematics. (2014). *Principles to actions: ensuring mathematical success for all*. Reston, VA: The National Council of Teachers of Mathematics, Inc.

National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Washington D.C.: National Governors Association Center for Best Practices, Council of Chief State School Officers. Washington, D.C.

Reeves, J. (2006). Secondary teacher attitudes toward including English-language learners in mainstream

TEACHERS' INSIGHTS ON EARLY MATH EDUCATION AS A WAY TO CONNECT WITH REFUGEE CHILDREN

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As classrooms become more diverse and include refugee children who bring a wide range of educational and life experiences, one enduring challenge is about how to meet the changing pedagogical and mathematics learning demands for all children. In this study, we draw on a RefugeeCrit framework and particularly focus on teachers' perspectives on the interplay of culture, language, and early mathematics in the teaching and learning of Syrian refugee children in Turkey. In our 2-year exploratory project MIGRA-MATH we focus on mathematical patterns and bring together four implementation components, each of which require the consideration of the needs of teachers working with refugee students. In the initial phase of our project, one preliminary trend we see is the teachers' ideas about what mathematical patterns as a content offer children's algebraic thinking, and teachers' previous multicultural teaching experience, along with their own personal histories as diverse students before, seem to shape the way they support refugee children in mathematics.

Keywords: equity in mathematics education; teacher education; refugee children

Introduction

As classrooms become more diverse and include refugee children who bring a wide range of educational and life experiences, one enduring challenge is about how to meet the changing pedagogical and mathematics learning demands for *all* children. Barton (2017) has highlighted the ethical responsibilities inherent in this challenge for mathematics educators, arguing, "The extent to which we free mathematics and mathematics education from society and culture is the extent to which we are absolving ourselves from responsibility to others and to our world. It frees us from social and cultural responsibility. Ultimately, this makes us amoral (p.3)." There is a need for timely research examining mathematics education practices situated in particular societal and cultural contexts with refugee students around the world to inform our work as mathematics educators and researchers. In this paper, we introduce the *MIGRA-MATH* Teacher Professional Development Project *(MIGRA-MATH: Supporting Teachers of Refugee and Immigrant Students with Respect to their Mathematics Education Professional Practices)* which addresses the current needs of early childhood teachers teaching mathematics in Turkey. Turkey hosts the world's largest community of displaced Syrians and currently has introduced several implementations to accommodate the educational needs of almost one and a half million school-aged Syrian children (United Nations High Commissioner for Refugees [UNHCR], 2018). In this study, we particularly focus on teachers' perspectives on the interplay of culture, language, and early mathematics in teaching and learning of Syrian refugee children in Turkey and the sociocultural role of mathematics education in refugee children's schooling process.

Purposes of the Study

The overall goal of the MIGRA-MATH project is to support the teaching of mathematics, in particular about mathematical patterns as a content area, for early childhood teachers (K-3) working with refugee students. The MIGRA-MATH project emphasizes the idea of multiple representations of mathematical knowledge by highlighting the visual representations of mathematical concepts, relations, and processes to make them more accessible to linguistically diverse children (Ainsworth, 2006; Lesh, Post, & Behr, 1987); the integration of collaborative methods in mathematics learning (Featherstone et al, 2013); and the promotion of school-family-community partnership (Civil, 2002), with a goal eventually to develop a pedagogical model based on the curriculum materials that are developed with teachers. This 2-year exploratory project brings together four implementation components, each of which require the consideration of the needs of teachers working with refugee students: (1) Teacher Knowledge Sharing Rounds; (2) Model Teaching and Classroom Observations; (3) Multivocal Video-Cued Interviews; (4) Family Math Workshops.

Theoretical Perspectives

In this paper we draw on refugee critical race theory (RefugeeCrit) (Strekalova-Hudges & Turner-Nash, 2017). RefugeeCrit is a newly evolving theoretical framework, which builds on critical race theory (Ladson-Billings & Tate, 1995). It offers a perspective to understand the unique lived experiences of children and families with refugee backgrounds, which is missing in current critical theoretical examinations. RefugeeCrit challenges the marginalized images of refugee children and families as suffering, needy and helpless, which ultimately serve as barriers to equitable education, and aims to produce counter-arguments about the problematic positioning of refugee children and families (Strekalova-Hudges, Erdemir & Turner-Nash, 2017). In our study with the teachers of Syrian refugee children in public schools in Turkey, the different tenets of RefugeeCrit supported us to interpret teachers' discourses in terms of how they might recognize the refugee status of children as a mechanism of oppression with a recognition of the intersectionality of children's experiences; how teachers see potential in early mathematics to create educational praxis for refugee children; and how they recognize the possible challenges for multilingual refugee children to learn mathematics through the current educational practices in Turkish schools.

Methods of Inquiry

We work with a total of forty K-3 grade teachers from five different cities which have the highest number of Syrian refugee children in the country. Teachers are randomly assigned to the project on a voluntary basis by the Ministry of National Education. In this paper, we report data from the first year of the project, particularly from the Teacher Knowledge Sharing Rounds where teachers participated in three different interactive sessions about (1) Multiple representations of mathematical knowledge with a particular emphasis on patterns; (2) Raising cultural awareness about refugee children and families; and (3) The interplay between language and mathematics. Throughout the project, we utilize a quasi-experimental mixed-method research design (Tashakkori and Teddlie, 2003) with a teacher action research component (MacIntyre, 2000). The knowledge sharing rounds phase of the research was also designed to incorporate teachers perspectives from the start in planning and designing the professional learning activities. We collected data during three Teacher Knowledge Sharing Rounds through semi-structured focus group interviews with teachers and we videotaped the three interactive sessions in each of these three Teacher Knowledge Sharing Rounds. The questions in the interviews particularly focused on how teachers view mathematics as a bridge to reach refugee children, and what their ideas about refugee children's learning of mathematics in the classrooms.

Results

Our preliminary analysis shows that teachers express struggles in teaching mathematics to refugee children resulting from both the structural barriers and their own professional readiness. However, teachers clearly position refugee children in their classrooms as capable learners of mathematics; and acknowledge the role that language plays in children's mathematical understandings and in the mathematical patterns as a content area. The strategies they developed to support children in mathematics vary. One trend we see is the teachers' ideas about what mathematical patterns as a content offers children's algebraic thinking and teachers' previous multicultural teaching experiences along with their own personal histories as diverse students seem to shape the way they support children in mathematics. Here we share examples of teachers' such notions generated in Teacher Knowledge Sharing Rounds:

Table 1. Teachers' Notions about Teaching Mathematics to Refugee Children

Overall Notions About Working with Refugee Children	One problem is the fear of to be left alone… We avoid differences because of the fear of being alienated from the rest of the society... The way we were taught to teach homogenizes the classrooms. We want all our children to learn mathematics very well, but the structure in the society does not allow for this. We want all students' success, but in real life there are structural barriers to achieve this.
	Because learning is not independent from language, you can't separate language from learning. However, a Kurdish child who does not speak Turkish and a Syrian child who does not speak Turkish are not the same. I am also Kurdish... A Kurdish child and I watch the same TV channels in Turkish language, but Syrian child doesn't have access to such media… Kurdish children (in his classroom) at least have family members who are fluent in Turkish but the Syrian child's family also can't speak Turkish.
	I told the family that not being able to speak Turkish will not prevent your child from learning [math]. I did my best to prevent marginalization in my classroom. The idea of we are all human beings, diverse languages are resources…. I started by assigning responsibilities to the refugee children in my classroom.
Mathematical Patterns: Content	Children learn patterns quickly, I, for example, show them tiles and see that they can get it, or I take them out to play hopscotch, we have shapes on the ground, you know, I make them count and jump… This is how I teach patterns... but when it comes to number patterns, I am not sure if they see the relation in 1,4,7
	I do love math. I do and I also tell my colleagues to engage children in skip counting a lot because it will set the foundation for other mathematical learning. For example, when we play a game, we do this (she produces a pattern by clapping), then I add a sound… It starts with sound, continues with ornamentation [refers to patterns in them] and then it lends itself to numbers, that's the direction it is supposed to go. We need to touch, we need to include real life materials. In this way you get amazing results.
Content Related Language	We do not know their language, for example I do have Syrian children in my classroom, we do not have a smooth communication and we are having difficulty in teaching mathematics.
	Math has a universal language. For example, I realize that when we do addition, Syrian children in my classroom show interest and attend. They do not need to know language.
	This is something I do not only observe with Syrian children but also with other students in my classroom. In a problem about patterns when a word like"in a row" appears, I see that children do not understand the meaning of "in a row". Then, I try to make it concrete for them and I make them to get up and stand like "in a row."

Conclusion

Considering children influenced by the current global refugee crises, there is a growing research literature focused on their access to general educational services (Uyan-Semerci & Erdogan, 2018; Tarim, 2018) and psychological implications for them (Sirin & Aber, 2008). However, we have much to explore in order to understand how content areas like mathematics can offer to support children's transitions to educational contexts in their host countries. Teacher professional development projects similar to one we share here help us to imagine new ways to support teachers and reveal their current exemplary practices in teaching mathematics to refugee students.

Acknowledgements

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References

Ainsworth, S. (2006). DeFT: A conceptual framework for considering learning with multiple representations. *Learning and Instruction, 16,* 183-198.

Barton, B. (2017). Mathematics, education, and culture: A contemporary moral imperative. In G. Kaiser (Ed.), *The Proceedings of the 13th International Congress on Mathematical Education* (pp. 35–43). New York: Springer.

Civil, M. (2002). Culture and mathematics: A community approach. *Journal of Intercultural Studies, 23*(2), 133–148.

Charmaz, K. (2006). *Constructing grounded theory: A practical guide through qualitative analysis.* Great Britain: SAGE publication.

Enochs, L., Smith, P. L. & Huinker, D. (2000). Establishing factorial validity of the mathematics teaching efficacy beliefs instrument. *School Science and Mathematics, 100*(4), 194–202.

Featherstone, H., S. Crespo, L. M. Jilk, J. A. Oslund, A. N. Parks, and M. B. Wood. (2013). *Smarter Together! Collaboration and Equity in the Elementary Math Classroom.* Reston, VA: National Council of Teachers of Mathematics.

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier, (Ed.), *Problems of Representations in the Teaching and Learning of Mathematics* (pp. 33-40). Hillsdale, NJ: Lawrence Erlbaum.

Macintyre, C. (2000). *The art of action research in the classroom.* Great Britain: David Fulton Publishers.

Maxwell, J. A. ve Miller, B. A. (2008). "Categorizing and connecting strategies in qualitative data analysis" (s. 461-477). Handbook of emergent methods. Editors: Hesse-Biber, S. N.,Nagy, S. ve Levy, P. New York: Guilford Press.

Sirin, S. & Aber, J.L. (2018) Increasing understanding for Syrian refugee children with empirical evidence, *Vulnerable Children and Youth Studies, 13*(1), 1-6, DOI: 10.1080/17450128.2017.1409446.

Strekalova-Hughes, E., Erdemir, E., Nash, K. (2017, October). Toward a Refugee critical race theory in education. International Reconceptualizing Early Childhood Education Conference, Toronto, Canada.

Strekalova-Hughes, E. & Nash, Kindel. (2017, September). Toward a Refugee critical race theory in early childhood education. European Early Childhood Education Research Association (EECERA) Conference, Bologna, Italy.

Tarım, S.D. (2017) Breaking Barriers with Collaborative Language Practices in a Multiethnic Classroom: A Potential Model for Immigrant Children, *European Education, 50(1), 27-41.*

United Nations High Commissioner for Refugees. (2018, January, 18). *Syria regional refugee response.* Retrieved from http://data.unhcr.org/syrianrefugees/country.php?id=224.

Uyan-Semerci, P. & Erdoğan, E. (2018) Who cannot access education? Difficulties of being a student for children from Syria in Turkey, *Vulnerable Children and Youth Studies,13,* (1,) 30-45, DOI: 10.1080/17450128.2017.1372654.

Ponterotto, J. G., Baluch, S., Greig, T., & Rivera, L. (1998). Development and initial score validation of the teacher multicultural attitude survey. *Educational and Psychological Measurement, 58*(6), 1002-1016.

Tashakkori, A. & Teddlie, C. (2003). *Handbook of mixed methods in social & behavioral research.* Thousand Oaks, CA: Sage.

ONE DISTRICT'S SYSTEMIC APPROACH TO DETRACK HIGH SCHOOL MATHEMATICS

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Systemic implementation of research-based instructional practices has persistently been a challenge in high school mathematics. The Escondido Union High School District is pursuing ambitious mathematics instructional reform with an aim to rehumanize the teaching and learning experiences for both students and teachers. I provide an overview of a four-year teacher curriculum that emerged during an ongoing ethnographic study of their work, and the many tensions provoked by the goal to rehumanize mathematics experiences. Specifically, I note the tensions of an effort to rehumanize the work of students and teachers, through a professional develop strategy undergirded by an effort to act teachers into new beliefs.

Keywords: Teacher Education-Inservice/Professional Development, Teacher Beliefs, High School Education, Equity and Diversity

Systemic implementation of research-based instructional practices has persistently been a challenge in high school mathematics (NCSM, 2014). There seem to be many barriers for the systemic and sustained shifts in high school mathematics organization and instruction. Understanding the navigation of these barriers in successful school systems will inform other schools and districts to make meaningful improvements to mathematics instruction.

The Escondido Union High School District (EUHSD) is pursuing ambitious mathematics instructional reform with an aim to rehumanize the teaching and learning experiences for both students and teachers. Realizing experientially and empirically (Cobb & Jackson, 2011) that lasting mathematics instructional improvement at scale is a challenge extending beyond teacher learning, mathematics leadership in EUHSD has doggedly moved forward to attend to both teacher and institutional learning. In this brief presentation, I will describe current state of a five(plus)-year project to shift mathematics instruction by working to redefine teacher's normative identities. From these efforts, I provide an overview of a four-year teacher curriculum that emerged during the work, and the many tensions provoked by the goal to rehumanize mathematics experiences. Specifically, I note the tensions of an effort to rehumanize the work of students and teachers, through a professional develop strategy undergirded by an effort to act teachers into new beliefs.

The findings I propose to share emerged from an ethnographic study of the district's efforts since they were launched in 2013. During that time, I served as an advisor and coach to district leadership, facilitator for leadership team projects, and instructor for some of the professional development workshops organized for mathematics teachers. I visit the schools in the district, teachers, principals, and district administrators five times each year for roughly 15 days. Further, I am in contact weekly with the district mathematics specialist and lead mathematics coach. I document these interactions and visits in a journal; text and email conversations are logged, and I collect and store documents and artifacts of the team's work.

EUHSD is a predominantly Latin@ district (72%) consisting of 5 high schools. The vast majority of the school and district leadership is white, as are the teachers—just over 7% of the district's teachers identify as Latin@. Student outcomes reflect the many challenges associated with urban schools. For example, in 2012 only 33.5% all 12th-grade graduates completed the specific college-preparatory coursework with a grade of "C" or better necessary to qualify to apply to the in-state public universities. Mathematics instruction in EUHSD prior to 2014 reflected unproductive beliefs about teaching, learning, assessment, and children (NCTM, 2014), including a focus on procedures and memorization, expectation for student to mimic teacher thinking, the notion that students possess different innate levels of ability in mathematics, unchangeable, and predicable by race or economic group. Such beliefs were held by many teachers, but even more strongly by many other members of the school community, including counselors, principals, students, and many parents. Similarly, unproductive beliefs could be found in structural elements of mathematics education in the district, including tracking students into more than six possible 9th grade mathematics classes, D-F rates in Algebra hovering around 50%, and student demographics in honors and AP classes not matching those of the district.

A determined effort to rehumanize the mathematics experiences for both students and teachers began in 2013, built on a theory of change to act district educators into new beliefs, to develop new normative identities (Cobb, Gresalfi, & Hodge, 2009). Since then, structural elements of mathematics instruction have shifted, including opening classrooms, detracking, and re-professionalizing teachers. The underlying design for system change was to draw upon a solid student curriculum to focus on developing new pedagogical strategies. Specific elements the design for change included:

1. Structuring schools for adult learning,
2. Curriculum as a lever for change,
3. Teacher beliefs,
4. Changing inequitable structures, and
5. New vision and measures.

Structures were utilized in conjunction with the teacher curriculum with an intention to act teachers into new beliefs. After initial teacher development efforts, an earnest four-year teacher curriculum was developed, grounded in work emerging from previous projects (Robinson et al., 2002), to support teachers to shift classroom structures to center on student-student discourse about student-generated ideas. The principles underlying the four-year teacher curriculum included:

1. Aligned to teacher implementation of new textbooks, sequentially;
2. Built upon (predictable) problems of practice;
3. Paired with expectations designed to act teachers into new beliefs; and
4. Foundation: people learn by thinking & interacting, detracked classrooms.

The foci in each year of the teacher curriculum include the list that follows. Table 1 provides a more complete summary of the four-year teacher curriculum that has emerged in the work at EUHSD.

- Year 1: group work and student discourse as the classroom norm
- Year 2: creating intellectual need through a launch, and Complex Instruction
- Year 3: disrupting deficit orientations, and assessment versus grading

Table 1: Four-Year Teacher Curriculum developed at EUHSD

	Problem of Practice	Opportunities	Structures	What was Learned
Y1	I'm thinking about teaching differently (SMPs), and I need to learn to teach from this new textbook.	Learn a new role for themselves in the classroom	Students in groups, students present, no supplementing	I have a very different role in this classroom; learning to listen to student's mathematics; there is value to kids thinking and interacting; I need to manage student discourse
Y2	I need to figure out how to make my groups work better	C.I. to address deficit orientation to children; explicit conversation about social inequalities replicated in the classroom; routines for supporting groupwork & discussion	C.I. Multiple Ability Treatment, status treatments; skill builders; whiteboards	I need to include all students in conversations; a way to reconcile their deeply held beliefs that all children are smart with why not expressed in the classroom
Y3	I now see learning and understanding differently. How do I grade?	Shift from answer-getting environment to focus on understanding	Invitation to grade differently, reorganize pacing around big ideas, collect and provide feedback on student work	Effort in Years 1 & 2 pay off (faith); some sort of competency-based grading & student ownership is important; and feedback important
Y4	This work is complex and interconnected. I understand equity issues differently.	Divergent equity interests; leadership; re-imagining assessment	Implement revised, district-wide unit exams (toward understanding instead of procedures)	

Alongside these efforts toward both institutional and teacher learning toward a pursuit of ambitious mathematics instruction come many tensions for those involved in the project. These can be classified into categories aligned with institutional and teacher learning, as well as deeply held societal beliefs that serve to maintain White Supremacy. Samples of these tensions include:

- Acting teachers into new beliefs: *Is this a coercive approach to change?*
- Deficit orientations and racism: *Are we doing enough to change hearts and minds? Where is the lever for change?*
- Curriculum as a lever for change: *Limitations of the chosen curriculum. Will we be able to move beyond this curriculum in ways that are more student centered?*
- Sustainability: *How can we ensure the work lives on and continues to develop?*
- Measures: *Will standardized testing corrupt our efforts to rehumanize school mathematics? What measures would reflect our values? And 'whose' values, if we are doing this work in a de-humanizing system*?

Across all categories underlies a challenge about beliefs, and although EUHSD may develop high quality mathematics teaching, deficit-oriented beliefs will continue to generate inequitable outcomes (Jackson, Gibbons, & Sharpe, 2017) and dehumanizing experiences.

References

Cobb, P., & Jackson, K. (2011). Towards an empirically grounded theory of action for improving the quality of mathematics teaching at scale. *Mathematics Teacher Education and Development, 13*(1), 6–33.

Cobb, P., Gresalfi, M. & Hodge, L. L. (2009). An interpretive scheme for analyzing the identities that students develop in mathematics classrooms. *Journal for Research in Mathematics Education, 40*(1), 40–68.

Jackson, K., Gibbons, L., & Sharpe, C. J. (2017). Teachers' views of students' mathematical capabilities: Challenges and possibilities for ambitious reform. *Teachers College Record, 119*(7), 1–43.

National Council of Supervisors of Mathematics. (2014). *It's TIME: Themes and imperatives for mathematics education.* Bloomington, IN: Solution Tree.

National Council of Teachers of Mathematics. (2014). *Principles to Actions.* Reston, VA: NCTM.

Robinson, E., Fendel, D., Fraser, S., & Klanica, J. (2002). *Elementary, secondary, and informal education: IMP for the 21st Century*. National Science Foundation Award #0137805. Washington, DC.

MEASURING TEACHER KNOWLEDGE OF TEACHING FOR ALGEBRA READINESS

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This study is based on a year-long professional development program aimed at improving teachers' knowledge for teaching algebra readiness. Thirty-one 7th and 8th grade teachers participated in the study, and survey data and artifacts were analyzed. Results suggest that (1) teachers' knowledge about algebra readiness indicators mostly aligned with research-based indicators, but they focus too much on pre 7th grade math concepts; (2) teachers have some misunderstanding about the content progression; and (3) teachers view teaching algebra readiness requires understanding students' level, standards, instructional approaches, use of resources, real world connections, and algebra content.

Keywords: Teacher knowledge, Algebra readiness

Objectives of the Study

Algebra has become a core component of school mathematics, and students are required to take algebra courses to meet high school graduation requirements. Consequently, helping middle school students prepare for Algebra has gained critical importance. Questions naturally arise about how best to prepare students for entry into Algebra and how middle school teachers can be prepared to have students ready to learn Algebra. The professional development (PD) program of this study is a year-long program aimed at deepening middle school teachers' content and pedagogical content knowledge for algebra readiness. The purpose of this paper is to explore the impact of the PD on teachers' knowledge for teaching algebra readiness with three guiding questions: 1) What skills and knowledge do teachers perceive as algebra readiness indicators?; 2) To what extent does teachers' understanding of pre-algebra and the algebra curriculum align with curriculum documents at the national/state level standards and frameworks?; and 3) What are teachers' perceptions about the knowledge needed for teaching algebra readiness?

Theoretical Framework

According to research, a conceptualization of learning improves a teacher's mathematical content knowledge and deepens their understanding of student learning, allowing them "to position their students' learning not only in relation to their current classes and the objectives for that cohort, but also in relation to prior and subsequent classes." (Heritage, 2008, p.3) The theoretical framework of the study is based on theories on teachers' knowledge for teaching algebra and frameworks on teachers' knowledge for algebra readiness.

Researchers have defined and categorized teachers' mathematical knowledge and endeavored to measure teachers' knowledge for teaching mathematics. Shulman has categorized teachers' knowledge (1986) into three areas: subject matter knowledge, pedagogical content knowledge, and curricular knowledge. Based on Shulman's categories, researchers have conceptualized and measured "the mathematical knowledge used to carry out the work of teaching mathematics" (Hill et al., 2005, p. 373) and called it Mathematical Knowledge needed for Teaching (MKT). MKT categorizes practicing teachers' knowledge (Blömeke & Delaney, 2012) into three areas: knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum.

As school mathematics deals with numerous topics, some researchers differentiate teachers' knowledge for teaching by math topics. Floden and McCrory (2007) focused on school algebra in grades K-12 and proposed three areas of knowledge of algebra for teaching (KAT): school (K-12) mathematics knowledge, advanced mathematics knowledge, and teaching mathematics knowledge (Huang, & Kulm, 2012). Unlike other researchers' broad knowledge categories, Even (1990) described teachers' knowledge of algebra in seven dimensions: essential features, different representations, alternative ways of approaching, the strength of the concept, basic repertoire, knowledge and understanding of a concept, and knowledge about mathematics. Agreeing with existing teachers' knowledge categories, Doerr (2004) added knowledge about student errors and misconceptions in algebra to Shulman's pedagogical content knowledge.

Such concepts, essential for success in an Algebra course, have been outlined by different entities across the nation. The Southern Regional Education Board (Bottoms, 2003) outlines both dispositional skills and mathematics content (readiness indicators) suggested for students to be successful in Algebra. While the state of Ohio, where this study was conducted, does not outline specific algebra readiness indicators, their standards, similar to those of Common Core State Standards for Mathematics (2010), have designated critical areas of focus for both 7th and 8th grade mathematics. These outlined concepts provide the foundation for teachers to determine mathematical content in which students should have strong foundational skills to successfully embark on an Algebra course.

Methods

This study was situated in the context of ongoing work of a year-long PD for 5 high-need school districts. The PD involved an intensive summer institute, follow-up blended PD with structured modules, on-site support from building coaches, site visits by PD providers, and professional learning dissemination at professional meetings. The PD activities covered investigating algebra readiness indicators and vertical progression of algebraic concepts, analyzing students' learning. Thirty-one 7th/8th grade teachers participated in the study. Of the 31 participants, 84 % were female and 97 % identified as white, non-Hispanic. The cohort was comprised mostly of experienced teachers, with 10 years the mean.

The study involved surveys, self-reflections, and PD artifact data. Due to the nature of the qualitative data, the analysis was based on categorizing to investigate emerging themes. The responses were first analyzed for common codes to create patterns, then the codes were finalized by comparison with existing studies. Throughout the data analysis process, common codes and patterns were tallied and percentages of participants' responses were calculated. The initial analysis outcomes of the surveys and PD artifacts were compared with theories and indicators discussed in the Theoretical Framework section.

Results

Teachers' Knowledge about Students

Teachers provided over 90 areas that they use to define student readiness for algebra: knowledge of math content (66%), math skills (16%), disposition (10%), and scores of standardized tests (8%). Math skills needed to be algebra ready were mathematical reasoning, critical thinking, and abstract thinking skills. In addition to skills and knowledge, approximately 10% of indicators were about students' disposition: willingness to learn, struggle, and complete challenging tasks. Some teachers rely on a student's grade and standardized test scores to determine readiness. Most algebra readiness indicators (66%) were specific mathematics concepts; number sense, computations with whole numbers and integers, variables and unknowns, one or two step equations, patters and rules, and (linear) graphs (Figure 2). Approximately 50% of math concepts were number sense and computational skills with whole numbers and integers; the other half was directly related to algebra standards.

Overall, there was no big gap between national/state recommended algebra readiness indicators and indicators discussed by the teachers. However, the indicators favored the younger grade level standards over 7th – 8th grade content. For example, teachers emphasized operations with whole numbers and integers but not so much with rational numbers, dealing with parentheses accurately, distinguishing unknown variables from units, etc. The lack of 7th and 8th grade algebra standards in the list was unexpected, because most teachers in the program were teaching 7th, 8th, and algebra classes. However, teachers' heavy focus on number sense and operations for algebra readiness may have to do with national/state recommendations, "The algebra readiness materials must also break these 16 standards into their component concepts and skills, with a primary focus on developing students' mastery of arithmetic." (Papa & Brown, 2007, p. 17)

Teachers' Knowledge about Algebra Curriculum

The teachers were asked to identify which concepts were indicators of Algebra readiness and to specify the grade level at which the content should be covered (pre-7th, 7th, or 8th grade). Some teachers teaching 8th grade or algebra course eliminated concepts around data and probability. Their perspective was that while all the indicators should be learned before algebra, the data and probability content was not indicative of a successful algebra student. Regarding grade level, teachers were in complete agreement for 11 of the 33 mathematical concepts. With regard to the specific mathematical concepts, the content teachers unanimously assigned were fundamental number and algebraic understanding while the outliers and disagreement upon content focused on non-traditional algebraic content (probability, data, and geometry). The data indicates that the teachers were not able to agree or accurately assign grade-level content to concepts that they are required, by the state, to teach at that level. Another finding suggests that teachers may have a misunderstanding of the number system. Teachers were in agreement about the need for fundamental skills associated with computation with fractions and irrational numbers, however, they assigned rational numbers to all grade levels. These findings may suggest that teachers need additional support with the rational number system. Disagreement over when a concept should be taught spanned over geometry, patterns, and number skills and concepts.

Teachers' Knowledge about Teaching Algebra.

The 67 knowledge areas needed for teaching algebra generated by teachers were categorized into 7 areas: understanding students' current level, their needs, and areas they struggle in (25%); general instructional approaches (21%); standards including content progression (19%), use of resources including manipulatives and technology (7%); and real world connections (6%). Only 10% of teacher comments specifically mentioned instructional approaches related to teaching algebra. Instead of providing specific algebra concepts, teachers simply said "content" knowledge (10%). As Doerr (2004) suggests, our teachers also believed that understanding students is a critical part of pedagogical knowledge. Some of the knowledge areas, such as general teaching approaches and math content, were supported by Even (1990). Unlike other studies, our teachers felt that using resources, making connections to real-world situations, and understanding content progression were important for teaching algebra readiness concepts.

Conclusion and Discussion

The focus of this study was unique in two ways: First, we aimed to deepen the teachers' understanding of teaching algebra readiness in a way that was different from what they experienced in their teacher preparation program or as part of professional development. Secondly, instead of focusing on particular algebraic concepts, we emphasized the nature of shifts in teachers' understanding of the progression of algebraic concepts and perspectives of teaching for algebra readiness.

Trends in data among teacher knowledge of students, algebra curriculum and teaching algebra, suggest that more efforts be made to support teachers in preparing students for algebra. When identifying indicators for student readiness to take Algebra, teachers largely pointed to mathematics content below their respective grade levels (number sense and skills). When this specific content was to be covered was in agreement by groups and aligned to the respective state standards, however, discrepancy existed among groups as the content progressed to higher levels of understanding, with disagreement in fundamental algebraic concepts such as rational numbers and patterns. Despite the lack of agreement on when the progression of content occurs, teachers failed to cite specific content knowledge as an additional teacher requirement in order to prepare students for Algebra. Teachers did identify a need for further understating of standards, but overall, suggested there is a greater need for teachers to be better prepared to assess and instruct on a general level.

Given the findings, focusing on building teacher knowledge of the algebra curriculum is not enough, as teachers identified disposition, mathematical thinking, general assessment strategies, and instructional approaches should also be addressed in a professional development program. This further supports the need for a well-rounded experience that builds teacher knowledge of: students, algebra curriculum, and teaching algebra.

References

Blömeke, S. & Delaney, S. (2012). Assessment of teacher knowledge across countries: A review of the state of research. *ZDM 44* (3), 223-247.

Bottoms, G. (2003). Getting students ready for Algebra I: What middle grades students need to know and be able to do. *Atlanta, GA: Southern Regional Education Board*.

Common Core State Standards Initiative. (2010). *Common Core State Standards for Mathematics*. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Doerr, H. M. (2004). *Teachers' knowledge and the teaching of algebra*. In K. Stacey, C. Helen, & K. Margaret (Eds.), The future of the teaching and learning of algebra, The 12th ICM Study (pp. 267–290). Boston: Kluwer.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics, 21*, 521–544.

Floden, R. E., & McCrory, R. (2007, January). *Mathematical knowledge for teaching algebra: Validating an assessment of teacher knowledge*. Paper presented at 11th AMTE annual conference, Irvine, CA.

Heritage, M. (2008). Learning Progressions: Supporting instruction and formative assessment. Washington, DC.: Council of Chief state School Officers. Retrieved from http://www.k12.wa.us/assessment/ClassroomAssessmentIntegration/pubdocs/FASTLearningProgressions.pdf

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. American Educational Research Journal, 42(5), 371–406. doi:10.3102/00028312042002371.

Huang, R. & Kulm, G. (2012). Prospective middle grade mathematics teachers' knowledge of algebra for teaching. *Journal of Mathematical Behavior, 31*, 417– 430

Papa, R., & Brown, R. (2007). The research base for algebra readiness. Macmillan/McGraw Hill Glencoe.

Shulman, L. S. (1986). Paradigms and research programs in the study of teaching: A contemporary perspective. In Wittrock, M. C. (Ed.), *Handbook of research in teaching*

K-8 TEACHERS' STORIES OF MATHEMATICS-RELATED TRANSFORMATION

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We describe our use of the frameworks of teacher transformation and narrative inquiry to investigate K-8 teachers' self-reported transformations related to mathematics and teaching, and aspects of the transformations that are associated with participation in Intel Math professional development. This report is part of a larger, ongoing investigation, but we are able, in the space of this report, to highlight the transformative journeys of two participating teachers.

Key words: Teacher Education-Inservice/Professional Development

In this brief report, we share preliminary analyses of elementary mathematics teachers' stories of transformational learning experiences related, in part, to participation in the Intel Math professional development (PD) program. We seek to address the following questions:

1. What narrative structures do teachers employ to describe their professional journeys and professional transformations, especially as they relate to the teaching of mathematics? How do teachers situate professional development within their narratives?
2. To what specific experiences do teachers attribute the transformation? How do they connect experiences to one another and across time?

The PD program includes 80 hours of instruction designed to deepen and expand K-8 teachers' knowledge of mathematics and evidence-based classroom practices. The program includes extended opportunities to learn in a professional community with other teachers, experience as a learner with student-centered instruction, and explicit links to children's thinking/learning – which are key components of effective professional development (e.g., Darling-Hammond, Hyler, & Gardner 2017). Participating teachers solve non-routine problems, connect multiple representations, and communicate and analyze their reasoning. Pedagogical techniques used by program instructors mirror practices that participating teachers can use in their classrooms. In addition to significantly increasing teachers' content knowledge related to elementary mathematics, the program positively impacts classroom practice with respect to richness of mathematics, and student participation in mathematics (Garet, et al., 2016). Although these findings are promising, we know that shifts in practice take time, are often gradual and accompanied by shifts in beliefs, and can be uneven in their implementation (e.g., Loucks-Horsley et al, 2010). At the same time, we have evidence that, for some teachers, the program is part of a transformative learning experience that significantly impacts practice (e.g., Broatch, 2011). Thus, we seek to identify these teachers and examine their professional trajectories, and their views of how the program has played a role in their professional transformations.

Theory and Methods

Our conceptualization of transformative learning is based on the work of Mezirow (1991) who identified states of transformation beginning with a "disorienting dilemma" in which adult learners "discover a need to acquire new perspectives in order to gain a more complete understanding of changing events" (Mezirow, 1991, p.3). Learners engage in critical self-examinations whereby they explore new ideas, roles, and relationships, acquire new knowledge, and experiment and build competence with new ideas (Mezirow, 1991). Building on the work of

Mezirow and others, King (2009) developed and validated a survey instrument to identify whether learners have undergone a perspective transformation linked to educational experiences, and which learning activities contributed to their transformation (King, 2009). We use an adapted version of the instrument to identify teachers who have experienced a perspective transformation linked to the PD program. We then invite teachers to be interviewed to further investigate their transformational experience(s).

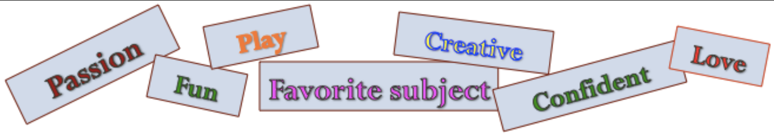
We use narrative inquiry methods in this research because it is "through narration, or storytelling, [that] individuals construct and present identities" (Temple, 2008, p. 3). Narrations are partial representations of experience (Riessman, 2008) that are jointly negotiated across speaker(s) and listener(s). "The plot of a narrative can be described as how narrators impose order on their experiences (Riessman, 1993) and . . . how analysts impose structure (Riessman, 2008)" (Ahmed, 2012, p. 235). "A structural narrative approach centering on plot can allow for the complexity of decision making processes, actions, and experiences . . . to be understood and placed in context" (Ahmed, 2012, p. 235). Consistent with a narrative inquiry perspective, we resist the urge to dissect teachers' stories into coded segments that would then be re-assembled to create meaning. Rather, our analysis of the audio-recorded stories involves careful (but not critical) examination of motives, meanings, and actions, and connections among these, as presented by the storyteller (teacher). To achieve this, we listen multiple times to the audio-recorded stories of the teachers, listening for, and making notes about, the ways teachers frame and build their stories, the turning points (explicitly described or implied) in their narrations, and the meanings they assign to transformational experiences. Once these have been identified, we re-present the teacher's experience in a narrative form based in written text. At this stage, we have interviewed eight teachers, with each interview lasting between approximately 30 minutes and one hour. From a narrative inquiry perspective, this is a very short time from which to develop a complete story of a teacher's experience, and we recognize this limitation in our work. We intend to conduct follow-up interviews with teachers, first to hear how teachers contextualize and assign new meanings to their transformational experiences over time, and also to share the stories we create to ensure that these stories resonate with teachers own meanings and experiences.

From the interviews we have conducted thus far, we have selected two to highlight in the space of this brief report. The first teacher, pseudonym Ginny, teaches second grade and has five years of teaching experience at the elementary level. The second teacher, pseudonym Fred, teaches third grade and has fourteen years of teaching experience at the elementary level.

Ginny

(all words are Ginny's unless in brackets or italics)

<table>
<tr><td colspan="2">I've always liked math, I always remember math being following steps, I've always been good at listening, and following directions, . . , but the second it came to reasoning and more critical thinking, I just didn't have a lot of experience, So I don't think I had any confidence with it,
I felt like this isn't math anymore, It's a different kind of thinking that I don't know how to do. That is why I started to shy away from it. . . . I enjoy teaching elementary math but it never made sense, it never had a place in my life anymore.</td></tr>
<tr><td colspan="2">Below, Ginny reflects on her experience in the PD program</td></tr>
<tr><td>I had on the shoes of the learner,
I experienced it.
You have to think,</td><td>I've heard before you need to ask higher order thinking questions and depth of knowledge and things like that,
I've heard about that, I've been taught about it before,</td></tr>
</table>

<table>
<tr><td>You have to think,
You have to figure it out.
Everyone has math to share!</td><td>plenty of times, but its never really like resonated with me before until I experienced it, . . . it was just like, Ok, this is how we think now. . . . I just came out thinking differently and I really don't know how that happened</td></tr>
<tr><td colspan="2">Below are words Ginny used to describe how she feels about math now</td></tr>
<tr><td colspan="2">
I have [had] the experience I want my kids to have. . . .[T]his year I looked back at [one] homework [from last year] . . . and I was like I can't even send that home, that makes no sense mathematically . . . that's a cheat, that's a shortcut . . . I'm just realizing that there was a lot of shortcuts I used last year and I was very test oriented . . . last year I'd have anchor charts with steps [for adding/subtracting on a number line] . . . this year its a lot more experimental and I have manipulatives out every day. . . . For the number line, instead of hops and walks, when they move ten on the number line we represent it with a ten rod so the kids are making connections between base ten blocks and counting on the number line because I understand that connection. I have a lot of new teachers on my team, and working with and explaining how I'm going to teach and how they are going to teach I realize I have a way deeper understanding than I did last year.</td></tr>
</table>

Fred

Fred hated math as a child. His memories of learning math include timed competitions in school and jealousy of the student who always won. "And he would walk up and get his two free books for winning that competition. And uh, I was always very frustrated by that." Fred's experiences as a student led him to think that math was a set of steps to be memorized. As a result, he still didn't like math as a beginning teacher, but wanted to teach it as best he could.

A few years later, Sarah Cruz (pseudonym) came to Fred's school as a new teacher. Sarah had been a finalist for a local educational foundation's math accolade. Fred figured that because of this distinction, she was probably pretty good at teaching math. "I actually went to her and I said, 'Listen, math has been a frustration for me for a long time, I'd really like to learn how to do it better.'" Sarah gave him some pointers and he started there, doing exactly what she had told him to do. That year, his students had the best improvement in math that he had ever seen. "I was just really quite delighted by the results. Because I actually enjoyed teaching math that year." That summer, Fred was invited to participate in the Intel Math program.

Fred: I went into it with the same mindset I had as a child. I was really nervous going in because I thought that it would really be a challenge for me. And that the other teachers who were around me were going to see right through me and notice that I'm not very good at this. But it was a real eye-opener for me because the math that they were doing was more logically based than it was memorization based. And, I'm pretty strong with logic. So I found myself, you know, essentially out-performing those who were at the same table from me—with me—and they were looking to me to get a better understanding of what was going on. Because a lot of these teachers were people who were teaching math the same way that I had learned math when I was a kid. I was obviously very very excited about it because a lot of the concepts that were being taught during this time, during the Intel Math training, were exactly what Sarah Cruz had been telling me the year before. And, so I realized that Sarah had, you know, a really good sense of what students needed in order to learn math. And, this Intel Math just really helped me take that to another level in my first year teaching third grade. And naturally, I had nice improvement with my math scores that year.

Fred's experiences learning from Sarah and participating in Intel Math helped him to reimagine math and math instruction as logically-based, instead of as a set of memorized facts.

Fred: I think the biggest thing that I have gotten out Intel Math is that if I really take it to heart and understand that each individual student is at a different place, and continue to work with my small groups … I can really, intimately understand where each of those students is, in their math learning. And I have a greater understanding of why my students are succeeding where others, who are learning how to do tricks with math—is what I call it now, when they learn to do those tricks, they're really not getting the fully number sense.

Fred's professional learning has been so transformative for his instructional practice, he uses the word "devastating" to describe a specific strategy he had previously taught.

Fred: One of the most devastating things that I did as a teacher—and I'm going back about 5 or 6 years now—was how I taught task analysis. Which was where the teacher says, 'These are the exact steps that you follow to solve this kind of math problem.' I took that as 'this is how you teach it.' However, task analysis is meant to be a guide. We need to give students choices in how they solve problems. To teach step-by-step procedures does not allow students to create their own ways of solving problems. And to me, that was just devastating. I think that might work for reading, but it just doesn't work for math because you're asking students to follow steps rather than teaching them to use their minds to figure out their own best way of solving these problems.

Conclusion

By creating a set of stories of teacher transformation, of which Fred and Ginny are two examples, we center teachers' voices, identities, and experiences, and the significance that they assign to different aspects of their professional journeys. At this stage in our research, we are intentionally experimenting with the forms we use to communicate these journeys. However, no matter the form, our goal is convey both the intellectual and affective domains of the stories, and ultimately consider how such stories can inform the work of teacher educators and others.

References

Ahmed, A. (2012). Structural narrative analysis: Understanding experiences of lifestyle migration through two plot typologies. *Qualitative Inquiry, 19*(3), 232-243.

Broatch, J. (2011). *Chandler Intel Math Academy (CIMA) Report.* Tempe, AZ: Arizona State University.

Darling-Hammond, L., Hyler, M. E., Gardner, M. (2017). *Effective teacher professional development*. Palo Alto, CA: Learning Policy Institute.

Garet, M. S., Heppen, J. B., Walters, K., Parkinson, J., Smith, T. M., Song, M., Garrett, R., Yang, R., & Borman, G. D. (2016). *Focusing on mathematical knowledge: The impact of content-intensive teacher professional development* (NCEE 2016-4010). Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education

King, K. P. (2009). The handbook of the evolving research of transformative learning based on the learning activities survey. Charlotte, NC: Information Age Publishing.

Loucks-Horsley, S., Stiles, K. E., Mundry, S., Love, N., & Hewson, P. W. (2010). *Designing professional development for teachers of science and mathematics*. Thousand Oaks: Corwin Press.

Mezirow, J. (1991). *Transformative dimensions of adult learning*. San Francisco: Jossey-Bass.

Riessman, C. K. (1993). *Narrative analysis* (Qualitative research methods series 30). Thousand Oaks, CA, SAGE.

Riessman, C. K. (2008). *Narrative methods for the human sciences*. Thousand Oaks, CA: SAGE.

Temple, B. (2008). Investigating language and identity in cross-language narratives. *Migrations& Identities*, *1*, 1-18.

UNDERSTANDING THE WAYS GRADUATE TEACHING ASSISTANTS LEARN TO TEACH THROUGH DISCOURSE IN PROFESSIONAL DEVELOPMENT

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Across the United States, there is increased national interest in improving the way mathematics departments prepare their Graduate Teaching Assistants (GTAs). We focus on GTAs engaged in professional development for active learning. We report preliminary results on the ways in which the understandings of GTAs of various teaching practices changed over a term. With this analysis, we contribute to a better understanding of how to support GTAs with their teaching.

Keywords: Graduate teaching assistants, professional development, teaching practices.

Many mathematics departments in the United States are changing the way they structure the teaching of the Calculus sequence based on the recommendations that emerged as a result of the Mathematical Association of America (MAA) study of successful Calculus programs (Bressoud, Mesa, & Rasmussen, 2015). One of the recommendations was to improve the professional development (PD) offered to the Calculus graduate teaching assistants (GTAs). Much of what we know about these PD programs arises from research focused on the various structures of PD programs, on the outcomes of the programs, or on a small, in-depth case study (e.g., Kung & Speer, 2009). In a review of the research, Speer, Gutmann, and Murphy (2005) stated the need for studies with longitudinal designs so as to "inform the design of exemplary programs that have a lasting influence on instructional practices" (p. 79).

We report on professional development at a large university in the southwestern United States whose mathematics faculty implemented several changes to the Calculus sequence, including to the structure of the PD program for the Calculus I and Calculus II GTAs. A large part of the PD involves formal meetings with mathematics education faculty and mathematics faculty course coordinators with a stated goal of supporting GTAs to teach with a more student-centred approach. In particular, the PD has a focus on supporting effective teaching practices, defined as "…what teachers do and think daily, in class and out, as they perform their teaching work" (Speer, Smith, & Horvath, 2010, p. 99).

The GTAs discussed teaching practices with each other, their course coordinators, and mathematics education faculty. We investigated the ways in which the discourse around various teaching practices evolved over time, so that we can better understand GTAs' learning. We share selected results in answer to: How do the GTAs' publicized understandings of various teaching practices change (through elaboration or transformation) over the course of a term? How do these publications highlight which teaching practices are most salient to their needs?

Background

There are a few studies done on the state of GTA professional development across the United States (e.g. Belnap & Allred, 2009; Bragdon, Ellis, & Gehrtz, 2017; Speer et al., 2010). In addition to these national level studies, there are also several case studies, with a focus on the structure of or the efficacy of the program (e.g., Griffith, O'Loughlin, Kearns, Braun, & Heacock, 2010; Marbach-Ad, Shields, Kent, Higgins, & Thompson, 2010). More recently, researchers are reporting on case studies that focus on changes to professional development and investigating the results, or focusing on what informs changes to the program (Beisiegel, 2017; Pascoe & Stockero, 2017; Wakefield, Miller, & Lai, 2017).

So, work is just beginning on the ways in which graduate teaching assistants develop their teaching practice. By examining their discourse, we better understand how the GTAs are appropriating, elaborating, or transforming various teaching practices in light of their own needs over time. Consequently, this work can inform our professional development work with GTAs.

Theoretical Perspective

The socio-cultural learning theory put forth by Vygotsky posits learning occurs through a reflexive relationship between the individual and the community in which the individual interacts (Vygotsky, 1987). We used a modified version of a framework known as the Vygotsky Space (Harré, 1983) to understand the ways in which the discussion evolved over time. With this framework, the understanding of a teaching practice can be tracked as it is appropriated and transformed by the GTAs throughout the term. See the modified Vygotsky space in Figure 1.

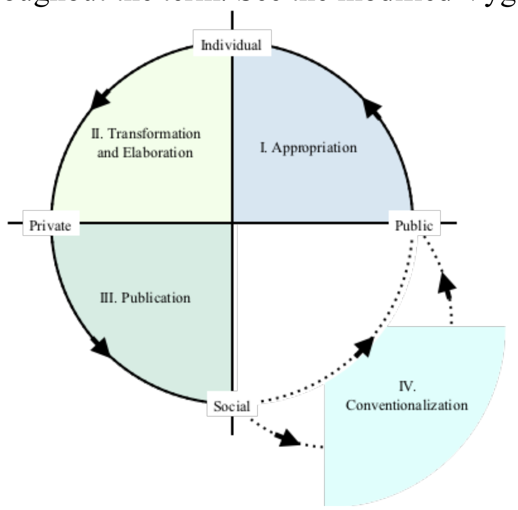

Figure 1. Diagram of the modified Vygotsky Space.

Within this diagram, there are two axes: Public-Private and Individual-Social. These two axes form four quadrants, which contextualize the four aspects of the Vygotsky space. For instance, *appropriation* is within the Public-Individual quadrant because it describes how a teaching practice comes to an individual from the public. *Transformation and Elaboration* is within the Individual-Private quadrant because it describes the way a person has possibly changed the meaning or motivation of a practice through making sense of it within their own context. In the third quadrant is *publication*, which describes how the person makes their own private understanding of a teaching practice known to the social group they are within where it can be then potentially be a resource for others. At this point, the teaching practice may go through several iterations of these three quadrants before it lands within the fourth quadrant, *conventionalization*, which represents that understanding of the teaching practice has become normalized within a community (Gallucci, DeVoogt Van Lare, Yoon, & Boatright, 2010).

Setting

At the university in this study, Calculus is taught in large lectures of approximately 160 students. The Calculus I and Calculus II students meet in break-out sessions twice a week in a class of approximately 35 students. In one of these meetings, GTAs focus on answering homework questions and addressing particular content of the lectures. In the other breakout session, the GTA leads group-based learning activities, typically an application of a topic from lecture. To get support for facilitating these active learning tasks, the GTAs participate in a three-day teaching seminar before classes begin in the Fall. The GTAs continue to meet approximately eight times throughout the term with mathematics education faculty. The GTAs also participate in weekly meetings with their course coordinator in which they talk about the activity for the following week and any administrative issues (e.g., grading of homework and exams).

Furthermore, the structure of the GTA program has been changed to include a Calculus I and II lead TA. The lead TA is a more experienced GTA who provides support in the form of professional development to his or her fellow GTAs, both before and throughout the term (Ellis, 2015). At two or three points in the term, the lead TA visits the break-out activity day sections of his or her fellow GTAs to observe the class and meets with the GTAs afterward to debrief.

Methods

Sixteen GTAs agreed to be part of the study, including both the Calc I and Calc II lead TAs, seven new GTAs, and seven GTAs returning from the previous year. The first author either audio or video-recorded each PD meeting, course coordinator meeting, debrief between the lead TA and a fellow GTA, and break-out sections observed by the lead TAs. We transcribed each of the video and audio recordings and coded each utterance about teaching practices using descriptive coding (Bakhtin, Emerson, & Holquist, 1986; Miles & Huberman, 1994). There were 137 sub-practices related to both traditional and active-learning teaching practices.

These sub-practices were introduced with a particular justification or motivation for their use. In the second phase of analysis, we determined the reasoning expressed for the use of a particular teaching practice. We could then see the ways in which the discourse around that practice transformed throughout the semester. If the meaning was changed, the practice was labeled as transformed. If the meaning had not changed but the practice was discussed in a new context, it was considered elaborated. These publications and transformations of the teaching practices, as well as who stated them and when, were tracked throughout the term to provide an overall portrait of how the discourse and inferred understanding evolved.

Results

There were three main results that were found in answering the research question given earlier: 1) GTAs do appropriate the language of mathematics education but may have different understandings of those practices than the PD leaders; 2) there is evidence that the GTAs found the practices of cognitive demand and lecturing challenging within the new context of active learning; and 3) there is evidence that the GTAs found the practices of asking for justification, checking for understanding, and precision for communication important to discuss but seemed to agree on their meaning and use. In this presentation, we will focus on the first result because of the theoretical and methodological implications it presents.

We took the criteria of conventionalization, the number of transformations, and what the transformations conveyed about the GTAs' understanding of the teaching practice to provide evidence for their appropriation of the language from mathematics education. The GTAs were appropriating some of the language of mathematics education but sometimes their publications revealed different meanings or motivations. This was particularly evident in a sub-practice related to *adding scaffolding* in which what became conventionalized was a transformed version of the practice. For example, math educators might think of scaffolding as in making connections to background knowledge to help the students gain access to the problem, rather than doing more of the problem for the students. Three other sub-practices had a transformed version approaching conventionalization. So, even though the GTAs were using the language of mathematics education, they could be using it in a different way than it was originally introduced or as understood by the mathematics education community.

Conclusion

The results from this study provide additional insight into the ways in which GTAs make sense of active learning teaching practices as they interact with each other and engage with professional development. With this information, the field can begin to understand how GTAs change their interpretations of teaching practice over time and ultimately improve the professional development offered to graduate students who are new to the practice of teaching.

References

Bakhtin, M., Emerson, C., & Holquist, M. (1986). *Speech genres and other late essays.* Austin: University of Texas Press.

Beisiegel, M. (2017). Exploring mathematics graduate teaching assistants' development stages for teaching. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1133–1139), San Diego, CA.

Belnap, J. K., & Allred, K. (2009). Mathematics teaching assistants: Their instructional involvement and preparation opportunities. *Studies in Graduate and Professional Student Development*, *12*, 11–38.

Bragdon, D., Ellis, J., & Gehrtz, J. (2017). Interaction, activities, & feedback: A taxonomy of GTA professional development. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 502–510) San Diego, CA.

Bressoud, D., Mesa, V., & Rasmussen, C. (Eds.). (2015). *Insights and recommendations from the MAA national study of college calculus*. Washington, DC: Mathematical Association of America.

Ellis, J. (2015). Three models of graduate student teaching preparation and development. In D. Bressoud, V. Mesa, & C. Rasmussen (Eds.), *Insights and recommendations from the MAA national study of college calculus*. Washington, DC: Mathematical Association of America.

Gallucci, C., DeVoogt Van Lare, M., Yoon, I. H., & Boatright, B. (2010). Instructional coaching: Building theory about the role and organizational support for professional learning. *American Educational Research Journal*, *47*(4), 919–963.

Griffith, L. M., O'Loughlin, V. D., Kearns, K. D., Braun, M., & Heacock, I. (2010). A pedagogy course's influence on graduate students' self-awareness as teacher-scholars. *Studies in Graduate and Professional Student Development*, *13*, 59–82.

Harré, R. (1983). *Personal being: A theory for individual psychology*. Cambridge, MA: Harvard University Press.

Kung, D., & Speer, N. (2009). Mathematics teaching assistants learning to teach: Recasting early teaching experiences as rich learning opportunities. *Studies in Graduate and Professional Student Development*, *12*, 133–152.

Marbach-Ad, G., Shields, P. A., Kent, B. W., Higgins, B., & Thompson, K. V. (2010). A prep course for graduate teaching assistants: Building a community. *Studies in Graduate and Professional Student Development*, *13*, 44–58.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis*. Thousand Oaks, CA: Sage Publications, Inc.

Miller, E., Wakefield, N., & Lai, Y. (2017). Opportunities to learn from teaching: A case study of two graduate teaching assistants. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education,* San Diego, CA.

Pascoe, A. E., & Stockero, S. L. (2017). Learning to notice and use student thinking in undergraduate mathematics courses. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 195–208). San Diego, CA.

Speer, N. M., Gutmann, T., & Murphy, T. J. (2005). Mathematics teaching assistant preparation and development. *College Teaching*, *53*(2), 75–80.

Speer, N. M., Smith, J. P., & Horvath, A. (2010). Collegiate mathematics teaching: An unexamined practice. *The Journal of Mathematical Behavior*, *29*, 99–114.

Vygotsky, L. (1987). *Thought and language*. Cambridge, MA: The MIT Press.

DESIGNING SPACES TO SUPPORT TEACHER LEARNING ABOUT TEACHING STATISTICS

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This study examined how participation in online professional development impacted 2,525 participants' perspectives about statistics and teaching statistics. The course design, participant engagement, impacts on perspectives, and influences on changes in perspective are described.

Keywords: Statistics, Teacher Professional Development, Online Learning

Introduction and Background

While statistics has gained prominence in school curricula (Franklin et al., 2007), it is often taught within a mathematics curriculum by mathematics teachers with limited experiences in statistics who are often trained in programs that do not support becoming effective statistics teachers (Zieffler, Garfield, & Fry, 2018). Since statistical thinking is inherently different than mathematical thinking (e.g., delMas, 2004), teachers need opportunities to develop their own statistical thinking and teaching practices. In part, this involves providing learning opportunities that challenge perspectives about the nature of statistics and teaching statistics. In this paper, we discuss how design of online professional development (OPD) that is free, open access, and can impact many educators across geographic boundaries (Kim, 2015).

Beliefs and perspectives about statistics include an educator's ideas about the nature of statistics, about oneself as a learner of statistics, and about the classroom context and goals for students' learning statistics (Pierce & Chick, 2011; Eichler, 2011). Certain beliefs likely result in different approaches to teaching. For instance, if an educator believes that statistics is a way of quantifying data, his or her teaching practices may favor statistical procedures without considering the context of the data, issues related to sampling, and making claims about uncertainty (Pierce & Chick, 2011). Furthermore, the focus of teachers' intended curriculum in statistics can be considered on a continuum from traditionalists (focused on procedures absent of context), to those wanting students to engage in an investigative process that is tightly connected to contexts of real data (Eichler, 2011). Professional development should move educators along this continuum towards a focus on investigative processes, which requires impacting educators' beliefs about the nature of statistics and statistical learning goals for students.

Theoretical Framework

We envision the *Teaching Statistics Through Data Investigations* [TSDI] course as an infinity space (Gee, 2005). In this space, educators with diverse backgrounds and skills come together to learn about strengthening their approaches to teaching statistics. An affinity space begins with content, or a *generator*. In the case of TSDI, the generator is an OPD course that emphasizes how teachers can use an investigation cycle to teach statistics and support students in exploring data to make evidence-based claims. *Portals* are ways individuals interact with content and engage with one another in an affinity space. Portals can also be generators. In our course, portals include the Moodle site that educators use to access the course materials and resources. Within this larger portal of designed resources, every unit includes opportunities for participants to engage in discussions. An affinity space also has an internal and external aspect; *internal grammar* characterizes the way a space is designed and organized, while *external grammar* signifies how people choose to interact with either the content or one another within the space.

According to Gee (2013), an advantage to an affinity space is that educators can access spaces and contribute in various ways with different people for different purposes. Further, educators can access these spaces to acquire resources and also share resources and approaches and strategies for teaching. Once accessed, there is flexibility in these spaces so that educators can interact with others by forming and re-forming groups. Gee suggests that the space is a form of emergent intelligence since various tools, different educators, and diverse skills are networked in ways that increases everyone's knowledge. In this study, we examined the following research questions: *How do educators engage in an OPD course focused on teaching statistics? How do educators' interactions in OPD impact their perspectives about the nature of statistics and the teaching of statistics? What portals influence changes in educators' perspectives?*

Design of the Course

TSDI (http://go.ncsu.edu/tsdi) does not focus on any particular grade band or specific statistical content. Rather, the course was designed to encourage participants to view statistics as an investigative process that incorporates statistical habits of mind, and view learning from a developmental perspective. One of the most important generators of the course, the Students' Approaches to Statistical Investigations [SASI] framework, was built on Franklin et al. (2007), illustrating the investigative cycle, students' statistical reasoning in each phase at three levels of sophistication, and an indication of productive habits of mind. Multiple portals focusing on the SASI framework are integrated throughout the course.

Important aspects of the internal grammar of the affinity space (i.e., the design) are based on design principles of effective OPD (Kleiman, Wolf, & Frye, 2015): (a) self-directed learning, (b) peer-supported learning, (c) job-connected learning, and (d) learning from multiple voices. TSDI was designed so that educators would have multiple opportunities to interact with the content and engage with others through numerous portals. *Learning from multiple voices* was enacted by providing ways to engage with videos of an Expert Panel discussion and classroom videos. *Self-directed* and *job-connected* learning opportunities included Dive Into Data experiences, where participants used free technology tools (e.g., Gapminder, Tuva, CODAP). Extensions in each unit include extra resources (e.g., data sets, lesson plans, applets, etc.). *Peer-supported* learning included two discussion forums where participants interact with others focusing on: 1) specific pedagogical aspects of teaching statistics, and 2) participant initiated discussions about teaching.

Methods

Participants

From Fall 2015 to Fall 2017, participants (n = 2,525) from 84 countries enrolled in six sections of the course. The majority resided in the US (79%), including all 50 states. The majority of participants were female (66%), held an advanced degree (75%), and identified themselves as classroom teachers (61%). Participants' experience as educators varied, with a mean of 14 years (20% had 1-5 years, 19% had 6-10 years, 18% had 11-15 years, 16% had 16-20 years, 21% had more than 20 years).

Data Sources and Analysis

Course activity was tracked through click logs, allowing us to examine trends in participants' engagement in the designed materials in the portal. A Tableau dashboard was used to visualize participants' engagement over time. Other data sources included: 1) discussion forum posts from two Unit 5 forums, 2) open-ended responses to end-of-courses surveys (administered at the end of Unit 5), and 3) a follow-up survey (administered approximately 6 months after course completion). In previous work that examined only Fall 2015 classroom teachers, Lee, Lovett, and Mojica (2017) reported four themes to describe changes in participants' beliefs/perspectives about statistics and teaching statistics. Thus, 721 Unit 5 posts from three sections of TSDI were coded for one of these themes. We also open coded, which led to new themes. Once we were saturated in data and no new themes emerged, we confirmed codes in the remaining sections of Unit 5 posts. End-of-course and follow-up surveys were analyzed using open coding.

Findings

Engagement in TSDI

TSDI was offered six times from Fall 2015 - Fall 2017 in Moodle. Each course remained active for about 10-15 weeks, and participants were able to access and view materials after the course closed. Amongst all who registered, 1,737 (69%) participants accessed any material, which is higher than what Jordan (2015) found across 59 different MOOCs with a typical "show up" rate near 50%. Across all sections, 1,441 (54%) of those who registered engaged in the Orientation Unit. Of those that participated in the Orientation Unit, a majority (71%) engaged in Unit 1. However, only 47% participated in Unit 2. Engagement in Units 3 and 4 were very similar, 38% and 33%, respectively. Only 26% of those who started the course completed Unit 5. Of those who registered, 959 (38%) engaged in a discussion forum. Across all sections, there was a total of 2,164 discussion forum threads, and 6,381 total posts, with an average of 6.65 responses per forum participant, with some participants posting 30-50 times.

Impacts on Perspectives about the Nature of Statistics and Teaching Statistics

Eight major themes emerged in relation to changes in participants' beliefs and perspectives about statistics and teaching statistics based on an analysis of discussion forum data: statistical thinking involves different processes than mathematical thinking; engaging in statistics should involve exploring data; posing good statistical questions and selecting an interesting/relevant context in relation to students is crucial in engaging students in statistical thinking; engaging in statistics is more than computations and procedures and should include investigative cycles and habits of mind; engaging in statistics is enhanced by the use of dynamic technology; engaging in statistics requires real (and messy) data; statistical thinking develops along a continuum; and, teachers do not have to know all answers when engaging students in the investigative cycle.

Findings from the follow-up survey support that engagement in the course impacted learning. Most participants (84%) indicated they acquired new knowledge and skills. Of those that indicated their knowledge/skills increased, 63% signified they had applied this in their practice. Only 45% reported newly acquired skills had an impact on student learning. Participants reported they used specific tasks, websites, technology tools, and other resources in their practice.

Influences on Changes in Perspectives

The SASI framework, expert videos, and videos of students engaged in statistics acted as portals that influenced changes in educators' perspectives to include a more robust vision of statistical thinking and teaching strategies for engaging learners in investigations. Discussion forum and end-of-unit surveys showed how participants often discussed what they learned from different multimedia resources related to the SASI framework. Frequently mentioned portals were the SASI diagram, videos with experts (panel discussion and development of concept of mean), and animated videos of students' work on a weighted die investigation.

Analysis of discussion forum data and end-of-course surveys indicate that the following portals provided learning experiences that impacted educators' perspectives about the nature of statistics and teaching statistics (in decreasing frequency): access to resources, technology tools, websites, and lesson plans; learning from videos of expert panel discussions; learning from videos of students and teachers work in classrooms; introduction to the SASI framework; focus on improving questioning, exploration, engaging students, and active learning; engaging in discussions with colleagues; appreciation for flexibility and learning at own pace; and being grateful for opportunity and inspired to learn more.

Discussion

Preparing and supporting educators to be effective statistics teachers is a challenge in teacher education. We suggest that designing spaces where educators have opportunities to challenge perspectives about the nature of statistics and teaching statistics through OPD is one way to support learning. Designing specific portals for educators can lead to implementation of new ideas in practice. Engagement in OPD can be a powerful experience in supporting many teachers' professional learning across geographic boundaries (Kim, 2015). The themes we identified and the self-reported changes in their practices suggest that the TSDI course, and perhaps other similar OPDs about teaching statistics, can be successful in shifting teachers' perspectives about the nature of statistics and their use of real world investigations in their teaching that align with Pierce and Chick (2011) and Eichler (2011).

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References

delMas, R. C. (2004). A comparison of mathematical and statistical reasoning. In D. Ben-Zvi & J. Garfield (Eds.), *The challenge of developing statistical literacy, reasoning and thinking* (pp. 79-95). Boston: Kluwer.

Eichler, A. (2011). Statistics teachers and classroom practices. In C. Batanero, G. Burrill, & C. Reading (Eds.). *Teaching statistics in school mathematics-challenges for teaching and teacher education* (pp. 175-186). Springer Netherlands.

Franklin, C., et al. (2007). *Guidelines for assessment and instruction in statistics education (GAISE) Report: A Pre-K-12 curriculum framework*. American Statistical Association.

Gee, J. P. (2005). Semiotic social spaces and affinity spaces. In D. Barton, & K. Tusting (Eds.), *Beyond communities of practice* (pp. 214–232). Cambridge: Cambridge University Press.

Gee, J. P. (2013). *The anti-education era: Creating smarter students through digital learning*. New York: Palgrave Macmillan.

Jordan, K. (2015). Massive open online course completion rates revisited: Assessment, length and attrition. *The International Review of Research in Open and Distributed Learning*, *16*(3).

Kim, P. (2015). *Massive open online courses: the MOOC revolution*. Routledge.

Kleiman, G., Wolf, M.A., & Frye, D. (2015). Educating educators: Designing MOOCs for professional learning. In Kim, P. (Ed.), *Massive Open Online Courses: the MOOC Revolution* (pp. 117-144). New York: Routledge.

Lee, H. S., Lovett, J. N., & Mojica, G. F. (2017). Characterizing impacts of online professional development on teachers' beliefs and perspectives on teaching statistics. In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 407-414). Indianapolis, IN: Hoosier.

Pierce, R., & Chick, H. (2011). Teachers' beliefs about statistics education. In C. Batanero, G. Burrill, & C. Reading (Eds.). *Teaching statistics in school mathematics-Challenges for teaching and teacher education* (pp. 151-162). Springer Netherlands.

Zieffler, A., Garfield, J., & Fry, E. (2018). What is statistics education? In D. Ben-Zvi, K., Makar, & J. Garfield (Eds.), *International handbook of research in statistics education* (pp. 37-70). Cham, Switzerland: Springer.

PREPARING ELEMENTARY MATHEMATICS SPECIALISTS (AS-SUBJECTS) AND RECONCEPTUALIZING TEACHER BELIEFS (AS-ENTANGLEMENT)

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This project was designed to help Prospective Elementary Mathematics Specialists (PEMSs) negotiate their beliefs about mathematics teaching and learning amidst all of the other aspects of teaching, addressing the messiness of teacher beliefs while navigating their role as mathematics teacher and teacher leader. To do this, I take up poststructural theories of subjectivity to consider PEMSs as subjects whose beliefs about teaching and learning mathematics are always already entangled, impossible to think as separate or pre-existing. This reconceptualization gave PEMSs a space to address and navigate the tensions of beliefs and enactment. These tensions have been crafted into conversations, an accessible contribution to the literature.

Keywords: teacher beliefs, teacher education – inservice/professional development

Introduction and Significance

For several decades now, teacher beliefs have been a focus in mathematics education research as well as the broader field of education. The basic rationale for this emphasis has been to understand how teacher beliefs about mathematics teaching and learning impact enacted instructional practices. In my thorough review of the extant literature, I found many studies that assert confirmation of an impact of teacher beliefs on practice; however, there is a growing body of empirical studies that complicates their influence (Fives & Buehl, 2012; Skott, 2015). In this dissertation study, rather than attempting to define teacher beliefs or dispute their impact, I reconceptualize teacher beliefs by writing a different story of beliefs as a *doing*, an entanglement, rather than a thing that can be reflected upon or distinguished. The purpose of this work is to produce new knowledge about teacher beliefs to help mathematics researchers address the documented difficulty in measuring the relationship between teacher beliefs and practice (Charalambous, 2015; Leatham, 2006; Schoenfeld, 2015). This project was designed to better help Prospective Elementary Mathematics Specialists (PEMSs) negotiate their beliefs about mathematics teaching and learning amidst all of the other aspects of teaching, addressing the messiness of beliefs (enacted) while navigating their role as mathematics teacher and teacher leader. To do this, I take up posthuman and poststructural theories of subjectivity and methodology to consider PEMSs as subjects (selves) whose beliefs about teaching and learning mathematics are *always already* entangled, impossible to think as separate or pre-existing (Derrida, 1967/1974). This reconceptualization gave PEMSs a space to address and navigate the *tensions* of teacher beliefs and their practice. These tensions have been crafted into conversations (Bridges-Rhoads, 2011; Davies, 2009; Shor & Freire, 1987), including creative analytical practices/processes (CAP) like narratives and poetry (Richardson, 1997), which is perhaps more accessible, contributing to the current conversation about teacher beliefs as well as extending it to address important issues of perspective and methodology (Koro-Ljungberg, 2015).

Rigorous preparation programs like the one in this study are designed to broaden EMSs' range of teaching practices, develop pedagogical competencies as described in the *Principles to Actions*: *Ensuring Mathematics Success for All* (NCTM, 2014), and prepare them to advocate for pedagogical shifts (AMTE, 2013). The preparation program is full of theory, teaching methods, and content knowledge intended to lead to enactment of such theory and methods in elementary classrooms. Addressing these goals requires a degree of clarity and certainty about the experiences of PEMSs in their enactment, identifying and distinguishing shifts in pedagogy and beliefs. During my own preparation and in my own enactment, naming (and performing) my beliefs led me to deeply question the coherence of my EMS-identity. A poststructural perspective on the subject allows for movement rather than stability during a time of transition and enactment for PEMSs. Therefore, this study is significant because of the impact of EMSs' (re)construction of teacher beliefs on themselves, their students, and others. More specifically, this study produces new knowledge for EMSs that might help them better navigate their role as teacher leaders, thinking about their beliefs about mathematics teaching and learning in a way that no longer limits their subject positions. Rather, EMSs can consider their instruction, pedagogy, and beliefs as intra-actions within the discourses that produce them.

Data Collection, Analysis, and Preliminary Findings

Three PEMSs engaged in this study, and data consisted of three classroom observations (two evaluative, one non-evaluative; nine total), two semi-structured individual interviews (six total), and three focus group interviews during a practicum course in a university K-5 Mathematics Endorsement program, as well as document analyses of observation protocols and professional portfolios both before and after completion by the university supervisor and PEMSs. Analysis consists of "thinking with theory" (Jackson & Mazzei, 2012). Such an analysis, which happens in writing (Richardson & St. Pierre, 2005), makes visible multiple ways of *being* and *becoming* subjects in mathematics education, enabling a story of teacher beliefs as entangled.

For this study, I use writing as a method of analysis, which broadly speaking, uses writing as a way of knowing, thinking, organizing, and creating. It involves thinking and rethinking, reading and rereading of data and theoretical texts side-by-side, writing and rewriting as a cyclical process. Writing in this way is rigorous, and while it looks differently for every researcher, it often involves setting a systematic writing schedule in which we read theories again and again while reading and/or listening to transcripts, documents, and other data. In this process of writing and making meaning, stories and themes come about on the page. These preliminary findings are not fixed or final but rather coming about in the midst of different stories. In this process thus far, I have created poems, conversations between speaking subjects, and narratives, as well as other types of prose, collectively written from transcripts, field notes, hallway conversations, theorizations, journals (mine), reflections (theirs), emails, facial expressions, readings, documents, curriculum, student work—anything that, as MacLure (2013) says, "glows" in the research, sparking my wonder. This is the rigorous process of writing as a method of inquiry, where all of this is welcome to the page in the systematic daily writing that happens alongside readings of data and theory.

I didn't set out with any particular format in mind, but what emerged were conversations, often about *tensions* teachers were feeling in their beliefs and enactment. I started to explore what stories could be told when taking up this format of a conversation. This process of writing a conversation has illuminated, for me, when words are not enough to tell the whole story—when things feel missing, like how their bodies are exhausted, their classrooms are dark and confining, and ways that hands and eyes might say more than spoken words. I am writing in these silences, using my voice to add analysis, commentary, stage directions, environment, citations, and more questions. Below, I present a small excerpt, a preliminary finding, of a conversation about tensions with beliefs and expectations:

Jill: I believe in the research. I believe in what I read and what we've experienced and what I'm seeing, but because our society in general is so test-driven, score-driven, competition-driven, I just want the evidence to be there to support that I'm making the right decision. I just don't want to mess this up and have all kinds of parents say that I ruined their children. They're going to be fine on the test…

Michael: Are they going to be fine on the test?

Jill: Can any of us confidently say 'oh, you just watch, at the end of the year my kids will be stronger, they will be better'? I'm not there yet.

Maggie: We have to tread lightly, making an effort to stick with it. Just reading a book or, you know, seeing an article or something you feel inspired, but not equipped.

Jill: But after going through this course, I do feel better equipped. One of my greatest concerns, though, is- what if I screw this up? What if I screw this up and I miss something and then they go on to fourth grade and they're like, 'whoever was in Jill's class does not know how to add.' I can't do that. So knowing that even though I'm not following the teacher curriculum guide page by page, I'm still getting at the skills, I'm still getting at the concepts, I'm still equipping them to solve problems. That is probably the most rewarding thing I've experienced in years. CGI is *powerful*, you all know that. But painting that picture and showing that to people who haven't seen it is *hard*.

[There is heat behind two of Jill's words– *powerful* and *hard*– that highlights the tensions she's feeling. She describes CGI as powerful, with such enthusiasm, her voice drawn out and deep, that word met with emphatic nods and closed eyes. *Yes, powerful*. You can hear stipulation in her voice. By the time she reaches her last word– *hard*– her body reacts as if it's just received a punch to the gut. The delivery of that word takes too long. Her shoulders slouch, her head hangs, her hands hit the table. Convincing people of that power, people with different expectations, is not just difficult; it's exhausting.]

Michael: Meeting everyone's expectations feels impossible.

These *tensions* are what is driving my reconceptualization of teacher beliefs as an entanglement that we negotiate again and again, impossible to disentangle. By paying attention to ways teacher beliefs are in-tension with these other aspects of teaching, I am not only extending the current conversation about teacher beliefs but also offering a different and responsive way for teachers (and teacher educators) to prepare for those tensions.

A preliminary summary of my findings includes a collection of conversations aimed at these ways I am seeing beliefs-entangled: (a) beliefs as effecting others, like students and colleagues, as well as practice and feelings; (b) beliefs as (not) mattering, feeling defeated by scripted curriculum, like their beliefs don't matter, while at other times it felt like their beliefs were all that mattered; (c) beliefs as exhausting on bodies and on attitudes, impacting things like perseverance and dedication to enactment; and (d) beliefs in tension with differing expectations. These reconceptualizations are not linear or separate but recursive and overlapping in my analysis and writing of conversations. They allow beliefs-entangled to move amidst the many tensions. In writing these results as conversations, I am producing texts that can be read by both researchers and teachers. My hope is that this accessibility can help prepare PEMSs and teachers to be responsive to these tensions, recognizing them in their beliefs-entanglement, and better negotiate and thus navigate their practice and enactment.

Initial Implications

I am finding some important stories that will contribute to the mathematics education research community. First, when preparing PEMSs, complicating the stability of teacher beliefs is allowing them to navigate the tensions they feel and see in their classrooms, schools, students, colleagues, and past experiences. Thus, this also has implications for preparing pre-service elementary teachers, as I myself came upon this problem of teacher beliefs because of my beliefs being questioned that first year I entered the classroom. Perhaps if pre-service teachers are prepared for those tensions and engage in conversations about beliefs as not static or concrete but moving and subject to surrounding discourses, they will be able to better navigate those early years, persevere in those struggles, and become better teachers for it. Further, perhaps by reconceiving of teacher beliefs as move-able and adaptable, teachers can feel more autonomy over their beliefs and practice, and less teachers will be pushed out of the classroom by feelings of inflexibility and inadequacy. These initial implications for the mathematics education research community suggest another implication—a call for different and innovative methodologies in mathematics education research. Perhaps studies like this one, by opening up different conversations, can offer a responsive call to mathematics teacher education.

References

Association of Mathematics Teacher Educators [AMTE]. (2013). Standards for Elementary Mathematics Specialists: A Reference for Teacher Credentialing and Degree Programs. San Diego, CA.

Bridges-Rhoads, S. (2011). *Pursuing responsibility: Writing and citing poststructural qualitative inquiry, social justice, and Christianity* (Unpublished dissertation). University of Georgia, Athens.

Charalambous, C. Y. (2015). Working at the intersection of teacher knowledge, teacher beliefs, and teaching practice: A multiple-case study. *Journal of Mathematics Teacher Education, 18*(5), 427-445.

Davies, B. (2009). Life in Kings Cross: A play of voices. In A. Y. Jackson & L. A. Mazzei (Eds.), *Voice in qualitative inquiry: Challenging conventional, interpretive, and critical conceptions in qualitative research* (pp. 197-220). New York: Routledge.

Derrida, J. (1974). *Of grammatology*. (G. Spivak, Trans.). Baltimore: Johns Hopkins University Press. (Original work published 1967).

Fives, H., & Buehl, M. M. (2012). Spring cleaning for the messy construct of teachers' beliefs: What are they? Which have been examined? What can they tell us? In K. R. Harris, S. Graham & T. Urdan (Eds.), *APA educational psychology handbook* (Vol. 2, pp. 471-499). Washington DC: APA.

Jackson, A. Y., & Mazzei, L. A. (2012). Thinking with theory in qualitative research: Viewing data across multiple perspectives. New York: Routledge.

Koro-Ljungberg, M. (2015). *Reconceptualizing qualitative research: Methodologies without methodology*. Thousand Oaks, CA: SAGE Publications.

Leatham, K. L. (2006). Viewing mathematics teachers' beliefs as sensible systems. *Journal of Mathematics Teacher Education, 9*, 91-102.

MacLure, M. (2013). The wonder of data. *Cultural Studies ↔ Critical Methodologies, 13*(4), 228-232.

National Council of Teachers of Mathematics [NCTM]. (2014). *Principles to actions: Ensuring mathematics success for all.* Reston, VA: Author.

Richardson, L. (1997). *Field of play: Constructing an academic life.* New Brunswick, NJ: Rutgers University Press.

Richardson, L., & St. Pierre, E. A. (2005). Writing: A method of inquiry. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (3rd ed.) (pp. 959-978). Thousand Oaks, CA: Sage.

Schoenfeld, A. H. (2015). What counts, when? –Reflection on beliefs, affect, attitude, orientations, habits of mind, grain size, time scale, context, theory, and method. In B. Pepin & B. Roesken-Winter (Eds.), *From beliefs to dynamic affect systems in mathematics education* (pp. 395-404). Switzerland: Springer International.

Shor, I., & Freire, P. (1987). *A pedagogy for liberation: Dialogues on transforming education*. Westport, CT: Bergin & Garvey Publishers, Inc.

Skott, J. (2015). The promises, problems, and prospects of research on teachers' beliefs. In H. Fives & M. G. Gill (Eds.), *International handbook of research on teacher beliefs* (pp. 13-30). New York, NY: Routledge.

NOTICING IN SMALL GROUPS: BEYOND MATHEMATICAL THINKING

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Small groups (SG) is a space where three aspects, such as mathematical thinking, status, and funds of knowledge, come into play. How teachers draw upon these three aspects in facilitating small group work is shaped by what teachers notice. Researchers have used the professional noticing framework (Jacobs, Lamb, & Philipp, 2010) to understand teachers' noticing of students' mathematical thinking. This paper proposes a teacher noticing framework based upon Louie (2017) and Jacobs and her colleagues (2010) to expand teacher noticing for facilitating SG.

Keywords: Equity and Diversity, Teacher Education-Inservice/Professional Development, Teacher Education-Preservice

Introduction

How do equity-oriented teachers make instructional decisions when facilitating small groups (SG) in mathematics classrooms? Our attempt to answer this question was predicated upon our experiences as former teachers when we recognized that we often noticed more than students' mathematical thinking when using SG. Through reviewing the literature on teacher noticing we learned that research-based descriptions of what teachers notice in SG contexts are limited and that current teacher noticing frameworks are not sufficient to understand teachers' instructional decisions when working with SG.

Recent research analyzes individual teachers' cognitive processes that contribute to noticing students' mathematical thinking along conceptual learning trajectories. SG, however, have both mathematical and social goals (Cohen & Lotan, 2014; Davidson, 1990). To achieve these goals, teachers may notice students' mathematical thinking, funds of knowledge, and status simultaneously while working with SG. As such, we argue that more work is necessary to help researchers understand how teachers balance these aspects. This paper proposes a teacher noticing framework that can help researchers more fully understand how teachers make instructional decisions when facilitating SG.

Theoretical Framework

The Noticing in Small Groups framework (NSG framework, Figure 1) is an attempt to conceptualize teacher noticing in the context of SG. Louie's (2017) concern about research on teacher noticing was that this research tended to emphasize individual teachers' cognitive process of students' mathematical thinking and ignore the contextual factors that may influence what teachers notice. Notwithstanding, we felt that focusing on teachers' cognitive process per se is not necessarily problematic in particular relation to SG. We agree with Louie (2017) that "the cognitive focus of existing literature obscures cultural and ideological obstacles to noticing students' mathematical thinking and strengths" (p. 56). When considering SG we concur with Louie (2017) that there is more to negotiate than students' mathematical thinking. The NSG framework highlights three aspects of SG to illustrate what equity-oriented teachers more likely notice when facilitating SG.

In the NSG framework, three aspects of SG are 1) mathematical thinking, 2) status, and 3) children's funds of knowledge. Prior work on professional noticing suggests that teachers' noticing competency of students' mathematical thinking holds promise for advancing students' mathematical understanding (Jacobs et al., 2010) and therefore we have kept this aspect in the NSG framework. In SG, students are expected to work together, while "asking questions, discussing ideas, making mistakes, learning to listen to others' ideas, offering constructive criticism, and summarizing discoveries in writing" (NCTM, 1998, p. 78). This quote brings into the imagination students working smoothly in SG with shared social and intellectual authority. In reality, however, students' status, "hierarchies where some members are more active and influential than others" (Cohen & Lotan, 2014, p. 27) may shape whose ideas are taken or ignored in SG (Cohen & Lotan, 2014). Funds of knowledge are "the historically and culturally based knowledge, skills, and practices found in students' homes and communities" (Turner & Drake, 2016, p. 32). Research has demonstrated that teachers' consideration in planning and use of students' funds of knowledge helps students to make sense of mathematical ideas (Turner & Drake, 2016). For these reasons we have added status and funds of knowledge to be included alongside mathematical thinking in the NSG framework. We think of these aspects to support one another, like in Figure 1, where each aspect of SG takes one side of each triangle. The triangle represents an interrelationship among these aspects because without one side the other two sides cannot stand. Thus, these aspects may be noticed simultaneously in order to support equitable teaching in the context of SG.

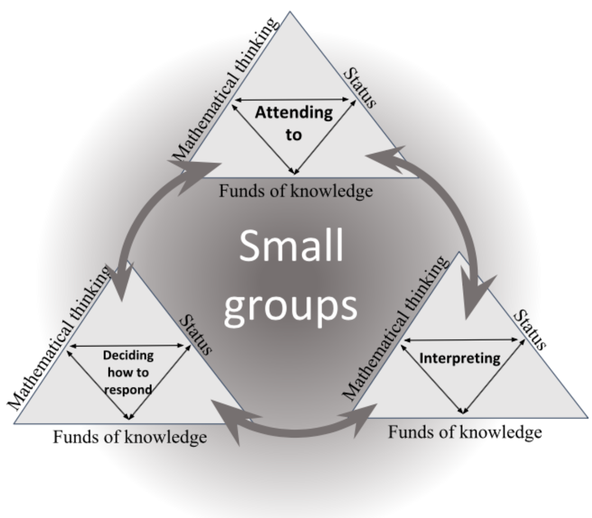

Figure 1: Noticing in Small Groups framework

The NSG framework draws upon professional noticing (Jacobs et al., 2010). Professional noticing of children's mathematical thinking consists of a set of the three components: (1) attending to children's strategies, (2) interpreting children's understandings, and (3) deciding how to respond on the basis of children's understandings. Turner, Drake, McDuffie, Aguirre, Bartell, and Foote (2012) adapted Jacobs and colleagues' professional noticing to address both mathematical thinking and children's funds of knowledge. The NSG framework builds on their adaptation in that it adds a third aspect, status, which influence the dynamics of SG (Cohen & Lotan, 2014).

The circular arrows in Figure 1 represent how the teachers' noticing process pertaining to intervention in small groups works in the NSG framework. These processes do not necessarily happen in any given order, such as, from attention, to interpretation, and to decisions about how to respond, nor begin with any specific component (i.e., attending may not always occur first).

For example, within the context of SG, a teacher may first interpret unequal participation as an issue of status; the teacher may then attend to the both status and funds of knowledge by eliciting from the low-status student an experience he/she has that will be beneficial to the group, which in turn assigns competence. If the unequal participation is first interpreted as related to mathematical thinking a different intervention is likely to occur. Alternatively, a teacher may first attend to a student's use of funds of knowledge and decide to highlight his/her mathematical thinking to position the student as competent among the group members. The inclusion of status and children's funds of knowledge into the NSG framework can invite teachers to be more responsive to students' multiple ways of being mathematically smart.

Illustrative Example

In order to understand the limitations of using professional noticing framework (Jacobs et al., 2010) and further understand the affordances of the NSG framework, we provide an illustrative example from existing research. While reading research regarding SG, we recognized that the in-the-moment decision making process for teachers' facilitation was absent. Although we recognized this absence among several research articles, the transcript from Wood (2013), below, highlights well what we recognized.

Wood (2013) studied the participation of one group of fourth grade mathematics learners to understand how students' mathematical identities can shift within and across lessons. She considered how the in-the-moment identities created or limited opportunities for mathematical learning. In this exchange the teacher asks a student, Jakeel, for his mathematical reasoning on the group's task of finding the area of a right isosceles triangle drawn on grid paper:

Teacher: What's the area of J? [The teacher looks at Jakeel.]
Jakeel: The sum is um [Jakeel points at figure K on his paper.]
Teacher: What is it? Tell me what it is.
Jakeel: Eight [The area of J is 8 squares.]
Teacher: How is that 8? I can't tell that's 8.
Jakeel: Because one, two, three, four five, six, seven, eight, nine, t–. [Jakeel points to the spaces in figure J as he counts. He points once at each space, including the small triangles in figure J, with the pink of his right hand. Because he counts each of the triangles as one, he arrives at a number bigger than 8.] Hold on. [Jakeel points at each space in K with his pinky. This motion suggests that he is silently counting.]
Rebecca: Can I tell him? [Rebecca talks to the teacher. The teacher then leaves.]
(Wood, 2013, p. 798).

Prior to this interaction among the teacher, Jakeel, and Rebecca, Wood described the way in which the students in Jakeel's group had relegated him to the role of a scribe which limited his opportunities for learning. Here, "Jakeel's shift back to mathematical explainer seemed to be tied to the teacher's positioning" (Wood, 2013, p. 798). We are left wondering, what did the teacher notice before posing the question to Jakeel? If the prior interactions between Jakeel and his group were noticed by the teacher, how were those events interpreted? Wood continues to explain that, "Jakeel's reidentification as mathematically capable was not an isolated event. During the remainder of the lesson, the teacher returned to the group three times, each time positioning Jakeel as a mathematical explainer by asking him questions about the area of the figures" (p. 799). Wood's interpretation suggests that the teacher was simultaneously noticing status issues occurring in the group and recognizing the mathematical contributions of the group members.

To be clear, Wood's intent of her analysis was not to better understand the decision-making process involved in the teacher's instructional moves; her purpose was to understand how students' opportunities for learning are mediated by the way they are positioned by peers and teachers. Our brief analysis demonstrates that a framework to understand teachers' facilitation of SG work that is not based solely on students' mathematical thinking can provide new insights.

Conclusion and Implications

To create the NSG framework we have looked across prior research focused on teacher noticing of students' mathematical thinking. Acknowledging that the teacher noticing framework is not enough to understand what teachers notice in the contexts of SG, we proposed the NSG framework as a way to better understand teachers' instructional moves by attending to what researchers document is important in mathematics teaching and learning: 1) mathematical thinking, 2) status, and 3) children's funds of knowledge. We propose this framework as a means to look ahead in order to initiate a discourse on the noticing practices of teachers when working with SG, including how the noticing practices during SG compare with the noticing practices during other modes of instruction.

The NSG framework will allow researchers to better understand how teachers attend to, or not, issues of status and children's funds of knowledge in order to explain the instructional move. Furthermore, the NSG framework may be useful for teacher educators to scaffold prospective and practicing teachers to facilitate SG while developing the dispositions to attend to and respond to small groups in productive ways. In this regard we see that the NSG framework may help develop visions of equitable mathematics teaching that go beyond content knowledge and pedagogical content knowledge, which will begin to address the concern of Louie (2017).

References

Cohen, E. G., & Lotan, R. A. (2014). *Designing Groupwork: Strategies for the Heterogeneous Classroom Third Edition*. Teachers College Press.

Davidson, N. (1990). Cooperative learning in mathematics: A handbook for teachers. Menlo Park, CA: Addison-Wesley.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for research in mathematics education*, 169-202.

Louie, N. L. (2018). Culture and ideology in mathematics teacher noticing. *Educational Studies in Mathematics*, *97*(1), 55-69.

National Council of Teachers of Mathematics. Commission on Standards for School Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Natl Council of Teachers of.

Turner, E. E., & Drake, C. (2016). A review of research on prospective teachers' learning about children's mathematical thinking and cultural funds of knowledge. *Journal of Teacher Education*, *67*(1), 32-46.

Turner, E. E., Drake, C., McDuffie, A. R., Aguirre, J., Bartell, T. G., & Foote, M. Q. (2012). Promoting equity in mathematics teacher preparation: A framework for advancing teacher learning of children's multiple mathematics knowledge bases. *Journal of Mathematics Teacher Education*, *15*(1), 67-82.

Wager, A. A. (2014). Noticing children's participation: Insights into teacher positionality toward equitable mathematics pedagogy. *Journal for Research in Mathematics Education*, *45*(3), 312-350.

Wood, M. B. (2013). Mathematical micro-identities: Moment-to-moment positioning and learning in a fourth-grade classroom. *Journal for Research in Mathematics Education*, *44*(5), 775-808.

MODELS AND MODELING PERSPECTIVE IN MÉXICO, THE MICHOACÁN FOREST

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This study describes the processes of solving a Model-eliciting activity called the Avocado Cultivation. The goal is to examine the potential of the activity in supporting mathematical knowledge and rethinking the mathematical learning in this new age. The theoretical framework was the Models and Modeling perspective. Eight Mexican teachers took part in this qualitative research. They were teaching at high school. The findings indicate that MEA provides opportunities for teachers to modify, extend and refine ways of thinking, validate the estimation to predict situations, and confront traditional ideas.

Keywords: Model eliciting activity, Teachers Knowledge, Linear and Exponential Models

Introduction

Creating a mathematical description or model to understand, explain, communicate, and predict a situation is not an easy task. Without basic knowledge and certain mathematical abilities and skills, people can make wrong decisions such as choosing the wrong credit card or a home loan, or even having difficulties to understand situations. What is the needed mathematical knowledge that education must provide to citizens in this new age?

The concepts of function and variation are useful to understand a variety of situations and phenomena. But many students have difficulty understanding them (Ärlebäck, Doerr, and O'Neil, 2013; National Council of Mathematics Teachers [NCTM], 2011), using algebraic notation, and connecting with representations such as graphs or data tables. A concept may not be learned in isolation from other concepts, phenomena and related processes. How should activities be designed to support the integration of knowledge and abilities? What characteristics should they have? According to Models and Modeling perspective [MMP], Model-eliciting Activities [MEA] allow the construction and integration of mathematical knowledge and abilities (Doerr and Lesh, 2003).

Some years ago, a group of researchers from México, began a project whose goals were modifying, extending and refining ways of thinking about teaching and learning process (Vargas, Cristóbal, Carmona, Reyes, & Alvarado, 2014). The interest was the development of knowledge and abilities about mathematical concepts as function and variation. The theoretical framework was Models and Modeling perspective (Lesh and Doerr, 2003). Researchers adapted and designed models-eliciting activities to Mexican contexts. They created workshops for teachers and students where they problematized the nature of mathematics learning.

In this study we present one of the MEAs designed to develop mathematical knowledge about variation, function and equation. The activity was about the problem of deforestation. We describe the teacher's processes of solving the MEA to know the potential of the activity. The research questions were the following. What happens when the math teachers solve the MEA? How do they engage in mathematical activity? How can we assess the potential of the activity using the six principles to design a MEA?

Theoretical framework

According to Models and Modeling Perspective learning mathematics is a process of developing models that are continuously modified during the interaction between the student and a problematic situation (Lesh & Doerr, 2003; Lesh, Yoon & Zawojewski, 2007; Doerr, 2016). The process of construction of models is a social process. The communication in the classroom between the students and the teacher permits that the conceptual systems or models can be shared, manipulated, modified and reused, to describe, interpret, construct, manipulate, predict or control systems.

The MMP proposes that students solve situations close to everyday life, including the so-called Model-Eliciting Activities, which are designed by following principles as the *Reality Principle*, the *Model Construction Principle*, *Model Documentation Principle*, the *Self-Evaluation Principle*, the *Model Generalization Principle*, the *Simple Prototype Principle* (Doerr, 2016). These activities require that students develop a mathematical interpretation, which require different ways of thinking about the situation.

The process demands posing questions, formulate conjectures, quantify information, organize and analyze data, do calculations, establish relationships, apply procedures, create representations, develop criteria, and evaluate results. The students have to express, test and refine their ways of thinking about the situation. They create sharable, manipulatable, modifiable, and reusable conceptual tools.

The product of learning is not the model but the process of its construction. “Learning mathematical content occurs through the process of developing an adequate and productive model that can be used and re-used in a range of contexts” (Doerr, 2016, p. 198). The models reside in the mind and in the representational media. The “meanings associated with a given conceptual system tend to be distributed across a variety of representational media” used to interpret the situation (Lesh and Doerr, 2003, p. 12).

Methodology

The research was qualitative. The data were collected from eight high school teachers (Teams M, N and P). They solved two MEAs during a workshop. The MEA discussed here is called: The Cultivation of Avocado (MEA1). The second MEA was called: Reforestation Proposal. The goal of MEA1 was to support the modification and extension of the teachers' conceptual system around the concepts of function, variation, equation, rate of change, and unknown; to put them in situations where they had to interpret, explain, justify, and evaluate the “goodness” of the models, as Models and Modeling perspective suggest. And finally, we designed the MEA to raise awareness of the problem of deforestation in Michoacán, México. The six principles of the MEAs construction were used to create the activity because we were interested in provoking that teachers a) created sharable, manipulative, modifiable, and reusable conceptual tools; b) expressed, tested and refined their ways of thinking about what learning mathematics and function, variation and equation concepts mean.

The MEA was about the effect of growth of avocado crop in Michoacán forest. We used information taken from Michoacan state’s official documents (*Comisión Forestal del Estado de Michoacán, 2014*) and national newspapers, such as Greenpeace journal. Data about the historical deforestation since 1976 was also included. Solving the problem required to estimate numbers because the data came from several sources, so there were different quantities.
The implementation of the MEA lasted four hours. The phases, following Lesh and Doerr (2003) recommendations, were: 1) Newspaper individual reading activity; 2) Resolution in small groups (M, N and P); 3) Group discussions of models; 4) Individual problem solving. The qualitative data sources included audio and video clips from the two sessions, letters with procedures to solve the activities, photographs of procedures performed on the blackboard, and teacher's notes.

Results and discussion

In this section, the models created by the small groups/teams of teachers are described. It is explained how students were engaged in the model-eliciting activities and the potential of the activity was assessed to get the meanings using the six principles to design a MEA.

Newspaper individual reading activity

Since the teachers were from Michoacan, and were near the forest, they knew that problem (MEA1) was a real-life situation. So, the eight teachers made sense of it based on their personal knowledge and experiences (*Reality Principle*).

Resolution in small groups

There were two types of models: linear algebraic model, and exponential tabular and graphic model. Teams M and N developed a description, explanation, and prediction of the forest deforestation situation (*Model Construction Principle*). They created models in paper and pencil environment. A variety of representation systems (concrete, graphic, and symbolic) were used to describe and illustrate relationships, operations and patterns. The concepts of function, variation, equation, rate of change, and unknown were used. However, the team P decided that the problem could not be solved because the information was not accurate.

The teachers of the team M created numerical and graphic representations.

> *Teacher M3:* What would I do here? Well… For example I would make a graph, in which it would…. Since when have you done this study?

They predicted that in 40 years there would be 147,000 hectares of avocado, almost equivalent to the existing forest. The exponential model was based on several assumptions.

The teachers of the team N created numerical and algebraic representation. The conclusion of the team N was: "the forest will disappear before the year 2189, but only if the deforestation does not increase year after year". They proposed the function $y = -800x + 145344$.

These responses revealed how they were thinking about the situation and how they used variables, linear and exponential functional relationships, graphical and algebraic representations, and equations (*Model Documentation Principle*).

Group discussions of models

The working group discussion highlighted the value of the models. The teachers modified, extended and refined their ways of thinking about the problem and the model. The eight teachers assessed the usefulness of alternative responses (*Self-Evaluation Principle*). They were able to judge for themselves when their responses were good enough.

The teacher M1 explained that the linear model could be applied to a broader range of situations (*Model Generalization Principle*).

> *Teacher N1: So, I checked the data that we have, and an exponential model cannot be done, can it?* It is a lineal model, and it is not good in a long term, but it gives us a good idea.

He mentioned that the process of estimating was a transferable and reusable way of thinking. The solution provided a useful prototype for interpreting a variety of other structurally similar situations (*Simple Prototype Principle*).

Individual problem solving activity

Six teachers described a linear model as procedure to solve the situation. One teacher wrote that it was not easy to elaborate a model from the data –he referred to build it with precision. The other teacher mentioned that the exponential model was the most adequate. All the teachers commented that undoubtedly, the forest would disappear, but it was difficult to know exactly when, due to the lack of precise data in the problem.

Conclusions

What happened when the math teachers solved the MEA? The teachers were confronted with traditional perceptions of solving routine problems in the classroom that needed a unique solution. The team P was an example, because they did not validate the process of estimation. However, the communication during the teamwork and the group discussion allowed modifying, extending, and refining mathematical knowledge (variation, function and equation), mathematical abilities, such as estimation, because the teachers had to choose and discriminate information. The six principles to design a MEA were fulfilled, we had enough information about the evolution of the teacher's knowledge and the eight teachers learned about the importance of inciting students to create models to learn mathematical knowledge and abilities. Although it is important to note that this was due to group work.

It was found that this MEA was interesting for all the teachers. They exhibited new ways of thinking about learning mathematics, and it emerged a strong interest in applying this kind of activities in the classroom to encourage mathematical learning (function, variation, equation, linear and exponential relations and estimation), to raise awareness of the problem of deforestation, and to develop abilities to interpret, explain, justify, and evaluate models.

Acknowledgments

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References

Ärlebäck, J. B., Doerr, H., & O'Neil, A. (2013). A Modeling Perspective on Interpreting Rates of Change in Context. *Mathematical Thinking and Learning*, 15(4), 314-336.

Comisión Forestal del Estado de Michoacán (2014). *Inventario estatal forestal y de suelos Michoacán de Ocampo 2014*. Morelia: Gobierno del Estado de Michoacán.

Doerr, H. M. (2016). Designing Sequences of Model Development Tasks. In C. R. Hirsch & A. R. McDuffie (Eds.), *Annual Perspectives in Mathematics Education 2016: Mathematical modeling and modeling mathematics* (pp. 197-205). Reston, Va: National Council of Teachers of Mathematics.

Greeno, J. G., Collins, A. M., & Resnick, L. B. (1996). Cognition and Learning. In D. C. Berliner & R. C. Calfee, (Eds.), *Handbook of Educational Psychology*. New York: Macmillan.

Lesh, R. & Doerr, H. M. (2003). Foundations of a models and modelling perspective on mathematics teaching, learning, and problem solving. In R. Lesh, & H. Doerr (Eds.), *Beyond constructivism. Models and Modeling perspectives on mathematics problem solving, learning and teaching* (pp. 3-34). Mahwah, NJ: Lawrence Erlbaum.

Lesh, R., Yoon, C., & Zawojewski, J. (2007). John Dewey revisited: Making mathematics practical versus making practice mathematical. In R. A. Lesh, E. Hamilton, & J. J. Kaput (Eds.), *Foundations for the future in mathematics education* (pp. 315-348). Mahwah, NJ: Lawrence Erlbaum Associates.

National Council of Teachers of Mathematics. (2011). *Developing essential understanding of expressions, equations and functions*. Reston, VA: National Council of Teachers of Mathematics.

Vargas, V., Cristóbal, C. Carmona, G. Reyes, J. & Alvarado, A. (2014). Campus Viviente Quintana Roo. In Ramos M. y Aguilera V. (Eds.), *Experiencias en la formación y operación de Cuerpos Académicos*, 22-30. México: Ecorfan.

TEACHER VIEWS OF USEFUL FEATURES OF MATHEMATICS PROFESSIONAL DEVELOPMENT AND HOW THE VIEWS INFLUENCED INSTRUCTION

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This study investigated what two high school teachers viewed as useful from a mathematics professional development (PD) program and how these views influenced their instructional practices. The teachers participated in PD that focused on using standards-based pedagogy and mathematical tasks with higher-level demands. The teachers commonly viewed mathematical tasks for use in the classroom and strategies for small group or whole class participation as useful features of the PD. Each of the PD features that teachers viewed as useful were enacted during instruction. These findings suggest that providers of PD for K-12 teachers of mathematics should learn teachers' views about the usefulness associated with PD programs to improve outcomes.

Keywords: professional development, academic standards, secondary mathematics

Research has documented that standards-based instruction in mathematics can be challenging for teachers to enact (e.g. National Council of Teachers of Mathematics (NCTM), 2014). Changes in teacher practice can be linked to teachers' perceptions about PD (Chapman, 2011; Martin & Gonzalez, 2017). This research investigated the question, what do teachers view as useful from a PD program designed to help teachers implement standards-based mathematics and how do these views influence their instructional practices?

Theoretical Framework

Stein, Remillard, and Smith (2007) describe the temporal phases of curriculum use. The written curriculum represents the design of materials, the intended curriculum represents teacher interpretations, and the enacted curriculum refers to what occurs during instruction. Learning opportunities are transformed among the phases due to teachers, students, and the context of teaching (Stein et al., 2007). The National Governors Association Center for Best Practices and the Council of Chief State School Officers (NGACBP and CCSSO) (2010) identified eight standards for mathematical practice that "describe varieties of expertise that mathematics educators at all levels should seek to develop in their students" (p. 6). The NCTM (2014) outlined eight mathematics teaching practices that "provide a framework for strengthening the teaching and learning of mathematics" (p. 9). The temporal phases of curriculum use and the sixteen mathematical practices were used as frameworks to classify curriculum use.

Methodology

This research is part of a larger study (Walker, 2016) that investigated factors that influenced mathematics instructional practice when participating in PD. A multiple-case study design (Merriam, 2009) was used to collect and analyze data on teachers' views of useful features of PD. The PD program for this research was Teaching Algebra with Practice Standards (TAPS). Fifteen middle and high school mathematics teachers participated in TAPS. Four of the fifteen teachers, all from one high school, volunteered to participate in this research. This summary focuses on two teachers who taught algebra.

Data Collection

Data sources used to describe the PD included field notes from the PD sessions and email interview responses from the PD providers. Data for instructional practices consisted of two observed and videotaped lessons for each teacher. The Wisconsin Longitudinal Study observation tool (Shafer, Wagner, & Davis, 1997), adapted to include the sixteen mathematical practices, was used for the observations.

The teachers were interviewed five times using adapted protocols (Shafer, Davis, & Wagner, 1997, 1998). The first interview took place in the summer after the PD to learn what each teacher found to be useful about the PD. The next two interviews took place during the first half of the school year. These interviews were before and after the first observed lesson to learn the teachers' views about the intended lesson, the enacted lesson, and how the PD related to the lesson. The final two interviews took place during the second half of the school year, before and after the second observed lesson. Influence on instruction was determined by comparing classroom observation notes with the portions of the PD that teachers identified as useful.

Data Analysis

Classroom observation data were used to describe the alignment between each teacher's enacted instruction and the sixteen practice standards. Lessons were classified as no evidence, sometimes, or yes. Reliability was checked with an independently classified lesson by calculating a Krippendorff's (2004) alpha value of 0.8223. Each of the teacher interviews were analyzed using an inductive approach of comparative pattern analysis to create a category coding system (Merriam, 2009). The reliability of the coding system was checked with independent coding by calculating the percent of agreement, which was 90%.

The PD Program

TAPS was a three-year, grant-funded mathematics PD program with two goals: (a) enrich teachers' knowledge and skills for teaching algebra, and (b) improve students' algebraic knowledge, skills, and disposition. Learning about the sixteen mathematics practice standards and finding ways to implement them in classrooms were the main teaching points for the PD. The PD also included the use of mathematical tasks with higher-level cogitative demand (Stein, Smith, Henningsen, & Silver, 2000). TAPS started with a ten-day institute in June 2015. Sample lessons and activities about patterns, relationships, and generalizations were used to help teachers learn about the mathematical practices. Three two-hour after school follow-up sessions took place during the school year. A total of eighty-six hours of PD were provided for the teachers.

Teacher 1: Doug Collins

Doug Collins was a male with thirteen years of teaching experience. This was his second year at the high school and he taught Algebra 1 and Geometry. During the interviews, Mr. Collins made five comments about the usefulness of receiving activities for his class. For example, when asked what was useful about the PD he stated, "For me I think it was really kind of learning, getting some of the different activities and things that you can do." Mr. Collins also made four comments about strategies for students working together as useful. For example, Mr. Collins was asked what features of the PD he would use and he replied, "Partner work, group work, giving [the students] a certain task."

Doug Collins: Enacted Curriculum

The first observed lesson was a task that Mr. Collins developed during the PD. Each student was given an algebraic expression on either a gold or a green piece of paper. Mr. Collins explained that a student with a gold sheet should find a student with a green sheet. They would set their algebraic expressions equal to each other and then find a value for the unknown that would make the equation true. The students worked in pairs on this activity for thirty-five minutes. They checked answers with each other and explained methods used to find an answer.

In the second observed lesson, Mr. Collins used a modified version of “Bridge Strength” (Lappan, 2005), which was presented during the PD. At the beginning of the lesson, he asked the students to find a partner and to gather materials for the lesson. They suspended paper bridges across two books, placed pennies on the bridges, and recorded the number of pennies required to collapse each bridge. When data collection was complete, the students made graphs and answered questions about the activity.

Doug Collins: Summary

The two observed lessons included features consistent with the practice standards like *making sense of problems and persevering in solving them*. These features were evident when the students worked in pairs to find solutions. Mr. Collins identified providing activities for his class and students working together in small groups as useful features of the PD. He used one mathematical task that he developed during the PD and another the PD providers presented. Both of the features that Mr. Collins identified as useful were observed as part of the enacted curriculum in his class.

Teacher 2: Ruth Lawrence

Ruth Lawrence was a female teacher with nine years of teaching experience, eight of them at the high school. At the time of this research, she was teaching Algebra 1 and Geometry. Ms. Lawrence made eleven comments about the PD providing activities for her class. For example, when she was asked what was useful about the PD she responded, “The tasks that you guys showed us were … they were just the best tasks that I have ever been exposed to.” There were four comments by Ms. Lawrence that strategies for students working in small groups was useful. When asked about using practice standards in her class Ms. Lawrence replied, “I feel like there was … one [standard] that referred to students discussing math together, like collaborative. I would definitely say that that would be done.”

Ruth Lawrence: Enacted Curriculum

The first lesson took place over two days. The lesson involved an “S-Pattern” task (Institute for Learning: Learning Research and Development Center, 2015) from the summer PD. Ms. Lawrence asked the students to get into pairs and answer questions about the patterns. The groups answered questions about representing the total number of squares algebraically until the end of the class. The next day, Ms. Lawrence asked different students to share the equations they created with the whole class.

The second lesson also took place over two days. The task was the “Painted Cube” (Lappan, Fitzgerald, & Fey, 2006) from the summer PD. Ms. Lawrence asked the students to get into small groups and complete a worksheet about painted cubes with edge length two, three, four, five, six, and any length “n.” The students used snap-cubes to build models and worked in groups to complete the worksheet. The next day Ms. Lawrence invited students to share algebraic expressions for a cube with an edge length of “n” with the whole class.

Ruth Lawrence: Summary

The two observed lessons included features consistent with the practice standards like *using and connecting mathematical representations*. These practice standards were evident when the students worked in small groups to find algebraic expressions to represent visual patterns. The two useful features of the PD identified by Ms. Lawrence were providing mathematical tasks for the class and students working together in small groups or as a whole class. Each of the observed lessons included a task from the PD and students were observed participating in small group work and whole class discussions.

Discussion and Implications for PD

The two teachers commonly identified mathematical tasks for use in the classroom and strategies for small group or whole class work as useful features from the PD. Each of the useful features were observed when the teachers enacted lessons aligned to the PD. For example, each teacher viewed mathematical tasks for their class as useful and both teachers used tasks from the PD during instruction. This finding is consistent with Martin and Gonzalez (2017) who found that teachers valued focusing on student mathematical thinking and reported increased wait time because of a PD intervention.

Teachers play a critical role in the outcomes of PD. What teachers identify as useful reflects their practical knowledge and situated perspective (c.f. Chapman, 2011). PD that addresses features identified as useful and is consistent with teachers' knowledge of classroom situations can result in instructional changes. PD providers should spend time learning about the practical knowledge of teachers and what teachers view as useful because these views influence how teachers will use the PD in their classrooms.

References

Chapman, O. (2011). Elementary school teachers' growth in inquiry-based teaching of mathematics. *ZDM Mathematics Education, 43*(6), 951-963.

Institute for Learning: Learning Research and Development Center. (2015). *Institute for learning mathematics: S-pattern task*. Retrieved from http://www.nctm.org/Conferences-and-Professional-Development/Principles-to-Actions-Toolkit/Resources/10-HS-Ziegler-LessonGuide-S-Pattern/

Krippendorff, K. (2004). Reliability in content analysis: Some common misconceptions and recommendations. *Human Communication Research, 30*(3), 411-433.

Lappan, G. (2005). *Thinking with mathematical models: Linear and inverse variation*. Boston, MA: Prentice Hall.

Lappan, G., Fitzgerald, W. M., & Fey, J. T. (2006). *Frogs, fleas, and painted cubes: Quadratic relationships*. Boston, MA: Pearson/Prentice Hall.

Martin, T. S., & Gonzalez, G. (2017). *Teacher perceptions about value and influence of professional development.* Paper presented at the 39th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Indianapolis, IN.

Merriam, S. B. (2009). *Qualitative research: A guide to design and implementation*. San Francisco, CA: John Wiley and Sons, Inc.

NCTM. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: NCTM.

NGACBP & CCSSO. (2010). *Common core state standards for mathematics*. Washington, DC: NGACBP, CCSSO.

Shafer, M. C., Davis, J., & Wagner, L. R. (1997). *Teaching and learning mathematics: Working paper #4*. University of Wisconsin–Madison.

Shafer, M. C., Davis, J., & Wagner, L. R. (1998). *Instructional planning and classroom interaction: Working paper #3*. University of Wisconsin–Madison.

Shafer, M. C., Wagner, L. R., & Davis, J. (1997). *Classroom observation scale: Working paper #6*. University of Wisconsin–Madison.

Stein, M. K., Remillard, J. T., & Smith, M. S. (2007). How curriculum influences student learning. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 319-369). Charlotte, NC: Information Age Publishing.

Stein, M. K., Smith, M. S., Henningsen, M., & Silver, E. A. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development*. New York, NY: Teacher College Press.

Walker, W. S., III. (2016). *Factors influencing teacher instructional practice in mathematics when participating in professional development.* Open Access Dissertations. Retrieved from https://docs.lib.purdue.edu/open_access_dissertations/1024

SUPPORTING AN INQUIRY STANCE WITH DOUBLE DEMONSTRATION LESSONS

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Effective professional development (PD) activities positively influence participants' classroom practices. However, ways in which teachers engage in PD activities vary widely. Farmer, Gerretson, and Lassak (2003) identified three levels of teacher appropriation for those involved in PD activities (practitioner, professional, and inquiry stances), with the inquiry stance indicative of teachers engaging in self-sustaining practices. We employed demonstration lessons in various forms as a way to explore classroom practices in an ongoing PD project. In the double demonstration lesson format, teachers took an active role in changing an observed lesson and then viewing impacts of those changes as a second lesson was taught. We offer evidence that this structure provided opportunities for teachers to engage in an inquiry stance on teaching and discuss implications for PD providers in structuring activities to foster an inquiry stance.

Keywords: Teacher Education-Inservice/Professional development

Professional development efforts should have the primary goal of supporting teachers in developing a vision for teaching that includes engaging students with challenging tasks and facilitating rich mathematical discourse (Loucks-Horsley, Stiles, Mundry, Love, & Hewson, 2010; National Council of Teachers of Mathematics [NCTM], 2014). It should further encourage applying knowledge gained to analyze mathematics lessons, and interpreting and using students' mathematical thinking to guide instruction (Sztajn, Borko, & Smith, 2017). Although a variety of PD models exist, demonstration lessons represent one form of PD that is situated within practice and, therefore, holds promise for meeting this goal (Loucks-Horsley et al., 2010).

In this report, we present evidence of teacher stances from a variation on demonstration lessons, *double demonstration lessons (DDLs)*, which we purposefully designed to engage teachers in an inquiry stance toward teaching. By engaging teachers who likely hold a practitioner or professional stance in the processes associated with DDLs, we offered them the opportunity to legitimately participate (Lave & Wenger, 1991) in the act of inquiry on teaching. This work is guided by the question: How might we design PD activities that support a teacher, who likely holds a practitioner or professional stance, in adopting an inquiry stance?

Theoretical Framework

To better understand the ways teachers appropriate knowledge gained from mathematics PD experiences, Farmer and colleagues (2003) offered a reflective three-level model, which included Practitioner, Professional, and Inquiry stances. They argued that although the levels consist of the same elements, how the pieces are viewed by the teachers creates the distinction among the levels. With a *practitioner's stance*, the teacher focuses on specific ideas that can be taken from the PD and used with little modification in his or her classroom. In contrast, the teacher with a *professional stance* approaches the PD as an opportunity for professional learning. Finally, teachers who have adopted an *inquiry stance* see themselves learning from the process of teaching. Farmer and colleagues (2003) noted that teachers who adopt an inquiry stance exhibit "self-sustaining changes in their mathematics instructional practices" (p. 331).

Introduced by Lave and Wenger (1991), situated learning theory espouses learning to be the result of "legitimate peripheral participation in communities of practice" (p. 31). Based on this theory, the community of practice represents a group of experts that possesses the knowledge to be learned by the novice. Legitimate peripheral participation allows the novice learner to engage in the practices of the community, constructing the knowledge of the community over time.

These two theories together form the conceptual framework that guides our work. We view teachers that have adopted an inquiry stance (Farmer et al., 2003) as the experts within a community of practice (Lave & Wenger, 1991), which is represented by our ongoing PD project. By designing PD activities that place novice practitioners (teachers who have adopted a practitioner or professional stance) alongside the experts in the community, the novice practitioners may engage in legitimate peripheral participation with the discussions of the community. The novice practitioners' legitimate engagement aids movement towards developing an inquiry stance. We argue that DDLs provide this opportunity for engagement.

Methodology and Context

This study took place during the fourth and fifth years of a six-year externally funded PD project serving over 150 K-8 mathematics teachers. The overarching goals of the project were to increase the mathematical content knowledge of the teachers and to increase their use of research-based instructional practices. Demonstration lessons were employed in a variety of forms (Strayer et al., 2017) to help teachers better understand how to blend deep learning of content with research-based instructional practices. Double demonstration lessons emerged as a meaningful context for engaging teachers in an inquiry stance. To gain insight into how the structure of the DDL supported teachers in this way, we gathered and analyzed data on a subset of teachers along with other general data sources collected during DDLs.

In a typical demonstration lesson, teachers were briefed on the lesson they were about to watch, watched a project faculty member teach a lesson to a class of students, and then participated in a debriefing of observations. We noticed in the early years of our project that teachers tended to view demonstration lessons with a practitioner's stance. Thus, we developed what we refer to as *DDLs* with a goal of supporting teachers in developing an inquiry stance.

Double demonstration lessons (Strayer et al., 2017) expand the typical demonstration lesson process to include revision of the lesson and an additional teaching and debriefing of the lesson. During the lesson briefing, teachers are made aware of the forthcoming opportunity to revise the demonstration lesson. Their observations of the demonstration lesson are supported through the use of an observation protocol. This combination of the protocol with the explicit goal of revising the lesson provides teachers with the opportunity to participate in processes associated with an inquiry stance, as they reflect on the initial lesson and collaboratively revise the lesson.

In this report, we draw on data from one grades 3-5 DDL. The lesson involved students looking for and making use of structure in a linear systems problem (Riddell, 2016). During the DDLs, we collected multiple writing prompts from all teachers (e.g., entrance/exit tickets, observations guides, debrief documents) and interviewed four purposefully selected teachers before and after the lesson. In addition, we video-recorded the debrief and used this data to document teachers' stances. We coded all data using Farmer and colleagues (2003) framework.

Findings

Written data documents revealed that teachers entered the PD with practitioner and professional stances toward the events of the day. For example, on entrance tickets teachers stated they hoped to learn "how to get students to persevere" (Tonya, Entrance Ticket) or "to gain insight on different teaching strategies" (Courtney, Entrance Ticket). The first evidence of a professional stance and the second a professional stance, both representative responses.

In contrast, we observed evidence of engaging in an inquiry stance in the debriefing discussions at the end of the day. The final group debriefing was prompted by the question, “What are the big ideas that you have in relation to today’s lessons?” Following the second lesson, teachers provided evidence of legitimately engaging in an inquiry stance, particularly as they addressed the role of scaffolding that was introduced as a revision to the lesson.

Tonya: Does the scaffolding always help all students? I think if the number had been higher they would have had to think about it longer, they just doubled. We didn’t think it made that big of a difference.

Peter: What if we had given them this (see Figure 1). Now it leads them to think the pieces are different values. Since we have just two copies of the same thing, I don’t think they had to think about the value of each one.

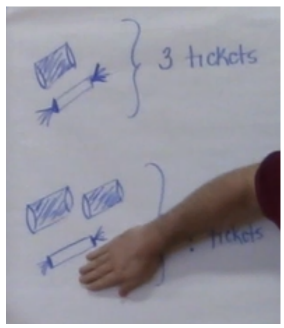

Figure 1. Scaffolding Question Revision

Tina: We wanted students to see the value of the model of all the candies not individual candies. We talked about making it more than 3 tickets, but we didn’t want to take up too much time. The purpose was to see that this group (the two candies that total to 3 tickets in Figure 1), can be pulled into and assigned a value together, instead of separating them apart. So that you don’t have to know the individual prices.

Tina and Peter continued to discuss the students in Lesson 1 and Lesson 2 and the differences in what the students noticed about the task in each fifth-grade class. During this discussion, the lead teacher exclaimed that the scaffolding question provided her exactly what she needed in the second lesson, because the scaffolding question gave her something to direct students’ attention so they were able to continue their thinking. Afterwards, the teachers shared more ideas.

Kim: I do think if the previous class had the scaffolding problem, they would have gotten further. They were struggling to make the 10, by grouping them, they weren’t connecting them because they were still thinking about different values.

Maggie: I really liked that everybody had an entry point. Every kid looked at that (the scaffolding question) and saw the structure, but it gave them something to hold onto that wasn’t beyond the limit.

Mary: Structure, I don’t use that word. I say powers or groupings, but that’s a word I need to make sure I am bringing out in my class. I don’t think the students understood the word structure.

Here, two of the PD instructors paused the discussion to elaborate on mathematical structure. As the PD team, we knew that the idea of structure was a complex concept, and we tried to attend to the difference between mathematical structure and organizational structure. We expressed that creating a structure (e.g., a table) to help notice patterns or numbers is different than the structure underlying the featured problem. A teacher expressed her wondering relating to structure, followed by another teachers' reflection about structure.

Mona: The big idea was wondering if when the students first came I wonder if they understand what structure means, but not just a term, but what structure are they looking for, structure in what?

Lindsay: It has taken a long time of teaching to pick up on structure myself and see how it relates to each component. Structure, as a whole, bridges across all content. It's just making those connections because not being taught that way, it's difficult for us as teachers. At the beginning (before Lesson 1), it's interesting to see when we all personally solved that problem how many of us actually used the structure that we hoped the students would actually pick up on. I bet a majority of us were using algorithms or some equation or something to get us there as opposed to structure, because that's not how most of us learned. Thinking about the use of structure is something I have to push myself to do.

Throughout these exchanges, teachers were exploring student thinking and viewing the demonstration lessons as a place for inquiry about student learning, mathematics teaching, and their own understanding of the mathematics involved.

Discussion

Developing opportunities to engage in levels of practitioner, professional, and inquiry stances is the first step towards encouraging teachers to fully develop an inquiry stance on teaching and become learners of their own practice. By design, DDLs provide teachers with the opportunity to achieve three essential PD outcomes: taking ownership of the teaching through a problem-solving vision; applying knowledge gained to analyze mathematics lessons; and interpreting and using students' mathematical thinking to guide instruction. Furthermore, our results suggested that the DDL model allowed the teachers to take initial steps towards ownership of an inquiry stance, positioning them to make sustained changes in their instructional practices (Farmer et al., 2003). If DDLs can support teachers' development of an inquiry stance during PD activities, then teachers will be better prepared to analyze mathematics lessons and interpret and use students' mathematical thinking to guide their instruction. Teachers will be empowered to implement mathematics instruction that supports the deep learning of mathematics for all students.

References

Farmer, J. D., Gerretson, H., & Lassak, M. (2003). What teachers take from professional development: Cases and implications. *Journal of Mathematics Teacher Education*, *6*, 331–360.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge: Cambridge University Press.

Loucks-Horsley, S., Stiles, K., Mundry, S., Love, N., & Hewson, P. W. (2010). *Designing professional development for teachers of science and mathematics* (3rd ed.). Thousand Oaks, CA: Corwin Press.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: Author.

Riddell, C. (2016). How many tickets? *Teaching Children Mathematics*, *23*, 128–131.

Strayer, J. F., Barlow, A. T., Lischka, A. E., Gerstenschlager, N. E., Stephens, D. C., Willingham, J. C., & Hartland, K. S. (2017). Meeting the needs expressed by teachers: Adaptations of the traditional model for demonstration lessons. *NCSM Journal of Mathematics Education Leadership, 18*(1), 18-26.

Sztajn, P., Borko, H., & Smith, T. M. (2017). Research on mathematics professional development. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 793–823). Reston, VA: National Council of Teachers of Mathematics.

SEEING GOALS "COME TO LIFE": TEACHING EXPERIMENTS AND TEACHERS' WORKGROUP CONVERSATIONS

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Shifts in student participation prompted by math teachers' experiments with a new teaching strategy created opportunities for the teachers to learn about student perseverance. The practice, in which teachers asked students stuck on a math problem to share three things they know, achieved one desired goal and got students "unstuck," but the teachers wondered whether the practice supported students to persevere in the sense of learning to draw on one another as resources. Through negotiation of a shared, new, and problematic experience in the classroom, the teachers problematized their prior conceptions of perseverance.

Keywords: Teacher Education-Inservice/Professional Development, High School Education, Instructional Activities and Practices.

Objectives

This study examines four workgroup conversations of a team of high school math teachers studying how to improve student perseverance in math class. The teachers agreed do a teaching experiment in which they tried a teaching strategy that they hoped would help students struggle productively, together. In their workgroup conversations, they reflected on what they noticed while trying the strategy and the implications of their noticings for their joint goal. I analyze their workgroup conversations to better understand: 1) How do the teachers frame their learning goal, "perseverance," across the meetings? and, 2) What do they notice about perseverance and its role in student learning of mathematics when they try their chosen teaching strategy? Preliminary analysis indicates that the shifts in student participation brought about by their use of this teaching strategy allowed problems of practice around perseverance to "come to life" for the teachers. Reflections on those problems of practice in workgroup conversations supported the teachers to negotiate teaching and learning mathematics.

This study indicates that teams of teachers with limited expertise in ambitious mathematics teaching can problematize and discuss their classroom practice by experimenting with teaching strategies that they hypothesize will create richer learning environments for students.

Background

Teachers learn in professional workgroups by negotiating notions of teaching and learning with peers (Little, 2002). Teachers have more opportunities to learn from workgroup conversations when they reflect on problems of practice raised through analysis of records of practice (Horn, Garner, Kane, & Brasel, 2016).

Different teaching experiences surface different problems of practice and, therefore, support different opportunities for teachers to learn. Horn and Kane (2015) found that teachers with more expertise in enacting ambitious mathematics teaching practices had richer workgroup conversations. Ambitious mathematics teaching puts teachers in complex problem-solving situations, in which students' experiences and perspectives are constantly considered (Lampert, 2010). Such situations should be conducive for teacher learning in the moment, as teachers grapple with problems, and in planning and reflection, as teachers consider the influence of their moves on students' experiences of math and math class.

But what about teachers who do not have expertise in ambitious mathematics teaching? Horn and Kane argue that such teachers are caught in a paradox of not being able to learn because they struggle to do the teaching they aim to learn-- teaching that would put them in problematic situations and give them fodder to talk about students' experiences of math and math class. This paper explores a possible way for leaders of professional development to support the workgroup conversations of such teacher: by engaging them in teaching experiments that introduce new, and potentially richer, forms of participation for their students.

Such a use of teaching experiments has precedent in teacher education. Pre-service math teacher educators have suggested introducing novice math teachers to ambitious teaching in part because they hypothesize that novices who use them will find themselves in teaching situations requiring problem-solving (McDonald, Kazemi, & Kavanagh, 2013). Kazemi and Franke (2004) found that when elementary math teachers elicited more thinking from their students, opportunities to learn in their workgroup deepened. These studies suggest that changing opportunities for students to participate can open learning opportunities for teachers even when teachers are not experts in ambitious teaching.

Methods

This paper describes preliminary analysis in an on-going study. A department of four high school math teachers (Cindy, Leo, Soledad, and Jamie) in an urban California district met every two weeks as part of the TRU-Lesson Study (TRU-LS) professional development project. TRU-LS engages departments of math teachers in Lesson Study, in which teachers iteratively research, plan, implement, and reflect on a lesson to learn more about a problem of practice, supported by the Teaching for Robust Understanding of Mathematics framework (Schoenfeld et al., in preparation). As part of TRU-LS, the four teachers engaged in bi-monthly cycles of inquiry leading up to their research lesson. The goal of these cycles was to iteratively build their *research theme*, a problem of practice, and *theory of action*, a hypothesis about what they could do in their research lesson to learn about their research theme. The teachers tried, collected data about, and reflected on an ambitious teaching strategy developed by the TRU-LS research team.

This analysis focuses on their first four meetings of TRU-LS. I co-planned the meetings with the facilitator, Elizabeth, and attended and video recorded the meetings as a participant-observer.

For my analysis, I isolated segments of video in which the teachers discussed what happened when they tried their chosen teaching strategy, up to twenty-five minutes of each hour-long meeting. I made activity logs of these segments, in which I identified episodes in which teachers shared records and problems of practice; negotiated uses of their chosen strategy; and/or revised their research theme and theory of action. I examined my logs for ways that teachers framed records of practice, problems of practice, and goals (Goffman, 1986) and patterns in their participation (Bannister, 2018). This analysis is preliminary.

Analysis

In their first department meeting, the teachers developed the research theme, "Building student perseverance/ capacity to struggle productively, together." In order to gather information about their theme and see it, as Elizabeth said, "come to life," the teachers agreed to try a strategy called Three Things: "If students are stuck on a problem, ask them to state three things they know about the problem and three things they are wondering about."

The teachers hypothesized that Three Things would help students who did not know how to get started on a problem realize how what they knew might be useful. The teachers also thought that Three Things might encourage students to explain their thinking. They hoped that it might become a routine that students could eventually use to get unstuck without teacher intervention. In this discussion, they conceptualized "perseverance" as students' capacity to continue working on a challenging problem after getting stuck. They acknowledged that collaboration and vocalizing thinking for peers are important for perseverance. What role talking with peers might play in supporting students to struggle productively was as of yet unarticulated, however.

In the second and third sessions, the teachers revised Three Things and their goals for its use based on their experiences with students. In the fourth session, three of the four teachers described how Three Things encouraged more students to participate. Cindy reported that when she used Three Things in a whole-class discussion about a challenging problem, students who did not typically turn in written work were encouraged to talk. Jamie and Soledad described how Three Things helped students get unstuck on math problems. From this discussion, it seemed that Three Things helped the teachers support students to persevere in both the sense of getting students unstuck and encouraging them to share their thinking.

Leo then challenged whether use of Three Things actually promoted collaborative productive struggle. He agreed that Three Things increased student participation, which was a positive step. But he argued that it shifted the burden to struggle away from students:

> "Um so I think that it *(Three Things)* increases student engagement. And that, sort of, decreases our opportunity to see persistence because they don't run into that wall? You know what I mean? So in my mind persistence is you get stuck, and then you try something else, or you—there's some strategy that you use. But we're providing a way to—to avoid that resistance, to avoid the wall. So we don't necessarily see the persistence."

Shifting the burden of coming up with a strategy to persevere from student to teacher, Leo argued, was contrary to their goal of supporting students to struggle productively, together.

This comment led the teachers to revise their definition of "persistence," which they used interchangeably with perseverance. They reflected on how persistence looks for students who experience math problems differently. Jamie summed up the distinctions they made:

> "Cause like, for some students, persistence means, I come to a grinding halt, I reassess, and I continue. For others it means, like, tinkering and continuing along? And for—and for others, they haven't, because of where the cognitive demand is currently, they haven't even come to that level of needing to persist."

Guided by the TRU framework, the teachers brainstormed experiences that might lead students to shut down. Elizabeth captured their thinking in this way:

> Cognitive Demand: "Why are we doing this? This is so hard. When do we get to the real math?"
>
> Equitable Access: Kids don't feel like they have a toolbox
>
> Agency/Ownership/Identity: Fixed mindset—"I'm not good at math."

Building on this thinking, Soledad suggested that Three Things gave students "more access to content, less access to perseverance" by taking away the opportunity for students to build and draw on a toolbox. The other three teachers gave a thumbs-up as Elizabeth recorded this. Her comment and the discussion that followed teased apart two goals that they had for student perseverance. They wanted students to persevere so that more students had access to challenging mathematical content and practices. But they also wanted students to develop the tools to move through struggle together, without relying on a teacher. Three Things was not helping them achieve this second goal for student perseverance. If perseverance means the capacity for students to struggle productively, together, having a teacher ask for three things did not help students develop that capacity.

As a result, the teachers chose a new strategy, “Ask what your partner said.” When the teachers approached a group stuck on a challenging problem, they would ask one student to share what another member of a group thought or said about the problem. The teachers hypothesized that this strategy would encourage students to turn to one another as resources when they got stuck. This new strategy launched them into another cycle of inquiry, armed with deeper understanding of the goals for student perseverance in math class.

Discussion

Noticing shifts in participation as a result of teaching experiments can open opportunities for teachers to learn during workgroup conversations. The teachers’ lack of experience with teaching practices that put students’ experiences at the forefront was mitigated by a teaching strategy that supported the teachers to notice new forms of participation from their students. Further research into the interplay between teachers’ learning in workgroups and experience in the classroom is needed to better understand the learning opportunities in this kind of teaching experiment.

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References

Bannister, N. A. (2018). Theorizing Collaborative Mathematics Teacher Learning in Communities of Practice. *Journal for Research in Mathematics Education*, *49*(2), 125–139.

Goffman, E. (1986). *Frame Analysis: An Essay on the Organization of Experience* (Later Reprint). Boston: Northeastern University Press.

Horn, I. S., Garner, B., Kane, B. D., & Brasel, J. (2016). A Taxonomy of Instructional Learning Opportunities in Teachers’ Workgroup Conversations. *Journal of Teacher Education*, *68*. https://doi.org/10.1177/0022487116676315

Horn, I. S., & Kane, B. D. (2015). Opportunities for Professional Learning in Mathematics Teacher Workgroup Conversations: Relationships to Instructional Expertise. *Journal of the Learning Sciences*, *24*(3), 373–418. https://doi.org/10.1080/10508406.2015.1034865

Kazemi, E., & Franke, M. L. (2004). Teacher Learning in Mathematics: Using Student Work to Promote Collective Inquiry. *Journal of Mathematics Teacher Education*, *7*(3), 203–235. https://doi.org/10.1023/B:JMTE.0000033084.26326.19

Lampert, M. (2010). Learning Teaching in, from, and for Practice: What Do We Mean? *Journal of Teacher Education*, *61*(1–2), 21–34. https://doi.org/10.1177/0022487109347321

Little, J. W. (2002). Locating learning in teachers’ communities of practice: opening up problems of analysis in records of everyday work. *Teaching and Teacher Education*, *18*(8), 917–946. https://doi.org/10.1016/S0742-051X(02)00052-5

McDonald, M., Kazemi, E., & Kavanagh, S. S. (2013). Core Practices and Pedagogies of Teacher Education A Call for a Common Language and Collective Activity. *Journal of Teacher Education*, 0022487113493807. https://doi.org/10.1177/0022487113493807

Schoenfeld, A. H., Dosalmas, A., Fink, H., Sayavedra, A., Weltman, A., Zarkh, A., & Zuniga-Ruiz, S. (in preparation). Teaching for Robust Understanding with Lesson Study. In R. Huang, A. Takahashi, & J. P. Ponte (Eds.), *Theory and Practices of Lesson Study in Mathematics: An international perspective*. New York: Springer.

INVESTIGATION INTO THE ROLE OF MATHEMATICS COACHES

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Hill, Rowan, and Ball (2005) explored the effects of a specific type of teacher knowledge, mathematics knowledge for teaching. They found that "teachers' mathematics knowledge for teaching positively predicted student gains in mathematics achievement during the first and third grade" (p. 399). From the results of their work, the authors called for identifying teachers who have a diminished knowledge in this area and provide them focused professional development.

To be effective, opportunities for teachers to grow in their understanding of mathematics knowledge for teaching must be persistent and long term (Garet et. al., 2001). One strategy for establishing long-term professional growth is the development of a specialized group of educators that situated in the school environment (Ansty & Clarke, 2010). Hull, Balka, and Miles (2009) describe an iteration of this specialized group as mathematics coaches, "an individual who is well versed in mathematics content and pedagogy and who works directly with classroom teachers to improve student learning of mathematics" (p. 3). Research suggests that mathematics coaches can have a positive impact on student achievement when there is "school-based professional interaction between coaches and teacher is of sufficient duration to permit emergence of coherent collective efforts marked by active learning and focused on content and pedagogy, as well as on student understanding." (Campbell & Malkus, 2011, p.451).

While mathematics coaches provide an opportunity to support effective mathematics instruction, it is less clear how mathematics coaches are currently being prepared and what their interactions are with practicing teachers (Campbell & Malkus, 2011). This study looked to gain a better understanding of the role of mathematics coaches by surveying superintendents and practicing mathematics coaches. Data gathered from superintendents focused on why school districts do or do not employ mathematics coaches, as well as the attributes they value in the position. Data from mathematics coaches focused on the ways in which they were prepared and utilized in their schools. Results suggest a positive perception of mathematics coaches form superintendents, but concerns related to financial limitations. Results from mathematics coaches suggest significant variation in both their preparation and utilization.

References

Anstey, L., & Clarke, B. (2010). Leading and Supporting Mathematics Teacher Change: The Case of Teaching and Learning Coaches. *Mathematics Teacher Education and Development*, *12*(2), 5-31.

Campbell, P. F., & Malkus, N. N. (2011). The impact of elementary mathematics coaches on student achievement. *The Elementary School Journal*, *111*(3), 430-454.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American educational research journal*, *42*(2), 371-406.

Hull, T. H., Balka, D. S., & Miles, R. H. (Eds.). (2009). A guide to mathematics coaching: Processes for increasing student achievement. Corwin Press.

Wayne, A. J., & Youngs, P. (2003). Teacher characteristics and student achievement gains: A review. *Review of Educational research*, *73*(1), 89-122.

FIVE COMPONENTS TO CONSIDER WHEN CHARACTERIZING MATHEMATICS TEACHERS' ACCESS TO PROFESSIONAL DEVELOPMENT

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Implementing changes in policy, such as new mathematics content standards, is a persistent challenge of large scale educational improvements. To support implementation efforts, professional development (PD) has been shown to be a determining factor for educators to make sense of new standards (Hill & Cohen, 2001). While a consensus exists about features of effective PD (Darling-Hammond et al., 2009), in practice, opportunities that teachers have to learn and improve their practice varies, is shaped by local education leaders, and is often not aligned with these research-based recommendations.

Following newly adopted state mathematics content standards, a research-practice partnership (Peneul et al., 2015) was formed between our state education agency and four universities in our state to co-design PD resources for standards implementation and research these efforts. To organize this work, we draw upon design-based implementation research (Fishman et al., 2013) to structure the development of materials and study the implementation. As part of our ongoing work, we developed a framework that identifies and describes key components of effective mathematics PD. The framework was created using a synthesis of literature identifying five key components of high quality mathematics PD: content, pedagogical model, teacher learning environment, attention to context, and structure. The framework includes both a description of each component and examples of activities and tools research-based mathematics PD programs draw upon with each component. Organizing the components in this way is not meant to be prescriptive, but rather to provide a link from observable characteristics of high quality mathematics PD to research evidence, offer examples, and serve as a boundary object (Wenger, 1998) for the partnership and the math leaders within. In doing so, we aim for the framework to serve as an implementation measure (Bryk et al., 2015) to gauge progress in improving implementation and opportunities teachers have to learn at scale and for district- and school-based mathematics leaders to use when adapting co-designed PD resources to their local contexts.

In this poster session, we will share the development of the framework, the ways it can be used as an implementation measure, and the ways in which districts and schools can use the framework when adapting co-designed PD resources for use in local contexts. Our hope is that this framework will support efforts to provide more equitable access to high quality mathematics professional development to teachers across the state.

References

Bryk, A. S., Gomez, L. M., Grunow, A., & LeMahieu, P. G. (2015). Learning to Improve: How America's Schools Can Get Better at Getting Better. Harvard Education Press. 8 Story Street First Floor, Cambridge, MA 02138.

Darling-Hammond, L., Wei, R. C., Andree, A., Richardson, N., & Orphanos, S. (2009). Professional Learning in the Learning Profession: A Status Report on Teacher Development in the US and Abroad. Technical Report. National Staff Development Council.

Fishman, B. J., Penuel, W. R., Allen, A. R., Cheng, B. H., & Sabelli, N. O. R. A. (2013). Design-based implementation research: An emerging model for transforming the relationship of research and practice. *National society for the study of education*, 112(2), 136-156.

Penuel, W. R., Allen, A. R., Coburn, C. E., & Farrell, C. (2015). Conceptualizing research–practice partnerships as joint work at boundaries. *Journal of Education for Students Placed at Risk (JESPAR)*, 20(1-2), 182-197.

THE EFFECTS OF A STATE-IMPLEMENTED CO-TEACHING TRAINING ON STUDENTS' MATHEMATICS ACHIEVEMENT SCORES

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Co-teaching is an instructional model in which special and general education teachers collaboratively educate a class made up of students with and without disabilities. Co-teaching has become an increasingly popular special education service delivery model over the past fifteen years, yet few studies have evaluated the effects of co-teaching on student achievement (McDuffie, Mastropieri, & Scruggs, 2009; Murawski, 2006; Murawski & Swanson, 2001).

The purpose of this study is to describe a state-sponsored co-teaching professional development project (the Co-Teaching Project), and analyze student mathematics achievement data from classrooms participating in the Co-Teaching Project. The Co-Teaching Project, in its sixth year of implementation, aims to create inclusive middle and high-school math classrooms by providing effective training and on-going coaching in co-teaching and other inclusive educational strategies. Co-teaching pairs, consisting of general and special education teachers who will be teaching together in the upcoming school year, receive training in the summer as well as on-going coaching throughout the school year. T-tests conducted on pilot pre- and post-test data from nine classes indicate that students without disabilities made significant ($p <.01$) improvements in eight classes, with an average Cohen's d effect size of 4.31 (range 0.67-11.45). Students with disabilities made significant ($p < .01$) improvements in seven classes, with an average Cohen's d effect size of 2.62 (range 0.8-6.23). While these results are promising, more sophisticated analysis methods are needed to thoroughly analyze and disaggregate student achievement results in relationship to the Co-Teaching Project.

In this follow-up study, we use Generalized Equation Estimating (GEE) to analyze and compare the mathematics achievement of approximately 1200 sixth- through ninth-grade students with and without disabilities in both co-taught (15 classes) and non-co-taught comparison classrooms (15 special education resource classes, 15 non-co-taught general education classes) to answer the following research questions: (1) Do students with (and without) disabilities achieve at higher rates in co-taught settings versus non-co-taught settings?; (2) Do students with disabilities achieve at comparable rates as students without disabilities in co-taught (and non-co-taught) settings?; (3) To what degree do student factors (other than disability/no-disability) affect the growth rates of students in co-taught and non-co-taught settings? These factors include: (a) Gender, Ethnicity, SES, language status, (b) Grade level, (c) Type of disability, (d) Prior Achievement level.

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References

McDuffie, K.A., Mastropieri, M., & Scruggs, T.E. (2009). Differential effects of peer tutoring in co-taught and non-co-taught classes: Results for content learning and student-teacher interactions. *Exceptional Children, 75*(4), 493-510.

Murawski, W.W. (2006). Student outcomes in co-taught secondary English classes: How can we improve?. *Reading & Writing Quarterly, 22*(3), 227-247.

Murawski, W.W., & Swanson, H.L. (2001). A meta-analysis of co-teaching research: Where are the data?. *Remedial and Special Education, 22*(5), 258-267.

COACHING THE 5 PRACTICES THROUGH LESSON IMAGING

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Keywords: Lesson imaging, coach press, 5 practices

Stein and Smith's seminal book, *5 Practices for Orchestrating Productive Mathematics Discussions* (2011)*,* outlines practices that can support teachers attempting a student-centered approach to instruction. Although teachers may understand these practices, it is another thing to implement them, especially if there is little mentoring support. We present data from a study in which a coach led a team of teachers to implement the five practices for the first time.

Participants and Data Collection/Analysis

Four 7th-grade teachers, one ELA school coach, two university faculty, and two doctoral students met once per week for a 1.5- hour planning meeting during which one faculty member coached. Our main research questions were, *how do teachers take up the five practices differently as they implement a reform textbook for the first time and what mentoring techniques support the change in their practices?* Transcriptions of the meetings were analyzed using the Constant Comparison Method (Glaser & Strauss, 1967) to identify themes as well as code for the types of supports the coach was giving.

Findings

The university coach used *lesson imaging* (Stephan et al., 2016) during planning meetings. Lesson *imaging* is the process of imagining the flow of the lesson, what the teacher might say during a launch, what questions she might ask, and how students might engage in the activities. Data indicate that the coach pressed teachers to a) image how students might solve the tasks in ways that result in both correct and incorrect answers, b) unpack the mathematical goal of the lesson, c) consider what parts of the launch students should understand in order to participate effectively in the lesson, and d) use learning progressions to determine whether the activity could be eliminated. As the project progressed, teachers started asking these questions and the coach became more of a participant. Our poster elaborates the imaging process and the university coach's role in pressing teachers to take the students' perspective as they imagine the flow of lessons. This can contribute to the knowledge base on coaching that tends to focus on model or co-teaching, video watching, and providing feedback.

References

Stephan, M., Pugalee, D., Cline, J., & Cline, C. (2016). *Lesson imaging in math and science: Anticipating student ideas and questions for deeper STEM learning*. Alexandria: VA, Association of Supervisors and Curriculum Designers.

Glaser, B. & Strauss, A. (1967) *The discovery of grounded theory: strategies for qualitative research*. Chicago: Aldine.

Smith, M. S., & Stein, M. K. (2011). *5 practices for orchestrating* productive mathematics discussions. Reston, VA: National Council of Teachers of Mathematics.

RURAL INSERVICE MATH TEACHERS' PERSPECTIVES FROM A THREE-YEAR SUMMER MATH INSTITUTE PROFESSIONAL DEVELOPMENT PROJECT

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Professional development opportunities can greatly impact teachers' pedagogical perspectives and content knowledge (Ball, 1990; Hill, 2007). Through a situated lens, teachers' learning depends on the social and cultural contexts in which that learning occurs (Lave & Wenger, 1991). This study examines the effects of a Summer Math Institute that is part of a three-year professional development project known as *Math Counts*. *Math Counts* focuses on supporting elementary teachers' learning in math pedagogy and content in the context of a rural Appalachian school district. In this analysis, we focus on the development of the participating teachers' pedagogical perspectives and content knowledge, drawing from surveys, content assessments, and interview data. The study informs the ongoing project and contributes to efforts that seek to support the mathematics learning of elementary students and their teachers in historically underserved communities. We examine the teachers' learning, key resources, and opportunities for learning during a Summer Institute in which we engaged teachers with math-focused tasks and utilized a *decentering* approach. This approach shifted attention to the elementary students taught by the participating teachers and the modification of tasks and supports based on the teacher's knowledge of students' math learning challenges and assets. We investigated the following research questions in this study:

1. What are teachers' perspectives on content knowledge, pedagogy, and student/teacher relationships?
2. How is the teachers' content knowledge affected by the Summer Institute?
3. What resources from the Summer Institute contributed to the teachers' learning from their perspective?

Findings indicate that teachers' content knowledge increased in most cases and that many teachers viewed the discussions about their students and task adaptation as resources for their learning. The findings of the study have implications for designing professional development that is effective in supporting teacher voice, content understanding, and task implementation.

References

Ball, D. L. (1990). Breaking with experience in learning to teach mathematics: The role of a preservice methods course. *For the Learning of Mathematics, 10*(2), 10-16.

Hill, H. C. (2007). Mathematical knowledge of middle school teachers: Implications for the No Child Left Behind policy initiative. *Educational Evaluation and Policy Analysis, 29*(2), 95-114.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Boston, MA: Cambridge University Press.

DEVELOPING SPECIAL EDUCATION TEACHER OBSERVATION INSTRUMENTS FOR MATHEMATICS INSTRUCTION

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This poster presents the development of teacher observation instruments focused on mathematics instruction for students with disabilities (SWDs). Looking back, there is an evidence base to support several instructional practices that develop the math abilities of SWD, but there remains the enduring challenge of translating the evidence base to practice. Looking forward, there is a need to integrate research conducted in different fields, specifically mathematics education and special education, to support special education teachers (SETs) in their content instruction for diverse learners. The aim of these observation instruments is to reliably evaluate SETs mathematics instruction and provide content-specific, actionable feedback that results in improved outcomes for SWDs (Hill & Grossman, 2013).

The poster will describe the use of Evidence-Centered Design to create a coherent assessment system (Johnson, Crawford, Moylan & Zheng, 2018; Mislevy, Steinberg, & Almond, 2003). It outlines the development process for several mathematics instruction rubrics: Developing Conceptual Understanding, Developing Understanding of Procedures, Practicing Procedures, and Word Problem Solving. Attention is placed on two phases of development intended to translate research-to-practice: a) creation of items that reflect the research base from special education and mathematics education, and b) development of performance-level descriptors derived from observation of actual practice. The poster will also present psychometric properties of one particular rubric, the Developing Understanding of Procedures rubric, studied through many-facet Rasch measurement (Eckes, 2011). We present findings from analysis in which 45 videos of mathematics instruction (15 teachers, 3 videos each) were scored with a partially-crossed, connected design by 12 raters. The poster will summarize reliability and fit statistics for four facets: teachers, lessons, raters and items. Finally, the poster will describe next steps in developing a validity argument for these observation instruments.

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References

Johnson, E. S., Crawford, A., Moylan, L. A, & Zheng, Y. (2018). Using Evidence-Centered Design to create a special educator observation system. *Educational Measurement: Issues and Practice, 37*(2), 35-44.

Eckes, T. (2011). *Introduction to many-facet Rasch measurement.* Frankfurt, Germany: Peter Lang.

Hill, H., & Grossman, P. (2013). Learning from teacher observations: Challenges and opportunities posed by new teacher observation systems. *Harvard Educational Review, 83,* 371-384.

Mislevy, R. J., Steinberg, L. S., & Almond, R. G. (2003). On the structure of educational assessments. *Measurement: Interdisciplinary Research and Perspectives, 1,* 3-67.

SUPPORTING SHIFTS IN CLASSROOM PRACTICE: LESSONS FROM A COLLABORATIVE PROFESSIONAL DEVELOPMENT

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We share a model of professional development and lessons learned from a two-year professional development project involving 60 teachers in grades K – 6 and 8 schools, addressing five goals: (1) increase students' understanding of and achievement in mathematics; (2) improve teachers' knowledge of mathematics; (3) strengthen teachers pedagogical skills; (4) improve teachers knowledge of uses of technology to support learning; and (5) strengthen teachers' dispositions to both reflect on their practice and collaborate to improve their teaching.

Our work was guided by the Leading for Mathematical Proficiency Framework (Bay-Williams & McGatha, 2014) to support teachers' shifts in classroom practice to implement the Standards for Mathematical Practice (CCSSO, 2010). We followed a "responsive and emergent" professional development curriculum (Confrey & Lachance, 2000, p. 244), centered around teachers' needs and research-based approaches to improve teaching skills. We formed a professional development team (PDT) including 2 mathematics educators, one mathematician, a math coach, and 2 mathematics education graduate students. A subgroup of seven lead teachers worked closely with the PDT to plan workshops, provide support to other teachers, and serve as a resource to other teachers so that project ideas are integrated into practice at their schools.

In order to contribute to the research on the changing mathematical and pedagogical demands on the preparation of teachers, we examined the shifts in classroom practice after two years of professional development, as well as the teachers' development of their mathematical knowledge needed for teaching. Data sources included an initial teacher inventory of classroom practices, lesson plans created by the teachers, and pre- and post-measures using the scales developed by the Learning Mathematics for Teaching (LMT) project (Hill, Ball & Schilling, 2004).

We are beginning to see the benefits of this approach. As we will share in our analysis, many teachers are incorporating number talks, formative assessment, and a Math Workshop model (Fosnot, 2007) into their lessons and experiencing shifts in classroom practice that display teaching skills that provide opportunities for students to demonstrate the Mathematical Practices (CCSSO, 2010).

References

Bay-Williams, J. M. & McGatha, M. B. (2014). *Mathematics coaching: Resources and tools for coaches and leaders* K-12. Boston: Pearson.

CCSSO (Council of Chief State School Officers). (2010). *Common core state standards*. Retrieved from http://www.corestandards.org/Math/

Confrey, J., & Lachance, A. (2000). Chapter 10: Transformative teaching experiments through conjecture-driven research design. In Lesh & Kelly (Eds.), *Handbook for design research in mathematics and science education* (pp. 231-266). New York: Lawrence Erlbaum.

Fosnot, C. T. (2007). Teaching and learning in a math workshop. In C. T. Fosnot, *Investigating Fractions Decimals and Percents Grades 4 – 6* (27- 41). Portsmouth, NH. Firsthand.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2004). Developing measures of teachers' mathematics knowledge for teaching. *Elementary School Journal*, *105*(1), 11-30.

TEACHER PERSPECTIVES ON FEEDBACK: A COMPARISON BETWEEN IMPLICIT THEORIES

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Keywords: Implicit Theories, Feedback, Mathematics Teacher Education

Many studies have shown feedback to be one of the most influential factors on student achievement; however, there are conflicting results and inconsistent patterns (Hattie & Timperley, 2007). This may be attributed to the variations of how feedback is given as well as the reasoning for why a teacher may provide feedback in a particular way (Shute, 2008). For example, teachers may provide feedback according to their initial judgement of their students' ability levels or based on whether the students demonstrate grit or perseverance (Dweck, 2006).

Personal experiences, cultural contexts, attitudes, or implicit beliefs held by teachers about learning mathematics may also contribute to a teacher's judgment when providing feedback (Brown, Lake, & Matters, 2011). In fact, Boaler (2016) claimed that out of all contributing factors, teachers' implicit beliefs may have the most influence on the information conveyed in the classroom. Often teachers believe they have little influence on their students' mathematical ability (Dweck, 2006; Dweck, Chiu, & Hong, 1995). As a result, teachers may only provide feedback that leads students to one method of solving a problem or only the correct solution. Teachers' underlying assumptions of students influence their decisions to provide feedback, even though it is unclear how or why (Dweck et al., 1995).

In order to be successful, students must be given opportunities to engage with meaningful mathematics and be given feedback to help them move forward in their learning and understanding (Boaler, 2015). Although implicit theories have been shown to be a mediator of students' actions in the classroom (Good, Aronson, & Inzlicht, 2003), teachers' implicit theories are often overlooked as a mediator of the feedback they provide.

My study focuses on why elementary mathematics teachers provide feedback during mathematics instruction. With very little existing research on why teachers provide different types of feedback (Rattan et al., 2012), this study will give insight into factors that may attribute to teachers' initial judgements when providing feedback. The results will serve to inform mathematics teacher educators who design professional development aimed at supporting teachers' feedback practices.

References

Boaler, J. (2015). *What's math got to do with it?* New York, NY: Penguin Books.

Brown, G. T. L., Lake, R., & Matters, G. (2011). Queensland teachers' conceptions of assessment: The impact of policy priorities on teacher attitudes. *Teaching and Teacher Education, 27*, 210-220.

Dweck, C. S. (2006). *Mindset: The new psychology of success.* New York, NY: Random House Inc.

Dweck, C. S., Chiu, C., & Hong, Y. (1995). Implicit theories and their role in the judgments and reactions: A world from two perspectives. *Psychological Inquiry, 6*, 267 – 285.

Good, C., Aronson, J., & Inzlicht, M. (2003). Improving adolescent's standardized test performance: An intervention to reduce the effects of stereotype threat. *Applied Developmental Psychology, 24*, 645-662.

Hattie, J., & Timperley, H. (2007). The power of feedback. *Review of Educational Research, 77*, 81 – 112.

Rattan, A., Good, C., & Dweck, C. S. (2012). "It's ok – Not everyone can be good at math": Instructors with an entity theory comfort (and demotivate) students. *Journal of Experimental Social Psychology, 48*, 731-737.

COMPONENTS OF WHOLE-CLASS DISCUSSIONS IN ELEMENTARY MATHEMATICS CLASSROOMS

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Whole-class discussion has been identified as a core practice of mathematics teaching. These discussions are often embedded in a lesson structure that includes a launch phase in which teachers introduce a mathematical task, an exploration phase in which students work on the task and the teacher circulates, and a discussion phase. The discussion can take many forms, but often includes the teacher displaying one or more pieces of student work from the students' engagement with the task and facilitating a conversation around the mathematical ideas in the strategies depicted in the student work (Stein, Engle, Smith, & Hughes, 2008). Despite the importance of discussions, implementing them remains a challenge. Much of the research on whole-class discussions has focused on the facilitation of discussions, while less attention has been given to other components of discussions.

To address this gap, I characterized components of whole-class discussions in the context of fraction instruction. Specifically, I used grounded theory to analyze 29 videos of classroom lessons documenting all potential components of the discussions. I looked for a variety of components identified in the literature, but also noted components that were not frequently mentioned. The lessons were taught by grades 3–5 teachers who were at various points in their participation in three years of professional development focused on children's fraction thinking and how teachers can build on that thinking during instruction. My study occurred within the context of a larger study, *Responsive Teaching in Elementary Mathematics*, and focused on the 29 teachers who posed an equal sharing problem with a fractional answer and chose to include a whole-class discussion during the lesson in which they were observed.

Analyses revealed an extensive list of whole-class discussion components, which were narrowed to identify four components that could be used to characterize the formats of discussions in elementary mathematics classrooms: (a) the start of the discussion; (b) how the strategy was displayed to the class, including the possible recreation of the strategy by the teacher; (c) the characteristics of student work shared; and (d) the end of the discussion. These components can enhance our understanding of the *discuss* phase of mathematics lessons and help support teachers in envisioning and implementing this challenging phase.

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References

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, *10*(4), 313–340.

EXPLORING CONVERSATION THEMES IN PROFESSIONAL LEARNING GROUPS

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Previous research into professional learning groups (PLGs) in mathematics has mainly focused on the benefits of the professional development on the teachers (e.g., Brahier & Schäffner, 2004). Since previous research on PLGs has pointed to a need for the professional development to be participant-driven (e.g., Gojmerac & Cherubini, 2012), what types of discussions are used within these groups in mathematics is important for discussions in creating effective and sustainable PLGs. The research project this poster is a part of focused on the discussions of the professional learning group in order to explore where teachers focused their own professional development by examining their conversations during the 17 individual (mainly half day) PLG meetings.

This research is a portion of a three-year narrative case study with grades 6-10 teachers who were board mandated to create a PLG to discuss the transitions from grade 8 to grade 9. The research data within the study included meeting tapes and transcripts, artefacts, and teacher interviews. The data was analyzed as part of the larger study; however, the focus of this portion was to determine themes related to the discussions of the teachers. In order to determine themes, the entire data set was explored to identify characteristics of the conversations. Conversations were only considered *if* there was an interaction between two or more participants using the framework of symbolic interaction (Blumer, 2004) as a basis to determine the portions that were important to the research. Next, these conversation characteristics were compiled in order to determine overarching themes. The themes were then combined as needed, and then the data set was again revisited to ensure that all of the themes were included in the final description.

In the end, eight themes were determined to be representative of the entire group conversations: factors outside of teacher control, topics related to group structure, classroom strategies, student-related conversations, mathematics-specific conversations, learning trajectories, and program differences. Although a smaller topic of conversation, it was noteable that the teachers did spend time discussing things like scheduling problems and whether or not there would be funding to continue the group meetings (factors outside of teacher control). The entire, larger research study concluded that this PLG was successful in supporting the development of the teachers within the group (to varying degrees). With this being a successful form of professional development, exploring the teacher conversations was important in exploring what teachers felt was important for improving their mathematics teaching in order to expand the conversation of how to operationalize the strengths of this group to other groups. Since PLGs are entirely teacher-driven (e.g., Gojmerac & Cherubini, 2012), a focus on the conversations within a group are essential to expand the research in mathematics education around how to support teacher growth through this type of professional development.

References

Blumer, H. (2004). *George Herbert Mead and human conduct*. (T. J. Morrione, Ed.). Walnut Creek, CA: AltaMira Press.

Brahier, D. J., & Schäffner, M. (2004). The effects of a study-group process on the implementation of reform in mathematics education. *School Science and Mathematics*, *104*(4), 170-178.

Gojmerac, A., & Cherubini, L. (2012). Connecting for greater purpose: An examination of a district professional learning community. *Journal of Teaching and Learning*, *8*(1), 43-58.

EXPLORING ASIAN AMERICAN MATHEMATICS TEACHERS' PERCEPTION AND IMPLEMENTATION OF CULTURALLY RELEVANT PEDAGOGY

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Keywords: Teacher Identity, Culturally Relevant Pedagogy, Asian American Teachers

This study explores Asian American mathematics teachers' perception and implementation of culturally relevant pedagogy through a multiple case study approach. Rubel (2017) described the concept of meritocracy, color-blindness, and inability to relate to students as "tools of whiteness" (p. 68) in mathematics education. The model minority myth (MMM) is an example of how those tools are manifested in society. Specifically, in education, the MMM ignores the structural obstacles that stem from race and socioeconomic class inequalities and emphasizes that hard work and a good work ethic are sufficient to be successful (Chen & Buell, 2017). The ideas embedded in the MMM directly oppose the principles of culturally relevant pedagogy (CRP). Contrary to the values of the MMM, culturally relevant pedagogy considers students' cultural and racial backgrounds as assets and as significant components of a curriculum (Ladson-Billings, 1994). Asian American mathematics educators, especially those who teach in urban and diverse contexts, are in the middle of this tension between the MMM and CRP.

Previous studies have argued that teachers' narratives as learners and reflections help to construct their teaching practices and professional identity (Drake, Spillane, & Hufferd-Ackles, 2001; Richardwon, 2003). Given this significance of personal histories and reflections, it is worthwhile to investigate Asian American teachers' narratives and how they perceive and implement culturally relevant pedagogy.

Interview data with three Asian American secondary mathematics teachers, along with an analysis of their lesson plans and sketches of themselves, led to a preliminary conclusion that teachers who have struggled with their personal identity construction as Asian Americans were more likely to foster students' personal identity construction through mathematics lessons. For example, one participant, Sunny, who defined herself as someone who "became Asian American" and that her identity "is still changing," planned a lesson that incorporated students' identity construction. Her students graphed an avatar of themselves using a computer program and then had to describe why and how that avatar represented them. Whereas another participant, Ann, who simply described herself as a "Korean American" and did not have much opportunity to discover her racial identity, did not perceive CRP as an effective pedagogy and turned to direct instruction. Findings suggest that reflections by pre-service and in-service are critical to the mathematics teacher's professional identity construction process, positive perceptions of CRP, and effective implementation of CRP strategies.

Key References

Chen, G. A., & Buell, J. Y. (2017). Of models and myths: Asian (Americans) in STEM and the neoliberal racial project. *Race Ethnicity and Education*, 1-19.

Drake, C., Spillane, J. P., & Hufferd-Ackles, K. (2001). Storied identities: Teacher learning and subject-matter context. *Journal of Curriculum Studies*, *33*(1), 1-23.

Ladson-Billings, G. (1994). The dreamkeepers: Successful teaching for African-American students.

Rubel, L. H. (2017). Equity-Directed Instructional Practices: Beyond the Dominant Perspective. *Journal of Urban Mathematics Education*, *10*(2).

WOMEN IN MATHEMATICS EDUCATION: AN ANALYSIS OF GENDER IN PROFESSIONAL ORGANIZATIONS GLOBALLY

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The underrepresentation of females in mathematics has been widely studied and analyzed. However, once women are retained in STEM fields including mathematics, further disparities are seen in their representation in leadership roles, especially in professional organizations (Welch, Parker, & Welch, 2013). Although efforts have been made to encourage the qualities necessary to hold leadership roles, this has been a more recent development (Dugan, J.P., Faith, K. Q., Howes, S. D., Lavelle, K. R., & Polanin, J.R., 2013). This begged the question as to if the disparities experienced by women in mathematical organizations in leadership roles was also experienced by women in mathematics education organizations. Historically, women have been more present in mathematics education when compared to mathematics (Safi, 2012). However, if this was reflected in leadership representation was unknown. Additionally, through these organizations, the representation of women's research in proceedings was questioned.

A theoretical framework of cultural feminism was used for the analysis in order to challenge not only the presence of women in leadership role but also to consider the general cultural attitudes toward women professionally as tied in to country or region of organization (Vasavada, 2012). Historical rosters were gathered from professional organizations for mathematics educators from six continents. The specific countries in these were chosen not only for their organizational presence but also for availability doctoral programs in mathematics education (Safi, 2014). Recent conference proceedings for select organizations were analyzed to determine relative proportions of accepted female presenters. It was found that, although research by females does tend to be featured at organizational conferences, representation at leadership levels is disproportionate in favor of males nearly uniformly across the globe. Further implications include funding initiatives to promote women's leadership is such organizations.

References

Dugan, J. P., Faith, K. Q., Howes, S. D., Lavelle, K. R., & Polanin, J. R. (2013). Developing the leadership capacity and leader efficacy of college women in science, technology, engineering, and math fields. *Journal of Leadership Studies, 7*(3), 6 – 23. doi: 10.1002/jls.21292

Pasha-Zaidi, N. & Afari, E. (2016). Gender in STEM education: an exploratory study of student perceptions of math and science instructors in the United Arab Emirates. *International Journal of Science & Mathematics Education, 14*(7), 1215 – 1231. doi: 10.1007/s10763-015-9656-z

Safi, F. (2014). A 100-year retrospective on research focus areas and doctoral programs in mathematics education. In Oesterle, S., Nicol, C., Liljedahl, P., & Allan, D. (Eds.) *Proceedings of the Joint Meeting of PME 38 and PME-NA 36, Vol. 6.*

Safi, F. (2012) Analyzing the evolution of the field of mathematics education: A historical perspective. *Proceedings of the 34th Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education – PME-NA*. Kalamazoo, Michigan.

Vasavada, T. (2012). A cultural feminist perspective on leadership in nonprofit organizations: A case of women leaders in India. *Public Administration Quarterly, 36*(4): 462 – 503.

Welch, E., Parker, M., & Welch E. W. (2013). Professional networks, science ability, and gender determinants of three types of leadership in academic science and engineering. *Leadership Quarterly, 24*(2), 332 – 348

REHEARSALS WITH IN-SERVICE TEACHERS: RECOGNIZING AND RESPONDING TO TEACHERS' CONFIDENCE

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Mathematics teacher education can be challenging to deliver in a way that it makes an impact on teachers' practice. Over a century ago, John Dewey emphasized teachers' need for laboratory experience (Dewey, 1904/1965). Similarly, Lampert et al. (2013) argue that situating the knowledge teachers gain in teacher education courses in actual teaching practice can help ensure that the knowledge translates to practice. They have argued for the use of teaching rehearsals, where teachers practice teaching a lesson to peers in a low stakes environment where a mathematics teacher educator (MTE) provides in the moment feedback. Rehearsals can be an answer to Dewey's century old call for laboratory experience. While this is a promising instructional technique, little is known about how in-service teachers engage in these rehearsals. This research focuses on conceptualizing in-service teachers' engagement in rehearsals so MTEs can better support teachers in productively engaging in these rehearsals.

In the summer of 2017 two MTEs engaged 14 middle school teachers in a week long professional development session using rehearsals. The professional development centered on how to teach a unit on algebraic generalization using figural patterns. During the week, there were six different rehearsals taught by six of the 14 participants. I chose three of the participating teachers who seemed to engage in different ways to focus my analysis on.

During the rehearsals, the MTEs or the other participants would pause the teaching to give in the moment feedback or discuss particular teaching moves. I analyzed these pauses, attending to the acting teacher's gaze and gestures as well as noting the purpose and topic of the pause. The following questions became salient in terms of characterizing the differences in the teachers' participation: Where does the teacher look for suggestions? And How does she engage with the suggestions? Teacher A tended to look to peers for feedback and generally accepted that feedback. Teacher B tended to look to the MTEs for feedback. Teacher C looked to both sources for feedback, but also seemed to see herself as a competent contributor.

This is significant because MTEs can use these results to better notice and support teachers' engagement. In particular, MTEs could specifically attend to where the teacher is looking for suggestions and how she is engaging with those suggestions. If an MTE was working with a teacher like Teacher A, they could create opportunities for that teacher to engage more critically with suggestions from others. If someone like Teacher B is looking to the MTEs for a suggestion, the MTEs could turn the question to the rest of the group. Thinking about who the teacher is asking for suggestions and how she is engaging with those suggestions can be helpful as MTEs decides how to facilitate productive engagement.

References

Lampert, M., Franke, M. L., Kazemi, E., Ghousseini, H., Turrou, A. C., Beasley, H., Cunard, A., & Crowe, K. (2013). Keeping It Complex: Using Rehearsals to Support Novice Teacher Learning of Ambitious Teaching. *Journal of Teacher Education*, *64*(3), 226–243. https://doi.org/10.1177/0022487112473837

Dewey, J. (1965). The relation of theory to practice in education. In M. Borrowman (Ed.), Teacher education in America: A documentary history (pp. 140-171). New York: Teachers College Press

ANALYZING ADJUNCT INSTRUCTOR'S ENGAGEMENT WITH A RESEARCH-BASED PRECALCULUS CURRICULUM

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Teachers play an important role in sustaining educational reform efforts by shaping how curriculum unfolds inside the classroom (Cohen & Ball, 1999). They need support to develop instructional capacity and influence their students' learning (Cohen & Ball, 1999). Support is important for teachers despite their appointment type, for example, full-time versus part-time faculty (Gappa, Austin, & Trice, 2007). Research points to a growing trend in higher education towards hiring part-time adjunct faculty (Mason, 2009; Curtis, 2014; Snyder & Dillow, 2015). This trend warrants research on adjunct instructors' experiences (Kezar & Sam, 2013). I provide an in-depth look at three adjunct instructors' experiences as they implemented a research-based Precalculus curriculum. Findings from this study have implications for designing professional developing programs for adjunct instructors.

To analyze adjunct instructors' experiences as they implemented a research based mathematics curriculum, I employed a sociocultural framework focusing on "mediated action" (Wertsch, 1998). I analyzed 3 adjunct instructors' engagement with the curriculum as they planned their instruction, enacted their lessons inside their classrooms, collaborated with their colleagues or reflected on their teaching experiences. Using the case study methodology (Yin, 2009) I describe adjunct instructors' engagement with the curriculum. Data was collected in the Fall 2016 and Spring 2017 semesters using semi-structured interviews, classroom observations and collaboration data from online meetings.

As the instructors engaged with the curriculum while planning, enacting, collaborating and reflecting, each mode of engagement influenced instructors' engagement through other forms. For example, planning was influenced by challenges faced while enacting the curriculum, by ideas discussed with other instructors and instructors' own reflection. Challenges arose while instructors planned their instruction and enacted their lesson. Collaboration and reflection took the form of supports for the instructors. Findings from this study adds to the limited body of research that focuses on mathematics adjunct instructors' experiences and can guide the development of professional development that is aligned with their needs.

References

Cohen, D. K. and Ball, D. L. (1999). Instruction, Capacity, and Improvement. *CPRE Research Reports.*
Curtis, J. W. (2014). The Employment Status of Instructional Staff Members in Higher Education, Fall 2011. Washington, DC: *American Association of University Professors, 2011-12.*
Gappa, J. M., Austin, A. E., & Trice, A. G. (2007). *Rethinking faculty work: Higher Education's Strategic Imperative*. Jossey-Bass.
Kezar, A., & Sam, C. (2013). Institutionalizing equitable policies and practices for contingent faculty. *The Journal of Higher Education, 84*(1), 56-87.
Mason, M. A. (2009). Is tenure a trap for women? *The Chronicle of Higher Education, 4.*
Snyder, T. D., & Dillow, S. A. (2015). Digest of Education Statistics 2013. (NCES 2015-011). *National Center for Education Statistics Working Paper.* Washington, DC: US Department of Education.
Wertsch, J. V. (1998). *Mind as Action*. New York, NY: Oxford University Press.
Yin, R. K. (2009). *Case Study Research, Design & Methods 4th ed*. Thousand Oaks, CA: SAGE.

HEAD START PRESCHOOL EDUCATORS' CONCEPTIONS OF MATHEMATICS LEARNING AND TEACHING

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Mathematics education leaders and national organizations have called for improved training and professional development for early childhood educators (Ginsburg, Lee, & Boyd, 2008; NAEYC & NCTM, 2010). To address these calls, our research group is designing and implementing a collaborative professional development (PD) model to support preschool mathematics teaching that builds on children's mathematical thinking and play. The PD involves 25 Head Start educators in whole-group learning sessions, video-club discussion, collaborative activity design, and classroom coaching.

Throughout the PD we are attending to teachers' development of new ways of thinking, being, and acting in classroom interactions. In this poster, we focus on two research questions: (a) how do these Head Start teachers conceptualize children's mathematics learning and development? And (b) how do these teachers conceptualize their role in supporting children's mathematical development?

We interviewed teachers at the beginning of the school year to explore their perspectives. The teachers discussed and provided examples of their mathematics teaching practice and their views of their roles and children's learning. We analyzed 11 transcribed interviews using an inductive coding approach. Teachers most often identified their roles in classroom activity as *Instructor* (64%) and *Nurturer* (55%). Four of the 11 teachers (36%) described their role as *Facilitator*; three (27%) made comments coded as *Observer*. Teachers made fewer comments overall related to children's learning, with 45% describing learning as *Engagement*, 36% learning as *Active*.

Teachers' emphasis of instructor and nurturer roles over facilitator and observer roles suggests assumptions about the locus of the generation of mathematical knowledge (Platas, 2015). Whether teachers assume a facilitative role or are active participants with children, teachers' self-positioning influences the structure of mathematical interactions in their classrooms (Graue et al., 2015) and can inform professional development efforts toward play-based pedagogy.

References

Ginsburg, H. P., Lee, J. S., & Boyd, J. S. (2008). Mathematics education for young children: What it is and how to promote it. *Social Policy Report*, *XXII*(I), 3–22.

Graue, E., Karabon, A., Delaney, K. K., Whyte, K., Kim, J., & Wager, A. (2015). Imagining a future in PreK: How professional identity shapes notions of early mathematics. *Anthropology and Education Quarterly*, *46*(1), 37–54.

NAEYC & NCTM. (2010). Early childhood mathematics: Promoting good beginnings. A joint position statement of the National Association for the Education of Young Children (NAEYC) and the National Council for Teachers of Mathematics (NCTM). Washington DC: NAEYC/NCTM.

Platas, L. M. (2015). The mathematical development beliefs survey: Validity and reliability of a measure of preschool teachers' beliefs about the learning and teaching of early mathematics. *Journal of Early Childhood Research*, *13*(3), 295–310.

ONLINE COMMUNITIES OF PRACTICE TO SUPPORT INQUIRY-ORIENTED INSTRUCTION

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At small colleges, only one professor may be equipped to teach certain courses, making content-based discussion with colleagues very difficult. Online support systems have emerged to fill this void. The online communities of interest here were created to support Inquiry-Oriented (IO) instruction, which requires different Mathematical Knowledge for Teaching (Ball, Thames, & Phelps, 2008) than other curriculum materials typically require (Johnson & Larsen, 2012). In addition, instructors are encouraged to use different instructional techniques than they may be used to. This shift in instruction can be taxing on instructors, with several challenges documented in the literature (e.g. Johnson & Larsen, 2012). This poster examines online groups of math faculty created to support one another while putting IO instructional materials into practice.

The members in the two Online Working Groups (OWGs) studied were mostly college faculty members. Data collection consisted of observations of two meetings from each OWG and 20-50 minute interviews of six OWG members. The field notes from OWG meetings and interviews were transcribed and coded using thematic analysis (Braun & Clarke, 2006). In order to analyze the OWGs, a community of practice (CoP) lens was used (Wenger, 1998). Wenger posited three dimensions of communities of practice: mutual engagement, a joint enterprise, and a shared repertoire. In order to concretize these dimensions, Wenger also provided fourteen indicators of a CoP; these indicators became the codes used for thematic analysis. Code prevalence was used to determine the extent to which each indicator of CoPs was present. This analysis addresses the research question: to what extent did these OWGs form a CoP?

Based on this analysis, eight of the fourteen indicators of a CoP given by Wenger to characterize CoPs were clearly present in the OWGs: shared ways of engaging in doing things together; substantial overlap in participants' descriptions of who belongs; knowing what others know, can do, and can contribute; mutually defining identities; specific tools, representations, and artifacts; jargon and shortcuts to communication; certain styles recognized as displaying membership; and shared discourse reflecting a certain perspective. The other indicators were present to a lesser extent. Thus, each OWG formed a CoP by the end of the semester.

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References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407. doi: 10.1177/0022487108324554

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77–101.

Johnson, E. M. S., & Larsen, S. P. (2012). Teacher listening: The role of knowledge of content and students. *The Journal of Mathematical Behavior, 31*, 117–129. doi: 10.1016/j.jmathb.2011.07.003

Wenger, E. (1998). *Communities of practice*. Cambridge, UK: Cambridge University Press.

RESULTS OF A COMMON CORE MATHEMATICS TRAINING PROGRAM: VETERAN TEACHERS' PERSPECTIVES

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Research shows implementation of the Common Core Mathematics Standards can be challenging (Bostic & Matney, 2013), especially for veteran teachers (Burks, et al, 2015). The NCTM underscores the importance of professional development in assisting mathematics teachers in this area (NCTM, 2013). Design of this teacher training program considered features of form, duration, active learning, and coherence (Birman, Desimone, Porter, & Garet, 2000) to address needs of experienced teachers struggling with implementing the CCSS-M. Research questions included: (1) In which ways, if any, will a one-year Common Core professional development training influence experienced elementary teachers' mathematics instruction? (2) What are the teachers' perspectives regarding influences of the training on their students' math performance and communication? (3) Which features of the training do the teachers report most useful in improving their mathematics instruction? Which additional features are recommended?

Training for teachers of grades 1-6 in one private girls' school included one session exploring mathematics practice standards using video demonstration and simulated collaborative problem solving as well as Common Core grade specific content applications. One model lesson targeting content and practice standards was conducted for each grade level, and follow-up consultations and classroom observations were conducted throughout the year. Seven veteran teachers participated in video recorded interviews for this study. Results were analyzed qualitatively.

Results and Implications

Initial results indicate all teachers reported influences of the training on their mathematics instruction, such as implementing more collaborative tasks, using deeper questioning, emphasizing strategy instruction and justification of solutions, and using more precise math language. Reported changes for students included more willingness to take risks, use of varied strategies, and improvement in mathematics communication skills. Teachers highlighted features of the training that helped shape their instruction, and recommended more consistent coaching for planning instruction in particular content areas. Findings of this project indicate ways in which mathematics professional development may influence teaching practices and students' learning. Features of this training project have import for researchers and practitioners designing mathematics teacher professional development.

References

Birman, B.F., Desimone, L., Porter, A.C., & Garet, M.S. (2000). Designing professional development that works. *Educational Leadership*, 28-33.

Bostic, J. & Matney, G. (2013). Overcoming a common storm: Designing professional development for teachers implementing the Common Core. *Ohio Journal of School Mathematics*, 67, 12-19.

Burks, B.A., Beziat, T.L.R., Danley, S., Davis, K., Lowery, H., & Lucas, J. (2015). Adapting to change: Teacher perceptions of implementing the common core state standards. Education, 136(2), 253-258.

National Council of Teachers of Mathematics. (2013). Supporting the common core standards for mathematics. Retreived from http://www.nctm.org/ccssmposition/.

NEW TEACHERS' VISIONS OF HOW GROUPWORTHY TASKS CAN HELP TO CREATE EQUITABLE MATHEMATICS CLASSROOMS

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Keywords**:** equity & diversity, teacher education-inservice/professional development

Theoretical Framework

All students need "access to high quality mathematics curriculum, effective teaching and learning, high expectations, and the support and resources needed to maximize their learning potential" (NCTM, 2014). Yet many of today's mathematics classrooms continue to place students in specific learning groups based on perceived ability (Alpert & Bechar, 2008). This practice, known as ability grouping, is of great concern, as it is often girls and diverse students that are most likely to be placed in lower-ability groups (Cohen & Lotan, 2014).

Featherstone et al. 2011 examined specific ways to eliminate ability grouping in the elementary mathematics classroom by creating student learning through "*groupworthy tasks.*" These are tasks that are complex in nature and offer multiple ways of thinking in order to be solved. Groupworthy mathematical tasks leverage the *mathematics smartness* of all students in mixed-ability groups, thereby providing opportunities for all students to gain deeper mathematical understandings.

Methods and Analysis

The participants in this study were sixteen newly credentialed elementary teachers who attended a weeklong professional development workshop focused on utilizing equitable mathematics teaching through groupworthy tasks. Data collected for this qualitative study included mathematics tasks, mathematics lesson plans, and individual reflections that centered on groupworthy tasks. An iterative analysis (Bogdan & Biklen, 2006) was used to demarcate the data sources that pertained to the teachers' views of groupworthy mathematics tasks.

Findings

All sixteen of the teachers expressed a commitment to provide their students with equitable mathematics learning experiences. Themes included 1) a classroom community built by students working together, 2) students supporting each other's ideas and strategies, 3) students trusting each other while struggling together to create mathematical understanding, 4) students taking risks while trying something different, 5) students being provided with multiple learning spaces and movement in the classroom, 6) teachers redefining what mathematics "smartness" looks like, and 7) teachers deconstructing the stereotypes of who can be a successful mathematics student.

References

Alpert, A. & Bechar, S. (2008). School organizational efforts in search for alternatives to ability grouping. *Teaching and Teacher Education*, 24, 1599–1612.

Cohen, E. & Lotan, R. (2014). *Designing groupwork strategies for the heterogeneous classroom*. NY, NY: Teachers College Press.

Featherstone, H., Crespo, S.,Jilk, L.,Oslund, J.,Parks, A.,& Wood, M. (2011). *Smarter Together: Collaboration and Equity in the Elementary Math Classroom.* Reston, VA: NCTM.

National Council of Teachers of Mathematics (2014). *Principles to Action: Ensuring Mathematics Success for All.* Reston, VA: NCTM.

UNPACKING A GEOMETRIC LEARNING TRAJECTORY THROUGH THE ANALYSIS OF A MATHEMATICAL TASK AND STUDENTS' STRATEGIES

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Keywords: Teacher Education-Inservice/Professional Development, Geometry, Learning Trajectories; Instructional Activities and Practices

This study examines teachers use of a learning trajectory to plan, implement, and debrief on student strategies to better understanding the developmental progression of geometric thinking during a Lesson Study. To guide the planning, we provided a lesson planning template called the Vertical Articulation to Unpack the Learning Trajectory (VAULT) that adapted from Confrey's (2012) five elements to unpack a learning trajectory. The VAULT planning guide asked the team to reflect on the cognitive demand of the task to be sure there was worthwhile mathematics and to develop conjectures on students' solution paths and barriers (See Figure 1). The Lesson Study team used the *Shapes and Properties* LT from TurnonCCMath.net to anticipate the student strategies, representations and misconceptions; develop questions to help probe student thinking, and look for sophistication of ideas to extend learning.

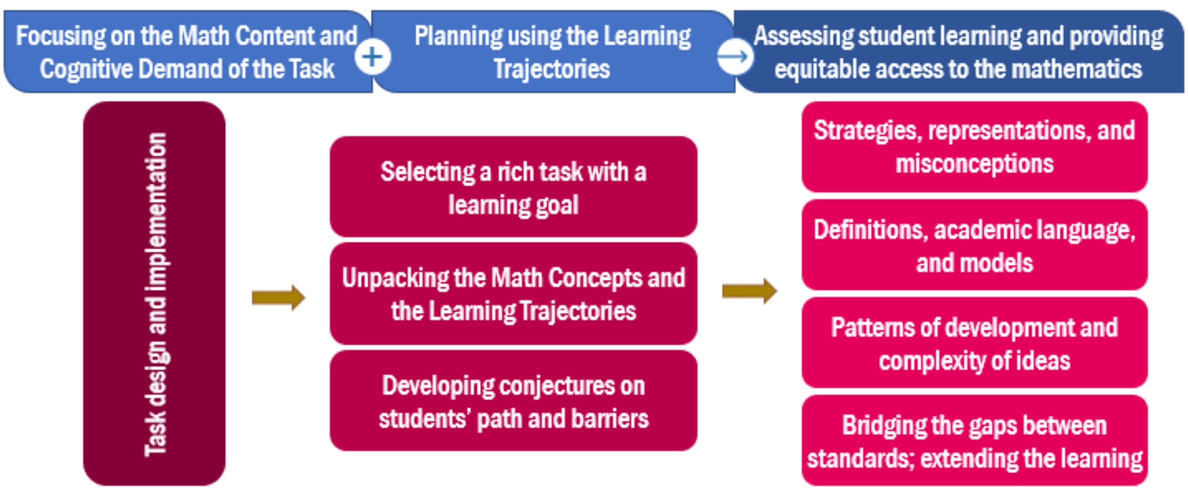

Figure 1. Vertical Articulation to Unpack the Learning Trajectory (VAULT) Planning Guide

When reflecting on student learning, the lesson study team analyzed student work and developed their own LT for comparing area, based on evidence from student strategies, which progressed from using shape relationships to order areas, to using fraction knowledge to be more precise in comparing areas, to using grid paper to assign area, and ending with using area formulas. Using an LT supports teachers in understanding student work as a progression of strategies that reflect conceptual development and intermediate understandings of mathematics.

References

Confrey, J. (2012). Articulating a learning science foundation for learning trajectories in the CCSS-M. In Van Zoerst, L. R. Lo, J. J. & Kratky, J.L. (Eds.). *Proceedings of the 34th annual meeting of the North American Chapter of the International Group for the Psychology Mathematics Education* (pp. 2-20). Kalamazoo, MI. Western Michigan University.

DESIGNING ONLINE PROFESSIONAL DEVELOPMENT MODULES THROUGH THE LENS OF ORGANIZATIONAL SENSEMAKING

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Keywords: Policy Matters, Equity and Diversity, Standards, Teacher Education - Inservice/Professional Development

Implementing changes in policy aimed at large scale educational improvements, such as new academic content standards, is a perennial challenge in education for both individual teachers and school districts. Most implementation efforts involve top-down approaches that expect implementing agents (i.e., teachers, mathematics teacher leaders at the school and district level) to change their practice to meet the goals of reform. However, researchers have consistently demonstrated this approach is problematic, highlighting that implementing agents typically notice superficial similarities to previous standards, attend to the familiar, have varied interpretations, and receive mixed messages about implementation vertically and horizontally throughout the system (Spillane, Reiser, & Reimer, 2002). Recognizing the challenges associated with standards implementation at scale, our research team partnered with our state education agency to develop online professional development (PD) modules to provide implementing agents with equitable access to opportunities to learn about standards and promote clear and coherent messages across the system.

In this poster session, we provide an overview of organizational sensemaking (Weick, 1995) as our theoretical perspective and use Sandoval's (2014) approach to conjecture mapping as a tool to guide our design of PD modules which were developed to be implemented across an organizational system (i.e., the state). We share the ways in which we drew on this perspective to develop module features that cued sensemaking processes, prompted intersubjective meaning making, and promoted action (Maitlis & Christianson, 2014; Weick, 1995). In presenting our poster, we will discuss our design process, including our conjecture maps, and provide examples of the online PD modules that embody the organizational sensemaking processes.

References

Maitlis, S., and Christianson, M. "Sensemaking in organizations: Taking stock and moving forward." *The Academy of Management Annals* 8.1 (2014): 57-125.

Sandoval, W. (2014). Conjecture mapping: An approach to systematic educational design research. *Journal of the Learning Sciences*, *23*(1), 18-36.

Spillane, J. P., Reiser, B. J., & Reimer, T. (2002). Policy implementation and cognition: Reframing and refocusing implementation research. *Review of Educational Research*, *72*(3), 387-431.

Weick, K. E. (1995). *Sensemaking in Organizations* (Vol. 3). Sage.

Chapter 6

Mathematical Knowledge for Teaching

Research Reports

Brief Research Reports

Posters

ANALYZING A DISCOURSE OF SCAFFOLDS FOR MATHEMATICS INSTRUCTION FOR AN ELL WITH LEARNING DISABILITIES

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In this case study, we examine the usage of language — how teachers used and regulated their language when teaching English language learners (ELLs) with learning disabilities (LD) how to solve mathematics multiplication problems. We focus on types of scaffolds used by teachers to identify how scaffolding helps ELLs with LD build better multiplicative reasoning. Using an exploratory case study, we find that more linguistic scaffolding and small group interactions are beneficial for ELLs with LD. In combination with kinesthetic scaffolding, they form an effective instructional method for improving multiplicative reasoning among ELLs with LD.

Keywords: Classroom Discourse, Number Concepts and Operations, Mathematical Knowledge for Teaching, Elementary School Education

Introduction

According to the section on English language acquisition in Title III of the Elementary and Secondary Education Act (ESEA) reauthorized by the Every Student Succeeds Act (ESSA) in 2015, schools are required to be accountable for the improvement of all children, including those with “disability, recently arrived ELLs, and long-term ELLs” (Non-Regulatory Guidance, 2016, p. 4). Students with limited English proficiency or what ESSA now refers to as English Learners (ELs) must also meet benchmark goals as a subgroup for passing achievement goals (pass/do not pass) and making adequate growth annually in mathematics. In order to meet district and school accountabilities requirements for dually classified ELLs (ELL and special education), it is necessary to provide appropriate support and interventions in a timely manner to promote their academic performance and address persistent achievement gaps in math (Zhou, 1997).

Math Problem-Solving Skills and Literacy Skills

Good literacy skills, which include reading, reading comprehension, and technical reading skills, play a significant role in students’ ability to solve math word problems efficiently, especially for students who have good calculation ability (Kyttälä & Björn, 2014). According to findings from previous studies, reading fluency predicts student performance in solving mathematical word problems (Vilenius-Tuohimaa et al., 2008; Kyttälä & Björn, 2014). In addition, Cummins et al. (1988) showed that children sometimes make mistakes on math word problems due to ambiguous language in the problem statements or miscomprehension of the verbal instructions, shaped by their level of English proficiency.

Content in an Academic Setting

ELL students experience a complex process with challenging academic content along with academic proficiency in language (Gerena & Keiler, 2012). Although ELLs may appear to be verbally fluent in English, they are still struggling with complex academic material that requires the production of specific academic discourse (Gerena & Keiler, 2012; Olsen, 2010) that differs from social language use.

Scholars in the field have researched and recommended the use of instructional scaffolds to convey meaning to students at varying levels of English proficiency, which include visual/graphic scaffolding, linguistic scaffolding, interactive scaffolding and kinesthetic scaffolding (Gibbons, 2014; Gottlieb, 2016). These scaffolds are important considerations in the planning of math instruction for dually classified ELLs (DC ELLs) (McGhee, 2011).

Scaffolding

In the teaching-learning framework, scaffolding is a central notion adapted from Gibbons (2002, 2014) which is supported by a constructivist theory of learning. Scaffolding is a support to "enable children to perform tasks independently that previously they could perform only with the assistance or guidance of the teacher" (Gibbons, 2002, p. vii). Scaffolding uses the theoretical framework that Halliday (1993) highlighted about registers of language through the classroom interaction of teachers and students working together to develop "new skills, concepts, and levels of understanding" (Gibbons, 2002, p. vii). Gibbons (2002, 2014) also suggested that scaffolding can be used for English language teaching to ELL students in mainstream classrooms, where they spend the majority of their school day.

Scaffolds are strategies that support the delivery of target content with an explicit inclusion of a given scaffold appropriate for each ELLs' level of English proficiency and, in this case, the added dimension of a learning disability. Gottlieb (2016) describes four types of instructional scaffolds that teachers can use and students can appropriate to create understanding around target content. These scaffolds include visual, linguistic, interactive and kinesthetic scaffolds (Gottlieb, 2016).

Visual scaffolding. Visual scaffolding helps ELL students by using drawings or photographs to connect English words to visual images and assists ELL students in learning the subject. This approach makes complex ideas feel more accessible to students and makes language more memorable, all while providing comprehensible input of the target content (McCloskey, 2005, p. 1). There are a variety of instructional supports that can build students' visual experience in the classroom, including manipulatives, real objects, and multimedia material (Carrasquillo & Rodrigues, 2002; Gottlieb, 2012).

Linguistic scaffolding. Linguistic scaffolding can be conceptualized according to the zone of proximal development (Vygotsky, 1978). Teachers must provide effective and responsive support for students' language output performance, which requires teachers to use language that is comprehensible to students when providing them with new and more sophisticated knowledge, including using a slower rate of speech or simplified vocabulary with consistent reinforcement of a target set of words (Gibbons, 2003; Bradley and Reinking, 2011).

Interactive scaffolding. As mentioned above, Heath (1982) described a "literacy event" as "any occasion in which a piece of writing is integral to the nature of participants' interactions and their interpretive processes" (p. 438). Moreover, Goffman (1993) put forward the idea of "interactionism," which relates "only to those aspects of 'context' that are directly observable and to such immediate links between individuals as their 'roles,' 'obligations,' 'face-to-face encounters,' and so on" (p. 439). An example of instructional support for both students and teachers is using active roles in pair work and small group work (Gibbons, 2008).

Kinesthetic scaffolding. Asher (1969) first introduced a strategy called *Total Physical Response Technique*, which directly relates to kinesthetic scaffolding. This approach requires the students to listen to a foreign language command and obey it using a physical action immediately with no expectation of speech production (Asher, 1969). Brand et al. (2012) suggested that students who use kinesthetic scaffolding can benefit from “sign language, translation into another language, gestures” during sessions (p. 139), while not being restricted from participating due to their lower levels of English proficiency.

In this paper, we apply four different kinds of scaffolding to analyze the mathematics instructional discourse exchanges between a teacher and an ELL student with LD within the context of a small group constructivist-oriented learning environment. We intended to answer following research questions:

1. What types of scaffolds do teachers and dually classified ELLs make in multiplicative reasoning during instruction and assessment activities?
2. How do teachers regulate language usage and scaffolding to facilitate the multiplicative reasoning of ELLs with LD?

Research Methodology

This exploratory case study investigates the interplay between teacher and student in mathematics instruction from a constructivist perspective of learning (Vygotsky, 1962). Constructivism is a philosophy of learning that focuses on individuals actively participating in learning rather than passively receiving knowledge (Gunning, 2010). In this perspective, the learning process can only occur when the learners are actively engaged in integrating new knowledge with existing knowledge (Morrow & Tracey, 2012). Therefore, constructivist theory will be the research framework for our analysis of teacher-student discourse.

Mode of Inquiry

We use an exploratory case study to examine the scaffolds used by teachers and appropriated by dually classified ELs. In light of the research process, Yin (2014) defined a case study as “an empirical inquiry that investigates a contemporary phenomenon (the ‘case’) within its real-life context, especially when the boundaries between phenomenon and context may not be clearly evident” (p. 16). The researcher-teacher (a math educator/ university professor) worked with an ELL student with LD and another native English speaker with LD in each session. The teacher used the constructivist teaching experiment method (Cobb & Steffe, 1983; Steffe, Thompson & von Glasersfelf, 2000) with the team of students.

Setting and Context of the Study

This study was conducted within the larger context of a National Science Foundation (NSF) funded project (Xin et al., 2008). This study took place at a local elementary public school resource room in the Midwestern United States. The participant attended 26 weekly teaching sessions of 25-35 minutes in pairs with another non-ELL student with LD. Each day, the math teacher worked with the pair of students together. Each lesson was designed based on an assessment of the student’s level of understanding of the given math content from the previous session. The study was conducted over a period of eight months. Each session, the instructor provided a pedagogical approach to promote the ELL’s progress toward multiplicative reasoning (Tzur et al., 2010) and problem solving (Xin, 2012).

Participants

The participants were selected from a local elementary school in the Midwestern United States. This study worked with students during an after-school program. The participants for this study were a fifth-grade ELL student with learning disabilities (Eliza) and a fifth-grade native English speaker with learning disabilities (Leslie). According to Eliza’s IEP, she was included in a general education class setting for 50% of the time, and received 45 minutes of math instruction in the resource room each day from different math instructors. Eliza was placed in a learning support classroom for reading, English language arts, and math. Eliza's intellectual functioning was in a very low range (IQ (OTIS) full scale is 69 with a verbal score of 69). Eliza has been placed in the special education program each of the past four years. The fifth-grade native English student (Leslie) worked as a group partner with Eliza during each session.

Data Sources

The sources of data were teaching videos and field notes taken during instructional observations. The teaching videos recorded the teacher and focal students, and the field notes were taken by graduate students. We included the transcripts and corresponding field notes for five out of the seven recorded teaching sessions. The rationale for including only those sessions was due to availability of the data.

Data Analysis

We coded both the instructor's discourse and the ELL student's problem solving and reasoning. The approach was coding the discourse moves of the discourse between the teacher and the student. The coding method we used was coding in terms of four different scaffoldings: visual/graphic scaffolding, linguistic scaffolding, interactive scaffolding and kinesthetic scaffolding. The purpose of this coding method was to answer the first research question and try to find the most successful scaffold that the math teacher used for ELL students with LD, which is the main purpose of this study.

Coding Scheme of Discourse Moves

We used NVivo 11 to transcribe and code the verbal and nonverbal mathematical communication for both the teacher and the pair of students (one of them, Eliza, is our participant) (Xin et al., 2016), as well as their behavior (e.g., using finger counting, creating the mathematical model on scratch paper). We did not transcribe unrelated mathematical verbal or nonverbal communication or behavior as it was not central to our inquiry.

Using the coding scheme, we coded each transcript by the type of scaffolding, including visual /graphic scaffold, interactive scaffold, linguistic scaffold or kinesthetic scaffold for both students and the teacher discourse (Table 1).

Moreover, in order to analyze the linguistic scaffold, we adopted the concordance software AntConc 3.4.3w (Windows) 2014. AntConc is a useful tool for analyzing a detailed corpus in linguistic research (Lei, 2016). After obtaining the organized discourse coding transcripts from Nvivo, we imported them into AntConc to analyze the frequency of the teacher's language in session transcripts by counting the four categories, such as "How many towers?" "How many cubes?" "How many more?" and "PGBM" (Please Go and Bring Me), which were the major activities involved in the constructivist-oriented learning of multiplicative reasoning (Xin, Tzur, and Si, 2008).

Table 1: Scaffolding Coding Scheme

Scaffolds	Teacher	Students
Visual/Graphic Scaffold	"Please generate a model of 5 towers of 9 on the grid ½ sheet."	"Can I use paper to double-check?"
Interactive Scaffold	The teacher helps E with the arithmetic and shows her the error she made—now E has 45.	The teacher asks S to help E and he does. S counts towers for E until S shows 5 with his hand.
Linguistic Scaffold	T: How many cubes do you already know are in a tower? L: 6 T: How many towers in all? E: 5	L: How many cubes in each tower? E: 5 L: How many towers? E: 6
Kinesthetic Scaffold	"Use my finger to keep track of it. And we can use our fingers if it is helpful. Here it is very helpful because you can keep track how many groups you have."	"I counted with my fingers."

In addition, we defined the interactive scaffolds by three characteristics: teacher-student interaction, student-student interaction and small group interaction (Table 2).

Table 2: Interactive Scaffolds

Teacher-student interaction	Student-student interaction	Small group interaction
The teacher helps E with the arithmetic and shows her the error she made—now E has 45.	The teacher asks S to help E and he does. S counts towers for E until S shows 5 with his hand.	T: How many cubes in all? E: 28 T: OK. What did you get on the calculator? (to L) L: 44 (with calculator)

Findings

In the first stage of analysis, we report the frequency results for the scaffolds used by the teacher and appropriated by the student (Eliza). The highest frequency of scaffolds used by the student and the teacher were kinesthetic scaffolds, while the second highest amount was interactive scaffolds (Figure 1). The teacher often used finger counting to help students do multiplication to solve the different types of problems, such as unit rate (UR) (e.g. "how many cubes in each tower"), composite units (CU) (e.g. "how many towers") and 1's (e.g. "how many cubes in all") (Tzur et al., 2010). Students in these sessions often used finger counting to show a finger trick for multiplication with numbers. Below is an example exchange between Eliza (E) and teacher (T).

Excerpt 1 (December 11, 2008)

E: 7 plus 7 equals 14 for 2 towers. She counted 15, 16, 17… (Finger counting) (Counted up to 34. She tries to keep track with her fingers and wanted to be at seven fingers when she had her answer.)

T: (prompt) Write down the number of cubes you got.

E: I lost count.

T: Try again. Do you want to use my fingers?

E: Yes. (counts the towers, 1, 2, 3, 4, 5, 6, 7, 8, 9, 1 finger/tower… 10, 11, 12 …

T: (Explains the trick to E use fingers)

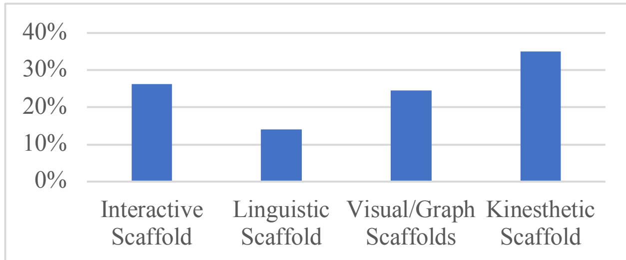

Figure 1. Frequency of Scaffolds across students and the teacher

As shown in this example, the teacher prompted to get Eliza's method and Eliza tried to use both her fingers and the teacher's fingers to solve the problem. We can find that in this situation, the teacher and the student had an effective interaction, and then the teacher could express the method that should be used for these types of problems. Therefore, Figure 2 shows the different types of interaction that the teacher and student used in sessions. It indicates that the teacher preferred to use small group interaction during the sessions, and students had more interaction during group work with both classmate and teacher. For example, the following excerpt is from a transcript between the teacher (T) and students Eliza (E) and Leslie (L).

Excerpt 2 (February 17, 2009)

T: Question number one
L: How many cubes in each tower?
E: 5
L: How many towers?
E: Six
T: Six what?
E: Six cubes.
L: How many cubes... in each
E: 5
L: How many in all?
T: How many what?
L: How many towers in all?
E: Six.
T: I think the question you're looking for is how many cubes in all. Can you ask it?
L: How many cubes in all?
E: 30

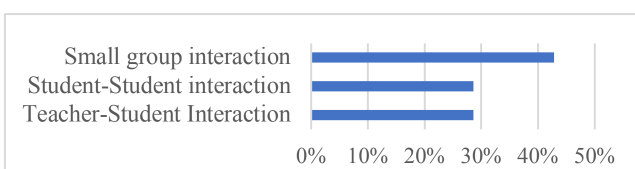

Figure 2. Frequency of scaffolding used by interaction types

The example above shows that the ELL student Eliza answered the native English speaker using different types of questions (Unit Rate [UR, Xin, 2012] and Composite Unit [CU]) in an interactive way to help each other understand the three basic elements (i.e., UR, # of Units, and Product) in elementary multiplicative problem solving. Also, the teacher was involved in the student-student interaction to ensure the accuracy of their linguistic usage and to check their understanding (such as "how many what?").

Using AntConc, we found that in the session transcripts the teacher used the phrase "how many" 111 times, while "how many towers" was used 18 times, and "how many cubes" was used 37 times. Another key word that the teacher frequently used was "PGBM" or "Please Go and Bring Me," which is the main task of a turn-taking 'platform' game PGBM (Xin et al., 2008). The authors created this game and used a simple language to name it and make it easier for ELLs with LD. The frequency of the language used by the teacher indicates that "PGBM" was used more and more often to engage the ELL in learning multiplicative reasoning and problem solving (e.g. "PGBM a tower of eleven," "PGBM six cubes").

Conclusions and Implications

In response to our research questions and in terms of the findings from our analysis, we draw the following conclusions:

1. The types of scaffolds that the teacher made in multiplicative reasoning to scaffold instruction for an ELL with LD are interactive, linguistic, visual/graphic and kinesthetic scaffolds. Among these, the kinesthetic scaffold was the most frequently used by the teacher. The teacher used finger counting as a method to show the student how to solve composite units (CU) and unit rates (UR). The second highest scaffold frequency was interactive scaffolding. We redefined and divided interactive scaffolding into three characteristics: student-student interaction, teacher-student interaction and small group interaction. The results show that small group interaction is the most effective and useful interaction that was used among the students and the teacher. Students, particularly Eliza, in the small group demonstrated a greater willingness and capacity to think and answer multiplication problems.
2. When the teacher taught multiplicative reasoning to the ELL with LD, he frequently used simple phrases such as "how many" and "PGBM." The rationale of using the linguistic scaffolding is that the teacher repeatedly used and also let students repeatedly use the simple phrase "how many" to illustrate the process of thinking and solving multiplication problems. In addition, "PGBM" characterizes the "platform" game used, which also benefits English language learners to get directions promptly and attend to multiplicative reasoning.

In general, we found that the four scaffolds in classroom discourse that the teacher frequently used with students can influence the multiplicative reasoning of the English language learner with learning disabilities and improve mathematical problem-solving achievement. We also found that kinesthetic scaffolding is the most direct method tied with helping the ELL with LD solving multiplicative problems. However, in order to better serve English language learners with LD, especially in the classroom environment, teachers should focus on better linguistic scaffolding usage within small group interactions. In our future research, we will analyze the level of intellectual work (Xin et al., 2016) done by the teacher and the students through determining how the four scaffolds promote students comprehend multiplicative reasoning at the abstract level, in particular, to meet the challenging math curriculum standards.

Acknowledgements

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References

Archer, A., & Hughes, C. A. (2011). *Explicit instruction: Efficient and effective teaching*. New York, NY: Guilford Publications.

Biber, D., Conrad, S., & Reppen, R. (1998). *Corpus linguistics: Investigating language structure and use*. Cambridge University Press.

Bishop, J. P., & Whitacre, I (2010). Intellectual work: The depth of mathematical discourse and its relationship to student learning. In P. Brosnan, D. B. Erchick, & L. Flevares (Eds.), *Proceedings of the 32nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 401–409). Columbus, OH: The Ohio State University.

Bradley, B. A., & Reinking, D. (2011). A formative experiment to enhance teacher-child language interactions in a preschool classroom. *Journal of Early Childhood Literacy*, *11*(3), 362-401.

Brand, T., S., Favazza, A. E., & Dalton, E. M. (2012). Universal design for learning: A blueprint for success for all learners. *Kappa Delta Pi Record*, *48*(3), 134-139.

Every Student Succeeds Act, Pub. L. No. Public Law 114-95 (2015 December 10, 2015).

Gerena, L., & Keiler, L. (2012). Effective intervention with urban secondary English language learners: How peer instructors support learning. *Bilingual Research Journal*, *35*(1), 76-97.

Gibbons, P. (2014). *Scaffolding language, scaffolding learning: teaching English language learners in the mainstream classroom* (2nd ed.). Portsmouth, NH: Heineman.

Gibbons, P. (2002). *Scaffolding language, scaffolding learning: Teaching second language learners in the mainstream classroom*. Portsmouth, NH: Heinemann. Wood, K. C., Smith, H., & Grossniklaus, D. (2008). Piaget's stages. Retrieved November, 20.

Gottlieb, M. (2016). *Assessing English Language Learners: Bridges to Educational Equity: Connecting Academic Language Proficiency to Student Achievement*, 2nd Ed. Thousand Oaks, CA: Corwin.

Gottlieb, M. (2013). *Essential Actions: A Handbook for Implementing WIDA's Framework for English Language Development Standards*. Madison, WiDA Consortium.

Gunning, T. G. (2010). *Creating literacy instruction for all students*. Boston, MA: Pearson.

Heath, S. B. (1982). What no bedtime story means: Narrative skills at home and school. *Language in society*, *11*(1), 49-76.

Kingsdorf, S., & Krawec, J. (2016). Assessing a multi-component math intervention within a cognitive-behavioral framework on the word problem-solving responses of a diverse group of third graders. *Cogent Education*, *3*(1), 1160638.

Kyttälä, M., & Björn, P. M. (2014). The role of literacy skills in adolescents' mathematics word problem performance: Controlling for visuo-spatial ability and mathematics anxiety. *Learning and Individual Differences*, *29*, 59-66.

Lei, Q. (2016). A research of English article errors in writings by Chinese ESL learners. *English Language Teaching Ideas Journal*. 2016.

McCloskey, M. L. (2005). Visual scaffolding to support ell reading. *Keeping the Promise*, 1-7.

Morrow, L. M., & Tracey, D. H. (2012). *Lenses on reading: An introduction to theories and models*. Guilford Press.

Street, B. V. (Ed.). (1993). *Cross-cultural approaches to literacy* (Vol. 23). Cambridge University Press.

Swales, J. (1990). *Genre analysis: English in academic and research settings*. Cambridge University Press.

Tzur, R., Xin, Y. P., Kenney, R., & Guebert, A. (2010). Students with Learning Disability in Math Are Left Behind in Multiplicative Reasoning? Number as Abstract Composite Unit is a Likely 'Culprit'. *Online Submission*.

Vygotsky, L. S. (1978). *Mind in society*: The development of higher mental process.

Vygotsky, L. S. (1962). Language and thought. *Massachusetts Institute of Technology Press*, Ontario, Canada.

Xin, Y. P. (2012). *Conceptual model-based problem solving: Teach students with learning difficulties to solve math problems*. The Netherlands: Sense Publishers.

Xin, Y. P., Liu, J., Jones, S. R., Tzur, R., & Si, L. (2016). A preliminary discourse analysis of constructivist-oriented mathematics instruction for a student with learning disabilities. *The Journal of Educational Research*, *109*(4), 436-447.

Xin Y. P., Tzur, R., & Si, L. (2008). Nurturing Multiplicative Reasoning in Students with Learning Disabilities in a Computerized Conceptual-Modeling Environment. National Science Foundation (NSF) funded project.

Yin, K. (2014). *Case study research: Design and methods* (5th ed.). Thousand Oasks, CA: Sage.

IDENTIFYING LATENT CLASSES OF MIDDLE GRADES TEACHERS BASED ON REASONING ABOUT FRACTION ARITHMETIC

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The purpose of this study was to examine distinct latent classes of middle grades mathematics teachers with respect to reasoning about fractions. Survey response data came from a nationwide sample of 990 in-service middle grades mathematics teachers. The survey focused on four components of reasoning about fractions in terms of quantities: referent unit, partitioning and iterating, appropriateness, and reversibility. The mixture Rasch model analysis detected three latent classes, each with strengths and weaknesses. Chi-square tests indicated significant relationships between latent class membership and various teacher characteristics such as gender, mathematics credential, grade-level experience, and highest grade-level certification. The results extend recent advances in measuring mathematical knowledge of teachers.

Keywords: Data Analysis and Statistics, Rational Numbers, Teacher Knowledge

Fractions are core content in the upper elementary and middle grades mathematics curriculum and are highly interconnected to whole-number multiplication and division, and ratios and proportional relationships (e.g., Vergnaud, 1988). Additionally, fractions are necessary for algebraic reasoning and further study in mathematics (Hackenberg & Lee, 2015). Although most teachers can multiply or divide two fractions by computing correctly, many studies acknowledged the difficulties that teachers experience in reasoning about products or quotients of fractions when they are embedded in problem situations (e.g., Ball, Lubienski, & Mewborn, 2001; Lee, 2017). Despite strong emphasis by recent curriculum standards such as the Common Core State Standards for Mathematics (CCSS-M; e.g., National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) on the necessity of developing reasoning about fraction arithmetic when solving problems embedded in situations, two main challenges exist in mathematics education. One main challenge is to foster teachers' reasoning about fractions in terms of quantities. A second main challenge is to understand how to use psychometric models for measuring teachers' fine-grained mathematical knowledge. Many recent applications of psychometric models to measure teachers' mathematical knowledge have relied on traditional item response theory (IRT) models. These efforts include the Learning Mathematics for Teaching (LMT) project (e.g., Hill, 2007), the Diagnostic Mathematics Assessments for Middle School Teachers (DTAMS) project (Saderholm, Ronau, Brown, & Collins, 2010), and the Knowledge of Algebra for Teaching (KAT) project (Senk, 2010). Traditional IRT models rely on the assumption that all examinees in a given sample belong to a single population. Some recent studies (e.g., Izsák, Orrill, Cohen, & Brown, 2010; Izsák, Jacobson, de Araujo, & Orrill, 2012), however, have demonstrated the existence of distinct latent classes of middle grades teachers. The presence of distinct latent classes violates local independence, a key assumption of traditional IRT models. To address this issue, the present study employs the mixture Rasch model (Rost, 1990), a combination of a latent class model and a traditional IRT model. When applying the mixture Rasch model, one can examine model fit for different numbers of latent classes. Each class is characterized by a distinct pattern of item responses, and differences in response patterns are thought to indicate different underlying cognitive strategies (Bolt, Cohen, & Wollack, 2001). For each examinee, the mixture Rasch model provides an estimate of ability (as does a unidimensional traditional IRT model) and a probability of class membership.

The present study used responses from a sample of 990 in-service middle grades teachers across the U.S. to the Diagnosing Teachers' Multiplicative Reasoning (DTMR) Fractions survey (Bradshaw, Izsák, Templin, & Jacobson, 2014). The survey measures teachers' capacities to reason about multiplication and division of fractions in terms of quantities. The purpose of this study was to identify distinct latent classes of middle grades teachers on reasoning about fractions and investigate the relationships between class membership and teacher characteristics such as gender, mathematics credential, grade-level experience, highest grade-level certification, and years of teaching experience. The following research questions were addressed:

1. Do distinct latent classes exist in the national sample of middle grades teachers?
2. If so, what areas of strength and weakness on reasoning about fractions distinguish the distinct latent classes?
3. Are there significant relationships between latent class membership and teacher characteristics including gender, mathematics credential, grade-level experience, highest grade-level certification, and years of teaching experience?

Theoretical Framework

The theoretical framework for this study considers middle grades teachers' reasoning about quantities and focuses on using drawings (e.g., area models and number lines) to learn and teach fraction arithmetic. We consider fine-grained components of reasoning as the property of an individual by following the constructivist perspective, in which the individual dynamically stores each component in his/her mind. We think that reasoning, which goes beyond computational fluency, requires a teacher to make sense of quantities in fraction arithmetic problems using drawings. From this standpoint, a teacher's capacity to reason about quantities with drawings can be increased by paying more consistent attention to distinct fine-grained components such as *referent units*, *partitioning and iterating*, *appropriateness*, and *reversibility* — the importance of which have been established in past research (e.g., Bradshaw et al., 2014; Izsák, Jacobson, & Bradshaw, in press) and explained later. Moreover, we take the stance that reasoning depends on context. That is, a teacher's performance about the use of one component in one situation might be appropriate, but his/her performance in another situation might be problematic, depending on the wording of the situation, the arithmetic operations required, or the representations provided. Hence, it is important to examine teachers' capacities to employ particular components across a range of problem situations.

Methods

Participants

The data consisted of survey responses from a sample of 990 in-service middle grades teachers across the U.S. (Table 1).

Table 1: Demographic information

Variables	%
Gender (*N = 976*)	
Female	75.0
Mathematics credential (*N = 974*)	
Yes	67.0
Grade-level experience (*N = 972*)	
K-5	46.0
Grades 6-8	29.0
Grades 9-12	25.0
Highest grade-level certification (*N = 971*)	
K-6	8.0
Grades 7-9	57.0
Grades 10-12	35.0
Years of teaching experience (*N = 966*)	
0-4 years	17.1
5-14 years	52.9
> 14 years	30.0

Instruments

The DTMR Fractions survey consists of two parts. The first part has 27 items (19 multiple choice and 8 constructed response) that measure four distinct components of reasoning about fractions including *referent unit*, *partitioning and iterating*, *appropriateness*, and *reversibility*. *Referent unit* deals with reasoning about units when numbers are embedded in problem situations and consists of three sub-components. *Norming* refers to the formation of standard units for measurement and occurs either in case of selecting a standard unit from alternate choices or in case of making at least two choices for a measurement unit in a given situation (i.e., renorming). *Referent unit for multiplication* and *referent unit for division* concern the problem situations that can be modeled by the equation $M \bullet N = P$ where M and N refer to different units. The second component, *partitioning and iterating*, refers to dividing a quantity into equal-sized pieces and concatenating unit fractions. It consists of three sub-components. *Partitioning in stages* refers to making a repartition to obtain a desired partition. *Partitioning using common denominators* and *partitioning using common numerators* refer to using common denominators or numerators to obtain common partitions. The third component, *appropriateness*, concerns identifying situations that can be modeled by multiplication and division and includes three sub-components: *identifying multiplication*, *identifying partitive division*, and *identifying quotitive division*. The fourth component, *reversibility*, deals with returning to a starting point after making some process. We conjecture that proficiency in use of these four components of reasoning across different problem situations enables teachers to solve fraction arithmetic problems in terms of reasoning with quantities.

Because the DTMR Fractions survey items are secure, we present one example item similar to an actual survey item (Figure 1). This item measures *referent unit* and *partitioning and iterating*. The correct choice is (b). A teacher who chose (a) or (c) would indicate confusion about the referent unit for 1/8. A teacher who chose (e) would indicate an incorrect partition (5 groups of 6 pieces that create 30ths). A teacher who chose (b) would demonstrate both the correct referent unit (the 1 meter) and the correct partition (5 groups of 8 pieces that create 40ths). The rest of the multiple-choice items were also constructed so that the different choices provided information about the four components of reasoning. A correct choice provided evidence for the components of reasoning intended for that item; incorrect choices simply indicated lack of evidence for the components of reasoning intended for that item. Constructed response items were also scored using rubrics for evidence of intended components of reasoning.

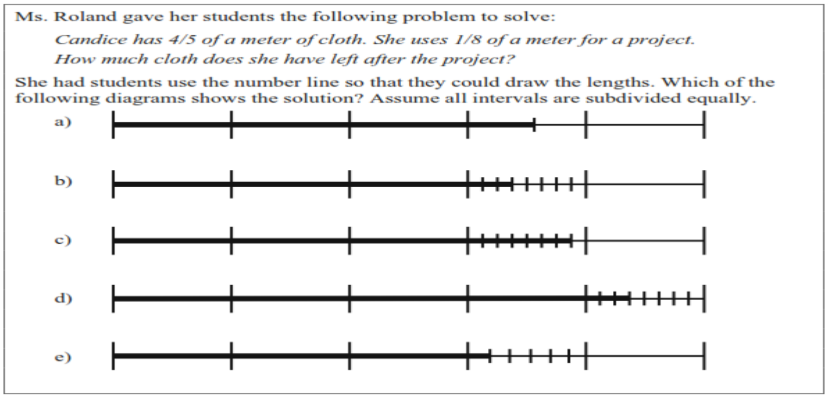

Figure 1: An item that measures referent unit and partitioning using common multiples of denominators. From Izsák et al. (2010).

The second part of the survey consists of a questionnaire to obtain information about various teacher characteristics including gender, mathematics credential, grade-level experience, highest grade-level certification, and years of teaching experience (see Table 1).

Data Analysis

We analyzed the data using the mixture Rasch model implemented in the computer program WINMIRA (von Davier, 2001). First, we estimated the mixture Rasch model with one, two, three, four, five, and six latent classes. Second, we compared three information indices to select the best fitting model: Akaike's information criterion (AIC), Bayesian information criterion (BIC), and consistent AIC (CAIC). With each of these criterion, smaller values indicate better fit. We selected the model with the smallest BIC values as the best fitting model (Li, Cohen, Kim, & Cho, 2009). Next, we analyzed the reasoning characteristics of each latent class by examining raw response data. In addition, we evaluated the relationships between latent class membership and teacher characteristics using analysis of variance (ANOVA) and chi-square tests across the latent classes.

Results

Checking Dimensionality

An exploratory factor analysis using maximum likelihood estimation as implemented in the SPSS 16.0 software (SPSS Inc., 2007) indicated eigenvalues of the first three factors as 5.1, 1.5, and 1.3, and the total variance explained by the first factor was 19%. Because the first eigenvalue was relatively large (Lord, 1980), a unidimensional model could be fit to the data.

Model Selection

Values for the three information indices are given in Table 2. Minimum values for BIC (30226.79) and CAIC (30312.79) indicated a three-class solution in the data.

Table 2: Model fit indices of the mixture Rasch model

Model	AIC	BIC	CAIC
One class	30526.03	30663.16	30691.16
Two classes	30011.27	30290.43	30347.43
Three classes	29805.59	**30226.79**	**30312.79**
Four classes	29706.11	30269.35	30384.35
Five classes	29672.67	30377.94	30521.94
Six classes	**29579.19**	30426.49	30599.49

Note. AIC = Akaike information criterion; BIC = Bayesian information criterion; CAIC = Consistent Akaike information criterion; the smallest information criterion index is bold.

Table 3 presents the descriptive statistics about raw scores for each of the three latent classes. Based on Table 3, Class-C is the least proficient latent class with the average score of 7.948 over the total score of 27, and 39% of the teachers (385 over 990 teachers) are members of this class. Moreover, Class-B is the middle proficient latent class with 19% of the teachers (184 over 990 teachers) in this class. And, Class-A is the most proficient latent class with 42% of the teachers (421 teachers over 990 teachers) in this class.

Table 3: Descriptive statistics for the latent classes.

	Class-A	Class-B	Class-C
Raw scores			
M	15.399	10.750	7.948
SD	4.165	4.265	3.529
N (%)	421 (42.5)	184 (18.7)	385 (38.8)

Analysis of Raw Response Data

To get the clearest view of the reasoning characteristics associated with each latent class, we narrowed analysis to the raw response data of the 649 teachers who were assigned to a latent class with a probability of .9 or higher. Table 4 lists the characteristics of the latent classes based on the percentage of teachers in each class who answered the survey items correctly.

Table 4: Characteristics of the latent classes (N = 649)

Component	Sub-component	Characteristic	Class-A	Class-B	Class-C
RU	Norming	Choosing a standard unit from alternate choices	Strong	Partial	Weak
RU	Norming	Renorming in the presence of proper fractions	Strong	Strong	Weak
RU	Norming	Renorming in the presence of improper fractions	Weak	Weak	Weak
RU	RU for Multiplication	Distinguishing part-of-a-part from part-of-a-whole	Partial	Partial	Weak

RU	RU for Multiplication	Reasoning when the whole is not present visually	Partial	Partial	Weak
RU	RU for Division	When quotient as a whole number	Partial	Partial	Partial
RU	RU for Division	When quotient as a fraction	Partial	Partial	Weak
PI	Partitioning in Stages	Partitioning in stages	Strong	Weak	Weak
PI	Common Denominator	Using common denominators	Strong	Weak	Weak
PI	Common Numerator	Using common numerators	Strong	Weak	Weak
APP	Identifying Multiplication	Identifying multiplication	Strong	Partial	Weak
APP	Identifying Partitive Division	Identifying partitive division	Strong	Strong	Strong
APP	Identifying Quotitive Division	Identifying quotitive division	Strong	Strong	Weak
REV	Reversibility	Reversibility	Strong	Strong	Weak

Note. RU=Referent Units; PI=Partitioning & Iterating; APP=Appropriateness; REV=Reversibility.

Finally, an exploratory examination of the latent classes obtained from the mixture Rasch analysis and an examination of the raw response data of the 649 teachers revealed the reasoning characteristics of each latent class (Figure 2). Based on this analysis, Class-C teachers perform well only in *identifying partitive division* (*Appropriateness*) problems, but have trouble in the remaining three components of fraction arithmetic (i.e., *referent units*, *partitioning and iterating*, and *reversibility*). On the other hand, Class-B teachers are found to perform well in *identifying multiplication* and *identifying quotitive division* (*Appropriateness*) problems, in addition to problems that involve *identifying partitive division* (*Appropriateness*), and in using *reversibility*. However, similar to Class-C teachers, Class-B teachers struggle with items that measure *referent units*, and *partitioning and iterating* such as *partitioning using common denominators* and *partitioning using common numerators*. In addition to having the strengths of Class-B teachers, Class-A teachers perform well in *partitioning using common denominators*, *partitioning using common numerators* and *partitioning in stages* (*Partitioning and iterating*). On the other hand, Class-A teachers have partial difficulty in *renorming* and *distinguishing part-of-a-part from part-of-a-whole* (*Referent units*). In this component, Class-C and Class-B teachers experience much more difficulty than Class-A teachers.

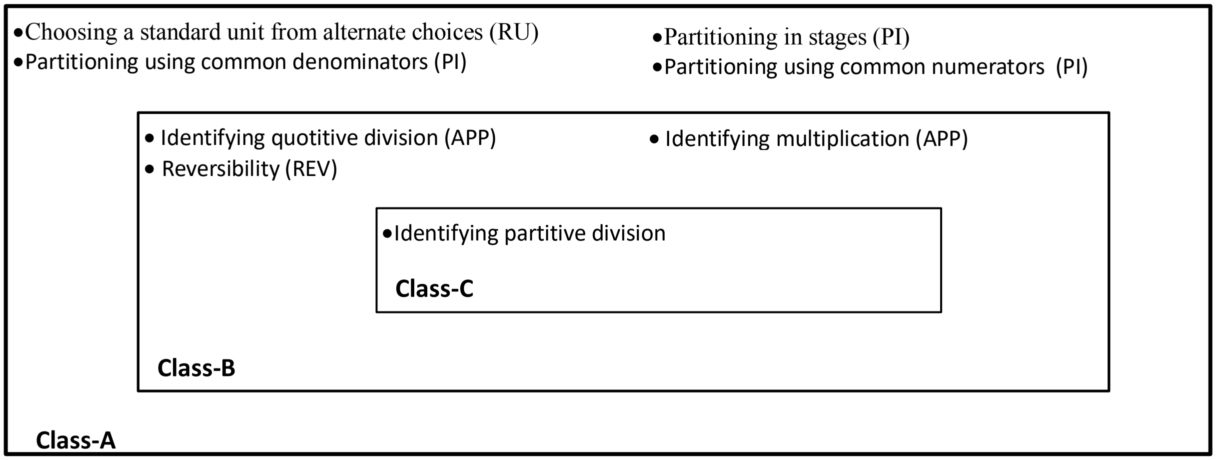

Figure 2: Identifying latent classes based on mixture Rasch analysis (N = 649).

The Relationships between Latent Class Membership and Teacher Characteristics

We also examined the relationships between latent class membership and various teacher demographic and professional history characteristics. First, we found significant differences for total mean raw scores based on ANOVA (F (2, 987) = 363.557, p = .00). The effect size (η^2) of the main effect was .42, indicating that the latent classes explained 42% of the variance in the total mean raw scores. Post hoc analyses using Scheffé's test showed significant differences of the three latent classes from each other (p = .00 for each comparison). This indicated teachers in Class-A scored significantly higher than those in Class-B, and teachers in Class-B scored significantly higher than those in Class-C. Second, a chi-square test for gender was significant ($\chi^2(2)$ = 25.40, p < .001), but Crámer's V statistic was .16, indicating a weak association between latent class membership and gender. Third, a chi-square test for mathematics credential was significant ($\chi^2(2)$ = 15.27, p < .001), indicating a relationship between having a mathematics credential and latent class membership. Fourth, the relationship between latent class membership and grade-level experience was significant ($\chi^2(4)$ = 25.07, p < .001). Fifth, the author(s) found a significant relationship between latent class membership and highest grade-level certification ($\chi^2(4)$ = 29.20, p < .001). Finally, we found a significant relationship between latent class membership and years of teaching experience ($\chi^2(4)$ = 11.92, p = .018). These results indicate that teachers who achieved higher scores, those who had a mathematics credential, those who had a high-school credential, and those who had more teaching experience tended to be in Class-A as opposed to other latent classes.

Discussion

Results of the present study demonstrate how combining research in mathematics education with psychometric models can reveal patterns in middle grades teachers' fine-grained reasoning about fractions. We used the mixture Rasch model to characterize differences in reasoning about fractions of middle grades teachers. Results for the first research question revealed three distinct latent classes. Results for the second research question indicated that teachers in the three latent classes were distinguished by their attention to norming and referent units for multiplication (i.e., *referent unit*), using common numerators (i.e., *partitioning and iterating*), identifying multiplication and division (i.e., *appropriateness*), and *reversibility*. These results extend those reported by Izsák et al. (2010, 2012). For instance, Izsák et al. (2010) found two latent classes in a convenience sample of 201 in-service middle grades teachers where teachers in one class were more proficient in *referent unit* than those in the second class. The present study used a more refined instrument and a much larger sample to refine the earlier results, found three latent classes instead of two, and identified differences across the latent classes based not only on *referent unit* but also on other components of reasoning including *partitioning and iterating*, *appropriateness*, and *reversibility*. For the third research question, we found significant relationships between latent class membership and the teacher characteristics such as mathematics credential, grade-level experience, highest grade-level certification, and years of teaching experience, as similar to the results of Hill (2007) and Izsák et al. (in press). Future studies should continue examining teachers' mathematical knowledge using innovative measures and applications of diverse psychometric models, including the mixture Rasch model.

References

Ball, D. L., Lubienski, S., & Mewborn, D. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 433–456). Washington, DC: American Educational Research Association.

Bolt, D. M., Cohen, A. S., & Wollack, J. A. (2001). A mixture item response for multiple-choice data. *Journal of Educational and Behavioral Statistics, 26*, 381–409.

Bradshaw, L., Izsák, A., Templin, J., & Jacobson, E. (2014). Diagnosing teachers' understandings of rational number: Building a multidimensional test within the diagnostic classification framework. *Educational Measurement: Issues and Practice, 33*(1), 2–14.

Hackenberg, A. J., & Lee, M. Y. (2015). Relationships between students' fractional knowledge and equation writing. *Journal for Research in Mathematics Education, 46*(2), 196–243.

Hill, H. C. (2007). Mathematical knowledge of middle school teachers: Implications for the No Child Left Behind policy initiative. *Educational Evaluation and Policy Analysis, 29*, 95–114.

Izsák, A., Jacobson, E., & Bradshaw, L. (in press). Surveying middle grades teachers' reasoning about fraction arithmetic in terms of measured quantities. *Journal for Research in Mathematics Education*.

Izsák, A., Jacobson, E., de Araujo, Z., & Orrill, C. H. (2012). Measuring mathematical knowledge for teaching fractions with drawn quantities. *Journal for Research in Mathematics Education, 43*(4), 391-427.

Izsák, A., Orrill, C. H., Cohen, A. S., & Brown, R. E. (2010). Measuring middle grades teachers' understanding of rational numbers with the mixture Rasch model. *The Elementary School Journal, 110*, 279–300.

Lee, M. Y. (2017). Pre-service teachers' flexibility with referent units in solving a fraction division problem. *Educational Studies in Mathematics, 96*, 327–348.

Li, F., Cohen, A. S., Kim, S. H., & Cho, S. J. (2009). Model selection methods for dichotomous mixture IRT models. *Applied Psychological Measurement, 33*(5), 353–373.

Lord, F. M. (1980). *Applications of item response theory to practical testing problems*. Hillsdale, NJ: Lawrence Erlbaum.

National Governors Association Center for Best Practices (NGA) & Council of Chief State school Officers (CCSSO). (2010). *Common core state standards for mathematics*. Washington, DC: Author. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf.

Rost, J. (1990). Rasch models in latent classes: An integration of two approaches to item analysis. *Applied Psychological Measurement, 14*, 271-282.

Saderholm, J., Ronau, R., & Brown, E. T. (2010). Validation of the Diagnostic Teacher Assessment of Mathematics and Science (DTAMS) Instrument. *School Science and Mathematics, 110*, 180–191.

Senk, S. (2010). *Using psychometrics to advance assessment in mathematics education. The knowing mathematics for teaching algebra (KAT) project*. Paper presented at the Research Presession of the Annual Meeting of the National Council of Teachers of Mathematics, San Diego, CA.

SPSS Inc. (2007). SPSS for Windows, Version 16.0. Chicago, SPSS Inc.

Von Davier, M. (2001). WINMIRA [Computer Software]. St. Paul, MN: Assessment Systems Corporation.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in middle grades* (pp. 141–161). Reston, VA: National Council of Teachers of Mathematics; Hillsdale, NJ: Erlbaum.

REFLECTED ABSTRACTION AS A MECHANISM FOR DEVELOPING PEDAGOGICAL CONTENT KNOWLEDGE

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Tallman (2015) argued that pedagogical content knowledge is a form of content knowledge with particular characteristics that endow it with pedagogical utility, and he conjectured that an essential characteristic of a teacher's content knowledge is her conscious awareness of the mental actions and conceptual operations that comprise her mathematical schemes. We report the results of a study that explored this conjecture by examining the pedagogical implications of engaging a pre-service secondary teacher in a mathematical intervention to support her construction of a powerful scheme for constant rate of change and to engender her conscious awareness of the mental processes that constitute this scheme.

Keywords: Pedagogical content knowledge; Reflecting abstraction; Mathematical knowledge for teaching; Pre-service teacher education

Researchers have long been concerned with conceptualizing the knowledge base that enables teachers to effectively support students' conceptual mathematical learning. Shulman's (1986, 1987) notion of *pedagogical content knowledge* (PCK), which he defined as a "special amalgam of content and pedagogy," represents a significant advance to this end (1987, p. 8). While researchers since Shulman have strived in various ways to elaborate the PCK construct, there is an aspect of Shulman's initial conceptualization that is almost never abandoned: the notion that PCK is a combination of content knowledge and pedagogical knowledge (Depaepe, Verschaffel, & Kelchtermans, 2013). Although many researchers follow Shulman in defining PCK as an amalgam of knowledge of content and knowledge of pedagogy (e.g., Ball, Thames, & Phelps, 2008; Fennema & Franke, 1992), the relationship between these distinct types of professional knowledge, as well as the specifics of their synthesis, remains elusive.

To investigate the relationship between teachers' content knowledge and their instructional actions, Tallman (2015, in submission) compared an experienced secondary mathematics teacher's knowledge of sine and cosine functions with the knowledge he enacted in the context of teaching. Through this comparison, Tallman identified the influences that affected the nature and quality of the subject matter knowledge the teacher leveraged in his lesson planning and instruction. Tallman's analysis revealed that the inconsistencies between the teacher's personal and enacted mathematical knowledge resulted from him possessing weak connections and multiple schemes for particular concepts related to trigonometric functions, as well as from his unawareness of the mental actions and conceptual operations that comprised these often powerful but uncoordinated cognitive schemes. Tallman concluded from these results that PCK is a form of content knowledge with particular characteristics that endow it with pedagogical utility; he argued that the pedagogical character of PCK derives from the specific ways in which a teacher's content knowledge informs her enactment of effective pedagogies. Additionally, Tallman conjectured that an essential characteristic of a teacher's content knowledge is the extent to which she is consciously aware of the mental actions and conceptual operations that comprise her own mathematical schemes.

In this paper, we report the results of a study that explored this conjecture by examining the pedagogical implications of engaging a pre-service secondary teacher in a mathematical intervention to support her construction of a powerful scheme for constant rate of change and to engender her conscious awareness of the mental processes that characterize this scheme. We leveraged Piaget's (2001) notion of reflected abstraction to stimulate such conscious awareness.

Theoretical Background

Piaget proposed abstraction as the mechanism of scheme construction and refinement and distinguished five varieties: *empirical*, *pseudo-empirical*, *reflecting*, *reflected*, and *meta-reflection* (Piaget, 2001). We discuss only *reflecting* and *reflected* abstraction because of their unique role in the construction and refinement of mathematical schemes and because these two types of abstraction served as design principles and analytical constructs in the present study. Reflecting abstraction involves the subject's reconstruction on a higher cognitive level of the coordination of actions from a lower level, and results in the development of *logico-mathematical knowledge*, or schemes at the level of *operative thought*. Reflecting abstraction is thus an abstraction of actions and occurs in three phases: (1) the *differentiation* of a sequence of actions from the effect of employing them, (2) the *projection* of the differentiated action sequence from the level of activity to the level of representation, or the reflected level, and (3) the *reorganization* that occurs on the level of representation of the projected actions (Piaget, 2001). A subject must differentiate (dissociate) actions from their effects before she can construct an internalized representation of them, what Piaget called projecting actions to the level of mental representation (i.e., the "reflected level"). Additionally, the subject must coordinate the actions that produced the effect before she can project and represent them on this higher cognitive level. Once a subject differentiates actions from their effect and coordinates them, she is prepared to project these coordinated actions to the reflected level where they are organized into cognitive structures, or schemes.

Reflected abstraction involves operating on the internalized actions that result from prior reflecting abstractions, which results in a coherence of actions and operations accompanied by conscious awareness. It is the act of deliberately operating on the actions and operations that result from prior reflecting abstractions that engenders such awareness. To consciously operate on actions at the level of representation suggests that one has symbolized coordinated actions at this higher level. Reflected abstraction thus relies on what Piaget called the *semiotic function*, or the subject's capacity to construct mental symbols to represent aspects of her experience. As a result of the conscious awareness of internalized actions that occurs as a byproduct of reflected abstraction, the subject's ability to purposefully assimilate new experiences to the reflected level provides evidence that she has engaged in reflected abstraction. Additionally, performing operations on the symbols the subject constructs to represent coordinated actions at the level of representation results in increasingly organized cognitive structures. Reflected abstraction is therefore the means by which systems of organized actions at the level of representation become progressively coherent and refined.

Conceptual Analysis of Rate of Change

As we mentioned above, we engaged a pre-service secondary teacher in a mathematical intervention to support her construction of a particular scheme for rate of change. In this section, we briefly describe the meanings we designed the intervention to support.

A mature rate of change scheme relies upon productive conceptualizations of ratio, rate, and continuous variation. A *ratio* is a multiplicative comparison of the measures of two constant (non-varying) quantities while a *rate* defines a proportional relationship between varying quantities' measures (Thompson & Thompson, 1992). Constructing a rate therefore involves images of smooth continuous variation (Thompson & Carlson, 2017), as well as the expectation that as two quantities covary, multiplicative comparisons of their measures remain invariant.

A rate is a reflectively abstracted constant ratio (Thompson & Thompson, 1992, p. 7), which means that constructing a rate involves internalizing the coordinated actions involved in multiplicatively comparing particular values of covarying quantities. An individual has done so if on the basis of her conceptualization of a particular quantitative situation she anticipates that subsequent multiplicative comparisons of the covarying quantities' measures will yield the same numerical value.

Rate of change is a quantification of two covarying quantities and results from a multiplicative comparison of changes in covarying quantities' measures. The rate of change is constant if changes in the quantities' measures are proportional. Conceptualizing changes in quantities' measures as quantities themselves involves a quantitative operation with attention to a point of reference (Joshua et al., 2015). Conceptualizing rate of change requires students to construct changes in quantities' values as quantities themselves and then to abstract an invariant multiplicative relationship between changes in the quantities' values.

Methods

This study's experimental methods proceeded in four phases. First, we video recorded all nine lessons from a pre-service secondary teacher's instruction of rate of change and slope in a 7th grade class during her student teaching semester. We then asked the participant, Samantha, to use the data analysis software Studiocode (Studiocode Version 5.8.4, Sportstec, Ltd., 2015) to identify segments of her first two lessons that exemplify high quality instruction as well as segments that indicate room for improvement. She wrote brief justifications for each selection. Samantha performed this analysis of her teaching videos independently and met with the research team after she had finished to answer clarifying questions and to provide additional rationale for her written responses. We then conducted a teaching experiment (Steffe & Thompson, 2000) during the third phase of the study to construct a model of Samantha's scheme for constant rate of change and to characterize the evolution of this scheme as she engaged in particular instructional experiences designed to promote reflecting and reflected abstractions. The teaching experiment consisted of eight teaching episodes that each lasted between 60 and 90 minutes. During the fourth phase of the study, we asked Samantha to analyze five teaching videos (including the two she analyzed prior to participating in the teaching experiment) using the codes "High Quality Instruction" and "Room for Improvement" and to provide written rationale for her selections. We again interviewed Samantha to probe her justifications for her selections and to ask focused questions about the segments of her instruction she identified as exemplars of high- or low-quality teaching. Our focus in the fourth phase of the study was to determine whether the criteria by which Samantha evaluated the quality of her instruction had changed, and to identify the role played by her content knowledge in making such evaluations.

Results

Due to space limitations, we discuss only our analysis of the data relevant to Samantha's first lesson on rate of change.

Samantha's Instruction of Rate of Change

Samantha began her teaching of rate of change by asking students to describe what they think of when they hear the word "rate" and to provide examples of rates. She then defined a rate as "a ratio of two quantities with different units" and described a unit rate as "a rate with a denominator of one." After defining these terms, Samantha asked students to interpret the meaning of a car traveling at a constant speed of 65 miles per hour. She validated a student's response, "You're traveling 65 miles in an hour" and then restated for the rest of the class the student's interpretation: "Every 65 miles that I travel, that's an hour of time; or if I travel for an hour, I've gone 65 miles." Samantha then asked students to "find the unit rate in miles per hour" if a car travels 155 miles in two hours. After students divided 155 by two, Samantha explained that the resulting 77.5 represents the number of miles the car travels each hour.

Samantha's definition of rate as a "ratio of two quantities with different units" supported students' understanding of constant rate as the change in some quantity's measure that corresponds to a one-unit increase in another quantity's measure. Her discussion of the meaning of a vehicle traveling at a constant rate of 65 miles per hour is a case in point. She encouraged a meaning based on a coordination of additive changes, instead of a proportional correspondence between continuously varying changes in accumulated distance and accumulated time.
Toward the end of the lesson, Samantha asked her students to compute the unit rate provided the values in Table 1. Her discussion of computing this unit rate supported a conception of rate as an additive comparison grounded in images of discrete variation. In response to the task, a student proposed dividing the weight of two books (6 lbs.) by the number of books—a reasonable strategy considering that just prior to this task the class had computed the unit rate of a car traveling 155 miles in two hours by dividing the car's distance traveled (155 miles) by two hours. Another student recognized that the number of pounds per book was evident in the third column of the table (3 pounds corresponds to one book). Samantha noticed that the two students were focusing on comparing the *values* in the table, rather than determining the *change* in the number of pounds that corresponds to adding one additional book. She subsequently directed students' attention to these changes.

Table 1: Find the unit rate.

Books	0	1	2	3
Pounds	0	3	6	9

Samantha's Pre-Intervention Video Analysis

After Samantha concluded her instruction of rate of change, but prior to the teaching experiment, we asked her to watch the videos of her first two lessons and to identify moments that exemplify high quality instruction as well as moments that suggest room for improvement. She categorized her opening discussion—in which she asked students to describe what they think of when they hear the word "rate"—as indicating room for improvement. In her written justification Samantha explained,

> I like that I ask students what they think of when they hear the word 'rate.' However, I think there is room for improvement because I tend to talk to one student (whoever answers the question) instead of addressing the whole class with the student's answer.

Samantha identified her explanation of the meaning of 65 miles per hour as representing high quality instruction and rationalized her selection as follows:

> I think this portion went really well because we really isolated what a unit rate is. I used speed limits because I knew it was something the students could understand. I liked that I asked the students what '65 miles per hour' means. When a student explained it using the same words found in the question, I asked if another student would repeat what he said but say it in their own words. I thought this was an effective way to get more than just one

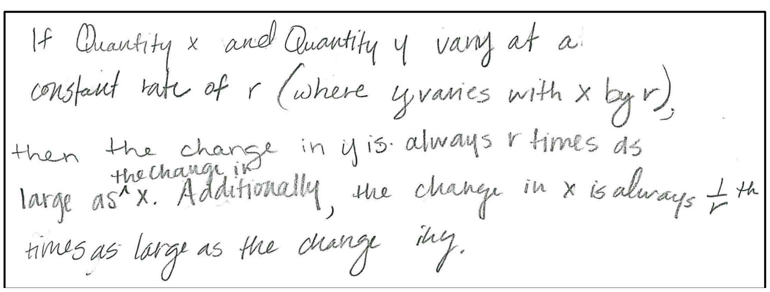
> student involved and to check that others understood the concept. Additionally, I liked that I had students give several examples. This helped me to see that they knew what a unit rate was and it kept them engaged.

We notice that Samantha prioritized students' engagement and participation as important criteria for evaluating the quality of her teaching. This focus on students and their activity was reflected in five of the six written justifications she provided for her selections. Samantha's appraisal of her teaching was based primarily on the extent to which she sustained students' interest and elicited their contributions. None of her written responses reflected a critical evaluation of the meaning of rate of change her instruction supported.

Teaching Experiment

We leveraged Piaget's notion of reflecting and reflected abstraction to design an instructional sequence that would enable Samantha to construct a scheme for rate of change that reflects the meaning we articulate in our conceptual analysis above. This instructional sequence unfolded over nine teaching episodes. On many occasions throughout the teaching experiment, Samantha repeatedly and convincingly demonstrated her understanding of constant rate of change as an invariant multiplicative relationship between changes in the measures of covarying quantities.

To engender reflected abstractions—that is, to support Samantha's awareness of the mental actions and conceptual operations that comprised her rate of change scheme—we provided Samantha with her written work to particular tasks in the instructional sequence and prompted her to compare the thinking required to correctly solve select pairs of tasks. We asked her to articulate her comparison in writing and then to respond to our clarifying questions about what she had written. Although the tasks in the instructional sequence varied substantially in terms of the information provided and the quantity Samantha was asked to compute or represent, all tasks in the sequence could be solved by leveraging a meaning for constant rate of change as a proportional correspondence between changes in covarying quantities' measures. After having recognized that similar or identical thinking is required to solve all 12 pairs of tasks from the instructional sequence we asked her to consider, we prompted Samantha to summarize in writing the meaning for rate of change that might enable a student to reason productively about all tasks in the instructional sequence. We include her response in Figure 1.



If Quantity x and Quantity y vary at a constant rate of r (where y varies with x by r), then the change in y is always r times as large as the change in x. Additionally, the change in x is always $\frac{1}{r}$ th times as large as the change in y.

Figure 1. Samantha's meaning for constant rate of change.

At the conclusion of the teaching experiment, Samantha was able to demonstrate her conception of constant rate of change as a proportional relationship between changes in covarying quantities' measures and to articulate the mental imagery, actions, and operations that comprised her meaning for constant rate of change. We were interested in assessing the implications of Samantha's sophisticated rate of change scheme and her awareness of its contents for the criteria she leveraged to evaluate her teaching and to propose alternative instructional actions.

Samantha's Post-Intervention Video Analysis

Just as she had done in her pre-teaching experiment video analysis, Samantha identified the opening discussion of the first lesson as indicating room for improvement. In this discussion, Samantha invited students to describe what they think of when they hear the word "rate" and to suggest some examples of rates. Prior to the teaching experiment, Samantha critiqued this episode of her first lesson based on her lack of success in getting more students involved in the discussion. The following is Samantha's justification for identifying this opening discussion as "room for improvement" after having participated in the teaching experiment:

> I think there is room for improvement here because I think there is a better way to introduce rates. I feel like asking students what they think of when they hear a word is a fine way to get them talking and engaged but it doesn't direct the conversation to the mathematical meaning of a rate. Instead, this question sort of just starts a guessing game between myself and the students rather than engaging in mathematical discourse. … I could've started the conversation off by posing a question like, 'Clara runs 7 miles per hour. How far will she run in 2 hours?'

We were interested in why Samantha suggested posing the question, "Clara runs 7 miles per hour. How far will she run in 2 hours?" so we asked her write a transcript of a hypothetical alternative to the opening discussion from the first lesson and to explain why she prefers this alternative. After having asked students to determine how far Clara will run in two hours, Samantha's transcript indicated her asking them to determine how far Clara will run in three hours, half an hour, and 4.3 hours. She then prompted students to generalize a (multiplicative) relationship between Clara's distance traveled (in miles) and the number of hours Clara had spent running. The following is Samantha's explanation for why she expected this alternate introduction to be more effective than how she actually began her first lesson:

> I think this interaction reflects an ideal unfolding of the conversation because we start with a basic example that is easy to understand (i.e. Suppose Clara runs 7 miles per hour. How far will she run in 2 hours?) This part gives students an easy starting point to use. Then I ask some follow up questions, like how far would she go in half an hour, another good benchmark. I move on to give an example that needs a little more explanation so that we can start thinking about how to break up the time (4.3 hours) and figure out what the respective number of miles is. It's not as obvious as 4 hours and I could have even chose something like 0.59 hours instead. … I think that opening with these examples fosters a conversation of how to find rate of change and what it means for those quantities to vary directly. The end of the conversation opens up many routes; we could talk about finding time and then add some initial values and dig into changes in quantities.

Samantha's alternate introduction to the first lesson reflects both her image of what it means to understand constant rate of change as well as her expected trajectory through which students must progress to construct this understanding. Rather than proposing instructional actions that simply elicit students' participation, Samantha's transcript and accompanying rationale demonstrate that her criteria for effective teaching had shifted to incorporate attention to the mathematical meanings her instruction supported.

The transformation in the criteria by which Samantha evaluated the quality of her teaching was evident in her analysis of other episodes from the first lesson. Regarding her definition of rate as a "ratio of two quantities with different units," Samantha commented:

> I'm not really a huge fan of doing a ton of definitions so I don't know why I did this. I don't think I would use this definition (the book definition) for rate again. Although it connects rates to ratios, it doesn't really give any meaning to rates. One could recite this definition without actually knowing how to do any problem that involves rates.

Samantha's appraisal of her discussion of what it means to say that a car travels at a constant speed of 65 miles per hour similarly reflected her attention to the understanding of rate she promoted:

> I think we are getting closer to what I would really like conversations about rate to look like. This is what I would call the first stage of understanding rates of change. The students seem to have an understanding that 65 miles per hour means that for every hour traveled, one goes 65 miles. I think this a great place to start conversations about rates of change. Maybe some other contexts could get the conversation moving further so we could generalize the idea by talking about changes in quantities.

Additionally, Samantha expressed contrition for asking students to determine the unit rate provided the values in Table 1. Specifically, she acknowledged that the context did not support students' conception of rate of change as conveying a relationship between *changes* in quantities' measures:

> I see what I am getting at here. I am trying to get at the idea of changes in pounds for changes in books and how that is a rate. Honestly, the examples I chose aren't the best for trying to make students think hard about changes in quantities. The problems are almost trivial, clearly every book is three pounds. Once students see that, it's hard to get them to talk about changes in quantities because the answer to the question has already been found and can be explained simply.

Samantha critically evaluated the mathematical meanings her instruction supported in 14 of the17 justifications she provided for the segments of her lessons that exemplify high quality instruction or indicate room for improvement. In Samantha's initial analysis of her teaching videos, none of her written responses suggested that she was attending to the understanding of rate of change her instruction promoted. This shift in the criteria by which Samantha evaluated her teaching quality did not escape her attention. When asked what she initially considered when evaluating her instruction, Samantha explained, "I think it was student engagement, using something the students care about or would be interested in, asking good questions, getting them to repeat back." Samantha acknowledged the importance of attending to issues of classroom management and students' participation, but then suggested that instruction viewed only through this lens does not enable one determine whether students are being provided with opportunities to construct productive ways of understanding. Having come to this conclusion, Samantha described the additional criterion that she considered after having engaged in the teaching experiment to assess the quality of her teaching:

> As we went over rate and what it means to be a rate together, I started thinking more about what am I doing to promote understanding. … Going back I think cared a lot more about classroom management versus when I went back and looked at these I acutely cared about, ok wait, is this promoting understanding. … I cared more about what they knew.

Discussion

Our findings expose a common assumption that underlies much of the literature on mathematical knowledge for teaching as well as mathematics teacher preparation and professional development programs: Mathematical knowledge for teaching is comprised of a variety of distinct but related knowledge "types" that coalesce in the practice of teaching. Mathematics teacher education programs based on this assumption focus on supporting teachers in developing these categories of knowledge without seriously attending to how knowledge in one domain informs or derives from knowledge in another. The central findings of our study suggest that teachers' image of effective teaching—and the pedagogies they enact to ensure that their practice conforms to this image—are influenced by having constructed powerful mathematical schemes and by having achieved an awareness of the mental actions and conceptual operations that comprise these schemes. The results of our study therefore support Tallman's (2015, in submission) hypothesis that engaging pre-service teachers in mathematical experiences designed to engender reflecting and reflected abstractions has positive implications for their image of effective teaching and the pedagogical actions they enact (or envision enacting) to achieve their instructional goals.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*, 389-407.

Corbin, J., & Strauss, A. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (3rd ed.). Los Angeles, CA: Sage Publications, Inc.

Depaepe, F., Verschaffel, L., & Kelchtermans, G. (2013). Pedagogical content knowledge: A systematic review of the way in which the concept has pervaded mathematics educational research. *Teaching and Teacher Education, 34*, 12-25.

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-164). New York, NY: Macmillan.

Joshua, S., Musgrave, S., Hatfield, N., & Thompson, P. W. (2015). Conceptualizing and reasoning with frames of reference. In T. Fukawa-Connelly, N. E. Infante, K. Keene & M. Zandieh (Eds.), *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education* (pp. 31-44). Pittsburgh, PA: RUME.

Piaget, J. (2001). *Studies in reflecting abstraction*. New York: Psychology Press.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1-22.

Steffe, L. P. & Thompson, P. (2000). Teaching experiment methodology: Underlying principles and essential elements. In Kelly, A. & Lesh, R. (Eds.) *Research design in mathematics and science education*. Amsterdam, Netherlands: Kluwer.

Studiocode (Version 5.8.4). [Computer Software]. (2015). Sydney, New South Wales, Australia: Sportstec, Ltd, DBA Studiocode Business Group.

Tallman (in submission). *Do teachers teach what they know? A comparison of a secondary teacher's personal and enacted subject matter knowledge.*

Tallman (2015). *An examination of the effect of a secondary teacher's image of instructional constraints on his enacted subject matter knowledge.* Unpublished Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University.

Thompson, P. W. & Carlson, M. P. (2017). *Variation, covariation, and functions: Foundational ways of thinking mathematically*. In J. Cai (Ed.), Compendium for Research in Mathematics Education (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.

Thompson, P. W. & Thompson, A. G. (1992). *Images of rate.* Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco.

INVESTIGATING HOW A MEASUREMENT PERSPECTIVE CHANGES PRE-SERVICE TEACHERS' INTERPRETATIONS OF FRACTIONS

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Learning fractions has been one of the challenges that mathematics educators face for few decades. Theoretical and empirical revisions of our common practices of teaching fractions are needed. This study investigates how re-examining fractions from a measurement perspective influences pre-service teachers' (PSTs) interpretations of fractions. Sixty-seven PSTs engaged in re-examining fractions from a measurement perspective during a 15-week semester. They also completed pre- and post-tests that assess their interpretations of fractions represented in discrete and continuous models. Findings show statistically significant increase in PSTs' scores on the post-test, specifically on questions that used continuous models. This study shifts investigating fractions learning from an exclusive focus on the partitioning perspective to the measurement approach and highlights its affordances.

Keywords: Teacher Knowledge, Number Concepts and Operations, Rational Numbers, Instructional activities and practices

Introduction

Supporting students' meaningful learning of fractions has been a prominent challenge of mathematics educators. Teachers struggle to help students from early ages to conceptualize fractions and know how to operate on them. Research on teachers' and students' learning of fractions and operations on fractions has been based on two ontological perspectives: partitioning and measurement. The partitioning perspective views a fraction as a relation established between parts of a whole partitioned into equal parts. This ontological perspective is rooted in a belief that counting is the basis for arithmetic and has led to the commonly accepted part/whole conception of fractions. For instance, it is how the Common Core State Standards (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) suggests introducing factions in the elementary school grades. Nevertheless, research indicates that this perspective seriously limits the robustness of students' understanding of fractions (Kerslake, 1986). It tends to support the erroneous idea that a fraction is composed of two different quantities and impedes understanding of improper fractions. Little research has investigated the known alternative source for fraction knowledge, the measurement perspective.

In response to this conceptual gap in the literature, we investigate how a measurement approach influences elementary pre-service teachers' (PST) interpretations of fractions. In contrast to a part/whole conception a measurement conception views fractions as numbers or ordered pairs that represent relations between two measurable, continuous quantities (Davydov & Tsvetkovich, 1991; Gattegno, 2009/1960). The relation is a multiplicative comparison of the two quantities measured by the same unit. The comparison is based on one quantity being a particular multiplier of the other. In our work with PSTs, we aim to investigate how their knowledge of multiple of representational models of fractions is influenced by their re-examination of fractions from a measurement perspective.

Theoretical Framework and Relevant Literature

Research on fraction learning indicates that both American students and teachers lack conceptual knowledge of fractions and operations on fractions. This deficiency coexists with current curriculum standards and textbooks grounded in known ineffective ontological and epistemological perspectives about fractions (Ni & Zhou, 2005). These perspectives and their corresponding curricular practices are based on partitioning approaches such as the part/whole method (Lamon, 2012). However, though the part/whole subconstruct and its partitioning approach are conceptually deficient, what is not known is how other definitional perspectives facilitate robust conceptual understanding of fractions. Measurement as an ontological source for fraction knowledge has been the object of limited research efforts. Informed by theoretical work of Davydov and Tsvetkovich (1991), we conceptualize the ontology of fractions and epistemology of fraction knowledge based on the objective source of fractions in measurement.

We view a *fraction* as a relation, a/b, a multiplicative comparison between two quantities measured by the same unit: Suppose A and B are two objects that have a common extensive attribute such as length expressed as quantities with a common unit, u, where A equals a units of u and B equals b units of u. Then, in the expression, a/b, b is understood to measure a and a/b is the relation of a measured by b. The quantities a and b have the same unit of measure.

Measurement as a material source of both whole numbers and fractions has been theoretically and pedagogically investigated by Davydov and others who have adapted Davydov's curriculum (Davydov & Tsvetkovich, 1991; Morris, 2000; Schmittau & Morris, 2004). The contributions of Morris (2000), Schmittau and Morris (2004), Brousseau (Brousseau et al., 2008), Davydov (Davydov & Tsvetkovich, 1991) and others to understanding affordances of a measurement perspective for whole number and fraction knowledge raises questions about what are possible approaches from a measurement perspective that can support students, younger than fourth graders, to develop number sense about fractions.

Methods

Our empirical study engages pre-service elementary teachers in an intervention about re-examining fractions from a measurement perspective to investigate changes in their knowledge of representational models of fractions. The intervention took place in an elementary mathematics methods course during a 15-week semester, fall 2017, and involved 67 PSTs. During the first week of the semester, the teachers completed a pre-test about fractions that involves expressing different fractions in discrete and continuous representations. In the last week of the semester, teachers completed the same assessment as a post-test. For approximately one hour every two weeks, the teachers used Cuisenaire rods to solve collaboratively fractions tasks. From a measurement perspective, the tasks intend to engage the PSTs in re-examining their number sense about magnitude, order, equivalence of fractions, and operations on fractions viewed as relations that express multiplicative comparisons between two measurable quantities measured by common unit.

Data for this study come from pre- and post-tests that PSTs completed during the first and last weeks of the semester. We adopted a fraction test created by Norton and Wilkins (2010) that involves representing fractions less than one and greater than in discrete and continuous models. The discrete problems ask PSTs to represent fractions of collections of dots. The continuous problems ask PSTs to represent fractions using rectangular and circular models. Each test includes 10 questions that involve only two of the representational models of fractions. We followed Boyce and Moss's (2017) method in grading PSTs' responses on the tests. They would receive either a one, 0.5, or a zero depending on the accuracy of their response. We used *t*-test to examine differences in responses between pre- and post-tests and among the different interpretations of fractions.

Results

Analyses of pre- and post-tests scores show an increase from before and after the intervention (see Table 1). On the pre-test, the PSTs scored an average of 2.96 out of 5 points on the five questions for each representational model. This indicates that PSTs have limited interpretations of fractions when expressed in different forms. The PSTs' scores on the post-test show statistically significant increase compared to their scores on the pre-tests. The scored an average of 3.53 points out of five total points. Even though this increase is significant, the scores indicate that some PSTs still need additional support to develop robust understanding of fractions.

Table 1: Pre- and Post-Test Scores

	Pre-test	Post-test
Mean	2.96	3.53
Median	3	4
Standard deviation	1.24	1.22
The t-value is 3.78762. The p-value is .000094. The result is significant at $p < 0.001$.		

When investigating PSTs' responses for each representational model (see Table 1), we can see that PSTs scored higher in the pre-test on questions that used circular models than questions that used discrete models or bars. This performance was also noticeable in the post-test. However, the most gain in scores between the pre- and post-tests was on questions that used bars to represent fractions. Using a *t*-test, we found that PSTs scores on questions that used bars increased significantly ($p < 0.001$) between pre- and post-test. Similarly, PSTs' scores on questions that used circular representations increased significantly ($p < 0.05$) between pre- and post-test. There was no statistically significant change in scores regarding questions that used discrete model to represent fractions.

Table 2: PSTs' Pre- and Post-Test Scores by Fractions Representational Models

	Bars		Circles		Dots	
	Pre	Post	Pre	Post	Pre	Post
Mean	2.59	3.56	3.47	3.71	2.79	3.34
Median	3	3.5	3	4	3	3.75
Standard deviation	1.30	1.27	1.02	1.12	1.25	1.28
Significance level (t-test)	$p < .001$		$p < .05$		not significant	

These results show that re-examining fractions using a measurement perspective can influence how PSTs interpret fractions represented in a rectangular or circular form. The rectangular model be a two-dimensional version of the three-dimensional manipulatives used during the intervention and this could explain the strong change in PSTs' scores on questions that used bars to represent fractions. The circular model, however, is mathematically different than the rectangular model. Circular models concern areas of circles and circular sectors where rectangular models concern the length of the bar or rectangle (only one dimension of the rectangle while holding the other dimension constant). Even though the intervention did not engage the circular models when re-examining fractions, PSTs were able to better interpret fractions when expressed using circles and circular sectors. This shows the potential that a measurement perspective can offer when introducing fractions.

Discussion

In this study, we examined the influence of re-examining fractions from a measurement perspective on PSTs' knowledge of fractions expressed through different representational models. Using Cuisenaire rods, we engaged 67 PSTs in mathematical tasks that allowed them to view fractions as multiplicative comparisons between two measurable, continuous quantities (length of different Cuisenaire rods). The PSTs completed a pre-test before the intervention and a post-test afterward. The pre- and post-tests included questions about interpreting fractions using rectangular, circular, and discrete forms. Findings show that, in general, PSTs scored significantly higher on the post-test. When examining the different representational models of fractions, PSTs scored significantly higher on the post-test for the questions that involve rectangular and circular representations of fractions. There was not significant change on PSTs' scores on post-test for questions that involve the discrete model. These findings show that re-examining fractions using a measurement perspective offers opportunities for PSTs to engage their multiplicative reasoning (Vergnaud, 1988) when conceptualizing fractions.

Our findings for the analysis of PSTs' scores on the pre-test, which are similar to the findings of Boyce and Moss (2017), highlight how PSTs' knowledge of fractions is limited, especially for fractions represented in continuous models. The significant changes in PSTs' scores on the post-test indicate that the affordances of measurement perspective can help learners overcome some of the challenges of conceptualizing fractions in different forms.

Our study is significant for practice. It supports teachers building deep understanding and flexible representational models of fractions, which, in turn, lead to improving students' fractions learning. In addition, this study contributes to the literature of teaching and learning of fractions by shifting from an exclusive focus on the partitioning perspective to an attention to the measurement approach to highlight its affordances.

References

Boyce, S. J., & Moss, D. (2017). *The role of representation in assessing prospective teachers' fractions' schemes*. Paper presented at the NCTM 2017 Research Conference, San Antonio, TX.

Davydov, V. V., & Tsvetkovich, Z. (1991). On the objective origin of the concept of fractions. *Focus on Learning Problems in Mathematics, 13*(1), 13-64.

Gattegno, C. (2009/1960). *Arithmetic: A teacher's introduction to the Cuisenaire-Gattegno method of teaching arithmetic* New York: Educational Solutions Worldwide.

Kerslake, D. (1986). *Fractions: Children's strategies and errors*. Windsor, England: NEER-NELSON.

Lamon, S. J. (2012). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers* (3rd ed.). New York: Routledge.

Morris, A. K. (2000). A teaching experiment: Introducing fourth graders to fractions from the viewpoint of measuring quantities using Davydov's mathematics curriculum. *Focus on Learning Problems in Mathematics, 22*(2), 32-83.

National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). *Common core state standards for mathematics* Retrieved from http://www.corestandards.org/wp-content/uploads/Math_Standards1.pdf

Ni, Y., & Zhou, Y.-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist, 40*(1), 27-52.

Norton, A., & Wilkins, J. L. (2010). Students' partitive reasoning. *The Journal of Mathematical Behavior, 29*(4), 181-194. doi:10.1016/j.jmathb.2010.10.001

Schmittau, J., & Morris, A. (2004). The development of algebra in the elementary mathematics curriculum of V.V. Davydov. *The Matheamtics Educator, 8*(1), 60-87.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 141-161). VA: National Council of Teachers of Mathematics.

KNOWLEDGE RESOURCES FOR PROPORTIONAL REASONING IN DYNAMIC AND STATIC TASKS

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In this study, we consider the knowledge resources invoked by middle school teachers as they solve two related proportional reasoning tasks. For one task, teachers were asked to think aloud about a situation in which an artist is asked to create a larger image. The other task was administered in a clinical interview setting and asked questions about the relationship in size between bears in a dynamic sketch in which one bear dilated. Analysis focused teachers' reasoning. Findings focus on the knowledge resources that were invoked. Implications focus on why the knowledge resources matter and opportunities for additional research.

Keywords: Teacher Knowledge, Technology, Number Concepts and Operations

Purpose and Background

In this study, we consider how teachers solved two mathematical tasks with similar underlying mathematics. One of the tasks relied on a dynamic sketch created in Geometers' SketchPad, while the other was a traditional paper and pencil task. Our interest in conducting this research came from noticing that teachers engaged with the mathematics differently in the dynamic environment than in the paper-based environment. Our research questions for this study were: (a) What knowledge resources do teachers invoke when they solve the dynamic task correctly?; (b) What knowledge resources do teachers invoke when they solve the dynamic task incorrectly?; and (c) How are the knowledge resources used in the dynamic environment similar to or different from those used in a paper and pencil task with similar content?

The tasks of interest in this study both required teachers to differentiate between situations that are and are not proportional. We built from a growing body of work (e.g., De Bock, Van Dooren, Janssens, & Verschaffel, 2002) investigating high school students' inappropriate use of proportionality. Across a number of studies, the researchers found that students overuse proportional approaches in situations that appear proportional but are not.

We rely on knowledge in pieces (KiP; diSessa, 1988, 2006) as our conceptual framework, which posits that knowledge is organized as fine-grained knowledge resources and that these resources are interconnected in ways that make them available in a wide range of settings. From this perspective, learning is achieved through the development of new knowledge resources, the refinement of existing resources, and/or the development of connections between resources.

Methods

We used a convenience sample of 32 in-service middle grades mathematics teachers from four states in the U.S. Their teaching experiences ranged from one to 26 years. Of the participants, 24 identified as female and eight as male. Five participants identified as Black, one as Biracial, and the rest as White. Participants taught at a variety of schools including urban, suburban, and rural schools that were public, private, or charter.

Participants completed two different types of interviews. The first was a think-aloud protocol that was mailed to them. The second interview was a one-hour clinical interview recorded using two video cameras to capture the participant's voice as well as any writing, pointing or use of Sketchpad during the interview.

Table 1: Most frequently occurring knowledge resources found in this study

Resource Type	Code	Description
Proportional Reasoning Resources	Constant ratio	Recognizes the invariant multiplicative relationship between two quantities.
	Scaling up/down	Uses multiplication to scale both quantities to get from one ratio in an equivalence class to another.
	Horizon knowledge	Demonstrates knowledge that extends into mathematics beyond proportions.
	Rule	Shares a verbal or written rule (e.g., Red = Blue - 2) stated in a way that conveys a generalizable relationship.
Pedagogical Resources	Anticipates or builds from others' thinking	Talks about what others would or might do in solving a problem or builds from the mathematical thinking of others.
Representation Resources (not unique to proportional reasoning)	Problem solving	Uses static or dynamic representations to support reasoning about and solving the problem. This includes sense-making. This is also exploratory and happens during problem solving.
	Justifying or communicating	Uses the representation to explicitly justify or explain a position already developed, rather than to solve a task.
	Conjecture testing	Makes a conjecture and then tries it out with the representation.

We developed a coding scheme (e.g. Charmaz, 2014) to identify the proportional knowledge resources being invoked by the participants. Two or more members of the team individually coded each utterance. Then the codes were discussed until 100% agreement was reached. Codes were applied based on mathematical reasoning used rather than correctness. The knowledge resources relied upon by the most teachers in the two tasks for this study are shown in Table 1. The resources fall into three categories: proportional reasoning, pedagogy, and representations.

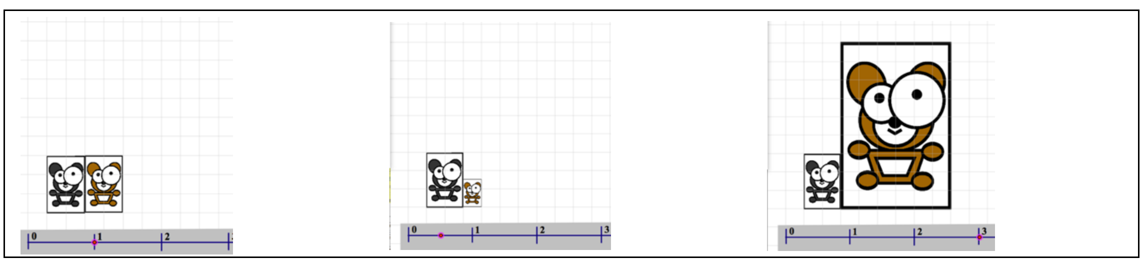

Figure 1. Screenshots of the bears task.

The Bear task presented the participants with a pair of bears (Figure 1). The bear on the left remained static, while the one on the right dilated as the point on the number line was dragged. During the clinical interview, participants were asked to interact with Bears in SketchPad Explorer to answer a series of questions.

The Santa Task was modified from De Bock et al.'s (2002) task to be more appropriate for teachers. The task asked teachers to consider that 56 cm high Santa was painted on a bakery door using 6 ml of paint. The painter was asked to create an enlarged version of the painting on another window. The new painting would be 168 cm high. The task asked how much paint was needed for the larger image. Three scaffolds were included to support appropriate reasoning.

Results

By the end of the series of questions around the Santa task, 13 of the 32 (41%) teachers correctly recognized that the area situation was not proportional and reasoned accurately. For the Bear task, 26 of the 32 teachers (81%) correctly identified proportional relationships.

Table 2. Number of participants using and frequency of use for each knowledge resource.

	Santa Task		Bear Task	
	# of teachers	# of occurrences	# of teachers	# of occurrences
Constant Ratio	1	1	17	25
Scaling Up/Down	21	33	22	55
Horizon Knowledge	5	7	20	34
Rule	2	2	21	58
Anticipates/Builds on others' thinking	16	26	2	2
Problem solving with representation	3	3	24	113
Justify/communicate with representation	6	8	29	97
Conjecture test with representation	2	3	15	30

The most common proportional reasoning knowledge resource used by participants in both tasks was Scaling Up/Down (see Table 2). Nearly two-thirds of the participants demonstrated evidence of this knowledge resource in both tasks. A typical Scaling comment in the Santa task was, "… if our figures are in fact proportional, this would have to be multiplied by three" (Ella, all names are pseudonyms). Here, Scaling was used to support an incorrect answer.

The only non-representation focused knowledge resource commonly used in the Santa task was Anticipates or Builds on the Thinking of Others (hereafter referred to as Anticipates). We conjecture that this pedagogically-focused code was commonly used in this task because all of the scaffolds asked the teacher to interpret students' thinking in some way. As shown in Table 2, this knowledge resource was used by 16 participants when solving the Santa task.

In the Bears task, participants used more knowledge resources. This was likely tied to the difference in interview format as there was an interviewer to ask follow-up questions here, but not for Santa. Perhaps because of the nature of the task, the role representations played in teachers' reasoning was more pronounced in the Bears Task with those three representation resource codes (in the gray rows of Table 2) being used commonly. The three representation resource codes in Table 2 were used by participants regardless of their reasoning (linear or area). For an utterance to be coded as one of these three codes, the participant had to use the sketch to explore a mathematical idea or explain their thinking to the interviewer by demonstrating the idea on the sketch. In addition, 21 participants were able to correctly generate a rule for the given dynamic situation. Constant Ratio was demonstrated by 17 of the participants while solving the Bear task. Horizon knowledge was evident for 20 participants while solving the Bear task.

Discussion and Implications

Scaling up and down was the only knowledge resource used consistently between the two tasks. This is notable because scaling can be accomplished through additive or repeated reasoning (e.g., Kaput & West, 1994) and suggests these teachers may have relied on additive reasoning rather than multiplicative reasoning in these tasks. In the Bears task, the same participants relied on two proportional reasoning knowledge resources: describing a rule for the relationship and reasoning with a constant ratio. In contrast to the common Scaling resource, reasoning with a constant ratio requires attention to the multiplicative relationships inherent in the proportion. While the format of the interview was different, we suspect the difference in presentation drove some of the differences in approach. This was corroborated by the high number of teachers who relied on the dynamic sketch to make sense of and solve the Bear task.

The use of particular knowledge resources matters because it highlighted the degree to which teachers reasoned about the situation rather than solving tasks. There is ample evidence that proportional reasoning is too often focused on solving a proportion (e.g., Lamon, 2007) rather than reasoning about the relationships between quantities. From the limited data presented here, we suggest that moving toward dynamic environments may support more reasoning about proportional situations. In our findings, we particularly note that over half the participants reasoned about one of the key structural aspects of a proportion, the constant ratio, in the dynamic environment. Similarly, the dynamic representation seemed to have opened avenues for teachers to draw on a broader body of knowledge as well as create meaningful generalizations about relationships in the Bear task. If our goal in engaging teachers in mathematical situations is to help them build connections between their knowledge resources, it appears that Bears was more effective in supporting that goal. We hypothesize the ability to generalize may be tied to the infinite instances inherent in the dynamic situation. This is an area for additional research.

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References

Charmaz, K. (2014). *Constructing grounded theory* (2nd ed.). Thousand Oaks, CA: Sage Publications Ltd.

De Bock, D., Van Dooren, W., Janssens, D., & Verschaffel, L. (2002). Improper use of linear reasoning: An in-depth study of the nature and the irresistibility of secondary school students' errors. *Educational Studies in Mathematics*, *50*(3), 311-334.

diSessa, A. A. (1988). Knowledge in pieces. In G. Forman & P. Pufall (Eds.), *Constructivism in the computer age* (pp. 49-70). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

diSessa, A. A. (2006). A history of conceptual change research: Threads and fault lines. In R. K. Sawyer (Ed.), *The Cambridge handbook of the learning* sciences (pp. 265-282). New York: Cambridge University Press.

Kaput, J. J., & West, M. (1994). Missing-value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel, & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 235-287). Albany, NY: State University of New York Press.

Lamon, S. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp.629-667). Reston, VA: National Council of Teachers of Mathematics.

HIGH SCHOOL TEACHERS' CONTENT KNOWLEDGE AND KNOWLEDGE OF STUDENTS' ERRORS ABOUT QUADRATIC FUNCTIONS

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This study examined forty high school algebra teachers' content knowledge as well as their knowledge of students' errors about quadratic functions through a teaching-scenario task. The teachers' responses were analyzed quantitatively and qualitatively. Analysis results show that the teachers have sufficient content knowledge and they performed better on translating among different representations of quadratic functions than on solving a real-world problem by using quadratic functions. Most of the teachers identified students' obstacles in translating from algebraic to graphic representations of quadratic functions and they tended to respond in a teacher-centered procedural way. The implications of this study for teacher education programs as well as professional developments are presented in accordance with the findings.

Keywords: Algebra and Algebraic Thinking, Mathematical Knowledge for Teaching

Algebra has long been regarded as a critical bridge to high school mathematics. National Council of Teachers of Mathematics [NCTM] (2000) highlighted the importance of algebra to all students. Among all the high school algebraic topics, quadratic functions are fundamental and essential (Even, 1990), given that it is a critical transition from straight lines to curves. To help students achieve both conceptual understanding and computational proficiency of the topic, teachers were suggested to have solid content knowledge (Harris & Sass, 2011; Hill, Rowan & Ball, 2005) as well as a variety of pedagogical content knowledge, which has been recognized directly relates to students' mathematics achievement and this relationship starts from upper-elementary grades (Campbell, et al., 2014). Among all the categories of teachers' pedagogical content knowledge, we explored teachers' knowledge of student errors because NCTM (2000) suggested that recognizing and responding to student errors appropriately is one of the main tasks of teachers in teaching mathematics. Additionally, researchers indicated that student errors provide teachers windows into student understanding that should be endeavored to help students develop conceptual basis of their errors (Ball, 1990; Borasi, 1994).

As for content knowledge, we examined how high school teachers apply quadratic functions to solve a real world problem and how they transform among different representations of quadratic functions. The reasons why these two aspects were investigated are that generating equations to represent the relationships in typical word problems has long been recognized as one of the major areas of difficulty for high school algebra students (Clement, 1982; Kieran, 1992) and that teachers should emphasize cultivating students' algebraic thinking in a way that enables "the use of any of a variety of representations that handle quantitative situations in a relational way" (Kieran, 1996, p. 4, 5). In terms of teachers' knowledge of students' errors, we studied how they interpret and respond to students' learning obstacles in translating from graphic to algebraic representations of quadratic functions. The specific research questions are:

1. How do high school teachers use quadratic functions to solve real world problems?
2. How do high school teachers transform among different representations of quadratic functions?

3. How do high school teachers interpret and respond to students' errors about translating from graphic to algebraic representations of quadratic functions?

Theoretical Framework

Zaslavsky (1997) summarized a few common obstacles related to students' understanding of quadratic functions. First, it is hard for students to imagine the parabola as extending forever. Second, students confused about the relation between quadratic functions and quadratic equations. They missed the fact that though $x^2 + 2x - 3 = 0$ is equivalent to $2x^2 + 4x - 6 = 0$, the function $F(x) = x^2 + 2x - 3$ is not equivalent to $F(x) = 2x^2 + 4x - 6$. Third, they prefer going from equations to graphs rather than from graphs to equations. Also, they prefer the standard form to the vertex or the factored form of quadratic functions. Later on, Eraslan (2005) systematically explored two honors Algebra II students' obstacles in learning quadratic functions. He found that the students struggled to translate quadratic functions from graphic to algebraic representation, tended to use the standard form over the vertex form, and failed to use quadratic model to solve problems given in real-world situations. In addition to high school students, preservice teachers also struggled to integrate algebraic and graphic representations of functions (Huang & Kulm, 2012). This relates to the current study, in that, we asked participants to respond to a students' error in a teaching scenario where the student is challenged to represent a parabola by algebraic equations.

Employing student errors as springboards to develop worthwhile mathematics inquiries, Borasi (1994) suggested that a consistent use of error analysis enables students to better understand the nature of mathematics, facilitates the learning of significant mathematical content, make students more proficient in doing mathematics and make them more confident in their ability to learn and use mathematics. In order to use students' errors constructively, Seifried and Wuttke (2010) suggested that teachers need to be competent in three aspects, including knowledge of possible error types, available strategies of reaction, and a constructive view on errors. Nevertheless, few research examined students' errors in learning mathematics, much less known is information about students' errors on specific and fundamental mathematics concepts, especially variable, equation, and function (Li, 2006).

Analyzing participants' responses and explanations to students' errors, we examined mathematical focus, pedagogical actions, form of address, degree of student error use and communication barriers (see Table 1).

Table 1: Analytical Framework for Teachers' Responses to Students' Errors

	Aspect	Categories
Global	1 Mathematical/ instructional focus	Conceptual vs. procedural
	2 Pedagogical action(s)	Re-explains, suggests cognitive conflict, probes student thinking, etc.
Local	3 Form of address	Show-tell vs. give-ask
	4 Degree of student error use	Active, intermediate, or rare
	5 With/without communication barrier	Over-generalization, a Plato-and-the-slave-boy approach, or a return to the basics, specific

Methods

Twenty Chinese and twenty U.S. high school teachers, who are currently teaching or have taught Algebra I topics before, joined the study and finished a questionnaire including two math problems and a teaching-scenario task. The first math problem was adopted from Chinese text *ShuXue*, the most frequently used text in China, to examine how the teachers abstract an algebraic expression of quadratic functions to solve a real world problem. The second math problem aiming to explore the participants' knowledge of transformation among different representations of quadratic functions was adopted from Vaiyavutjamai's study (2009). The teaching-scenario task, where a student was required to write an algebraic equation for a given parabola but she didn't know where to start, was from Eraslan's work (2005).

To answer the research questions, this study applied grounded theory inquiry (Strauss & Corbin, 1994). As for teachers' content knowledge, we checked the correctness and explanations of the teachers' responses. In terms of responses to students' errors, the authors individually analyzed and coded the participants' responses according to the analytical framework while looking for new categories. We reached a satisfying agreement rate between 75% and 90%.

Results

Teachers' Content Knowledge of Quadratic Functions

We found that the teachers performed well on using quadratic functions to model real-life situations and further to solve problems. More than half of the teachers have chosen to apply the vertex form of quadratic functions, which is a wise choice because the vertex of the quadratic function is given and in this way the calculation process becomes easier. While most of the teachers used the vertex form, twenty-five percent of the teachers employed the standard form. Although both the standard and vertex forms of quadratic functions can be used to solve the problem, the teachers who applied vertex form showed deeper procedural knowledge than the teachers who employed the standard form because they were flexible in terms of choosing an appropriate form to represent functions and they did this based on the real-world situation. Six percent of the teachers showed conceptual knowledge of the problem in the way that they attempted the problem by using different methods as well as explained the solution by using different representations of the quadratic functions and analyzed the effects of parameters on the graphs of the functions (we plan to show examples in the presentation).

Transforming from graphic to algebraic representations of quadratic functions, all the teachers solved the problem correctly. Similar to their performance on modeling a real-life situation, more than half of the teachers applied the vertex form that showed deep procedural knowledge. Three teachers demonstrated their conceptual knowledge by identifying the operations and transformations played on the parent function $y = x^2$ to get the formula for the transformed parabola.

Teachers' Knowledge of Students' Errors

The teachers identified the student's (Amy) learning obstacles in reading graphs, choosing appropriate form of quadratics, and applying the method of undetermined coefficients. Among the teachers who identified Amy's knowledge deficiency, half of them recognized more than one piece of her learning obstacles. When the participants responded to Amy, they chose either to focus on conceptual knowledge that help Amy to solve this type of problems or to emphasize showing her the problem solving procedures. However, procedural explanation is a dominated method, which is not a surprise since most of the teachers solve problems procedurally themselves. With respect to procedural knowledge, most of the teachers demonstrated how to find three points from the given graph and plug the coordinates of the points into the standard form of the quadratic function to get three equations. Among the teachers who tried to build conceptual knowledge for Amy, some suggested to use the relationship between graphs and algebraic expressions, function transformation knowledge while the others explained how it would be more helpful to use the vertex or the factored form over the standard form.

Providing guidance to help Amy finish the problem, most of the teachers responded in a teacher-centered way. Since Amy did not make any mistake, the teachers barely used her errors but directly taught her how to solve the problem. More than sixty percent of the teachers guided Amy to finish the problem in a specific way, which means there is no communication barrier between the teacher and Amy and she is expected to finish the problem independently later on. Also, around thirty percent of the participants gave general instructions that provide a direction to solve this type of problems, for example, some of the teachers explained the rationale of the method of undetermined coefficients. However, this type of explanation will not help Amy solve the problem directly and probably causes more confusions for her.

Discussion and Conclusions

We found that the high school teachers have sufficient content knowledge about quadratic functions and they did better on translating among different representations of functions than on using quadratic functions to solve real world problems. Most of the teachers showed deep procedural and procedural knowledge and they responded to students' errors in a procedural way. A few teachers demonstrated conceptual understanding of quadratic functions but not all of them explained to Amy conceptually. Except knowledge focus, teachers who showed conceptual knowledge and those who presented procedural knowledge are consistent in terms of the pedagogical strategies used to respond to Amy. This study provides implications to teacher educators and professional developers. The teachers' emphasis on providing teacher-centered instruction points to the need for teacher educators to give preservice teachers opportunities to practice pedagogical strategies about responding to students' errors. Professional developers should consider content intensive sessions to develop teachers' content knowledge on problem solving, especially on abstracting a mathematical model from real world situations and using the mathematical model to solve problems.

References

Campbell, P. F., Nishio, M., Smith, T. M., Clark, L. M., Conant, D. L., Rust, A. H., ... & Choi, Y. (2014). The relationship between teachers' mathematical content and pedagogical knowledge, teachers' perceptions, and student achievement. *Journal for Research in Mathematics Education*, 45(4), 419-459.

Eraslan, Ali, "A Qualitative Study: Algebra Honor Students' Cognitive Obstacles as They Explore Concepts of Quadratic Functions"(2005). *Electronic Theses, Treatises and Dissertations.* Paper 557.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational studies in mathematics, 21*(6), 521-544.

Huang, R., & Kulm, G. (2012). Prospective middle grade mathematics teachers' knowledge of algebra for teaching. *The Journal of Mathematical Behavior*, *31*(4), 417-430.

Kieran, C. (1992). The learning and teaching of school algebra, in D. A. Grouws(ed.), The Handbook of Research on Mathematics Teaching and Learning. Macmillan, NewYork, pp.390-419.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Son, J. W. (2013). How preservice teachers interpret and respond to student errors: ratio and proportion in similar rectangles. *Educational Studies in Mathematics*, 1-22.

Zaslavsky, O. (1997). Conceptual Obstacles in the Learning of Quadratic Functions. *Focus on Learning Problems in Mathematics, 19*(1), 20-44.

ANALYZING THE DEVELOPMENT OF MKT IN CONTENT COURSES

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To improve teaching and learning in content courses for secondary mathematics teachers, we take the approach of supporting faculty who teach these courses – often mathematics faculty – in developing their own mathematical knowledge for teaching (MKT) at the secondary level. We describe a framework that has informed the design of educative curricula for a set of these courses. This framework integrates theory for knowledge development, empirical work on dimensions of knowledge used in teaching, and findings on observable behaviors in teaching that reveal prospective secondary teachers' knowledge development.

Keywords: Mathematical Knowledge for Teaching, Teacher Education – Preservice

Mathematics teacher preparation programs for all levels aim to provide opportunities for prospective mathematics teachers (prospective secondary teachers) to develop mathematical knowledge for teaching (MKT) (e.g., CBMS, 2012). Yet, mathematicians, who often teach content courses for secondary preparation programs (Murray & Star, 2013), who may have much to offer in the way of mathematical knowledge, often do not have experience teaching in K-12 settings. They may not be positioned to notice nuances in the development of prospective secondary teachers' MKT at the secondary level (Lai, 2016), especially on tasks that situate mathematics in pedagogical contexts. This problem exacerbates existing tensions between intended outcomes of secondary preparation programs and teachers' perceptions that their mathematical preparation is irrelevant to their teaching (e.g., Goulding, Hatch, & Rodd, 2003).

One approach to this problem is developing educative curricula (Davis & Krajcik, 2005). We take this approach, focusing on curriculum supports for mathematics faculty to learn MKT at the secondary level and ways to observe and analyze the development of prospective secondary teachers' MKT, particularly in enactments of approximations of practice (Grossman et al., 2009). In this paper, we discuss a framework that has informed the design of educative curricula for mathematics faculty teaching content courses for secondary level prospective secondary teachers. This framework integrates theory for knowledge development, empirical work on dimensions of knowledge used in teaching, and findings on observable behaviors in teaching that reveal prospective secondary teachers' knowledge development. Such a framework is potentially a critical resource for supporting mathematics faculty in teaching MKT. Our purposes in this paper are to describe this novel framework, and how it can support mathematics faculty in learning MKT and providing substantive feedback to prospective secondary teachers about their MKT, and serve as a resource to refine research on learning and teaching MKT.

Conceptual Foundations

To construct the framework, we integrated Rowland and colleagues' Knowledge Quartet framework for observing MKT and Ader and Carlson's (2018) framework for analyzing and observing teaching with Silverman and Thompson's (2008) developmental framework for MKT. We take MKT to be "practice-based theory of knowledge for teaching" (Ball & Bass, 2003, p. 5) and follow Thompson, Carlson, and Silverman (2007) in taking MKT to include coherent and generative understandings of key ideas that make up the curriculum.

The empirical analysis of mathematics teaching at the elementary and secondary level underlying the Knowledge Quartet suggests four dimensions of observable MKT: Foundation (knowledge of mathematics and its nature), Transformation (presentation of ideas to learners), Connection (sequencing of material for instruction), and Contingency (the ability to respond to unanticipated events) (Rowland, Thwaites, & Jared, 2016).

Ader and Carlson's (2018) framework for analyzing instructional interactions identifies levels of understanding and acting on student thinking in terms of teachers' mental actions and observable behaviors. They characterized teachers' executions of responses to student thinking in terms of Piaget's (1977/2001) notions of decentering and reflective abstraction.

The Knowledge Quartet and Ader and Carlson's framework can be used to articulate Silverman and Thompson's (2008) framework for use in educative curricula for teacher education, as we detail in the next section. Silverman and Thompson used Simon's (2006) idea of key developmental understandings (KDUs) in combination with decentering and reflective abstraction to construct stages of MKT development. Although they described components of development, but they did not elaborate where these components might be observed in actual teaching practice or in approximations of practice, or how one instance of decentering or reflective abstraction may be more sophisticated than another.

Framework for Observing and Analyzing the Development of MKT

The framework we use is based on the work described above and then refined based on the analyses of 15 secondary prospective secondary teachers' responses to an approximation of practice used in a mathematics content course, which has been used at three different institutions in different states in multiple years. The responses analyzed are representative.

Table 1: Framework for Observing and Analyzing the Development of MKT

Developmental component[a]	Knowledge dimension[b]	Mental actions[c]	Level (L), in terms of observable behaviors[d]
Personal KDU	Foundation	Reflective abstraction on personal math. knowledge	L1: Performs procedures within topic L2: Describes procedures accurately L3: Connects isolated features to underlying concepts L4: Connects structure of procedure to underlying concepts
Decentering applied to Activities and Analyzing Potential for Student KDU	Transformation	Reflective abstraction on student thinking	Gives explanations, representations, and examples that: L1: Describe only procedures L2: Describe own way of thinking L3: Attempt to change students' current thinking L4: Enhance students' understanding
	Connection		Poses questions that: L1: Focusing on procedures or echoing key phrases L2: May reveal student thinking, but then gives explanations while not asking students to provide reasoning L3: Attempts to move students' reasoning L4: Supports advancing students' reasoning
	Contingency		Responses to student thinking: L1: Do not act in any visible way upon the thinking L2: Evaluate but do not use the thinking in teaching L3: Directly use the thinking L4: Frames questions or explanations in terms of students' thinking, to make connections and deepen understanding

Note: Levels here depend on the KDU. This is just one possible example of how levels may appear. Constructs for each column were refined through our analysis and based on: (a) Silverman and Thompson, 2008; (b) Rowland, Thwaites, & Jared, 2016; (c) Piaget, 1977/2001; and, (d) Ader and Carlson, 2018

Discussion: Uses of Framework in Teacher Education

The framework presented in this paper is a resource for implementing and writing educative curricula in teacher preparation, as well as for future research in the learning and teaching of MKT. In terms of implementing curricula, we used the framework to inform guides for faculty to use when interpreting and responding to prospective secondary teachers' work. Differentiating between the dimensions of MKT used can support faculty in noticing different kinds of knowledge in teachers and providing more explicit feedback. Foundation, Connection, and Transformation emphasize potential differences among a prospective secondary teacher's display of personal knowledge, providing explanations to students, and posing questions that elicit student reasoning. Although faculty may not traditionally make these distinctions in providing feedback in a mathematics course, these distinctions are also ones that may be familiar to faculty and may well be informative educative for their own teaching practice (e.g., Pascoe & Stockero, 2017).

It is worth noting that our analysis indicated that the dimensions of knowledge were independent in the context of prospective secondary teachers' enactments of approximations of practice, a context for which the dimensions had not been previously analyzed. This suggests that development of MKT may well proceed along these dimensions in different ways. One prospective secondary teacher in our dataset explained the connection between a definition and procedure as a rationale for a task they would assign to their students (Foundation, L4), but only posed questions that focused on echoing key phrases (Connection, L1), proposed only explanations of procedures to the students (Transformation, L1), and did not acknowledge any of the sample student thinking provided by approximation of practice (Contingency, L1). Another prospective secondary teacher began with using the provided sample student work to make a specific mathematical point about a definition (Contingency, L4) then did not provide any subsequent examples or explanations to connection of procedure and definition (Transformation, L1).

In our own work writing approximations of practice for use in content courses and hearing initial feedback from mathematics faculty, articulating how observables may correspond to knowledge dimensions has prompted us to think more clearly about the opportunities presented by approximations of practice. For instance, in an early draft of an approximation of practice, we asked prospective secondary teachers to respond to student thinking, but we did not give a clear mathematical goal for the teaching situation. This left unspecified the Foundation knowledge we were aiming to elicit, which impacted the Transformation and Connection knowledge visible in prospective secondary teachers' responses.

Finally, the framework supports validating and refining the articulation of the development of MKT. We view this framework as a set of testable hypotheses grounded in known results. At the same time, we have drawn from work in emergent stages. We see great opportunity in using this framework, which unifies work from different groups, to push researchers' understanding of the development of MKT. Future work made visible by this framework includes investigating how well these codes hold up to responses to approximations of practice across content courses; whether the interpretation of levels for the purpose of providing feedback to teachers improved teaching and learning outcomes; and the independence or dependence of levels and dimensions. As Morris and Hiebert (2009) argued, a professional knowledge base – such as that for teaching mathematics courses for prospective secondary teachers – can only advance with shared goals and artifacts that articulate those goals for the professional community in ways that can be observed. We propose a framework that articulates the goal of developing teachers' MKT in terms of actions observable in approximations of practice used in content courses. Our framework simultaneously leverages theory for the development of MKT, empirical analyses of teaching, and empirical analyses of approximations of practice used in content courses.

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References

Ader, S. B., & Carlson, M. P. (2018). Observable manifestations of a teacher's actions to understand and act on student thinking. Paper presented at the annual meeting of the Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education. San Diego, CA, 22-24 February 2018.

Ball, D.L., & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In B. Davis & E. Simmt (Eds.), *Proceedings of the 2002 Annual Meeting of the Canadian Mathematics Education Study Group*, (pp. 3-14). Edmonton, AB: CMESG/GCEDM.

Conference Board of the Mathematical Sciences (2012). *The mathematical education of teachers II.* Providence RI and Washington DC: American Mathematical Society and Mathematical Association of America.

Davis, E. A., & Krajcik, J. S. (2005). Designing educative curriculum materials to promote teacher learning. *Educational Researcher, 34*(3), 3-14.

Goulding, M., Hatch, G., & Rodd, M. (2003). Undergraduate mathematics experience: its significance in secondary mathematics teacher preparation. *Journal of Mathematics Teacher Education*, *6*(4), 361-393.

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. (2009). Teaching practice: A cross-professional perspective. *Teachers College Record*, *111*(9), 2055–2100.

Lai, Y. (2016). Lacking confidence and resources despite having value: Mathematicians teaching prospective secondary teachers. *Proceedings of the 19th Annual Conference on Research in Undergraduate Mathematics Education*. Pittsburgh, PA.

Morris, A. K., & Hiebert, J. (2009). Introduction: Building knowledge bases and improving systems of practice. *The Elementary School Journal, 109*(5), 429-441.

Murray, E., & Star, J. R. (2013). What do secondary preservice mathematics teachers need to know? Content courses connecting secondary and tertiary mathematics. *Notices of the American Mathematical Society, 60*(10), 1297-1300.

Pascoe, A. E., & Stockero, J. (2017). Learning to notice and use student thinking in undergraduate mathematics courses. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 812–820). San Diego, CA.

Piaget, J. (1977/2001). *Studies in reflecting abstraction* (R. L. Campbell, Trans.). Sussex, UK: Psychology Press.

Rowland, T., Thwaites, A., & Jared, L. (2016). Analysing secondary mathematics teaching with the Knowledge Quartet. Paper presented at the 13th International Congress on Mathematical Education. Hamburg, Germany, 24-31 July 1, 2016.

Silverman, J., & Thompson, P. W. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education*, *11*, 499–511.

Simon, M. (2006). Key developmental understandings in mathematics: A direction for investigating and establishing learning goals. *Mathematical Thinking & Learning, 8*(4), 359–371.

Thompson, P. W., Carlson, M., & Silverman, J. (2007). The design of tasks in support of teachers' development of coherent mathematical meanings. *Journal of Mathematics Teacher Education*, *10* (4–6), 415–432.

FRACTALIZATION AS A METAPHOR FOR MATHEMATICAL KNOWLEDGE FOR TEACHING TEACHERS: SYNTHESIZING RESEARCH AND EXPLORING CONSEQUENCES

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We aim to develop a theoretical foundation for the knowledge needed by mathematics teacher educators to support prospective teachers in developing the knowledge needed for teaching. We have approached this goal by exploring existing frameworks that not only generalize conceptualizations of teacher knowledge, but also position teacher knowledge as a subdomain of the knowledge needed for teaching teachers. We envision this generalization to teacher educator knowledge as a "fractalization" of teacher knowledge. Applying this concept, we propose our own fractalization model for the knowledge of mathematics teacher educators and discuss some resulting consequences.

Keywords: Mathematical Knowledge for Teaching, Teacher Knowledge, Post-Secondary Education, Teacher Education-Preservice

Mathematics teacher educators (MTEs) have long considered the knowledge needed for teaching mathematics to be distinct from the knowledge needed for doing mathematics. Much work has been done in explicating the domains of Mathematical Knowledge for Teaching (MKT, e.g. Ball, Thames, & Phelps, 2008), but researchers are only beginning to explore the knowledge needed by MTEs to help prospective teachers (PTs) develop the knowledge needed for teaching. We aim to develop a theoretical foundation for the mathematical knowledge for teaching teachers (MKTT) by analyzing, synthesizing, and extending existing literature on MTE knowledge. In this paper, we will discuss several proposed frameworks that conceptualize teacher educator knowledge as a generalization of teacher knowledge. That is, these frameworks not only position teacher knowledge as a subdomain of teacher educator knowledge but also echo the structure of teacher knowledge. Our analysis of these frameworks led us to use "fractalization" as a metaphor for this generalization, as the visualizations of many of these frameworks resemble part of a fractal. We propose a model of MTE knowledge that applies this fractal metaphor to the MKT framework (Ball et al., 2008), begin to conceptualize the subsequent domains of MKTT knowledge, and discuss some consequences of framing MKTT as fractalization of MKT.

Frameworks of MTE Knowledge that Use Fractalization

In synthesizing the extant frameworks for understanding teacher educator knowledge, we noticed that several frameworks generalized teacher knowledge in a way that we envisioned as following a "fractalization" model (Welder, Prasad, Superfine, & Olanoff, 2017). Below we will explore four examples of such frameworks and discuss how they follow this model of fractal generalization.

In his 1985 presidential address at the meeting of the American Educational Research Association, Lee Shulman identified three aspects of teacher knowledge: *content knowledge*, *curricular knowledge*, and *pedagogical content knowledge*, which he called the "missing paradigm" in the study of teaching (Shulman, 1986). Chauvot (2009) has expanded Shulman's (1986) framework to develop a knowledge map for MTEs. In this model, she identifies the knowledge base for "Math Teacher Educator-Researchers" as being comprised of *subject matter content knowledge*, *pedagogical content knowledge*, and *curricular knowledge*. Her model fractalizes Shulman's work by placing all three of his domains of knowledge for teaching within the *subject matter content knowledge* of MTEs. Her map also integrates Grossman's (1990) *knowledge of context* to consider how the context in which MTEs work (i.e., university-based teacher preparation, school-based professional development, etc.) affects what they need to know.

Similarly, Zaslavsky and Leikin (2004) have fractalized Jaworski's (1992) model for the practice of teaching mathematics to better understand the practice of teaching teachers of mathematics. Jaworski's model, known as the "teaching triad of mathematics teachers," highlights the interactions among three important components of teaching: *challenging content for students* (mathematics), *the management of students' learning*, and *sensitivity to students*. Zaslavsky and Leikin created analogous terms for aspects important to the teaching of teachers: *challenging content for mathematics teachers*, *the management of mathematics teachers' learning* and *sensitivity to mathematics teachers*. In developing their "teaching triad of MTEs," Zaslavsky and Leikin consider Jaworski's "teaching triad for math teachers" to be contained within the component of *challenging content for mathematics teachers*.

Perks and Prestage (2008) propose a "teacher-educator knowledge tetrahedron," as an expansion of their "teacher knowledge tetrahedron." Both frameworks highlight the interactions among four aspects of classroom practice: *teacher knowledge*, *learner knowledge*, *professional traditions*, and *practical wisdom*. However, in the teacher-educator knowledge tetrahedron, the learners in question are prospective or practicing teachers; thus, in this meta-framework, the teacher knowledge tetrahedron is entirely contained within the learner knowledge portion of the teacher-educator knowledge tetrahedron.

Another example of fractalization can be noted in Ball's (2012) proposal to expand a model she and her colleagues offered for the instructional dynamics that affect student learning (Cohen, Raudenbush, & Ball, 2003). This expanded framework models the instructional dynamics that affect the learning of teachers. Both models highlight the ways in which *teachers* and *students* interact with each other and how they interact with the *content* of interest. In the case of teacher education, the *teachers* are teacher educators, the *students* are teachers, and the *content* of interest now encompasses the entirety of the original model.

Fractalization of the MKT Framework

Applying the idea of fractalization, we propose a model of MKTT that extends the MKT framework developed by Ball and her colleagues (Ball et al., 2008) to a set of analogous domains of teacher educator knowledge (see Figure 1). We are considering all aspects of MKT as being comprised within the overall *subject matter knowledge* needed by MTEs. On the other hand, MTEs' *pedagogical content knowledge* encompasses the mathematical knowledge needed for working with prospective and practicing teachers (i.e., adult learners, rather than K-12 students). We envision this knowledge to encompass the ways in which PTs interact with the mathematical content they will be teaching and effective methods for facilitating their relearning of mathematics in a way that will support their development of MKT (Castro Superfine, Welder, Prasad, Olanoff, & Eubanks-Turner, in press). While we have a general idea of what this conception of MKTT requires of MTEs, characterizing the specific aspects of the subdomains of MKTT requires further exploration.

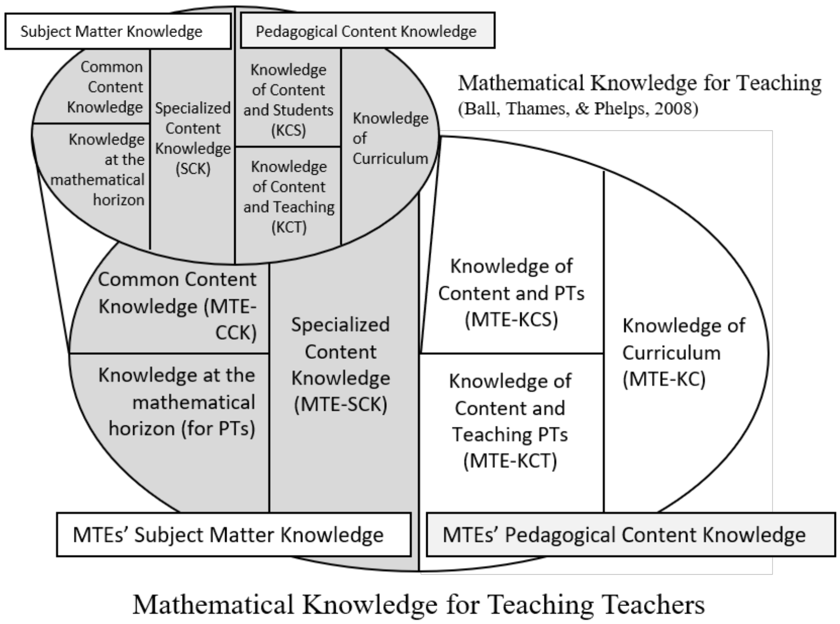

Figure 1. Our "Fractalized" Model for Mathematical Knowledge for Teaching Teachers

Discussion

Conceptualizing MKTT as a fractalization of MKT provides a basis for a framework for the knowledge needed for teaching teachers. We believe further exploring fractalization models will allow researchers an entry point into theorizing about the unique aspects of MKTT that exist separately from MKT. It is important to note that one consequence of this conception is that MKT becomes a subset of MKTT. This implies that MTEs' knowledge must include all aspects of MKT at the level of mathematics teaching for which they are preparing PTs. Given the varied academic and experiential backgrounds of MTEs (including research mathematicians and mathematics educators who have not had experience teaching in K-12 contexts), this requirement may be logistically impossible. On the other hand, if facilitating PTs' development of MKT is considered a goal of teacher preparation, it could be argued that the MTEs teaching courses for PTs would need to possess such knowledge.

Another aspect of MKTT that needs to be further explored within this fractalization model is the role that MTEs' knowledge of research plays in their practice of teaching teachers. There is no clear analogue between a domain of MKT and MTEs' knowledge of research, as most frameworks for teacher knowledge do not explicate knowledge of research. Yet, research is an important component of the work of many MTEs and is thus part of their knowledge base. Of the frameworks we have discussed, Chavot's (2009) is the only one that makes explicit reference to knowledge of research, situating it uncomfortably in the middle of her framework for MKTT.

In this report we have explored a subset of frameworks for MTE knowledge that utilize a fractalization model of teacher knowledge. However, different frameworks of teacher knowledge exist, and may also lend themselves to conceptualizations of MKTT, perhaps through a form of fractalization. Additionally, there are frameworks for teacher-educator knowledge that are not conceptualized as fractalizations of teacher knowledge. Such frameworks may have different consequences for the development of a knowledge base for teacher educators, and there is further work to be done in exploring them.

References

Ball, D. L. (2012). Afterword: Using and designing resources for practice. In G. Gueudet, B. Pepin & L. Trouche (Eds.), From text to 'lived' resources: Mathematics curriculum materials and teacher development. Dordrecht, Netherland: Springer.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Castro Superfine, A., Welder, R. M., Prasad, P. V., Olanoff, D., & Eubanks-Turner, C. (in press). Exploring mathematical knowledge for teaching teachers: Supporting prospective teachers' relearning of mathematics. *In A. Appova, R. M. Welder, and Z. Feldman, (Eds.) Special Issue: Supporting Mathematics Teacher Educators' Knowledge and Practices for Teaching Content to Prospective (Grades K-8) Teachers, pp. x-xx. The Mathematics Enthusiast.*

Chavout, J. (2009). Grounding practice in scholarship, grounding scholarship in practice: Knowledge of a mathematics teacher educator-researcher. *Teaching and Teacher Education, 25*, 357-370.

Cohen, D., Raudenbush, S., & Ball, D. (2003). Resources, instruction, and research. Educational Evaluation and Policy Analysis, 25(2), 1–24.

Jaworski, B. (1992). Mathematics teaching: What is it? For the Learning of Mathematics, 12(1), 8-14.

Perks, P. & Prestage, S. (2008). Tools for learning about teaching and learning. In B. Jaworski & T. Wood (Eds.), The international handbook of mathematics teacher education, vol. 4: The mathematics teacher educator as a developing professional (pp. 31–56). Rotterdam: Sense.

Shulman, L. S. (1986). Those who understand: A conception of teacher knowledge. *American Educator, 10*(1), 9-15, 43-44.

Welder, R. M., Prasad, P. V., Castro Superfine, A. & Olanoff, D. (2017). Developing a framework for mathematical knowledge for teaching teachers. In E. Galindo & J. Newton (Eds.) *Proceedings of the 39th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* (p. 637). Indianapolis, IN.

Zaslavsky, O., & Leikin, R. (2004). Professional development of mathematics teacher educators: Growth through practice. *Journal of Mathematics Teacher Education, 7*, 5-32.

EXAMINING TEACHER KNOWLEDGE RESOURCES FOR PROPORTIONAL REASONING VISUALLY

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In this paper, we rely on Epistemic Network Analysis to examine the connections between knowledge resources to understand the coherence of teachers' knowledge resources used in solving proportional reasoning tasks. Our findings show that teachers who scored lower on the LMT tended to use more knowledge resources and to coordinate knowledge resources in less systematic ways than those teachers who scored higher on the LMT. We also noted a shift from strategies suggest additive reasoning to strategies that focus on the multiplicative nature of proportions. Implications for teacher development and for further research are included.

Keywords: Cognition, Teacher Knowledge, Mathematical Knowledge for Teaching

Introduction

Research suggests that, like students, teachers struggle with proportional reasoning (e.g., Harel & Behr, 1995). However, the research on teachers' knowledge is still somewhat limited. We have ample evidence that teachers have trouble solving some kinds of tasks (e.g., Harel & Behr, 1995; Pitta-Pantazi & Christou, 2011) and some studies that provide in-depth descriptions of how one teacher or a small number of teachers approached particular tasks (e.g., Riley, 2010). We sought to better understand how teachers use their existing knowledge resources across a set of tasks so we could determine which tasks may need to be further developed. To better understand how teachers reason about a variety of proportional situations we asked:

1. What proportional reasoning knowledge resources do these middle school teachers use to solve a diverse set of proportional tasks?
2. Are there differences in the resources used by teachers who exhibit higher or lower levels of knowledge on a multiple-choice assessment?

Theoretical Framework

This research is grounded in the Knowledge in Pieces perspective (KiP; diSessa, 2006). KiP asserts that people develop understandings as fine-grained knowledge resources that can be refined and connected to other resources to create a more coherent set of understandings. Having well-connected knowledge resources allows access to more ideas *in situ*. Thus, expertise could be seen as developing more knowledge resources and more connections between and among those knowledge resources. This is consistent with research on expertise (Bédard & Chi, 1992).

Methods

Data were collected from 32 in-service middle school teachers (grades 6-8) from four states. The teachers reported having between one and 26 years of teaching experience. Twenty-four were female and eight were male. Five participants identified as Black, one as Biracial, and 26 as White. The participants taught in a variety of schools including urban, suburban, and rural schools as well as public, private, and charter schools.

We collected three kinds of data from each participant. The first was the *Learning for Mathematics Teaching* (LMT, 2007) assessment for Proportional Reasoning. For each teacher, a single score was calculated that reports the logit score for the teacher in standard deviations from the mean. Teachers' scores ranged from -0.563 to 2.657 on the LMT. The second data source was a one-hour think aloud interview protocol that was mailed to each teacher along with a LiveScribe pen that captured their utterances and any inscriptions at the same time. The third data source was a video-recorded one-hour clinical interview with each teacher. The tasks for both interviews asked the teachers either to solve a proportional situation or to make sense of hypothetical students' or colleagues' mathematical thinking.

Both the clinical interview and think-aloud protocol were transcribed verbatim. We coded each using a code set that was developed using grounded theory techniques that allowed us to build from the literature as well as from the trends emerging in the data (Charmaz, 2014). Each utterance was coded using a binary approach to determine whether each code was present in the data. In all, there were 18 codes specifically focused on proportional reasoning concepts for which each utterance was coded (Weiland, Orrill, Brown, Nagar, & Burke, 2016). Every interview was coded by at least two members of the research team.

To understand the knowledge resources used, we relied on Epistemic Network Analysis (ENA; Shaffer, Collier, & Ruis, 2016) to understand in what ways the knowledge resources were connected. ENA is a computational modeling technique that focuses on the connections between elements. ENA was developed to model connections between elements (knowledge resources) and to show the strength and association among elements in a network (Shaffer et al, 2016). Using ENA, an equiload graph was generated to examine teachers' responses based on the group means, where groups were defined by LMT performance. The equiload graph shows each knowledge resource as a node. The lines connecting the nodes indicate that two resources were used in the same task multiple times (a proxy for connectedness of resources, e.g., Chesler, et al., 2013). These equiload graphs were developed using ENA Webkit software (http://www.epistemicnetwork.org/).

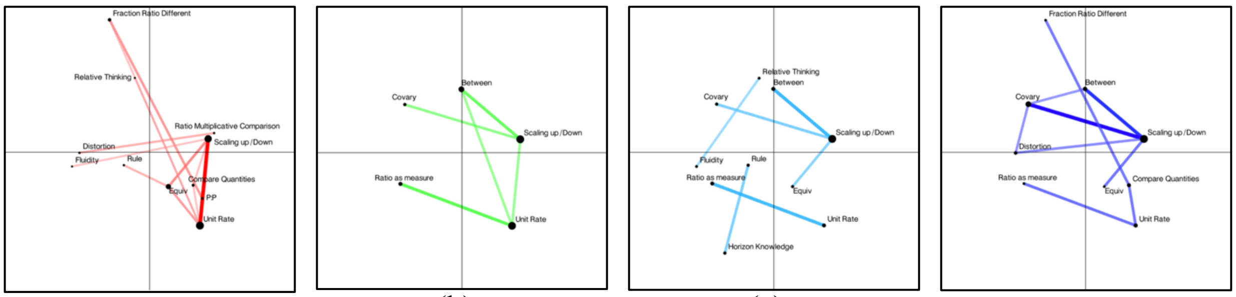
Findings

To understand the knowledge resources used by the teachers in this study, we organized teachers into groups based on LMT scores: (a) teachers scoring -1 to 0 (*n*=3; referred to as Group A), (b) teachers scoring between 0 and 1 (*n*=11; referred to as Group B), (c) teachers scoring from 1 to 2 (*n*= 9; referred to as Group C), and (d) teachers scoring between 1 and 2 (*n*=9; referred to as Group D). Figure 1 presents the mean equiload graphs for each of these groups. In the equiload graphs, it is clear that some co-occurrences of knowledge resources were much more frequent than others. For example, in Figure 1a, the co-occurrence of Scaling Up/Down and Unit Rate is much bolder than the other co-occurrences, indicating that it was a more common pairing than some of the others. For Figure 1a, the co-occurrence of Unit Rate and Scaling Up/Down occurred in a total of 14 tasks across the three participants in this group. Given that there are 35 tasks in all, this means that the co-occurrence of Scaling Up/Down and Unit Rate occurred in 12% of the tasks for this subgroup. Similarly, in Figure 4d, the heavier line between Scaling Up/Down and Covary indicates relatively more co-occurrences between these resources. In fact, across the 9 participants in Group D, there were 33 co-occurrences of this pairing (10%), with many of those pairings occurring not just within the same task, but also within the same utterance.

Expertise as Knowing More and Having More Connections

Consistent with the KiP perspective, the participants in this study exhibited a number of different connections between knowledge resources. Further, within each group we see several resources and different kinds of connections between those knowledge resources. If we use the

LMT scores as a measure of teachers' knowledge, then the weakest teachers in this study would be those scoring between -1 and 0 on the LMT (Group A). Group A participants relied on the connection between Scaling Up/Down and Unit Rate, which are additive approaches. The reliance on this pairing was lessened in Group B (see Figure 1b), where Ratio as Measure was paired with Unit Rate and Scaling Up/Down was paired with Between Measure Space reasoning. We also note that Group B has fewer connected resources showing in the mean equiload graph, indicating that their knowledge resources are potentially better organized.



(a) (b) (c) (d)

Figure 1. Mean equiloads for four teacher groups (A, B, C, and D) based on LMT scores.

Overall, the members of Groups B, C, and D relied on Covary, a structural aspect of proportions, co-occurring with other resources. Similarly, these groups paired Ratio as Measure with Unit Rate, suggesting participants were not only invoking 'how much per x' reasoning, but also have solidified that reasoning within the context of the situation by identifying the abstractable quantity that results from the proportion.

Teachers who performed the strongest on the LMT (Group D) and those who performed the weakest on the LMT (Group A) both had connections between Fractions and Ratios are Different and Comparing Quantities. Similarly, only Groups A and D had connections between Distortion and another code. This is important because it shows a connection between geometric ideas of similarity and dilation to numeric scaling (Group D) and to an understanding there is a fixed multiplicative relationship between the two quantities in a ratio (Group A).

The knowledge resources exhibited by participants in this study changed between the groups of teachers based on their performance on the assessment. In the mean equiload graph for Group A, we see many more knowledge resources used together, but fewer clear patterns in the thick lines. In Group B, we see fewer resources than in Group A and more consistent connections – suggesting more consistency within the group in their patterns of use of knowledge resources. In Group C, additional knowledge resources emerge, but there are still several connections that are consistently prevalent. Then, in Group D, the participants performing the highest on the LMT, we see a clear growth from Group C to continue using several of the connections, but also bringing in additional connections. The relative sophistication between groups is also notable with Group A relying on unit rate and scaling, which are two ideas that can be used with additive or multiplicative reasoning. In Group B, we see more explicitly multiplicative ideas (e.g., Covariation and Between Measure Space Reasoning). In Group C, the participants brought in fluidity in symbolic manipulation as well as continuing to attend to clearly proportional ideas including Ratio as Measure and Covariance. Then, in Group D, geometric ideas were included in the participants' discussions of Covariance and Scaling, suggesting that their knowledge is organized in such a way that they have easy access to similar ideas from multiple strands of mathematics across a variety of proportional reasoning tasks.

Implications and Conclusions

In this paper, we set out to understand what knowledge resources teachers used to solve a set of proportional reasoning tasks and whether there were differences in the patterns of knowledge resources used between teachers exhibiting different levels of performance on an assessment of proportional reasoning. As shown in Figure 1, each group of teachers relied upon a number of reasonable knowledge resources to solve the tasks in both interviews. Figure 1 shows there are clear differences between the groups of teachers.

This work is still in an exploratory phase. However, it can already contribute in meaningful ways to the discourse about supporting teacher understanding of proportional reasoning in mathematics education. From a professional development or teacher education perspective, it is clear that there are some knowledge resources that teachers have and can use in pairings with other knowledge resources. This provides guidance on ways to shape PD to be most effective.

Acknowledgments

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References

Bédard, J., & Chi, M. T. H. (1992). Expertise. *Current Directions in Psychological Science*, *1*(4), 135-139.

Charmaz, K. (2014). Constructing grounded theory (2nd ed.). Thousand Oaks, CA: Sage Publications Ltd.

Chesler, N. C., Arastoopour, G., D'Angelo, C. M., Bagley, E. A., & Shaffer, D. W. (2013). Design of a professional practice simulator for educating and motivating first-year engineering students. *Advances in Engineering Education, 3*(3), 1-29.

diSessa, A. A. (2006). A history of conceptual change research: Threads and fault lines. In R. K. Sawyer (Ed.). *The Cambridge handbook of the learning sciences* (pp. 265-282). New York: Cambridge University Press.

Harel, G., & Behr, M. (1995). Teachers' solutions for multiplicative problems. *Hiroshima Journal of Mathematics Education*, *3*, 31–51.

Learning Mathematics for Teaching (2007). *Survey of teachers of mathematics: Form LMT PR-2007.* Ann Arbor, MI: University of Michigan.

Pitta-Pantazi, D., & Christou, C. (2011). The structure of prospective kindergarten teachers' proportional reasoning. *Journal of Mathematics Teacher Education, 142), 149-162*, doi 10.1007/s10857-011-9175-y

Riley, K. R. (2010). Teachers' understanding of proportional reasoning. In P. Brosnan, D. B. Erchick, & L. Flevares (Eds.), *Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1055-1061). Columbus, OH: The Ohio State University.

Shaffer, D. W., Collier, W., & Ruis, A. R. (2016). A tutorial on epistemic network analysis: Analyzing the structure of connections in cognitive, social, and interaction data. *Journal of Learning Analytics, 3*(3), 9-45. http://dx.doi.org/10.18608/jla.2016.33.3

Weiland, T., Orrill, C. H., Brown, R. E., Nagar, G. G., & Burke, J. P. (2016, April). *Formulating a robust understanding of proportional reasoning for teaching*. Roundtable presented at the Annual Meeting of the American Educational Research Association, Washington, DC.

IN-SERVICE TEACHERS' ABILITIES TO MAKE SENSE OF FIXED NUMBER OF VARIABLE SIZED PARTS TASKS

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In this study, we interviewed 32 practicing middle grades teachers to determine whether they were able to invoke reasoning about a fixed number of variable-sized parts in a proportional situation. To this end, we presented them with a sample student response to an item that relied on such reasoning to determine whether teachers understood the approach and whether they thought it was appropriate. Findings suggest that teachers who did not invoke this reasoning were unable to make sense of the student's reasoning, thus limiting their ability to make sense of students' work.

Keywords: Teacher Knowledge, Middle Grades Mathematics, Mathematical Knowledge for Teaching

Introduction

Ratios and proportional relationships are important topics in school mathematics (e.g., Lamon 2007). Reasoning with ratios and proportions serves as a foundation for more advanced topics (e.g., Lobato & Ellis, 2010). Additionally, it has been well document that these topics can be difficult for both teachers and students (e.g., Lamon, 2007).

While there is a considerable body of research focused on students' understandings of ratios and proportions, research on teachers is more limited. The literature on teachers highlights both how future teachers and in-service teachers solve proportional situations (e.g., Beckmann & Kulow, 2018; Harel & Behr, 1995; Riley, 2010) and the ways in which they make sense of students' thinking in proportional situations (e.g., Jacobson, Lobato, & Orrill, 2017).

This study contributes to both of these categories of research on teachers' understanding of proportions. We consider one aspect of proportional reasoning (described below) to determine whether teachers can make sense of it and whether they believe a student's work is useful.

Theoretical Framework

Variable Parts Perspective

Beckmann and Izsák (2014; 2015) identified two distinct but complementary perspectives on how quantities vary together in proportional relationships. One perspective on proportional relationships is the *multiple batches* perspective. This approach relies on coordinating the two quantities in a ratio and iterating them. It relies on composed unit reasoning, which has been written about elsewhere (e.g., Lobato & Ellis, 2010).

The second perspective on proportional relationships is the *fixed number of variable parts* perspective ("Variable Parts"). In this perspective a proportion is conceived of as a ratio made of a fixed number of equal sized parts in which the size of the part can vary. This is different than the multiple batches perspective where the number of groups vary and the size of the parts are fixed. For example, in a relationship of 2 parts of oil to 5 parts of vinegar in a salad dressing recipe, those parts can be conceived of as teaspoons, cups, half-gallons, or any other unit. The flavor of the batch of salad dressing will remain unchanged as those as the ratio of the units is held constant.

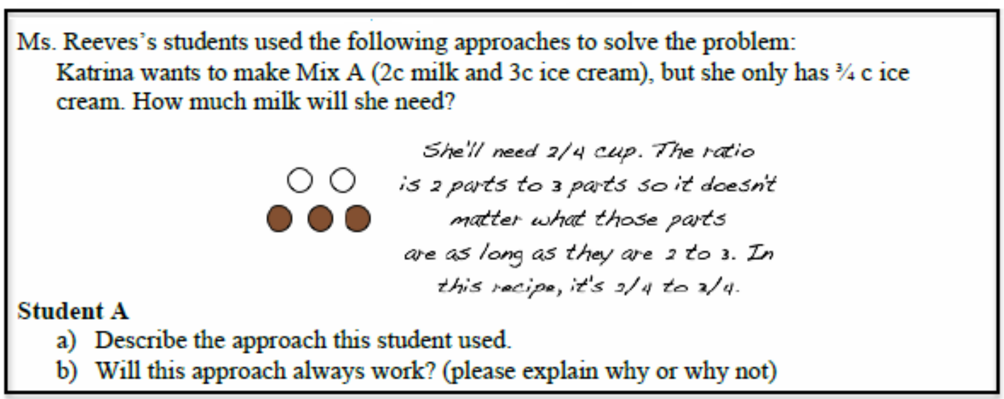
The existing body of variable parts research has mainly been conducted with preservice teachers enrolled in a course in which they are explicitly taught to reason using both perspectives (Beckmann & Izsák, 2014; Beckmann, Izsák, & Ölmez, 2015; Beckmann & Kulow, 2018; Ölmez, 2016). Most of the research has focused on mixture tasks. In this paper, we extend the existing research by considering the knowledge resources used by in-service teachers who have not received instruction in Variable Parts reasoning. Specifically, we considered what knowledge resources teachers used to make sense of a student's variable parts solution to a mixture task.

Knowledge in Pieces

Our study relies on knowledge in pieces (KiP; diSessa, 1988). KiP posits that understandings are developed as fine-grained knowledge resources. Through perturbations, those knowledge resources can be refined or added to. They can also be connected in myriad ways. As a person develops deeper understandings of a domain, they would be expected to develop a more robust set of knowledge resources, but even more, they would be expected to have more connections between and among those resources. We argue that because teaching requires considerable engagement with novel ideas, teachers need a robustly connected set of knowledge resources to support the widest variety of students.

Methods

For this study, we asked 32 in-service teachers to complete a think-aloud protocol by mail. Each teacher was provided with a LiveScribe pen to use so their utterances and inscriptions were recorded and coordinated. The think-aloud protocol included a number of questions about seven contexts, one was the milkshakes context, which included the task used in this study (Figure 1).


Ms. Reeves's students used the following approaches to solve the problem:
Katrina wants to make Mix A (2c milk and 3c ice cream), but she only has ¾ c ice cream. How much milk will she need?

She'll need 2/4 cup. The ratio is 2 parts to 3 parts so it doesn't matter what those parts are as long as they are 2 to 3. In this recipe, it's 2/4 to 3/4.

Student A
a) Describe the approach this student used.
b) Will this approach always work? (please explain why or why not)

Figure 1. Milkshake Task used for this study

Data were coded from transcripts. Each interview was analyzed by at least two team members who coded independently, then met to identify and solve any coding discrepancies. In cases where agreement was not readily possible, the utterance was brought to the full research team, a decision was made and documented in the definition for the code to ensure consistency across all coding. The codes were developed from both the literature and from emergent coding of the data (Charmaz, 2014). Each code identified a knowledge resource used to solve the task. The coding set included 18 knowledge resource codes related to proportional reasoning, including one identifying whether Variable Parts reasoning was present.

We separated participants into a Variable Parts group and a group that did not use Variable Parts. We then returned to the original transcripts and, when necessary, recordings from the interviews. While analyzing the transcripts, we used memoing (Corbin & Strauss, 2014) as well as the original coding to identify knowledge resources used to solve a task that suggested variable parts reasoning.

Results

In this section, we briefly describe findings from the Milkshake task shown in Figure 1. Our goal was to understand whether teachers made sense of a student's solution that relied on Variable Parts reasoning and whether they believe that reasoning was useful.

Variable Parts Group Approaches

For this task, we found that 13 teachers (41%) used Variable Parts reasoning while 19 teachers (59%) did not. Teachers who relied on Variable Parts reasoning made sense of the students' reasoning by saying things like:

> This was very clever. The student recognized that the ratio of milk to ice cream is 2 to 3, so, yeah. It doesn't matter. As the student said, it does not matter what those parts are, as long as they are 2 to 3. So that's a wonderful… a wonderful approach. Doesn't… doesn't have to be concerned that it's three quarters of a cup. It's just 3/4ths… 2/4ths. Excellent. (Charlotte, all names are pseudonyms)

Similarly, these teachers were able to make sense of the student's drawing saying things like, "So, they said each, instead of each part being a full cup they considered each part to be ¼ of a cup. So instead of having 2 full cups to 3 full cups they have 2, ¼ cups to 3, ¼ cups. I like it." (Heather) Interestingly, though, some of the teachers who understood the Variable Parts perspective and valued it as an approach, were unsure how they might teach it. For example, Georgia noted, "I think this approach would always work, I really like this approach. I'm not sure how to teach this type of understanding. But if they had that type of abstract understanding this would be a very good approach or way to show the work."

Non-Variable Parts Group Approaches

The 19 participants who did not rely on Variable Parts reasoning struggled to make sense of the student's reasoning. For example, Tonya explained:

> I'm not sure of the approach the student used. I see that the ratio is 2 parts milk to 3 parts ice cream. I'm not understanding why they think they will need a 2/4th cup if I need 3/4ths of milk…I'm not sure of what the approach is.

And, 12 of the 19 teachers who did not rely on Variable Parts were either confused by the drawing or created an alternate drawing to make sense of the task. Christine said:

> But I don't understand how…I don't understand her writing, how her written description correlates to her drawing…this is kind of baffling to me because I…I just don't understand…I don't understand her writing.

Not making sense of the work resulted in nine participants determining that the student's approach was not useful. They stated that this approach was for a particular type of student, predicted that the student would struggle with this this type of reasoning in the future, or in some cases the teachers doubted the students' understanding of proportional relationships. For example, Bridgette uttered:

> If they were to do this in the test for me I would ask for a little more explanation as to how they got the denominator four. Because from that picture I don't know if I would easily conclude how they got the number four.

In short, teachers who did not invoke Variable Parts reasoning did not understand this approach and dismissed it as being limited or not useful for students.

Scholarly Significance of the Study

The most notable implications for this work surround the teachers' abilities to make sense of student reasoning. As a community, teacher educators know that there is tremendous value in teachers listening to students (e.g., Darling-Hammond, 2006). However, these participants appeared unable to make sense of the student work unless they invoked the Variable Parts knowledge resource. We do not have adequate data to determine whether the teachers understood variable parts reasoning or whether this task simply did not invoke that reasoning for them. But, we do see compelling evidence that not accessing Variable Parts reasoning correlated to struggling to make sense of student thinking and working the task a way other than the student's way. This suggests to us that having access to and being able to invoke variable parts reasoning may be an important aspect of a teacher's robust understanding of proportional reasoning and, thus, should be given explicit attention in teacher preparation and professional development.

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References

Beckmann, S., & Izsák A. (2015). Two perspectives on proportional relationships: Extending complementary origins of multiplication in terms of quantities. *Journal for Research in Mathematics Education, 46*(1), 17-38.

Beckmann, S., Izsák, A., & Ölmez, I. B. (2015). From multiplication to proportional relationships. In X. Sun, B. Kaur, & J. Novotna (Eds.), *Proceedings of the twenty-third ICMI study: Primary mathematics study on whole numbers* (pp. 518–525). Macau, China: University of Macao.

Beckmann, S., & Kulow, T. K. (2018). How future teachers reasoned with variable parts and strip diagrams to develop equations for proprtional relationships and lines. In L. Yeping, J. W. Lewis, & J. Madden (Eds.), *Mathematics Matters in Education: Essays in honor of Roger E. Howe* (pp. 117–148). Cham.: Springer.

diSessa, A. A. (1988). Knowledge in pieces. In G. Forman & P. Pufall (Eds.), *Constructivism in the computer age* (pp. 49-70). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Harel, G., & Behr, M. (1995). Teachers' solutions for multiplicative problems. *Hiroshima Journal of Mathematics Education*, 3, 31-51.

Jacobson, E., Lobato, J., & Orrill, C. H. (2017 online first). Middle school teachers' use of mathematics to make sense of student solutions to proportional reasoning problems. *International Journal of Science and Mathematics Education*. doi: 10.1007/s10763-017-9845-z

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 1, pp. 629–667). Charlotte, NC: Information Age Publishing

Lobato, J., & Ellis, A. B. (2010). *Developing essential understanding of ratios, proportions, and proportional reasoning for teaching mathematics: Grades 6–8*. Reston, VA: National Council of Teachers of Mathematics

Ölmez, İ. B. (2016). Two distinct perspectives on ratios: Additive and multiplicative relationships between quantities. *Elementary Education Online*, *15*(1), 186–203.

Riley, K. J. (2010). Teachers' understanding of proportional reasoning. In P. Brosnan, D. B. Erchick, & L. Flevares (Eds.), *Proceedings of the 32nd annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (pp. 1055-1061). Columbus, OH: The Ohio State University.

HIGH SCHOOL TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING EXPONENTIAL FUNCTIONS: MKT AS DECISION-MAKING

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We propose expanding traditional conceptions of teachers' MKT to include decision making. We explore this idea using the decisions two teachers make while implementing an exponential task to create an equation from context using a table. This framing better allows us to understand the mathematical reasoning teachers know to be relevant for students to complete a mathematical process and the actual actions they choose to engender this reasoning.

Keywords: Mathematical Knowledge for Teaching, Research Methods

The field widely accepts that teachers' mathematical knowledge for teaching (MKT) is strongly related to their effectiveness and has long-endeavored to describe and investigate this specialized way of knowing. However, more research is needed to advance understanding of how to conceptualize, develop, and assess MKT (National Resource Council, 2012). Current descriptions treat MKT as declarative knowledge and have not emphasized the thinking processes that contribute to teachers integrating and enacting the knowledge (Borko, Roberts, & Shavelson, 2008). Focusing on describing MKT in action may contribute to understanding how teachers develop and utilize this special way of knowing. Taking the view of decision making as the core work of teaching (Borko et al.), this brief report describes the affordances of using decision making to examine the nature of teachers' action-based MKT for exponential functions.

Theoretical Framing

Because we view MKT as intertwined with the process of teaching (Rhoads & Weber, 2016), we seek to describe MKT in action. That is, we want to describe the body of mathematics knowledge teachers draw upon to make decisions when teaching and the active process of making decisions within a particular social context for a specific purpose targeted to influence the mathematics learning opportunities afforded to their students.

We borrow from enactivism (Maturana & Varela, 1998) which indicates knowing is doing and doing is knowing. That is, knowing is enacting behaviors the individual deems effective for a particular purpose in a particular context. Knowledge is the body of effective behaviors and the underlying cognition that engender one to perceive the situation and categorize which behaviors are effective. Concerning teaching mathematics, we view knowing as making decisions the teacher deems effective in affording students' opportunities to reason about mathematics. We term this knowing *enacted MKT*.

Decisions have two components: a decision outcome (i.e., what was decided, the observable behavior) and a decision rationale (i.e., why the decision outcome was chosen). The work of teaching involves such an enormous quantity of decisions it is impossible to identify them individually. During instruction, though, a teacher's actions are the overall results of their decision outcomes and the overall impact of those decisions is observable. Characterizing the teacher's actions, then, is a way of capturing the gestalt of the teacher's decision outcomes. A teacher cannot explain the rationale for each decision given the sheer volume of decisions made during instruction and the time distance between instruction and providing explanations. Over time, an experienced teacher will have built up a body of coordinated behaviors he or she deems effective for particular contexts and purposes (Brown & Coles, 2011). An expert teacher will be able to engage in a deliberate analysis of their behavior to construct a plausible rationale (Brown & Coles, Herbst & Chazan, 2011) for the actions taken. These rationale can point to an underlying propensity for making certain decisions (Herbst & Chazan). We seek to describe our teachers' decisions and their propensity for making decisions related to teaching exponential functions.

Part of a larger study, this work targeted the following research question: What is the nature of a teacher's exponential functions enacted MKT when viewed through the lens of decision-making? Answering this question involves describing decisions the teacher makes and the underlying rationale for those decisions.

Methods

For the larger study, we collected data from seventeen high school teachers engaged in teaching units related to exponential functions. We report findings from two teachers. Ms. M taught an honors section of Algebra II that tracks into an IB program; Ms. L taught College Algebra. Both teachers had taught for over ten years, had obtained their master's degrees, and had participated in teacher leadership programs. For each teacher, we video recorded three to five lessons. We conducted interviews prior to and following the teaching of each lesson as well as an overall introductory interview and a final reflecting interview over the set of lessons observed.

We transcribed the observations and partitioned each lesson into smaller segments called *episodes* and *sequences* (Wells, 1996) based on transitions between classroom tasks. For each segment, we described the overarching mathematical purpose. We formed hypotheses about the decision outcomes that contributed to the development of the intended mathematical purpose. Using the identified decision outcomes and mathematical purposes, we coded relevant sections of the interview transcripts to provide insight into the teacher's rationale for their decisions.

Preliminary Findings

To illustrate how we use a teacher's decisions to formulate a description of her KMT, we present examples of decisions made by Ms. M and Ms. L around the purpose of generating an exponential expression from a context. These cases exemplify how our framing allows for making explicit what teachers know for teaching exponential functions by identifying decisions around a mathematical learning purpose. Both teachers enacted tasks where students generated table of values to develop an exponential expression. We note the decision outcomes (actions) of the teacher to enact the purpose and their described rationale as taken from the interviews.

The Case of Ms. M

Modifying *The Domino Skyscraper Task* (http://threeacts.mrmeyer.com/dominoskyscraper/), Ms. M presented a situation where dominos were placed next to each other so that each successive domino is 1.5 times the height of the previous domino. Students worked in small groups to determine the number of dominos needed to equal the height of a given skyscraper. Ms. M intended students to create a table and then develop a formula for the n^{th} domino.
Ms. M encouraged students to generate and use tables of values to develop the explicit formula for the situation. Ms. M guided the students to explore how the output values of their tables

changed as they *moved up* (i.e., starting with the height of the skyscraper and decreasing in size) or *moved down* (i.e., starting with the initial height of the domino and increasing in size) so students would notice *repeated multiplication by a constant ratio* was relevant to the explicit formula. She further pushed students to consider *skipping ahead* (i.e., consider the change in height from one domino to another domino several stages away) to encourage moving away from generating the table values recursively and to facilitate the use of an exponent. "I was trying to get them to realize that … if I needed to skip steps, then I had to multiply by 1.5 multiple times, how do I write that? So I was trying to get them to realize that it was an exponential rather than a multiplication problem." Ms. M asked students to check their generated equation by comparing the values in the table against outputs of their equation and to modify the equation if the outputs did not match. Through this checking, some, not all, students found that they needed to modify the exponents of their equations based on how they numbered their dominos.

The Case of Ms. L

Ms. L chose to engage her students in the Social Media Task which asked students to find the number of users in 2006, 2007, and 2008 given that the number of social media users in 2005 is 3.2 million and the number of users is tripling every year. Students worked alone or in groups on this task. Then Ms. L facilitated a discussion of the student solutions and how to use those solutions to generate an expression to represent the context.

During the whole-class discussion of the task, Ms. L solicited and organized student answers (Table 1). Although students consistently described their procedure for generating the number of users in a specific year in terms of multiplying the previous number of users by three (e.g., to get the number of users in 2007, take 9.6 million and multiply by 3), Ms. L pressed students to consider the number of times one must multiply 3.2 million by 3 and to represent that relationship in the form of an exponential expression (see the middle column in Table 1). The decision to focus on the middle column suggested that Ms. L wanted students to attend to the exponential expressions that highlight the multiplicative pattern inherent in the context.

Table 1: Developing an Exponential Expression from a Context

2005	3.2 million	3.2×3^0	4	Numbers in the fourth column represent the order in which the expressions were created
2006	9.6 million	3.2×3	1	
2007	28.8 million	$3.2 \times 3 \times 3 = 3.2 \times 3^2$	2	
2008	86.4 million	3.2×3^3	3	
2052		3.2×3^{47}	5	

Ms. L then asked students towards formulating an equation to represent the situation, but quickly changed to have students find an expression for the number of users in year 2052. After recording the expression for year 2052, Ms. L asked students to (re-)consider writing an expression for the number of users in year x. Ms. L documented a student's solution of $3.2 \times 3^{x-2005}$ on the board.

Discussion and Conclusions

The second and third columns of Table 2 describe the choices teachers made to facilitate student learning to create an exponential expression that represents a context. The sequences of actions and the teachers' described rationale suggest Ms. M and Ms. L know a trajectory of mathematical reasoning for this process (first column of Table 2). Note the shared trajectory of thinking for T1-T4 with slightly varied enactment. Consider the points of diversion in rows T5 and T6. Ms. L provided a separate, explicit opportunity for students to reason about the necessary exponent value utilizing a specific input prior to writing the expression. Ms. M did not explicitly provide that opportunity except in cases where the students made an error in their initial attempts to generalize the expression.

Table 2: Implied Reasoning Trajectory with Associated Decision Outcomes

Trajectory of reasoning		Ms. M decision outcomes	Ms. L decision outcomes
T1	See how the value at a point can be determined from recursive multiplication	Have students generate tables of data values by multiplying the previous domino's height by 1.5	Have students generate data values by multiplying the previous year's data by 3
T2	Notice the explicit expression will account for the multiplicative relationship	Have students explore moving up and down the tables	Write values as a repeated multiplication by 3 using the initial value
T3	See how the values at any point can be generated from the initial value	Have students explore skipping ahead	
T4	See how the data can be re-represented to highlight the repeated multiplication as exponentiation	Have students re-represent thinking from T3	Rewrite the repeated multiplication expressions with exponents
T5*	Connect the exponent value in the expression to the input value	---------	Formulate the expression for some value in the future
T6*	Generalize the expression for any input value	Have students utilize T4 reasoning, attempt an expression, then modify expression by checking table values	Have students utilize T1-T5 reasoning to find an expression for year x

The combination of the decision outcomes and the implied trajectory allow us to describe the mathematical reasoning the teachers know to be relevant for students to complete a mathematical process and the steps and moves chosen to facilitate this reasoning. Comparing the decisions between the two teachers enabled further explication of the relevant decisions. Such descriptions of teacher enacted MKT can push the field to understand how teachers develop and utilize this knowing in specific contexts for specified purposes.

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References

Borko, H., Roberts, S.A., & Shavelson, R. (2008). Teachers' decision making: from Alan J. Bishop to today. In P. Clarkson & N. Presmeg (Eds.), *Critical Issues in Mathematics Education: Major Contributions of Alan Bishop* (pp. 37-67). Springer. Available from http://www.springer.com/us/book/9780387096728.

Brown, L., & Coles, A. (2011). Developing expertise: How enactivism re-frames mathematics teacher development. *ZDM Mathematics Education,* 43, 861-873.

Herbst, P., & Chazan, D. (2011). Research on practical rationality: Studying the justification of actions in mathematics teaching. *The Mathematics Enthusiast, 8*(3), 405-462.

Maturana, H., & Varela, F. (1998). *The tree of knowledge: The biological roots of human understanding*. Boston: Shambala. Revised edition.

National Research Council (2012). *Monitoring progress toward successful K-12 STEM education: A nation advancing?* Washington, DC: The National Academies Press.

Rhoads, K., & Weber, K. (2016). Exemplary high school mathematics teachers' reflections on teaching: A situated cognition perspective on content knowledge. *International Journal of Educational Research,* 78, 1-12.

Wells, G. (1996). Using the tool-kit of discourse in the activity of learning and teaching. *Mind, Culture, and Activity, 3*(2), 74-101. doi: 10.1207/s15327884mca0302_2

A MULTIPLE CASE STUDY EXPLORING THE RELATIONSHIP BETWEEN MODEL-ELICITING ACTIVITIES (MEAS) AND PROSPECTIVE SECONDARY TEACHERS' (PSTS) MATHEMATICAL KNOWLEDGE FOR TEACHING (MKT) ALGEBRA TOPICS

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Through the years, mathematical modeling has been shown to help develop mathematical competencies that support students' skills in solving problems arising in everyday life, society, and the workplace. With the implementation of the Common Core State Standards of Mathematics (CCSSM) in the majority of the states in the U.S., mathematical modeling has come into the forefront of the learning and teaching of mathematics. As a result, teachers are expected to support their students' classroom modeling experiences. Nonetheless, teachers have not experienced modeling thoroughly themselves in their training, because mathematics teacher education programs seldom integrate mathematical modeling in their curricula. Research studies begin to reveal how learners engage in mathematical modeling and how mathematics is supported through the exploration of modeling tasks, however, there is a gap in looking at how mathematical modeling can help support PSTs' MKT. It is important to explore modeling in relation to MKT because many studies have linked teachers' MKT to the effectiveness of mathematics instruction, and achievement gains in students. Therefore, this study aims to address a gap in the field by exploring the relationship between mathematical modeling and PSTs' MKT algebra topics and answer the following research question—what is the nature of the relationship between MEAs and PSTs' MKT algebra?

To explore this question, the researchers used a multiple case study design in order to provide a detailed understanding of three cases and a comparison of these cases to illustrate the topic being examined. Out of the 30 participants in the study, three PSTs were selected using a maximum variation purposeful sampling method. Specifically, the PSTs were given a pre-assessment to measure their MKT using a Learning Mathematics for Teaching (LMT) algebra scale. Those selected had scores belonging to one of the following three categories: (a) less than 25th percentile range, (b) between 25th and 50th percentile range, and (c) greater than 50th percentile. The participants were video and audio recorded as they explored three MEAs, artifacts pertaining to their explorations were collected, and interviews were conducted to gather extensive information about the PSTs' mathematical modeling experiences and MKT algebra topics. We developed a conceptual framework using a thorough review of the literature. Seminal work in the MKT and the Models and Modeling Perspective fields were used to guide the research, and the data analysis of the study. The results of the study demonstrate that MEAs can be used to elicit PSTs' MKT algebra through the implicit characteristics of planning, analyzing, writing, forming connections, and reflecting on the mathematical content and processes.

TRACING THE EVOLUTION OF MATHEMATICAL KNOWLEDGE FOR TEACHING FROM A LONGITUDINAL PERSPECTIVE

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Keywords: Mathematical Knowledge for Teaching, Teacher Knowledge, Number Concepts and Operations

Across various disciplines in mathematics and other sciences, numerous efforts have documented the manner in which each field has developed throughout time, and the institutions that have led the way in this process (Kilpatrick, 1992). Beginning with the first doctorate granted in mathematics education in 1906, academic institutions have produced thousands of mathematics teacher educators who have played a vital role in shaping policy decisions which have impacted classroom instruction, research and the preparation of future mathematics educators. Previous efforts have looked at key doctoral institutions and their influences on producing doctorates in the field of mathematics education (Reys, Teuscher & Nevels, 2008). Efforts have also been made to look at the preparation of doctorates as future educators and researchers (Shih, Reys & Engledowl, 2016).

This poster will illustrate a historical perspective of the evolution of mathematics education to demonstrate certain connections between research areas, academic institutions, and influential figures in the mathematics education community. An extensive data-driven approach has been used to compile information spanning over a century for this ongoing study. A longitudinal analysis of the data indicates influential figures, and institutions that have played a role in the development of various research trends. More specifically, study of the influential figures in the generations following specific mathematics educators shed a light on the development of research relating to conceptual knowledge, and content knowledge for teaching, which has been influential in curriculum development for K-12 mathematics, as well as in the preparation of pre-service teachers. In order for mathematics education to continue developing in a robust fashion, it is important for both current and future mathematics teacher educators to be aware of the development of research areas throughout time and the role that institutions have played/ are currently playing in this process. In this manner, mathematics teacher educators and doctoral programs can not only maintain a historical perspective, but also utilize these research findings to impact the development of future mathematics teacher educators and the research areas that have yet to be fully explored for generations to come.

References

Kilpatrick, J. (1992). A history of research in mathematics education, In D. Grouws (Ed). *Handbook of Research on Mathematics Teaching and Learning:* A project of the National Council of Teachers of Mathematics, (pp. 3-38). New York, NY, England: Macmillan Publishing Company.

Reys, R., Teuscher, D., & Nevels, N. (2008). Doctoral Production in Mathematics Education in the United States: 1960-2005. 1-15.

Shih, J., Reys, R. E., & Engledowl, C. (2016). Profile of research preparation of doctorates in mathematics education in the United States. *Far East Journal of Mathematics Education,* 16(2), 135-148.

ELEMENTARY TEACHERS' RECOGNITION OF ARITHMETIC PROPERTIES

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Teachers require more than an implicit understanding of the properties of arithmetic in order to take advantage of instructional opportunities and to engage students in productive discourse (Ball et al, 2001). Elementary teachers need to have the ability to recognize and use structure in order to recognize and promote this mathematical practice in their students. Ball et al observe that practicing teachers' inability to recognize the contributions of arithmetic properties to related content interferes with their ability to support their students' sense-making activities. In this study, we ask: what aspects of mathematical structure in arithmetic equations do EPTs attend to when asked to sort equations that have been designed to represent the different arithmetic properties and arithmetic operation definitions into categories? What can EPTs' explanations for how they categorize such equations tell us about the learning progression for recognizing mathematical structure?

Data were collected in two class sessions of two sections of a first content course for EPTs at a Western rural regional university. 39 to 47 EPTs participated. Students were asked to sort 20 equations on individual strips of paper into categories based on what change they observed between one side of the equation to the other. The following are representative examples:

- (2 x 3 groups) x 4 items/group = 2 x (3 groups x 4 items/group) [Associative]
- 8 groups x 4 items/group = (4 groups x 4 items/group) + (4 groups x 4 items/group) [Distributive]
- 3 x (10 groups x 4 items/group) = (10 groups x 4 items/group) x 3 [Commutative]

The categories and rationales for categories were collected and analyzed on (1) the explanations that the EPTs gave for the categories that they formed and (2) the clustering of the sentences. The open-coding of the individual EPTs' explanations yielded five types of explanation that correspond to three types of abstraction.

The resultant codes for the EPTs' explanations mapped closely to Piaget's types of abstraction (Campbell, 2001) At the level of empirical abstraction were explanations representing the use of the same numbers or same operation in the members of the equations. Two levels of pseudo-empirical abstraction were observed: (1) explanations representing recognizing equations whose members evaluated to the same numerical amount, and (2) observation that different sets of numbers were in parentheses on one side of the equal sign than on the other. At the level of reflective abstraction was EPTs' recognition that there were patterns of changes in number relationships from one member of an equation to the other--closest to corresponding to recognizing one or more structural properties. The findings of the study suggest a learning progression: (1) recognizing superficial features of equations such as the numbers and symbols used, (2) evaluation of equations, (3) recognizing the organization of arithmetic processes, (4) observing patterns of number relationships.

References

Ball, D. L., Lubienski, S. T., & Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. *Handbook of research on teaching*, *4*, 433-456.

Campbell, R. L. (2001). Reflecting abstraction in context. In J. Piaget (Ed.), *Studies in reflecting abstraction* (R. L. Campbell, Trans. and Ed.). (pp.1-27). Philadelphia: Psychology Press.

FUTURE TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE IN MATHEMATICAL MODELING INSTRUCTION

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It is agreed that teaching mathematical modeling is challenging, especially for teachers new to the process (Anhalt & Cortez, 2015). In addition to relevant mathematics to solve a problem, teachers have to possess an instructional system to facilitate students' mathematical thinking process (Blum, 2011). While ample studies have documented future teachers' knowledge for teaching various mathematics concepts, little is known about knowledge for teaching mathematical modeling. This poster details findings of our research on secondary preservice teachers' pedagogical content knowledge for teaching mathematical modeling.

Ferrini-Mundy and colleagues (2005) conceptualized three types of pedagogies that draw explicitly on mathematical knowledge domain of teachers: *Decompressing*, *Trimming*, and *Bridging*. Teachers' *decompressing* is the process of selecting or modifying tasks based on teachers' own analysis of students' mathematical understanding. *Trimming* involves transforming mathematical ideas from a more advanced form to one that preserves the essence, but that will be accessible to students based on their knowledge. *Bridging* encompasses the various kinds of connections that teachers do such as linking one area of school mathematics to another. Relying on this framework we examined preservice teachers' knowledge of teaching mathematical modeling using case based interviews.

Three secondary preservice teachers participated in the study. Selection of participants was deliberate, targeting variability among mathematical backgrounds and self-efficacy towards doing and teaching mathematical modeling. The participants were given three mathematical modeling tasks and each one was interviewed three times. In each interview, the participants worked on a modeling task and then commented on how they would implement the same task, challenges they anticipated regarding learners' difficulties and their plans for addressing them.

Analysis revealed that while the participants managed the demands of solving the modeling tasks, they experienced difficulty in the *decompressing* process to anticipate how school learners might perform on the same tasks. They viewed mathematical modeling as a tool for *bridging* mathematics to other subject areas. However, despite the fact that they were able to anticipate challenges that learners might encounter relative to defining variables or identifying parameters when working on tasks, the participants were unable to define ways to address challenges. For example, *trimming* was only meant to simplify the modeling problem into an application problem by adding some numerical variables. The study suggests that exposing preservice teachers to mathematical modeling scenarios might not be sufficient in helping them develop pedagogical approaches to facilitate classroom implementation.

References

Anhalt, C. O., & Cortez, R. (2016). Developing understanding of mathematical modeling in secondary teacher preparation. *Journal of Mathematics Teacher Education*, *19*(6), 523-545.

Blum, W. (2011). Can modelling be taught and learnt? Some answers from empirical research. In *Trends in teaching and learning of mathematical modelling* (pp. 15-30). Springer, Dordrecht.

Ferrini-Mundy, J., Floden, R., McCrory, R., Burrill, G., & Sandow, D. (2005). Knowledge for teaching school algebra: Challenges in developing an analytic framework. *American Education Research Association. Montreal, Quebec, Canada.*

TASK DESIGN AND REVISION AS A VEHICLE FOR DEVELOPING MATHEMATICAL KNOWLEDGE FOR TEACHING TEACHERS

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In what ways are the tasks of teaching prospective teachers (PTs) different from the tasks of teaching K-12 students? In this poster, we present a portrait of a few aspects of *mathematical knowledge for teaching teachers* (MKTT) that highlight some of the distinctions between it and mathematical knowledge for teaching children (MKT; Ball, Thames, & Phelps, 2009). We conceptualize the process of designing and revising tasks for PTs as a distinct form of mathematical knowledge for teaching, one that is specific to mathematics teacher educators. In this poster we share insights and reflections on how our MKTT guided and was enriched by the design and revision of lessons on five mathematical concepts.

Using a form of lesson study outlined by Berk and Hiebert (2009) and attending explicitly to the iterative process of task design outlined by Liljedahl et al. (2007), we developed five mathematical tasks for a mathematics content course for elementary PTs. The process of task design and revision highlighted the ways in which teaching PTs differs from teaching K-12 students, as PTs need to *relearn* mathematics (Zazkis, 2011); that is, PTs must reflect on previously-held conceptions and unpack their knowledge of procedures and problem-solving strategies to reveal the underlying conceptual foundations.

In the portraits of task design presented on this poster, the idea of uncovering and addressing PTs' preconceptions was essential to the MTEs' design and revision of each task. We contend that the centrality of this goal in the MTEs' thinking during this process makes these portraits examples of MKTT in action. For example, a task that presents growing visual patterns to children may focus on developing students' abilities to continue a visual pattern or on introducing functional thinking (Warren & Cooper, 2008); a task that presents growing visual patterns to PTs must do those things, as well as address any previous conceptions surrounding linear growth, algebraic generalization, or sequences that PTs may have. Thus, the purpose of designing tasks for PTs differs fundamentally from the purpose of designing tasks for students.

References

Ball, D.L., Thames, M.H., & Phelps, G. (2008). Content knowledge for teaching: what makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Berk, D. & Hiebert, J. (2009). Improving the mathematics preparation of elementary teachers one lesson at a time. *Teachers and Teaching: Theory and Practice, 15*(3), 337-356.

Liljedahl, P., Chernoff, E., & Zazkis, R. (2007). Interweaving mathematics and pedagogy in task design: A tale of one task. *Journal of Mathematics Teacher Education, 10*(4-6), 239-249.

Warren, E. & Cooper, T. (2008). Generalising the pattern rule for visual growth patterns: Actions that support 8 year olds' thinking. *Educational Studies in Mathematics, 67*, 171-185.

Zazkis, R. (2011). Relearning mathematics: A challenge for prospective elementary school teachers. Information Age Publishing: Charlotte: NC.

Chapter 7

Mathematical Processes

Research Reports

Brief Research Reports

Posters

HOW DO UNDERGRADUATE STUDENTS MAKE SENSE OF POINTS ON GRAPHS IN CALCULUS CONTEXTS?

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The purpose of this study is to examine the characteristics of students' thinking about graphs while evaluating statements from Calculus. We conducted clinical interviews in which undergraduate students evaluated mathematical statements using graphs to explain their reasoning. We report our classification of students' thinking about aspects of graphs in terms of value-thinking and location-thinking, which emerged from our data. These two ways of thinking were rooted in students' attention to different attributes of points on graphs we provided: either the input and output values represented by the points or the location of the points in space. Our findings indicate that students' thinking about aspects of graphs accounts for key differences in their understandings of mathematical statements.

Keywords: Geometry and Geometrical and Spatial Thinking, Post-Secondary Education, Cognition, Reasoning and Proof

The purpose of this study is to characterize students' thinking about aspects of graphs of real-valued functions and to investigate its role in understanding and evaluating statements from Calculus, such as the Intermediate Value Theorem (IVT). Through analyzing students' evaluations and interpretations of these statements using graphs, we seek to address the following research questions: *What are characteristics of undergraduate students' thinking about aspects of graphs related to statements from Calculus contexts?* Specifically,

(1) How do students interpret outputs of a function on a graph, points on a graph, and a graph as a whole?
(2) How do various types of student thinking about graphs of real-valued functions affect students' understanding and evaluation of the Intermediate Value Theorem and similar statements?

Literature Review

Undergraduate Calculus courses, from elementary through advanced Calculus, are comprised of many definitions and theorems about real-valued functions. Often, these statements are accompanied by visual representations in the form of graphs of relevant functions. For example, the Intermediate Value Theorem (IVT) is one such statement commonly associated with a visual representation (e.g., Briggs, Cochran, & Gillett, 2011; Finney, Thomas, Demana, & Waits, 1994; Larson, Hostetler, & Edwards, 1994; Stewart, 2012). Although research has called for the inclusion of such visual representations in mathematics instruction (e.g., Arcavi, 2003; Davis, 1993; Dreyfus, 1991; Hanna & Sidoli, 2007), few empirical studies have been conducted to look at undergraduate students' thinking about graphs of real-valued functions.

While it is hoped that students focus on the details of a provided graph that highlight the intended concept, some students may construe other properties of the given graph rather than the intended ones. For example, Moore and Thompson (2015) found that some students treat graphs as an object itself, and infer details of a situation from the shape of a graph, rather than coordinating the numerical values represented by the points of the graph. If students' interpretations of graphs differ from what is intended, students' interpretations of provided graphs might hinder their subsequent mathematical activities, such as rigorous proofs (Alcock & Simpson, 2004). Although several studies have looked at students' understanding of graphs as a whole (Monk, 1992; Moore & Thompson, 2015; Moore, 2016), it is not widely known what meaning students have for various aspects of graphs, such as the input, output, and points on graphs.

Theoretical Perspective

This study, including the data collection, data analysis, and development of the theoretical framework, is grounded in a constructivist perspective. We adopt von Glasersfeld's (1995) view that students' knowledge consists of a set of action schemes that are increasingly viable given their experience. This perspective also implies that we, as researchers, do not have direct access to students' knowledge and can only model their thinking about graphs based upon their observable behaviors.

We adopt components of Arcavi's (2003) definition of visualization for this study, which he states as "The ability, the process, and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper, or with technological tools" (p. 217). While Arcavi's description of visualization is broad, in this paper, we focus on investigating how *students* interpret, use, and think about aspects of graphs of real-valued functions and relations.

Methodology

As part of a larger study, we conducted two-hour clinical interviews (Clement, 2000) with nine undergraduate students from a public southwestern university in the United States. We selected three undergraduates who had just completed one of the following three mathematical courses that may cover the IVT: Calculus I, Introduction to Proof, and Advanced Calculus. During the interview, the interviewer asked students to evaluate each of the four mathematical statements in Table 1 and to provide justification for their evaluations. The second statement, the Intermediate Value Theorem (IVT), was the only true statement we presented. The remaining three statements (1, 3, and 4), all of which are false, were created from the IVT by reordering the quantifiers (for all, there exists) and/or the variables (N, c).

Table 1: Statements Presented to Students

Statement 1	Suppose that f is a continuous function on $[a, b]$ such that $f(a)\neq f(b)$. Then, for all real numbers c in (a, b), there exists a real number N between $f(a)$ and $f(b)$ such that $f(c)=N$.
Statement 2 (IVT)	Suppose that f is a continuous function on $[a, b]$ such that $f(a)\neq f(b)$. Then, for all real numbers N between $f(a)$ and $f(b)$, there exists a real number c in (a, b) such that $f(c)=N$.
Statement 3	Suppose that f is a continuous function on $[a, b]$ such that $f(a)\neq f(b)$. Then, there exists a real number N between $f(a)$ and $f(b)$ such that for all real numbers c in (a, b), $f(c)=N$.
Statement 4	Suppose that f is a continuous function on $[a, b]$ such that $f(a)\neq f(b)$. Then, there exists a real number c in (a, b) such that for all real numbers N between $f(a)$ and $f(b)$, $f(c)=N$.

After students evaluated the four statements, the interviewer presented six graphs. The graphs were chosen to represent a spectrum of possible functions, relations, and relevant counterexamples and included: a polynomial with extrema beyond the endpoints of the displayed function, a vertical line segment, a continuous sinusoidal function, a monotone increasing function, a constant function, and a function that is discontinuous on $[a, b]$. The interviewer asked if the students could use any of these graphs to explain their evaluation of each statement, which they could change at any time. Students were also asked to explain how they interpreted various aspects of each graph and to label relevant points and values on the graphs where appropriate.

Our data analysis was consistent with Corbin and Strauss' (2014) description of grounded theory, in which our categories of students' thinking about the graphs emerged from the data analysis. We began preliminary analysis during and immediately following each interview to note relevant findings. After all the interviews were conducted, we employed open coding (Corbin & Strauss, 2014) to document students' interpretations of aspects of the graphs they worked with. We refined these categories and re-coded the video interview data using axial coding (Corbin & Strauss, 2014). Through this process, we finalized two codes, *value-thinking* and *location-thinking*, to broadly characterize student thinking about the graphs.

Results

In our data, we found two distinct ways that students thought about graphs, specifically outputs, points, and the graph as a whole. We observed that some students primarily focused on the *values* represented by the coordinates of points on graphs while others primarily attended to the spatial *location* of these points. To describe these findings, we use the term *value-thinking* to refer to student thinking which focused on the *values* represented by the coordinates of a point. We use the term *location-thinking* to refer to thinking that primarily attends to the spatial *location* of the point. We detail characteristics of both categories of thinking in Table 2 by listing the meanings for aspects of the graph for each category and observable behaviors indicative of these meanings.

Table 2: Comparison of Value-Thinking and Location-Thinking

<table>
<tr><th colspan="2"></th><th colspan="2">Value-Thinking</th><th colspan="2">Location-Thinking</th></tr>
<tr><th colspan="2"></th><th>Interpretations</th><th>Evidence</th><th>Interpretations</th><th>Evidence</th></tr>
<tr><td rowspan="3">Aspects of a Graph</td><td>Output of Function</td><td>The resulting value from inputting a value in the function</td><td>▪ Labels output values on the output axis
▪ Speaks about output values</td><td>The resulting location in the Cartesian plane from inputting a value in the function</td><td rowspan="3">▪ Labels outputs on the graph
▪ Labels points as outputs
▪ Speaks about points as a result of an input into the function (e.g., “an input is mapped to a point on the graph”)</td></tr>
<tr><td>Point on Graph</td><td>The coordinated values of the input and output represented together</td><td rowspan="2">▪ Labels points as ordered pairs
▪ Speaks about points as the result of coordinating input and output values</td><td>A specified spatial location in the Cartesian plane</td></tr>
<tr><td>Graph as a Whole</td><td>A collection of coordinate pairs of values of the input and output</td><td>A collection of spatial locations in the Cartesian plane associated with input values</td></tr>
</table>

Value-Thinking

By *value-thinking*, we mean thinking about graphs that focuses on the input and output values represented by the coordinates of points on the graph of a function. Figure 2 contains two labeled graphs from one of the students, Jay, whom we considered to be engaged in value-thinking. One of the key characteristics of value-thinking is distinguishing between the outputs of a function and the points on a graph. Students who engaged in value-thinking labeled points as ordered pairs (see Figure 2, left), and spoke about points as representing both input and output values simultaneously. When considering the output of a function, students who thought in this way tended to label relevant output values on the output axis of the graph of a function (see Figure 2, right), and specifically spoke about the *values* of the output only. Students engaged in value-thinking thus treated graphs as a collection of ordered pairs that relate corresponding input and output values. In terms of the four statements, these students interpreted N in the phrase "N between $f(a)$ and $f(b)$" as referring to *values* between two values $f(a)$ and $f(b)$. They typically indicated the N values on the y-axis, as shown in Figure 1, right.

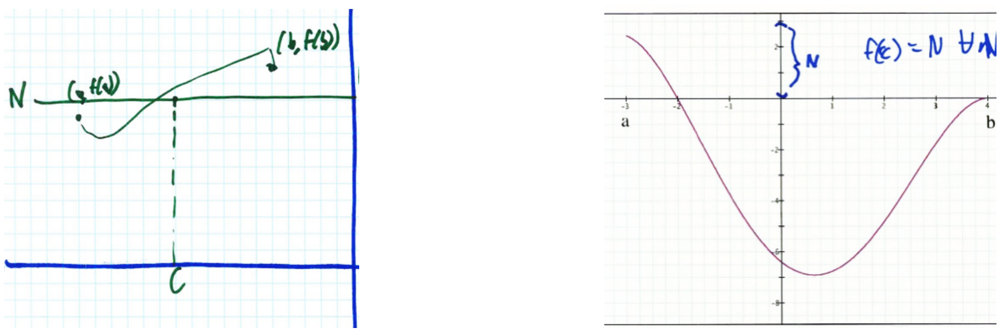

Figure 1. Jay's graph labels, characteristic of value-thinking

Location-Thinking

By *location-thinking,* we mean thinking about graphs that relies on the spatial locations of points in a Cartesian plane. Students who engaged in location-thinking focused on the location of points, while the values of the coordinates for the points were either in the background of their reasoning or absent from it. In contrast with value-thinking, one of the key characteristics of location-thinking we observed was treating the output of the function as indistinguishable from the location of the point. Accordingly, students who engaged in location-thinking often labeled points on the graph as outputs, rather than ordered pairs, and spoke about points in terms of their location in the coordinate plane. While students who engaged in value-thinking labeled outputs on the output axis, students who engaged in location-thinking frequently placed the output label at the location of the point on the graph. Instead of speaking about output *values*, these students speak about *points on the graph* as the result of an input value. Students engaged in location-thinking treated graphs as a collection of locations in space associated with inputs.

Outputs as locations. Zack was one such student who labeled $f(a)$, N, and $f(b)$ not on the y-axis, but on the graphs that he drew, as shown in Figure 3. As a result, Zack did not label points as ordered pairs.

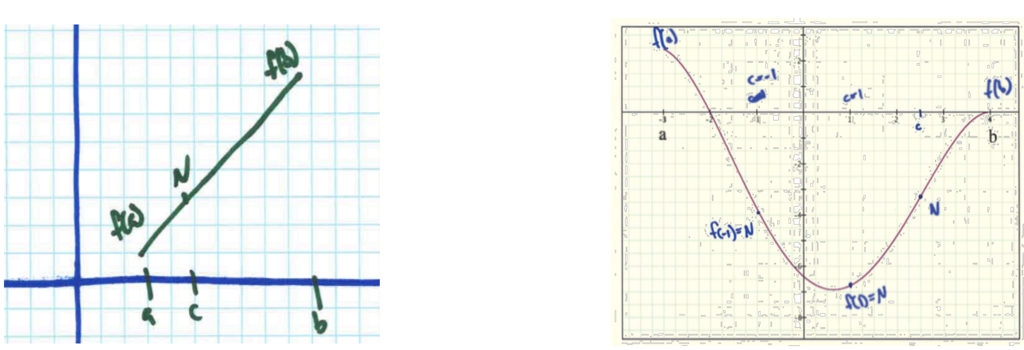

Figure 3. Zack's labels of points as outputs, a common characteristic of location-thinking

Zack's placement of an output label at a location on the graph, rather than on the y-axis, indicates that he considered these outputs to be locations on the graph, rather than values on the y-axis. Additionally, Zack called the endpoints of the graph in Figure 3, right "f(a)" and "f(b)," again indicating that he conceived of outputs as locations on the graph. Furthermore, when Zack referenced possible N's between f(a) and f(b), he swept along the entire graph of the function, rather than along the y-axis. His gesture along the graph when describing N's between f(a) and f(b) is also consistent with his conception of outputs as locations along the graph. We thus take Zack's label on the graph, words, and gestures as evidence of his consideration of outputs of the function as locations, indicative of location-thinking.

Points as locations. In addition to his graph labels of outputs at points in Figure 3, more striking evidence of Zack's location-thinking was observed when he was working with a constant function and claimed that f(a) is not equal to f(b). When the interviewer presented Zack with the graph of a constant function, Zack confirmed that the function is continuous on the interval [a, b], pointed to the endpoints of the graph, and stated that f(a) is not equal to f(b). Zack read off the remainder of Statement 3, and again pointed to the endpoints of the graph when reading the phrase "N between f(a) and f(b)." Next, he pointed to a spot on the graph, which he explained was an example of N between f(a) and f(b), plotted a dot there, and labeled the dot on the graph as "N." He also labeled the endpoints of the graph as f(a) and f(b), respectively, as shown in Figure 4.

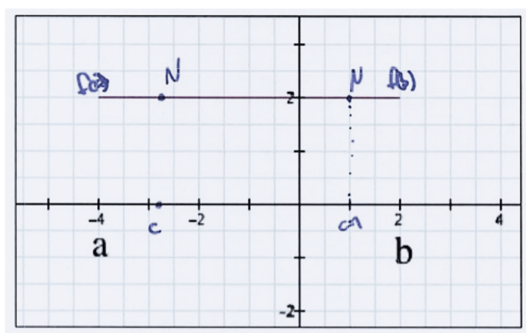

Figure 4. Zack's labeling of *f*(*a*), *f*(*b*), and possible *N*'s when he claimed *f*(*a*)≠*f*(*b*).

We take Zack's claim that *f*(*a*) and *f*(*b*) are not equal for the constant function *f* as evidence that he attended to the different spatial *locations* of the endpoints, rather than the pairs of input and output *values* represented at each point. For Zack, the point *N* that he labeled on the graph was in between the *locations* of the endpoints, which he referred to as *f*(*a*) and *f*(*b*). We also note that Zack labeled points as *f*(*a*), *f*(*b*), and *N*, rather than as ordered pairs. In essence, for Zack, there was no difference between outputs of the function and points on the graph, as both referred to spatial locations on the graph. We thus conclude that Zack conceived of *points as locations*, an indication of location-thinking.

***N* as a location between *f*(*a*) and *f*(*b*).** Like Zack and other students who engaged in location-thinking, Nate also considered outputs as locations, rather than values, and points as locations, rather than ordered pairs. We highlight Nate's meaning for the phrase "*N* between *f*(*a*) and *f*(*b*)," which was indicative of location-thinking. When working with one of the provided graphs, Nate first labeled the endpoints of the graph as *f*(*a*) and *f*(*b*), respectively. Then, Nate explained that for every *c* on this axis, he could find an *N* on the curve that *c* maps to. He also motioned from the *x*-axis vertically to the graph when describing that *c*'s mapped to *N*'s *on the* *graph*. Similarly, when describing *N*'s, he swept along the entire graph of the function from what he marked as *f*(*a*) to *f*(*b*). Nate's graph labels are shown in Figure 5 below.

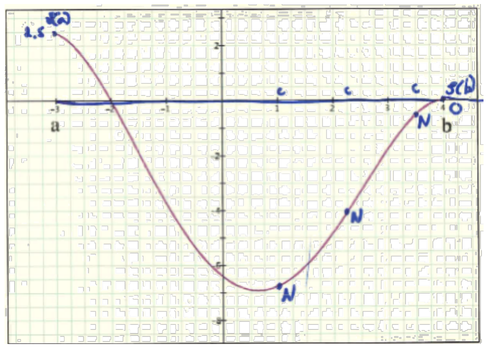

Figure 5. Examples of N's Nate claimed were between f(a) and f(b)

Nate labeled possible *N*'s on the graph that are, from our perspective, not between *f*(*a*) and *f*(*b*). Noticing Nate's placement of *N* labels and his sweeping motion along the graph, we infer that Nate interpreted "*N* between *f*(*a*) and *f*(*b*)" to mean *all* the points on the graph between the points that he labeled *f*(*a*) and *f*(*b*).

To further examine Nate's meaning for this phrase, the interviewer extended the graph to the right and marked a point on this extension of the graph, at approximately (5, 1), whose output value, 1, is between the values of *f*(*a*) and *f*(*b*) (see Figure 6). The interviewer then asked Nate if this output was between *f*(*a*) and *f*(*b*). After thinking about the question briefly, Nate stated that the output was *not* between *f*(*a*) and *f*(*b*).

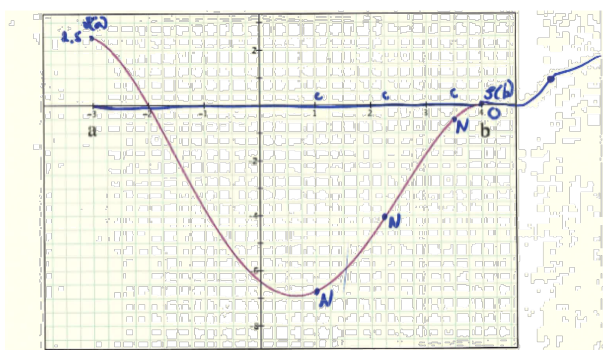

Figure 6. A point at which Nate claimed the output was not between f(a) and f(b)

The interviewer's prompt allowed Nate to more carefully consider his meaning for "*N* between *f*(*a*) and *f*(*b*)." Nate explained that there are two possible interpretations of this phrase, which he described as a "number interval" and a "function interval." By "number interval," Nate referred to the set of all output values between 0 and 2.5. In contrast, Nate used "function interval" to refer to the set of all points on the graph between the endpoints which Nate labeled *f*(*a*) and *f*(*b*). While Nate said the point that the interviewer marked was not between *f*(*a*) and *f*(*b*), he acknowledged that this point was "in the number interval" between 2.5 and 0 (the values of *f*(*a*) and *f*(*b*)). Nate clarified that although the output value of this point was between 0 and 2.5, the point was not between *f*(*a*) and *f*(*b*) because it was not "in the function interval." As he described the "function interval," Nate motioned along the entire graph between the points which he had labeled *f*(*a*) and *f*(*b*). Although Nate acknowledged the numerical interval of output values, [0, 2.5], he considered his notion of the "function interval" as more relevant for interpreting the phrase "N between $f(a)$ and $f(b)$." We thus take Nate's interpretation of "N between $f(a)$ and $f(b)$" in terms of the *spatial location* of the points, rather than the values of the outputs, as indicative of *location-thinking*.

Value-Thinking, Location-Thinking, & Students' Evaluations of Statements

We found that students' interpretations of aspects of graphs, whether in terms of value-thinking or location-thinking, were related to their evaluations of the four statements we presented. Among the five engaged in value-thinking, three students evaluated all four statements correctly. In contrast, no student engaged in location-thinking evaluated all four statements correctly. Table 3 reports our classification of each student's thinking (Value-Thinking or Location-Thinking), along with each student's mathematical level (Calculus, Introduction to Proof, and Advanced Calculus), and final evaluations of the four statements (True, False, or Sometimes True).

Table 3: Students' Final Evaluations of Statements

Students Observed Engaging in…	Student Name	Math Level	Final Student Evaluations S1(F)	S2(T)	S3(F)	S4(F)
Value-Thinking	Jay	Advanced Calculus	F	T	F	F
	James	Advanced Calculus	F	T	F	F
	Nikki	Introduction to Proof	F	T	F	F
	Ron	Introduction to Proof	T	T	F	F
	Mike	Introduction to Proof	F	F	F	F
Location-Thinking	Zack	Calculus	ST	ST	ST	ST
	Nate	Advanced Calculus	T	T	F	F
	Hannah	Calculus	T	T	T	T
	Marie	Calculus	T	T	T	T

Shaded cells indicate mathematically incorrect evaluations.

Although all students who correctly evaluated the statements engaged in value-thinking, not all the students who engaged in value-thinking evaluated the statements correctly. Ron, a student who engaged in value-thinking, failed to attend to the restriction on the values of N, which led him to evaluate statement 1 as true. Mike, another student who engaged in value-thinking, evaluated all four statements as false due to unconventional meanings for the logical quantifiers (for all, there exists) involved in the statements. Thus, we view value-thinking as necessary, but not sufficient, for correctly evaluating the IVT and similar statements. In contrast, no student who engaged in location-thinking evaluated all four statements correctly. We take this as an indication that location-thinking does not support students in correctly evaluating the statements we presented. Even Nate, a student with a more advanced mathematical background, incorrectly evaluated Statement 1. His location-thinking was the main factor in his incorrect evaluation.

Conclusion & Discussion

Our findings in this study reveal critical distinctions in students' interpretations of aspects of graphs, namely in terms of value-thinking and location-thinking. In our study, some students interpreted and labeled points as pairs of input and output values, while others interpreted points as locations in space and labeled them with output notation. Our results highlight and explain significant aspects of students' interpretations of graphs not previously accounted for by current theories and studies on students' thinking about graphs. Thus, the use of our constructs of value-thinking and location-thinking may progress the depth of analysis in the field of students' understanding of graphs.

These two distinct ways in which students interpreted points, and thus graphs, have significant implications for how students understand important mathematical ideas, such as the Intermediate Value Theorem (IVT), as we observed with our students. We acknowledge that, in this study, value-thinking supported students in correctly evaluating the IVT and similar statements. However, in other contexts, such as diagrams in geometric settings, location-thinking may be preferable. Additionally, other contexts, like graphs of parametric curves, may require both location and value-thinking for interpreting various aspects of the same image. Ideally, students should possess the ability to think in both ways, focusing on the values represented at a point and the point's spatial location, along with the ability to discern when it is appropriate to use each way of thinking. To support students in recognizing these two ways of thinking, instructors and curriculum developers may consider providing students with opportunities to think both ways and bring to light this distinction. We hope that our findings increase practitioners' awareness of the subtle yet significant details of students' interpretations of aspects of graphs and may thus inform decisions in curriculum design and instruction.

References

Alcock, L., & Simpson, A. (2004). Convergence of sequences and series: Interactions between visual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics*, *57*(1), 1-32.

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, *52*(3), 215-241.

Briggs, W., Cochran, L., & Gillett, B. (2011). *Calculus: Early transcendentals*. Boston, MA: Addison-Wesley.

Burn, H., & Mesa, V. (2015). The calculus I curriculum. In D. Bressoud, V. Mesa, & C. Rasmussen (Eds.), *Insights and recommendations from the MAA national study of college calculus.* (pp. 45-57). Washington, DC: Mathematical Association of America.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In R. Lesh & A. Kelly (Eds.), *Handbook of research methodologies for science and mathematics education* (pp. 547-589). Hillsdale, New Jersey: Lawrence Erlbaum.

Corbin, J., & Strauss, A. (2014). *Basics of qualitative research: Techniques and procedures for developing grounded theory*. Newbury Park, CA: Sage publications.

Davis, P. J. (1993). Visual theorems. *Educational Studies in Mathematics*, *24*(4), 333-344.

Dreyfus, T. (1991). On the status of visual reasoning in mathematics and mathematics education. In *Proceedings from 15th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 1 (pp. 33-48). Assisi, Italy.

Finney, R. L., Thomas, G. B., Demana, F. D., & Waits, B. K. (1994). *Calculus: Graphical, numerical, algebraic*. Reading, MA: Addison-Wesley Publishing Co.

Hanna, G., & Sidoli, N. (2007). Visualisation and proof: A brief survey of philosophical perspectives. *ZDM*, *39*(1-2), 73-78.

Larson, R. E., Hostetler, R. P., & Edwards, B. H. (1994). *Calculus with analytic geometry* (5th ed.). Lexington, Massachusetts: D. C. Heath and Company.

Monk, S. (1992). Students' understanding of a function given by a physical model. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy*, *25*, 175-194.

Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In T. Fukawa-Connelly, N. E. Infante, K. Keene & M. Zandieh (Eds.), *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education*, (pp. 782-789). Pittsburgh, PA: RUME.

Moore, K. C. (2016). Graphing as figurative and operative thought. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.), *Proceedings of the 40th Conference of the International Groups for the Psychology of Mathematics Education,* Vol. 3, (pp. 323-330). Szeged, Hungary: PME.

Stewart, J. (2012). *Calculus: Early transcendentals* (7th ed.). Stamford, CT: Brooks/Cole Cengage Learning.

COMPUTATIONAL AND INFERENTIAL ORIENTATIONS: LESSONS FROM OBSERVING UNDERGRADUATES READ MATHEMATICAL PROOFS

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This paper presents selected findings from an assessment of university students' moment-by-moment reading of mathematical proof. This method, adapted from an assessment of narrative reading validated by psychologists, yields novel insights into the strategies students use to construct meaning for the equations in a proof text. In particular, we present evidence that novice readers constructed meanings for the equations using mathematical practices familiar from non-proof oriented courses – substituting and solving for variables – while more experienced readers drew upon practices native to proof-oriented mathematics – inferring properties of quantities. We refer to these as computational and inferential orientations, respectively. We interpret this mismatch of practices in terms of Systemic Functional Linguistic's notion of textual metafunction and briefly discuss its implications for proof-oriented instruction.

Keywords: Reasoning and Proof, Post-Secondary Education, Cognition

Researchers of language in mathematics education have long noted that there exists a *mathematical register* (e.g. Halliday, 1975; Pimm, 1987) constituting a subdomain of spoken English (in the English-speaking world). Halliday (1975) explains that the mathematical register constitutes much more than technical terms and the extensive use of symbols. He explains, "It is the meanings, including the styles of meaning and modes of argument, that constitute a register, rather than the words and structures as such" (Halliday, 1975, p. 65). Many have noted that students' ability to participate in mathematics depends on developing competencies within the mathematical register (e.g. Schleppegrell, 2007; Morgan, 1998), but relatively few studies have investigated the nature of those competencies. This paper helps address this gap in the literature.

This report presents findings stemming from a study of undergraduate students' reading of mathematical proofs. We adapted a moment-by-moment think aloud protocol that has proven valuable for assessing narrative text reading competencies (Magliano & Millis, 2003; Magliano, Millis, Team, Levinstein, & Boonthum, 2011). Adapting this methodology to mathematical proof has yielded many insights into novice and experienced student reading behaviors. Hereafter, we highlight one difference between some novice and experienced readers' sense-making about equations in proofs. We refer to their strategies as *computational* and *inferential orientations*.

Studies of reading behavior

Two mathematics education studies particularly investigated undergraduate students' reading of mathematical text by comparing it to expert reading behaviors. Shepherd and van de Sande (2014) observed that mathematics faculty and graduate students read (i.e. articulated written text) equations in terms of their conceptual structure while undergraduates read them symbol-by-symbol. Inglis and Alcock (2012) used eye-tracking technology to compare undergraduate and mathematician reading of mathematical proofs. They found that experts focused much more on the text surrounding equations, which encoded the logical links in the proof, while undergraduates focused much more on equations themselves. However, eye-tracking methodology did not provide any explanation for why this difference emerged.

As mentioned above, we adapted our methodology from an assessment for narrative reading comprehension developed by Magliano, Millis, and colleagues (Magliano & Millis, 2003; Magliano et al., 2011). That assessment (called RSAT) presents students with narrative text one line at a time and asks students to think aloud or to respond to particular content-related prompts. The validated assessment distinguishes novice and expert readers based on the types of connections they make (not the quality). Students who connect the text to prior information in the text or outside knowledge tend to be more competent readers than those who simply paraphrase or restate the given line. One group of psychologists (Fletcher, Lucas, & Baron, 1999) adapted this method to reading high school proofs. They found that reading such text was more effortful than reading narrative and allowed new types of connections such as anticipating later lines of the text. The findings from moment-by-moment assessment techniques correlate with end-reading comprehension tests, but they reveal a different set of behaviors and insights into the reading process itself. We accordingly adapted the methodology to gain new insights that might not be yielded from existing assessments of proof end-reading comprehension (Mejia-Ramos, Lew, de la Torre, & Weber, 2011) or proof validation (e.g., Alcock & Weber, 2005).

Conceptualizing the linguistic functions of proof

Most studies of language learning in mathematics education draw upon Halliday's (1975; 1984) theory of Systemic Functional Linguistics (SFL). This theory is particularly useful because it is rooted in language use, it focuses on speakers' and hearers' choices by which they make meaning in language, and it emphasizes the role of linguistic function to explain language use. As Schleppegrell (2004, p. 137) explains, "It is important for students to develop academic register options in different disciplines because particular grammatical choices are functional for construing the kinds of knowledge typical of a discipline." SFL posits that three metafunctions are always operative in the construction or interpretation of a text: the *ideational* metafunction concerns what is being talked about, the *interpersonal* metafunction concerns who is involved in the discourse and how they are positioned, and the *textual* metafunction concerns what type of text is being constructed and what features are necessary or appropriate for such text.

Consider the following excerpt from one of our proof tasks:

For every primitive Pythagorean triple (a, b, c) there exist some numbers s and t with no common factors such that $s > t \geq 1$ where $a = st$, $b = \frac{s^2 - t^2}{2}$, and $c = \frac{s^2+t^2}{2}$.

1. Let (a, b, c) be a primitive Pythagorean triple.
2. Then $a^2 = c^2 - b^2 = (c + b)(c - b)$.
3. We want to show that $(c + b)$ and $(c - b)$ are both squares and share no common factors.
4. Suppose that d is a common factor of both $(c + b)$ and $(c - b)$.
5. Then d is also a factor of $(c + b) + (c - b) = 2c$ and a factor of $(c + b) - (c - b) = 2b$.

Figure 1. Excerpt from the primitive Pythagorean triples proof (Rotman, 2013).

Ideationally, the proof is about characterizing primitive Pythagorean triples. Many portions of the proof are dedicated to deducing numeric properties such as sharing common factors or being perfect squares. *Interpersonally*, mathematics text tends to hide any direct reference to human participants by expressing relations about mathematical objects. In this text, the subjects of most clauses are quantities to which properties are attributed. *Textually*, the symbols and equations mark the text as clearly mathematical. They conform to the genre of mathematical proof by drawing (and justifying) inferences. The conflict between the claims in lines 3 and 4 further suggest to an expert reader the beginning of a proof by contradiction (or proof that $d = 1$).

A key function that distinguishes mathematical proof from other texts is the emphasis on both making and justifying claims. There is an implicit expectation for the reader to identify the implied warrant in line 5: "the sum and difference of multiples of d are also multiples of d." Students must parse the equation in multiple ways to construct the intended meaning. They must recognize that both sides of the equation represent the same quantity, which is the object of the clause "d is a factor of." They must parse the left sides as the sum and difference of the two compound quantities discussed in line 4, which is supported by parenthetical grouping. Finally, a reader should determine that the property of being a multiple of d is being transferred from two quantities known to share the property to a pair of new quantities $2b$ and $2c$. Though equations are symmetric, this inference is directed (from left-hand quantities to right-hand quantities).

While the various claims about the normative or intended construal of this text are cued in subtle ways in these three lines, an expert reader in many ways constructs its meaning using textual expectations and devices. Ultimately, the text is far from explicit regarding all that it intends to convey. This is not a particular defect of mathematical proof, but rather,

> Inferencing on the basis of background assumptions plays a central role in the interpretation of all texts. Highly complex and abstract background assumptions which are not spelled out are often necessary for the interpretation of written language, especially in school contexts. (Schelppegrelle, 2004, p. 11)

With this background we may now better articulate the key question investigated in this report: what textual assumptions and functions do novice readers use to construct meaning for equations in mathematical proofs and how do these differ from those used by more experienced readers?

Methodology

One contribution of this study is the adaptation of the moment-by-moment think-aloud reading assessment protocol to advanced mathematical proof. Our study involved interviewing mathematics students ranging in experience from no proof-oriented courses to beginning graduate study. To accommodate such diverse experience, we sought out proofs that (a) had at least 10 lines of text, (b) required no knowledge of mathematical concepts with which novices were not familiar, and (c) were relatively outside the curriculum such that experts were less likely to have seen the exact proof being presented. We used four such proofs in our interviews, though due to space limitations we only report on the proof quoted from in Figure 1.

Adapting moment-by-moment read aloud protocol to proof

Similar to RSAT (Magliano & Millis, 2003), each proof was presented one line at a time. Unlike RSAT, the previous lines of our proofs remained visible. For three of the four proofs students first read relevant definitions and theorems. These remained available for their reference on sheets of paper that doubled as scratch paper. Students were instructed to read each line aloud and think aloud before the interviewer might provide more specific response prompts. We developed the response prompts by listing all the connections that we expect expert readers could make for each line. These included (a) relations to previous lines, (b) implicit warrants, (c) use of definitions, (d) elements of proof frames such as universal generalization, case structure, proof by contradiction, or induction, (e) opportunities to anticipate future lines, and (f) goal statement or achievement. We then decided for which lines we would provide targeted prompts such as "why is this line justified," "what do you expect in the following lines," or "why did the author introduce d." The interviewers explained the meaning of terms or statements upon request, though we tried to withhold judgment about the students' interpretations. We note that this methodology does not elicit a naturalistic reading because inviting students to think aloud and respond to targeted prompts in many ways mimics the self-explanation training that Hodds, Alcock, and Inglis (2014) found to improve proof comprehension. Rather, we interpret our data as representing an optimal reading for each student at that time. Such readings still elicit the students' capabilities as readers of mathematical proof.

Participants and analysis

Participants were recruited from a large public university in the Southwestern United States and a mid-sized public university in the Midwestern United States. Volunteers were solicited from courses ranging from differential equations, introduction to proof, real analysis, and topology (the latter two were dual-listed as undergraduate and graduate courses). To date, 17 students completed interviews for which they were given minor monetary incentives. Three students recruited from differential equations had no collegiate proof coursework and three other participants were recruited from real analysis, which was their first completely proof-oriented course. We classify all six as *novice readers*. The nine other undergraduate students had completed at least one proof-oriented course and were classified as *experienced readers*. Finally, two mathematics graduate students participated who were considered *expert readers*.

Coding began by forming narrative note files for each interview that described students' reading behavior for each line of text. From these line-by-line notes, we formed categories of noteworthy behaviors and patterns in student reading behavior. The two authors independently coded two novice reader interviews in this way and met to discuss and compare the findings. By comparing the various categories of reading behaviors across the four proof tasks, we identified five meta-categories that organized the primary range of categories. We then independently coded the rest of the interviews, meeting regularly to discuss our interpretations and coding. These five meta-categories remained relatively stable, though nuanced to accommodate further cases. The meta-categories are computational versus inferential orientation, depth of encoding for proof claims, logic structure and language, meanings for relevant mathematical concepts, and sense-making activities. Due to space concerns this paper only presents findings about the first category, computational versus inferential orientation.

Results

The meta-category computational versus inferential orientation most directly addresses students' reading behaviors relevant to the textual metafunction characterized by SFL. In particular, it concerns the mathematical practices that students called upon to make sense of the proof's use of equations. We briefly introduce the two constructs with reference to line 2 in Figure 1. Every study participant could identify and justify the algebraic manipulations the line expressed. This type of reorganization of equations is very common in computational mathematics courses and thus familiar to all of these students. The proof goes on to use this equation in two other ways. First, because d is defined as a factor of $(c + b)$ and $(c - b)$, this equation supports the inference that d is a factor of a. The property "d is a factor" (more precisely d^2 is a factor) is transferred from the right side of the equation to the left. Then, once it is established that $(c + b)$ and $(c - b)$ share no common factors, the equation is used to infer that both of those quantities are perfect squares. In this case, the property "is a perfect square" is transferred from the left side of the equation to the two components on the right. This latter use of the equation as a vehicle for inferring properties about the quantities within is not native to early collegiate non-proof-oriented courses. As a result, we noticed much greater discrepancies in students' abilities to construct these meanings for the equations.

We thus define two orientations as follows:

- *Computational orientation*: Mathematics is about computing and solving for numbers, so proofs use equations for counting, evaluating, computing, and solving for quantities.
- *Inferential orientation*: Mathematics is about relating mathematical properties, so proofs use equations for stipulating, inferring, and denying properties of quantities.

We posit a few points about these constructs before presenting some instances of student data that exemplify how these orientations operated in our data. First, the two are not mutually exclusive since both are accurate reflections of mathematical practices. Both of these claims express resources that students may use to interpret linguistic choices in a proof, specifically ideational and textual choices. The main concern is when a proof was constructed according to one orientation and is interpreted according to another. All students recognized that line 2 could be justified in terms of computational practices. However, not all students recognized the uses of line 2 in terms of inferential practices. Also, such orientations are implicit and are merely enacted by students in reading. We now present two cases of novice reading behavior that exemplify reading inferential proof texts using a computational orientation.

Nov5 – Explaining operations rather than inferring properties

Nov5 was a chemistry major taking differential equations when he participated in the study. He had no collegiate proof-oriented coursework. He spent about 45 minutes reading the Pythagorean triples proof (17 lines). He expressed low confidence in his interpretations and confusion about what the interviewer meant by questions about "justifying" lines of the proof. His responses to such questions most often focused on explaining the author's reasoning or on trying to explain why each step in the proof was chosen. He explained line 3 (that introduced subgoals for the proof) saying "I guess they are saying that the focus is going to be on this part of their equation." When he read line 4 that introduced the quantity d, he explained:

N5: Ok. I guess they are gonna put in a variable d that I guess is a factor of both, so it's both divisible by d [rising tone as if a question].

I: What is the purpose of introducing d here?

N5: Uhh. Probably to somehow come out with the quotient for b and c being 2.

I: When you say quotient, what were you pointing at?

N5: Just the denominator here [reads b and c equations in theorem] just showing where that 2 is coming from. [reads line 5]

I: So why is this true?

N5: Well I am pretty sure what they did was set the parentheses equation up to the $c^2 - b^2$ and then attempted to solve and then since you would have a square root to solve, you have to have the positive and the negative version of it. I feel like I am wrong, but. […] Yeah. Multiply it out here. [He multiplies $(c + b)(c - b)$ on paper.] Well I guess that proves number two for sure. Let's see what they. I think my idea is off. Ok, now I am not 100% sure where that came from. d is a common factor.

I: So tell me something about what you are thinking here.

N5: I am thinking maybe possibly that, that these here the $2b$ and $2c$ are derivatives. Because the derivative of c^2 would be $2c$ and the derivative of b^2 would be $2b$. I am just trying to think where they got the addition from.

N5: [Later, he reads line 7, which says] "If d is a factor of both b and c, then d is also a factor of a." Ok yeah cause that would match their equation.

I: Which equation are you looking at?

N5: Pythagorean theorem because if you mess with one side of the equation you also have to do it with the other side of the equation or else it's not equal, so I guess, yeah.

We notice that Nov5 drew on multiple resources to make meaning from the text. He did not explain the introduction of d in terms of the goals in line 3, which he only interpreted as focusing the reader's attention. Rather, he hoped to explain why the equations for b and c in the theorem both had a denominator of 2. He next cited the algebraic procedure of introducing positive and negative possibilities when taking the root of an equation to explain why the two equations in line 5 differed by addition and subtraction. When he could not recreate the inferred algebraic manipulations, he connected the right sides of the line 5 equations to the derivatives of terms in the line 2 equation. In each case, we notice that Nov5 drew upon computational practices (substituting, solving, and taking derivatives) to make meaning of the text.

Finally, we note that Nov5 was able to make sense of the property inference in line 7, but he explained it in terms of operations performed on each side of the equation rather than in terms of the equality between the quantities. On one hand, we may celebrate his linguistic improvisation for conveying a meaning that he did not seem to have ready tools to express. On the other, his explanations give the sense that he understood "d is a factor of" as an operation to be performed rather than a static property of a number. We expect that an inferential orientation and the language to express its meanings are interdependent and co-emerge (Schleppegrell, 2004).

Nov2 – "What is the goal of this equation?"

Nov2 was an electrical engineering major recruited from differential equations who had not taken any proof-oriented courses. She also spent about 45 minutes reading the Pythagorean triples proof. The following represents her reading of the fourth and fifth lines:

I: So what is the purpose of introducing d?

N2: Umm, d would, d would show the correlation between the number b and c, to see how they, to figure out what c and b is possibly.

I: Ok. Anything else that comes to mind about this line?

N2: I mean if you multiply everything out and add d to the equation, you could solve for one of them, if you knew one of the numbers.

I: Ok let's see what comes up [shows line 5].

N2: [Reads line 5]. Umm. [Long pause.] I'm just seeing how d fits into everything, I guess?

I: My question about this is why is this line justified? Why does this have to be true?

N2: Well if you, obviously if you do the math you would figure out that. Sorry I am not good at putting this into words.

I: So when you say "do the math" can you tell me more specifically what you are thinking about?

N2: Well if. So are we trying to figure out what b and c is? What is the goal of this equation? What are we trying to figure out? Just figuring out what the missing variables are?

N2 showed direct awareness that she was confused about the purpose of introducing d and the equations in line 5. She inferred that the purposes were to substitute and solve for variables, consistent with the practices in non-proof-oriented courses. She used the parlance "do the math" to refer to algebraic operations, suggesting an identification between the two. When asked to explain, she posed four questions about the purpose and goal of the equations in line 5. These convey a clear sense that her current understanding did not help her construe the text coherently, but she did not know what other goals or practices to infer beyond solving for variables.

By claiming that N2 interpreted using a computational orientation rather than an inferential orientation, we do not claim she did not draw inferences and try to justify them. For instance, after the proof establishes that d cannot be a factor of b and c, line 10 asserts that "d is 1 or 2." N2 responded saying:

N2: I am guessing it's 2 because, being a factor of 1 is being a factor of everything, right?

I: Yeah, 1 is a factor of every number.

N2: So since d is not a factor of b and c, it can't be 1. So it would have to be 2.

It is important to note that the proof's references to the property "share no common factors" up to this point left the caveat "factors greater than 1" unstated. Some other study participants made the same observation that N2 did, but concluded that if 1 is a factor of every number then "share no common factors" must implicitly exclude 1. However, from N2's current construal of the text that d is not a factor of b and c, her inference that $d = 2$ is justified. As a result, we argue that N2 was clearly capable of drawing (deductive) inferences from the proof, but her inferences did not always match the intended inferences. This further nuances what we mean by students reading with a computational orientation. It describes the resources that students use to make meaning from the text, not their capabilities for engaging in particular mathematical reasoning.

N2 was perturbed enough by the end of reading this proof that she sought to explain herself:

> I am really awful at proofs. […] I am more of a plug and chug person, so I am good at integrating and all that stuff. Word problems like this where I have to think about it and why it equals something, it is not my forte. […] For me I want to solve, I don't really care about like why is it this, why is it that. I just wanna solve for it. I am just so used to solving, I have never even thought about why is this this and this. It is a little new to me.

Our view of N2's learning capabilities is higher than what she expressed here. We maintain that reading capabilities such as an inferential orientation can be learned through experience with proofs and her difficulty with reading can be overcome. However, her explanation corroborates our claim that her extensive experience with computational practices in mathematics led her to use them to make sense of the proof, even when different practices were intended.

Inferential orientation readings of line 5

The majority of study participants, specifically all of those with more proof experience, constructed meaning for line 5 in terms of attributing the property "d is a factor of" to two quantities. Furthermore, they interpreted the interviewers' request for justification in a more normative manner rather than simply trying to explain why the author chose to add and subtract the quantities they did. We still observed a range of reading behaviors for justifying the inference. Some students imagined substituting some expression including d in place of $(c + b)$ and $(c - b)$ and noted that d could be factored out, verifying that the sum and difference should also have d as a factor. We consider this as essentially *constructing a mini-proof* of the implicit warrant. This type of proof still drew upon algebraic meanings for factor, namely to substitute and "factor out." Other students – primarily more experienced readers – searched for a general form of the particular inference to evaluate its validity. For instance, they might paraphrase the inference as "if d is a factor of two numbers, then it is a factor of the sum or difference of those numbers." This pattern may be called *generalizing the inference*. Finally, one of the graduate student readers justified line 5 by citing that any linear combination of multiples of d would also be multiples of d. In this case, she *recalled a warrant* to justify the inference. Thus we recognize that within an inferential orientation students may exhibit a range of meaning-making behaviors.

Discussion

Using observations of students' moment-by-moment reading behavior, we characterize two textual orientations that students drew upon to make meaning for the equations in a proof text: *computational orientations* and *inferential orientations*. We claim this contributes to the existing literature in two ways. First, our findings could explain Inglis and Alcock's (2012) eye-tracking finding that novice readers focused much more on the equations in a proof rather than the surrounding text. Their calculus student participants may have read the provided proofs using a computational orientation that led them to draw upon the wrong kinds of practices, while the expert readers focused on the inferences expressed by the surrounding text. Second, it begins the process of characterizing some of the reading capabilities that students need to develop to interact productively with advanced mathematical proof texts. Our findings highlight the ways in which students draw upon their knowledge of mathematical practices to read a text. We expect that their ability to understand new practices (such as proving properties of quantities in number theory), to read texts intending to express those practices, and to use language to convey those practices are interdependent and co-emergent in student learning. As noted above, interpreting inferential text in terms of computational mathematical practices was observed only among our least experienced study participants. This could be interpreted in one of two ways. Either basic proof-oriented instruction was successful in fostering an inferential orientation or those who did not construct an inferential orientation did not succeed in proof-oriented courses before we recruited participants. Further data gathering would be required to understand the interaction between these orientations and instruction. We expect that further study should reveal significant variation among the reading behaviors within the inferential orientation. These could be used to understand advanced mathematical reading competencies and how they develop.

References

Alcock, L. & Weber, K. (2005). Proof validation in real analysis: Inferring and checking warrants. *Journal of Mathematical Behavior*, 24(2), 125-134.

Fletcher, C.R., Lucas, S., & Baron, C.M. (1999) Comprehension of Mathematical Proofs. In S.R. Goldman, A.C. Graesser, & P. van den Broek (Eds.) *Narrative Comprehension, Causality, and Coherence: Essays in Honor of Tom Trobasso* (pp. 195-207). Mahwah, NJ: Lawrence Erlbaum Associates.

Halliday, M. (1978) 'Some aspects of sociolinguistics', in *Interactions Between Linguistics and Mathematics Education*, (pp. 64-73). Copenhagen: UNESCO.

Hodds, M., Alcock, L., & Inglis, M. (2014). Self-explanation training improves proof comprehension. *Journal for Research in Mathematics Education*, *45*(1), 62-101.

Inglis, M., & Alcock, L. (2012). Expert and novice approaches to reading mathematical proofs. *Journal for Research in Mathematics Education, 43*(4), 358−390.

Magliano, J. P., & Millis, K. K. (2003). Assessing reading skill with a think-aloud procedure and latent semantic analysis. *Cognition and Instruction*, *21*(3), 251-283.

Magliano, J. P., Millis, K. K., The RSAT Development Team, Levinstein, I., & Boonthum, C. (2011). Assessing comprehension during reading with the Reading Strategy Assessment Tool (RSAT). *Metacognition and learning*, *6*(2), 131-154.

Mejía-Ramos, J. P., Lew, K., de la Torre, J., & Weber, K. (2017). Developing and validating proof comprehension tests in undergraduate mathematics. *Research in Mathematics Education*, *19*(2), 130-146.

Morgan, C. (1998). Writing Mathematically: The Discourse of Investigation. London, UK: Falmer.

Rotman, J. J. (2013). Journey into mathematics: An introduction to proofs. Courier Corporation.

Schleppegrell, M. J. (2004). The Language of schooling: A functional linguistics prespective. Mahwah, NJ: Lawrence Erlbaum.

Schleppegrell, M. J. (2007). The linguistic challenges of mathematics teaching and learning: A research review. *Reading & Writing Quarterly*, *23*(2), 139-159.

Shepherd, M. D., & van de Sande, C. C. (2014). Reading mathematics for understanding—from novice to expert. *The Journal of Mathematical Behavior*, *35*, 74-86.

A FINE-GRAINED ANALYSIS OF PROOF SUMMARIES: A CASE STUDY OF ABSTRACT ALGEBRA STUDENTS

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In this paper I explore eleven undergraduate students' comprehension of a proof taken from an undergraduate abstract algebra course. My interpretation of what it means to understand a proof is based on a proof comprehension model developed by Mejia-Ramos, et al. (2012). This study in particular examines the extent to which undergraduate students are able to summarize a proof using the proof's higher-level ideas. Additionally, eleven doctoral students in mathematics were asked to provide a summary of the same proof that the undergraduate students received. Undergraduates' holistic comprehension of the proof was then analyzed in light of summaries that the doctoral students provided. The main finding of the study is that undergraduates' comprehension of the proof was overall inadequate—notably, they demonstrated limited skills in summarizing a proof via the proof's key ideas. Moreover, undergraduates failed to recognize the scope of the method used in the proof.

Key words: Proof, Proof Comprehension, Abstract Algebra.

In advanced undergraduate mathematics courses, students are expected to spend a significant portion of their time reading proofs. Although proof comprehension is a fundamental aspect of undergraduate mathematics education, studies by Conradie and Frith (2000), Rowland (2002) and Weber (2012) suggest that mathematicians rarely measure their students' comprehension of proofs. For example, Weber (2012) reports that mathematicians assess their students' comprehension of proofs by asking students either to reproduce the proof or prove a similar or a trivial consequence of proofs they presented in class. However, Conradie and Frith (2000) suggests that asking students to reproduce a proof may not be a pedagogically useful way of assessing students' understanding of a proof because students can and do correctly reproduce a proof by simply memorizing it word for word, with virtually no understanding at all.

Despite its importance in undergraduate mathematics education, research on undergraduates' comprehension of proofs is limited. In fact, much of the proof literature focuses on students' aptitude to construct or validate proofs and less on their ability to comprehend proofs (Mejia-Ramos et al., 2012; Mejia-Ramos & Inglis, 2009). Mejia-Ramos and his colleagues (2009) systematically investigated a sample of 131 studies on proofs and they found that only three studies focused on proof comprehension. They hypothesize that the scarcity of the literature on proof comprehension is perhaps due to the lack of a model on what it means for an undergraduate student to understand a proof. In this study, I adopt an assessment model for proof comprehension that was developed by Mejia-Ramos, et al. (2012) to explore undergraduates' comprehension of proofs. In particular, this study seeks to address the following research questions.

To what extent do undergraduates comprehend a proof? More specifically, to what extent do undergraduates:

- summarize a proof using its high-level ideas,
- recognize and appreciate the scope of a method used in a proof?

Theory: Assessment Model for Proof Comprehension

Mejia-Ramos, et al. (2012) proposed that one can assess undergraduates' comprehension of a proof along seven facets. These seven facets are organized into two overarching categories: local and holistic. A local understanding of a proof is an understanding that a student can gain "either by studying a specific statement in the proof or how that statement relates to a small number of other statements within the proof" (p.5). Alternatively, undergraduates can develop a *holistic* comprehension of a proof by attending to the main ideas of the proof. Below, I will elaborate on what it means to understand a proof holistically.

Assessing the Holistic Comprehension of a Proof

According to the proof comprehension model, the holistic understanding of a proof consists of being able to: (1) summarize the proof using the proof's main ideas, (2) identify the modular structure of the proof, (3) recognize and extend the method used in the proof, and (4) illustrate the method of the proof using a specific example or diagram. They developed these four facets of holistic comprehension of proofs based on: (a) mathematicians' perspectives on how and why they read and present proofs and what it means for them to understand a proof; (b) the proof literature on the role of proof; and (c) the recommendations by mathematicians and mathematics educators on proof presentations that would presumably improve students' proof comprehension. Below, I elaborate on (1) and (3).

Summarizing a proof via its high-level ideas. Mejia-Ramos, et al. (2012) state that "one way that a proof can be understood is in terms of the overarching approach that is used within a proof" (p.11)." Being able to summarize a proof via its high-level ideas entails understanding the proof's "top-level overview" or "big idea". A good summary of a proof may include what Raman (2003) describes as a proof's *key ideas.* Raman (2003) defines *key ideas* as "heuristic ideas which one can map to a formal proof with appropriate sense of rigor" (p. 323). *Key ideas* provide "a sense of understanding and conviction why a particular claim is true" (Raman, 2003, 323). Mathematicians can evaluate their students' understanding of this aspect of a proof in at least two ways. They can, for instance, directly ask students to provide a brief summary of the proof that includes the proof's higher-level ideas. Alternatively, they can provide students with a few summaries of the proof and ask them to choose which summary best captures the main ideas of the proof (Mejia-Ramos et al., 2012). In this study, I asked undergraduates to provide a proof summary using the proof's main ideas.

Transferring the general ideas or methods to another context. Mejia-Ramos, et al. (2012) suggested that identifying the scope of a method or technique used in a proof is an important aspect of proof comprehension. Mathematicians can assess this aspect of proof comprehension by asking students to (a) identify methods or techniques without which the proof would have collapsed, or (b) prove new claims by applying methods similar to those used in the original proof. In this study, undergraduates were asked questions to elicit their understanding of the scope of a method used in a proof.

Review of the Literature

As noted earlier, educational research on proof comprehension in undergraduate mathematics has received little emphasis. Osterholm (2006) was among the first to look into student's comprehension of mathematical texts. He conducted a quantitative study of reading comprehension of abstract algebra students (he compared texts with one including symbols and another one not). He concludes that "mathematics itself is not the most dominant aspect affecting the reading comprehension process, but the use of symbols in the text is a more relevant factor" (p.325). His contention is based on the fact that the group of students who had almost no symbols and notations in their reading assignment outperformed, in a reading comprehension test, those whose reading assignments involved mathematical symbols and notations. Although Osterholm (2006) does point to difficulties student encounter while reading mathematical texts, it should be noted that his study only asked students to read mathematical texts and not proofs specifically.

Research also suggests that undergraduates are not successful in gleaning understanding from the proof they see during lecture (Selden & Selden, 2012; Lew et al, 2015). For example, students interviewed in Lew et al.'s (2015) study fail to comprehend much of the content the instructor desired to convey, including the method used in the proof. Students interviewed in Selden and Selden's (2012) study also failed to understand a proof holistically since they were fixated on verifying each line and put little emphasis in attending to the overarching methods used in the proof. One purpose of this study is to build on the growing body of research on proof comprehension.

Methods

Research Settings

This study took place in a large public university in the northeastern United States. The content of the proof used in this study come from an introductory abstract algebra course. In the chosen research setting the standard textbook used is *Abstract Algebra: An introduction* by Hungerford (2012). The goal of the course (as stated in the syllabus) is to introduce students to the theory of algebraic structures such as rings, fields, and groups in that order.

Participants

Undergraduate student participants. Since the main purpose of this study is to explore undergraduates' comprehension of proofs—in particular, proofs that appear in an introductory abstract algebra course—I personally approached undergraduates who had taken or were enrolled in an introductory abstract algebra course. Eleven undergraduates agreed to participate in this study and were assigned pseudonyms S1-S11. At the time of the study, six of the eleven undergraduate participants (S3, S5, S6, S7, S8, and S9) were enrolled in an introductory abstract algebra course. Seven participants—S1, S2, S3, S5, S6, S7, S8, and S9—were pursuing a major in secondary mathematics education and said they intended to be high school mathematics teacher. The remaining four students were mathematics majors. Furthermore, each participant had taken a minimum of three proof-based courses and all but three (S1, S3, and S6) said they received an A or B in their introduction to proof course. Participants' responses on a background survey suggest that each participant spent at least two hours per week reading proofs outside of class.

Doctoral students. Eleven doctoral students at the aforementioned research site agreed to participate in this study. I used doctoral students to analyze undergraduates' summaries of a proof. At various times, I asked the doctoral students to provide, in writing, a summary of a proof using the proof's key or main ideas. To avoid confusion, in the remainder of this paper I will refer to these doctoral student participants as experts.

Materials and Research Procedures

In this study undergraduates were asked to read a proof that shows that any finite integral domain is a field. This proof is given in appendix 1. I chose this proof because (a) it nicely draws connection between two important topics covered in abstract algebra: integral domains and fields, and (b) it uses a proof technique that I speculated most participants probably have not seen.

The principles I employed to write this pedagogical proof—a proof that is geared toward undergraduates for the purpose of pedagogy—is based on the Lai, Weber, & Mejia-Ramos (2012) study. Lai et al. (2012) report that mathematicians valued pedagogical proofs that (1) made assumptions and conclusions of the proof explicit, (2) centered on important equations to emphasize the main ideas, and (3) did not contain "true but irrelevant statements." (p.94). Also, when writing the proof, I consulted mathematics professors and made appropriate modifications.

Participants were asked to read the proof until they felt they understood it and were encouraged to write and/or highlight on the proof paper as well as to think out loud while reading. Once a participant finished reading the proof, she/he was asked to:

- to provide a good summary of the proof including the proof's main or higher-level ideas
- to indicate assertion(s) in the proof that would fail if $\boldsymbol{R}$ was infinite.

Analysis

Recall that eleven doctoral students in mathematics were asked to write a summary of the proof using what they think are the main ideas of the proof. Doctoral students were also asked to describe the key ideas of each proof. First, I carefully studied doctoral students' summaries of the proof. I then developed a synthesized summary for the proof. This synthesized summary, which will hereafter be referred to as the expert's summary, also incorporated all the key ideas that doctoral students described for each proof. That process resulted in the following summary of the proof.

> The proof shows that a finite integral domain R is a field by showing that any non-zero element of R has a multiplicative inverse. Let a be any non-zero element of R. The absence of zero divisor in R taken together with the fact that R is finite gives us that left multiplication by a defines a bijective map from the integral domain to itself ($f_a: R \to R$ given by $f_a(x) = ax$ is a bijective). Surjectivity of this map guarantees that a has a multiplicative inverse.

The key ideas identified in the expert's summary of the proof are:

- Overarching method: given an arbitrary non-zero element $a \in R$, show that there exists $b \in R$ such that $ab = 1_R$
- Approach: define a left multiplication by a non-zero element $a \in R, f_a: R \to R, x \mapsto ax$. Using kernel of f_a one can show that it is injective.
- The finiteness of R and the injectivity of f_a to show that f_a is a surjective map from R to R.

The above expert's summary of the proof was eventually verified by two experienced researchers in mathematics education as to whether or not it indeed incorporated all the *key ideas* of the proof that doctoral students discussed; modifications were then made to the expert's summary, as needed.

Finally, using a rubric, two researchers, both with a master's degree in mathematics, independently conducted a comparative analysis of undergraduates' summaries of the proof against expert's summaries. When disagreement emerged, w engaged in discussion until a consensus was reached.

Results and Discussion Results on Undergraduates' Summaries of the proof

Recall that undergraduates' summaries of the proof were analyzed in comparison to the expert's using a rubric that is omitted here for a shortage of space. Nearly all participants, nine out of eleven, provided a summary of the proof that suggested a limited proof comprehension. In particular, their responses indicated that they either *poorly* or *very poorly* understood the proof. The results of students' summaries of the proof is given in Table 1. Note, in table 1, that a majority, six undergraduates, provided a *very poor* summary of the proof, which implies that their summary failed to highlight the main ideas of the proof that was described in the expert's summary.

Table 1: Undergraduate students' summaries of the proof

Evaluation	Undergraduate students
Very poor	S3, S5, S6, S8, S9, S11
Poor	S1, S2
Satisfactory	S4, S7, S10
Good	None

In order to give you a sense of a *very poor summary*, I will provide a detailed description of summaries given by S5, S8, and S9. When asked to provide a good summary of the proof including the proof's main ideas, S5 offered the following summary:

> *R* is an integral domain and field. It is both surjective and injective to the kernel of the function that defines it. There is also $1_R \in R$ that allows it to have multiplicative inverse, thus units. I know a field's non-zero elements all make units, a field.

S5's summary consists of incomplete sentences and phrases that are either mathematically incorrect or appear to have been copied from the proof word for word. For instance, S5 begins by supposing the thing that needs to be shown—that *R* is a field. Also, there is evidence of a misunderstanding about injectivity and surjectivity of a map. This is evident when he asserts that *R*, as opposed to f_0, is both surjective and injective. Finally, S5's summary doesn't mention how crucial assumptions in the proof such as the finiteness of *R* are used in the proof. Thus, S5's summary is deemed to be *very poor*, which means it was considered to be very different from the expert's as it did not highlight main ideas of the proof. S8, likewise, provides a *very poor summary*:

> The proof basically gave a less textbook traditional explanation of a way to prove that all finite integral domains are fields. It used the understandings of kernals [sic], bijections, injection, surjections in order to prove facts about rings, where usually you learn the facts about rings before being introduced to functions

Her summary above makes no mention of the proof's high-level or *key ideas* and thus does not suggest a satisfactory comprehension of the proof. While she enumerates topics that are used in the proof, her summary does not illustrate how they were employed in the proof. S8's summary, for example, indicates that the concept of bijectivity is employed in the proof, but she does not explain how it is used. Along the same lines as S5, and S8, S9 also supplied a summary of the proof that did not suggest a satisfactory comprehension of the proof. For example, S9 provided the following summary:

> Given a finite integral domain, you can prove that it is a field by showing it has a multiplicative inverse, no zero divisors, injective and surjective, kernels, and if there is a multiplicative inverse such that $ax = 1_R$ and $a \neq 1_R$ then R is a field.

S9's summary above says very little beyond restating the claim. S9 essentially repeats phrases that appeared in the proof verbatim. He does not draw any connection between *key ideas* described in the expert's summary of the proof. Also, S9's summary states that *R* is first shown not to have zero divisors, but this is neither necessary nor true; *R* is assumed to be an integral domain and therefore it does not have zero divisors. Overall, S9's summary fails to mention the proof's key ideas and consequently shows limited comprehension of the proof.

While the majority, six out of eleven students, provided what is considered to be a *very poor summary* of the proof, S1 and S2 supplied a *poor summary* of the proof. S2, for example, provided the following summary of the proof:

> The aim of the proof is to show that if R is a finite integral domain, then R is a field. It then wants to show that R has a multiplicative inverse, then that the kernel of $f_a: x \rightarrow ax$ to be trivial. The prof then shows that $x = 0$ since if $ax = 0$ then a or x is 0 but a is not, thus $\ker f_a = \{0\}$ so f_a is injective. Thus, $|R| = |f_a(R)|$, which shows it is surjective. It then proves a has a multiplicative inverse, so R is a field.

S2's summary is incoherent and appears to duplicate some parts of the proof word for word. Moreover, her summary includes way too much unnecessary information; for example, she repeats the argument for the triviality of f_0. While S2's summary does mention some *key ideas* that are noted in the expert's summary, it doesn't make the logical connection between those *ideas*. In fact, S2 appears to have the logic of the proof backward, as she seems to think the existence of multiplicative inverse is what guarantees the triviality of the kernel of f_0.

While no one provided a *good summary* of the proof, three students—S4, S7, and S10—provided a *satisfactory summary* of the proof. S7, for instance, summarized the proof as follows:

> First it is important to show each nonzero element of *R* has a multiplicative inverse. Then we consider a nonzero element $a \in R$ and the map of f_a. We use the kernel of f_a to prove f_a is injective. Then from the fact that f_a is injective and therefore $|R| = |f_a(R)|, f_a$ is also surjective. Finally, we show $f_a(x) = 1_R$ hence a has a multiplicative inverse and therefore *R* is a field.

Evidently, S7's summary above has significant resemblance to the expert's summary of the proof. In particular, S7 does mention some *key ideas* of the proof. However, she did not indicate the fact that the surjectivity of f_0 depends on the finiteness of *R*, which was a crucial idea that was noted in the expert's proof. Furthermore, the last line of her summary is incorrect in the sense that f_a is not identically equal to the identity 1_R. Also, based on what is stated at the very end of her summary, S7 does not seem to have understood how a (the fixed nonzero element) has a multiplicative inverse. However, S7's summary overall does bear some resemblance to the expert's summary and suggests a satisfactory understanding of the proof. On the whole, while nearly all students provided a *poor* or a *very poor* summary of proof, no one provided a *good summary.* Indeed, only S4, S7, and S10 managed to provide a satisfactory summary of the proof.

Results on Undergraduates' Ability to Recognize or Appreciate the Scope of a Proof's Method

No undergraduate demonstrated why $\boldsymbol{R}$ must be finite for the proof to be valid. When asked why the method of the proof would fail if *R* was assumed to be infinite, six out of eleven undergraduates offered no response or said "I don't know…" The other five students pointed incorrectly to an assertion that would fail if *R* is infinite. Table 2 below illustrates the various responses they provided:

Table 2 Reasons undergraduates provided for why R must be finite

Reason why R must be finite	Participants
R would not be commutative (line two)	S1
There wouldn't be exactly be the same number of	S3 elements in each
f_a would not be injective (line seven)	S2, S8, S5
No response/I don't know/Not sure	S4, S6, S7, S9, S10, S11

Table 2 shows that participants failed to pinpoint a specific assertion in the proof that would fail—the argument in line eight—if *R* was infinite. That is to say, if *R* is infinite, $f_a(R) \subseteq R$ and $|R| = |f_a(R)|$ taken together would not guarantee that $f_a(R) = R$, which would not make f_a surjective.

To summarize, Mejia-Ramos and colleagues (2012) maintain that being able to provide a good summary of a proof is a key indicator of comprehension. However, undergraduates in this study showed a limited comprehension of the proof. In particular, in their proof summary undergraduates failed to highlight the proof's main idea. For a large number of participants, their proof summary was essentially a replica of a few sentences that appeared in the proof. Further research is needed to identify why undergraduates fail to summarize a proof using the proof's main ideas.

Appendix 1

Claim: Let *R* be a finite integral domain. Then *R* is a field.

Proof.

1. Let *R* be a finite integral domain whose multiplicative identity is 1_R and whose additive identity is 0_R.
2. Since *R* is a commutative ring, it suffices to show that every nonzero element in *R* has a multiplicative inverse.
3. Let *a* be a fixed nonzero element of R $(a \neq 0_R)$. Consider the map $f_a: R \to R$ defined by $f_a: x \to ax$. We first show that the kernel of f_a is trivial.
4. Note that kernel of $f_a = \{x \in R: f_a(x) = 0_R\} = \{x \in R: ax = 0_R\}$.
5. Since R has no proper zero divisors, $ax = 0_R \Longrightarrow a = 0_R$ or $x = 0_R$. But, $a \neq 0_R$ thus $x = 0_R$.
6. Therefore kernel of $f_a = \{0_R\}$ and so f_a is injective.

7. Next, note that $|R| \geq |f_a(R)|$. Since f_a is injective, it follows that $|R| = |f_a(R)|$.
8. Because $f_a(R) \subseteq R$ and $|R| = |f_a(R)|$, we have that f_a is surjective.

References

Conradie, J., & Frith, J. (2000). Comprehension tests in mathematics. *Educational Studies in Mathematics, 42*(3), 225-35.

Lai, Y., Weber, K., & Mejia-Ramos, J. (2012). Mathematicians' perspectives on features of a good pedagogical proof. *Cognition and Instruction, 30*(2), 146-169.

Maxwell, J. A. (2013). *Qualitative research design: An interactive approach: An interactive approach* (3rd ed.) Sage.

Mejia-Ramos, J. P., & Inglis, M. (2009). Argumentative and proving activities in mathematics education research. In F. Lin, F. Hsieh, G. Hanna & d. M. Villiers (Eds.), *Proceedings of the ICMI study 19 conference: Proof and proving in mathematics education* (Taipei, Taiwan ed., pp. 88-93)

Mejia-Ramos, J. P., & Weber, K. (2014). Why and how mathematicians read proofs: Further evidence from a survey study. *Educational Studies in Mathematics, 85*(2), 161-173.

Mejia-Ramos, J., Fuller, E., Weber, K., Rhoads, K., & Samkoff, A. (2012). An assessment model for proof comprehension in undergraduate mathematics. *Educational Studies in Mathematics, 79*(1), 3-18.

Raman, M. (2003). Key ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics, 52*(3), 319-325.

Osterholm, M. (2006). Characterizing reading comprehension of mathematical texts. *Educational Studies in Mathematics, 63*(3), 325-346.

Selden, J., & Selden, A. (2009). Teaching proving by coordinating aspects of proofs with students' abilities. *Teaching and Learning Proof Across the Grades: A K-16 Perspective*, 339-354.

Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education, 34*(1), 4-36.

Weber, K. (2012). Mathematicians' perspectives on their pedagogical practice with respect to proof. *International Journal of Mathematical Education in Science and Technology, 43*(4), 463-482.

STUDENT UNDERSTANDING OF THE GENERAL BINARY OPERATION CONCEPT

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In this paper, we address the variety of ways in which students conceive of binary operations and the metaphors they might leverage when working with binary operations in group theory. We use open-ended surveys paired with interviews to qualitatively explore student's conceptions of binary operation. Through this analysis, we identified three central metaphors (function, arithmetic, and structure), as well as a number of attributes related to binary operation that students perceive as critical. This work has implications for our treatment of binary operation in advanced mathematics, and the messages we send about operation in the K-12 setting.

Keywords: Advanced Mathematical Thinking, Algebra and Algebraic Thinking, Post-Secondary Education

Binary operations are threaded throughout the mathematics curriculum starting with a focus on arithmetic, and then expanding to operations such as matrix multiplication and function composition. Although binary operations are prolific throughout mathematics in K-12, it is not until advanced courses in undergraduate mathematics that binary operation is formally defined. In courses such as group theory, students are put in a position to generalize the binary operation concept to better understand the structures that underlie our mathematical systems. Despite this occurrence, and the prevalence of binary operation mathematics, little research attention has been given to the general binary operation. This is disconcerting for two reasons: (1) according a survey of expert instructors, binary operation is considered to be one of the most important topics in group theory (Melhuish & Fasteen, 2016); (2) binary operations are one of the core concepts that can be connected back to the K-12 curriculum (Melhuish & Fagan, in press).

However, the survey of expert instructors also revealed a possible cause for the lack of treatment. They identified binary operation as a topic with low difficulty level (Melhuish & Fasteen, 2016). Yet, we found evidence to suggest that students' conceptions of binary operations can significantly interfere with performance on group theory tasks (Melhuish & Fasteen, 2016). This evidence was the impetus to study student understanding of binary operation more directly. In this paper, we address the variety of ways in which students conceive of binary operations and the metaphors they might leverage when working with binary operations across a number of task types. We present a qualitative study, guided by variation theory, in which we elaborate on the critical attributes of binary operations that students attend to while engaging with related tasks.

Literature Review

Binary operation has uneven treatment throughout the literature. Before we unpack this literature, we define what we mean by binary operation. Informally, binary operations can be thought of as rules that combine two elements within a set and produce a single element of the same set. For instance, addition is a binary operation on the set of integers because addition can be thought of as a rule for combining two integers to produce a single integer. Formally, binary operations are often defined as follows:

A *binary operation* $*$ on a set S is a function mapping $S \times S$ into S.

This formal definition situates operation as a type of function. We therefore outline literature across three categories: (1) arithmetic and other specific operation literature, (2) function literature, and (3) literature pertaining to the general binary operation.

As most literature on student mathematical thinking connects to operation in some way, we focus on a few key theoretical contributions (that go beyond the specifics of a particular arithmetic operation.) One such example is the *operation sense* framework developed by Slavit (1998), which unpack a series of stages built around familiarity with standard arithmetic operations. In this framework, operations represent a standard process, such as the process of combining groups underlying the operation of addition. Slavit's work documented a path of increasing sophistication that could support students' in developing operation sense that moved beyond particular arithmetic process metaphors to an understanding of operation without a need for concrete referents. Such understanding can support students' transition from arithmetic to algebra. Gray and Tall (1994) provided a different view on this abstractness focusing on operation as a *procept*. That is, an operation can be conceived as a process and a concept. An expression such as "3+2" is both the object (the sum) and the process (adding the two numbers.) In light of this literature base, we identify (1) students may struggle to move beyond arithmetic operations on concrete referents and (2) students need to conceive of operations as processes and objects (which is non-trivial.)

We next turn to the literature on function to identify ways in which function understanding may be relevant to binary operation. First, there is a large amount of existing literature documenting the complexities of function (Oehrtman, Carlson, & Thompson, 2008). Students have been shown to possess numerous conceptions throughout their K-16 education. Due to space limitations, we note two that provide insight beyond arithmetic literature. First, students may interpret functions as necessarily having a written rule (Vinner & Dreyfus, 1989). Such an interpretation limits students' ability to leverage or make sense of non-symbolic representations. Additionally, function conceptions are tied to a wide range of metaphors (Zandieh, Ellis, and Rasmussen, 2017). One such metaphor is the ubiquitous input/output machine metaphor (Tall, McGowen, & DeMarois, 2000). In light of binary operations being a special case of function, we conjectured that similar representation and metaphor preferences might exist.

Finally, we attend to the literature on the general, abstract, binary operation concept. While majority of research related to abstract algebra and linear algebra implicitly treat binary operations (e.g., Larsen's (2009) work having students reinvent groups), few studies have explicitly focused on binary operation. These few studies are primarily theoretical breakdowns (with empirical instantiations) that are part of a larger attempt to map student conceptions. Brown, DeVries, Dubinsky, and Thomas (1997) and more recently Wasserman (2017) presented genetic decompositions of binary operations in which individuals may have an action (requiring concrete actions with individual inputs), process (seeing a binary operation as a general process on a domain), or object (seeing binary operation as something that can be acted on) conception of binary operations. Alternately, Novotná, Stehlíková, and Hoch (2006) use lens of *structure sense* to produce a framework of binary operation capturing shifts of attention where students can first recognize binary operation in familiar settings, then make sense of properties like closure, engage with unfamiliar operations, and compare differing representations. Other notable results include students have limited representational flexibility (Ehmke, Pesonen, & Haapasalo, 2010), and have limited example spaces (Zaslavsky & Peled, 1996). While these contributions provide some insight into student conceptions around binary operation, many questions remain.

As a whole, the body of research points to substantial reasons to identify binary operation as a potentially troubling topic with its nature as a function and operation. In the next section, we outline a theoretical lens that has aided us in further parsing students' activity and conceptions around binary operaiton: variation theory (Marton & Booth, 1997).

Theoretical Framework

One way to conceptualize learning is the perception of new attributes of a phenomenon or experiencing a phenomenon in a new way. Such an approach leverages a variation theory (Marton & Booth, 1997) lens. For a given individual, their understanding of a concept reflects which attributes of the concept are foregrounded for them. Students become aware of attributes through experiences of concepts (and non-examples of concepts) where attributes may be foregrounded via contrast. If a student never experiences a non-example of a binary operation with one or three inputs, they may not perceive two inputs as a critical attribute of binary operation. Alternately, if all examples students encounter have certain commonalities such as associativity, students may perceive associativity as a critical attribute of binary operation. In this way, variation theory ties learning tightly with the role of experiences and particularly contrasting experiences where students may discern attributes of a particular object.

From a variation theory stance, we then leverage several key ideas including: *object of learning*, *critical attributes*, and *metaphors*. Variation theory provides a complimentary tool to traditional cognitive or social investigations of learning via shifting from cognition to perception (Dahlin, 2001). This shift leads to questions about: What are the critical attributes of a given object of learning? What opportunities do students have to perceive (through variation) the critical attributes of an object? And ultimately, what critical attributes are in the *lived* object of learning for a student?

Vikström (2008) has expanded the use of variation theory to also attend to metaphors as the means for which students structure and communicate their lived object of learning. As students become aware of critical attributes of a phenomenon, metaphors are the mechanism to integrate and connect with the abstract notion of a particular concept. When students engage in tasks, they then rely on metaphors to connect a given situation to their greater understanding of a concept.

In accordance with this view on learning, the research questions guiding our study are: (1) What are the metaphors leveraged by students when communicating about binary operation?; (2) Which attributes are students attending to when engaged in binary operation related tasks?

Methods

Participants and Data Collection

The participants in this study consisted of students enrolled in three introductory, undergraduate-level abstract algebra classes (n =9, n = 12, and n = 12 respectively). Additionally, we interviewed six students, four of which come from the surveyed class, two from another class (that received a shortened survey that is not reported here.) Classes span two institutions and three unique instructors. Surveys were created to cover a variety of binary operations and non-binary examples. In particular, the questions were crafted in accordance with the following activity domains:

1. *Is or is not*. Determining if a given instantiation is an example of a concept (e.g. Ehmke , Pesonen, and Happasalo, 2011)
2. *Same or different*. Determining if two instantiations are mathematically the same (e.g. Novotna, Stehlikova, and Hoch, 2006)

3. *Properties*. Determining what properties an example may or may not have. (e.g. Dubinsky Dautermann, Leron, & Zazkis, 1994)
4. *Generating*. Creating an example meeting some criteria (e.g. Zaslavsky & Peled, 1996).

Table 1: Example Survey Prompts

Category	Is or Is not	Same or Different	Property	Generating
Prompt (shortened)	$(a) = a^2$, on $\boldsymbol{R}$	$\boldsymbol{Z}_3$ and $\boldsymbol{Z}_6$ (presented tabularly)	Is ½(a+b) associative?	Define a binary operation on {1,2,4}
Purpose	Critical attribute: closure; Missing: 2 inputs	Calls on attribute: element-wise defined	Missing varying attribute: *E-O-E* formatting	Calls on varying attribute: non-symbolic representation

Each of the questions was open-ended and provided a prompt for the students to explain their reasoning. Across the classes, the surveys contained ten common questions, along with several variations designed to test the robustness of response types. For the scope of this report, we focus on the common questions.

In addition to the surveys, six semi-structured interviews were conducted to gain deeper insight into the students' mathematical thinking as they worked through the survey. The purpose of the interviews was twofold: (1) to validate our interpretations of written responses and (2) allow for a closer inspection of metaphors that are leveraged in verbal communication (and not always immediately apparent in written responses.)

Analysis

The analysis of the survey data was driven by methods of phenomenographic content analysis (Trigwall, 2006). Student responses were disconnected from the individual. The goal was to identify patterns in the different ways that students experience a given phenomenon. Each task was then analyzed for patterns in responses beginning with broad open codes. These were condensed to categories reflecting critical attributes of binary operation as experienced by students. We then independently coded all survey responses resolving all disagreements via discussion. After this analysis, we determined the degree to which a given student's attention to attributes was consistent across tasks. In the results section, we share the dominant critical attributes from student responses, as well as reports on consistency.

For the interview data, we analyzed transcripts to identify the metaphors they leveraged to communicate about binary operation. For each transcript, we identified each instance that a student communicated directly about binary operation noting varying patterns in language. We then returned to the literature to categorize metaphor clusters: arithmetic, function, and structure. We used these metaphors to code the transcripts to determine which of the three metaphors occurred in a given individual's communication.

Results

We begin by discussing the three metaphors that underlined students' communication.

Metaphors

Function Metaphor Cluster. We define the function metaphor cluster as being enacted whenever a student communicates about binary operation as if it were a function, that is, a mapping with inputs and outputs. Indicators include: reference to a domain and range, input/output language, mapping language, and referencing function specific properties such as one-to-one. An example of a student using this metaphor is the following description of a binary operation, "[the binary operation] takes two inputs and returns an output based on the operation." Notice this language evokes traditional function metaphor language where there is something (akin to a function machine) that takes inputs and produces outputs. Other examples of using the function metaphor include: "[The binary operation] does take two of the numbers and it will return a third," and "The division binary operation is going to take me from the Reals to the Reals." Such metaphors occurred across all of the interviews (see Table 2).

Table 2: Interviewed Students' Metaphors for Binary Operation

Metaphor	Function	Arithmetic	Structure
Student 1 (Class B)	x	x	x
Student 2 (Class B)	x	x	x
Student 3 (Class C)	x	x	x
Student 4 (Class C)	x	x	x
Student 5 (Class D)	x	x	x
Student 6 (Class D)	x	x	

Arithmetic Metaphor Cluster. We define the arithmetic metaphor as being enacted whenever a student communicates about binary operation as if it was an arithmetic operation. In arithmetic contexts, we tend to leverage a combining metaphor. For example, when adding 2 and 3, one metaphor would be combining collections of 2 items and 3 items to produce 5 times. This action is internal to the inputs rather than taking inputs and mapping to something new (as in a function metaphor.) Indicators of this metaphor include language around "answer" and "combining." For example, a student described the action: "Two numbers, take them, combine them". Another described comparing two operations on particular elements with: "They are not the same answer." As in the case of function, these metaphors occurred in all interviews (see Table 2).

Structure Metaphor Cluster. We define the structure metaphor cluster as being enacted whenever a student communicates about a binary operation as a mechanism for structuring a set. The structure may be noticed based on prior exposure to mathematics (such as a binary operation on a set forming a group) or the student may spontaneously create the structure (such as noticing a pattern or explicitly searching for a pattern within a set of outputs). Indicators include language about overall behavior, patterns, or structural properties. The structure metaphor tended to emerge in tasks in which students were asked to compare two binary operations to determine whether they were the same. For example, one student asked, "What pattern do we have going on there?" and another commented, "These two groups behave rather similarly." This metaphor can be quite productive when considering structural properties of a given set. Structure metaphors were frequent, occurring in five of six interviews.

These results reflect that the three metaphors are not a mutually exclusive categorization of a student's thinking about binary operations. A student can enact any of three metaphors depending on the task they are engaged with, and in fact it was common of the students to enact all three at some point. Furthermore, the same metaphors can be supportive or unsupportive depending on the critical attributes the metaphor is structuring.

Critical Attributes

In this section, we unpack some of these critical attributes. Our focus is on the survey results, but we note that are interview subjects provided similar profiles in terms of critical attributes. We cover two definitional properties (closure, two input elements), one additional critical attribute (binary operations are defined at the element level), and two non-critical representational attributes (symbolic rule and element-operator-element (E-O-E) formatting).

Table 3: Survey Responses when Determining if an Instantiation is a Binary Operation

	Attended to Closure			Attended to Two Input Elements		
	Always	Inconsistently	Never	Always	Inconsistently	Never
Class A	5 (62.5%)	2 (25.0%)	1 (12.5%)	0 (0%)	0 (0%)	8 (100%)
Class B	5 (41.7%)	5 (41.7%)	2 (16.7%)	1 (8.3%)	1 (8.3%)	10 (83.3%)
Class C	4 (33.3%)	2 (16.7%)	6 (50.0%)	6 (50.0%)	3 (25.0%)	3 (25.0%)
Total	14 (43.8%)	9 (28.1%)	9 (28.1%)	7 (21.9%)	4 (12.5%)	21 (65.6%)

Closure. The first of the critical attributes is closure. It was common for students to attend to closure of a binary operation on a set when asked to determine if a given operation was binary or not (see table 3). For example, when asked if $*(a) = a^2$ defined on $\mathbb{R}$ is a binary operation, one student responded, "yes, if a is real, the square of a is also real."

Two input elements. The second critical attribute is that a binary operation must be defined on two elements (or ordered pairs.) Some students perceived this critical attribute (as seen in Table 3). For example, when asked to determine if $\sqrt{a}$ on $\mathbb{Z}$ is a binary operation, one student responded, "No, only a is used, not both inputs." However, it was far more common for students to ignore this critical attribute and only attend to closure (as in the prior example).

Table 4: Survey Responses when Determining if two Given Operations were the same

	Attended to Element-Wise Differences		
	Across Prompts	Element-Defined Prompt Only	Never
Class A	4 (50%)	3 (37.5%)	1 (12.5%)
Class B	1 (8.3%)	8 (66.7%)	3 (25%)
Class C	6 (50%)	4 (33.3%)	2 (16.7%)
Total	11 (34.4%)	15 (46.9%)	6 (18.8%)

Defined element-wise. A particular binary operation is determined by where any two elements are mapped. However, students inconsistently treated this as a critical attribute of binary operation. This became clear on tasks comparing two operations. For example, when provided Cayley tables representing addition modulo 3 and 6 and asked if they were the same binary operation, one student responded they were not because whenever they "plug in 1 and 2 into both these operations [they] will get different results" reflecting attention to individual elements. In contrast, another student claimed that the binary operations were the same because the elements "behave the same." Over half of the students either never attended to element-definedness or only when provided a binary operation already defined that way (see Table 4).

Table 5: Survey Responses when Asked to Create a Binary Operation on {1,2,4}

	Non-symbolic Representation	Symbolic Representation	No Answer
Class A	0 (0.0%)	6 (66.7%)	3 (33.3%)
Class B	1 (8.3%)	4 (33.3%)	7 (58.3%)
Class C	9 (75.0%)	3 (25.0%)	0 (0%)
Total	10 (39.4%)	10 (39.4%)	13 (30.3%)

Symbolic Format. Students across the dataset reflected a preference for symbolic representations. In fact, the most commonly skipped tasks across all prompts where those in alternate representations. This preference was most notable in the final prompt asking students to generate a binary operation for set {1,2,4}. As seen in table 5, well over half of the students did not create a binary operation. Instead they often attempted to build a symbolic rule without any success. This desire for a symbolic rule is well documented in terms of functions, and it seems that students perceive a similar attribute as critical to binary operation.

Table 6: Survey Responses when Treating Associativity of ½(a+b) binary operations

	Focused on EOE Portion(s) of Operation (addition/multiplication)	Focused on Entire Operation
Class A	4 (44.4%)	5 (55.6%)
Class B	5 (41.2%)	7 (58.3%)
Class C	9 (75.0%)	3 (25.0%)
Total	18 (54.5%)	15 (45.5%)

E-O-E Format. The desire for a symbolic rule manifested further with students' perceiving a particular format as critical. This preference was seen throughout the survey responses but was most prominent when addressing the prompt to determine if ½(a+b) was associative. As seen in table 6, when placed in a situation without this formatting, students often defaulted to focusing on addition and multiplication as separate binary operations rather than holistically. Through the follow-up interviews, students who responded this way indicated that they could not discern the operation "rule" or that the operation was the "plus" portion reflecting the perception that E-O-E symbolism was critical.

Discussion and Conclusion

In this paper, we have addressed a variety of ways in which students conceive of binary operations in terms of critical attributes and identified metaphors they might leverage when working on tasks with binary operations. In particular, we found three metaphors that were commonly enacted by students: the function metaphor, the arithmetic metaphor, and the structure metaphor. Such metaphors can serve to be productive when they are robust and developed. However, that requires students to also have perceived the critical attributes of binary operation that can be structured and communicated via the metaphors. When leveraging a function metaphor, students may naturally attend to function attributes (such as the need for inputs), and not perceive a binary operation specific attribute (needing exactly two inputs). Similarly, students with an arithmetic metaphor may generalize from arithmetic examples and perceive attributes like E-O-E formatting as critical for general binary operations. As variation theorists have noted, variations are perceived not just synchronically, but also "by remembering earlier related experiences (diachronic simultaneity)" (Vikström, 2009, p. 212). As students structure their knowledge using metaphors, they are likely integrating attributes perceived from many prior experiences related to function and arithmetic in their earlier mathematics experiences.

While the links to function and arithmetic are immediate, the structure metaphor's link to prior experience is a less direct. Notably, the structure metaphor is a dominant metaphor for determining when mathematical structures are isomorphic. However, this is a new treatment at the undergraduate level. Yet many structural treatments exist throughout education such as finding patterns when adding even numbers, or connecting operations like multiplication and division. In fact, this played out robustly in our data. Many students identified multiplication and division as the same operation because of their similar behavior. A structural metaphor can support attending to important varying features (such as a structural property), but can also limit students' attention to the critical attribute of binary operations being element-defined.

These results have several implications. First, binary operations are not a trivial despite instructors classifying it as such. Instructors may wish to increase attention to binary operation in classes with this focus (such as abstract algebra and linear algebra.) Second, students may not perceive many critical attributes such as the need for two elements, and may perceive non-critical attributes as critical such as E-O-E formatting. From a variation standpoint, a potential solution is to expose students to examples, non-examples, and non-standard examples so that they have opportunities to contrast instantiations with varying critical and non-critical attributes. This type of activity can foreground these features so that students may perceive them. From a research standpoint, we may want to consider the degree to which these three metaphors, and attention to specific attributes may underlie student activity across the K-16 spectrum as they engage with common binary operations such as number arithmetic or function composition.

References

Brown, A., DeVries, D. J., Dubinsky, E., & Thomas, K. (1997). Learning binary operations, groups, and subgroups. *The Journal of Mathematical Behavior*, 16(3), 187-239.

Dahlin, B. (2001). The primacy of cognition–or of perception? A phenomenological critique of the theoretical bases of science education. *Science & Education*, 10(5), 453-475.

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics*, 27(3), 267-305.

Ehmke, T., Pesonen, M., & Haapasalo, L. (2011). Assessment of university students' understanding of abstract binary operations. *Nordic Studies in Mathematics Education (NOMAD)*, 15(4), 25-40.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Educatio*n, 25(2), 116-140.

Larsen, S. (2009). Reinventing the concepts of group and isomorphism: The case of Jessica and Sandra. *The Journal of Mathematical Behavior*, 28(2), 119-137.

Marton, F., & Booth, S. A. (1997). *Learning and awareness*. Psychology Press.

Melhuish, K. & Fagan, J. (Accepted / In Press). Connecting Group Theory Concept Assessment Results to Core Concepts at the Secondary Level. In N. Wasserman (Ed.), *Connecting abstract algebra to secondary mathematics, for secondary mathematics teachers (Research in Mathematics Education Series)*. Netherlands: Springer.

Melhuish, K, & Fasteen, J. (2016). Results from the Group Concept Inventory: Exploring the Role of Binary Operation in Introductory Group Theory Task Performance. In T. Fukawa-Connelly, N. Infante, M. Wawro, & S. Brown (Eds.), *19th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1098–1103). Pittsburgh, PA.

Novotná, J., Stehlíková, N., & Hoch, M. (2006). Structure sense for university algebra. Paper presented at the Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education.

Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. Making the Connection: Research and Teaching in Undergraduate Mathematics Education, 27, 42.

Slavit, D. (1998). The role of operation sense in transitions from arithmetic to algebraic thought. *Educational Studies in Mathematics,* 37(3), 251-274.

Tall, D., McGowen, M., & DeMarois, P. (2000). The function machine as a cognitive root for the function concept.

Trigwell, K. (2006). Phenomenography: An approach to research into geography education. *Journal of geography in higher education*, 30(2), 367-372.

Wasserman, N. H. (2017). Making Sense of Abstract Algebra: Exploring Secondary Teachers' Understandings of Inverse Functions in Relation to Its Group Structure. *Mathematical Thinking and Learning*, 19(3), 181-201.

Vikström, A. (2008). What is intended, what is realized, and what is learned? Teaching and learning biology in the primary school classroom. *Journal of Science Teacher Education*, 19(3), 211-233.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, 356-366.

Zandieh, M., Ellis, J., & Rasmussen, C. (2017). A characterization of a unified notion of mathematical function: the case of high school function and linear transformation. *Educational Studies in Mathematics*, 95(1), 21-38.

Zaslavsky, O., & Peled, I. (1996). Inhibiting factors in generating examples by mathematics teachers and student teachers: The case of binary operation. Journal for Research in Mathematics Education, 67-78.

REPRESENTATIONAL SAMENESS AND DERIVATIVE

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This study focuses on students' understanding of multiple representations of functions. It examines student responses to a task in which calculus students are asked to evaluate the derivative at a point of the cubing function when represented piecewise. Results suggest that attending to the graph of the piecewise function does not improve students' ability to differentiate it. Results also suggest that students tended to view a piecewise-defined function not as a singular function, but as a set of instructions for which function to use.

Keywords: function, derivative, multiple representations

Introduction and Literature Review

Multiple representations of functions play an important role in mathematics and mathematics education. There is a body of literature addressing college students' difficulties linking multiple representations of functions, and some studies suggest that post-secondary students struggle translating between different representations (Even, 1998; Gagatsis and Shiakalli, 2004; Chinnappan and Thomas, 2001). The literature on multiple representations tends to focus on translation between multiple *types* of representations (e.g., graphical, analytical, and verbal), rather than multiple representations of the same type. However, working with multiple representations of the same type is also a crucial part of mathematics. Although there is no body of literature specifically devoted to within-representation-type translation, some authors have highlighted the importance of this for specific types of representations. For example, Moore and Thompson (2015) make the point that math students should be able to move flexibly between different coordinate systems, being able to recognize when the same graph has two different visual representations.

I argue in Mirin (2017) that students' understandings of sameness of representation of function, by which I mean student assessments of which function representations represent the same function, are inextricably linked to their concept of function. We can see how one's concept of sameness-of-representation-of-function and the function concept itself are interlinked when we consider how a student might view derivative. If a student views a derivative as operating on a function, then his concept of function is inextricably tied to his concept of derivative. For example, his criteria for determining whether two function representations share a derivative might be influenced by his criteria for determining whether those representations refer to the same function. This leads us to the following research question: What are students' understandings of multiple analytic representations of a single function as it relates to derivative?

We follow Thompson's (1982) constructivist approach of being sensitive to student understanding by asking, "What is the problem that this student is solving, given that I have attempted to communicate to him the problem I have in mind?"(p.153). This is akin to Harel, Gold, and Simon's (2009) description of the "interpreting" mental act; in analyzing students' responses to a task, we, as researchers, pay careful attention to how students interpret a task. Thompson makes the point that, when referring to representations *of* something, we ought to be clear about *to whom these are representations* of whatever "something" is (Thompson & Sfard, 1994). In the case of this study, there is a representation of the cubing function *to us* (as mathematicians), but to students, it may not be. So, we ought to be sensitive to the fact that a student might agree with the assertion that two representations of the same function share a derivative, but these students might have non-standard understandings of what "same function" is. In fact, this is precisely the sort of reasoning a particular student used to determine that sharing a graph was not sufficient for sameness of functions; she concluded that two particular representations of functions share the same graph but do *not* share a derivative, leading her to conclude that, to be the same function, having the same ordered pairs on the graph is not sufficient (Mirin, 2017).

Task Design and Methodology

This study arose from an anecdote that Harel and Kaput (1991) share in which calculus students, when prompted to differentiate the function g defined piecewise by $g(x) = \sin x$ if $x \neq 0$ and $g(x) = 1$ if $x = 0$, answered with $g'(x) = \cos x$ if $x \neq 0$ and $g'(x) = 0$ if $x = 0$, appearing to use the constant rule. To these students, the only aspect of the representation as relevant for determining the value of $g'(0)$ is the second line of the piecewise function definition. It seems reasonable to believe that, if the definition of g were modified to instead have $g(x) = 0$ if $x = 0$ (resulting in a nonstandard representation of the sine function) students would answer identically. However, given the anecdotal nature of Harel and Kaput's claim, there is no data available to substantiate how common such errors are or why they occur.

This paper undertakes the task of studying this phenomenon more systematically. I designed the following task to address this issue:

Let f be the function defined by

$$f(x) = \begin{cases} x^3 & \text{if } x \neq 2 \\ 8 & \text{if } x = 2 \end{cases}$$

Evaluate $f'(2)$, and provide an explanation of your answer.

Figure 1. The Task on which This Study is Based (Quiz A)

Henceforth, the task of evaluating $f'(2)$ for f defined piecewise as above will be referred to as "The Task". Notice that the function f is simply the cubing function, but represented in a non-standard way. Whether students see it that way is part of the investigation.

Subjects and Methods

Initially, The Task, exactly as pictured in Figure 1, was given to 240 introductory calculus students during the last week of the semester at Anonymous State University (ASU). Referred to as "Quiz A", it was administered in an exam environment by course instructors as part of the course, where students were required to work silently and independently. I conducted follow-up interviews of eight individual students. I collected and coded their responses before performing the interviews, and, informed by the interviews, re-coded the responses to reflect students' rationales as suggested by the interviews.

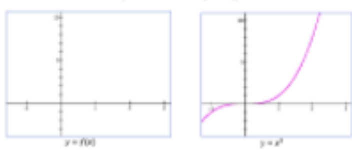
The interviews each lasted 60-90 minutes. The interviews operated according to clinical interview methodology (a la Clement, 2000) and served as establishing students' rationale for their responses to The Task. Additionally, students were given similar problems, as well as asked to graph the function *f* and asked to make sense of their answer to The Task with their graph of *f*.

Results, Quiz A vs. Quiz B

Results of Quiz A suggest that Harel and Kaput (1991)'s anecdote is indeed indicative of a wider phenomenon, as only 18.3% of students gave the correct answer. Like in Harel and Kaput's story, the majority (53.6%) of students claimed that the answer was "0". Further, many students (41.2% of total) explicitly cited the constant rule.

Now that we have established that there is a larger phenomenon, the next natural question to ask is, "why"? It is possible that some students erred due to inattention or carelessness, rather than a major misconception. That is, they might have simply seen the "8" and applied the constant rule out of habit or simply not realized that 2^3 is 8 and that the given function is in fact continuous. This would explain why some students answered "undefined," and it is also consistent with some of the graphs that students volunteered (graphs with removable discontinuities). Further, it might not have occurred to students to compare the graph of *f* with that of the cubing function - as discussed earlier, the piecewise-defined *f* is a representation of the cubing function to *us*, but perhaps not to students.

Suppose *f* is the function defined by $f(x) = x^3$ if $x \neq 2$, $f(x) = 8$ if $x = 2$.
1. Calculate 2^3
2. Below to the right is a graph of $y = x^3$. On the blank axes provided, graph $y = f(x)$ for the function *f* defined above.



3. Evaluate $f'(2)$ for f defined above. Please provide a 1-3 sentence explanation of your answer. (a)0, (b)8 (c)12, (d)undefined, (e)0 if $x = 2$, and 12 if $x \neq 2$, (f) 0 if $x = 2$, and undefined if $x \neq 2$.
4. Suppose g is a function and h is a function. What does it mean for g and h to be the same function?

Figure 2. Quiz B (Visually Condensed)

Accordingly, informed by Quiz A results, Quiz B (Figure 2, above) was created to test this possibility that inattention accounts for student responses. Quiz B involves The Task (multiple choice form), except, prior to attempting The Task, students are prompted to calculate 2^3 and to graph $y = f(x)$ aside a provided graph of $y = x^3$. Also included on Quiz B is a task asking students to state their definition of when a function *g* is the same function as a function *f*. If inattention accounts for student responses, then the following hypotheses should hold:

(1) Overall, students will perform significantly better on The Task in Quiz B than Quiz A, leaving open the possibility that inattention or carelessness could account for students' tendency to do poorly on The Task in isolation. Students might, because of the prompting, be more likely to compare *f* to that of the cubing function.
(2) Quiz B students who answered "12" would be more likely than Quiz A students who answered "12" to provide a justification involving the comparison of *f* with the cubing function.
(3) Students who provided a mathematically normative definition of function sameness (Problem 4) would be more likely to answer "12" than students' who did not.

As discussed earlier, a student's criteria for determining whether two function representations share a derivative might be influenced by his or her criteria for determining whether those representations refer to the same function. If a student believes that having the same set of ordered pairs is sufficient for function h to be the same as function g, then it seems she is more likely (than a student who does not believe this) to conclude that because g and h are the same function, their respective derivatives are the same function. These hypotheses center on the idea that if students are positioned to compare the ordered pairs on $y = f(x)$ to that of $y = x^3$, they are more prone to answer The Task correctly.

Table 1: Student Answers to Quiz A and Quiz B

	0	**8**	**12**	**Undefined**	**Multiple answers**	**Other**	**Blank**	**Total**
Quiz A %, n	56.3%, 135	4.6%, 11	18.3%, 44	5.8%, 14	(8.3%, 20) 0&12: (5.8%, 14) 0&undef: (2.5%, 6)	5.4%, 13	1.3%, 3	100%, 240
Quiz B %, n	40.2%, 41	7.8%, 8	23.5 %, 24	9.8%, 10	(18.7%, 19) e: (12.8%, 13) f: (5.9%, 6)	N/A	0%, 0	100%, 102

The data reveal no evidence to support that inattention could account for student responses. Although there was a slight improvement in correctness rate from Quiz A to Quiz B (see Table 1), this improvement was not statistically significant (χ^2=1.21, p>.05), contrary to (1). In other words, prompting students to compare the graph of $y = f(x)$ to that of $y = x^3$ *did not* appear to cause improvement, suggesting that students did not err simply due to inattention to the function's graph. Moreover, the Quiz B students who answered "12" were no more likely than the Quiz A students who answered "12" to draw an explicit comparison between *f* and the cubing function (4.2% of Quiz A students who answered 12 did, whereas only 2.9% of Quiz B students did so), contrary to (2). Also, the students who provided a mathematically normative definition of function sameness were no more likely to answer "12" than those who did not, contrary to (3). These results suggest that, contrary to my hypotheses, prompting students to compare *f* to the cubing function did not appear to encourage them to infer that *f* and the cubing function share a derivative at 2. This naturally led to the emergent question: if inattention to the graph of *f* does not account for students' tendency to answer incorrectly, then why are students answering the way they are answering?

Phenomena and Student Rationale

To answer this question, we turn to the student graphs together with the student interviews. Normatively, two graphs (of functions) are the same if and only if they consist of the same ordered pairs. It seems reasonable to believe that some students might *not* have this criterion for sameness of graph; indeed, interviews suggested that some students viewed a graph of x^3 with an extra "dot" placed at (2,8) as different from a graph of x^3 without one. Some students referred to the point (2,8) as "separate". Accordingly, a sub-category (category B) of "correct" was created: mathematically normative graphs that highlighted (2,8) in the sense that they had a dot on (2,8) that was more prominent than any other dots. The remaining "correct" graphs - those that were correct but indicated nothing special about (2,8) - were grouped together as category. The remaining graphs were classified as follows: those with a single dot at (2,8) (2.0%), those with just a graph of $y = 8$ (6.9%), those with a removable discontinuity at x=2 (6.9%), blank (4.9%), those with graphs of both $y = 8$ and $y = x^3$ (2.9%), and other (8.8%). Among the correct graphs (A and B), graph A students were more likely than graph B students to answer "12" on The Task (χ^2=3.932, p<.05), suggesting some sort of difference (in the graph B students' minds) between *f* and the cubing function. How students understand their graphs in relation to The Task will be further elaborated below.

Now we turn to the qualitative data: the student interviews, which shed light on why students answer the way they do. Musgrave and Thompson's (2014) construct of "function notation as idiom" was useful in accounting for student responses. A student views function notation idiomatically when he or she views "$f(x)$" in its entirety as a name for a function (Musgrave & Thompson, 2014). Such students might view "$f(x)$" as no more than another name for "*y*" (Thompson, 2013). It appeared that many students used this way of thinking when evaluating $f'(2)$, as students seemed to view "*8*" and "x^3" as names of functions with "$f(x)$" referring to both of these functions. Another common theme, appearing both on the written quizzes and in the interviews, was the viewpoint that the "if $x \neq 2$" served as a *restriction* on the domain for students, rather than as a condition.

Since there is not space to discuss every student in detail, we provide insight from the interviews that is consistent with various student responses. The following subsections should be viewed as *descriptions and illustrations* of student thinking that explain students' answers, rather than rigorous evidence of such phenomena. Moreover, we discuss only the parts of the interviews that explain why students answered the way they did originally, rather than elaborating on the in-depth portions that were more exploratory. Each subsection begins with a direct, written quote from a student, which provides a concise summary of the way of thinking described in the subsection. We also discuss how, for the students, the point (2,8) was special and the way students made sense of their graphs. Additionally, we discuss how students' ways of thinking are reflected in their responses to the interview prompt to find *h'(5)* for the function *h* defined by $h(x) = x^3$ if *x≠5*, $h(x) = x^2 + 100$ if *x=5*.

Students who answered "0"

"When the graph is at the point *x=2*, the function is determined by the piecewise part '*8*'. So, $f(x)$ *itself* equals *8*. When *8* is derived, it becomes 0" [Pete, Quiz B student (emphasis added)]. The rationale summarized by Pete appears to exemplify a common way of thinking amongst students who answered "0". For these students, the "$f'(2)$" tells them that they are in the situation "$x = 2$," which serves as an instruction to use the function "*8*". Here, the "*8*" serves as a name of a function rather than a particular output, suggesting an idiomatic conception. Many of these students provided a category B graph of *f* (graph of $y = x^3$ but a special dot at (2,8)) and found no issue with the fact that they couldn't "see" that $f'(2) = 0$ in their graph; when asked to explain graphically, they would provide a graph of $y = 8$ and explained why its derivative at 2 is 0.

Interviewed students extended this way of thinking to evaluating *h'*(*5*) for the function *h* defined by $h(x) = x^3$ if *x≠5*, $h(x) = x^2 + 100$ if *x=5*. It was common for students to answer "10" by evaluating the derivative of $x^2 + 100$ as 2*x* and substituting *x*=5 to result in 10, with the rationale that "Um I used this part, the part that makes the parabola [$y = x^2 + 100$]. Because we're interested in the time when x equals 5. And that's kind of the rule here, when *x* equals 5 to use the parabola" [Jennifer, Quiz A student]. She elaborated: "The derivative of *h* when *x* equals 5 is gonna be *2x* um…..if *x* were to equal some number other than 5, you would use this (underlines x^3) function up here, but because *x* is 5 we use this one." Jennifer's rationale exemplifies the way of thinking that led students to answer "$f'(2) = 0$": viewing the conditions on a piecewise-defined function as instructions for which function to use, and a piecewise-defined function involving two different functions.

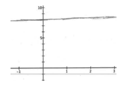
$f'(2) \rightarrow x = 2 \rightarrow y = 8 \rightarrow$  $\rightarrow y' = 0$

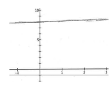
Figure 3. Student rationale for answering "0"

Many of these students (Pete included) provided a graph of *f* that was like $y = x^3$ but with a dot at (2,8) (category B graph). It seems that students viewed the dot at (2,8) as separate or independent from the rest of the graph. For example, one student recreated his graph during the interview, explaining his reasoning as follows: "At the point (2,8) I draw a circle to show there is an opening there, there's a gap. I'm excluding that point from what it is we are talking about in this point in time." He elaborated: "So the *two*...they're existing on the same coordinate system but *existing independent of each other*" (emphasis added).

Students who answered both "12" and "0"

"If *f(x)* does not equal 2, the function is x^3. The derivative of x^3 equals $3x^2$, then substitute 2 for *x,* $3(2)^2=12$. However, if *x* is allowed to be 2, then the derivative of *8*=0" [Carlos, Quiz A student]. The case of Carlos illustrates how a student can reason idiomatically to get the answers 0 and 12. In the interview he reiterated his reasoning: "If *x* isn't 2 then the function is x^3. The derivative of x^3 is $3x^2$. Then substitute 2 for *x* here and you get 12. However, if *x* is allowed to be 2, then the derivative of *8* is 0". For Carlos, the "if *x=2*" condition told him that he was in the case in which "the function" is the function "$f(x) = 8$," and that the "if $x \neq 2$" condition told him he was in the case in which "the function" is x^3. Carlos did not even make the connection that the "2" in "$f(2)$" told him he was in the case where "$x = 2$"; for him, the "$f(x)$" was just a shorthand for "*y*". When prompted to graph *f*, he provided a graph of (what he thought was) *y=8* as well as a graph of $y = x^3$, indicating that he viewed himself as graphing two separate functions. When asked how $f'(2)$ can be 12 while he had said prior that it was 0, he explained: "this is an *entirely different function"*, indicating that the conditions on the piecewise function were instructions about which function to use.

Two cases: $x = 2,\ x \neq 2$

$x = 2 \rightarrow y = 8 \rightarrow$  $\rightarrow y' = 0 \rightarrow f'(2) = 0$ if $x = 2$

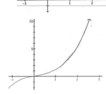
$x \neq 2 \rightarrow y = x^3 \rightarrow$  $\rightarrow y' = 12 \rightarrow f'(2) = 12$ if $x \neq 2$

Figure 4. Student Rationale for answering "0 if $\boldsymbol{x = 2}$, 12 if $\boldsymbol{x \neq 2}$"

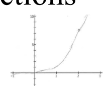
Carlos' way of thinking was confirmed when he was asked to calculate *h'*(*5*) when *h* is defined by by $h(x) = x^3$ if $x \neq 5$, $h(x) = x^2 + 100$ if *x*=*5*. He graphed $y = x^3$ and $y = x^2 + 100$ on the same axes (Fig.4). When prompted to find the value of *h'*(*5*), he differentiated x^3 and plugged in 5 to get 75, and then he differentiated $x^2 + 100$ and plugged in 5 to get 10. When asked which was the value of *h'*(*5*), he exclaimed confidently, "both! 75 and 10!".

Students who answered both "0" and "undefined"

"If just looking at $f(x) = 8$, the derivative of a constant would make $f'(2) = 0$. If just looking at $f(x) = x^3$, the derivative would be undefined because $f(2)$ is not on the graph of x^3.

There is a hole at x=2" [Eric, Quiz B]. Eric's reasoning exemplifies how students could have come to select choice "f" in Quiz B. A different student, Sarah, explained her reasoning in detail in the interview. Sarah initially answered that "both" are undefined, but during the interview, she revealed that she interprets "0" to mean the same thing as "undefined" (which was a common trend in student responses). Like Carlos, she viewed two functions as being involved, which was again confirmed when she was asked about the piecewise-defined function "h". She appeared to reason about two different functions, and calculated $f'(2)$ by treating the first function as "$y = x^3, x \neq 2$", and the second function as "$y = 8, x = 2$". She interpreted the "$x \neq 2$" as a restriction on the *first* function, and the "$x = 2$" a clarification that such a restriction did not exist on the *second* function. Thus, for the first function, $f'(2)$ is undefined, and for the second function, $f'(2)$ equals 12.

Two cases, two functions

$y = x^3, x \neq 2 \rightarrow$  $\rightarrow f'(2)$ undefined if $x \neq 2$.

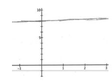
$y = 8, x = 2 \rightarrow y = 8$ and x can be 2 $\rightarrow$  $\rightarrow f'(2) = 0$ if $x = 2$.

Figure 5. Student Rationale for answering "0 if $x = 2$, undefined if $x \neq 2$"

Conclusions and Discussion

The results of Quiz A showed us that Harel and Kaput's (1991) anecdote is indeed indicative of a larger phenomenon: many students appeared to differentiate a piecewise function formally by differentiating each expression as a separate function. By comparing the results of Quiz A to Quiz B, we confirmed that this phenomenon cannot be attributed merely to inattention. The interviews, together with the Quiz B results, suggest that a non-normative understanding of piecewise function notation, stemming from a view of function notation as idiom and the conditions on the domain as either instructions or as restrictions, accounts for many students' responses.

This study shows that students do not view the same function, represented in two different analytic ways, as sharing a derivative at a particular value. However, this last sentence was ambiguous; when we say "a function", we are not being clear if students view these function representations as referring to the *same* function. Students might, for example, consider it possible for two *distinct* functions to share a graph, and we can ask: do students believe that same graph implies same derivative? The answer to this appears to be "no," as many Quiz B students provided normative graphs of f yet did not evaluate $f'(2)$ normatively. Yet, we run into another ambiguity: what students view as "same graph" might not be consistent with the normative notion of "same graph," as suggested by students' insistence that the point (2,8) being highlighted. This means that, although it is tempting to conceptualize this study as one about students' understanding of derivative, its results highlight how students think about function notation. To illustrate this point consider the way of thinking that accounted for students answering "0." It arose from a misconception of function notation: no matter how strong of a meaning the student has of "derivative", the student was still reasoning with the graph of "$y = 8$", leading to an answer of "0".

As discussed earlier, it seemed reasonable to hypothesize that students who provided normative definitions of what it means for functions g and h to be the same (Problem 4, Quiz B) would be more likely to correctly evaluate $f'(2)$; this is because it seems these students would be more likely to assess piecewise-defined *f* and the cubing function as "the same," positioning them to infer that *f* and the cubing function share a derivative. In light of the interviews and students' ways of thinking, the counter-intuitive result – that this hypothesis did not hold – makes sense. This is because, to students, *f* was not a function in the same way that the cubing function is; instead, *f* was *two functions*. Having a strong criteria for sameness of functions did not help students evaluate $f'(2)$ because *f* was not in the category of "functions" to which sameness can apply!

References

Chinnappan, M., & Thomas, M. O. J. (2001). Prospective teachers' perspectives on function representations. In J. Bobis, B. Perry, &M. Mitchelmore (Eds.), *Proceedings of the 24th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 155–162). Sydney, Australia.

Clement, J. (2000) Analysis of clinical interviews: Foundations and model viability. In Lesh, R. and Kelly, A., *Handbook of research methodologies for science and mathematics* education (pp. 341-385). Hillsdale, NJ: Lawrence Erlbaum.

Even, R. (1998). Factors involved in linking representations of functions. *The Journal of Mathematical Behavior*, 17(1), 105–121.

Gagatsis, Athanasios & Shiakalli, Myria. (2004). Ability to Translate from One Representation of the Concept of Function to Another and Mathematical Problem Solving. *Educational Psychology Review*, 24(5), 645–657.

Harel, G., Gold, B., & Simons, R. (2009). What is Mathematics? A Pedagogical Answer to a Philosophical Question. In B. Gold & R. Simons (Eds.), *Proof and other Dilemmas* (pp. 265–290). Washington DC: The Mathematical Association of America.

Harel, G, & Kaput, J. (1991). Conceptual Entities. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 82–94). Kluwer Academic Publishers.

Mirin, A. (2017). Function Sameness to 'Function' Meaning. In *Proceedings of the 20th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education*. San Diego, CA: SIGMAA/RUME.

Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In T. Fukawa-Connelly, N. Infante, K. Keene, & M. Zandieh (Eds.), *18th Annual Conference on Research in Undergraduate Mathematics Education (pp. 782–789). Pittsburgh, PA: West Virginia University.*

Musgrave, S., & Thompson, P. W. (2014). Function Notation as Idiom. *Proceedings of the Joint Meeting of PME 38 and PME-NA 36*, 281.

Thompson, Patrick W. (1982). Were lions to speak, we wouldn't understand. *Journal of Mathematical Behavior*, 3(2), 147-165.

Thompson, P. W., & Sfard, A. (1994). Problems of reification: Representations and mathematical objects. in D. Kirshner (Ed.) Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education -- *North America, Plenary Sessions* (Vol. 1, pp.1-32). Baton Rouge, LA: Louisiana State University.

Thompson, P. W. (2013). Why use f(x) when all we really mean is y. *OnCore, The Online Journal of the AAMT*, 18–26.

BUSINESS CALCULUS STUDENTS' INTERPRETATIONS OF MARGINAL CHANGE IN ECONOMIC CONTEXTS

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Despite the overwhelming amount of research on students' interpretations of rates of change in real-world physical science contexts, similar studies in real-world economic contexts are sparse. Contributing towards addressing this gap in knowledge, this study reports on how 12 pairs of business calculus students interpreted marginal change (marginal cost and marginal revenue) in different economic contexts and function representations while solving two optimization tasks. Analysis of task-based interviews conducted with the students revealed that a majority of the students interpreted marginal change as an amount (the difference) and not as a rate (the difference quotient). A few students interpreted marginal change as the derivative. Implications for instruction are discussed.

Keywords: rates of change, marginal change, business calculus, optimization problems

Much research has reported on students' difficulties with interpreting rates of change (average rate of change and instantaneous rate of change) in real-world physical science contexts such as in fluid dynamics, kinematics, and thermodynamics (e.g., Carslon, Jacobs, Coe, Larsen, & Hsu, 2002; Prince, Vigeant, & Nottis, 2012). However, research on how students, especially at the undergraduate level, interpret rates of change (herein referred to as marginal change) in real-world economic contexts is limited, which is the motivation for this study. According to Gordon (2008), more than 300,000 students take business calculus in the United States each year. Understanding marginal change in a business context is vital in fields such as marketing, managerial accounting, supply chain management, finance, and economics. This study reports on business calculus students' interpretations of marginal change while reasoning about two optimization tasks (shown in the methods section), which are situated in two different economic contexts that have different function representations. Our study was guided by this research question: How do business calculus students interpret marginal change when solving optimization problems that have different function representations and are situated in different economic contexts?

The term marginal change, as used in this study, refers to marginal cost or marginal revenue. Marginal cost is the cost per additional unit produced and marginal revenue is the revenue generated per additional unit sold. Mathematically, marginal change can be calculated as an average rate of change where the length of the interval of change is one unit. Marginal change can be approximated using the instantaneous rate of change.

Related Literature

Rates of Change in Context

Several studies have reported on students' tendency to conflate the rate of change of a quantity with either the amount of the quantity or the amount of change of the quantity when solving application problems that are situated in real-world physical science contexts (e.g., Lobato, Hohensee, Rhodehamel, & Diamond, 2012; Prince et al.,2012; Rasmussen & Marrongelle, 2006). A majority of the 373 engineering students in Prince et al.'s (2012) study had difficulty distinguishing between the rate of change and the amount of change of a quantity in a thermodynamics context. Rasmussen and Marrongelle (2006) reported on students who failed to make "a conceptual distinction between rate of change in the amount of salt and amount of salt" (p. 408) in a differential equations course. In a remote-controlled airplane task where the amount of change had the same numerical value as the rate of change (as it is in the economic context of marginal change), Lobato et al. (2012) reported that a majority of the 24 students who participated in their study confused the distance traveled by the airplane in the task with the speed of the airplane. The study reported in this paper extends the body of research on students' reasoning about rates of change from the physical sciences to economic contexts by examining students' interpretations of marginal change in two different profit maximization contexts.

Rates of Change in Multiple Function Representations

The importance of the concept of function in students' learning of calculus cannot be overemphasized. Oehrtman, Carlson, and Thompson (2008) argued that "the concept of function is central to undergraduate mathematics, foundational to modern mathematics, and essential in related areas of the sciences" (p. 27). In his review of research literature on students' understanding of functions, Thompson (1994c) argued that "tables, graphs and expressions might be multiple representations of functions" (p. 23) to instructors and researchers, but that they are not "multiple representations of anything to students" (p. 23). Thompson's argument is well supported by evidence from research on students' understanding of rates of change in multiple function representations (e.g., Klymchuk, Zverkova, & Sauerbier, 2010; Villegas et al., 2009). Much of the existing research on students' understanding of rates of change in multiple representations is limited to graphical, algebraic, and textual representations of functions. This study extends the literature on students' understanding of rates of change by reporting on students' interpretations of rates of change in two functional situations, one that is based on a numerical table of values and another that is algebra-based.

Theoretical Perspective

The current study draws on the theory of quantitative reasoning (Thompson, 1993; Thompson, 1994b; Thompson, 2011). Quantitative reasoning is the analysis of a situation in terms of the quantities and relationships among the quantities involved in the situation (Thompson, 1993). Thompson (2011) described three tenets (a quantity, quantification, and a quantitative operation) that are central to the theory of quantitative reasoning. A quantity is a measurable attribute of an object. Thompson (1993) distinguished between a quantity and a numerical value: a quantity has a unit of measurement, and a numerical value does not. Examples of quantities in this study include marginal cost and marginal revenue.

Quantification is the process of assigning numerical values to quantities (Thompson, 1994d). A quantitative operation is the process of forming a new quantity from other quantities (Thompson, 1994b). In economics, for example, comparing (by way of finding the difference) marginal revenue and marginal cost with the intent to find the excess of marginal revenue against marginal cost otherwise known as marginal profit is a quantitative operation known as a quantitative difference. We designed two mathematical tasks (shown in the next section) that provided students with opportunities to reason about quantities and relationships among quantities. The interview protocol used during data collection engaged pairs of students in reasoning about relationships among quantities. We believe that having pairs of students share ideas while solving the tasks helped to reveal students' interpretations of the concept of marginal change in detail, something that could have been harder to achieve when interviewing individual students. The interviewing of pairs of students further shifted the students' focus from the researcher to the tasks.

Methods

Setting, Participants, and Data Collection

The study participants were 24 undergraduate students at a research university in the United States who had recently completed a business calculus course. The students were chosen based on their willingness to participate in the study, their major (business or economics), and their prior exposure to the ideas of marginal cost and marginal revenue through the course textbook and course lectures. The cumulative grade point averages (GPAs) of the 24 students had a mean of 3.43 on a 4.0 scale, a standard deviation of 0.37, and a range of 1.56. All but two of the students had earned at least a B grade in their business calculus course. The students were recruited from five different sections of a business calculus course.

We remark that the concept of marginal change was poorly presented in the course textbook and in course lectures. Specifically, in both the textbook and in course lectures, marginal change was defined as a rate (the difference quotient) and interpreted as an amount (the difference) (Mkhatshwa, 2016). As we will show in the results section, we argue that defining marginal change as a rate and interpreting it as an amount as was done in course lectures and in the textbook had an influence on students' reasoning about the units of marginal change in Task 2 (and likely in the other tasks we gave to the students) that appears in the following section. Part of the difficulty is that the numeric value of the difference is the same as the numeric value of the difference quotient since the denominator is one in the case of marginal change, a difficulty also noted by Lobato et al. (2012) in a kinematics context. In many cases the "per additional unit" may have been held implicitly by the students when giving units of marginal change in dollars instead of dollars per additional unit. We further remark that students' opportunities to reason about marginal change in course lectures and via the course textbook were mainly limited to the use of algebraic tasks.

Data for the study consisted of (1) transcriptions of audio-recordings of task-based interviews (Goldin, 2000) conducted with 12 pairs of students and (2) work written by the 12 pairs of students during each task-based interview session. Each interview lasted for about one hour and fifteen minutes. The interviews covered four economic tasks, and this study reports on two of the tasks (herein referred to as Task 1 and Task 2). Both tasks are situated in a profit maximization context. In addition, while both tasks are textually represented, Task 1 is also algebraic and Task 2 is numeric (table-based). We describe the design of each of the two tasks:

Task 1 (Haeussler, Paul, & Wood, 2011, p. 617): A manufacturer can produce at most 120 units of a certain product each year. The demand equation for the product is $p = q^2 - 100q + 3200$ and the manufacturer's total cost function is $c = \frac{2}{3}q^3 - 40q^2 + 10{,}000$, where q denotes the number of units that are produced and sold. Find the maximum profit.

We designed Task 1 to examine students' reasoning about optimization problems similar to those given in their textbook and in course lectures. We used this task to examine students' interpretation of marginal change in a profit maximization context when given a task that has an algebraic representation. We also designed this task to examine students' reasoning about relationships among four quantities: the number of units produced and sold, total cost, total revenue (which can be obtained by multiplying the demand equation given in the task by q, the number of units that are sold), and profit (which can be obtained by finding the difference between total revenue and total cost).

Task 2: The following table shows the marginal revenue (MR) and marginal cost (MC) at various production and sales levels (q) for SciTech, a company that specializes in producing and selling computer chips. The company knows that total revenue is greater than total cost at all the production and sales levels shown on the table.

q (units)	400	401	402	403	404	405
MR (marginal revenue)	58	56	55	54	53	51
MC (marginal cost)	52	54	55	57	60	62

What advice can you give to the management of the company about when to increase or decrease production and sales of computer chips?

We designed Task 2 to examine students' reasoning about relationships among four quantities: the number of computer chips produced and sold, marginal cost, marginal revenue, and profit. We used this task to examine students' interpretations of marginal change (e.g., the cost of producing the 401st computer chip) in a profit maximization context when given a task that has a numeric (table-based) representation. We also used Task 2 to examine students' reasoning about the units of marginal change, that is, as a rate (the difference quotient) or as an amount (the difference).

Data Analysis

Data analysis was done in two stages. In the first stage, we carefully read through each interview transcript and coded instances where pairs of students interpreted marginal change (marginal cost or marginal revenue) while they reasoned about each of the two tasks. Some of the codes that emerged from the data (through an inductive process) include interpreting marginal change as an amount i.e. the difference (e.g., saying the units of marginal cost in Task 2 are dollars), interpreting marginal change as a rate i.e. the difference quotient (e.g., saying the units of marginal cost in Task 2 are dollars per unit), and interpreting marginal change as the derivative (e.g., saying the derivative of the total cost function in Task 1 is marginal cost).

In the second stage of our analysis, we did a comparison across tasks to see how interpretations of marginal change given by pairs of students changed (or did not change) across the representations and contexts of the tasks they were given. For example, if a pair of students interpreted marginal change as the derivative in Task 1 (a task with an algebraic representation) and marginal change as an amount in Task 2 (a task with a numerical representation), we concluded that the students' interpretations of marginal change varied with the representation and context of the task they were given. The comparison made in the second stage provided answers to our research question: - How do business calculus students interpret marginal change when solving optimization problems that have different function representations and are situated in different economic contexts?

In light of the tenets of the theory of quantitative reasoning, and as we would show in the next section, some of the students mentally performed quantitative operations such as by conceptualizing the quantity of marginal cost as the derivative of the total cost function given in Task 1. We state as a remark that a majority of the students in this study performed a quantitative operation by conceptualizing a measure of profit as the amount of excess revenue above cost when creating a formula for the quantity of profit. These students engaged in the process of quantification (assigning a numerical value for the quantity of profit) by evaluating the profit formula at a particular production and sales level to find the amount of profit at that level. In our previous work (Mkhatshwa & Doerr, 2018), we showed that some of the students in this study performed a quantitative operation by conceptualizing a measure of the quantity of marginal profit as the difference between marginal revenue and marginal cost while working on Task 2.

Results

Analysis of interview transcriptions and work written by students revealed three findings. Because of space limitations, this paper reports on two of the findings. First, nearly all the pairs of students interpreted marginal change as an amount (the difference) and not as a rate (the difference quotient). Second, three pairs of students interpreted marginal change as the derivative.

Interpreting Marginal Change as an Amount

With the exception of one pair of students (Yuri and Kyle), all the other pairs of students indicated that the units of the marginal cost (MC) and marginal revenue (MR) values given in Task 2 would be in dollars. When asked to justify why the units would be in dollars, a majority of these students indicated that "since we are in the United States," the units must be in dollars. We interpreted the students' claim that the units would be in dollars to mean that they interpreted marginal change as a change (the difference), and not as a rate of change (the difference quotient). A critical reader may argue that this may have been less of a conceptual issue and more of a language issue. For example, it is common for people to speak of marginal cost in units of dollars while holding the "per unit" implicitly. However, these students consistently referred to marginal cost in terms of units of a change in total cost (i.e. dollars) and never mentioned the "per additional unit" while reasoning about marginal cost in other tasks. While this may have been a language issue, we argue that because of the consistency at which the students referred to units of marginal cost as dollars and not as dollars per additional unit that the students were interpreting marginal cost as an amount (the difference) and not as a rate (the difference quotient). As noted earlier, part of the difficulty may have been that that the numeric value for the difference and the numeric value for the difference quotient are equal since the denominator in the difference quotient is one in the case of marginal change.

Yuri, however, stated that the units of the MC and MR values in the table shown in Task 2 would be in "dollars per unit" suggesting that he interpreted marginal change as a rate (the difference quotient). The following excerpt illustrates Yuri and Kyle's reasoning about the MC and MR values in Task 2.

Researcher: What are the units of these numbers [pointing at the MR and MC values in Task 2]?
Kyle: Dollars
Researcher: Yuri?
Yuri: Dollars per unit
Kyle: or cents
Researcher: Yuri, why did you say dollars per unit?
Yuri: Marginal revenue is additional, extra revenue per unit
Researcher: Tell me more about that

Yuri: [Silence]

Yuri's statement that "marginal revenue is additional, extra revenue per unit" in the above excerpt suggests that Yuri interpreted marginal change as a rate, the rate [R(q+1)-R(q)]/1, for a total revenue function R(q) where q is the number of units sold. Kyle's initial response that the units of the marginal cost and marginal revenue values are "dollars" suggests that he was interpreting marginal cost and marginal revenue as total cost and total revenue respectively. This was confirmed later in the interview when Kyle stated that the company mentioned in Task 2 breaks even (i.e., total cost equals total revenue) at a production and sales level of 402 computer chips in the table shown in the task when, in fact, marginal cost equals marginal revenue. However, when he added "or cents" after Yuri had said that the units of marginal cost and marginal revenue would be "dollars per unit," Kyle was either agreeing with Yuri that the units were dollars per unit or cents per unit, or he was still interpreting marginal cost as total cost. Regardless of the units given by each of the 24 students for marginal cost and marginal revenue, the fact that these students assigned units to the marginal cost and marginal revenue values suggests that they interpreted marginal cost and marginal revenue as quantities and not as numerical values.

Interpreting Marginal Change as the Derivative

In this study, interpreting marginal change as the derivative refers to an understanding of the quantity of marginal cost (or marginal revenue) as the result of differentiating a cost function (or a revenue function) and not as a quantity that has rate-related units such as dollars per unit. Three pairs of students interpreted marginal change as the derivative while reasoning about Task 1 and Task 2. Two of these pairs of students interpreted marginal change as the derivative only in an algebraic representation (Task 1) and the other pair of students interpreted marginal change as the derivative only in a tabular representation (Task 2). None of these three pairs of students consistently interpreted marginal change as the derivative in both tasks (Task 1 and Task 2) or even in the other two tasks reported in the larger study (Mkhatshwa, 2016). Alan and Sarah are representative of the two pairs of students who interpreted marginal change as the derivative while reasoning about how to solve the problem posed in Task 1. The following excerpt, which occurred early in the interview, illustrates how Sarah and Alan reasoned about what they needed to do in order to answer the question posed in Task 1.

Sarah: Take the derivative of the demand equation $[p = q^2 - 100q + 3200]$
Researcher: What do you get when you take the derivative of the demand equation?
Alan: Is it the marginal?
Sarah: That would be the marginal cost
Researcher: What is marginal cost?
Sarah: The derivative of the total cost [function], right?
Alan: Yah yah, you are right

In the above excerpt, Alan wondered if taking the derivative of the demand equation would give them "the marginal" while Sarah stated that by taking the derivative of the demand equation what they will get "would be the marginal cost." Alan's wondering about the derivative of the demand equation being "the marginal" and Sarah's assertion that marginal cost is "the derivative of the total cost" were taken by the researchers to be the students' interpretations of the derivative of the demand equation by Alan (or the total cost function by Sarah) as marginal cost. Alan and Sarah, did not reason any further about the idea of marginal cost in solving the problem posed in Task 1. It would appear that these students associated the act of taking the derivative of an equation (e.g., the total cost function in the case of Sarah) with the term "marginal."

Joy and Nancy are the only pair of students who interpreted marginal change as the derivative while reasoning about the units of the MC values and MR values in Task 2. The following excerpt, which occurred towards the end of Task 2, illustrates how Joy and Nancy reasoned about the units of the MC and MR values in Task 2.

Researcher:	What do you think are the units of these numbers [pointing at the MR and MC values in Task 2]?
Nancy:	Oh, dollars.
Joy:	Dollars.
Researcher:	How do you know it's dollars?
Joy:	Because revenue and cost is dealing with money
Researcher:	But that's marginal cost and marginal revenue, is it the same thing?
Nancy:	Yah
Researcher:	Joy?
Joy:	I think, if you like take the, like if you take the derivative of the revenue it gives you the marginal revenue…

Joy's statement that "if you take the derivative of the revenue, it gives you the marginal revenue" suggests that she was interpreting the marginal revenue as the derivative of the revenue function. There is, however, no evidence that Nancy also thought the same way even though she did not object to Joy's statement about the derivative of the revenue function being the marginal revenue. Joy and Nancy went on to calculate differences between the marginal revenue values and marginal cost values at each production and sales level shown in the table that appears in Task 2. They referred to these differences as "profit," suggesting that these students also interpreted marginal cost as total cost and marginal revenue as total revenue as profit is generally defined as the difference between total revenue and total cost, and this is how profit was defined in the textbook used by the students and during course lectures.

Discussion and Conclusions

Nearly all the pairs of students who participated in this study interpreted marginal change as an amount (the difference) and not as a rate of change per unit of one (the difference quotient). These students stated that the units of marginal cost and marginal revenue in Task 2 would be dollars instead of dollars per unit. Throughout their reasoning about marginal change in other tasks (including two other tasks not included in this paper), the students consistently referred to marginal change in units of dollars and not in units of dollars per unit. We argue that part of the difficulty in this may have been the fact that in the case of marginal change, the numeric value for the difference and that of the difference quotient are equal as the denominator in the difference quotient is one. Only one student interpreted marginal change as a rate of change per unit of one. This student stated that the units of marginal cost and marginal revenue in Task 2 would be dollars per unit. As noted earlier, similar results were reported by Lobato et al. (2012) in a kinematics context. To some extent, the students' interpretation of marginal change as both a difference and a difference quotient can be attributed to the opportunities in the textbook they had to learn about marginal change. Specifically, marginal change was defined as a rate per unit of one (the difference quotient) and interpreted as an amount (the difference) in the textbook used by the students (Mkhatshwa, 2016). Also, the presentation of marginal change in course lectures closely followed the presentation of marginal change in the textbook (Mkhatshwa, 2016).

A majority of the students' interpretations of marginal change tended to change within and across the tasks they were given. For example, Sarah and Alan interpreted marginal change as the derivative in Task 1 but then they switched to interpreting marginal change as an amount of change with units of dollars in Task 2. Given that the tasks had different function representations and were situated in different economic contexts, these results suggests that the students in our study had weak understandings of the idea of marginal change in different situations. Specifically, these results suggest that students' interpretations of marginal change varied in different economic contexts and representations of economic situations. As noted in our previous work (Mkhatshwa, 2016), opportunities for students to interpret marginal change (e.g., marginal cost) in the textbook and in course lectures were limited. We argue that by providing more opportunities for students to interpret marginal change in different economic situations and function representations, business calculus instructors and business calculus textbook authors might be able to support students towards developing a robust understanding of marginal change.

References

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 352-378.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517-545). Mahwah, NJ: Laurence Erlbaum Associates.

Gordon, S. P. (2008). What's wrong with college algebra? *Problems, Resources, and Issues in Mathematics Undergraduate Studies*, *18*(6), 516-541.

Haeussler, E. F., Paul, R. S., & Wood, R. J. (2011). *Introductory mathematical analysis for business, economics, and the life and social sciences* (13th ed). Upper Saddle River, NJ: Pearson Prentice Hall.

Klymchuk, S., Zverkova, T., Gruenwald, N., & Sauerbier, G. (2010). University students' difficulties in solving application problems in calculus: Student perspectives. *Mathematics Education Research Journal*, *22*(1), 81–91.

Lobato, J., Hohensee, C., Rhodehamel, B., & Diamond, J. (2012). Using student reasoning to inform the development of conceptual learning goals: The case of quadratic functions. *Mathematical Thinking and Learning*, *14*(2), 85-119.

Mkhatshwa, T. P. (2016). *Business calculus students' reasoning about optimization problems: A study of quantitative reasoning in an economic context*. (Doctoral dissertation). Syracuse University, New York.

Mkhatshwa, T., & Doerr, H. (2018). Undergraduate students' quantitative reasoning in economic contexts. *Mathematical Thinking and Learning, 20*(2), 142-161.

Oehrtman, M. C., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' understandings of function. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics* (pp. 27-42). Washington, DC: Mathematical Association of America.

Prince, M., Vigeant, M., & Nottis, K. (2012). Development of the heat and energy concept inventory: Preliminary results on the prevalence and persistence of engineering students' misconceptions. *Journal of Engineering Education*, *101*(3), 412-438.

Rasmussen, C., & Marrongelle, K. (2006). Pedagogical content tools: Integrating student reasoning and mathematics in instruction. *Journal for Research in Mathematics Education*, *37*, 388-420.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 25*(3), 165–208.

Thompson, P. W. (1994b). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics*, *26*(2), 229-274.

Thompson, P. W. (1994d). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), The development of multiplicative reasoning in the learning of mathematics (pp. 179–234). Albany, NY: SUNY Press.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education.* WISDOMe Mongraphs (Vol. 1, pp. 33- 57). Laramie, WY: University of Wyoming.

Villegas, J. L., Castro, E., & Gutierrez, J. (2009). Representations in problem solving: A case study with optimization problems. *Electronic Journal of Research in Educational Psychology, 7*(1), 279–308.

SPONTANEOUS GENERALIZATIONS THROUGH EXAMPLE-BASED REASONING IN A COLLABORATIVE SETTING

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This study explores the progression from student justification to generalization in the course of example-based reasoning. Data was collected through group interviews with high school students who were working collaboratively on a task of determining connections between perimeter and area of tile shaped patterns. The task called for making and justifying conjectures regarding patterns of specific number of tiles. Our findings show that the task elicited collaborative example-based reasoning that evoked spontaneous generalizations about patterns of any number of tiles. The findings point to the importance of collaboration in generalizing, as well as to the intuitive nature of generalizations.

Keywords: Reasoning and Proof, High School Education, Middle School Education, Geometry and Geometrical and Spatial Thinking

Background

The study reported in this paper is part of a larger study on the roles of examples in learning to prove (Knuth, Zaslavsky, & Ellis, 2017). The main goals of this study were to examine ways in which the use of examples may facilitate students' reasoning and proving. This approach reflects a shift from focusing on students' overreliance on examples as a stumbling block to learning to prove (Healy & Hoyles, 2000; Harel & Sowder, 2007), to exploring productive ways of using examples for proving. By example-based reasoning we refer to justifications that use examples to convince one's self or others regarding a certain assertion (Rissland, 1991; Zaslavsky & Shir, 2005). The data set of the study consists of transcriptions of videotaped task-based interviews, mostly with individual students and some with groups of two to three students. In this paper we focus on the group interviews and examine the interplay between example-based reasoning and generalization in a collaborative setting.

Conceptual Framing

Generalization is considered to be an important form of algebraic reasoning and an instrumental component of mathematics education reform (Kaput, 1999; National Council of Teachers of Mathematics, 2000; Ellis, 2007; Ellis, 2011). Early theories and frameworks characterized generalization as an individual act (Ellis, 2011), focusing on theories related to the static nature of transfer in which knowledge remains unchanged during the transfer process (Ellis, 2007). However, researchers have more recently considered the dynamic nature of transfer through an actor-oriented perspective in which generalization may be influenced by instructional environments and social interactions (Jurrow, 2004; Ellis, 2007; Ellis, 2011).

Jurrow (2004) and Ellis (2011) show that students can generalize through peer interaction. Jurrow (2004) describes *linking*, the process of creating and applying classification systems, as a way in which students use talk and interaction to generalize. Ellis (2011) identifies seven categories of generalizing-promoting actions within a collaborative setting, noting how student interactions support and shape the generalizing activities. Ellis (2007) also developed a taxonomy for categorizing generalizations in which a distinction is drawn between generalizing actions (*relating, searching, and extending*) and reflection generalizations (a student's ability to identify or use an existing generalization). Vinner (2011) emphasizes that most generalizations made by students are intuitive in that they are immediate, spontaneous, and rely on global impressions, rather than an analytical thought process. He points to this as one reason why generalizations often fail.

Students construct new knowledge through generalizing activities (Ellis et al, 2017). The generalization schema sorts through similarities and differences, forming a generalization through the similar examples (Vinner, 2011). Examples have been identified as the preferred way in which students justify the truth of their conjectures (Stacey, 1989; Ellis, 2007; Vinner, 2011). However, students may not scrutinize the similarities found in examples, leading to false generalizations (Stacey, 1989; Vinner 2011). Justification is an integral aspect of generalization (Kaput, 1999) helping students to establish conviction in their generalization, as well as to prove the generalization (Ellis, 2007); but students attempt to justify through the use of *empirical examples* rather than *generic examples* (Lannin, 2005). *Empirical* example-use is most often characterized by treatment that confirms or contradicts a given conjecture with no consideration of general structure, while *generic* example-use illuminates the general case through a particular case and may convey the main idea(s) of the relevant proof (Mason & Pimm, 1984; Stylianides, 2008; Leron & Zaslavsky, 2013). Ellis et al (2017) developed the Criteria-Affordances-Purposes-Strategies framework to characterize the ways in which students think with examples while investigating and proving conjectures. The CAPS framework identifies *generalization* as one of the affordances of example-use (Ellis et al, 2017).

Although Ellis (2007) contends that the connection between generalization and justification is bidirectional, existing literature is thin with regards to how students progress from example-based justification to develop a generalization. Our study contributes to this area of research.

The Study

In this part of the study we consider the questions: (*i*) How might students' example-based justification of a particular mathematical conjecture promote generalizations? (ii) What roles do peer interactions play in this process?

Research Instrument

The research instrument for this part of the study was "The Tile Task" - a task designed to elicit example-based reasoning in a geometric context. Participants were given the following information: "A design company offers Tile-Patterns in different sizes and shapes. Each tile is a square with a 1-unit length side. Tile-Patterns are constructed by combining some number of tiles so that each tile shares one full side with at least one other tile. No 'holes' are permitted, and no two tiles are allowed to share only part of a side." The following examples of acceptable tile-patterns were provided (Fig. 1):

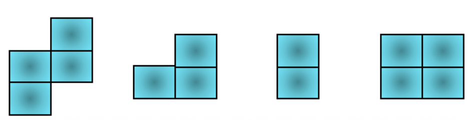

Figure 1. Acceptable tile patterns.

The two parts of the Tile Task that were analyzed for this paper (Question 4 and Question 5) were the same except for the number of tiles used. Question 4 asked: "For a tile-pattern that uses 8 tiles, what would be a shape of the pattern with the smallest perimeter? With the largest perimeter? How do you know?" Question 5 asked the same for a tile-pattern that uses 7 tiles.

While each of these questions could have been answered by using a previous part of the task in which students determined if the perimeter would increase, decrease, or remain the same by adding one tile, students were not directed to do so.

Data Collection

Students were randomly grouped for participation in four separate one-hour-long semi-structured, task-based clinical interviews. There was one individual interview ('Group 1' – one female), two dyad interviews ('Group 2' – one male one female; 'Group 3' – two females), and one triad interview ('Group 4' – three males). All interviews were recorded and transcribed, and any written work was also collected. Participants used Livescribe pens that captured audio, as well as pen strokes that can be replayed in real time.

Findings

This part of the study revealed that when asked to justify their answer, students attempted to generalize their findings to all tile-patterns regardless of the number of tiles used, with varying degrees of success. Students referred back to the examples they had generated when doing so. Two such instances, described below, occurred after student had identified the shapes of the pattern that would result in the largest and the smallest perimeters for tile-patterns using 8 and 7 tiles, respectively.

Case 1

S1, a 10th grade male, and S2, a 9th grade female, are working as a pair on the Tile Task. After reading Question 4, S1 immediately speaks in general terms about which tile-pattern would make the smallest and largest perimeters. As he speaks he sketches the tile-patterns as in Fig. 1.

> *S1*: Okay. Okay well I guess the largest, the smallest perimeter I guess would be the one that shows the least amount of sides, so...oh. Yeah so so if you have a really long one then that would be the most I think because you're showing as much of the tile as possible for each one and let me see.

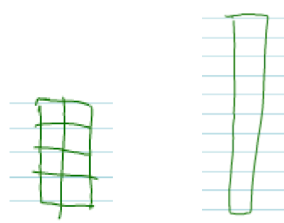

Figure 2. Tile pattern

S2 agrees with this assessment; they count the sides to determine that the smallest perimeter is 12 units and the largest is 16 units (this group incorrectly counted the largest perimeter as 16 while it really is 18 units). Once Question 4 is answered, they move onto Question 5.

> *S2:* I think the largest is still just a long one.
>
> *S1:* Yeah and this (Fig. 3) comes out to perimeter of twelve like this, so I don't know. I don't-

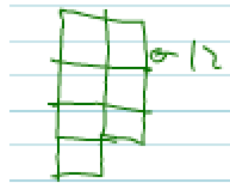

Figure 3. Another tile pattern

S2: I don't think there's anything you can do to make it smaller

S1: Yeah okay so that's smallest and the long one is largest.

Before leaving the Tile-Task, the interviewer asks S1 and S2 to talk about the shapes they used in Questions 4 and 5:

S1: Well the shortest one, or sorry the one with the smallest perimeter is usually in general stout, stouter I don't know how to describe it. Its length, its, the ratio of its length to width was smaller, no larger. I'm not, you put its length over its width it would be closer to one than if you put this length over this width is what I'm saying. So essentially when the closer to a square this gets or the closer to a figure where the ratio of length to width is one, the smaller its perimeter is**.** I think. Does that sound? (refers to Figure 2)

S2: Yeah

S1: So I guess yeah, if we needed to do a formula you could say as l over w approaches 1, the perimeter gets smaller. Or the less the, the less, the more the ratio, or I guess you say the ratio of length over width is inversely proportional to the perimeter. Yeah yeah okay because as the length over width, the ratio itself gets closer to one which is the biggest it can be, the perimeter goes down so yeah.

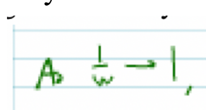

Figure 4. Formula

In describing the shapes used to determine the smallest perimeter, S1 attempts to generalize the relationship between a tile-pattern's length and width with regards to the tile-pattern's perimeter. By doing so, S1 is moving beyond the particular cases concerning tile-patterns that use 7 and 8 tiles into the general structure for all cases, regardless of the number of tiles used. Although S1 explicitly refers to the smallest perimeter, his attempt to generalize also captures all cases for the largest perimeter.

Case 2

The students below are both female, S3 is in 7th grade and S4 is in 9th grade. Each student begins working independently, drawing Figures 5 and 6.

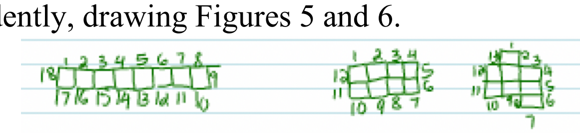

Figure 5. One student's work

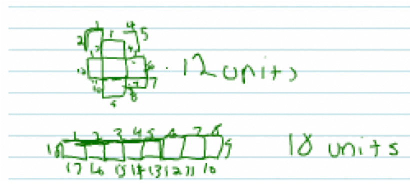

Figure 6. Another student's work

S3: What do you? I said smallest is 12 and longest is 18.

S4: Oh yeah. That's what I got too. But, yeah. I think that, I think that's, is that right? Let me, I'll just try something (Fig. 7).

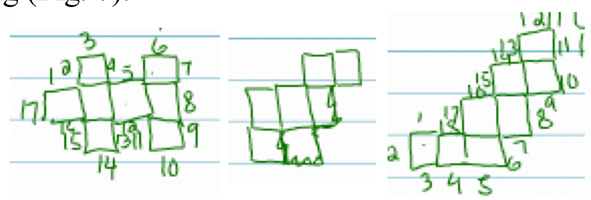

Figure 7. Student's work

S3: Yeah, lemme try a square (Fig. 8).

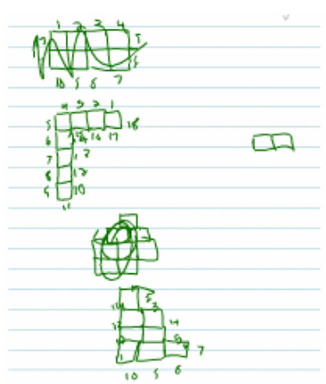

Figure 8. Student's work

S4: Yeah. I think the largest amount is 18.

S3: I'm good with that.

S4: Alright. I think our original answer is justified**,** um, the largest is 18 units and the smallest is twelve units, right?

The interviewer then asks S3 and S4 how they know that these are the largest and smallest perimeters.

S4: Um, I just personally, I experimented and I found that in um most structures most of the tiles had to only um be exposing to, or, 'cause I kept trying to make it so that each tile was exposing three sides, but um, it didn't work out because you have to plonk the other tile somewhere.

In trying to justify her results for Question 4, S4 talks about structural similarities found in the shapes of the various examples generated. The pair then moved onto Question 5, generating several tile-patterns using 7 tiles. After S3 and S4 were in agreement that the smallest perimeter was 12 and the largest 16, the interviewer asked: “How do you know?” The response of S3 is below:

> *S3:* Um. So I tried a different shape than what I did before and it still came in as 12. I feel like if the, um, like if it's larger than I guess 6 maybe, it's going to be, 12 I feel like is the smallest it can go if there's only like 6 tiles, like 5 tiles, the smallest it can go is 12 for the perimeter and then the highest it can go is 2 plus there like, you do 7 plus 7, it's 14 and it's 16, so like the perimeter would be 2 units more than what the number of tiles plus the number of tiles is. (refers to Fig. 9)

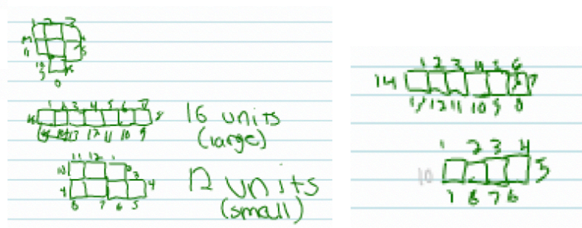

Figure 9. Student tried a different shape

The interviewer probes S3 further, asking if S3 is thinking about the largest perimeter in terms of any number of tiles; S3 replies: “yeah.” The following conversation between S3 and S4 ensued:

> *S4:* Then how would you split up the tiles when you're adding them together?
> *S3:* What do you mean, like?
> *S4:* If, if we had 7 tiles, how would you add them together? How would you split them to add them together?
> *S3:* To get a small perimeter or a large perimeter?
> *S4:* A large.
> *S3:* I just did a line and...
> *S4:* But, like applying your rule, how would you do it?
> S3: Um, I'm not really sure, but like it's the same thing as the 8 tiles. It came out to 18 for the largest and that's 2 plus 8 plus 8, which is 16, so it's 2 plus, that makes sense.
> *S4:* So it's the number, the number of
> *S3:* Tiles
> *S4:* tiles times two
> *S3:* Yeah.
> *S4:* plus two?
> *S3:* Yeah.
> *S4:* Ok.

Although S3 and S4 did not discuss the shape that would yield the largest perimeter, S3 sought to generalize the numerical value of the largest perimeter, regardless of the number of tiles used. S4 then engaged S3 in a conversation clarifying how to use the “rule” to find the largest perimeter, which further developed the algebraic generalization.

Discussion

The Tile Task was designed with the goal of characterizing students’ use of examples in conjecturing and proving when engaged in a task of a geometric nature. As expected, students generated their own examples in an effort to answer various questions regarding the size of the perimeter. The spontaneous example-use served to highlight what came naturally to students (Aricha-Metzer & Zaslavsky, 2017) with a goal of building an intuition regarding proving or disproving a conjecture (Zaslavsky, 2017). In addition to this, and as suggested by Vinner (2011), students spontaneously began to investigate the similarities found in their examples in order to make general statements beyond the particular case being considered. Students in each of the four groups, at some point during the Tile Task, engaged in a form of generalization that went beyond the scope of the question asked. This could reflect the intuitive nature of generalizations (Vinner, 2011).

The findings also reveal that students encounter difficulty when shifting from pattern recognition to pattern generalization (Ellis et al, 2017). According to Ellis (2011), students are aided in this shift by sharing ideas with their peers. While individual students initiated the attempt to generalize, the shift from pattern recognition to pattern generalization occurred when students shared their initial ideas with their peers. In case 1, this is evident when both students began to discuss S1's sketches in terms of exposed sides, which lead to an insight of how the length-to-width ratio can be used to determine the shape of a tile-pattern that results in the maximum or minimum perimeter. In case 2, S3 and S4 work individually (e.g. S3: "I tried… S4: "I just personally…") and then share their results. Through this sharing, S3 reveals that she recognizes a pattern between the numbers of tiles used and the largest perimeter of the tile-pattern. S4 propels the generalization forward by asking S3 how she would "split up the tiles" when adding them together. This discussion leads to a general algebraic formula that determines the size of the largest perimeter for a tile-pattern that uses a given number of tiles. Through a collaborative effort, each group volunteered a generalization related to Question 4 and Question 5.

It is also worth noting that while Question 4 and Question 5 discussed in the cases above could have been addressed by using information gained from Question 3, no group drew on this prior knowledge. Question 3 asked if the perimeter of a tile-pattern would increase, decrease, or remain the same following the addition of one tile. All three cases are possible depending on where the additional tile is placed: a) the perimeter decreases when one tile is added to a cavity, b) the perimeter increases when one tile is added to the end of another, and c) the perimeter remains the same when one tile is added to a corner. While only two of the four groups had concluded that the perimeter could decrease, all concluded that the other two cases were possible. However, none of the four groups used the information regarding the increased and constant perimeter when thinking about a shape that would yield the largest and smallest perimeters. The disjointed manner in which students advanced through the Tile Task warrants further exploration.

Pedagogical Implications

The results of this study have pedagogical implications with regards to example-based reasoning and student collaboration. This study shows how both can aid students with generalization. This study also points to the importance of engaging students in rich mathematical tasks.

Acknowledgement

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References

Aricha-Metzer, I., & Zaslavsky, O. (2017). The nature of students' productive and non-productive example-use for proving. *Journal of Mathematical Behavior*. http://dx.doi.org/10.1016/j.jmathb.2017.09.002

Ellis, A. B. (2007). Connections between generalizing and justifying: Students' reasoning with linear relationships. *Journal for Research in Mathematics Education*, *38*(3), 194-229.

Ellis, A. B. (2011). Generalizing-promoting actions: How classroom collaborations can support students' mathematical generalizations. *Journal for Research in Mathematics Education*, *42*(4), 308–345.

Ellis, A., Lockwood, E., Tillema, E., & Moore, K. (2017). Generalization across domains: The relating-forming-extending generalization framework. In Galindo, E., & Newton, J., (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 677-684). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Ellis, A. B., Ozgur, Z., Vinsonhaler, R., Dogan, M. F., Carolan, T., Lockwood, E., … Zaslavsky, O. (2017). Student thinking with examples: The criteria-affordances-purposes-strategies framework. *Journal of Mathematical Behavior*. https://doi.org/10.1016/j.jmathb.2017.06.003

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In (2nd ed.) F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning, Vol. 2* (pp. 803–842). Greenwich, CT: Information Age Publishing.

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education, 31*, 396–428.

Jurow, S.A. (2004). Generalizing in interaction: Middle school mathematics students making mathematical generalizations in a population-modeling project. *Mind, Culture, and Activity* 11(4), 279-300.

Kaput, J. (1999). Teaching and learning new algebra with understanding. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York: Macmillan Publishing Company.

Knuth, E., Zaslavsky, O., & Ellis, A. (2017). The role and use of examples in learning to prove. *The Journal of Mathematical Behavior*. http://dx.doi.org/10.1016/j.jmathb.2017.06.002

Lannin, J. K. (2005). Generalization and justification: The challenge of introducing algebraic reasoning through patterning activities. *Mathematical Thinking and Learning*, 7(3), 231-258.

Leron, U., & Zaslavsky, O. (2013). Generic proving: Reflections on scope and method. *For the Learning of Mathematics, 33*(3), 24–30.

Mason, J., & Pimm, D. (1984). Generic examples: Seeing the general in the particular. *Educational Studies in Mathematics*, *15*(3), 277-289.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Rissland, E. L. (1991). Example-based reasoning. In J. F. Voss, D. N. Parkins, & J. W. Segal (Eds.), *Informal reasoning in education* (pp. 187-208). Hillsdale, NJ: Lawrence Erlbaum Associates.

Stacey, K. (1989). Finding and using patterns in linear generalising problems. *Educational Studies in Mathematics*, *20*(2), 147–164.

Stylianides, G. J. (2008). An analytic framework of reasoning-and-proving. *For the Learning of Mathematics*, *28*(1), 9-16.

Vinner, S. (2011). The role of examples in the learning of mathematics and in everyday thought processes. *ZDM - International Journal on Mathematics Education*, *43*(2), 247–256. https://doi.org/10.1007/s11858-010-0304-3

Zaslavsky, O. (2017). There is more to examples than meets the eye: Thinking with and through mathematical examples in different settings. *Journal of Mathematical Behavior*. http://dx.doi.org/10.1016/j.jmathb.2017.10.001

Zaslavsky, O., & Shir, K. (2005). Students' conceptions of a mathematical definition. *Journal for Research in Mathematics Education*, *36*(4), 317-346.

VARIOUS MEANINGS A STUDENT USES FOR QUANTIFIED VARIABLES IN CALCULUS STATEMENTS: THE CASE OF ZACK

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This study investigates one Calculus student's meanings for quantifiers in Calculus statements involving multiple quantifiers. The student was asked in a two-hour long clinical interview to evaluate and interpret the Intermediate Value Theorem (IVT) and three other statements whose logical structure was similar to the IVT except for the order of both the quantifiers and their attached variables. Four different meanings for variables attached to universal and existential quantifiers emerged from his responses at various moments of the interview. In this paper we detail these four meanings with empirical evidence and discuss implications of our findings to research and teaching of quantified variables in Calculus statements.

Keywords: Reasoning and Proof, Postsecondary Education, Cognition, Advanced Mathematical Thinking

The purpose of this study is to investigate students' mental processes associated with quantified variables. In particular, we focus on a Calculus student, Zack, and his quantifications for variables in the Intermediate Value Theorem (IVT) and similar statements as we answer the following research question: *What are the meanings for quantified variables that a student uses when interpreting statements from Calculus contexts?*

Literature Review

Quantifiers such as "for all" ($\forall$) and "there exists" ($\exists$) may be used to state important definitions and theorems in Calculus. For example, the Intermediate Value Theorem (IVT), the Mean Value Theorem (MVT), and definitions of limits and continuous functions may be stated with multiple quantifiers (Bartle & Sherbert, 2000; Stewart, 2003). Several studies have reported various student tendencies to mistreat quantifiers when analyzing quantified statements involving either the universal or existential quantifier (Barkai, Tsamir, Tirosh, & Dreyfus, 2002; Epp, 1999). For example, Barkai et al. (2002) reported that some participants of their study tended to suggest that a few examples are sufficient to prove that a statement involving a universal quantifier, in the form 'For all x, P(x),' where P(x) is a statement about x, is true. In the case of a statement involving an existential quantifier, in the form 'there exists x such that P(x),' students also rejected the notion that one example would suffice for proving such a statement (Tirosh & Vinner, 2004). One such explanation for why students tend to have these particular tendencies in their interpretations of quantified statements is that they may confound colloquial language with mathematical language (Epp, 1999). For example, colloquially we may state that "Every book on the bookshelf is French," and we would assume that there is at least one French book on the specified bookshelf. However, in mathematics, we could consider the case where the statement is vacuously true. Colloquial language may explain some student difficulties with quantification, but there may be other reasons why these tendencies exist that have yet to be noted in the literature that may be explained by classifying students' meanings for quantified variables.

Theoretical Perspective

In this section, we define some terms that we will use to describe one student's meanings for quantified variables. *Quantifiers* are phrases (e.g., "for all" and "there exists") used to indicate the number of elements, x, in the domain of discourse satisfying a predicate, $P(x)$. For example, consider the mathematical statement, "Every isosceles triangle x has congruent base angles." This statement is a quantified statement where the phrase "every" is a quantifier for the elements x within the domain of discourse, which is the set of isosceles triangles, satisfying the predicate, "x has congruent base angles." We refer to x as a *quantified variable*. We use the term *quantify* to mean that an individual is mentally searching for (or anticipating searching for) a specific number of, or quantity of, values of the variable x in the domain of discourse that satisfy the predicate $P(x)$. By *quantification*, we refer to an individual's mental search processes (or anticipated search processes) for a specific number of elements that satisfy the predicate.

Our definitions for the terms *quantifying* and *student quantification* align with constructivist views of meaning, as each individual constructs and reinforces his own quantifications through his own experiences (Thompson et al., 2014). We situate these definitions from a constructivist perspective partially because students may not share conventional meanings for quantifiers in mathematics (Sellers, Roh, & David, 2017; Dubinsky & Yiparaki, 2000; Epp, 1999). Student quantification can be regarded as a meaning for quantifiers as described by Thompson et al. (2014) because quantification is comprised of an *individual's* mental actions or schemes that are easily triggered as a result of the person's understanding (or assimilation to a scheme). In particular, our definition for student quantification emphasizes that a student quantifies in ways that *he* deems necessary based on *his own* interpretation of a given statement, which also aligns with a constructivist view of meaning. Thus, when we refer to a *student's meanings for a quantified variable*, we refer to a student's own constructed quantification for a specific variable in the given statement from the *student's* perspective.

Methodology

We conducted a two-hour long clinical interview (Clement, 2000) with a Calculus student, whom we call Zack, in the spring of 2016. Zack had completed a first semester Calculus course and was currently enrolled in a second semester Calculus course at the time of the interview. We report Zack's interview because of the various meanings for quantified variables that we found across different moments of his interview and our ability to triangulate his words, gestures, and markings on graphs.

Tasks

We asked Zack to interpret and evaluate several complex mathematical statements, shown in Table 1. We did not present Zack with the symbolic representations of the conclusion of each of the statements in Table 1, but we provide these representations in this paper to display the logical differences in the statements. Statement 2 is the IVT and the only true statement. The other three statements reorder the quantifiers and variables in the IVT, but maintain the hypothesis.

Table 1: Statements Presented in Clinical Interviews

Statements	Symbolic Representations
Statement 1 (S1): Suppose that f is a continuous function on the closed interval $[a, b]$, where $f(a) \neq f(b)$. Then, for all real numbers c in (a, b), there exists a real number N between $f(a)$ and $f(b)$, such that $f(c)=N$.	$\forall c(\exists N\, f(c) = N)$

Statement 2 (S2): Suppose that f is a continuous function on the closed interval $[a, b]$ where $f(a) \neq f(b)$. Then, for all real numbers N between $f(a)$ and $f(b)$, there exists a real number c in (a, b), such that $f(c)=N$.	$\forall N(\exists c\, f(c) = N)$
Statement 3 (S3): Suppose that f is a continuous function on the closed interval $[a, b]$, where $f(a) \neq f(b)$. Then, there exists a real number N between $f(a)$ and $f(b)$, such that for all real numbers c in (a, b), $f(c)=N$.	$\exists N(\forall c\, f(c) = N)$
Statement 4 (S4): Suppose that f is a continuous function on the closed interval $[a, b]$ where $f(a) \neq f(b)$. Then, there exists a real number c in (a, b), such that for all real numbers N between $f(a)$ and $f(b)$, $f(c)=N$.	$\exists c(\forall N\, f(c) = N)$

We first presented each of the statements shown in Table 1 one at a time and asked Zack to explain in his own words the meaning of each statement and also asked him to determine if each statement was true or false. Zack was asked to justify his evaluations and we allowed Zack to draw his own graphs to explain his thinking about his evaluation of each statement. After Zack evaluated all four statements, we provided him with several graphs and asked him if he could use the graphs to support his evaluations of each of the four statements. We allowed (and often asked Zack to) highlight the variables in a given statement on the given graph after he referenced them. These markings as well as his gestures of sweeping or pointing on the graphs were used in the data analysis as an indicator of characteristics for his meanings for the quantified variables in that statement in a given moment.

Data analysis

Our analysis was conducted in the spirit of grounded theory (Strauss & Corbin, 1998). The use of grounded theory allowed new categories to emerge from our data that have not yet been described in the literature regarding students' meanings for quantifiers. We noticed that Zack's meanings for a particular quantified variable with the same statement changed at different moments in the interview. Thus, we also employed Thompson et al.'s (2014) construct of "meanings in the moment" to analyze Zack's meanings for quantified variables. We marked a new moment when Zack was presented with a new interview prompt or task, when he changed his evaluation of a statement, or if he provided a different meaning for a quantified variable while working with the same statement. Every time we found a new meaning for quantified variables in a moment, we added this new meaning into our overall coding system. We also compared Zack's meanings against other students' meanings we interviewed. From this process, similarities and differences in student meanings for quantified variables were refined into four categories. Finally, using the four meanings for quantified variables, we re-analyzed all student interviews and refined our previous coding for each student moment as necessary to ensure that these categories were reliable for all moments, as well as with other students.

Results

We found Zack used four different meanings for quantified variables, which we call MQ1-MQ3 and NQ. Evidence of these meanings came from moments across different moments of Zack's interview. In the subsections that follow, we explain each of the four meanings for quantified variables and provide examples from different moments of the interview when Zack used each of the four meanings.

MQ1: Checking the predicate holds for at least one element

We found that in some moments, Zack described his imagined process of checking the predicate of the statement for *at least one* element of x. We classified his meaning as MQ1 whenever he appeared to strategically search for at least one value of x, within his domain of discourse, that satisfies the predicate.

Although MQ1 is consistent with the mathematical convention for existentially-quantified variables, in the following moment, Zack used MQ1 for a variable attached to a *universal* quantifier in the given statement. Zack quantified the variable N in Statement 2 ($\forall N(\exists c\, f(c) = N)$) as follows: "I have read the first part of the second sentence, 'for all real numbers N between $f(a)$ and $f(b)$.' So that made me think, or realize, that there exists a number N between my output variables $f(a)$ and $f(b)$ on this curve." Although Statement 2 contained the phrase "for all N," Zack's interpretation indicates that he was looking for (at least) one N-value as he stated that, "there *exists* a number N."

MQ2: Checking the predicate holds for exactly one element

In some other moments, Zack emphasized that he was looking for *exactly one* value satisfying the predicate. We use MQ2 to refer to Zack's meaning when he appeared to mentally search (or suggested he imagined searching) through every element in the domain of discourse to ensure that *exactly one* value satisfies the predicate. MQ2 follows the mathematical convention for variables attached to "there exists a unique." However, as we see in the moment below, Zack used MQ2 despite the absence of the word "unique" in the provided statements.

We highlight one moment with Zack where he appeared to use MQ2 for the variable N in Statement 4 ($\exists c(\forall N\, f(c) = N)$). Zack first stated that the graph in Figure 1 was "still a function because [Figure 1 is] still in a parabola shape." He then labeled $f(a)$ and $f(b)$ on the graph, and then proceeded to chose a specific value of c, drew the vertical line shown on the graph, and wrote $f(c)$=N at two different points on the curve. Next, he concluded that Statement 4 was false for this graph and explained his reasoning in the following transcript.

Zack: I don't know if I would use this graph to prove this statement […] I'm getting two output variables from the *c*. So if it was […] a regular parabola, I would say […] If I choose *c* in between *a* and *b*, I am only gonna get one *N*. […] But […] I get two outputs for a *c*.

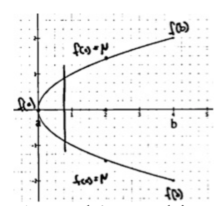

Figure 1. Zack's markings on provided graph for S4.

Zack's explanation above suggests that he anticipated finding one value of N (Lines 3-5). Yet, he found two specific N-values that satisfied the predicate for his chosen c-value (Lines 2, 5), which led him to complete his search process for *exactly one* N-value that satisfies the predicate. Zack evaluated Statement 4 as false because *more than one* element of N satisfied the predicate for the given graph. Zack ultimately determined that this statement was false for this graph because there was more than one N-value for his chosen c, and thus, we claim that Zack utilized MQ2 for N in Statement 4 in the moment above.

MQ3: Checking the predicate holds for all elements

In contrast to the moments in which Zack employed MQ1 or MQ2, we found some moments when he described his imagined process of checking the predicate $f(c)$=N for *all* values of either c or N. We classified this type of student quantification as MQ3.

We classified Zack's meaning for a quantified variable as MQ3 whenever we observed the following behaviors: (1) he chose a value or values from his own identified domain of discourse and determined whether or not his chosen value(s) satisfy the predicate and (2) he repeated (or imagined repeating) checking the predicate for all elements within his domain of discourse.

MQ3 is akin to the mathematical convention for universally-quantified variables. However, students may use MQ3 for existentially-quantified variables, as shown in the moment with Zack below. In this moment, we focus on Zack's quantification for the variable c in Statement 4 ($\exists c(\forall N\, f(c) = N)$. Even though c is an existentially-quantified variable in the given statement, Zack's quantification for c does not follow convention in this moment.

Zack: If I were to input c, whatever number c may be, and that's just arbitrary. By choosing that number, I know that I am gonna get N […] So, in this case c equals $^{-}1$ (*marks* c=$^{-}1$ *on x-axis*) […]
Int: Is there any particular reason why you picked this c […]?
Zack: No. I could have […] represented c as 2 (*points to the number 2 on the x-axis*) […] I would say c only represents one input-output relation at one time. […] I think what I am trying to say is I can choose […] any number between a and b for an input variable, and that could be c. So yeah if I choose again if I choose c to be -1 then I know that my N would be this number right here (*points to N on the curve*). If I represent any other x number to be c, then that output would have to be unique to that input.

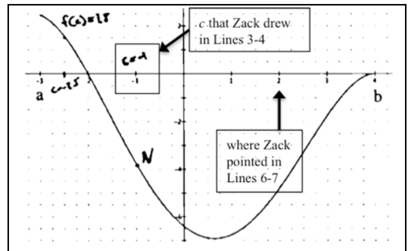

Figure 2. Zack's markings on provided graph for S4.

The transcript above indicates that Zack's domain of discourse was the interval (a, b) (Lines 9-10) and he checked the predicate for a specific value of c in his domain of discourse (Lines 2-3). Initially, he checked the predicate for just this value (Lines 2-3), but also referred to his choice of c as *arbitrary* (Line 2). Although Zack chose to check one specific value of c, he did not claim that this was the *only* value for c that could have been chosen. Instead, he accepted multiple values of c in his consideration of the predicate (Line 6). Furthermore, Zack also used words such as *whatever number c* (Line 1) and *any number between a and b* (Line 9), which indicate that Zack considered not only multiple values of c, but *all* the values of c within his chosen domain of discourse, (a, b). He also discussed his stipulations for checking the predicate for *any* value for c (Line 8). Since Zack explained the satisfaction of the predicate for *any other* values of c (Line 11), we conclude that his language suggests an imagined search through all values of c, and thus, we conclude that he quantified c with MQ3.

NQ: No quantification

Thus far, we have detailed several different ways that Zack quantified variables. In some other moments, Zack did not quantify a variable in the given statements (see Table 1). Regardless of the presence of the quantifier words in a statement, in these moments Zack focused on certain attributes of *x other than the quantity* and often attended to properties of a variable without attending to the number of elements in the domain of discourse. Indeed, Zack did *not* search for a specific number of elements of x that satisfied the predicate in these moments. We refer to this type of student quantification (no quantification) as NQ. We illustrate characteristics of NQ from a moment in which Zack first analyzed Statement 3 ($\exists N(\forall c\, f(c) = N)$, without being given any graphs. In this moment, we highlight Zack's quantification for both variables c and N.

Zack: If I draw this line…I am now going to write *c* in between these two [*a* and *b*] such that for a real number *c* in interval (*a*, *b*), if I put […] *c* into the function, […] then I get my number *N*. So, right here (*marks N on the line*). […] I will get my output variable *N*.
Int: […] In your own words how would you explain this? […]
Zack: So I know that I can only work between my *x* values of *a* and *b* on a closed interval. […] I know that *c* lies in the interval between *a* and *b*. So I am just gonna put it right in the middle. If I put this *c*-value into the function, I get *f*(*c*), which is equal to my number *N* in between *f*(*a*) and *f*(*b*).

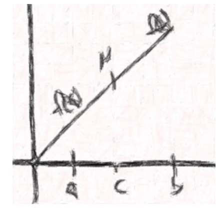
Figure 3. Zack's hand-drawn graph for S3.

Zack stated that he knew that *c* has to be in the interval (*a*, *b*) and also stated that *N* was between *f*(*a*) and *f*(*b*) (Lines 2, 9). Thus, he recognized a domain of discourse for *c* and *N*, respectively. However, Zack does not refer a specific number of *c*-values or *N*-values that he is checking to ensure that *f*(*c*)=*N*. Rather, he stated that he was drawing his graph in such a way that his *c would* yield *N* (Lines 1-4). We take Zack's words as indicative that he is *interpreting* the predicate and drawing his graph in such a way to ensure that he gets *N*-values that satisfy the predicate, *f*(*c*)=*N*. Thus, we conclude that Zack used NQ for both *c* and *N* in this moment.

Conclusion

Zack used four different meanings for quantified variables throughout his interview, which we refer to as MQ1-MQ3 and NQ. MQ1-MQ3 are akin to mathematical conventional uses of the existential, existential unique, and universal quantifier meanings, respectively. Zack also exhibited one other meaning for quantified variables, which we categorized as "No quantification" (NQ). NQ is not characteristic of any conventional mathematical meaning for quantified variables, and we conjecture that in moments where Zack used NQ, he lacked a mental search for a search for a number of elements of either variable *c* or *N* that satisfied the predicate. In these moments where he used NQ, he appeared to only interpret the meaning of the predicate instead of checking the validity of the predicate.

The four categories of meaning are highlighted in Table 2. In this table, we provide descriptions of evidence that we considered in our classifications and the mental actions, which we theorize comprise each meaning for quantified variables. We utilize *x* as an arbitrary quantified variable and X as a generic domain of discourse in Table 2, as these meanings applied to either variable *c* or *N*. We categorized Zack's meaning for a quantified variable based on crucial observable behaviors that distinguished his meaning from another meaning, even if he did not exhibit all observable behaviors listed. These crucial behaviors are italicized in Table 2.

Table 2: Student Meanings for Quantified Variables

Meaning	*Mental Actions*	*Observable Evidence*
MQ1: Checking the predicate holds for at least one *x* in X	1. Identify the domain of discourse, X. 2. Choose (or imagine choosing) one value (x_0) for *x* from X, then check if the predicate is satisfied by x_0 (i.e. $P(x_0)$ is true). 3. Repeat (or imagine repeating) this mental action until at least one value of *x* in X is found that satisfies the predicate. May complete without exhausting all values of *x*.	• Marks off the domain of discourse, X. • Marks one value of *x*, x_0, in X & explains or illustrates whether or not this value of *x* satisfies the predicate. • May mark more values of *x* & may explain or illustrate that at least one value of *x* in X satisfies the predicate. • *Uses phrases such as there is, some, or at least one to refer to the values of x in X that satisfy the predicate.*

MQ2: Checking the predicate holds for exactly one *x* in X	1. Identify the domain of discourse, X. 2. Choose (or imagine choosing) one value (x_0) for x from X, then check to determine if the predicate $P(x)$ is satisfied by this value of x. 3. Repeat (or imagine repeating) step 2 until all the elements of x in X are exhausted to ensure that exactly one value of x in X satisfies the predicate.	• Marks off the domain of discourse, X. • Marks one value of x, x_0, that satisfies predicate. • *Claims that the value, x_0, is the only value that satisfies the predicate.* Explains or illustrates that other elements of x do not satisfy the predicate. • Uses phrases such as there is exactly one to refer to the values of x in X that satisfy the predicate.
MQ3: Checking the predicate holds for all *x* in X	1. Identify a domain of discourse, X. 2. Choose one value (x_0) for x from X, then check if the predicate is satisfied by x_0 (i.e. $P(x_0)$ is true). 3. Repeat (or imagine repeating) step 2 until all the values of x in X are exhausted.	• Marks off the domain of discourse, X. • Marks one value of x, x_0, in X & explains or illustrates whether or not this value of x satisfies the predicate. • *Explains or illustrates that the predicate holds for every x.* One possible illustration may be sweeping along X. • Does not use the phrases there is, there exists, or for some, but may use words all, every, each, any or arbitrary to refer to values of x in X that satisfy the predicate.
NQ: No quantification	1. Identify the domain of discourse, X. 2. Choose (or imagine choosing) one value (x_0) for x from X & interpret predicate $P(x)$ using the chosen x_0. (i.e. $P(x_0)$ means…)	• Marks off the domain of discourse, X. • May mark a value of x in X. • *Interprets the given predicate $P(x)$ in their own words, but does not explain how many values of x satisfy $P(x)$.*

For all four categories, including MQ1-MQ3, Zack often used meanings for quantified variables in unconventional ways, i.e. he used these meanings *regardless of the given quantifier in the given statement*. For example, Zack quantified a variable with a meaning that is more akin to a mathematical meaning for a universally-quantified variable even though the variable is existentially quantified in the given statement and vice versa in different moments.

Discussion

All four meanings for quantified variables that emerged from our study are applicable to analyze students' meanings for quantified variables in other mathematical contexts. Our discovered four meanings may explain findings in previous research, and they have potential to guide teaching in many content domains.

The four categories of meaning that emerged in this study have explanatory power for previous research findings. Students' reasoning from a given statement may not be perceived as erroneous if that reasoning is based on a different interpretation of the variable given in the statement. As previously mentioned, prior research has found that students may determine that a few examples are sufficient to prove a universally-quantified statement (Barkai et al., 2002). This behavior could be explained if the student interpreted the given universal statement with MQ1, checking the predicate for at least one value of x in X. If the student perceives that the statement is implying that there should be at least one x that satisfies the predicate, then this meaning explains their acceptance of few examples to prove the statement true. As another example, other studies have noted that some students state that one example is insufficient for an existentially-quantified statement (Tirosh & Vinner, 2004). This type of reasoning is not an erroneous argument if a student's current meaning for x is MQ3, searching through all elements of x to ensure that all values of x satisfy a given predicate. Thus, if a part of a quantified statement leads a student to believe that a variable should be quantified in a way different than intended, then students' arguments may also deviate from convention.

These categories for student meanings for quantified variables may also be used by a variety of undergraduate mathematics teachers to aid them in characterizing their own students' meanings. All undergraduate mathematics courses involve quantified statements, and as such, all undergraduate mathematics instructors should be attuned to students' types of quantification. Beyond having a variety of meanings, different parts of a mathematical statement may cause students to quantify in unconventional ways. Students may even skip over quantifier words altogether and interpret pieces of these statements rather than quantifying variables. Thus, we view one of the primary uses of our findings as a tool for educators to identify students' meanings and address their current meanings through questioning and activities that promote reflection of differences in types of quantified variables. Our findings suggest that quantification as a mental process for students is much more complicated than researchers and teachers may expect, and Calculus students in particular could benefit from learning opportunities that would help them confront their various meanings for quantified variables. We hope that teachers and curriculum writers will consider thoughtful questions and activities that will give students an opportunity to become aware of their meanings and address inconsistencies in their meanings for quantified variables.

References

Barkai, R., Tsamir, P., Dirosh, T., & Dreyfus, T. (2002). Proving or refuting claims: The case of elementary school teachers. In A.D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th International Conference on the Psychology of Mathematics Education.* Vol. 2 (pp. 57-64). Norwich, England.

Bartle, R. G., & Sherbert, D. R. (2000). *Introduction to Real Analysis*. United States: John Wiley & Sons.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In R. Lesh & A. Kelly (Eds.), *Handbook of research methodologies for science and mathematics education* (pp. 547-589). Hillsdale, NJ: Lawrence Erlbaum.

Dubinsky, E., & Yiparaki, O. (2000). On student understanding of AE and EA quantification. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *CMBS issues in mathematics education* (pp. 239-289). Providence, RI: American Mathematical Society.

Epp, S. (1999). The language of quantification in mathematics instruction. In L. V. Stiff & F. R. Curcio (Eds.), *Developing mathematical reasoning in grades K-12, 1999 Yearbook* (pp. 188-197). Reston, VA: National Council of Teachers of Mathematics.

Sellers, M. E., Roh, K. H., & David, E. J. (2017). A comparison of calculus, transition-to-proof, and advanced calculus student quantifications in complex mathematical statements. *Proceedings of the 20th annual conference on RUME.* (pp. 285-297). San Diego, CA.

Stewart, J. (2003). *Single variable calculus: Early transcendentals*. United States: Brooks/Cole.

Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: techniques and procedures for developing grounded theory, 2nd ed*. Thousand Oaks, CA: Sage.

Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: An hypothesis about foundational reasoning abilities in algebra. In K. C. Moore, L. P. Steffe & L. L. Hatfield (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing.,* WISDOMe Monographs (Vol. 4, pp. 1-24). Laramie, WY: University of Wyoming. http://bit.ly/1aNquwz.

Tirosh, C., & Vinner, S. (2004). Prospective teachers' knowledge of existence theorems. In M. J. Høines, & A. B. Fuglestad (Eds.), *Proceedings of the 28th International Conference on the Psychology of Mathematics Education*, Vol. 1 (p. 360). Bergen, Norway.

GENERALIZATION OF AN INVARIANT RELATIONSHIP BETWEEN TWO "QUANTITIES"

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In this report, we present an analysis of two prospective secondary mathematics teachers' generalizing actions in quantitative contexts. Specifically, we draw from a teaching experiment to report how Lydia and Emma engaged in different generalizing processes for the same task. Based on these differences, we found Lydia's generalizing actions (i.e., coordinating quantities) to be a more productive generalization than Emma's (i.e., slopes of tangent lines). The former was extendable to a new case, instance, or situation, whereas the latter was constrained to a specific representational system.

Keywords: Cognition; Algebra and Algebraic Thinking; Design Experiment

Several researchers have illustrated that teachers' covariational reasoning is critical to supporting their students in understanding major pre-calculus and calculus ideas (Ellis, 2007a; Confrey & Smith, 1995; Thompson, 1994; Thompson, 2011). Moreover, Ellis (2007a) reported that quantitative reasoning is an integral part of students constructing productive and global generalizations. In this paper, we characterized two pre-service teachers' (PSTs') generalizing actions during a teaching experiment focused on graphing and modeling covariational relationships. We give specific attention to how two PSTs' generalizing actions differed when reasoning about the rate of change between the distance a rider has traveled (i.e., arc length) around a Ferris wheel and the rider's vertical distance from the horizontal diameter of the Ferris wheel (i.e., height). More specifically, this paper reports (a) the generalizing actions of two PSTs, one which coordinated quantities, and one which focused primarily on steepness of tangent lines and (b) implications of those generalization actions on their later activities.

Theoretical Framework

We investigated PSTs' generalizing actions about relationships between quantities in dynamic situations. We use *quantity* to refer to a conceptual entity an individual constructs as a measurable attribute of an object (Thompson, 2011). Relatedly, Ellis, Tillema, Lockwood, and Moore (2017) introduced a generalization framework (Table 1) involving three major forms of students' generalizing—relating, forming, and extending—by building on Ellis' (2007b) taxonomy of generalizations, in which students' generalizing actions were viewed as different than students' final statements of generalization. In this paper, we are using this framework to illustrate two PSTs' generalizing actions by situating "generalization" within the perspective of the learner and with sensitivity to their reasoning about quantities. In other words, we do not take mathematical correctness as a necessary criterion of generalization, instead focusing on the process of how individuals identify their own patterns/similarities determined across cases.

Ellis et al.'s (2017) framework has two major forms of generalizing—intra-contextual forms and inter-contextual forms—in which students' generalizing actions occur within one context, task, or situation in the former and across situations, contexts, or tasks in the latter. Intra-contextual forms of generalizing include students searching for and identifying similar elements, patterns, and relationships across cases, numbers, or figures in order to form a similarity or regularity and extend it to new cases, instances, situations, or scenarios. Inter-contextual forms of generalizing include students forming similar relationships across contexts, problems, or situations and transfer. In this study, we illustrate several intra- and inter-contextual generalizations and their relationships to forms of abstraction presented in the work of two PSTs.

Table1: Ellis et al. (2017) generalization framework

Intra-contextual forms of generalizing		Inter-contextual
Forming	Extending	Relating
Relating Objects • Operative • Figurative • Activity Search for Similarity or Regularity Identify a Regularity • Extracted • Projected Isolate Constancy	Continuing Operating • Near • Projection Transforming • Constructing a Quantity • Recursive Embedding Removing Particulars	Relating Situations • Connecting Back • Analogy Invention Relating Ideas or Strategies (Transfer)

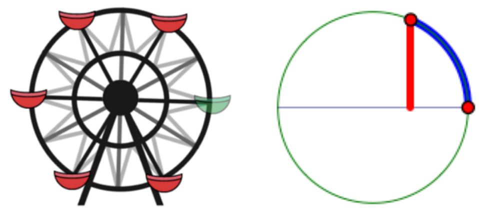
Methods

The data we present and analyze is from a teaching experiment (Steffe & Thompson, 2000) conducted over the course of a spring semester at a large public university in the southeastern U.S. with two PSTs. Our goal in the teaching experiment was to investigate the PSTs' ways of thinking and create models of their mathematics. Specifically, we explored how PSTs conceived of situations quantitatively and represented particular quantitative relationships under particular coordinate system constraints. In this paper, we focus on Lydia and Emma, who, at the time of the study, were two junior undergraduate students enrolled in both content and pedagogy courses. They had completed at least two additional courses beyond a traditional calculus sequence with at least a C as their final grade.

Lydia and Emma participated in 12-videotaped sessions (interview and teaching sessions) and we digitized their written work. Each session was approximately 1-2 hours in length. The second author served as the teacher-researcher (TR), and at least two project members observed each teaching session. We analyzed PSTs observable and audible behaviors (e.g., talk, gestures, and task responses) in details by relying on the generative and axial methods (Corbin & Strauss, 2008) combined with conceptual analysis techniques (Thompson, 2008). We drew on Ellis et al.'s (2017) analytic framework for categorizing students' generalizing actions and reflection generalizations in order to identify forms of generalization and abstraction.

Analysis and Findings

In this section, we analyze Lydia and Emma's activities during two tasks—Taking a Ride Task and Circle Task—from the teaching experiment to illustrate their generalizing actions and implications of those generalizing actions. For the Ferris wheel task (Figure 1a), we draw attention to how Lydia and Emma used the same verbal statement to describe their identified regularity, but these verbal statements signified different reasoning and generalizing actions. Then, for the Circle Task (Figure 1b), we show implications of those generalizing actions for Emma and how those generalizing actions influence Emma's activity in a later session.



(a) (b)

Figure 1. (a) Ferris Wheel and (b) simplified version used during the Circle Task.

Taking a Ride Task

The Taking a Ride Task occurred during the first session of the teaching experiments. The PSTs watched an animation of a Ferris Wheel rider that was indicated by a green bucket (see Figure 1a) who travels at a constant speed counterclockwise starting from the 3 o'clock position (Desmos, 2014). Then, we asked them to describe how the height of the rider above the horizontal diameter of the Ferris wheel changes in relation to arc length it has traveled. Both Lydia and Emma described the height as increasing less and less for the first quarter of a rotation. However, their generalizing actions entailed marked differences, which we will illustrate below.

Lydia's generalizing actions. In the Taking a Ride Task, Lydia, first, *identified a regularity* in directional changes of height with respect to the arc length it traveled (i.e., height is increasing or decreasing as the arc length increases). In other words, we inferred from her activity she identified a predictable stable feature of the riders' height from horizontal diameter in relation to arc length it traveled for each quarter turn. We note that, for the second quarter turn, she continued her generalizing actions with a *minor accommodation*. That is, she did not fully extend her identified regularity of directional changes of height in relation to arc length to the next case (i.e., second quarter turn); she conceived that the height began to decrease as the arc length continued to increase.

The TR then asked Lydia if she could provide more information—in addition to her observation that the arc length increases and the height increases during that first quarter—about how the height and arc length change in relation to each other. Lydia used the spokes of the Ferris wheel (i.e., each of the black bars connecting the center of the wheel to its edge) to partition the Ferris wheel into equal arc lengths. This partitioning activity was to become an important tool for Lydia to *establish a way operating* with respect to identifying how height changes in relation to arc length.

(a) (b) (c)

Figure 2. Lydia engaging the Ferris wheel task.

After Lydia used the spokes to create equal partitions in arc lengths (see the green dots in Figure 2a), she drew associated heights (see the green bars in Figure 2a). Then, with support from the TR's questioning, Lydia constructed successive amounts of change in height corresponding to successive equal changes in arc length (see Figure 2b, circled in blue). That is, Lydia's partitioning activity led her to *construct a quantity*, namely change in height. The TR next asked Lydia if she noticed anything to describe the relationship between height and arc length in the first quarter of trip. The following excerpt demonstrates her response.

Lydia: Um, the distance from A to B and then B to – the distance from – Okay, so this is A, B, and C [*labels as shown in Figure 2c*], from B to C the distance is less than from A to B.

TR: So what distance is less?

Lydia: Um, from point B to C [*motions along arc*], um, that distance [*indicates circled vertical distance from Figure 2c*] from – is less than the – from A to B [*unclear if pointing along arc or along vertical distance*], I feel like.

TR: Like when you're saying distance, what do you mean? Do you mean like –

Lydia: Vertical distance.

TR: Vertical distance, okay, gotcha.

Lydia: Yes. And then also if we were to say at the top this is D [*labels pi/2 on circle d*], then from C to D [*motions along arc then vertically down to the center*] is less than B to C [*unclear motion*].

We inferred from her activity that she was *establishing a way of operating* that involved the construction of a new quantity (i.e., change in height) and associated partitioning activity. With that quantity constructed, she was then able to *search* for regularity or pattern in that quantity's variation (e.g., decreasing increases). After conceiving that change in height decreased along with those equal partitioning in arc length as shown in Figure 2, Lydia concluded that "as the arc length is increasing... [the] vertical distance from the center is increasing ... but the value that we're increasing by is decreasing." We interpreted this to indicate that Lydia had *identified the regularity* in how height changes in relation to arc length in the first quadrant. To say more, we inferred that she *extracted* this regularity because she identified a common feature of a change in one quantity (i.e., height is increasing less and less) across multiple cases relying on a change in another quantity (i.e., with respect to equal increase in arc length). We did not have evidence that she could generalize her regularity to any size of equal partitions (i.e., a *projected* regularity).

Emma's generalizing actions. In the Taking a Ride task, Emma began similarly to Lydia. She initially drew several consecutive vertical line segments (see red line segments in Figure 3) and discussed the directional variation of these segments with respect to traveling along the arc. Thus, Emma also *identified a regularity* in directional changes in the height of the rider (i.e., the height is increasing or decreasing). She specifically explained, "you're going to be increasing your heights or your distances from the center until you get to 3 pi over 2, and then, you're going to go back to decreasing your heights." As such, she made a *minor accommodation* to her identified regularity in the directional change in height with respect to the arc length for the first quarter rotation in order to extend it to other quarter turns.

Noticing that Emma used the phrase "slowly increasing" to describe the height of the rider above the horizontal diameter, the TR asked Emma to explain more. Emma drew a blue line segment from the 12 o'clock to the 3 o'clock position (see Figure 3) and responded, "there isn't like a linear trend that the lines [*the vertical red segments in the Figure 3 representing the different heights of the rider along the ride*] follow…. Like the lines [*referring to the vertical red* *segments in Figure 3*] don't increase at the same rate." We inferred that she used an *analogy invention* by creating a new situation (i.e., a linear trend by drawing a linear line) with the purpose of comparing and contrasting the two situations in order to show the increase in height was not linear. Although her associating the current situation with an analogous situation did not provide an explanation of why the height increased "slowly" over the given interval, she determined the increases in height were not constant or "linear".

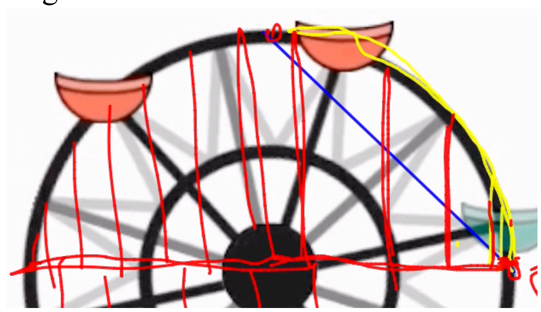


Figure 3. Emma engaging the Ferris wheel task.

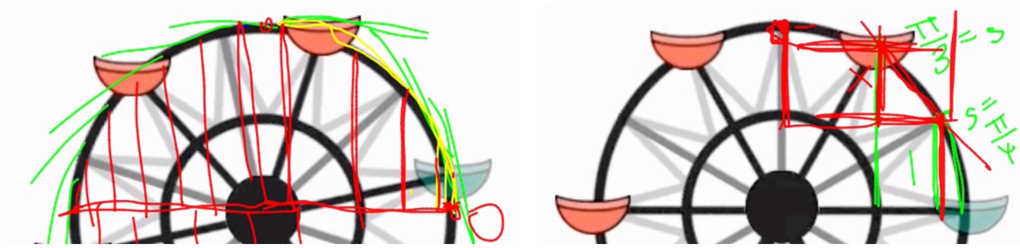
Continuing, Emma explained, "The lines increase … more steeply, and then, they slowly increase less and less [*makes a curved shape with her hand*] as you reach that inflection point at the top [referring to the point at the 12 o'clock position]." For the second quarter, she said, "They [*i.e., the heights*] start decreasing at a slower rate and then begin to decrease at a faster rate as you reach pi." It is important to note that, in contrast to Lydia, we did not have evidence that Emma coordinated the quantity of arc length beyond imagined movement in order to investigate how the increase or decrease in height changes. Although she imagined the movement at the beginning as height increasing steeply, her overall reaction to the first quarter in height was "increasing less and less", which was akin to Lydia's statement. The underlying generalizing activity of this statement, however, was quite different than that of Lydia's, which we illustrate below.

The TR returned Emma's attention to the first quarter turn and asked her to discuss how she saw the height increasing at a steeper and then slower rate. She referenced "the concavity" of the Ferris wheel in the first quarter, saying, "Well, you can see that the concavity of the – of like, I guess the arc length, the line that follows the circle…". An explanation for Emma's actions is that they were influenced by a prior context or information (i.e., *transfer*) such as presented relationships between the concavity of a curve and classifications of rates of change (e.g., concave down means increasing or decreasing at a decreasing rate).

As Emma continued, she transitioned to comparing the steepness of tangent lines that she drew along the Ferris wheel, starting at the 3 o'clock position and ending at the 9 o'clock position, as seen in Figure 4a. She said,

> …like, if these are the tangent lines showing the slopes and then you start to hit a slope that's more horizontal [*moves to the top of the circle and draws tangent lines*] as you reach your maximum point where you start decreasing … more steeply as you get back down, so like you – like these are horizontal [*motioning to the top of the wheel*]; these are vertical [*motioning to the left side of the wheel*]. So, you can see the steepness of your increasing or decreasing….

Here, Emma compared the steepness of tangent lines, concluding the tangent lines were becoming horizontal or less steep in the first quarter turn and they were getting vertical or steeper in the second quarter turn. She related these features to the height increasing at a decreasing or increasing rate. Similar to her reference to concavity, an explanation of her actions is that they were influenced by prior knowledge (i.e., *transfer*) relating the slope of lines tangent to a function graph to classifications of rate. Thus, by means of the generalizing action of the *transfer*, Emma *identified a regularity* in the steepness of tangent lines as the rider rotated around the Ferris wheel. We inferred that Emma began to identify regularity by *extracting* a few cases (i.e., multiple tangent lines that were getting steeper/horizontal, see Figure 4a), and then she immediately described a predictable feature of the entire interval, which was a *projected* identification of a regularity.



(a) (b)

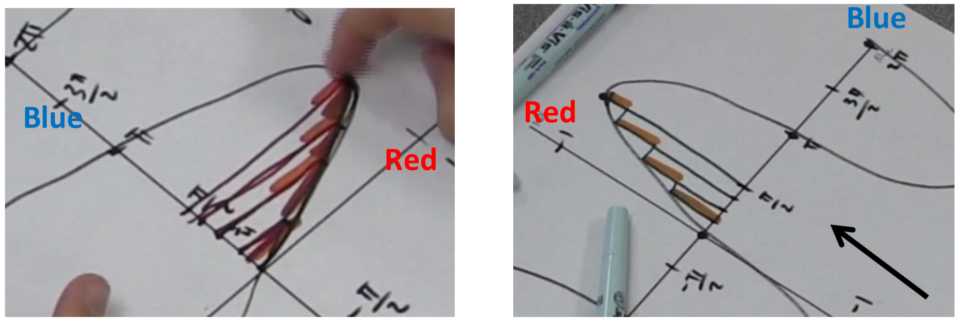
Figure 4. (a) Emma draws the tangent lines; (b) Emma identifies the amount of change in height.

It is important to note that when we asked Emma to talk about height increasing slowly (or at a decreasing or increasing rate), she persistently described tangent lines or some other idea vaguely connected to the tangent lines (e.g., ambiguous values of derivative functions as the values of the slope of tangent lines). In fact, when the TR guided her in identifying amount of changes in height in relation to each equal arc length increments along the Ferris wheel as shown in Figure 4b, she neither spontaneously noted that changes in height were decreasing, nor did she relate the changes in height to her slope of tangent lines. She instead reverted back to discussing the steepness of tangent lines and associating those features with increasing or decreasing “rates” of height. We inferred that her image of tangent lines and their steepness has little to do with explicitly coordinating changes in height with systematic changes in arc length.

The Circle Task

In the Circle Task, which occurred during a later teaching experiment session, PSTs watched an animation of a point moving along the circle and without any physical context like an amusement park ride (see Figure 1b). The TR asked them to graph the relationship between the height of the point above the horizontal diameter of the circle (i.e., the red segment in Figure 1b) and the arc length (i.e., the blue arc in Figure 1b). In this section, we only provide Emma’s data, because she engaged in generalizing actions including both coordinating quantities (i.e., Lydia above) and the steepness of the tangent lines (i.e., Emma above) in order to show how the former generalization enabled her to accommodate to new situations; however, the latter generalization did not. In addition, due to space constraints and because her activity was consistent with that described above, we will not describe Emma’s generalizing actions in a great detail. Therefore, we will only provide implications of those two different generalizing actions as it relates to Emma considering a new situation.

Emma graphed the relationship between quantities on a Cartesian coordinate system in which the vertical axis was labeled “blue” and the horizontal axis was labeled “red” (Figure 5a). She then described how the red segment changed in relation to the blue arc by coordinating quantities. She equally partitioned the arc length on the horizontal axes and determined the corresponding changes in the red segment and indicated those changes by orange line segments (Figure 5a). She explained, “… over the same arc lengths, my heights are, um, increasing but at a decreasing rate, so like, the rate that they're increasing is less and less each time." Verbally, this is nearly the same statement that she used in the Taking a Ride task in order to refer to how the height changes, but in this case it stemmed from her coordinating quantities and their changes.



(a) (b)

Figure 5. Emma's graphs in two differently-oriented Cartesian coordinate system.

Next, we asked Emma to graph the same relationship in a differently-oriented coordinate system (i.e., the vertical axis labeled as red and horizontal axis labeled as blue). She graphed the relationship as shown in Figure 5b. Importantly, she identified and illustrated (Figure 5b) that the amount of change in red was decreasing with sensitivity to the alternative orientation of the coordinate axes. Because Emma had briefly mentioned comparing tangent lines at the onset of the problem, we raised this idea in order to determine if she connected it to her partitioning activity. Referring to the graph in Figure 5b, she explained, "it's increasing at a decreasing rate, so like, the slope follows this change." She further explained that the slopes became more horizontal and she estimated the slope of the tangent lines in each increment by estimating the vertical change. In this case, her generalizing activity relative to the steepness of the tangent lines was compatible with her generalizing activity relative to coordinating quantities.

Turning to Figure 5a, Emma conceived the slopes of the tangent lines to indicate something different than that from coordinating quantities. She claimed the red quantity "is increasing at a decreasing rate", when considering the orange line segments (Figure 5a). She also claimed, "Slope is increasing at an increasing rate", when considering slopes of the tangent lines (e.g., becoming steeper and estimated the slope of the tangent lines). The TR drew her attention to these differing claims, which did not perturb Emma. Rather, these conclusions existed independent of each other. She transformed her coordinating quantities to take into account a different coordinate orientation, but her image of slope and associated generalizations were specific to a particular orientation. We do not mean to imply a deficiency in Emma's reasoning, but rather point out that due to its roots in "steepness", her generalizations regarding tangent lines and "rate" did not take into account different orientations (e.g., dy/dx versus dx/dy).

Discussion

In summary, after both reasoning about directional change in height, Lydia and Emma used similar verbal statements to describe their identified regularity in the relationship between height and arc length. Lydia's initial generalization was connected to coordinating the quantities of height, arc length, and their changes. Emma's initial generalization, along with her scheme and mental actions, primarily involved associating the steepness of tangent lines along the circle with the "rate" of height. We did not have evidence that the quantity of arc length and its variations were central to Emma's reasoning. Thus, we can claim that Emma and Lydia made similar generalized statements, but the objects that were associated and what regularity they identified were different.

Byerley and Thompson (2017) reported teachers holding different meanings for a slope, and illustrated that some of these meanings are useful in limited circumstances. For instance, they identified slope as an index of steepness as such a meaning, which correspond to Emma's case. Specifically, in Emma's case, her generalization slope as an index of steepness in association with "rate" did not enable her to conceive invariance among graphs in different coordinate orientations. With respect to Figure 5a, her meaning led her to conclude "increasing at an increasing rate." With respect to Figure 5b, her meaning led her to conclude "increasing at a decreasing rate." On the other hand, when coordinating quantities, Emma was able to conceive invariance in "rate" among the two graphs; both represented red increasing at a decreasing rate with respect to blue. Ellis (2007a) argued that generalizations rooted in coordinating quantities are more powerful, and we interpret Emma's activity as such in that her generalization rooted in quantities was more flexible than that rooted in the perceptual steepness of a tangent line.

Acknowledgments

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References

Byerley, C., & Thompson, P. W. (2017). Secondary mathematics teachers' meanings for measure, slope, and rate of change. *The Journal of Mathematical Behavior, 48*(Supplement C), 168–193.

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352–378.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66–86.

Corbin, J. M., & Strauss, A. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (3rd ed.). Thousand Oaks, CA: Sage.

Ellis, A. B. (2007a). The influence of reasoning with emergent quantities on students' generalizations. *Cognition and Instruction, 25*(4), 439–478.

Ellis, A. B. (2007b). A taxonomy for categorizing generalizations: Generalizing actions and reflection generalizations. *Journal of the Learning Sciences, 16*(2), 221–262.

Ellis, A., Tillema, E., Lockwood, E., & Moore, K. (2017). Generalization across domains: The relating-forming extending generalization framework. In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 677–684). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Desmos. (2014, 2016). Function carnival. Retrieved from https://www.desmos.com

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267–307). Hillsdale, NJ: Erlbaum.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics*. Albany, NY: SUNY Press.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sépulveda (Eds.), *Plenary Paper presented at the Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 31–49). Morélia, Mexico: PME.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education*, WISDOMe Monographs (Vol. 1, pp. 33–57). Laramie, WY: University of Wyoming Press.

ON HOW PARTICIPATION IN A MODELING COMPETITION OCCASIONS CHANGES IN UNDERGRADUATE STUDENTS' SELF-EFFICACY REGARDING MATHEMATICAL MODELING

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Though scholars have long called for applications and modeling to be explicitly added to classroom agenda (Niss, Blum, & Galbraith, 2007), opportunities for undergraduates to engage in modeling in the classroom remain scarce. We share the efforts of a national organization (SIMIODE) to provide extra-curricular opportunities for undergraduate STEM majors to engage in authentic, open-ended modeling tasks using differential equations through a modeling competition. In this preliminary report, we document changes in twenty-one undergraduates' self-efficacy regarding their own modeling competencies, develop hypotheses about what aspects of the competition occasioned those changes, and how these changes may benefit students.

Key words: self-efficacy, mathematical modelling, differential equations

Educators have increasingly turned to mathematical modeling to resolve the *relevance paradox* of mathematics in the curriculum. The relevance paradox refers to the disparity between the objective relevance of mathematics for society and the subjective irrelevance of mathematics perceived by many students who study it (Niss & Hojgaard, 2011). Even at the post-secondary level, interest in mathematics is associated with seeing its practical relevance (Liebendörfer & Schukajlow, 2017). Since Mathematical modeling promotes interdisciplinary thinking (Bliss et al., 2016), fosters mathematical reasoning as a basis of decision making (OECD, 2017), and develops communication skills (Niss & Hojgaard, 2011), many instructors endeavor to engage students in mathematical modeling. Rising demand for instructional materials for the advanced mathematics of STEM majors has led organizations like the Systemic Initiative for Modeling Investigations & Opportunities with Differential Equations (SIMIODE) and the Community of Ordinary Differential Equations Educators (CODEE) to develop authentic modeling experiences that engage students in problems from (for example) mathematical biology or systems engineering. Because opportunities to engage in authentic modeling in STEM classrooms remain scarce, SIMIODE began hosting an extra-curricular mathematical modeling competition that would introduce students to authentic modeling problems in differential equations and build their self-confidence to carry out modeling. The present study investigated changes in student self-efficacy from before to after participating in the event. In this paper, we report on preliminary results of the pilot run and develop hypotheses for a large-N nationwide competition that will be hosted in April 2018.

Cognitive Perspective on Modeling and Empirical Background

From multiple perspectives on modelling, we adopt the cognitive perspective which views modeling as an iterative process (Kaiser, 2017). In this view, modeling is studied as a suite of interrelated competencies that transform a real-world problem into a mathematical problem by building a cognitive correspondence between the two. The competencies include identifying relevant variables, simplifying relevant quantities and their relationships, estimating parameters, and expressing the quantities and relationships mathematically (Maaß, 2006). Other modeling competencies include validating the appropriateness or reasonableness of the model and communicating results, findings, or consequences of using the model.

In post-secondary classrooms, models are foundational to STEM disciplines, often augmenting or replacing "real world situations" as the object of study. For example, one studies the "heat equation" or the "particle in a box." The former stands for a range of imaginary situations where heat energy transfers between an object and its environment while the latter is an imaginary physical system determined by second-order differential equation. The complexity of mathematical theory and procedures in advanced STEM courses, along with departmental pressures sometimes overwhelm instructors' intentions to use modeling to motivate course content. Within differential equations, scholars have focused largely on student achievement of curricular aims. For example, Czocher (2017) found that consistent emphasis on mathematical modeling principles, even in lecture, could positively impact engineering students' learning of differential equations. Others have shown that drawing on "experientially real" starting points to instruction can positively impact student learning of content (e.g., Rasmussen & Blumenfeld, 2007). However, the complexity of mathematical theory and procedures in advanced mathematics courses, along with time constraints, departmental pressures, and beliefs about non-lecture approaches, sometimes overwhelm instructors' desires to use research-based innovations in their classrooms (Johnson, Keller, & Fukawa-Connelly, 2017). These facts point to the need to document the advantages of modeling at the post-secondary level.

Affective factors such as interest, motivation, and self-efficacy are also important to developing modeling competencies. We take self-efficacy to be a problem-specific assessment of an individual's confidence in their ability to successfully address a mathematics problem (Hackett & Betz, 1989, p. 262). Schukajlow et al. (2012) found that for ninth graders, student-centered instruction using modeling tasks was the most beneficial for increasing student affect and Zbiek and Conner (2006) reported that a modeling approach deepened prospective secondary mathematics teachers' motivation to learn new mathematics content. Since self-efficacy plays a role of all aspects of goal-setting, persistence, and effort (Bandura, 2006), we interpret the literature to mean that finding ways to increase STEM students' self-efficacy in modeling may go hand-in-hand with increasing their modeling competencies and their willingness to engage in activities designed to improve their modeling competencies. Thus, there is a pressing need to gain a sense of whether and how engaging in mathematical modeling might impact undergraduate students' self-efficacy about modeling competencies.

Methods

Twenty-six students participated in teams of 2 or 3 led by an instructor-coach in an intensive week-long modelling competition. Each team selected one of three challenging problems framed to be about *changes* in relevant quantities, necessitating the use of differential equations. The teams created a written overview of their models and argued for its merits. At the end of the week, teams gathered at the competition site where coaches offered critiques and additional factors to consider in their models. The teams amended their models and the competition culminated in a 10-minute peer-reviewed presentation of their solutions. The final models were subsequently ranked by the judges and winners were notified following the competition.

We explored changes in student self-efficacy regarding modeling competencies (Maaβ 2006), including: identifying important variables, estimating parameters, making simplifying assumptions, mathematization, validation, communication. We developed a self-efficacy scale (Bandura, 2006), implemented as a matched pair of online surveys. Survey items asked students to rate their confidence in carrying out the modeling competencies for a problem about modeling the spread of solar panel adoption in the 21st century. They responded on a 100-point scale, with 10-unit intervals from 0 (“Cannot do”); through intermediate degrees of assurance, 50 (“Moderately certain can do”); to complete assurance, 100 (“Highly certain can do”). The pre-event survey items in Table 1 were changed to past tense for the post-event survey and the context for Item 1 was changed to modeling the spread of smart home appliances in the 21st century. The survey also collected demographic information about the students’ majors. Twenty-one students completed both the pre- and post-event surveys. Nine students were mathematics majors, three were applied mathematics majors and the rest were a mix of computer science, statistics, and engineering majors. A set of six dependent sample t-tests was performed on the mean difference between pre- and post- scores at significance level of $\alpha = .05$ for each item (a post-hoc Bonferroni correction was planned, but not needed).

Results and Discussion

Participants collectively showed gains in their self-efficacy in all 6 questions (Table 1). Students' self-efficacy for mathematizing $(t = 4.503, df = 20, p = 0.000)$, identifying important variables $(t = 2.832, df = 20, p = 0.027)$, and ability to communicate the usefulness of their model $(t = 2.104, df = 20, p = 0.048)$ has increased where as their self-efficacy in simplifying $(t = 1.330, df = 20, p = 0.198)$, validating $(t = 1.722, df = 20,\ p = 0.101)$and estimating parameters $(t = 0.377, df = 20,\ p = 0.710)$ remain statistically unchanged.

Table 1 Self-efficacy items (pre-event survey) and targeted modeling competency

Rate your level of confidence by recording a number from 0 to 100 using the scale given below: **0 10 20 30 40 50 60 70 80 90 100** **Cannot do at all — Moderately can do — Highly certain can do**	**Competency**	**Mean Difference (post – pre)**
1. Create a differential equation model for the spread of the use of electric powered vehicles in the United States during the twenty-first century.	Mathematizing	19.524
2. In (1) identify the important variables leading to a reasonably accurate prediction.	Identify variables	10.000
3. In (1) make simplifying assumptions to reduce the number of important variables.	Simplify	6.190
4. In (1) consult appropriate resources to check whether your model was reasonable.	Validating	7.143
5. In (1) convince a solar power car manufacturer that your model would be useful for its purpose.	Communicating	10.952
6. Estimate the parameters α and β in the following differential equation which describes the time rate of formation of material A, $'(t) = \alpha A(t)^{\beta}$, given a set of data from observations for time, t, and amount of material A, at time t, i.e. A(t).	Estimating	1.905

The competition reproduced some of the favorable factors leading to success in modeling observed by Carlson, Larsen, & Lesh (2003), such as externalizing and communicating reasoning, providing and receiving feedback from their peers and others, revising their initial models, and having extended time to fully engage with the problem. Though we cannot say what aspect(s) of the modeling competition may have contributed causally to any changes in self-efficacy, we can hypothesize that these factors, taken together, contributed to the students’ increased self-efficacy in mathematical modelling. Participant self-selection led to two methodological complications: (1) participants with high self-efficacy may not show gains and

(2) the student sample was exceptional in their self-assessment of mathematical content knowledge (nearly all stated they usually get A's in mathematics). Another limitation of this study is that the survey used an *atomistic* (Boesen et al., 2014) approach to assessing students' self-efficacy by asking them about each modeling competency independently. Future research should develop parallel measures for holistic self-assessments. We anticipate that the larger sample will reveal stronger relationships among student beliefs about the relevancy of mathematics, their personal achievements during the competition, and self-efficacy regarding subsets of the modeling competencies which are emphasized in the competition. Middleton and Spanias (1999, p. 81) argued that it is better to develop students' intrinsic motivation in mathematics, rather than providing extrinsic incentives for achievement. We hypothesize that the modeling competition, which addresses the relevance paradox through providing opportunities for students to engage in challenging and authentic discipline-specific problems, may be one avenue for doing so.

References

Czocher, J. A. (2017). How can emphasizing mathematical modeling principles benefit students in a traditionally taught differential equations course? The Journal of Mathematical Behavior 45, 1-17.

Bandura, A. (2006). Guide for constructing self-efficacy scales. In F. Pajares & T. Urdan (Eds.), *Self-efficacy beliefs of adolescents* (Third, pp. 307–337). United States of America: Information Age Publishing.

Bliss, K., et al. (2016). *Guidelines for assessment & instruction in mathematical modeling education*. Consortium for Mathematics and Its Applications and the Society for Industrial and Applied Mathematics.

Boesen, J., Helenius, O., Bergqvist, E., Bergqvist, T., Lithner, J., Palm, T., & Palmberg, B. (2014). Developing mathematical competence: From the intended to the enacted curriculum. *The Journal of Mathematical Behavior*, *33*, 72–87.

Carlson, M., Larsen, S., & Lesh, R. (2003). Integrating a models and modeling perspective with existing research and practice. *Beyond Constructivism: Models and Modeling Perspectives on Mathematics Problem Solving, Learning, and Teaching.*, (1998), 465–478.

Hackett, G., & Betz, N. E. (1989). An Exploration of the Mathematics Self-Efficacy/Mathematics Performance. *Journal for Research in Mathematics Education*, *20*(3), 261–273.

Johnson, E., Keller, R., & Fukawa-Connelly, T. (2017). Results from a Survey of Abstract Algebra Instructors across the United States: Understanding the Choice to (Not) Lecture. *International Journal of Research in Undergraduate Mathematics Education*.

Kaiser, G. (2017). The teaching and learning of mathematical modeling. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 267–291).

Liebendörfer, M., & Schukajlow, S. (2017). Interest development during the first year at university : do mathematical beliefs predict interest in mathematics ? *ZDM*, *49*(3), 355–366.

Maaß, K. (2006). What are modelling competencies? *ZDM*, *38*(2), 113–142.

Middleton, J. A., & Spanias, P. A. (1999). Motivation for achievement in mathematics: Findings, generalizations, and criticisms of the research. *Journal for Research in Mathematics Education*, *30*(1), 65–88.

Niss, M., Blum, W., & Galbraith, P. (2007). Introduction. In W. Blum, P. L. Galbraith, H. W. Henn, & Mogens Niss (Eds.), *Modelling and applications in mathematics education* (pp. 3–32). New York: Springer.

Niss, M., & Hojgaard, T. (2011). Competencies and Mathematical Learning: Ideas and inspiration fo rthe development of mathematics teaching and learning in Denmark. (M. Niss & T. Hojgaard, Eds.). Roskilde.

OECD. (2017). PISA 2015 Assessment and Analytical Framework. Paris: OECD Publishing.

Rasmussen, C., & Blumenfeld, H. (2007). Reinventing solutions to systems of linear differential equations: A case of emergent models involving analytic expressions. *The Journal of Mathematical Behavior*, *26*(3), 195–210.

Schukajlow, S., Leiss, D., Pekrun, R., Blum, W., Müller, M., & Messner, R. (2012). Teaching methods for modelling problems and students' task-specific enjoyment, value, interest and self-efficacy expectations. *Educational Studies in Mathematics*, *79*(2), 215–237.

Zbiek, R. M., & Conner, A. (2006). Beyond motivation: Exploring mathematical modeling as a context for deepening students' understandings of curricular mathematics. *Educational Studies in Mathematics*, *63*(1), 89–112.

DIFFICULTIES OF LOGICAL NATURE IDENTIFIED IN STUDENTS IN A FIRST COURSE OF CALCULUS

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This paper reports some results of an investigation about the difficulties of logical nature that students have in a first course of Calculus. The instrument used was designed with questions of the basic contents of Calculus and was applied to students who had already taken a first course of it. It was found that difficulties of a logical nature persist in most of the students such as modus ponens, affirmation of the consequent, handling of quantifiers, denial of statements, which complicate the comprehension of basic concepts of Calculus.

Keywords: Advanced Mathematical Thinking, Reasoning and Proof.

The study of Calculus in a mathematics career at the university is really a course on its foundations. This brings up two important elements: the language accompanied by mathematical symbology and logical reasoning. Regarding the latter, we emphasize the students' difficulties in distinguishing between necessity and sufficiency in a condition, as well as in the interpretation of the universal and existential quantifiers and, finally, in the denial of propositions.

In relation to the above, we pose the following question: what are the difficulties of a logical nature in the courses of Calculus that students have during their first year in a major in mathematics? This report gives account of three common situations in the course of Calculus that illustrate the difficulties mentioned above.

Theoretical Framework

M. Artigue classifies the difficulties of access to Calculus in three categories: (1) those associated with the complexity of the basic objects of Calculus, (2) difficulties associated with the conceptualization and formalization of the concept of limit and (3) difficulties related to the necessary rupture between algebraic reasoning and calculus reasoning (Artigue, 1995). In the same sense, R. Moore identifies seven difficulties in the comprehension of a mathematical proof in students with little experience (Moore, 1994). We emphasize that logic impacts each of the seven difficulties. Dubinsky & Yiparaki (2000) and Epp (2003) showed that a significant number of students confuse the management (and order of placement) of universal and existential quantifiers in a mathematical proposition.

On the other hand, when asked for a proof of a contradiction theorem, a large number of students do not identify either the reason for the beginning of the proof or when it has ended. According to U. Leron, proofs by contradiction are difficult for students to digest, they represent a psychological problem that involves mathematical thinking (Leron, 1985).

Finally, investigations in logical reasoning account for the complex paradigm of negations (Evans & Handley, 1999). Also, it has been reported that the verbal and mathematical contexts can influence when making a logical inference (Stylianides, Stylianides, & Philippou, 2004); for axample, when we ask about the existence of the limit of a quotient sequences whose denominator tends to zero, prior knowledge of the prohibition of dividing by zero can act as a "distractor" to infer that, necessarily, the limit of the quotient does not exist.

Conceptual framework

In this research we deal with the term *difficulty* as it is exposed in the researches and theoretical contributions on the difficulties of access to the Calculus of Artigue (1995) and in understanding a proof Moore (1994). For the difficulties in logical reasoning, we rely on Evans & Handley (1999), Evans, Handley, Neilens & Over (2007), and Inglis & Simpson (2008).

We will understand by logical reasoning (conditional) the inferences that can be made from a rule in the form "$P \rightarrow Q$" and a given premise (Evans *et al,* 2007; Inglis & Simpson, 2008). In this report we focus only on the valid inference *modus ponens* (MP) and on the invalid inference *affirmation of the consequent* (AC).

Methodology

The research instrument consisted of a questionnaire of 15 multiple-choice questions and interviews. The questionnaire was accessed during a 90-minute online session. The instrument was applied to 73 students from two of the most important educational institutions in Mexico. The students who participated in our research had just passed their first Calculus course. Each question was designed within the context of the Calculus and is supported by the research findings of Moore (1994), Artigue (1995) and in the investigations on logical reasoning Evans & Handley (1999), Evans *et al* (2007) and Inglis & Simpson (2008).

Discussion and Results analysis

We report the analysis and discussion of questions 3, 13 and 15, of our questionnaire. Question 3 presents a wrong process of solving an equation. It starts with the equation and, through a succession of implications, it ends with a result. The student was asked to decide if the procedure was correct or not. Of course, the student had the option of verifying that the result obtained was not a solution to the equation, so he could suspect that the process was wrong; nevertheless, approximately 50% of the participants concluded that the process was correct. In general, the other half argued that the supposed solution did not satisfy the equation, for example, one of them argued: "algebraically everything seems to me to be correct, but when substituting the value, we conclude that 4 = 1". Another student said, "I do not remember the reason well, but, according to me, there is no solution within the real numbers (sic)". In fact, few found the wrong step in the resolution process, which resulted from a non-reversible implication. This reveals a lack of understanding in the reasoning of the type MP and AC. The conditional "if-then" and "only if" are usually interpreted individually as the double conditional, "if and only if" (Epp, 2003).

Question 13 refers to the following statement:

> If $a \in \mathbb{R}$, such that $0 \leq a \leq \xi$, for all $\xi > 0$, then $a = 0$, of which the following statement is made: the statement is contradictory, since a particular case of the inequality $a \leq \xi$, is $a = \xi$, with ξ a positive number. However, the statement concludes that $a = 0$.

Almost half answered that the statement is true, that is to say that the statement is contradictory. One of the justifications for this choice was obtained in an interview.

> *Teacher:* Regarding question 13, I'm interested in knowing what you thought when you answered the question for the first time.
>
> *Student:* It seemed logical to me to think about this example. If you set to epsilon with value equal to 3 you can take any a other than zero that is less than or equal to 3.

To expand this question, a complementary activity was carried out in the classroom, which consisted of completing the following statement: let $a, b \in \mathbb{R}$, if for any $\varepsilon > 0$ it is satisfied that $0 \leq |a\text{-}b| < \varepsilon$, then __________.

Some students tried to solve the inequality, which reveals a lack of familiarity with statements of the MP form. Others related the statement with the concept of upper bound; the majority related it with the definition of limit or continuity of a function, which is illustrated in Figure 1, 'it has to comply with some continuity, but I do not know how to express it'. Others related it to the definition of integral that involves partitions of superior and inferior sums. This suggests the influence of their previous knowledge in his reasoning, as suggested by Stylianides *et al* (2004), although the statement to complete had nothing to do with functions, nor partitions. Only 13% of participants in this complementary activity were able to deduce $a = b$.

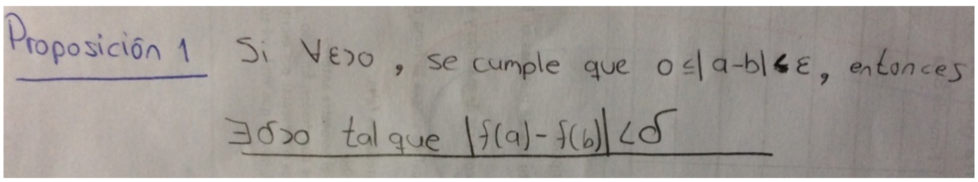

Figure 1. Student's answer in the complementary activity.

Both the online questionnaire and the complementary activity corroborate the difficulty that students have to understand the meaning and interpretation of the universal quantifier in a particular context of the Calculus, in a form of reasoning MP.

Our last example (question 15) has the purpose of identifying the difficulties that students present to recognize the end of a proof by contradiction (Moore, 1994). This reasoning entail, three difficulties: to understand the process of a demonstration by contradiction (Leron, 1985), to deny the statement (Evans & Handley, 1999) and to identify the contradiction to finish the proof.

The proposition to prove was: For all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \varepsilon$. We provided the following answers to conclude the proof: (a) there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, $0 < n \leq {}^{1}/_{\varepsilon}$, (b) the natural numbers are bounded superiorly, (c) there exists $\varepsilon > 0$ and there exists $n \in \mathbb{N}$ such that $0 < {}^{1}/_{n} < \varepsilon$, (d) there exists $n \in \mathbb{N}$ such that $0 < {}^{1}/_{\varepsilon} < {}^{1}/_{n}$ for some $\varepsilon > 0$.

Note that answers (a) and (b) are equivalent and correct. The first is the result that is obtained, immediately, of denying the statement, and the second is its translation to the concept of bounded set. The purpose of putting these two correct answers was to find out if they manage to recognize the contradiction in different versions.

Several students chose option (c); their arguments were: "because we wanted to proof that, for a given epsilon, there was such a number with such property" or "because we will have found then an n for the given epsilon that fulfills the thesis (sic)". In other cases, we found these justifications: "because it is the opposite of what I want to demonstrate" or "it is the negation of the statement", which allows us to conclude that these students conceive the contradiction obtained with the denial of the statement.

Of those who answered correctly, most chose option (b), which is a translation of the contradiction. In contrast, we mention the case where some students chose item (a), but their arguments do not reflect an understanding of what a contradiction proof means: "the conclusions must be related to the initial proposition" and "the rest of the paragraphs do not make sense because they say different things from the proposition".

Conclusions

The difficulties of a logical nature were manifested in most of the participants in different aspects, for example, in the use of existential and universal quantifiers, in some forms of conditional reasoning and in the denial of propositions. All this prevents the understanding of basic concepts of Calculus, because it is not necessary to advance too much in these courses so that these difficulties arise.

Although the participants were students in the second semesters of the major in mathematics, these difficulties of a logical nature persist in a significant number of them.

Based on the results of the investigations related to the logical aspects and in our own study, we consider it convenient to look at the elaboration, design and application of didactic strategies that promote and favor a better performance in the logical reasoning of the students in the context of the Calculus, and so in these courses, special attention is paid to the type of logical difficulties reported**.**

References

Artigue, M. (1995). La enseñanza de los principios del cálculo: problemas epistemológicos, cognitivos y didácticos. Gómez, P. (Ed.). *Ingeniería didáctica en educación matemática*. 97–140.

Dubinsky, E., & Yiparaki, O. (2000). On student understanding of AE and EA quantification. *Research in Collegiate Mathematics IV*, 275–282. https://doi.org/10.1007/0-306-47231-7_25

Epp, S. (2003). The role of logic in teaching proof. *American Mathematical Monthly 110*, 886–899.

Evans, J. S. B. T., & Handley, S.J. (1999). The role of negation in conditional inference. *Quarterly Journal of Experimental Psychology, 52A*(3) , 739–769.

Evans, J. St. B. T., Handley, S. J., Neilens, H., & Over, D. E. (2007). Thinking about conditionals: A study of individual differences. *Memory & Cognition, 35*(7), 1772–1784.

Inglis, M. & Simpson, A. (2008). Conditional inference and advanced mathematical study. *Educ Stud Math, 67*, 187–204. doi: 10.1007/s10649-007-9098-9

Leron, U. (1985). A direct approach to indirect proofs. *Educational Studies in Mathematics 16*(3), 321–325.

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics, 27*, 249–266.

Stylianides, A. J., Stylianides, G. J., & Philippou, G. N. (2004). Undergraduate Students' understanding of the contraposition equivalence rule in symbolic and verbal contexts. *Educational Studies in Mathematics, 55*, 133–162.

DIFICULTADES DE NATURALEZA LÓGICA IDENTIFICADAS EN LOS ESTUDIANTES EN UN PRIMER CURSO DE CÁLCULO

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En este artículo reportamos algunos resultados de una investigación acerca de las dificultades de naturaleza lógica que tienen los estudiantes en un primer curso de Cálculo. El instrumento empleado en la experimentación fue diseñado con preguntas en el contexto de los contenidos básicos de Cálculo, y se aplicó a estudiantes que ya habían acreditado un primer curso del mismo. Encontramos que en la mayoría de los estudiantes persisten dificultades de naturaleza lógica —tales como modus ponens, afirmación del consecuente, manejo de cuantificadores, y negación de enunciados— que impiden la comprensión de conceptos básicos del Cálculo.

Keywords: Pensamiento matemático avanzado, Razonamiento y prueba.

El estudio del Cálculo en una carrera de Matemáticas en la Universidad es realmente un curso sobre sus fundamentos. Con ello surgen dos elementos importantes: el lenguaje acompañado de la simbología matemática y el razonamiento lógico. Al respecto destacamos las dificultades de los estudiantes para distinguir entre los significados de las condiciones de necesidad y las condiciones de suficiencia en una proposición, así como en el manejo de los cuantificadores universal y existencial y en la negación de proposiciones.

En relación con lo anterior, nos planteamos la siguiente pregunta: ¿cuáles son las dificultades de naturaleza lógica en los cursos de Cálculo que tienen los estudiantes durante su primer año en una carrera universitaria de matemáticas? En este reporte, damos cuenta de tres situaciones en el Cálculo que ilustran las dificultades antes mencionadas.

Marco Teórico

M. Artigue clasifica las dificultades de acceso al Cálculo en tres categorías: (1) aquellas asociadas a la complejidad de los objetos básicos del Cálculo, (2) dificultades asociadas con la conceptualización y la formalización del concepto de límite y (3) dificultades vinculadas con las rupturas necesarias entre el razonamiento algebraico y el razonamiento en cálculo (Artigue, 1995). En el mismo sentido, R. Moore identifica siete dificultades en la compresión de una prueba matemática en estudiantes con poca experiencia (Moore, 1994), destacamos que la lógica impacta en cada una de las siete dificultades. Dubinsky & Yiparaki (2000) y Epp (2003) muestran que un número considerable de estudiantes confunden el manejo (orden de colocación) de los cuantificadores universal y existencial en una proposición matemática.

Por otra parte, al realizar en clase una prueba de un teorema por contradicción, algunos estudiantes no tienen claro el porqué del inicio de la prueba, ni cuándo ha finalizado. De acuerdo con U. Leron, las pruebas por contradicción resultan difícil de digerir para los estudiantes, representan un problema psicológico que involucra pensamiento matemático (Leron,1985).

Finalmente, investigaciones en el razonamiento lógico dan cuenta del complejo paradigma de las negaciones (Evans & Handley, 1999). También, se ha reportado que los contextos verbal y matemático pueden influir a la hora de realizar una inferencia lógica (Stylianides, Stylianides & Philippou, 2004); por ejemplo, al preguntarnos por la existencia del límite de un cociente de sucesiones cuyo denominador tiende a cero, el conocimiento previo de la prohibición de dividir entre cero puede fungir como un "distractor" para inferir que, necesariamente el límite del cociente no existe.

Marco conceptual

En esta investigación manejamos el término *dificultad* como se expone en las investigaciones y aportaciones teóricas sobre las dificultades de acceso al Cálculo de Artigue (1995) y en la comprensión de una prueba Moore (1994). Para las dificultades sobre el razonamiento lógico nos basamos en Evans & Handley (1999), Evans, Handley, Neilens & Over (2007), e Inglis & Simpson (2008). Entenderemos por razonamiento lógico (condicional) a las inferencias que se pueden realizar a partir de una regla en forma "$P \to Q$" y una premisa dada (Evans *et al*, 2007; Inglis & Simpson, 2008). En este reporte nos enfocamos sólo en la inferencia válida *modus ponens* (MP) y en la inferencia inválida *afirmación del consecuente* (AC).

Metodología

El instrumento de investigación consistió en un cuestionario de 15 preguntas de opción múltiple y entrevistas. El acceso al cuestionario fue durante una sesión de 90 minutos en línea. El instrumento se aplicó a 73 estudiantes de dos de las instituciones educativas más importantes de México. Los estudiantes que participaron en nuestra investigación recién habían aprobado su primer curso de Cálculo. Cada pregunta fue diseñada dentro del contexto del Cálculo y están apoyadas en los hallazgos de las investigaciones de Moore (1994), Artigue (1995) y en las investigaciones sobre el razonamiento lógico Evans & Handley (1999), Evans *et al* (2007) e Inglis & Simpson (2008).

Análisis y discusión de resultados

Reportamos el análisis y la discusión de las preguntas 3, 13 y 15 de nuestro cuestionario.

La pregunta 3 presenta un proceso erróneo de resolución de una ecuación. Se parte de ésta y, mediante una sucesión de implicaciones, se llega a un resultado. Se le pidió al estudiante que opinase si era o no correcto el procedimiento. Por supuesto, tenía la opción de verificar que el resultado obtenido no era solución de la ecuación, de manera que podía sospechar que el proceso era erróneo; no obstante, aproximadamente la mitad de los participantes concluyeron que el procedimiento era correcto. Los estudiantes restantes expresaron que la supuesta solución no satisfacía la ecuación, por ejemplo, algunos argumentaron: "algebraicamente todo me parece correcto, pero al sustituir el valor llegamos a que 4 = 1". Otro expresó "no recuerdo bien el motivo, pero, según yo, no tiene solución dentro de los reales". En realidad, pocos encontraron el paso erróneo en el proceso, mismo que resultaba de una implicación no reversible. Esto revela falta de comprensión en los razonamientos del tipo MP y AC. Los condicionales "si-entonces" y "sólo si" suelen interpretarse de manera individual como la doble condicional, "si y sólo si" (Epp, 2003).

La pregunta 13 se refiere al siguiente enunciado:

> Si $a \in \mathbb{R}$, es tal que $0 \leq a \leq \xi$, para toda $\xi > 0$, entonces $a = 0$. En este caso se le pide al estudiante que opine sobre la veracidad o falsedad de la afirmación: "el enunciado es contradictorio, pues un caso particular de la desigualdad $a \leq \xi$, es $a = \xi$, con ξ un número positivo. Sin embargo, el enunciado concluye que $a = 0$".

Casi la mitad contestó que la afirmación es verdadera, es decir que el enunciado es contradictorio. Una de las justificaciones de esta elección la obtuvimos en entrevista:

Profesor: Con respecto a la pregunta 13, me interesa saber lo que pensaste al responder la pregunta por vez primera.

Estudiante: Me sonó lógico pensar en este ejemplo. Si fijas a épsilon con valor igual a 3 puedes tomar cualquier *a* diferente de cero que sea menor o igual a 3.

Para profundizar en esta pregunta, se realizó una actividad complementaria en el salón de clases, la cual consistía en completar el siguiente enunciado:

Sean $a, b \in \mathbb{R}$, si para toda $\varepsilon > 0$ se cumple $0 \leq |a\text{-}b| < \varepsilon$, entonces _________.

Algunos estudiantes intentaron resolver la desigualdad, esto revela una falta de familiaridad con enunciados de la forma MP. Otros relacionaron el enunciado con el concepto de cota superior; la mayoría, lo relacionó con la definición de límite o de continuidad de una función, lo que se ilustra en la Figura 1, “tiene que cumplir algo de continuidad, pero no sé expresarlo”. Otros tantos, lo relacionaron con la definición de integral que involucra particiones de sumas superiores e inferiores. Esto sugiere la influencia de su conocimiento previo en su razonamiento, Stylianides *et al* (2004), aunque el enunciado a completar nada tenía que ver con funciones, ni particiones. Sólo el 13% de participantes de esta actividad lograron deducir $a = b$.

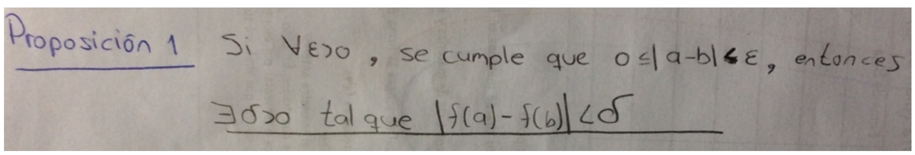

Figura 1. Respuesta de estudiante en actividad complementaria.

Este par de actividades corroboran la dificultad que tienen los estudiantes para comprender el significado e interpretación del cuantificador universal en un contexto particular del Cálculo, en una forma de razonamiento del tipo MP.

Nuestro último ejemplo (pregunta15) tiene como propósito identificar las dificultades que los estudiantes presentan para reconocer el final de una prueba por contradicción (Moore, 1994). Este tipo de razonamiento conlleva tres dificultades: comprender el proceso de una demostración por contradicción (Leron, 1985), negar el enunciado (Evans & Handley, 1999), e identificar la contradicción con la que finaliza la prueba.

La proposición por demostrar fue: Para toda $\varepsilon > 0$, existe $n \in \mathbb{N}$ tal que $0 < \frac{1}{n} < \varepsilon$. Propusimos las siguientes respuestas para dar por concluida la prueba: (a) existe $\varepsilon > 0$, tal que para toda $n \in \mathbb{N}$, $0 < n \leq 1/\varepsilon$, (b) los números naturales están acotados superiormente, (c) existe $\varepsilon > 0$ y existe $n \in \mathbb{N}$ tal que $0 < 1/n < \varepsilon$, (d) existe $n \in \mathbb{N}$ tal que $0 < 1/\varepsilon < 1/n$, para alguna $\varepsilon > 0$.

Notemos que las respuestas (a) y (b) son equivalentes y son las correctas. La primera es el resultado que se obtiene de manera inmediata de la negación del enunciado y la segunda es su traducción al concepto de conjunto acotado. La finalidad de poner estas dos respuestas correctas fue para averiguar si logran reconocer la contradicción en diferentes versiones.

Varios estudiantes eligieron la opción (c); sus argumentos fueron del tipo: “porque se quería demostrar que, para una épsilon dada, existía dicho número con tal propiedad” o “porque se habrá hallado entonces un *n* para la épsilon dada que cumpla la tesis (sic)”. En otros casos encontramos justificaciones del tipo: “porque es lo contrario de lo que quiero demostrar” o “es la negación del enunciado”, lo que nos permite concluir que estos estudiantes conciben la contradicción obtenida con la negación del enunciado.

De los que respondieron correctamente, la mayoría escogió la opción (b), que es una traducción de la contradicción. En contraste, mencionamos el caso donde algunos estudiantes eligieron el inciso (a), pero sus argumentos no reflejan una comprensión de lo que significa una prueba por contradicción: “las conclusiones deben estar relacionadas con las observaciones iniciales” y “los demás incisos no tienen sentido porque dicen cosas distintas a la proposición”.

Conclusiones

Las dificultades de naturaleza lógica se manifestaron en la mayoría de los participantes en diferentes aspectos, por ejemplo, en el uso de los cuantificadores existencial y universal, en algunas formas de razonamiento condicional y en la negación de proposiciones. Todo esto impide la comprensión de conceptos básicos del Cálculo, pues no es necesario avanzar demasiado en estos cursos para que estas dificultades surjan. No obstante que los participantes fueron estudiantes de segundo semestre de una carrera de matemáticas, estas dificultades de naturaleza lógica persisten en una cantidad significativa de ellos.

Con base en los resultados de las investigaciones relacionadas con los aspectos lógicos y en nuestro propio estudio, consideramos conveniente mirar hacia la elaboración, diseño y aplicación de estrategias didácticas que promuevan y favorezcan un mejor desempeño en el razonamiento lógico de los estudiantes en el contexto del Cálculo, y así en estos cursos se preste especial atención en el tipo de dificultades lógicas reportadas.

Referencias

Artigue, M. (1995). La enseñanza de los principios del cálculo: problemas epistemológicos, cognitivos y didácticos. Gómez, P. (Ed.). *Ingeniería didáctica en educación matemática*. 97–140.

Dubinsky, E., & Yiparaki, O. (2000). On student understanding of AE and EA quantification. *Research in Collegiate Mathematics IV*, 275–282. https://doi.org/10.1007/0-306-47231-7_25

Epp, S. (2003). The role of logic in teaching proof. *American Mathematical Monthly 110*, 886–899.

Evans, J. S. B. T., & Handley, S.J. (1999). The role of negation in conditional inference. *Quarterly Journal of Experimental Psychology, 52A*(3) , 739–769.

Evans, J. St. B. T., Handley, S. J., Neilens, H., & Over, D. E. (2007). Thinking about conditionals: A study of individual differences. *Memory & Cognition, 35*(7), 1772–1784.

Inglis, M. & Simpson, A. (2008). Conditional inference and advanced mathematical study. *Educ Stud Math, 67*, 187–204. doi: 10.1007/s10649-007-9098-9

Leron, U. (1985). A direct approach to indirect proofs. *Educational Studies in Mathematics 16*(3), 321–325.

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics, 27*, 249–266.

Stylianides, A. J., Stylianides, G. J., & Philippou, G. N. (2004). Undergraduate Students’ understanding of the contraposition equivalence rule in symbolic and verbal contexts. *Educational Studies in Mathematics, 55*, 133–162.

MAGNITUDE REASONING: CHARACTERIZING A PRE-CALCULUS STUDENT'S QUANTITATIVE COMPARISON BETWEEN COVARYING MAGNITUDES

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This paper discusses a student coordinating and comparing amounts of change in two covarying quantities' magnitudes (as opposed to numerical values). We describe not only his coherent system of mental actions involved in coordinating magnitudes, but also how this system affords his reasoning about graphical concavity and a quotient between magnitudes.

Keywords: Cognition, Algebra and Algebraic Thinking, Design Experiments

Quantity, Covariation, and Quotient

Many researchers have attempted to understand how students and teachers coordinate and compare quantities (Carlson, Jacobs, Coe, & Hsu, 2002; Johnson, 2015). We adopt Thompson's (1994) conceptualization of a quantity that entails reasoning about a quantity's *magnitude* (i.e., amount-ness) while anticipating infinitely many associated value-unit pairs, and thus does not require physically carrying out a measurement process to obtain a quantity's values. This distinction between a quantity's magnitude and its value enables us to account for reasoning about a quantity's variations that are not constrained to the availability of values; one can imagine a varying segment as entailing a magnitude (i.e., lengthiness) varying in flux.

Imagining variations in a quantity's magnitude positions a person to reason covariationally. Carlson et al. (2002) specified levels of mental actions involved in coordinating quantities covariationally, among which students' coordination of amounts of change in one quantity with respect to changes in another (i.e., MA3) is central to our work. To illustrate, a student reasoning about covarying quantities B and K can envision the magnitude ||B|| (denoted in blue, Figure 1a) accumulating in equal accruals, construct the magnitude ||K|| (denoted in red, Figure 1a) accumulating in corresponding accruals, and conceive ||K|| increasing by decreasing amounts (denoted in light blue, Figure 1b) with respect to ||B||. One can also re-present these variations on two orthogonally-oriented bars and a Cartesian coordinate system (Figure 1c-d).

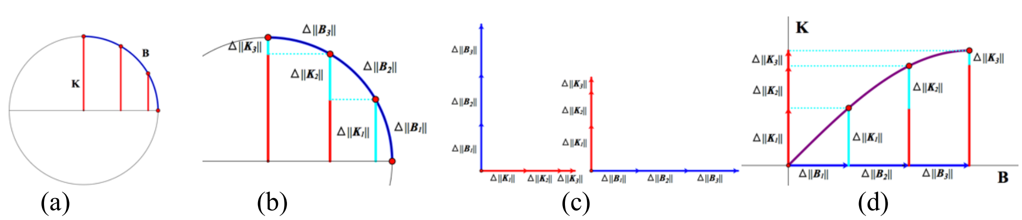


Figure 1. ||K|| increases by decreasing amounts for equal increases in ||B||, which can be represented on (a-b) a circle, (c) two orthogonally-oriented bars, (d) and a coordinate system.

Although coordinating amounts of change is fundamental to rate of change reasoning, Thompson and Carlson (2017) identified that students' construction of rate of change was also related to conceptualizations of ratio, quotient, and proportionality. A productive meaning of rate of change entails a quantitative meaning for quotient as a multiplicative comparison of two quantities' magnitudes with the intention of determining their relative size (Byerley & Thompson, 2017). Relevant to the aforementioned example, a quantitative quotient meaning regarding amounts of change magnitudes involves making additive comparisons of two accumulated magnitudes to determine accruals in those magnitudes (e.g., conceiving $\Delta||B_1||$ and $\Delta||K_1||$) and multiplicatively comparing those magnitudes (e.g., conceiving $\frac{\Delta||B_1||}{\Delta||K_1||}$ or $\frac{\Delta||K_1||}{\Delta||B_1||}$).

Methods

We conducted fourteen teaching experiment (Steffe & Thompson, 2000) sessions with a pre-calculus student, Caleb, who, at the time of the study, was a sophomore majoring in music education. Our overall goal was to characterize his mental actions involved in reasoning with dynamic situations from a quantitative and covariational reasoning perspective. We used session videos, transcripts, and digitized copies of Caleb's written work for both on-going and retrospective conceptual analyses (Thompson, 2008) to develop working models of his thinking.

In the tenth teaching session, we designed and used the Changing Bars Task (Figure 2) to gain insights into Caleb's magnitude reasoning. On the circle (left, Figure 2), the red segment represents a dynamic point's height magnitude above the horizontal diameter and the blue segment represents the point's counterclockwise arc magnitude traveled from the 3 o'clock position. A student can move the point along the circle between the 3:00 and 12:00 position. With respect to the orthogonally oriented bars (right, Figure 2), while a student dragging an end-point of a red bar to manipulate its length, the blue bar varies accordingly. For an increasing red bar from zero to the radius length of the circle, the blue bar in pairs (1)-(2), (3)-(4), and (5)-(6) increases at a constant, increasing (i.e., the red bar increases at a decreasing rate with respect to the blue bar; the normative solutions to the task), and decreasing rate with respect to the red bar, respectively. The only difference between the corresponding pairs is the orientation of the bars. We asked Caleb to choose which, if any, of the orthogonal pairs accurately represents the relationship between the red and blue segments on the circle, and to graph the relationship.

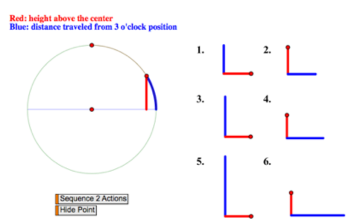
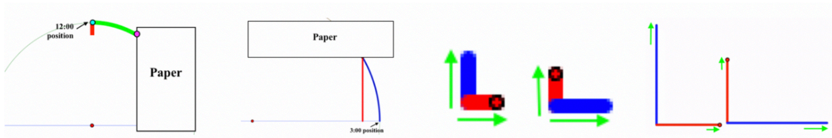


Figure 2. Changing Bars Task (numbering of the six pairs was edited for readers)**.**

Results

Comparison Between Amounts of Change in Two Quantities' Magnitudes

Caleb first focused on the circle to identify the appropriate relationship between height and arc length. Drawing attention to the variation near the 12:00 position, he used a paper to cover a portion of the circle's circumference and observed that the amount the height increased was "small" (see the red segment denoted in Figure 3a) but the arc traveled was "considerable" (see the green arc denoted in Figure 3a). In contrast, he observed that, near the 3:00 position, the change in magnitude of red and blue was "almost equal" (Figure 3b). We interpret that Caleb was not only comparing amounts of change in height and arc magnitude, but was also aware that the results of such comparisons were not constant as the dynamic point traveled.



(a) (b) (c) (d)

Figure 3. Annotated recreation of Caleb's using paper to compare amounts of change in arc length and height at around (a) the 12:00 position and (b) the 3:00 position; Caleb's choice of two pairs of bars at (c) the beginning state and (d) the final state (the direction and length of each arrow indicates the direction and magnitude of change respectively).

Turning his attention to the orthogonal pairs, he made use of this observation to select two pairs (the normative solutions) in which the blue bar changed by "almost equal" amount as the red near their minimal amounts (Figure 3c) and changed by noticeably larger amounts than the red near their maximum amounts (Figure 3d). Transitioning to a graphical representation, Caleb produced a graph (Figure 4a) and explained that the graph should start with having "a slope of 1" to re-present the amount the red segment changed being "almost equal" to the blue and should gradually level out to re-present the blue segment increasing by a larger amount than the red. He was able to mentally coordinate and re-present amounts of change magnitudes in blue and red on the horizontal and vertical axis respectively and to anticipate how joining these orthogonally-oriented magnitudes would result in a graph with particular concavity. Together, we interpret Caleb's ways of comparing and coordinating magnitudes of quantities to be coherent and stable across multiple representations (i.e., circle situation, orthogonal bar pairs, and graph).

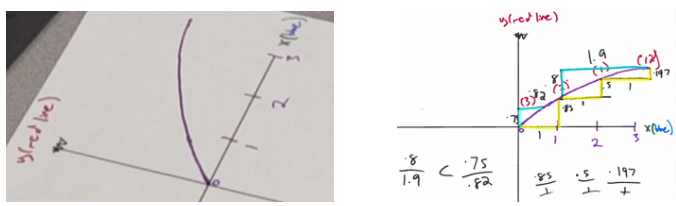

(a) (b)

Figure 4. Caleb's (a) initial graph and (b) graph with estimation of quotient values.

Quotient as Relative Size of Two Quantities' Magnitudes

Observing that Caleb consistently attended to changes in two intervals only (one close to the 3:00 position and one close to the 12:00 position), we prompted him to describe the relationship throughout the entire first quarter circle. He then drew in successive horizontal and vertical segments (denoted in yellow in Figure 4b) to represent equal changes in the blue and decreasing changes in the red, respectively. He also estimated a value for each increment and wrote three quotients (i.e., ".85/1," ".5/1," ".197/1"), concluding that the ratios of how much the red increases to how much the blue increases decreased. He explained that his choice of making the change in blue a size of 1 for each increment was only for convenience in that it allowed direct additive comparisons, saying "if you divide these out [*scratching out the '1's in the denominators as seen in Figure 4b*] it would just be 0.85, 0.5, 0.197, you see them decreasing." He further explained that he could have used contrasting sizes for those intervals. He illustrated by drawing in the segments denoted in blue in Figure 4b and wrote down an estimated number for each segment, claiming that, although he did not know the values for ".8/1.9" and ".75/.82", he knew that the former is greater than the latter due to his previous conclusions. We interpret that Caleb was multiplicatively comparing amounts of change in two quantities' magnitudes and anticipating each quotient as representing a relative size of the magnitude of change in blue with respect to that in red. Although he made use of numerical values, his reasoning was not constrained by those values; rather, he used numbers to convey particular features of the relationship between the two magnitudes. Additionally, he could hold the covariational relationship in mind and anticipate variation of quotient values across *any* intervals.

Discussion

Contributing to Thompson et al.'s (2014) call for more detailed characterization of students' magnitude thinking, our analysis suggests that a student's mental actions involving magnitudes can be generative for productive mathematical meanings that include graphical concavity and quotient. In contrast to the teacher participants in Thompson et al.'s (2014) study who had poor understandings of magnitude, quotient, and rate of change, Caleb's disposition of reasoning with magnitudes afforded his reasoning about quotient (and potentially rate of change) productively. Specifically, his ability to compare magnitudes multiplicatively supported his quotient understandings that entailed flexible partitions or intervals, which were critical for constructing rate of change meanings. Our findings also contribute to the covariation frameworks of Carlson et al. (2002) and Thompson and Carlson (2017). Carlson et al. (2002) suggested that coordination of uniform successive amounts of change in quantities is fundamental to understanding average and instantaneous rate of change (i.e., MA3, MA4, and MA5). In this study, we identify a student whose ability to coordinate quantities does not originate from constructing successive changes in quantities (including their magnitudes). Rather, his way of thinking aligns with Thompson and Carlson's (2017) *chunky continuous covariation* that entails reasoning with covariations regarding intervals of any sizes. We call for future investigations to focus on the extent to which a student can reason with smooth variations across interval chunks, which may provide further insights into the constraints and affordances of students' magnitude reasoning regarding rate of change and other more advanced concepts.

References

Byerley, C., & Thompson, P. W. (2017). Secondary mathematics teachers' meanings for measure, slope, and rate of change. *The Journal of Mathematical Behavior, 48*(Supplement C), 168-193.

Carlson, M. P., Jacobs, S., Coe, E. L., Sean, & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-379.

Johnson, H. L. (2015). Together yet separate: Students' associating amounts of change in quantities involved in rate of change. *Educational Studies in Mathematics*, 1-22.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-307). Hillside, NJ: Erlbaum.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY: SUNY Press.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sépulveda (Eds.), *Plenary Paper presented at the Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 31-49). Morélia, Mexico: PME.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.

Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In L. P. Steffe, K. C. Moore, L. L. Hatfield, & S. Belbase (Eds.), *Epistemic Algebraic Students: Emerging Models of Students' Algebraic Knowing., WISDOMe Monographs* (Vol. 4, pp. 1-24). Laramie, WY: University of Wyoming.

AN ANALYSIS OF STUDENT CHALLENGES WITH ALGEBRA IN CALCULUS

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National data reveals that more than half of freshmen who declared STEM majors at the start of college left these fields before graduation. Our research suggests that students' difficulties with algebra cause significant problems in many first-year math courses. This paper presents our analysis of student challenges with algebra in calculus and the nature of their errors.

Keywords: Algebra, Calculus, Common Error

Background

Over the past several decades, much attention has been focused on the need for improved mathematics and science teaching and learning in the United States. The pressure for global competition and changing demands of the workforce have propelled the conversation regarding learning outcomes in science, technology, engineering, and mathematics (STEM) forward. Given that calculus courses are often the gatekeeper to disciplines in STEM, at least one calculus course is typically required for all STEM majors. Bressoud, Mesa, & Rasmussen (2015) suggest that the requirement of a calculus course is often an insurmountable obstacle or one that discourages students from pursuing STEM majors. In fact, research has shown more than half of students are deterred from a career in STEM due to challenges completing calculus (Crisp, Nora, & Taggart, 2009; Mervis, 2010). Recent studies by Stewart (2017) and Stewart and Reeder (2017) suggest that college students' weaknesses with algebra play a major role in their success in calculus. Stewart (2017) further suggests college instructors face the challenge of working with students everyday who can seemingly make sense of complex mathematical concepts but are unable to solve problems related to those concepts due to their difficulties with algebraic procedures.

The goal of this research is to analyze students' difficulties with algebra and how these difficulties impact their work in calculus problems. We assert that the unresolved high school algebra knowledge wrapped inside the calculus problems are major cause of failure for many students when taking calculus. While resolving algebra deficiencies that students bring with them will be challenging, "it cannot be simply ignored and remain as an everyday accepted or out of our hands part of teaching university level mathematics courses" (Stewart, 2017, p. 15).

The following research questions guided this study: (a) What were the most common algebra errors in both the algebra and calculus problems? (b) To what extent where the algebraic errors made on the algebra and calculus problems aligned and (c) What were the students' perceptions of their challenges with algebra and calculus related to these problems?

Method

This qualitative research study involved 275 university Calculus I students at the end of their 16-week course. Students were asked to first solve the Calculus I problems (Stewart, 2014) and then the algebra problems, identify what caused them the most challenge, algebra or calculus, and provide a brief response about what challenged them while solving the problems (see Table

1). Collected data were de-identified and incomplete data sets were removed. The result was N = 84 complete sets of data. The research team analyzed each problem to develop initial codes. The initial codes were used by researchers to code data independently for both calculus mistakes in the calculus problems and algebra mistakes in both the calculus and algebra problems. A second meeting of the research team established the codebook after assuring inter-coder reliability. Using our codebook (see Table 2), each set of problems were analyzed and coded independently by two researchers who then met to review the codes and establish 100% agreement.

Table 1: The Calculus and Algebra Problems

Calculus Problems	Algebra Problems
1. Implicitly differentiate. $\sqrt{xy} = 1 + x^2y$ 2. Find the critical numbers of the function $f(t) = t\sqrt{4 - t^2}$ 3. Evaluate the limit. $\lim_{t\to 0} \frac{\sqrt{1+t} - \sqrt{1-t}}{t}$	1. Solve for y. $5 + xy = 10 + x^2y$ 2. Solve for y. $\frac{1}{2\sqrt{5x}}(5 + xy) = 10x + x^2y$ 3. Solve for t. $\frac{1}{\sqrt{t+1}} - \frac{1}{t} = 0$ 4. Solve for y. $\frac{2y^2}{2\sqrt{y^2 - 9}} + \sqrt{y^2 - 9} = 0$
My main problem with the test was: Algebra ☐	Calculus ☐
Please write a comment relevant to your experience in taking this test.	

Table 2: Errors for Calculus and Algebra Contexts

Calculus Errors	Algebra Errors	Other Errors
1. Power Rule	8. Convert radical to exponent	23. Blank
2. Product/Quotient rule	9. Exponent Operations	24. Incomplete Calculus
3. Chain Rule	10. Balance Points	25. Incomplete Algebra
4. Process of Implicit	11. Distributive Property	26. Computation
5. Interpret Critical Numbers (set =0)	12. Combining Like Terms	27. Avoiding Algebra
6. Undefined points are Critical	13. Cancelling	28. Avoid Calculus
7. Taking the limit	14. Factoring	29. Miscellaneous
	15. Simplifying nested fractions	30. Isolating Variables
	16. Sign error	
	17. Operations with radicals	
	18. Finding Common Denominators	
	19. Recognizing undefined values	
	20. Conjugating Rational Fractions	
	21. Quadratic Functions	
	22. Operations with Fractions	

Results

Our results are presented in three sections which correspond to our research questions on common errors, comparing calculus and algebra contexts, and student perceptions.

Common Errors

Analysis of the algebra and calculus problems revealed that the student errors were numerous and significant with algebra in both sets of problems and calculus related errors were frequent in the calculus problems as well. The most common algebra errors made in both sets of problems were errors working across the balance point in equations, cancelling, operations with radicals, appropriate application of the distributive property, and incomplete algebra (work that was not completed due to confusion). While the students' work with the calculus problems were replete with algebra errors, they also made many calculus errors. The most common among these were incorrectly taking the derivative implicitly, using the product rule improperly, failing to identify undefined points as critical, incorrectly taking the limit, and avoiding algebra.

Comparing Calculus and Algebra Contexts

Of the 84 students in the sample, 58 of the students made a total of 102 algebra errors on the calculus questions; the other 26 students made no algebra errors. Thus, 69% of the students in the sample made some type of algebra error on the calculus problems. However, of the 102 algebra mistakes made on calculus problems, only 26 times (25.5%) students made the equivalent mistake on the associated algebra problem. In fact, only 24 of the 58 students who made algebra errors on the calculus problems repeated their mistakes on the algebra problems. Yet, on the algebra problems, these 58 students made a total of 130 algebra errors on the algebra problems; that is, they made more algebra mistakes on the algebra problems than they had on the calculus problems from which the algebra problems were derived. Further, the patterns of the mistakes made differed (see Figure 1 below).

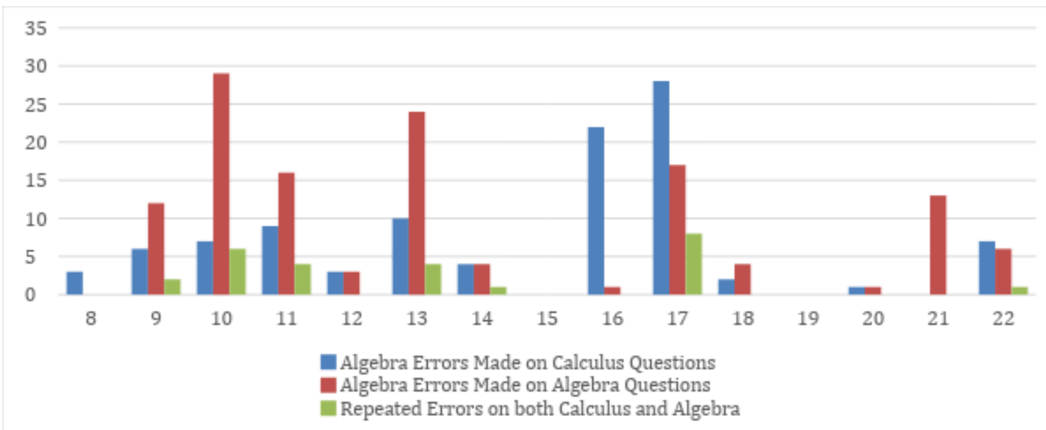

Figure 1. The algebra errors of the 58 students who made algebra errors on calculus problems.

These students did not tend to repeat the same type of algebra errors on both the calculus and algebra problems. Students were much more likely to make errors in with the balance point (error code 10) or cancelling (error code 13) on the algebraic problems than on calculus based problems. Students were much more likely to have sign errors (error code 16) on calculus problems. Errors while working with radicals (error code 17) were also more likely to appear on calculus problems than algebra problems. Yet, of the 28 students who made an error with radicals on the calculus problems, only 8 of these students repeated the error when presented with an algebra problem dealing with a radical; another nine students had a radical error on an algebra problem when they had not made the error in the calculus portion of the exam.

Student Perceptions

Our final research question aimed to provide insight on the students' perceptions about their abilities with algebra and calculus as presented in the problems they were asked to solve. Of the 84 participants who completed all the mathematical problems, 73 chose to provide a response to our short-answer item. When asked to select which gave them more challenge, algebra, calculus, or both, 57% indicated algebra, 31% indicated calculus, while 12% indicated both. Their comments overwhelmingly expressed recognition that algebra causes them difficulties, frustration, anxiety, and in some cases, hopelessness about their abilities to succeed in mathematics.

Discussion and Implications

We believed most students would solve algebra problems correctly when these problems were in isolation from calculus, but make predominantly algebraic mistakes in the context of calculus problems with algebra problems embedded. However, we found that students struggled with the isolated algebra problems as well as the calculus problems.

While the problems presented students challenges with both calculus and algebra, the student responses overwhelmingly indicated they had frustration and concerns with their algebra abilities. In the words of one student "I knew how to start the problem, but could not finish because of the difficulty of the algebra involved." So, while these students may have the prerequisite knowledge of algebra, they may not have had the opportunity to develop the strong procedural fluency necessary for algebra to function as a versatile tool in calculus as expected or required. Their challenges with algebra create difficulties for us, and them, as we try to make sense of what calculus they are learning. The challenges and misconceptions students bring with them to their university mathematics courses are difficult to remedy, even if identified. As Smith et al. (1993, p. 152) suggest, the misconceptions developed by students are "… persistent and resistant to change are likely to have especially broad and strong experiential foundation."

References

Ashlock, R. B. (2010). *Error patterns in computation: Using error patterns to improve instruction.* Allyn & Bacon.

Brown, G., & Quinn, R. J. (2006). Algebra students' difficulty with fractions: An error analysis. *Australian Mathematics Teacher, 62* (4), 28-40.

Bressoud, D., Mesa, V., & Rasmussen, C. (Eds.) (2015). *Insights and Recommendations from the MAA National Study of College Calculus*. Washington, DC: The Mathematical Association of America.

Crisp, G., Nora, A., & Taggart, A. (2009). Student characteristics, pre-college, college, and environmental factors as predictors of majoring in and earning a STEM degree: An analysis of students attending a Hispanic serving institution. *American Educational Research Journal*, *46*(4), 924-942.

McGowen, M. (2017). Examining the role of prior experience in the learning of algebra. In S. Stewart (Ed.) *And The Rest is Just Algebra*. (pp. 19-43). Springer International Publishing.

Mervis, J. (2010). Better intro courses seen as key to reducing attrition of STEM majors. *Science*, *330*(6002), 306-306.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and Standards for School Mathematics.* Reston, VA.

Smith, J., diSessa, A., & Roschelle, J. Misconceptions Reconceived: A Constructivist Analysis of Knowledge in Transition. The Journal of the Learning Sciences, Vol. 3, No. 2 (1993 - 1994), pp. 115-163

Stewart, J. (2014). *Calculus* (8th ed.). Boston: Cengage Learning.

Stewart, S. (Ed.) (2017). *And the rest is just algebra.* Springer International Publishing.

Stewart, S., & Reeder, S. (2017). Common Algebraic Errors in Calculus Courses. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, and S. Brown (Eds.), *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education*, (pp. 937-944). San Diego, CA.

EL EFECTO DE LOS ESTADOS EPISTÉMICOS (CERTEZA, DUDA) SOBRE EL APRENDIZAJE. MICROANÁLISIS DE UNA INTERACCIÓN

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La investigación se centra en fenómenos del convencimiento de hechos de las matemáticas (H) que surgen en el ámbito educativo. Se aportan evidencias empíricas de que la seguridad en H puede inhibir el aprendizaje de las matemáticas y la duda en H, impulsarlo. Las evidencias provienen de un caso de interacción entre una pareja de estudiantes de secundaria que intercambian argumentos en torno a la resolución de una tarea de comparación de razones.

Keywords: Affect, Emotion; Cognition; Reasoning and Proof

Antecedentes y Preguntas de Investigación

La investigación se centra en fenómenos del convencimiento (seguridad, confianza, duda… los que aquí se denominan *estados epistémicos de convencimiento* o *ee*) de hechos de las matemáticas que surgen en el ámbito educativo. Expertos (e.g., Foster, 2016) han aportado evidencias de la influencia (positiva o negativa) de los ee en el aprendizaje de las matemáticas: un contenido matemático que no es asimilado con convencimiento, aunque sea comprendido, posteriormente no se usa (Fischbein, 1982); la sobre-confianza o seguridad en un argumento inhibe la posibilidad de considerar un argumento distinto (Fischbein, 1982; Inglis, Mejía-Ramos & Simpson, 2007). En continuidad con esos trabajos, en éste se plantea: ¿Los ee tienen influencia sobre el aprendizaje de las matemáticas? ¿Cómo inciden y qué tipo de repercusión ejercen? Aquí se muestra que los ee tienen un efecto como impulsores o como inhibidores del aprendizaje. El argumento se sustenta en el caso de una interacción entre un par de alumnos de secundaria que intercambian argumentos en torno a una tarea de comparación de razones.

Marco Teórico Interpretativo

Apoyados en conceptos provenientes de la neurobiología de Damasio, la autora del presente escrito y sus colaboradores consideran a los ee como parte del ámbito afectivo, específicamente como un cierto tipo de emociones (Rigo & Martínez, 2017). En tanto emociones, los ee se expresan corporal y cognitivamente; en la Tabla 1 aparecen criterios para identificarlos.

Tabla 1. Criterios para la identificación del convencimiento de una persona

Discurso verbal y corporal	La persona: -Recurre a enfatizadores del lenguaje (modo indicativo de los verbos) (e.g., tengo; mitigadores: convendría); -Utiliza un tono de voz firme y enfático y posiblemente más alto de lo usual; -Acude a enfatizadores corporales, como gestos y movimientos del cuerpo que reafirman sus ideas.
Determinación	La persona manifiesta de manera espontánea y determinada su adhesión a la veracidad de un enunciado matemático. Busca convencer a otros, aún cuando están en su contra.
Consistencia	La persona muestra consistencia en sus distintas intervenciones.

Sobre la teoría del aprendizaje significativo de Ausubel

Para Ausubel, Novak y Hanesian (1978), una condición para el aprendizaje significativo consiste en que los nuevos contenidos de conocimiento sean potencialmente significativos para el aprendiz, es decir, que puedan ser vinculados de modo no arbitrario y sustancial (no al pie de la letra) con lo que el alumno ya sabe. Otra condición para el aprendizaje significativo consiste en que el material de aprendizaje sea lógicamente significativo, es decir, que se pueda conectar de manera sustancial y no arbitraria con cualquier estructura cognoscitiva apropiada. Ausubel sugiere otras variables de tipo afectivo, motivacional y social, que también influyen en el aprendizaje significativo.

Metodología y métodos de recuperación de datos empíricos

En el estudio se aplicó en forma individual un cuestionario a alumnos pertenecientes a escuelas con estándares altos de calidad. En el cuestionario se planteó una tarea de corte realista basada en la comparación de razones. En la tarea se proponen tres ofertas para la compra de videojuegos: en una se ofrece pagar 2 y llevarse 3 (3x2); en otra, se ofrece el segundo videojuego a mitad de precio y en la otra se ofrece el 70% de descuento en el segundo videojuego. La pregunta es ¿cuál de las ofertas es la que más conviene? (tomada de Gómez et al., 2013). Se formaron parejas de alumnos, cada uno integrada por estudiantes que en el cuestionario sostuvieron una postura distinta y explícita. En la interacción se solicitó a los participantes que intentara convencer a su compañero de que su resolución a la tarea era la adecuada; uno de los investigadores actuó como coordinador del diálogo. Cada interacción fue videograbada y luego transcrita. Aquí se presenta un microanálisis -cualitativo- del diálogo de una de las parejas. Los datos empíricos y las transcripciones provienen de Gómez (2017).

Análisis interpretativo de los datos empíricos y algunos resultados

El análisis se centra en el alumno V, quien expresó nítidamente sus ee, los cuales surgieron en la interacción con J y el investigador. En la Tabla 2 se exponen los fragmentos más significativos de sus participaciones. En la columna central de esa Tabla aparecen fragmentos de la transcripción de la interacción. En su lado derecho, están los argumentos a favor de la oferta 70%, así como los ee que experimentaron los estudiantes en torno a dichos razonamientos. En el lado izquierdo, aparece la misma información pero relacionada con la oferta 3x2.

Table 2: Interacción V-J, con análisis de los argumentos y estados epistémicos

<table>
<tr><th>RESPUESTA: 3X2
Argumentos y Estados epistémicos</th><th>Participación</th><th>RESPUESTA: 70%
Argumentos y Estados epistémicos</th></tr>
<tr><td>Argumentos
A & EM
Alta presunción:
Ausencia de mitigadores</td><td>Cuestionario Inicial (ci) de V
$1000 por juego.<table><tr><th>Oferta</th><th>Costo de dos juegos</th><th>Costo de tres juegos</th></tr><tr><td>1 1/2</td><td>1500</td><td>2500</td></tr><tr><td>70% en 2°</td><td>1300</td><td>2300</td></tr><tr><td>3x2</td><td></td><td>2000</td></tr></table>R: 3x2 es la mejor oferta porque te regalan uno.</td><td></td></tr>
<tr><td>Argumentos: A, EM
Certeza:
Habla sin mitigadores. Golpea pizarrón; tono de voz alto. Determinación: argumenta de dos formas. Interés: es sistemático e informativo.
Es consistente en sus respuestas.</td><td>1ª participación de V (1ªP)<table><tr><th>Oferta</th><th>J1</th><th>J2</th><th>J3</th><th>Total</th><th>Ícono</th></tr><tr><td>3x2</td><td>1000</td><td>1000</td><td>¡Gratis!</td><td>2000</td><td>☺</td></tr><tr><td>1 1/2</td><td>1000</td><td>500</td><td>1000</td><td>2500</td><td>☹</td></tr><tr><td>70% en 2°</td><td>1000</td><td>300</td><td>1000</td><td>2300</td><td>😕</td></tr></table>"La oferta 3x2… y el tercero te sale gratis (golpea con el plumón en la zona del pizarrón donde dibujó los tres juegos y señaló al tercero con 'gratis') ¡gratis!... (y así sigue con el análisis de las otras ofertas, golpeando a veces enfáticamente el pizarrón para reafirmar su dicho) … está mejor éste (3x2) porque pagas menos por tres juegos"</td><td></td></tr>
<tr><td>Argumento: EM
Muy alta presunción:
Habla sin mitigadores; golpea pizarrón; voz alta. Determinación: argumenta para sustentar su respuesta. Consistencia: refrenda su postura.</td><td>2ª participación de V (2ªP)<table><tr><th>Oferta</th><th>J1</th><th>J2</th><th>J3</th><th>J4</th><th>Total</th></tr><tr><td>3x2</td><td>1000</td><td>1000</td><td>Gratis</td><td>1000</td><td>3000</td></tr><tr><td>1 1/2</td><td>1000</td><td>500</td><td>1000</td><td>500</td><td>3000</td></tr><tr><td>70% en 2°</td><td>1000</td><td>300</td><td>1000</td><td>300</td><td>2600</td></tr></table>"Ok… Pero nadie se va a esperar tanto tiempo, yo sé que van a escoger ésta" (golpea el pizarrón donde está marcada la oferta 3x2)</td><td>Argumento: A
Presunción baja:
Mitigadores del lenguaje. Ausencia de enfatizadores corporales. Tono de voz bajo. Actúa siguiendo instrucciones del Investigador</td></tr>
</table>

<table>
<tr><td></td><td>3ª participación de V (3ªP) (a solicitud del investigador)
“Entonces en la primera por seis serían 4000 morlacos… Seis, aquí son cuatro (analiza la segunda oferta) … entonces son 4500… ¡sí! mmm… (analiza la tercera oferta) son 2600 por cuatro… ya, 3900… “
(si quiero llevarme seis juegos conviene más) la tercera”</td><td>Estrategia: Building up (Bu)
Presunción baja
Ausencia de enfatizadores corporales y verbales.</td></tr>
<tr><td></td><td>Intervención de José
“… supongamos que cada videojuego sale en 1000 pesos, entonces tenemos 3x2,, entonces sólo pagaría 2, o sea 2000. Pero si los divides entre tres, que es el número de juegos que te llevas, o sea, … no te van a dar nada gratis. Entonces serian 666 por cada uno de los videojuegos … 650 por juego es menor que 666 por juego”</td><td></td></tr>
<tr><td>Argumento: EM
confianza media o ¿duda?
Mitigadores del lenguaje; no hay enfatizadores corporales
Determinación:
Consistencia: refrenda su postura</td><td>4ª participación de V (4ªP)
Investigador: ¿Te conviene lo que hace tu compañero?
V: Nnnn …
Investigador: ¿No te convence la respuesta de José?
V: ¡No! no dije que no. Sí (me convence la respuesta de José) pero, ¿y si nada más quieres tres?... O sea, qué tal si no tienes la posibilidad económica para comprarte eso (parece retomar el caso de 6j propuesto por el investigador). … (Si nada más quisieras tres juegos) te convendría ese” (señala la oferta 3x2).</td><td>Confianza media
Acepta respuesta con un monosílabo. Es muy lacónico y parco.</td></tr>
<tr><td>Confianza (¿duda?)</td><td>5ª participación de V (5ªP)
Lo dejamos en un empate</td><td>Confianza</td></tr>
<tr><td></td><td>Cuestionario final de V (cf)
Se infiere propuesta de precio: $1000
<table><tr><th>Oferta</th><th>No. de juegos</th><th>Costo</th><th>Costo por juego</th></tr><tr><td>4x3</td><td>4</td><td>3000</td><td>750</td></tr><tr><td>1 1/4</td><td>4</td><td>2500</td><td>625</td></tr><tr><td>70% en 2°</td><td>4</td><td>2600</td><td>650</td></tr></table>Por obvias razones es mejor un %75 que un 70% solamente.</td><td>Argumento
Tipo relativo
Certeza
Enfatizadores del lenguaje: “por obvias razones”</td></tr>
</table>

V poseía conocimientos previos que le permitieron diseñar una estrategia de tipo aditivo A: para la comparación de parejas de magnitudes se fija primero una de ellas y luego se compara aditivamente la otra. Además, V tenía experiencias previas en las compras, las que dejó ver a través de argumentos extra-matemáticos (EM). En la interacción, V acudió a esa estrategia A (v. columna derecha Tabla 2: en ci, 1ªP y 2ªP) de la que derivó la respuesta 3x2, la cual reforzó con justificaciones EM (en 1ªP, 2ªP y 4ªP). Cuando V, a solicitud del investigador, obtuvo correctamente otra respuesta (70%) mediante A (en 2ªP) o a través de otra estrategia (Building up, 3ªP), él de todos modos regresó a su respuesta 3x2, apelando siempre a justificaciones EM. Lo mismo hizo ante el argumento de José (en 4ªP). Es hasta el cf que V adoptó una nueva forma de argumentar, con perspectiva relativa, y una nueva solución, omitiendo los argumentos EM.

Durante la interacción, V no parece haber construido nuevos aprendizajes significativos que tuvieran repercusiones visibles (en el corto plazo); muestra de ello es que a lo largo de toda la interacción (con excepción de 5ªP) siempre regresó a su respuesta inicial y a sus argumentos EM. Es hasta el cf que V muestra haber conseguido nuevos aprendizajes, al resignificar la comparación de parejas de magnitudes a partir de una perspectiva relativa.

Aquí surgen 2 preguntas. ¿Por qué V no parece haber logrado nuevos aprendizajes durante la interacción (sino sólo hasta el final)? ¿Por qué sí consigue un aprendizaje significativo en su cf?

La tarea era lógicamente significativa para estudiantes del nivel de V; además, la tarea era potencialmente significativa para V. No fueron entonces factores cognoscitivos los que impidieron que, durante la interacción, V progresara en sus aprendizajes. Quedan entonces los factores afectivos; éstos pudieron haber influido en su desempeño, pero no hay evidencia de que fueran factores tan definitivos. Aquí se argumenta que es probable que haya sido la confianza que V experimentó durante la interacción, la que impidió de manera importante que él haya conseguido un aprendizaje significativo durante la mayor parte del intercambio; y que fue la confianza moderada o la duda, lo que, en buena medida, posibilitó al final ese aprendizaje.

V dejó ver desde el inicio una sobreconfianza en A, en sus argumentos EM y en la respuesta 3x2 que se derivó de A y de EM. Muestra de esa sobreconfianza son los indicadores que aparecen en la primera columna de la Tabla 2, sobre todo 1ªP. Era natural que V tuviera esa confianza, ya que sus argumentos (A y EM) le eran muy familiares y A estaba basado en principios aritméticos básicos. Pero estos ee tuvieron sus efectos negativos en el aprendizaje de V. Si bien V aplicó correctamente estrategias distintas a A (e.g, Bu) o derivó otras soluciones (70%), no parece haberlo hecho, de entrada, con toda la convicción necesaria para considerar estos nuevos contenidos, ponderarlos y compararlos objetivamente con lo propuesto inicialmente por él; indicadores de la falta de esa convicción aparecen descritos en la columna derecha de la Tabla 2. Es posible que esa falta de convicción haya sido resultado de la sobreconfianza en A, EM y 3x2, la cual lo llevó a centrarse firmemente en esos contenidos y le impidió, de entrada, asumir compromisos de seguridad con otras opciones distintas (Fischbein, 1982; Inglis et al, 2007); y es posible que esa escasa convicción en Bu y 70% impidió, a su vez, que esa nueva estrategia y solución tuvieran un uso posterior (Fischbein, 1982). No obstante, y a pesar de todo esto, sus actividades en 2ª y 3ªP fueron sensibilizando a V, aunque imperceptiblemente, en esas otras estrategias y resultados y fue así disminuyendo su sobre-confianza en su propuesta inicial, hasta llegar a un estado de confianza o incluso de duda en torno de esa resolución inicial (4ªP). Estos cambios en los ee -decremento en la confianza en respuesta 3x2 (v. columna izquierda Tabla 2) y apertura de credibilidad a otras estrategias y respuestas (v. columna derecha Tabla 2)- permitieron a V ser sensible a la propuesta matemática de J. Y esa confianza o duda, más cautelosa (4ª P), en su propuesta inicial, acompañada de una confianza en la estrategia relativa (5ªP), le permitieron a V hacer en el cf una comparación más objetiva entre todas las estrategias disponibles -lo que no pudo hacer durante la interacción- y optar por una estrategia matemática de tipo relativo hacia la que mostró certeza y en la que ya no necesitó de los argumentos EM.

Consideraciones finales

El análisis expuesto ofrece evidencias empíricas de que, al igual que las variables cognoscitivas, los estados epistémicos (que aquí se conciben como pertenecientes al ámbito afectivo) sí pueden contribuir (o por el contrario, pueden frenar) al surgimiento, precisión, integración y otros aspectos cualitativos de los nuevos aprendizajes; en particular, se deja ver empíricamente que la certeza ejerce un efecto inhibidor de la edificación o establecimiento de los nuevos contenidos, y que la duda actúa como impulsor del aprendizaje.

References

Ausubel, D., Novak, J. D., & Hanesian, H. (1978). *Educational Psychology: A Cognitive View.* New York: Holt, Rinehart and Winston.

Fischbein, E. (1982). Intuition and Proof. *For the Learning of Mathematics, 3*(2), 9-24.

Foster, C. (2016). Confidence and competence with mathematical procedures. *Educational Studies in Mathematics, 91*(2), 271-288.

Gómez, H. D. (2017). *El papel del convencimiento y la interacción entre pares durante el intercambio de resoluciones de un problema de comparación de razones*. Tesis de Maestría en Ciencias, Especialidad Matemática Educativa. Centro de Investigación y de Estudios Avanzados del I. P. N. México.

Gómez, B., Monje J., Pérez-Tyteca, P. & Rigo, M. (2013). Performance on ration in realistic discount task. Un B. Ubuz, C. Haser & M. A. Mariotti (Eds.), Proceedings of the CERME 8 (pp. 293-302).

Inglis, M., Mejia-Ramos, J. P., & Simpson, A. (2007). Modelling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics, 66*(1), 3-21.

Martínez B. & Rigo M. (2017). Confidence and comprehension building processes regarding mathematical content. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.), *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 217-224). Singapore: PME.

Rigo-Lemini, M., Martínez-Navarro, B. (2017). Epistemic States of Convincement. A Conceptualization from the Practice of Mathematicians and Neurobiology. In U. Xolocotzin (Ed.), *Understanding Emotions in Mathematical Thinking and Learning*. India: Elsevier.

THE EFFECT OF THE EPISTEMIC STATES (CERTAINTY, DOUBT) ON LEARNING. MICRO ANALYSIS OF AN INTERACTION

The research focuses on mathematical facts (H) convincing phenomena that arise in the educational field. Empirical evidence is provided that the certainty in H can inhibit the learning of mathematics and the doubt in H, boost it. The evidence comes from a case of interaction between a couple of high school students who exchange arguments around the resolution of a ratio task.

Keywords: Affect, Emotion; Cognition; Reasoning and Proof

Antecedents and Research Questions

The research focuses on convincing phenomena (security, confidence, doubt ... those that are here called epistemic states of convincement or ee) of mathematical facts that arise in the educational field. Experts (e.g., Foster, 2016) have provided evidence of the influence (positive or negative) of the ee in the learning of mathematics: a mathematical content that is not assimilated with conviction, even if it is understood, later it is not used (Fischbein, 1982); the over-confidence or security in an argument inhibits the possibility of considering a different argument (Fischbein, 1982, Inglis, Mejía-Ramos & Simpson, 2007). In continuity with these works, in this one it is posed: Do ee have influence on the learning of mathematics? How do the ee affect learning and what kind of effect do they have? Here it is shown that ee have an effect as drivers or inhibitors of learning. The argument is based on the case of an interaction between a couple of high school students who exchange arguments around a ratio task.

Theoretical Interpretive Framework

Based on concepts from Damasio's neurobiology, the author of this paper and her collaborators consider the ee as part of the affective field, specifically as a certain type of emotions (Rigo & Martínez, 2017). As emotions, the ee express themselves bodily and cognitively; in Table 1 there are criteria to identify them.

Table 1: Criteria for the identification of the person's convincement

Verbal and corporal speech	The person: -Appeals to language emphasizers (indicative mode of the verbs) (e.g., I have, mitigators: would agree); -Uses a firm and emphatic tone of voice and possibly higher than usual; -Employs body emphasizers, as gestures and body movements that reaffirm their ideas.
Determination	The person manifests spontaneously and determined his adherence to the veracity of a mathematical statement. He seeks to convince others, even when they are against him.
Consistency	The person shows consistency in their different interventions.

On Ausubel's Theory of Significative Learning

For Ausubel, Novak and Hanesian (1978), a condition for significative learning is that the new knowledge contents are potentially significative for the learner, that is, they can be linked in a non-arbitrary and substantial way (not at the bottom of the letter) with what the student already knows. Another condition for significative learning is that the learning material is logically significative, that is, can be connected in a substantial and non-arbitrary way with any appropriate cognitive structure. Ausubel suggests other variables of affective, motivational and social type, which also influence significative learning.

Methodology and methods of empirical data recovery

In the study, a questionnaire was applied individually to students belonging to schools with high quality standards. In the questionnaire a realistic task was proposed based on the comparison of mathematica ratios. The task proposes three offers for the purchase of video games: one offers to pay 2 and take 3 (3x2); in another, the second videogame is offered at half price and in the other, 70% discount is offered in the second videogame. The question is, which of the offers is the best one? (taken from Gómez et al., 2013). Pairs of students were formed, each composed of students who in the questionnaire held a different and explicit position. In the interaction the participants were asked to try to convince their partner that their resolution to the task was appropriate; One of the researchers acted as coordinator of the dialogue. Each interaction was videotaped and then transcribed. Here a microanalysis -qualitative- of the dialogue of one of the couples is presented. The empirical data and the transcriptions come from Gómez (2017).

Interpretive analysis of the empirical data and some results

The analysis focuses on student V, who clearly expressed his ee, which emerged in the interaction with J and the researcher. Table 2 shows the most significant fragments of their participation. In the central column of that Table appear fragments of the transcription of the interaction. On the right side, there are the arguments in favor of the 70% offer, as well as the ee that the students experienced regarding said reasoning. On the left side, the same information appears but related to the 3x2 offer.

Table 2: V-J interaction, with analysis of the arguments and epistemic states

<table>
<tr><th>ANSWER: 3X2
Arguments and
epistemic states</th><th>Participations</th><th>ANSWER: 70%
Arguments and
epistemic states</th></tr>
<tr><td>Arguments
A & EM
High presumption:
Absence of mitigators</td><td>V's Initial questionnaire (iq)
$1000 per game.
<table><tr><th>Offer</th><th>Cost of two games</th><th>Cost of three games</th></tr><tr><td>1 1/2</td><td>1500</td><td>2500</td></tr><tr><td>70% in 2°</td><td>1300</td><td>2300</td></tr><tr><td>3x2</td><td></td><td>2000</td></tr></table>R: 3x2 is the best offer because they give you one.</td><td></td></tr>
<tr><td>Arguments: A, EM
Overconfidence:
Speak without mitigators. Hit blackboard; high voice tone. Determination: argue in two ways. Interest: is systematic and informative; is consistent with his answers.</td><td>V's 1st participation (1P)<table><tr><th>Offer</th><th>1G</th><th>2G</th><th>3G</th><th>Total</th><th>Icon</th></tr><tr><td>3x2</td><td>1000</td><td>1000</td><td>¡free!</td><td>2000</td><td>☺</td></tr><tr><td>1 1/2</td><td>1000</td><td>500</td><td>1000</td><td>2500</td><td>☹</td></tr><tr><td>70% in 2nd</td><td>1000</td><td>300</td><td>1000</td><td>2300</td><td></td></tr></table>The 3x2 offer ... and the third you get for free (hit with the down in the area of the board where he drew the three games and pointed to the third with 'free') for free! ... (and so continues with the analysis of the other offers, sometimes hitting the board emphatically to reaffirm his saying) ... this one is better (3x2) because you pay less for three games "</td><td></td></tr>
<tr><td>Argument: EM
Very high presumption: Speak without mitigators; hits blackboard; aloud. Determination: argues to support his answer. Consistency: endorses his position.</td><td>V's 2th participation (2P)<table><tr><th>Offer</th><th>1G</th><th>2G</th><th>3G</th><th>4G</th><th>Total</th></tr><tr><td>3x2</td><td>1000</td><td>1000</td><td>Gratis</td><td>1000</td><td>3000</td></tr><tr><td>1 1/2</td><td>1000</td><td>500</td><td>1000</td><td>500</td><td>3000</td></tr><tr><td>70% in 2°</td><td>1000</td><td>300</td><td>1000</td><td>300</td><td>2600</td></tr></table>"Ok ... But nobody is going to wait so long, I know they're going to choose this one" (hits the board where the 3x2 offer is marked)</td><td>Argument: A
Low confidence:
Language mitigators. Absence of body emphasizers. Low voice tone Acts following the Researcher's instructions</td></tr>
<tr><td></td><td>V's 3th participation (3thP) (3P) (at the request of the researcher)
"Then in the first six there would be 4000 'morlacos' ... Six, here are four (analyze the second offer) ... then there are 4500 ... yes! mmm ... (he analyzes the third offer) they are 2600 by four ... already, 3900 ... "
(if I want to take six more games suits) the third "</td><td>Argument: Building up (Bu)
Low confidence
Absence of corporal and verbal emphasizers.</td></tr>
</table>

<table>
<tr><td></td><td>J’s participation
"... suppose that each video game comes out in 1000 pesos, then we have 3x2, then it would only pay 2, that is 2000. But if you divide them by three, that is the number of games that you take, that is, ... you are not going to give nothing for free Then there would be 666 for each of the games ... 650 per game is one amount less than 666 per game "</td><td></td></tr>
<tr><td>Argument: EM
Medium confidence (or doubt?)
Language mitigators; there are no body emphasizers
Determination;
Consistency: endorses his position</td><td>V’s 4th participation (4P)
Researcher: Does what your partner does suit you?
V: Nnnn ...
Researcher: Does not José's response convince you?
V: No! I did not say no. Yes (José's answer convinces me), but what if you only want three? ... That is, what if you do not have the economic possibility to buy that (seems to take up the case of 6g proposed by the researcher). ...
(If you only want three games) that would be better for you "(points to the 3x2 offer).</td><td>Average confidence
Accept response with a monosyllable. He is very laconic and sparing.</td></tr>
<tr><td>Confidence (or doubt?)</td><td>V’s 5th participation (5P)
We leave it in a tie</td><td>Confidence</td></tr>
<tr><td></td><td>V’s final questionnaire (fq)
Price proposal inferred: $ 1000
<table>
<tr><th>Offer</th><th>Number of games</th><th>Cost</th><th>Cost per game</th></tr>
<tr><td>4x3</td><td>4</td><td>3000</td><td>750</td></tr>
<tr><td>1 1/4</td><td>4</td><td>2500</td><td>625</td></tr>
<tr><td>70% in 2°</td><td>4</td><td>2600</td><td>650</td></tr>
</table>
For obvious reasons, 75% is better than 70% only.</td><td>Argument
Relative type
Certainty
Emphasizers on language: "for obvious reasons"</td></tr>
</table>

V had previous knowledge that allowed him to design an additive type strategy (A): for the comparison of pairs of magnitudes, one of them is fixed first and then the other is additively compared. In addition, V had previous experiences in shopping, which he showed through extra-mathematical arguments (EM). In the interaction, V resorted to strategy A (v. Right column Table 2: in iq, 1stP and 2thP) from which he derived the 3x2 response, which he reinforced with EM justifications (in 1stP, 2thP and 4thP). When V, at the request of the researcher, correctly obtained another answer (70%) through A (in 2thP) or through another strategy (Building up, 3thP), he anyway returned to his 3x2 response, always appealing to EM justifications. He did the same with José's argument (in 4thP). It is until fq that V adopted a new way of arguing, with a relative perspective, and a new solution, omitting the EM arguments.

During the interaction, V does not seem to have built significative new learning that would have visible repercussions (in the short term); proof of this is that throughout the entire interaction (with the exception of 5thP) he always returned to his initial response and his EM arguments. It is up to fq that V showed to have obtained new learnings, by re-signifying the comparison of pairs of magnitudes from a relative perspective.

Here two questions arise. Why V does not seem to have achieved new learning during the interaction (but only until the end)? Why does he get significative learning in his fq?

The task was logically significative for students at the V level; in addition, the task was potentially significative for V. It was not then cognitive factors that prevented, during the interaction, V from progressing in their learning. There are then affective factors; these may have influenced their performance, but there is no evidence that they were such definitive factors. Here it is argued that it is likely that was the confidence that V experienced during the interaction, which impeded significantly that he has achieved significant learning during most of the exchange; and that it was moderate confidence or doubt, which, to a great extent, made his learning possible in the end.

V showed from the beginning an overconfidence in A, in his EM arguments and in the 3x2 response that was derived from A and from EM. Sample of this overconfidence are the indicators that appear in the first column of Table 2, especially 1stP. It was natural for V to have that confidence, since his arguments (A and EM) were very familiar to him and A was based on basic arithmetic principles. But these ee had their negative effects on the learning of V. While V correctly applied strategies other than A (e.g., Building up, in 3thP) or derived other solutions (70%, in 2ndP), it does not seem to have done it, right away, with all the conviction necessary to consider these new contents, weigh them and compare them objectively with what was initially proposed by him; indicators of the lack of that conviction are described in the right column of Table 2. It is possible that this lack of conviction was the result of the overconfidence in A, EM and 3x2, which led him to focus firmly on these contents and it prevented, at the outset, assuming security commitments with other different options (Fischbein, 1982, Inglis et al, 2007); and it is possible that that little conviction in Bu and 70% prevented, in turn, that this new strategy and solution had a later use (Fischbein, 1982). However, despite all this, his activities in 2ndP and 3rd P were sensitizing V, although imperceptibly, in these other strategies and results and thus reducing their over-confidence in their initial proposal, until he reached a state of confidence, even doubt, about his initial resolution (in 4th P). These changes in ee - decrease in confidence in 3x2 response (see left column Table 2) and openness of credibility to other strategies and responses (see right column Table 2) - allowed V to be sensitive to J's mathematical proposal. And that more cautious confidence or doubt (4th P) in his initial proposal, accompanied by a confidence in the relative strategy (5th P), allowed V to make a more objective comparison between all the available strategies, in the fq - what he could not do during the interaction - and opt for a mathematical strategy of a relative type towards which he showed certainty and in which he no longer needed the EM arguments.

Final Remarks

The above analysis offers empirical evidence that, like cognitive variables, epistemic states (which are conceived here as belonging to the affective sphere) can contribute (or on the contrary, can stop) to emerge, precision, integration and others qualitative aspects of the new learnings; in particular, it is empirically shown that certainty exerts an inhibiting effect on the construction or establishment of new contents, and that doubt acts as a driver of learning.

References

Ausubel, D., Novak, J. D., & Hanesian, H. (1978). *Educational Psychology: A Cognitive View.* New York: Holt, Rinehart and Winston.

Fischbein, E. (1982). Intuition and Proof. *For the Learning of Mathematics, 3*(2), 9-24.

Foster, C. (2016). Confidence and competence with mathematical procedures. *Educational Studies in Mathematics, 91*(2), 271-288.

Gómez, H. D. (2017). *El papel del convencimiento y la interacción entre pares durante el intercambio de resoluciones de un problema de comparación de razones*. Tesis de Maestría en Ciencias, Especialidad Matemática Educativa. Centro de Investigación y de Estudios Avanzados del I. P. N. México.

Gómez, B., Monje J., Pérez-Tyteca, P. & Rigo, M. (2013). Performance on ration in realistic discount task. Un B. Ubuz, C. Haser & M. A. Mariotti (Eds.), Proceedings of the CERME 8 (pp. 293-302).

Inglis, M., Mejia-Ramos, J. P., & Simpson, A. (2007). Modelling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics, 66*(1), 3-21.

Martínez B. & Rigo M. (2017). Confidence and comprehension building processes regarding mathematical content. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.), *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 217-224). Singapore: PME.

Rigo-Lemini, M., Martínez-Navarro, B. (2017). Epistemic States of Convincement. A Conceptualization from the Practice of Mathematicians and Neurobiology. In U. Xolocotzin (Ed.), *Understanding Emotions in Mathematical Thinking and Learning*. India: Elsevier.

THE PARALLELOGRAM PROBLEM: SUPPORTING COVARIATIONAL REASONING IN THE CONSTRUCTION OF FORMULAS

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I studied preservice teachers' uses of covariational reasoning with a dynamic situation to construct a formula to represent the covariational relationship between one of the interior angles of a dynamic parallelogram and its area. I propose a definition for reasoning covariationally with formulas, provide a first order model of what it means to do so, and then provide the results of the analysis of two students' activities with this task, highlighting the bidirectional relationship between students' images of a situation and their representation of it.

Keywords: Algebra and Algebraic Thinking, Cognition, Design Experiments

Students' understandings of modeling problems, variables, and equations have long been identified as a source of difficulty in research on mathematics education (Stephens, Ellis, Blanton, & Brizuela, 2017). Students interpret symbols as static unknowns (Dubinsky, 1991) and fixed, given referents (Gravemeijer, Cobb, Bowers, & Whitenack, 2000). Thompson and Carlson (2017, p. 425) describe this reasoning as one with constants (i.e., person envisions a quantity as having a value that does not vary ever) versus parameters (i.e., person envisions the quantity as having a value that can change from setting to setting but does not vary within a setting) or variables (i.e., person envisions that the quantity's value varies within a setting). I attempted to support preservice teachers' construction of variables through interviews using area formula contexts by incorporating ideas of covariational reasoning—students conceiving of situations as composed of quantities that vary in tandem (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). I propose covariational reasoning with formulas to be simultaneously imagining two quantities represented via symbols as variables that represent covarying quantities in a situation. I report on the relationship between students' perceptions of a dynamic situation and symbols in formulas.

Background

In this study, I adopt the radical constructivist perspective (von Glasersfeld, 1995) that individuals actively construct quantities—measurable attributes (Thompson & Carlson, 2017, p. 425)— and that an individual's image of a situation is projected from their mental organization of sensory data. My definition of image is rooted in Piaget's (1967) descriptions of images as shaped by mental operations individuals perform. Thompson notes that "the image is shaped by the operations, the operations are constrained by the image, for the image contains vestiges of having operated, and hence results of operating must be consistent with the transformations of the image" (Thompson, 1996). Similarly, Thompson and Carlson (2017, p. 448) emphasized the idea that students' constructions of symbolic expressions are constrained by the quantitative structures they construct about a situation. The formulas I intend for students construct stem from reasoning with dynamic area situations, a context other researchers are exploring as a productive way for K-8 students to construct formulas (e.g., Matthews & Ellis, in press).

Methods

In this study, four volunteer preservice teachers from a large public university in the southeastern U.S. partook in a videotaped sequence of 3-5 individual clinical (Clement, 2000) and exploratory teaching interviews (Steffe & Thompson, 2000) lasting 1.5-2 hours each over the course of four weeks during the summer. The students had just completed their first or second semester in a four-semester secondary mathematics education program and had completed a Calculus sequence and passed at least two other upper level mathematics courses. The students had also completed a secondary mathematics topics course designed from the *Pathways Curriculum* (Carlson, O'Bryan, Oehrtman, Moore, & Tallman, 2015) and thus had opportunities to reason quantitatively and covariationally with dynamic situations. I only focus on two students' second interview session, their first exploratory teaching session. I encouraged the students to think aloud (Goldin, 2000). The goal of my questioning was to build viable models of the *students' mathematics* (Steffe & Thompson, 2000). I conducted open (generative) and axial (convergent) analyses (Strauss & Corbin, 1998) to inform these second order models.

First Order Model of Parallelogram Problem

I presented students with a manipulative (Figure 1a) and following prompt: "Describe the relationship between the area inside the shape (shape formed by two pairs of parallel lines) and one of the interior angles of the parallelogram (up to a straight angle)." In the following first order model, I will use $\overline{AB}$ to indicate the physical segment from point A to B, AB_{unit} to indicate the length of $\overline{AB}$ measured in the unit specified, and AB as a measurement parameter for the situation (i.e., I anticipate measuring $\overline{AB}$ but have no specific unit of measure in mind).

Figure 1 offers a sequence of mental actions for a student's construction of a formula that relies on conceiving of two quantities, area within $ABCD$ and $\angle DAB$ (Figure 1b), and conceiving of them as variables within a formula through covariational reasoning. First, the student conceptualizes the area of a parallelogram as equivalent to the area of a corresponding rectangle through a translation (Figure 1c-e). Second, the student makes the following conclusion about covariational relationship between the angle measure and the rectangle: For equal changes in angle measure from $m\angle DAB = 0 \text{ to } \pi/2$ radians, the amount by which the area increases for each successive equal change in angle measure is less each time (Figure 1f-h). The student understands the height of the parallelogram as the height above the center of a circle measured in radii that they assimilate to this situation (Figure 1i) and assimilates it to the sine relationship (see Moore (2014) for details). Using $\sin(\theta)$ to denote the height assumes the unit is the radius of the circle (i.e., AB_{radii}) because sine outputs a result measured in radii. The radius, the length AB, can be measured with any other unit length, u, making the measurement for the height using that same unit $AB_u \sin(\theta)$, where AB_u is the measurement of the radius using any unit. Thus, the final formula for the area of the parallelogram that relates the area to the angle measure is $Area{=}AD{\bullet}AB_u \sin(\theta)$. This conception of a formula treats $Area$ and θ as variables and AB_u as a parameter based on the unit choice chosen for the situation for θ and AD.

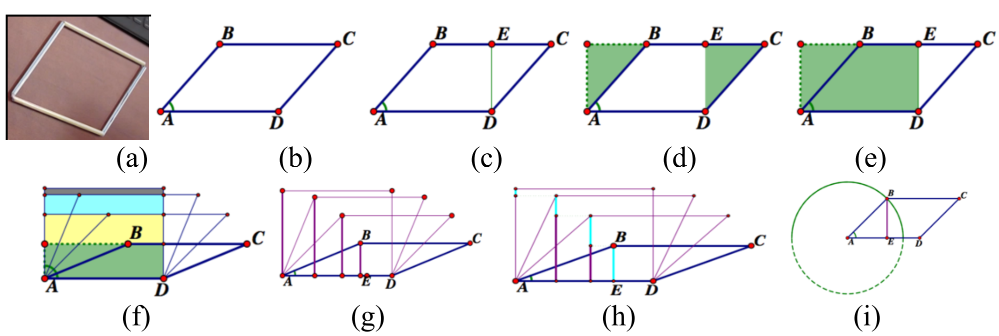

Figure 1. Sequence of mental actions to construct formula in Parallelogram Problem.

Results: Students' Image Structuring of the Parallelogram Problem

The following categories illustrate how different images informed students' conceptions of the relationship between angle measure and area and a formula for the area of the shape.

Unstructured Initial Images of Directional Covariational Relationships Between Quantities

Three of the four students stated at some point that the area of the parallelogram remained constant as they changed the angle measure. In their justifications, they made note of properties of the manipulative such as the constant lengths of the sides of the parallel lines (Charlotte, Alexandria) or noting that as two of the interior angle measures decrease, the remaining two interior angles increase by the same amount (Charlotte). Alexandria's image changed when she considered small angle measures; Charlotte was perturbed when she did so, saying, "It's like to a certain point I believe [the area remains constant], and then it's hard to believe it."

Comparison of Resulting Measurements from Formulas

Charlotte constructed two formulas, one for a rectangle and one for a parallelogram at another angle. She wanted to compare the measures of the area for the shape as a rectangle by multiplying *XC* (Figure 2, left) to the measure of the area of the shape as a parallelogram with a given angle measure (Figure 2, middle). This comparison of static figures was problematic for her when she realized she did not know which values to use to compare to one another.

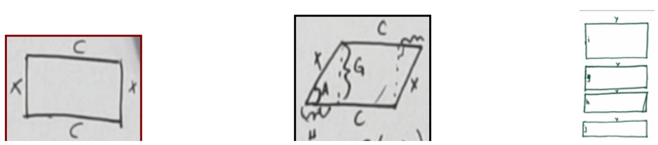

Figure 2. Charlotte's two instances (left, middle) of the situation from which she constructed two formulas and her construction of rectangles from parallelograms at "extreme" angles.

Equating Known Formulas with Images

Charlotte later noticed she could move her triangle on the left (Figure 2, middle) to the right side and form a rectangular shape (as in Figure 1d-e) and concluded that to find the area of the parallelogram shape, she should also multiply *CX* (i.e., the side lengths) "because I think that that's just a rectangle like looked at a different way" resulting in constant area for changing angles. This reasoning is problematic if the parallelogram is not precisely a rectangular shape.

Using Knowledge of Relationships between Area Formulas for Rectangles and Parallelograms to Reason with a Dynamic Image

With prompting, Charlotte carefully drew each rectangle in Figure 2(right) she was imagining forming separate from the corresponding instance of a parallelogram. She was explicit about constructing the *y* length to remain constant and the heights to decrease. When asked if it now looked like the rectangles were getting smaller, she said, "Yep. Now they do. It's easier to see like through the rectangle than when it's slanted. Especially since like I got something to compare. So like, here [*pointing to the parallelogram*], I can't necessarily see that *y* is the same, but since I like made a conscious effort to draw these lines the same, I can tell that these are-the distances between both of them are getting smaller." At this point, the quantitative structure Charlotte constructed was sufficient for her to reason using directional covariation.

Alexandria similarly constructed an image of directional covariation, and she also thought the growth in area was proportional to the angle measure and spent a considerable amount of her activity on this problem trying to find the scale factor to construct a formula. Her image changed when she constructed the corresponding heights for each marked angle measure in her diagram (similar to Figure 1g), and highlighted the segments that represented the change in height for each successive equal change in angle measure (similar to Figure 1h). She concluded, "The change in height increases" and abandoned her idea that relationship was proportional. She proceeded to construct a formula using reasoning the previous section outlined.

Conclusions and Discussion

Alexandria successfully illustrated the potential of reasoning covariationally with quantities constructed in dynamic situations to construct formulas that treat symbols as variables. However, both students' operations were still constrained by their images of both the situation and their formulas in the moment, which perturbed students if their justifications for the relationships they conceived were rooted in (i) images of other quantities changing or remaining constant in the situation (i.e., other angle measures and side lengths) and gross visual comparisons of area at random angle measures (ii) images of formulas as solely representations of measurement processes for static figures. These images impacted their understandings of their formulas.

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References

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378.

Carlson, M. P., O'Bryan, A. E., Oehrtman, M. C., Moore, K. C., & Tallman, M. A. (2015). *Pathways algebra and precalculus* (5 ed.). Kansas City, KS: Rational Reasoning.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 547-589). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. O. Tall (Ed.), *Advanced mathematical thinking* (pp. 95-123). Dordrecht, Netherlands: Kluwer.

Gravemeijer, K., Cobb, P., Bowers, J., & Whitenack, J. (2000). Symbolizing, modeling, and instructional design. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design* (pp. 225-273). Mahwah, NJ: Lawrence Erlbaum Associates.

Matthews, P. M., & Ellis, A. B. (in press). Natural alternatives to natural number: The case of ratio. *Journal of Numerical Cognition*.

Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education, 45*(1), 102-138.

Piaget, J. (1967). *Six psychological studies*. New York: Random House, Inc.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-307). Hillside, NJ: Erlbaum.

Stephens, A. C., Ellis, A. B., Blanton, M. L., & Brizuela, B. M. (2017). Algebraic thinking in the elementary and middle grades. In J. Cai (Ed.), *Compendium for research in mathematics education*. Reston, VA: National Council of Teachers of Mathematics.

Thompson, P. W. (1996). Imagery and the development of mathematical reasoning. In L. P. Steffe, B. Greer, P. Nesher, & G. A. Goldin (Eds.), *Theories of learning mathematics* (pp. 267-283). Hillsdale, NJ: Erlbaum.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Washington, D.C.: Falmer Press.

STUDENTS' CONSTRUCTION OF MATHEMATICAL GENERALIZATION IN A TECHNOLOGY-INTENSIVE INSTRUCTIONAL ENVIRONMENT

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Generalizing is widely considered to be essential to the learning and doing mathematics. Using design experiment methodology this study investigates middle school students' construction of mathematical generalization when learning geometric transformations in a technology-intensive instructional environment. The study reported different types of mathematical generalizations students constructed when engaged in instructional activities about geometric transformations.

Keywords: Mathematical Generalization, Dynamic Geometry Software

Making generalization is fundamental to mathematics and a crucial component of mathematical thinking. Developing learners' capacities towards making and justifying generalizations about mathematical relationships has been a pivotal goal of mathematics education for decades (Davydov, 1990). Indeed, three of the eight Standards (i.e., reason abstractly and quantitatively, look for and express regularity in repeated reasoning, and look for and make use of structure) for Mathematics Practices in Common Core State Standards for Mathematics directly relate to generalizing activities. Despite this, literature continues to highlight challenges associated with developing K-16 learners' abilities to generalizing mathematical ideas. To develop instructional activities that help students become more proficient at constructing mathematical generalization, it is important to understand the forms that the constructive process might take and the conditions that might affect the ways that individuals engage in the construction of mathematical generalizations. Meanwhile, researchers have provided evidence that mathematics-specific cognitive technologies influence both the technical and conceptual dimensions of a mathematical activity (Zbiek, Heid, Blome & Dick, 2007). The current study aimed to understand students' construction of mathematical generalizations in a technology-intensive instructional environment and was guided by one questions: (1) what mathematical generalizations do students construct when learning geometric transformations in a technology-intensive instructional environment?

Theoretical Framework

When studying students' construction of a mathematical generalization, we need to consider what comes before generalizing, what becomes generalized, and what generalizing yields. Pirie and Kieren (1994) perceived mathematical understanding as dynamic, leveled but nonlinear, recursive process. According this model, *Primitive Knowing* is the starting place for the growth of any new mathematical understanding. When *Image Making*, a learner engages in specific activities aimed at helping him to develop initial conceptions and ideas for the meaning of a mathematical object. By the stage of *Image Having*, a learner can carry out these specific activities with a general mental plan. At the layer of *Property Noticing*, a learner reflects on his mental image and recognizes attributes and features of it. When *Formalizing*, a learner abstracts a method or common quality from the previous image and has a class-like mental object that is not dependent on meaningful images. When *Observing*, a learner reflects on and coordinate such formalizing activity and express such coordination as theorems. *Structuring* occurs when a learner is aware of how a collection of theorems is inter-related and calls for justification of statements through logical argument. Pirie and Kieren's model provides a useful framework to trace the development of a mathematical generalization in the continuum of mathematical understanding.

Methodology

This study was frame by a design experiment methodology (Cobb et al., 2003). The goal of this design experiment was to explore how students construct mathematical generalizations when interacting with the teacher and dynamic geometry software in learning geometric transformations and what sorts of interventions become important in facilitating students generalizing practices. The participants were 7 middle school students who enrolled in a summer enrichment program for two weeks. Prior to this design experiment, a pre-assessment was designed to gather information about students' understanding of geometric transformations. Van Hiele-like levels of learning geometric transformations (Soon, 1989) was used to guide the design of the pre-assessment. Data from the pre-assessment indicated all the participants were able to recognize geometric transformations and perform transformation motions. This data was used in designing activities for the planned experiment. The transformation unit in the design experiment focus on examining properties of geometric transformations, identifying defining parameters of geometric transformations, and exploring composition of transformations.

Data processing and analysis consisted of three phases. In Phase I, each piece of video data was segmented based on transitions in mathematical activities. Generalizing episodes were then identified within those mathematical activities. In particular, an instructional activity was identified as a generalizing episode if the participants were engaged in at least one of the following four activities: (1) identifying a property of a mathematical object, (2) extending one's reasoning beyond the range in which it originated, (3) searching for a procedure, and (4) deriving a generality from known properties. Each generalizing episode was then transcribed verbatim. The transcript included descriptions of the participants' interactions with the teacher and technology. In Phase II, Pirie and Kieren's (1994) model was used to trace the development of each generalizing episode. Critical moments of interactions among the participants, the teacher, and the technology were included in each event map. In Phase III, all of the observations, conjectures, and generalizations which the participants shared in each generalizing episode were identified and categorized based on their level of generality and connections to a broader mathematical structure.

Results

The participants in the study constructed a variety of mathematical generalizations when engaged in activities related to geometric transformations. Based on their levels of generality and attentiveness to broader mathematical structures, these generalizations were categorized into three types: property-based generalizations, construction-based generalizations, and theory-based generalizations. *Property-based generalizations* are mathematical properties abstracted from empirical cases and extended beyond the cases in which the properties originated. *Construction-based generalizations* are construction procedures that allow learners to produce desired configurations through using a series of primitive construction commands. *Theory-based generalizations* are mathematical statements learners deduce from definitions, properties, and theorems through chains of logical reasoning. In this study, 44 generalizations were shared in 14 generalizing episodes. Among these generalizations, 24 were property-based generalizations, 8 were construction-based generalizations, and 12 were theory-based generalizations. Table 1 summarizes the frequencies of the different types of generalizations and the number of generalizing episodes in which each type of generalization was observed. Note that the total number of generalizations reported in Table 1 only includes those that the participants shared during whole-class discussions. Analysis of participant artifacts revealed that some of the generalizations that the participants constructed were not shared during whole-class discussions. Therefore, the total number of generalizations reported in the table may not be representative of the total number of generalizations participants produced during the study. Further, the participants often produced more than one generalization within one generalizing episode.

Table 1: Types of Mathematical Generalizations

Type of Generalization	Number of Generalizations	Number of Generalizing Episodes
Property-based generalization	24	10
Construction-based generalization	8	7
Theory-based generalization	12	4
Total	56	14

Although property-, construction-, and theory-based generalizations vary in their levels of generality and attentiveness to broader mathematical structures, they are closely linked. Property-based generalizations can be assembled to form a construction-based generalization form a geometric construction. Construction-based generalizations rely on properties of geometric objects, which might come from property-based or theory-based generalizations. A figure produced through a geometric construction is essentially a conceptual entity, although the figure might mobilize perceptual activities. In a dynamic geometry environment, a geometric construction can lead to the discovery of a new generalization or extension of the domain of validity of an existing generalization. Theory-based generalizations are generalizations that learners construct through reasoning about properties of mathematical objects. Explicit use of mathematical structures to deduce new knowledge characterizes theory-based generalizations. A theory-based generalization might build on a property-based generalization and it can also be used to form a geometric construction. Numerical and visual-graphical evidence plays an important role in the discovery of the properties and relationships of geometric objects.

Analysis of event maps of the generalizing episodes revealed patterns of distribution of the different types of generalizations along the layers of mathematical understanding. *Property-based generalizations* occurred frequently at the *Property Noticing* and *Formalizing* stages. *Construction-based* and *theory-based generalizations* were advanced at the *Formalizing* layer. The distribution of these types of generalizations suggested that the *Formalizing* stage included property-, construction-, and theory-based generalizations. Therefore, Property Noticing and Formalizing were the two main layers of understanding at which generalizations became observable. The fact that the Formalizing level included three types of generalizations revealed that mathematical ideas constructed at the same layer of understanding might have different natures and levels of generality. Although property-, construction-, and theory-based generalizations became observable only at the *Property Noticing* or *Formalizing layers*, *Image Making* and *Image Having* activities were crucial in the construction of generalizations, especially when the participants could not immediately develop a useable mental image for solving the problem at hand. Indeed, event maps of the exemplary generalizing episodes illuminated that Image Making and Image Having activities were prominent in many generalizing episodes.

Generalizing Episode 1 illustrates the findings from this study. In this generalizing episode, the teacher created an image of a 180° rotation and an image of a reflection to demonstrate the concept of orientation preserving and then asked students whether or not the image of the rotation could be produced by a reflection. The students made a conjecture that the composition of two reflections could produce the same image as the 180° rotation. After two attempts, the two images still did not match. The teacher then engaged the students in *Image Reviewing* and pointed out that the two images shared the same orientation. In response to the *Image Reviewing* activity, another student suggested translating the image of the composite reflection to the image of the 180° rotation. The student then made the two images match through continuously dragging and adjusting the two segments (*Image Making*). After the students made the two images match, the teacher asked what they noticed about the two reflection lines. The students observed that the two segments seemed to be perpendicular and verified their observation with measuring tool in GSP (*Property Noticing*). The teacher then asked the students how to find the two reflections if the angle of rotation is 75°. Students made a conjecture that the angle of the two reflection lines might be 37.5° since, in the previous case, 90° was one half of 180° ((*Property Noticing*). In an attempt to find the two reflection lines, the students folded back to *Image Having* and asked the teacher to draw two segments whose angle was 37.5°. The teacher drew the two segments, adjusted their angle to 37.5°, and executed the two reflections. It turned out that the two images did not match. The students then folded back to *Image Making* and tried to move the two reflection lines to match the two images. The teacher pointed out that the two segments shared an endpoint which might cause the problem (*Image Reviewing*) and then demonstrated how to construct two reflection lines with a fixed angle measure (*Primitive Knowing*). After the teacher constructed the image of the composite reflection, the students noticed that the center of rotation was the point of intersection of the two reflection lines when the two final images matched (*Property Noticing*). The teacher then asked students to find the two reflections whose composition would produce the same image as that given by a 125° rotation with a minimum number of guesses and checks. In doing so, the teacher moved the students in the direction of formalizing a general procedure for finding two reflections. The students suggested drawing two segments passing through the center of rotation with a fixed angle of 62.5° (*Formalizing*). This indicates that the students were starting to formalize the procedure. The teacher then synthesized this formal procedure and asked students to draw a picture to capture this generalization.

References

Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational researcher*, 32(1), 9-13.

Davydov, V. V. (1990). *Types of Generalization in Instruction: Logical and Psychological Problems in the Structuring of School Curricula* (Vol. Soviet Studies in Mathematics Education). Reston, Virginia: National Council of Teachers of Mathematics.

Dörfler, W. (1991). Forms and Means of Generalization in Mathematics. In A. J. Bishop, S. Mellin-Olsen, & J. Dormolen (Eds.), *Mathematical Knowledge: Its Growth Through Teaching* (pp. 61-85). Springer.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational studies in mathematics*, 26(2-3), 165-190.

Soon, Y. (1989). *An investigation of van Hiele-like levels of learning in transformation geometry of secondary school students in Singapore*. Unpublished Doctoral Dissertation, The Florida State University.

Zbiek, R.M., Heid, M.K., Blume, G., & Dick, T.P. (2007). Research on technology in mathematics education: A perspective of constructs. In F.K. LesterJr., (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 1169–1207). Charlotte, NC: NCTM and Information Age Publishing.

MIDDLE GRADE STUDENTS' UNDERSTANDING OF MEDIAN AND MODE

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Keywords: Middle School Education, Learning Trajectories (or Progressions)

Measures of Central Tendency (MOCT), which include mean, mode and median, are important mathematical/statistical concepts; hence, they are typically included in mathematics curriculum as early as middle grades (Amiruzzaman & Kosko, 2016). However, most research related to MOCT has generally focused on the teaching and learning of arithmetic mean, with a little study of median or mode (Amiruzzaman & Kosko, 2016; Amiruzzaman, Kosko, & Amiruzzaman, 2017). In the past, a few studies focused on students' conceptions of median and/or mode have found that high school and college-level students tend to have more difficulty calculating median and mode than the arithmetic mean (Amiruzzaman & Kosko, 2016; Amiruzzaman, Kosko, & Amiruzzaman, 2017). This study focused on a small group of middle grades (i.e., sixth-grade) students, especially to perform more in depth analysis of students' initial understanding and how their understanding has developed through actions of working with mathematical problems and manipulative. In this study, a classroom based teaching experiment was used to build our models of participants' conceptions of median and mode. Total of six students (5 males and 1 female) participated in the study, and the study consisted of 14 teaching experiment episodes (approximately 45-50 minutes/episode) across 14 weeks in spring 2015. The initial tasks focusing on median and mode for the initial episode were prepared ahead of time; however, tasks for following episodes were prepared based on the analysis of the prior episode. This study presents data from episodes 6 to 8 to illustrate preliminary findings for learning trajectories of the participating students, Alice and Bob.

In episode 6, both Alice and Bob described the median as the middle value of a data set, and generally were able to identify the median with data sets with an odd number of elements. However, when Alice and Bob were asked to find the median for data with an even number of elements, both of them that claimed there was no median. In episode 7, both Alice and Bob described that a dataset may have only one mode. However, in episode 8, it was observed that a number-line manipulative helped Alice and Bob to understand the concepts of median and mode better. The findings of this study indicate that middle grade students' conceptions of median and mode may require a more visual manipulation of the data elements. Working with data and seeing how the data elements are distributed helps to understand the concepts of median and mode better. One of the implications of this study is that students' conception of the median and mode are more complex than previously recognized. In regards to practice, these findings suggest that practitioners (i.e., mathematics teachers) should use manipulative, so that students could manipulate data and understand its distribution before they learn about the definitions of these concepts.

References

Amiruzzaman, M., & Kosko, K. W. (2016). Exploring students' understanding of median. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter for the Psychology of Mathematics Education* (p. 660-663). Tucson, AZ: The University of Arizona.

Amiruzzaman, M., Kosko, K. W., Amiruzzaman, S. R. (2017). Preservice mathematics teachers' understanding of mode. In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter for the Psychology of Mathematics Education* (pp. 625-628). Indianapolis, IN.

COLLEGE ALGEBRA STUDENTS' CONCEPT IMAGES OF PARABOLAS

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Keywords: Cognition, Post-secondary Education

Concept image is the set of all mental pictures associated with the concept name and the properties associated with the concept, acquired through experiences with examples and nonexamples of the concept (Tall & Vinner, 1981). Given the importance of representations in developing understanding of mathematical concepts, this study documents the concept images of parabolas held by students in a college-level Algebra class, and identifies potential influences on students' understandings. Students' concept images of parabolas are of interest because parabolas are both mathematical objects and representations of symbolic algebra.

Clinical interviews were conducted with two students from a college-level Algebra course at a large mid-Atlantic public university. The cognitive tasks during the interview aimed to activate different parts of students' concept image of parabola. A representational fluency analytical framework was used to characterize the coherence of students' concept images, as the level of coherence of an individual's concept image impacts their ability to successfully navigate between the elements. In addition to clinical interviews, select sessions of the college Algebra course were observed and the assigned textbook was examined, providing insight to potential influences on students' concept images.

Findings suggest that participants' concept images are not coherently connected, as demonstrated by their low levels of sophistication in representational fluency. Both participants relied on the algebraic definition of parabola, as a graph of a quadratic function, throughout the tasks. Participants were preoccupied with remembering rather than reasoning. Despite awareness of processes and procedures regarding parabolas and quadratic functions, participants were unsuccessful in transposing within and translating between representations. Participants acted inconsistently across the different tasks, suggesting that different elements of their concept images were activated. Participants showed a lack of understanding of the parabola's vertex, possibly due to the over-emphasis on solving for only one coordinate of key points when graphing parabolas in both the course textbook and lecture. Although participants in this study completed advanced mathematics courses at the secondary level, their intellectual behaviors during the clinical interviews reflect those of secondary students who are learning about parabolas or quadratic functions for the first time, as seen in related research. Thus this research suggests that emphasis on symbolic manipulation and algorithms at the secondary and early college level may fail to facilitate students' visualization of concepts and create the relational connections between symbolic and graphical representations that are necessary for conceptual understanding.

References

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151-169.

CONTENT VALIDITY EVIDENCE FOR NEW PROBLEM-SOLVING MEASURES (PSM3, PSM4, AND PSM5)

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Keywords: assessment and evaluation, elementary school, measurement, problem solving

Instrument development should adhere to the *Standards* (AERA et al., 2014). "Content-oriented evidence of validation is at the heart of the [validation] process" (AERA et al., 2014, p. 15) and is one of the five sources of validity evidence. The research question for this study is: What is the evidence related to test content for the three instruments called the PSM3, PSM4, and PSM5? The study's purpose is to describe content validity evidence related to new problem-solving measures currently under development. We have previously published validity evidence for problem-solving measures (PSM6, PSM7, and PSM8) that address middle grades math standards (see Bostic & Sondergeld, 2015; Bostic, Sondergeld, Folger, & Kruse, 2017).

We chose a design-science based methodology to develop the PSM series. This methodology is useful for measure development, gathering data from the measure, drawing reasonable conclusions from the data, revising the measure, and repeating the cycle. Three forms of data were collected sequentially to explore test content validity evidence. Our data analysis approach used traditional methods (Sireci & Faulkner-Bond, 2014). Broadly speaking, all reviewers on the expert panel agreed the items were open, complex, and realistic. Mathematicians confirmed that each item could be solved in two or more ways. Students expressed that the items were complex, solvable, and realistic.

Results indicated that items were both representative and relevant of the construct. Supplementing this conclusion with the knowledge that the definition is sufficiently bounded and test construction followed the *Standards* (AERA et al., 2014), leads to a conclusion that the PSMs have adequate test content validity evidence. This is the initial step in building a validity argument; next steps are to gather evidence related to response processes, relations to other variables, internal structure, and consequences from testing.

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References

AERA, APA, & NCME. (2014). *Standards for educational and psychological testing*. Washington, DC: American Educational Research Association.

Bostic, J., Sondergeld, T., Folger, T. & Kruse, L. (2017). PSM7 and PSM8: Validating two problem-solving measures. *Journal of Applied Measurement, 18*(2), 151-162.

Bostic, J., & Sondergeld, T. (2015). Measuring sixth-grade students' problem solving: Validating an instrument addressing the mathematics Common Core. *School Science and Mathematics Journal, 115, 281-291*.

Sireci, S. & Faulkner-Bond, M. (2014). Validity evidence based on test content. *Psichothema, 26*(1), 100-107.

TEACHING RATE OF CHANGE TO STUDENTS WITH DISABILITIES USING AN INTEGRATED CONCRETE-REPRESENTATIONAL-ABSTRACT APPROACH

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Rate of change is a challenging area of mathematics spanning multiple grades and courses, and is related to functional skills such as accurately reading maps and calculating interest (Herbert & Pierce, 2012). To date, no research has been conducted to determine effective ways to teach rate of change to students with disabilities. Across multiple grade and ability levels, instruction incorporating an integrated concrete-representational-abstract (CRA-I) approach increases math achievement (Strickland & Maccini, 2012). CRA-I is often paired with cognitive strategies instruction, such as problem-solving heuristics, as well as student verbalizations to help build conceptual understanding (Watt, Watkins, & Abbitt, 2016). Writing To Learn (WTL) may provide a platform for incorporating problem-solving heuristics and written explanations of student thinking into CRA-I interventions, with the goal of increasing students' conceptual and procedural understanding of rate of change (Gersten et al., 2009).

In this study, we used a single-subject multiple-probe design across participants to evaluate whether an intensive intervention consisting of CRA-I, a problem-solving heuristic, and writing to learn helped ninth-grade students with disabilities improve their understanding of rate of change. The participants included four ninth-grade students with disabilities below state proficiency levels in math. Results indicated that the multi-component intervention, implemented across seven weeks, may improve students' with disabilities understanding of rate of change. All four students improved their scores on math assessments, and maintained improvements on post-instruction assessments administered one to seven weeks following the completion of the intervention. Two of the four students also moderately improved their scores on writing assessments. The results of this study support the use of instructional strategies, such as CRA-I, heuristics, and writing, that build both conceptual and procedural knowledge. Additionally, the results of this study indicate the importance of incorporating writing activities in math to help students develop the ability to explain and justify their reasoning.

The research questions explored include: (1) What is the effect of implementing an integrated concrete-representational-abstract (CRA-I) instructional sequence incorporating writing to learn and cognitive strategies on students with disabilities' proficiency in solving rate of change problems, and (2) Do students with disabilities find the multi-component (CRA-I + writing to learn + cognitive strategies) intervention to be socially acceptable?

References

Gersten, R. Chard, D.J., Jayanthi, M., Baker, S. K., Morphy, P., & Flojo, J. (2009). Mathematics instruction for students with learning disabilities: A meta-analysis of instructional components. *Review of Educational Research, 79*(3), 1202-1242.

Herbert, S., & Pierce, R. (2012). Revealing educationally critical aspects of rate. *Educational Studies in Mathematics, 81*, 85-101.

Strickland, T.K., & Maccini, P. (2012). The effects of the concrete-representational-abstract integration strategy on the ability of students with learning disabilities to multiply linear expressions within area problems. *Remedial and Special Education, 34*(3), 142-153.

Watt, S.J., Watkins, J.R., & Abbitt, J. (2016). Teaching algebra to students with learning disabilities: Where have we come and where should we go?. *Journal of Learning Disabilities, 49*(4), 437-447.

AN ANALYTIC FRAMEWORK FOR ASSESSING STUDENTS' UNDERSTANDING OF PROOF COMPONENTS

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Keywords: Reasoning and Proof, Assessment and Evaluation, High School Education

Researchers have used a variety of frameworks to evaluate students' work that differ in focus but all yield a single score or level (e.g., Healy & Hoyles, 2000; Knuth, Choppin, & Bieda, 2009; Lee, 2016). By assessing students' work holistically, none of the frameworks are able to differentiate, for example, between instances whether an empirical solution was produced due to the student's content knowledge or to their understanding of proof (e.g., Knuth et al, 2009). For students who are still learning to construct proofs, attention to multiple characteristics of their work can serve to not only characterize students' current understanding but also suggest possible areas for growth in their mathematical and proof understanding.

The proposed framework (Table 1), based on Stylianides (2007) definition of proof, attends to multiple aspects of students' work, including categories that assess whether students demonstrated an awareness of a proof component (e.g., attention to the justification requirement) and categories that assess students' ability to correctly apply the given component (e.g., uses mathematically accurate justifications). The poster will include examples of 9th graders' coded work, produced at the end of an introduction-to-proof design research study, selected to illustrate the framework and the distinctions between it reveals. As the field seeks to move the teaching and learning of proof in K-12 classrooms forward, it is important that we look for ways of identifying the skills and understanding students have developed, which can then be leveraged in future instruction.

Table 1: Analytic Framework for Assessing Students' Understanding of Proof Components

Stylianides (2007) definition	**Framework Categories**	
A connected sequence of assertions	Assertions follow a logical flow	Identifies and uses the correct hypothesis
Uses set of accepted statements	Attention to the justification requirement	Uses mathematically accurate justifications
Modes of argumentation	Attention to the generality requirement	Overall type of argument
Modes of argument representation	Forms of expression used	Relationship between written text and other forms of expression, if applicable

References

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. Journal for Research in Mathematics Education, 31(4), 396–428.
Knuth, E. J., Choppin, J., & Bieda, K. N. (2009). Middle school students' production of mathematical justifications. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), Teaching and Learning Proof Across the Grades: A K-16 Perspective (pp. 153–170). New York: Routledge.
Lee, K. (2016). Students' proof schemes for mathematical proving and disproving of propositions. The Journal of Mathematical Behavior, 41, 26–44.
Stylianides, A. J. (2007). Proof and proving in school mathematics. Journal for Research in Mathematics Education, 38(3), 289–321.

UNDERGRADUATE STUDENTS' CONSTRUCTION OF GENERAL STATEMENTS

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Introduction

Construction of general statements is essential mathematical activity (Mason, 2008). Advanced mathematical study should support students' development of abilities to generalize and conjecture (CUPM, 2015). This is particularly important for undergraduate students who self-select to pursue careers in studying and teaching mathematics. Understanding how students construct such statements can help instructors plan to support students' development of mathematical practices. This presentation shares findings of empirical research into the question: How do undergraduates in mathematics-intensive degrees, near the end of their degree programs, approach the construction of general statements?

Method

Six secondary mathematics education majors and four mathematics majors, each within three semesters of completing their mathematics program of study, were recruited from a large Mid-Atlantic public university. Data were collected via individual task-based interviews. Tasks were *populating tasks,* defined as tasks that present a set of *general properties* that could be true for some, but not all, elements in a broad collection. The broad collection is the *universe* of the task. The task prompts are to identify a *domain,* i.e., a subset of the universe, describing cases that have the general properties. Using inductive, qualitative methods (Corbin & Strauss, 2008), responses were sorted by similarities in ways of using cases, descriptions of sets within the universe, and inductive or deductive reasoning. Essential characteristics of each category were identified and used to characterize the category as an *approach.*

Findings and Implications

Approaches fell into three broad categories: Generalizing, Specializing, and Direct. Generalizing approaches involve forming an initial domain case-by-case, then expanding that domain. Approaches differed in the method of expansion: either by inductively describing a set containing the cases, or by systematic testing of new cases to eliminate nonessential characteristics, thus expanding the domain. Specializing approaches involve generating sets to satisfy a subset of general properties, then reducing the initial sets to a subset satisfying all of the general properties. A Direct approach involve deducing logically necessary characteristics for a case to have the general properties. The use of specific cases and inductive reasoning by these students, though useful, has implications for those who go on to further study in areas of mathematics in which specific cases are difficult to construct and for those whose future careers involve shepherding the growth of deductive reasoning in others.

References

Corbin, J., & Strauss, A. (2008). *Basics of qualitative research* (3rd ed.). Thousand Oaks, CA: SAGE.

Mason, J. (2008). Making use of children's powers to produce algebraic thinking. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 57–94). New York, NY: Erlbaum.

Schumacher, C. S., Siegel, M. J. (Co-chairs) & Zorn, P. (Ed.). (2015). *2015 CUPM curriculum guide to majors in the mathematical sciences.* Washington, D.C.: The Mathematical Association of America.

QUANTIFYING ANGULARITY: CIRCLES OPTIONAL

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Keywords: Cognition, Geometry and Geometrical and Spatial Thinking, High School Education

Angle measure is a pervasive topic in school mathematics and is often introduced in the elementary grades. Extant literature (e.g., Moore, 2013) and standards (e.g., CCSSM) have emphasized quantifications (Thompson, 2011) of angularity whereby multiplicative comparisons of circular arcs are used in service of measuring angles. Although circular quantifications of angularity are productive, mathematics educators cannot take for granted that all students can make the requisite multiplicative comparisons, much less generalize them as holding invariant across all circles. Therefore, examining other quantifications of angularity is paramount.

I present two non-circular quantifications of angularity, which I abstracted from my conceptual analysis (von Glasersfeld, 1995) of two ninth-graders' activities throughout a yearlong teaching experiment (Steffe & Thompson, 2000) focused on quantifying angularity. Rather than measuring angles by measuring arcs, Bertin and Kacie each developed quantifications of angularity entailing extensive quantitative operations (Steffe & Olive, 2010) enacted on the interiors of angles (e.g., angular iteration). Additionally, both students established re-presentable templates for familiar angles (e.g., right, straight, and full angles), and permanently assigned these templates dedicated degree measures (e.g., a right angle is a 90° angle). Bertin and Kacie could make and measure other angles by subdividing these familiar templates into equiangular parts and performing corresponding numerical computations on the dedicated measures. A distinguishing factor in Kacie's and Bertin's quantifications was how each student mentally subdivided angles into smaller equiangular parts. Bertin indicated the equipartitioning operation (i.e., the *simultaneous* insertion of same-sized parts intended to exhaust a whole). In contrast, Kacie indicated the equisegmenting operation (i.e., the *sequential* insertion of same-sized parts intended to exhaust a whole). Although both students constructed powerful quantifications of angularity, neither student developed a circular quantification of angularity during the teaching experiment; therefore, I hypothesize extensive quantifications of angularity necessarily precede circular quantifications of angularity. As such, students in elementary grades may be better supported if teachers begin instruction in angle measure with the intent of fostering extensive quantitative operations on angles (e.g., iterating and partitioning). At the very least, students who develop non-circular quantifications of angularity should be celebrated in mathematics classrooms.

References

Moore, K. C. (2013). Making sense by measuring arcs: A teaching experiment in angle measure. *Educational Studies in Mathematics, 83*(2), 225–245.

Steffe, L. P. & Olive, J. (2010). *Children's fractional knowledge*. New York: Springer.

Steffe, L. P. & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267–307), Hillside, NJ: Erlbaum.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^e* (pp. 35–57). Laramie, WY: University of Wyoming.

von Glasersfeld, E. (1995). *Radical Constructivism: A way of knowing and learning*. London: Falmer.

THE RELATIONSHIP BETWEEN COGNITIVE RESOURCES IN MATHEMATICAL MODELING AND MATHEMATICS ACHIEVEMENT

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Keywords: mathematical modeling, cognitive resources, mathematics achievement

Mathematical modeling (Lesh & Doerr, 2003) is a thinking process in which students need to apply a wide range of cognitive attributes – one component of intellectual resources related to knowing-how (Bailin, 2002). Students can develop cognitive attributes in mathematical activities and some of those cognitive attributes are used when students solve test problems. Common cognitive attributes used in mathematical modeling and solving test problems – *analyze*, *select*, *compute*, and *represent* in this research (adopted from the TIMSS 2011 assessment framework; Mullis, Martin, Ruddock, O'Sullivan, & Preuschoff, 2009) – allow us to connect mathematical modeling as classrooms practices and mathematics achievement. Thus, the purpose of this research is to quantitatively and empirically investigate the relationships between students' development of mathematics cognitive attributes and their achievement.

We collect data of students from grades 4 to 8 (over 15,000 students at each grade) who took the Iowa Assessment from 2006 to 2012. We apply the generalized DINA (GDINA; de la Torre, 2011) to students' responses, which produces a database about sets of cognitive resources individual students are expected to attain. In the secondary analysis, students' national standard scores are matched to their mastery patterns. Then, the linear regression analysis is employed to show which cognitive resource contribute to better mathematics achievement while effect sizes of each cognitive attribute are estimated.

The findings show that the four cognitive attributes together are positively correlated to mathematics achievement. In addition, students who master *select* or *compute* are likely to have higher achievement scores with large degrees of effects. Master of *analyze* is also important only with a combination of mastery of *select*. Longitudinally, *select* and *analyze,* which are used at the beginning stages of mathematical modeling, have maximum effects on achievement at grade 7. The effect sizes of *compute* seems monotone increasing from grade 4 to 8 with a remarkable drop at grade 7. The effect sizes of *represent* are generally small showing a gap between middle and elementary school levels.

We suggest several implications based on the findings: students need to have opportunities to develop the four cognitive attributes based on the positive relationships between mathematical modeling and achievement. However, teachers need to have instructional emphases on different stages of mathematical modeling depending on grade levels: representing a solution at elementary-school levels because of the effect size gap, analyzing a problem situation and selecting strategies at middle-school levels, and computation across grades 4 to 8.

References

Bailin, S. (2002). Critical thinking and science education. *Science & Education, 11*, 361–375.

de la Torre, J. (2011). The generalized DINA model framework. *Psychometrika, 76*(2), 179–199.

Lesh, R., & Doerr, H. M. (Eds.). (2003). *Beyond constructivist: A models & modeling perspective on mathematics teaching, learning, and problems solving*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Mullis, I. V. S., Martin, M. O., Ruddock, G. J., O'Sullivan, C. Y., & Preuschoff, C. (2009). *TIMSS 2011 assessment frameworks*. Chestnut Hill, MA: TIMSS & PIRLS International Study Center Lynch School of Education, Boston College.

ELEMENTARY STUDENTS' ATTEMPTS TO IDENTIFY AND JUSTIFY PATTERNS WITH A GRAPH THEORY PROBLEM

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Keywords: Elementary School Education, Problem Solving, Reasoning and Proof

Elementary students are not typically encouraged or given the opportunity to provide generalizable arguments in mathematics classes (Carpenter, Franke, & Levi, 2003). When they are given the opportunity, they generally limit their argument to giving an example. However, it has been documented that elementary students are capable of making conjectures and developing mathematical claims (Ball, 1993). Many researchers have recommended that all students in all grade levels should engage in mathematical proof and reasoning (e.g., Carpenter et al., 2003; NCTM, 2014). In this study, I explored whether students were able to provide justifications or state generalized patterns of a graceful label for different types of tree graphs.

Ten Grade 4 and 5 students participated in seven semi-structured, task-based interviews (Goldin, 2000). The interviews each lasted between 35 and 45 minutes and took place at an after-school program in the Midwest. During the interviews, students worked on the Graceful Tree Conjecture. They examined graceful labelings of star graphs, path graphs, and caterpillar graphs. All interviews were video recorded, transcribed, and analyzed. For analysis, I documented each instance where students found patterns or made a mathematical argument.

I found that all of the students were able to describe patterns for gracefully labeling star graphs. Three students found and described patterns for path graphs, but all ten students were able to find a graceful label. For caterpillar graphs, three students described a pattern they found through trial-and-error, and three students used their pattern from path graphs to develop a pattern for caterpillar graphs.

When students present an argument, they either appeal to authority, use an example, or give a generalized argument (Carpenter et al., 2003). I found because of the nature of the problem—graph theory—the students had no prior experience and thus they were not able to appeal to authority. The most common way students tried to justify their pattern was through providing an example. When asked how they knew that a labeling was graceful, the students would explain their pattern using a one specific example. One student even went as far as explaining her labeling would work for a graph containing 20 nodes. Several times the students would attempt to give a generalizable argument, but they struggle with the knowledge of how to do that and did not have the appropriate language.

References

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal, 93*(4), 373–397. doi:10.1086/461730

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.

Goldin, G. A. (2000). A scientific perspective on structures, task-based interviews in mathematics education research. In A. E. Kelley & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517–545). Mahwah, NJ: Erlbaum.

National Council of Teachers of Mathematics. (2014). *Principles to action: Ensuring mathematical success for all*. Reston, VA: Author.

Chapter 8

Preservice Teacher Education

Research Reports

Brief Research Reports

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.

Posters

EXAMINING TEACHER CANDIDATES' RESPONSES TO ERRORS DURING WHOLE-CLASS DISCUSSIONS THROUGH WRITTEN PERFORMANCE TASKS

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Effectively leading whole-class mathematics discussions is made more difficult when students' contributions are incomplete, imprecise, or not yet correct, and not easily correctable—what we call "errors." Through purposefully designed opportunities to investigate, enact, and reflect on teaching, teacher candidates (TCs) can develop skill to productively respond to errors in whole-class discussions. We investigate how TCs respond to errors when engaging in a written performance task that calls for TCs to play out discussions in response to a classroom scenario. We consider what the performance task reveals about TCs' practice and perspectives, with implications regarding theory and practice.

Keywords: Teacher Education-Preservice, Instructional Activities and Practices

Whole-class mathematics discussions are critical spaces in which students can participate authentically in mathematics and develop a broad set of mathematical proficiencies and practices (Kilpatrick, Swafford, & Findell, 2001). Effectively leading discussions is complex work comprised of eliciting, responding to, and building upon student contributions toward mathematical goals (Boerst, Sleep, Ball, & Bass, 2011). Contributions that are incomplete, imprecise, or not yet correct present an additional challenge for teachers wanting to keep student reasoning central, make progress toward identified goals, and not convey erroneous ideas. We contend that, through purposefully designed opportunities for representing, decomposing, and approximating practice (Grossman et al., 2009), teacher candidates (TCs) can develop skill to productively respond to student errors in whole-class discussions. We share our use of written performance tasks in which TCs produce dialogues and rationales for their dialogues in response to a classroom scenario. Our analyses provide a window into TCs' practice and perspectives and offer implications regarding theory and practice of mathematics teacher education.

Literature Overview

All mathematics teachers have the opportunity to respond to student errors as part of the work of teaching, and there is a robust literature base exploring the role of errors in the mathematics classroom. Understanding teacher responses to student errors requires first defining the term "error." Nesher (1987) argues that making errors are the way in which students contribute to their own learning process. This perspective is echoed in later literature, which emphasizes viewing students as sense-makers and identifying opportunities to build on student thinking (e.g., Van Zoest et al., 2017). Following Brodie (2014), we define errors as incomplete, imprecise, or not yet correct contributions that are more complex than mistakes that are easily correctable. Errors occur "among learners within and across contexts" (Brodie, 2014, p. 223), and move beyond mistakes such as misspeaking or accidently doing an incorrect computation. Taking this view on the role of errors makes it imperative to understand the ways in which teachers respond to errors. One common approach is for teachers to make corrections (e.g., Tulis, 2013), quickly highlighting the error and introducing correct ideas into the discussion. Another approach is to avoid errors (e.g., Bray, 2011; Santagata, 2005) and steer the conversation toward correct contributions. These actions potentially remove the opportunity for students to make sense of errors. If students make sense of mathematics at least in part through their errors, then productive teacher responses ought to support that sense-making. Teachers must notice students' mathematical reasoning (Jacobs, Lamb, & Philipp, 2010) and navigate the in-the-moment work of building understanding (Bray, 2011; Leatham, Peterson, Stockero, & Van Zoest, 2015).

An added layer of complexity in responding to errors results from the context in which the error occurs. In whole-class discussions, the teacher must navigate the needs of the student who contributed the error along with the needs of the rest of the class, who may or may not share that student's conception. Leading whole-class discussions requires responding to student reasoning and making student contributions central to the mathematical work of the class (Boerst et al., 2011). In this context, teachers must find ways to navigate making student contributions central at the same time as a desire to have the conversation focus on correct responses. We extend the work on both errors and whole-class discussion in secondary mathematics by considering whole-class discussions as the particular context for responding to errors. We examine the ways in which teachers may respond to errors through analyzing representations of their practice.

Theoretical Framework

We take the perspective that, to develop skilled practice, TCs must not only have opportunities to think about and reflect on teaching using representations of practice such as observation, written vignettes, and video, but also the opportunity to approximate the work (Grossman et al., 2009). This can occur when TCs meaningfully engage in interactive and contingent aspects of teaching in settings of reduced complexity and authenticity. Through opportunities to enact teaching practices in response to student contributions, TCs demonstrate and further develop adaptive skill that coordinates pedagogical approaches, the goals of a particular approach, and a vision of mathematics teaching (Ghousseini, Beasley, & Lord, 2015). These opportunities also provide a lens for teacher educators and researchers to assess TCs' developing practice and their coordination of approaches and goals.

This theoretical perspective frames our use of pedagogies such as coached rehearsals (Lampert et al., 2013), and as teacher educators and researchers we strive to find multiple approximations and strategies for assessing TCs' practice. We use written performance tasks (e..g, Bray, 2011) as an additional way to put TCs in the position to make sense of and respond to student reasoning. In designing these tasks, we draw on research around scripting classroom interactions (Crespo, Oslund, & Parks, 2011; Zazkis, 2017). TCs are presented with a realistic classroom scenario involving whole-class discussion and student errors. TCs demonstrate, through written dialogue, how they might continue the discussion. These dialogues represent, in part, TCs' imagined response to a particular student contribution. They also represent TCs' sense of how students might contribute further, giving insight into TCs' view of what is reasonable or desirable in a classroom episode. TCs also provided a rationale for why they continued the discussion the way they did. In this paper, we explore what these performance tasks say about TCs' developing practice, building on preliminary work (Campbell, Baldinger, Selling, & Graif, 2017). We ask: (1) what are the features of TCs' dialogues written in response to a student error made during a whole-class discussion; (2) what are the features of TCs' rationales for their dialogues written in response to a student error made during a whole-class discussion; and (3) in what ways do TCs' dialogues relate to their rationales, and what do these relationships suggest about TCs' coordination of approaches and goals in their responses to student errors in whole-class discussions?

Methods

We describe our analyses of one performance task we designed, with a scenario centered on the use of a card sorting activity (Baldinger, Selling, & Virmani, 2016) intended to elicit and refine the following definition of a polygon: “a 2-dimensional (plane) figure, where each side is a straight-line segment that intersects exactly one other side at each endpoint.” In the scenario, the whole-class discussion began after students had worked in small groups. The teacher asked for students to name cards that they easily knew were polygons. One student, Rosalia, offers Shape Q (see Figure 1) and, after a back-and-forth with the teacher, shares that it is a polygon because “it is a square” and that “all the sides are straight lines.” After the teacher asked for another example of a polygon, Jessie offers Shape J (see Figure 1), stating that it was square, like Shape Q. TCs dialogues began following Jessie’s contribution. After writing their dialogues, TCs wrote rationales describing why they continued the discussion as they did.

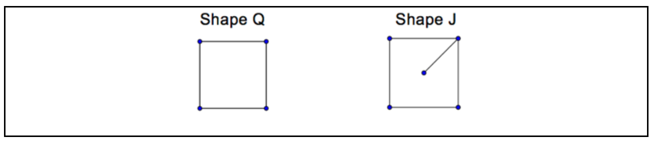

Figure 1. Two cards presented in the scenario as examples of polygons

We collected responses from 25 secondary mathematics TCs in methods courses at two large, public research institutions. Seventeen participants were engaged in a yearlong post-baccalaureate licensure program at one institution. Eight participants from a second institution were enrolled in a shared methods course across multiple licensure programs. The methods courses in both programs incorporated practice-focused teacher education pedagogies, such as coached rehearsal. The full performance assessment (including two scenarios) was administered using Qualtrics in October 2016. TCs completed the assessment individually during the methods class. Response times for the full assessment ranged from 11:24 (minutes and seconds) to 42:37, with a median duration of 25:34.

We used a priori and emergent codes to describe features of TCs’ response to student errors in the dialogues. Examples of codes included talk moves used by TCs, such as re-voicing (Chapin, O’Connor, & Anderson, 2013), asking funneling or leading questions, and a focus on comparing the two shapes. We used similar codes to describe the features of TCs’ rationales. Examples of codes for the rationales included the rationale mentioning wanting to resolve the error, or the rationale mentioning the need to recognize and/or value student contributions. Using these codes for the dialogues and the rationales, we developed a series of yes or no questions to ask of each dialogue and rationale. These questions fell into three main categories: attending to the error, attending to the mathematics, and attending to students. Two authors independently answered the questions for each dialogue and rationale. Inter-rater reliability was assessed and disagreements were resolved through discussion.

Next, we engaged in a process of analytic memo writing and theme building (Miles, Huberman, & Saldaña, 2013) to identify the most salient features in the dialogues and in the rationales. As we looked at counts of dialogues or rationales with particular features, we kept in mind that the absence of evidence was not the same as evidence of absence. For example, though 15 TCs explicitly mentioned that Jessie’s contribution contained an error, other evidence demonstrated that all participants understood Jessie’s error. To address the relationships between TCs’ rationale and dialogue, we utilized a matrix highlighting relationships among dialogue features and rationale features (Miles et al., 2013). We used the three categories of questions in our analyses of the dialogues and the rationales to explore connections between the two. This allowed for a coordination of TCs' approaches to responding to errors, as demonstrated through the dialogue, and TCs' goals for responding to errors, as demonstrated through the rationale.

Findings

We report findings for each research question around the three analytic themes: (1) attending to the error, (2) attending to the mathematics of the task, and (3) attending to the students.

Theme: Attending to the Error

RQ1: Features of TCs' dialogues. One striking feature of the dialogues was the way in which the error was resolved or corrected in a few lines. This occurred in nine of 25 dialogues. In five cases, the original student corrected their own error, as in the example below:

> Teacher: It does look like a square but what is different about Shape J and Shape Q?
> Student: There is a line from one corner of the square to the center.
> T: Correct, what do you think we can conclude by noticing the line from the corner?
> S: We can conclude that this is not a polygon, because they are not all connected.

In the remaining four cases, another student corrects or resolves the error. Six additional dialogues included a last turn of talk in which the teacher posed a question that was potentially leading toward resolving or correcting the error through the next student response. These 15 cases illustrate the common approach of resolving errors quickly, and often in a one-on-one exchange between teacher and student, or as the result of a pointed question from the teacher.

In contrast, some TCs wrote dialogues in which the error was not resolved. One TC used a "tabling" move—explicitly pausing the conversation without resolving the error. Five dialogues ended with the teacher asking for a student to contribute a new card to the discussion, which signaled the discussion moving on without immediate resolution of the original error. One TC did not explicitly address Jessie's contribution and centered the dialogue around a third shape:

> T: Does anyone have another example of a polygon?
> S: Shape M [a parallelogram].
> T: Why do you think shape M is a polygon?
> S: All the sides are straight.
> T: Before I get another example, I want us to take a look at what we have up here. How are these shapes up here different from one another? Do we all believe they are all polygons?

This set of cases suggest a recognition of approaches, such as eliciting new ideas, that can continue a discussion while a student error is not necessarily resolved in the moment.

RQ2: Features of TCs' rationales. An explicit part of a majority of the rationales (15 of 25) was an acknowledgement that Jessie has made an error (or, in some TCs' word, a "mistake"). This indicates that: (1) TCs recognized the mathematical issue at hand, and (2) TCs' dialogues were constructed in a way where responding to an error was a key factor. Five of these 15 rationales mentioned wanting Jessie to correct her own error, and six of the 15 mentioned wanting other students to recognize Jessie's error. The different ways in which TCs characterize Jessie's contribution ("misconception", "holes in [her] logic", a key consideration is missing) could impact the way in which TCs conceive of how to respond and, furthermore, what might be convincing to Jessie. These differences could result in different approaches to the dialogue.

RQ3: Relationships between TCs' dialogues and rationales. Of the 15 TCs who mentioned the error in their rationales, six resolved the error in their dialogues. An additional four dialogues approached resolution in their final talk turn. The remainder responded to the error in other ways, such as moving on to discuss a new card. This indicates that for the TCs who focused on the error in their rationales, the typical response was to move toward resolving it, with a smaller number of TCs addressing the error through continued discussion.

It is also interesting to note what is not included in the rationales, given the features of the dialogues. For example, no TCs mentioned that Jessie's error needed to be or could be resolved quickly. While many of the TCs who acknowledged that Jessie's contribution was an error also suggested it needed to be resolved at some point, no one suggested that this would necessarily happen in a few turns of talk. Yet, in many dialogues, the error was resolved or was soon-to-be resolved. We see this incongruity as potentially highlighting an implicit expectation that some conclusion be reached in the assigned five to eight lines of dialogue. It may also be that TCs do not wish to correct an error with their immediate response, but feel that what one TC referred to as, "a few well-placed questions," could be a sufficient way to respond to and resolve an error.

Another example is that no TCs mentioned any external factors, such as time, curriculum, or testing, as influencing their dialogues. While, as an approximation of practice these performance tasks are intended to reduce complexity, it is interesting to see how TCs' dialogues could still have less-than-productive features that are commonly explained by appealing to these constraints. This represents another incongruity, in which TCs do not explicitly appeal to common classroom constraints, yet still put forward a dialogue in which student reasoning and mathematical meaning are not kept central. Overall, these tensions between TCs' approaches and goals highlight the support TCs need around productively responding to errors.

Theme: Attending to the Mathematics of the Task

RQ1: Features of TCs' dialogues. TCs attended to mathematics in their dialogues in several different ways. Almost all dialogues (23 of 25) included a teacher move drawing attention to the difference between Shapes J and Q. Of these 23, 17 dialogues had this as the first teacher move. This approach to the mathematics is necessarily dependent on the particular task being discussed, and some dialogues contained approaches that might be used in multiple contexts. For example, six dialogues used the word "definition". This shows attending to the specific features of a polygon under consideration. Seven dialogues introduced an additional shape to the discussion (as shown in the dialogue above), with the shapes chosen reflecting attention to the features of the definition requiring exploration. The approaches of focusing on the definition and introducing a new example are teaching moves that can be used across contexts. The two TCs who did not explicitly address the differences between shapes J and Q introduced new shapes to the discussion, indicating that all 25 dialogues attended to key mathematical ideas.

RQ2: Features of TCs' rationales. Half (13 of 25) the TCs mentioned moving toward the mathematical goal—establishing and clarifying the definition of a polygon—in their rationales. In some cases, this was fairly general, with references to the "definition" or "objective." In other cases, TCs highlighted specific mathematical features of the goal. For example, one TC explained that their dialogue was constructed in order to support Jessie to, "realize the other defining feature of a polygon, that sides can only meet one other side at the endpoints."

In addition to the 13 TCs who mentioned the goal, eight TCs wrote about a desire to have students consider the differences between Shapes J and Q, though they did not explicitly connect that idea to pursuing the goal of the activity. Five of those eight further specified the key difference that Shape J has an "extra segment" that intersects with two other segments at one endpoint and does not intersect with another segment at the other endpoint. Finally, one other TC generally mentioned students needing to consider "one more property of a polygon."

Altogether, 22 of 25 TCs referenced something related to the mathematics of the activity. There was a strong emphasis on an aspect of the provided definition of polygons that does not come out in either student's contribution—that each side is a line segment that intersects exactly one other side at each endpoint. We use these analyses as evidence that TCs attended to the mathematics of the scenario, which has implications for the dialogue they constructed.

RQ3: Relationships between TCs' dialogues and rationales. The prevalence of attention in TCs' rationales to the mathematics of the scenario is closely connected to our finding that almost all the TCs focused the discussion in some way on the differences between Shapes J and Q. It is possible that focusing students' attention on the differences between the shapes (with a pointed prompt, for example) is seen as the way to make progress toward the goal of the activity.

We identified two interesting exceptions to this trend. The two TCs who did not draw any attention to the differences between Shapes J and Q in their dialogue either elicited a new card from students or offered a new shape that was not represented on another card. These cases illustrate how a discussion, in response to this scenario, can focus on key mathematical ideas without an explicit prompt about the differences. Given the role that introducing a new shape played in dialogues that did not result in a quick resolution of the error, we consider this approach productive, assuming the decisions made about the new shapes are purposeful.

Another exception is represented in the second dialogue above, where the teacher starts by asking for a new card and ends with a question about the differences among the three cards. This TCs' rationale was one of the few that made no explicit reference to the mathematics of the task. Instead, the rationale emphasized a goal that students maintain "skepticism" and be able to "speak up when something does not seem right to them." This provides an interesting case of how a focus on providing space for students to reason, even without mentioning particular mathematical ideas, may give rise to productive approaches to responding to errors.

Theme: Attending to the Students

RQ1: Features of TCs' dialogues. TCs brought students into their dialogues in multiple ways. Nine introduced a new student, sometimes using a new name. Three others reintroduced Rosalia. Some TCs used moves such as orienting students to one another (5), re-voicing (9), recording student thinking (2), and using positive language to describe student thinking (8). "Telling" in the discussion was also sometimes used as a way in which TCs attended to students, and nine dialogues included some form of telling. Many dialogues attended to students using more than one of these approaches, and 21 of 25 dialogues attended to students in at least one of these ways. Of the four remaining dialogues, all included some form of leading or funneling questions, and several were not written as a true dialogue (i.e., not in transcript form), which showed little explicit attention to including student voice. Among the 21 dialogues attending to students, all had a teacher talk turn between every student talk turn. These results highlight the variety of approaches TCs use in imagining responses in these hypothetical scenarios.

RQ2: Features of TCs' rationales. A key feature of the rationales was a desire to respond in a way that valued students' opportunities to reason mathematically. Nine TCs made specific mention of approaches and goals such as: "exploring", "discovery", "comparing", "critical thinking", or "connecting" in reference to student thinking. Four other TCs discussed wanting to value or draw on Jessie's or Rosalia's reasoning. This signals a set of principles in which student reasoning should be supported and made central.

Other TCs' rationales focused on the specific moves they used in their dialogue. Two TCs highlighted "orienting" moves in their dialogue (e.g., having a student re-voice or reason about a peer's contribution). In four other cases, TCs mentioned what we called "telling" moves (e.g., restating and summarizing ideas, giving new examples), and recording ideas. While moves such as these do not necessarily explicitly draw upon student reasoning in the way that orienting moves do, they are powerful moves that support students' sense making and reasoning. Other mention of moves in rationales raised questions for us. For example, some TCs mentioned the use of "guiding questions," positioning this approach as productive. However, as we interpreted these cases, we inferred "guiding" to mean "leading" or "funneling" and not as something that was necessarily responsive to or supportive of student thinking. In total, 18 TCs mentioned valuing student reasoning or using specific teaching moves related to student thinking.

RQ3: Relationships between TCs' dialogues and rationales. We focus here on the four dialogues that were not coded as attending to students. What is striking in these cases is that all four of the related rationales mentioned attending to or valuing students in different ways. For example, one TC wrote that they wanted to "use Jessie's mistake" to "help the class understand." However, this TC did not produce a true dialogue. Rather, they listed questions and answers without any evidence of student voice. Another TC mentioned wanting students to do the mathematical work, saying "The kids need to come up with the other rules and they will end up sorting that out." The associated dialogue in this case has six talk turns focused on the differences between Shapes J and Q, and then a note from the TC that they would "move on from here and see how everything else was sorted to see if we can add to our rule." These findings point to a need to be cautious what taking what a TC says about their approaches and goals for teaching, even in the context of a specific scenario, and making claims about skilled teaching practice.

Discussion and Conclusion

Our use of a written performance task provided a lens for investigating TCs' developing practice. We gained insight into their coordination of approaches and goals related to responding to student errors in whole-class discussion. In both the approaches (represented by the dialogues) and the goals (represented by the rationales), three themes emerged that characterized TCs' perspectives on responding to errors: attending to the error, attending to the mathematics, and attending to students. At times, dialogues and rationales presented approaches and goals in alignment with one another. However, in several notable cases, we observed incongruities between TCs approaches and goals around responding to errors.

Our results suggest several potential areas for growth around teacher learning. In cases where approaches and goals are mostly congruent, we observed TCs wrestling with the dilemmas highlighted in earlier literature around responding to student errors. In the cases where approaches and goals were not congruent, we saw misalignment in two directions. Some TCs wrote about productive goals for responding to errors in their rationales but their dialogues did not have features associated with productive responses. Other TCs wrote dialogues that included productive responses to student errors, but rationales that did not clearly articulate productive goals for their responses. This suggests that part of the work of teacher education is to provide TCs with productive approaches to responding to errors in concert with developing their understanding about why those approaches are productive. Approximations of practice support this work, providing opportunities for TCs to demonstrate and further develop adaptive skill in interactive and contingent moments of teaching.

Next steps for this work include examining the ways in which TCs' responses to the performance tasks change over the course of their participation in the methods course. This will allow us to explore how coordinating approaches and goals changes over time. Additionally, we plan to explore how the responses to the written performance task relate to features of TCs' rehearsals and enactments focused on leading whole-class mathematics discussions.

References

Baldinger, E. E., Selling, S. K., & Virmani, R. (2016). Supporting novice teachers to lead discussions that reach a mathematical point: Defining and clarifying mathematical ideas. *Mathematics Teacher Educator*, *5*(1), 8–28. http://doi.org/10.5951/mathteaceduc.5.1.0008

Boerst, T. A., Sleep, L., Ball, D. L., & Bass, H. (2011). Preparing teachers to lead mathematics discussions. *Teachers College Record*, *113*(12), 2844–2877.

Bray, W. S. (2011). A collective case study of the influence of teachers' beliefs and knowledge on error-handling practices during class discussion of mathematics. *Journal for Research in Mathematics Education*, *42*(1), 2–38. http://doi.org/10.5951/jresematheduc.42.1.0002

Brodie, K. (2014). Learning about learner errors in professional learning communities. *Educational Studies in Mathematics*, *85*(2), 221–239. http://doi.org/10.1007/s10649-013-9507-1

Campbell, M. P., Baldinger, E. E., Selling, S. K., & Graif, F. (2017). Responding to students during whole-class discussions: Using written performance tasks to assess teacher candidate practice. In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 977–980). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Chapin, S. H., O'Connor, C., & Anderson, N. C. (2013). *Classroom discussions in math: A teacher's guide for using talk moves to support the Common Core and more, Grades K-6* (3rd ed.). Sausalito, CA: Math Solutions Publications.

Crespo, S., Oslund, J. A., & Parks, A. N. (2011). Imagining mathematics teaching practice: Prospective teachers generate representations of a class discussion. *ZDM Mathematics Education*, *43*(1), 119–131. http://doi.org/10.1007/s11858-010-0296-z

Ghousseini, H. N., Beasley, H., & Lord, S. (2015). Investigating the potential of guided practice with an enactment tool for supporting adaptive performance. *Journal of the Learning Sciences*, *24*(3), 461–497. http://doi.org/10.1080/10508406.2015.1057339

Grossman, P. L., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. W. (2009). Teaching practice: A cross-professional perspective. *Teachers College Record*, *111*(9), 2055–2100.

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169–202. http://doi.org/10.2307/20720130

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. *Adding it up*. Washington, DC: National Academy Press.

Lampert, M., Franke, M. L., Kazemi, E., Ghousseini, H. N., Turrou, A. C., Beasley, H., … Crowe, K. (2013). Keeping it complex: Using rehearsals to support novice teacher learning of ambitious teaching. *Journal of Teacher Education*, *64*(3), 226–243. http://doi.org/10.1177/0022487112473837

Leatham, K. R., Peterson, B. E., Stockero, S. L., & Van Zoest, L. R. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. *Journal for Research in Mathematics Education*, *46*(1), 88–124. http://doi.org/10.5951/jresematheduc.46.1.0088

Miles, M. B., Huberman, A. M., & Saldaña, J. (2013). *Qualitative data analysis: A methods sourcebook* (3rd ed.). Thousand Oaks, CA: Sage Publications.

Nesher, P. (1987). Towards an instructional theory: The role of student's misconceptions. *For the Learning of Mathematics*, *7*(3), 33–40.

Santagata, R. (2005). Practices and beliefs in mistake-handling activities: A video study of Italian and US mathematics lessons. *Teaching and Teacher Education*, *21*(5), 491–508. http://doi.org/10.1016/j.tate.2005.03.004

Tulis, M. (2013). Error management behavior in classrooms: Teachers' responses to student mistakes. *Teaching and Teacher Education*, *33*, 56–68. http://doi.org/10.1016/j.tate.2013.02.003

Van Zoest, L. R., Stockero, S. L., Leatham, K. R., Peterson, B. E., Atanga, N. A., & Ochieng, M. A. (2017). Attributes of instances of student mathematical thinking that are worth building on in whole-class discussion. *Mathematical Thinking and Learning*, *19*(1), 33–54. http://doi.org/10.1080/10986065.2017.1259786

Zazkis, R. (2017). Lesson Play tasks as a creative venture for teachers and teacher educators. *ZDM Mathematics Education*, *49*(1), 95–105. http://doi.org/10.1007/s11858-016-0808-6

PROFESSIONAL NOTICING IN COMPLEX MATHEMATICAL CONTEXTS: EXAMINING PRESERVICE TEACHERS' CHANGES IN PERFORMANCE

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This paper examines the implementation of an instructional module on Preservice Elementary Teachers' (PSETs) professional noticing of children's mathematical thinking as defined by Jacobs, Lamb, and Philipp (2010). The module focuses on professional noticing skills through the content focus of early algebraic reasoning and uses complex video vignettes from whole class instruction in authentic elementary mathematics classrooms. It was found that two of the three components of professional noticing (attending and interpreting) showed statistically significant increases in a treatment group that did not occur in a comparison group. The deciding component remains a challenge that warrants further research.

Keywords: Teacher Education-Preservice, Teacher Knowledge, Professional Noticing, Algebra and Algebraic Reasoning, Instructional Activities and Practices

Introduction

Developing responsive mathematics teaching practices is a persistent challenge for teacher educators. Over the past decade, teacher noticing and variants of such noticing (i.e., professional noticing of children's mathematical thinking, etc.) have risen in prominence as a construct of great interest among mathematics education researchers (Schack, Fisher, & Wilhelm, 2017; Sherin, Jacobs, & Philipp, 2011). Much has been learned about teachers' development and performance of noticing practices. However, vexing questions remain regarding the fundamental nature of such noticing and the extent to which the practice should be considered a net (capturing as much of the activity as possible) or a filter (distilling the activity to key moments to act upon) (Thomas, 2017). These questions are exacerbated in complex environments featuring multiple actors, converging mathematical topics, and varied avenues for sound instructional decision-making. While research suggests that productive experiences help teachers engage in more sophisticated forms of noticing, in more complex contexts, the impact of such experiences (and even the nature of noticing itself) becomes less clear (Castro-Superfine, Fisher, Bragelman, & Amador, 2017). For this study, we focus on the impact of a professional learning experience conducted with preservice elementary teachers (PSETs) aimed at developing their noticing capacities in a complex mathematical environment. Given our prior focus on noticing performance with respect to individual students' counting/arithmetic strategies (Schack, Fisher, Thomas, Eisenhardt, Yoder, 2013), we elected to contextualize noticing within mathematical activities of multiple children and their early-algebraic reasoning. The research question guiding this inquiry was*: To what extent can teacher educators facilitate the development of PSETs' professional noticing of children's mathematical thinking in the context of algebraic thinking in whole class settings?*

Theoretical Framework

Teacher Noticing and Professional Noticing

Variants of teacher noticing have been the focus of research for some time (Mason, 2002; Sherin et al., 2011; Schack et al., 2017). While teacher noticing has been organized around two related components, “attending to particular events in an instructional setting” and “making sense of events in an instructional setting” (Sherin et al, p.5), Jacobs et al. (2010), posit a third related component, *deciding*, which refers to teachers’ responses ostensibly built upon interpretations of children’s activities. These interpretations are, themselves, “derived from events and behaviors to which teachers had attended” (Thomas, 2017, p. 508). The assemblage of attending, interpreting, and deciding as interrelated component skills has been referred to as professional noticing of children’s mathematical thinking, hereinafter as simply professional noticing (Jacobs et al., 2010). Further, professional noticing is typically considered a complex and challenging practice to develop (van Es, 2011), and, as with most complex practices, there are varying perspectives on how such practices may be implemented. In some instances, professional noticing may “be used as a filter to identify only the most impactful moments” while from other perspectives, noticing may be “focused on capturing and interpreting as much of the instructional landscape as possible” (Thomas, 2017, p. 508). While the former perspective is represented by the Mathematically Significant Pedagogical Opportunities to Build on Student Thinking (MOST) analytic framework (Leatham, Peterson, Stockero, & Van Zoest, 2015), the latter is typified by the research of Wells (2017) and Schack et al. (2013). It is this latter perspective of more inclusive professional noticing that we use for this study.

Returning to the notion of professional noticing as a complex practice, studies have demonstrated that preservice teachers at different levels of practice (i.e., elementary, middle level, secondary) can engage in productive professional noticing (Floro & Bostic, 2017; Krupa, Huey, Leisseg, Casey, & Monson, 2017). However, there is some emerging conjecture that professional noticing considerations and development may vary somewhat depending upon grade level (Krupa et al., 2017). Germane to this study, though, are inquiries of noticing performance within elementary grades. In a previous study, we found evidence that PSETs may advance their practice of professional noticing as measured via growth in the component skills of attending, interpreting, and deciding (Schack et al., 2013). Using a video-based instrument, we found statistically significant performance gains in each of the component skills (attending, interpreting, and deciding) with the largest gains occurring in the deciding component. Note, though, that this study was conducted in the context of a single child’s mathematical activity along a mathematical progression of counting and arithmetic reasoning (Steffe, von Glaserfeld, Richards, & Cobb, 1983; Thomas and Tabor, 2012). There has been some study of preservice teachers’ professional noticing performance that have grounded such noticing in complex pedagogical domains such as the enactment of specific mathematical practices (Floro & Bostic, 2017). However, this study is unique in that we have adopted an inclusive professional noticing perspective to examine PSET performance within a complex mathematical domain – namely early algebraic reasoning.

Early Algebraic Reasoning as a Complex Domain

Mentioned earlier, our previous research focused on the professional noticing of an individual child’s mathematical thinking in the area of counting and arithmetic strategies. We relied upon a highly descriptive framework detailing children’s early arithmetic strategies and changing conceptions of unit (Steffe et al., 1983; Wright, Martland, & Stafford, 2006). While this context proved fruitful for the development of foundational noticing capacities (Schack et al., 2013), we concluded that such contexts did not accurately represent the "blooming, buzzing confusion of sensory data" that occurs when multiple children are acting and interacting upon a convergence of mathematical topics (Sherin & Star, 2011, p. 69).

Representative of such mathematical convergence, early-algebraic reasoning comprises and intertwines several mathematical domains including properties of operations, equality, patterning, symbolic representation, and functions (Kaput, Carraher, & Blanton, 2007; Russell, Schifter, & Bastable, 2011; Warren, 2005). As with other domains such as counting and arithmetic reasoning (Steffe, 1992; Thomas and Tabor, 2012), each of the converging domains of early-algebraic reasoning may be thought to have some manner of developmental progression. Further, these progressions have been defined via empirical study for some of these domains (Clements & Sarama, 2009). However, the convergence of these individual progressions and resultant *conglomerate trajectory* may be rightly considered a significantly complex area of mathematical content rife with possibilities and pitfalls regarding the myriad aspects to which one might attend, interpret, and respond instructionally. While there are many other domain convergences and conglomerate trajectories within the mathematics education landscape, early-algebraic reasoning should be one of the first encountered by children and enacted by their teachers. As such, this represents a rich mathematical context for the study of professional noticing.

Methodology

Participants

Participants in the study included 296 preservice elementary teachers (PSETs) enrolled in an elementary mathematics methods course at one of five universities in the south central United States. Among the total participants, 171 completed the measures as part of an intervention group and the remaining 125 were in the comparison group. Participants with any missing scores on the professional noticing measure were removed, thus resulting in an intervention sample size of 147 and comparison sample size of 121.

Instructional Module

PSETs in the intervention group experienced an instructional module focusing on professional noticing and early algebraic thinking. The in-class module was taught by three professors at two of the institutions in the study. The three-session module gradually introduces each of the three components of professional noticing. The first session of the module, which focuses on just the attending and interpreting components, begins with a discussion of a prerequisite reading on the early understanding of equality and a review of the three components of professional noticing. Then, the PSETs analyze two videos with class discussion, match equality tasks to the Common Core State Standards for Mathematics, use their professional noticing skills to analyze children's written work, and complete a play-by-play storyboard from a video of whole class completing an algebraic task. The second session adds deciding and includes video analysis and discussions of authentic classroom experiences as well as analysis of children's written work. The final session includes all three components but focuses more deeply on productive decision making through the video analyses and discussion. All three sessions contain homework consisting of readings, mathematical tasks, or video analyses focused on algebraic thinking.

Measurement and Scoring

All PSETs in the comparison and intervention sites completed pre- and post-assessments to measure their professional noticing skills. All pre-assessments were administered near the beginning of the semester and the post-assessments were administered at the end of the instructional module for the intervention sites and near the end of the semester for the comparison group. The professional noticing measure consisted of a 74 second video involving a group of second grade students grappling with the task “10 + 10 = ___ + 5” where they were discussing the missing number in the mathematical sentence. In the video, four children provided different answers and explanations for the task. The explanations for each child are provided in Table 1.

Table 1. Transcripts of Individual Student Responses

Student	Answer	Transcript of Student Explanation
1	25	“I added 10 and….I added both 10’s and the 5. Ten plus 10 is 20 and add a 5 equals 25.”
2	20	*<No explanation - blurts answer out of turn>*
3	4	“Cause if you add a 5…and then you count up by 5’s up to 4, it’d be 20.”
4	15	“Because I counted 15, 16, 17, 18, 19, 20” (*using his right hand to track the count*). Teacher (in background) asks: “Why did you start with 15?” Student stated: “Because I thought to myself 5 plus 10 equals 15, but then I thought of adding 15 with the 5 and I counted up and it was 20.”

PSETs are asked to view the video and describe what they would do next if they were the teacher (deciding), what mathematical thinking and actions they observed (attending), and what the children understood about mathematics that influenced their answers (interpreting). Decision trees were used to aid in the scoring process. All responses were scored on a scale from 0 to 3 with a score of 3 representing the most advanced responses (See Table 2 for examples). All responses were scored using a double-blind process and any scores not in agreement between the two scorers were discussed and negotiated. In general, PSETs were assigned higher scores when they thoughtfully addressed the diversity in responses instead of focusing on one response or their own assumption of what the intended outcome should be in this situation.

Table 2. Sample PSET Responses

Component	Score	Sample Response
Attending	0	The children were explaining how they got the answer to the problem.
	2	Some of the kids added the first side of the problem, then added 5 and thought that was the answer. Other students were able to understand that the equal sign meant both sides had to be the same, and they were able to move numbers around to make them equal.
	3	The children were given the equation 10+10=___+5. The first child to respond was adding all the numbers together, he said the answer was 25. He was under the impression that the blank should be the sum of all the numbers. Another student said he thought the number should be 4 because it would take 4 5's to make 20 which is what 10 and 10 made. The last student came to the conclusion that it should be 15. He explained that 15 and 5 added together made 20 which is what 10 and 10 made.
Interpreting	0	I would make sure that the students understood what the equal sign meant. I would use manipulatives to show this.

	1	I was surprised to find such an interesting method for finding the answer. It was obvious that he understood the relation of counting by 10s and 5s.
	3	Some students were getting the correct answer (15) but he did not know where he got it from, or he could not explain it. Another student was adding all the numbers together to get the blank number so he needed to break down the number sentence into smaller parts to better understand it.
Deciding	0	I would have the children come up to the board and list a few different ways and ideas that work and don't work.
	1	Next I might show a true expression that looks different on both sides. I might ask, "Are both of these sides equal to each other?" Then I would use student responses to guide the discussion. Some sample questions I may ask along the way would include: How did you get that answer? Are both sides the same? Are both sides different? How are both sides the same? How are both sides different?
	2	I would continue to ask students for more strategies. I would write those strategies on the board. Then I would try some of the strategies on the board to show which ones were correct and which ones were incorrect. I would emphasize that there are multiple ways to solve this problem. I would explain that many of the students had the right answer, but got to it in different ways. I would encourage the students who solved it incorrectly towards the correct answer. It seems like they are thinking in the correct way, but that they had a few miscalculations. I would address those, so that the student still feels successful.
	3	I would have other students see if they can explain the last students thinking since he is on the right track. This would get all the students involved without saying that the other students who shared their opinions were wrong.

Results

When comparing the mean scores for the three professional noticing components (See Table 3), we found that the attending and interpreting scores increased from pre- to post-assessment in the intervention group, but slightly decreased, but not significantly, in the deciding component. The comparison participants decreased in all three components. Not all intervention participants improved, however, and Table 4 outlines the number of participants in each group that increased, stayed the same, and decreased from pre- to post-assessment.

Table 3: Mean Scores of Professional Noticing Components

	Treatment		Control	
	Pre	Post	Pre	Post
Attending	1.020	1.340	0.959	0.826
Interpreting	0.769	1.218	0.876	0.661
Deciding	1.361	1.340	1.554	1.446

Table 4: Noticing Score Changes

Component	Change	Treatment (%; N = 147)	Control (%; N = 121)
Attending	Increase	38	23
	Same	41	47
	Decrease	20	30
Interpreting	Increase	46	18
	Same	37	55
	Decrease	16	26
Deciding	Increase	29	28
	Same	39	32
	Decrease	31	40

In order to determine whether the increases from pre- to post-assessment in attending and interpreting and the decrease in deciding were statistically significant, Wilcoxon signed rank tests were conducted. The test was significant for attending ($Z = -3.219$, $p = .001$) and interpreting ($Z = -3.961$, $p < .001$), but not for deciding ($Z = -.384$, $p = .701$). Noting that an effective intervention should raise scores to above the level of the control group (which scored higher on interpreting and deciding at pre-test than the treatment group did), Mann-Whitney U tests were conducted to determine whether the treatment group's post-assessment scores were higher than the maximum (i.e., pre-) scores from the control group. The test was significant for attending ($U = 6913$, $p = .001$) and interpreting ($U = 7172$, $p = .004$). The test was also significant for deciding, but in the other direction; the control group received higher deciding scores than the treatment group ($U = 7654$, $p = .039$).

Discussion and Conclusion

Previous research by this team revealed that professional noticing was a teachable skill when taught in a one-on-one teacher to student setting (Schack et al., 2013). In a similar study using an intervention and control group, but within a one-to-one teacher to student setting, PSETs in the intervention group showed statistically significant growth in all three areas of professional noticing (Fisher, Thomas, Schack, Jong, & Tassell, 2017). However, in that same study, the comparison group did not show statistically significant increases in attending and interpreting, but they did show a statistically significant increase in the deciding component. This opens the discussion to the pedagogy taught in the mathematics methods course and how it could potentially impact the deciding component scores.

To further advance the research, this video vignette of a whole-class discussion focused on an algebraic concept was used to determine if the professional noticing skills were still teachable when expanded to a more complex setting. It is interesting to note that attending and interpreting skills both still revealed a statistically significant increase in the intervention group, but both components decreased in the comparison group. These results indicate that attending and interpreting are both responsive to intervention in the varying settings (one-on-one vs. whole class) but deciding needs further research and discussion.

Many questions still remain within the deciding aspect of professional noticing; However, we believe there is still much to learn from the results. Perhaps more research should be conducted on what constitutes effective deciding. While defined mathematical practices provide some organization for considering such decisions, there are myriad decisions a teacher might make in each moment and the productivity of such decisions likely varies greatly according to many different factors (e.g., instructional goals, students' mathematical needs, availability of resources, etc.). This variability is further compounded when considering potential decisions within complex mathematical domains such as the early-algebraic convergence.

While we recognize that many limitations exist with this study, such as the short timeline for learning and assessing these skills (4-6 weeks for implementation sites) or the less-than-diverse sampling that can be very difficult to overcome in a study of elementary education majors, perhaps the most troublesome limitation is the quantification of qualitative responses. For example, scorers credited more highly PSET responses that formally addressed each of the student's thinking in the video vignette, while realizing the difficulty of doing so in an authentic setting.

However, many times, responding to each child may have diminished the robustness of the decision. There remains a delicate balance in how to best quantify two different, yet both important, aspects in this situation: teaching to all and productive decision-making. We believe there should be a place for both, but these results suggest we may not have found the best way to measure their intersection as productive decision-making can be quite circumstantial and adequate measurements for those circumstances do not exist.

In summary, we remain optimistic that teacher educators can facilitate the development of PSETs' professional noticing skills within the context of algebraic thinking in whole class settings. We are most hopeful that this facilitation exists at least within the attending and interpreting components. More difficult to demonstrate, however, is the deciding component, thus we encourage the development of measurement tools that consider the complexities of classroom contexts when measuring productive decision-making. Tools that could potentially be used for both research purposes and teaching more productive decision-making for preservice and inservice teachers would be most desirable.

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References

Castro Superfine, A., Fisher, A., Bragelman, J., & Amador, J. (2017). Shifting perspectives on preservice teachers' noticing of children's mathematical thinking. In Schack, E.O, Fisher, M.H., Wilhelm, J (Eds.) (2017). Teacher Noticing – Bridging and Broadening Perspectives, Contexts, and Frameworks. In, *Research in Mathematics Education* series. Springer. New York: NY.

Clements, D. H., & Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach*. New York: Routledge.

Fisher, M. H., Thomas, J., Schack, E. O., Jong, C., Tassell, J. (2017*, online first*). Noticing Numeracy Now! Examining Changes in Preservice Teachers' Noticing, Knowledge, and Attitudes. *Mathematics Education Research Journal.*

Floro, B. & Bostic, J.D. (2017). A case study of middle school teachers' noticing during modeling with mathematics tasks. In Schack, E.O, Fisher, M.H., Wilhelm, J (Eds.) (2017). Teacher Noticing – Bridging and Broadening Perspectives, Contexts, and Frameworks. In, *Research in Mathematics Education* series. Springer. New York: NY.

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169–202.

Kaput, J., Carraher, D. W., & Blanton, M. (Eds.). (2007). *Algebra in the early grades.* Hillsdale/Reston: Erlbaum/The National Council of Teachers of Mathematics.

Krupa, E.E., Huey, M., Leisseg, K., Casey, S. & Monson, D. (2017). Investigating secondary preservice teacher noticing of students' mathematical thinking. In Schack, E.O, Fisher, M.H., Wilhelm, J (Eds.) (2017). Teacher Noticing – Bridging and Broadening Perspectives, Contexts, and Frameworks. In, *Research in Mathematics Education* series. Springer. New York: NY.

Leatham, K.R., Peterson, B.E., Stockero, S.L., & Van Zoest, L.R. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. *Journal for Research in Mathematics Education, 46*(1), 88 – 124.

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. New York: Routledge.

Russell, S.J., Schifter, D., & Bastable, V. (2011). Developing Algebraic Thinking in the Context of Arithmetic. In J. Cai & E. Knuth (Eds.). *Early Algebraization: Advances in Mathematics Education* (pp. 43-69). New York: Springer.

Schack, E. O, Fisher, M. H., Thomas, J., Eisenhardt, S., Tassell, J., Yoder, M. (2013). Prospective Elementary School Teachers' Professional Noticing of Children's Early Numeracy. *Journal of Mathematics Teacher Education (16),* pp. 379-397.

Schack, E.O, Fisher, M.H., Wilhelm, J (Eds.) (2017). Teacher Noticing – Bridging and Broadening Perspectives, Contexts, and Frameworks. In, *Research in Mathematics Education* series. Springer. New York: NY.

Sherin, M. G., Jacobs, V. R., & Philipp, R. A. (Eds.). (2011). *Mathematics teacher noticing: Seeing through teachers' eyes*. New York: Routledge.

Sherin B.L. & Star, J. (2011). Reflections on the study of teacher noticing: Seeing through teachers' eyes. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.). *Mathematics teacher noticing: Seeing through teachers' eyes*. (pp. 66-78). New York: Routledge.

Steffe, L. P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application*. New York: Praeger Scientific.

Steffe, L. P. (1992). Learning stages in the construction of the number sequence. In J. Bideaud, C. Meljac, & J. Fischer (Eds.). *Pathways to number: Children's developing numerical abilities* (pp. 83–88). Hillsdale, NJ: Lawrence Erlbaum.

Thomas, J. (2017). The Ascendance of Noticing: Connections, Challenges, and Questions. In Schack, E.O, Fisher, M.H., Wilhelm, J (Eds.) (2017). Teacher Noticing – Bridging and Broadening Perspectives, Contexts, and Frameworks. In, *Research in Mathematics Education* series. Springer. New York: NY.

Thomas, J., & Tabor, P. D. (2012). Differentiating instruction to facilitate quantitative mental imagery. *Teaching Children Mathematics, 19*, pp. 174-183.

van Es, E. A., (2011). A framework for learning to notice student thinking. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.). *Mathematics teacher noticing: Seeing through teachers' eyes*. (pp. 117-133). New York: Routledge.

Warren, E. (2005). Patterns supporting the development of early algebraic thinking. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, Melbourne. Building connections: Research, theory and practice* (Vol. 1, pp. 759–766). Sydney, Australia: MERGA.

Wells, K. J. (2017). Noticing students' conversations and gestures during group problem-solving in mathematics. In Schack, E.O, Fisher, M.H., Wilhelm, J (Eds.) (2017). Teacher Noticing – Bridging and Broadening Perspectives, Contexts, and Frameworks. In, *Research in Mathematics Education* series. Springer. New York: NY.

Wright, R. J., Martland, J., & Stafford, A. (2006). Early numeracy: Assessment for teaching and intervention (2nd ed.). London, UK: Paul Chapman Publications/Sage.

TEACHERS' INCOHERENT CALCULATIONS IN PROPORTIONAL TASKS

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We investigated how pre-service teachers (PSTs) interpret their calculations in proportional tasks. A written questionnaire was administrated to 199 PSTs and an inductive content analysis approach used for data analysis. We found that one item that asked PSTs to interpret the meaning of their results had unusually low success; open coding on the responses revealed several common themes. We argue these common errors cannot be dismissed as simple unit or rounding mistakes; they are a reflection on how respondents think about quantities, story problems, and the nature of mathematics itself. We end with suggestions on how to address this problem.

Keywords: Preservice Teacher Education, Teacher Knowledge, Proportional Reasoning, Problem Solving

Introduction

Proportional reasoning has been widely investigated as a key type of reasoning that both students and teachers in K-12 mathematics struggle with (Beckmann, 2015; Byerley, 2017; Langrall, 2000; Son, 2013; Tourniaire, 1985) and is also cited in the Common Core as one of the eleven mathematical domains that span mathematics from elementary to high school (National Governors Association Center for Best Practices, 2010). We surveyed 199 preservice elementary school teachers on 10 tasks that involve proportional reasoning, and the responses revealed that the PSTs struggled to interpret the meanings of their own calculations. This issue was most clearly illustrated in the task that required students to interpret their own calculations in order to give an answer. In this report we will share the major themes we found amongst PSTs that had difficulty answering this task. Our research questions include: (1) How do PSTs solve a proportional reasoning problem including a unit rate? (2) What kind of challenges in reasoning could cause a PST to struggle with answering a proportional reasoning question?

Theoretical Perspective

We approach our research questions and data analysis from a theoretical framework of quantitative reasoning (Smith & Thompson, 2007; P. W. Thompson, 1993, 2011). A quantity is a measurable attribute (such as length, elapsed time, volume, etc.) of an object (such as a car, a person, the Earth, etc.), and a "person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the possibility of measuring it" (P. W. Thompson, 1993). This way of thinking in word or story problems stands in marked contrast to more procedural ways of reasoning such as key-word approaches (replacing "and" with "+" and "less than" with "-", etc.) Quantities as defined by Thompson occur only in the mind of a thinker, who conceptualizes them by making sense of a quantitative situation. Quantitative reasoning, then, is "the analysis of a situation into a quantitative structure--a network of quantities and quantitative relationships" (P. W. Thompson, 1993). A person is reasoning quantitatively when he or she is reasoning about quantities, instead of numbers, undefined variables, or memorized procedures. In the task and responses we present below, the prompt deliberately did not ask for a numerical answer; instead, it asked a yes/no question that required respondents to engage in three steps: a) decide what calculations would be relevant and useful to answering the question, b) carry out those calculations accurately, and c) interpret the meaning of their calculated results to answer the prompt. Our analysis focuses mainly, though not exclusively, on the written work that illustrates the third part of this process in our PST population.

Methodology

The proportional reasoning questionnaire was given to 199 elementary preservice teachers over 3 semesters at a large Southwestern university in the U.S. All participants were juniors who enrolled in an elementary mathematics content course covering patterns, functions, and modeling. They were on average 1.5 years away from being becoming full-time elementary school teachers including grades K through 8.

The participants completed a written questionnaire including 10 problems about proportional reasoning and we focused one of the problems (see Figure 1). In the questionnaire, PSTs were explicitly asked to solve all problems using quantitative approach (e.g., using pictures) first rather than using a standard algorithm and to provide their work along with clear justification. The written questionnaire including the problem in Figure 1 was administrated to all PSTs in six sections of a mathematics concept course in the middle of spring 2015, fall 2015, and spring 2016 and it was assigned as take-home homework right after teaching proportional reasoning.

One author came up with a first draft coding strategy for the problems based on the format of the questions. The other author then coded over half the data and looked at how well the coding strategy worked, suggested modifications, and the two authors together agreed on the next draft to the coding strategy. All the data was then coded but several smaller coding strategy changes were also implemented and those sections were recoded. The other author then coded a random subset of responses to verify that the strategy was being implemented uniformly. There was 100% agreement on the coding of 95% of the examples.

Once the initial coding was done we decided that we would like to focus on a specific phenomenon: the cases where preservice teachers either incorrectly interpreted the results of their own calculations, or did not interpret their results at all. We therefore chose to focus on our data on the "Dr. Lee's Car" item out of the ten problems.

Dr. Lee drove 156 miles and used 6 gallons of gasoline. At this rate, can he drive 561 miles on a full tank of 21 gallons of gasoline? Solve this problem and justify your reasoning.

Figure 1. "Dr. Lee's Car" Item

Results & Discussion for "Dr. Lee's Car" Item

We decided that since the prompt asked for a decision of Dr. Lee's driving ability, and not for a numerical answer, that a correct answer would say that Dr. Lee could not drive 561 miles on 21 gallons of gasoline. 152 preservice teachers made correct final statements such as "No", "He cannot go 561 miles", "He can go only 546 miles", "He has miles left over", or "He would need more than 21 gallons"; 9 of these teachers still had problems interpreting their own results despite their correct final answer. 36 preservice teachers answered "Yes" or "He can go 561 miles"; only 4 were due solely to arithmetic errors and the other 32 teachers had problems interpreting their own results. 11 preservice teachers gave no final answer, but 5 of them completed all of the necessary calculations yet did not interpret them to answer the question. Overall, 46 out of 199 teachers, or 23.11% of the teachers, demonstrated some problem with interpreting the results of their own calculations.

Table 1: Summary of all responses

	Correct answer: "He can't make it"	Incorrect answer: "He can make it"	No final answer	TOTAL
Total responses:	152	36	11	199
Responses with interpretation errors:	9	32	5	46

Amongst the 143 teachers that made relevant and accurate computations *and* correctly interpreted those results to answer the prompt, there were three solution techniques: to compute how many miles the car could go with 21 gallons, to compute the gas efficiency needed for both trips, or to compute the gallons needed to complete the second trip. Examples of each strategy are displayed below.

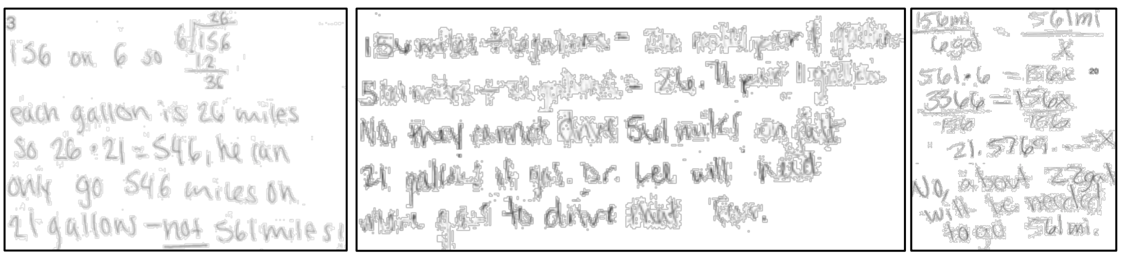

Figure 2. 3 Correct Solution Methods

Among the 46 teachers who had problems interpreting their own calculations, we found several themes.

Whole Number Bias (8.5% of respondents)

By far the most common problematic interpretations revolved around a strong bias towards either only calculating whole numbers, or rounding all values to whole numbers. 17 of the 45 preservice teachers who struggled to interpret their own work did so at least in part because they chose not to reason with decimal numbers. We hypothesize that these teachers looked for similarities in the results of their calculations without attending to the measures that they represented.

5 of the 199 teachers rounded both gas efficiency rates to 26 miles per gallon and concluded that Dr. Lee could make his trip. If a preservice teacher wanted to compare miles per gallon rates to evaluate whether Dr. Lee can make the trip or not, it is clearly relevant that 26.7 mpg is not the same as 26 mpg. Therefore, a teacher that rounded to 26 mph made a problematic interpretive decision when transferring work from calculator to paper, or when deciding to end their long division after finding the whole number part of their response.

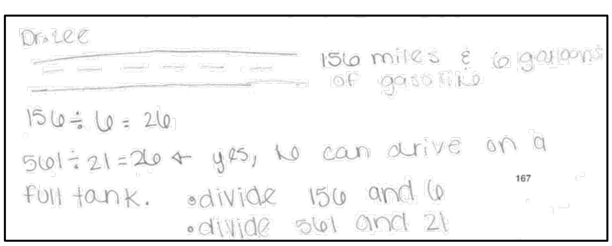

Figure 3. All calculations end in whole numbers

8 out of the 199 teachers found that the gas efficiency rates of both trips was approximately the same, and concluded that Dr. Lee could complete his trip; 4 teachers qualified their answers by saying that Dr. Lee could just barely make it. In doing so they neglected to keep track of the meaning and significance of their own calculations. The equality of both efficiency rates is irrelevant; rather, it matters whether the needed gas efficiency of the hypothetical trip is *less than or equal to* the known efficiency of the car.

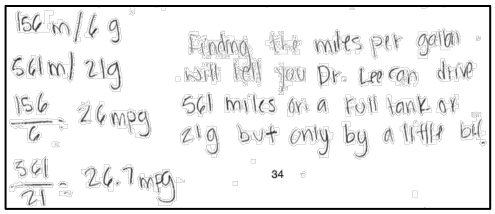

Figure 4. Yes, but only by a little bit [of what?]

5 of the 199 teachers found that it would require approximately 21.5 gallons for Dr. Lee to complete his trip, and concluded that his trip was possible. What matters is whether the gas the trip requires is *less than or equal to* the gas that Dr. Lee has available to him in a full tank, not whether they are approximately equal.

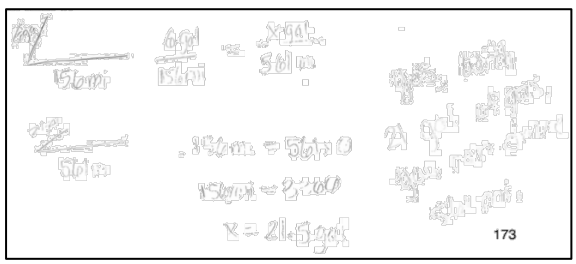

Figure 5. Yes, but only by a little bit [of what?]

Mixed Up Quantities (7.5% of respondents)

The next most common mistake is that preservice teachers did not keep track of the quantitative meaning of their results. These mistakes cannot simply be dismissed as writing down the wrong unit if we start with a presupposition that every calculation should have meaning to the person doing the calculating. We hypothesize that 16 out of the 45 preservice teachers who struggled to interpret their own work did so at least in part because they carried out operations on numbers without following them with operations on quantities.

9 out of 199 teachers calculated values and then ascribed the wrong quantitative meaning to them. For example, they calculated the gallons needed to complete a trip of 561 miles (21.5 gallons) but then wrote down 21.5 miles, calculated the gas efficiency of the shorter trip (26 miles per gallon) but then wrote down 26 gallons, or calculated the relative size of the trips in both gallons and miles (561 miles is 3.5 times as large as 156 miles, and 21 gallons is 3.3 times as large as 6 gallons) and then concluded that the difference meant that "Dr. Lee will be pushing [the car] .2 miles!".

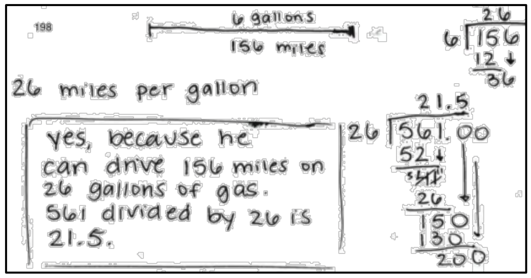

Figure 6. Mixed Up Quantities

6 out of the 199 teachers found values that would enable them to answer the question and interpreted the quantitative meaning of that value correctly, but then did not keep track of the meaning of a difference. 4 teachers correctly set up a proportion to find a value of 546, but then said that this means Dr. Lee *can* make the longer trip; 2 of those teachers interpreted the result of the calculation 561-546=15 to mean that Dr. Lee could drive an extra 15 miles. They did not keep track of the meaning of the difference as the miles that Dr. Lee *could not* drive on a full tank. 2 teachers calculated other differences (in the gas efficiencies of each trip, and gallons of gas used in each trip) and also concluded that Dr. Lee could make the trip.

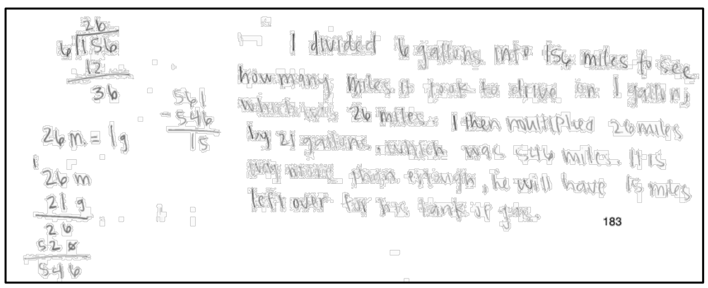

Figure 7. Directionless difference

Values Are Answers, No Meaning (3.0% of respondents)

5 of the 199 teachers reached the values they needed, of either the number of miles Dr. Lee could travel on 21 gallons, or the gas efficiency of both trips, but simply did not answer the question. While this may be an oversight, it may also illustrate a belief that mathematics is about finding numerical answers.

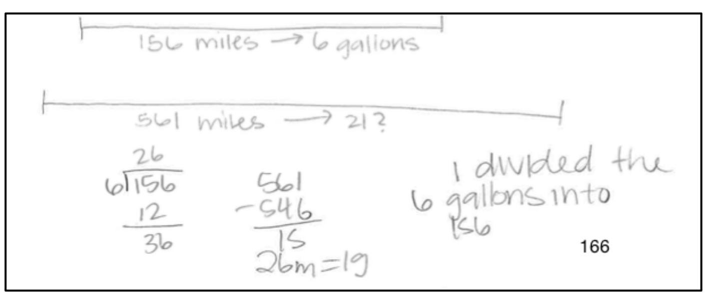

Figure 8. No answer

Chunky Gas (2.0% of respondents)

4 out of 199 teachers reasoned only in chunks of 6 gallons. 3 of the teachers concluded that Dr. Lee could not make the trip, and 1 concluded that he could, but all only calculated the miles

that 6, 12, 18, and 24 gallons would enable Dr. Lee to drive. We commonly see such chunky thinking in more abstract contexts like linear equations, slope, and accumulation in calculus (Castillo-Garsow, 2013; P. W. Thompson, & Carlson, M. P., 2017), but we were surprised to find it in such a concrete example.

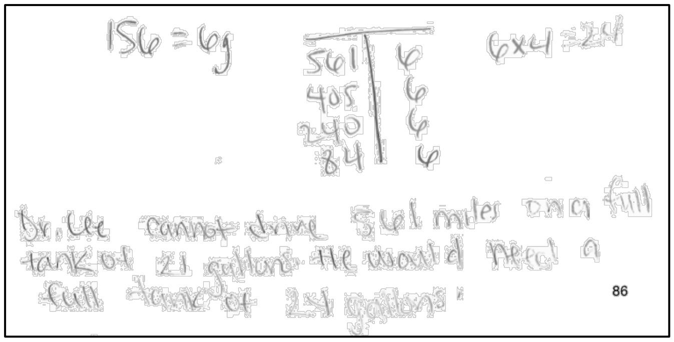

Figure 9. Gas comes in 6 gallon chunks

Equality is Everything (1.5% of respondents)

3 of the 199 teachers used cross-multiplying and looked for the equality of both sides. This strategy is appropriate when checking whether two fractions are equal, but not an appropriate strategy for determining whether one unit rate or gas efficiency is greater than or equal to another. Moreover, the resulting numbers (3276 and 3366) do not have clear quantitative meanings, as can be seen from their incoherent units of “dollar-gallons”.

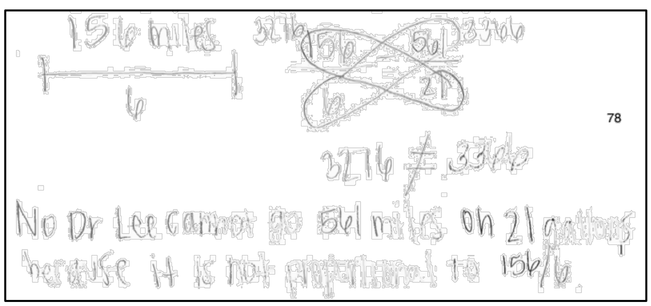

Figure 10. Equality is Everything

Conclusions & Further Thoughts

One of the enduring problems of mathematics education research is how to improve mathematics education in a way that benefits students both inside and outside the classroom. The days when a human calculator was valued and useful are over; now as technology develops, a person’s ability to interpret the significance of mathematical outputs has been more emphasized than simple calculation. In the Common Core, the ability to interpret the meaning of one’s answer is central to at least five of the eight Mathematical Practices: make sense of problems, reason quantitatively, construct viable arguments, model with mathematics, and attend to precision (National Governors Association Center for Best Practices, 2010). It is deeply concerning that over three semesters at a large university’s Teacher’s College, almost a fourth of PSTS struggled to interpret the meaning of their own calculations on a sixth-grade level task. There are several implications for students in such a teacher’s classroom. Teachers that struggle to assign meaning to their calculations will certainly also struggle to impart that skill to students, but there are also further concerns. Such a teacher is also likely to avoid an area he/she feels weak in, which can be reflected in the problems he/she chooses to assign to students or to do with students in class. Additionally, such choices can also convey beliefs to students about what mathematics entails or what kinds of problems should be present in a mathematics class. We also want to note that quantitative reasoning is not only applicable to real world or story problems; for example, students need to reason about the abstract quantities represented by the independent and dependent variables in order to make sense of functions.

The only foreseeable solution to this problem lies in making teachers' *quantitative reasoning* a core focus of both preservice teacher education and inservice professional development. Such a solution would be best implemented by refining the structure and content of current methods courses and professional development opportunities; a separate course might imply that quantitative reasoning is a special and separate topic only applicable to a narrow section of mathematics. The work that has been done on mathematical knowledge for teaching (MKT) shows how crucial it is to student achievement that teachers have a deep understanding of the meaning underlying the procedures they carry out (Hill, 2005; Silverman & Thompson, 2008). We have four specific recommendations for teachers of teachers from our experiences in analyzing this data set: (1) Discontinue the use of the cross-multiplying procedure (where the intermediate results have no clear quantitative meaning and incoherent units such as dollar-gallons) in favor of finding unit rates, or finding the relative size of one pair of measurements and then applying it to the other. (2) Focus explicitly on reasoning with non-integer measurements of quantities. (3) Tasks necessitating proportional reasoning should be given interspersed with tasks that sound similar but do not involve proportional quantities, so that teachers need to genuinely assess whether proportional reasoning is appropriate for each task. Looking back, this was a missing piece in our own data collection. (4) Most significantly, tasks that require teachers to ascribe meaning to their own calculations should be the norm and not the exception. The overall arching problem identified in this data set is that teachers are adept at carrying out calculations without having coherent meanings for their own results. A teacher who cannot make sense of his or her own calculations has no chance of helping students to understand their own.

References

Beckmann, S., & Izsak, A. (2015). Two perspectives on proportional relationships: Extending complementary origins of multiplication in terms of quantities. *Journal for Research in Mathematics Education, 46*(1), 17-38.

Byerley, C., & Thompson, P. W. (2017). Teachers' meanings for measure, slope, and rate of change. *Journal of Mathematical Behavior, 48*, 168-193.

Castillo-Garsow, C., Johnson, H. L., & Moore, K. C. (2013). Chunky and smooth images of change. *For the Learning of Mathematics, 33*(3), 31-37.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American educational research journal, 42*(2), 371-406.

Langrall, C. W., & Swafford, J. (2000). Three balloons for two dollars: Developing proportional reasoning. *Mathematics teaching in the middle school, 6*(4), 254.

National Governors Association Center for Best Practices, C.o.C.S.S.O. (2010). Common Core State Standards: Mathematics Standards.

Silverman, J., & Thompson, P. W. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education, 11*(6), 499-511. doi:10.1007/s10857-008-9089-5

Smith, J., & Thompson, P. W. (2007). Quantitative reasoning and the development of algebraic reasoning. In D. C. J. Kaput, & M. Blanton (Eds.) (Ed.), *Algebra in the early grades* (pp. pp. 95-132). New York: Erlbaum.

Son, J. W. (2013). How preservice teachers interpret and respond to student errors: ratio and proportion in similar rectangles. *Educational Studies in Mathematics, 84*(1), 49-70.

Thompson, P. W. (1993). Quantitative Reasoning, Complexity, and Additive Structures. *Educational Studies in Mathematics, 25*(3), 165-208. doi:10.1007/BF01273861

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S.C.S.B.E.L.L. Hatfield (Ed.), *New perspectives and directions for collaborative research in mathematics education WISDOMe Monographs (Vol. 1)* (pp. pp. 33-57). Laramie, WY: University of Wyoming Press.
Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.
Tourniaire, F., & Pulos, S. (1985). Proportional reasoning: A review of the literature. *Educational Studies in Mathematics, 16*(2), 181-204.

PRESERVICE SECONDARY MATHEMATICS TEACHERS' PERCEPTIONS OF PROOF IN THE SECONDARY MATHEMATICS CLASSROOM

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Despite the recognized importance of mathematical proof in secondary education, there is a limited but growing body of literature indicating how preservice secondary mathematics teachers (PSMTs) view proof and the teaching of proof. The purpose of this survey research was to investigate how PSMTs in Australia and the United States perceive of proof in the context of secondary mathematics teaching and learning. PSMTs were able to outline various mathematical and pedagogical aspects of proof, including: purposes, characteristics, reasons for teaching, and imposed constraints. In addition, PSMTs attended to differing, though overlapping, features of proof when asked to determine the extent to which proposed arguments constituted proofs or to decide which arguments they might present to students.

Keywords: Reasoning and Proof, Teacher Education-Preservice, Mathematical Knowledge for Teaching, High School Education

Introduction

The importance of proof in the discipline of mathematics, and subsequently in the teaching and learning of mathematics, is recognized in the literature worldwide (e.g., Hanna & de Villiers, 2008; Stylianou, Blanton, & Knuth, 2009), and is echoed in recent policy documents and national curriculum (e.g., ACARA, 2017; Common Core State Standards Initiative, 2010). This importance largely rests on the notion that engaging in proof helps students reason about mathematical ideas as they critique arguments or construct their own logically sound explanations or justifications. In fact, both the Australian and United States mathematical proficiencies and practices reference the importance of the reasoning abilities of students, with emphases placed on constructing, justifying, and communicating arguments. However, research has revealed that secondary mathematics teachers often hold a limited view on the purpose of proof instruction and its appropriateness for all students (Bergqvist, 2005; Knuth, 2002). Specifically, teachers often relegate proof to verifying formulas in high school geometry, neglecting the explanatory role proof can play in the learning of mathematics at all levels (Hanna, 2000; Knuth, 2002). Moreover, teachers often focus on the structure rather than substance of a proof and have difficulty evaluating proofs presented pictorially (Dickerson & Doerr, 2014; Tsamir, Tirosh, Dreyfus, Barkai, & Tabach, 2009).

A recent study by Dickerson and Doerr (2014) suggested that teachers' ability to accept and promote less formal modes of proof representation (e.g., visual & concrete models) develops over time, with more veteran teachers less likely to impose strict standards for mathematical language and format. While both experienced and veteran teachers in this study appreciated the explanatory role of proof (Hanna, 2000), espousing proof as a vehicle to build student understanding of specific mathematics content and generalized thinking strategies, they differed considerably in terms of the value they afforded to explicit logic, detail, and precise language in writing proofs (Dickerson & Doerr, 2014). Views of proof as a formalistic mechanism are magnified for prospective teachers (Boyle, Bleiler, Yee, & Ko, 2015; Varghese, 2009) whose most recent experiences with proof are often in the context of upper-level mathematics courses. For example, in a study by Varghese (2009), secondary level student teachers expressed uneasiness with respect to teaching proof, with many indicating that proof should only be introduced to students planning to study advanced mathematics. Without a more complete understanding of the nature of proof and its role in learning mathematics, it is unlikely that beginning teachers will be equipped to enact proof instruction for all students as envisioned by mathematics education policies.

Purpose of the Study

The chief aim of our study was to investigate how preservice secondary mathematics teachers (PSMTs) in Australia and the United States perceive the importance and purpose of proof in the context of secondary mathematics teaching and learning. The overarching research question guiding this study was: *What are preservice, secondary mathematics teachers' conceptions of proof and proof teaching in a secondary classroom context?*

Theoretical Perspective

Our study is grounded in a situative perspective that takes into account what, when, and how mathematical knowledge is required in various practices such as teaching (Adler & Davis, 2006; Putnam & Borko, 2000). Specific to our study, Knuth (2002a, 2002b) demonstrated how teachers' conceptions of proof in the discipline of mathematics were sometimes at odds with their views on the role of proof in mathematics teaching and learning. This perspective also underlies the various frameworks that have been developed to delineate mathematical knowledge that would support the work of teaching proof (e.g. Lesseig, 2016; Steele & Rogers, 2012). For instance, Lesseig's (2016) Mathematical Knowledge for Teaching Proof (MKT for Proof) framework describes Common and Specialized Content Knowledge related to constructing and understanding proof, explicit knowledge of proof components, and the functions of proof that would support teachers' classroom work with proof. This particular framework provided a lens through which to analyze our data and grounded our subsequent interpretations of PSMTs' conceptions of proof and proof evaluations.

Methods

Participants and Contexts

The purpose of this study was to investigate PSMTs' conceptions of proof in the context of secondary mathematics teaching and learning. Data for this paper come from the results of a survey distributed to students enrolled in secondary mathematics teacher preparation programs across 6 different universities in the United States and Australia. Twenty-two PSMTs completed the survey, with exactly half of the participants from each country. The 11 PSMTs from Australia were all enrolled at the same university whereas the 11 PSMTs from the United States were split amongst five different universities. Half of the participants were in their first year of their preparation program, and the other half of the participants were in their second to third year of the program. All PSMTs had taken at least 3 college-level mathematics courses, with the majority of PSMTs (13) having taken 10 or more college-level mathematics courses. Exactly half had taken a course focusing on proof.

Data Collection

The survey was created in Qualtrics and was completed electronically. The three-part survey was qualitative in nature, with the majority of questions being open-ended.

Part I. The first part of the survey focused on PSMTs' conceptions of the nature and role of proof in mathematics and in teaching mathematics. This section included four open-ended questions: (1) What purpose(s) does proof serve in mathematics? (2) What makes an argument a proof? (3) If proofs are to be taught to students, what are your reasons for teaching proof? and (4) What will be the constraints, if any, on teaching proofs?

Part II. The second part of the survey focused on what secondary preservice teachers attend to when evaluating whether or not an argument is a proof. For five different statements, PSMTs evaluated between one and three student-generated arguments, decided whether or not each argument constituted a proof, and provided an explanation to support their decision. The five statements were drawn from the content areas of geometry, algebra, elementary number theory, and infinite geometric series. Student-generated arguments varied in clarity, generality, and approach. The survey purposefully included both symbolic and visual representations as well as deductive and empirical modes of argumentation.

Part III. The third part of the survey focused on what PSMTs attend to when deciding what kinds of arguments are most helpful for a group of students working on mathematical ideas underlying each of the five statements from Part II.. PSMTs were prompted to provide an explanation to support their selection of argument(s) they would share with students.

Analysis

Part I Coding. Once all survey responses were collected, the first two authors separately read through the Part I responses provided by PSMTs from their respective countries. Initial codes were drawn from the literature. For example, the codes VERIFY, EXPLAIN, SYSTEMATIZE, DISCOVER, and COMMUNICATE, adapted from de Villiers (1990), provided an initial lens to analyze questions 1 and 3. In addition, codes for question 2 such as LOGIC, THEOREM, and GENERAL were determined *a priori*, as they captured essential proof understandings one would expect from those familiar with mathematics, including future teachers (Lesseig, 2016). After the first pass of coding, researchers met virtually to discuss other themes that arose throughout the analysis and refined codes to incorporate these additional themes as well as to remove themes that were not applicable to certain questions. On the second pass of Part I, the first two authors analyzed the data from both countries using the agreed-upon codes. Once completed, they met virtually to discuss similarities and differences in their analyses and came to consensus on codes for each PSMT response to each question. The inter-rater reliability (IRR) was 85% for Part I and was calculated as the number of PSMT responses for which there was initial agreement on one or more codes (as more than one code could be used per response), divided by the total number of PSMT responses. A total of 60 responses were coded for Part I.

Parts II and III Coding. The codes used for the open-ended responses in Parts II and III were discussed and decided upon after researchers had read through all PSMT responses from their respective country. The first two authors then separately coded responses from both countries. After separately coding all data, the researchers discussed responses that did not initially have full agreement and came to consensus. The IRR was calculated at 83% for Part II, and 86% for Part III. There was a total of 134 responses coded for Part II, and a total of 50 responses coded for Part III.

Results

Findings for each of the three parts of the survey are presented below. For each of the survey parts, a table of researcher-generated codes has been provided to indicate the five most common responses offered by each cohort (see Tables 1-6). Overall, responses from both Australian and US cohorts appeared strikingly similar with little variation.

Part I – Conceptions of Proof and Proof Teaching

What purpose(s) does proof serve in mathematics? Participants indicated that proof establishes an axiomatic system to formalize mathematical knowledge, provides verification that a mathematical statement is true (or false), and helps to explain why a statement is true or makes sense. In this way, the notion of proof acting as a 'failsafe' during the teaching and learning of mathematics underpinned many statements regarding the purpose of proof. This response from Grant (Aus) describes the role proof plays in verification and systemization, "(proof serves) to establish fact, it provides the means where fundamental truths can be established which then provide the foundations for further understanding to be built."

Table 1: The purpose of proof in mathematics (PSMTs' conceptions)

Code	Code Description	PSMT (AUS)	PSMT (USA)	PSMT (Total)
SYSTEM	To establish an axiomatic system, referring to the formalization of mathematical knowledge	4	5	9
VERIFY	To provide verification that a statement is true or false	5	2	7
EXPLAIN	To explain why a statement is true or makes sense	2	4	6
BUILD-U	To build or deepen understanding of the mathematical concepts underlying the proof	1	2	3
COMM	To communicate mathematics	0	2	2

What makes an argument a proof? For the second survey question, participant responses aligned closely with the criteria identified as *Essential Proof Understandings* in Lesseig's (2016) MKT for Proof framework. Those criteria stipulate that a theorem has no exceptions, a proof must be general, a proof is based on previously established truths, and the validity of a proof depends on its logic structure. The most frequently elicited responses were that a proof is based on accepted statements, follows a logical structure, and removes any doubt about the veracity (i.e. truth or falsehood) of the statement.

Table 2: What makes an argument a proof? (PSMTs' conceptions)

Code	Code Description	PSMT (AUS)	PSMT (USA)	PSMT (Total)
THEOREM	It is based on accepted statements or theorems	2	7	9
LOGIC	It follows a logical structure	3	3	6
INFALLIBLE	It removes any doubt about the truth or falsehood of the statement	2	3	5
GENERAL	It proves the statement in general by covering all cases within the domain	2	2	4
VERIFY	It provides verification that a statement is true or false	2	0	2

If proofs are to be taught to students, what are your reasons for teaching proof? According to the proffered responses for survey question three, participants feel that proof should be taught to students to impart a variety of mathematical skills (e.g. logical reasoning), to build an understanding of the mathematical concepts underlying the proof, and to increase student agency. Walter's (US) response illustrates ways in which PSMTs' saw proof as valuable in promoting reasoning and argumentation skills that were useful beyond mathematics as well as a vehicle for both building and assessing understanding:

> The ability to convey reasoning and to justify in writing is a vital skill in the real world. In any job, you have to persuade others (often through proving that your way is right). Proofs also press for deep understanding and makes students' thinking visible to the teacher.

Table 3: Reasons for teaching proofs to students (PSMTs' conceptions)

Code	Code Description	PSMT (AUS)	PSMT (USA)	PSMT (Total)
T-SKILLS	To teach skills in logical reasoning, argumentation and problem solving	5	5	10
BUILD-U	To build or deepen understanding of the mathematical concepts underlying the proof	3	3	6
S-AGENCY	To increase or build student agency	2	4	6
DISC	To discover or explore an idea or create new knowledge	0	3	3
ASSESS-U	To assess student understanding	0	2	2

What will be the constraints, if any, on teaching proofs? Most respondents tended to assert that proof is a difficult topic, skill, or body of knowledge to teach. PSMTs' responses reflected their perceptions of the complexity of proof by way of questioning students' prior knowledge or ability to deal with the abstract nature of proof. PSMTs identified additional resource constraints such as a lack of time, curricular emphasis, or adequate preparation for teaching proof.

Table 4: The constraints on teaching proofs (PSMTs' conceptions)

Code	Code Description	PSMT (AUS)	PSMT (USA)	PSMT (Total)
S-KNOW	Students may not have the requisite mathematical content knowledge or skills for proof	3	4	7
TIME	Takes too much time to teach properly	2	4	6
CURRIC	There is not enough clear direction in standards or curriculum	3	3	6
ABSTRACT	Proof is too abstract for this age group	0	3	3
T-KNOW	Teachers may not have the knowledge and skills necessary to teach proof	1	2	3

Part II - Proof Evaluation

The purpose of Part II in the survey was to determine the extent to which PSMTs felt that proposed arguments constituted a proof, and ascertain the features of proof to which PSMTs attended. Participants attended to a variety of features in describing why they felt a proposed argument constituted a proof or identifying what they felt was missing in the argument. The most common rationales provided were in relation to whether (or not) the arguments proved the statement in general (i.e. they covered all cases within the domain), were based on accepted statements or theorems, and followed a logical structure, aligning with the top five characteristics of proof outlined in Part I of the survey. To a lesser degree, participants argued (in the affirmative or the negative) that the proofs were error-free and were easy to follow or understand. Table 5 below displays the counts for these top five characteristics evident in responses from US and Australian participants. Counts for two additional characteristics frequently cited in Part III are included for comparison.

Table 5: Characteristics PSMTs attend to when evaluating a proof.

	General	**Logic**	**Theorem**	**Correct**	**Clear**	Representatio	Accessible
AUS	23	20	9	15	8	4	0
US	26	17	16	9	10	6	0
TOTAL	49	37	25	24	18	10	0
	25.1%	19%	12.8%	12.4%	9.2%	5.1%	0%

*Percentages in Tables 5 and 6 are calculated based on total number of codes applied across responses. There were a total of 195 codes applied to the proof evaluation questions (Table 5) and 90 total codes in response to why an argument would be helpful (Table 6). Also, the last two columns have been included to facilitate comparison with the five most popular responses in Table 6.

Part III - Identification of proof features

Our intent in Part III of the survey was to identify features of proof that PSTs attend to when evaluating arguments or deciding which arguments to present in a classroom. In an identical manner to Part II, we assigned codes to responses regardless of whether PSTs referenced a particular feature in a positive or negative manner.

When deciding which arguments they might present to a group of students, PSMTs not only valued arguments that were correct and general (characteristics mentioned in Part II), but also considered pedagogical features such as whether the proof was accessible to students and how different representations might support student learning. As shown in Table 6, the mode of representation and clarity of the argument were the two most commonly mentioned characteristics across all four questions (20% & 18.9% respectively), yet when compared with findings from Part II, logic and theorem were not commonly espoused features. Several PSMTs attended to how accessible the argument might be for different levels of students.

Table 6: Helpful characteristics of proofs for students (PSMTs' perceptions)

	Representation	**Clear**	**Correct**	**Accessible**	**General**	Logic	Theorem
AUS	7	9	4	4	7	4	1
US	11	8	5	5	0	0	1
TOTAL	18	17	9	9	7	4	2
	20%	18.9%	10%	10%	7.8%	4.4%	2.2%

Discussion and Conclusion

Overall survey responses reveal that many PSMTs still hold rather formalistic views of proof. This is perhaps not surprising given participants' current status in teacher education programs. Participants' most recent experiences with proof have been in the context of taking pure mathematics courses wherein opportunities to consider proof from a teacher perspective are uncommon. However, we claim that to carry out the work of mathematics teaching wherein reasoning and proof remains central, PSMTs need to consider pedagogical aspects of proof. Specifically, our findings highlight the need to enhance PSMTs' views of the role of proof and extend their understanding and acceptance of other forms of proof that may be more accessible to secondary students.

PSMTs from both Australia and the US cited systematization and verification as the primary roles of proof. Proof was most commonly described as a mechanism for establishing truths in mathematics and demonstrating that a result is true beyond a reasonable doubt. Previous researchers (e.g., Knuth, 2002; Varghese, 2009) have discussed how this dominant view has the potential to limit teachers' perspectives on when and how proof should be utilized in secondary mathematics. However, in our case we see this not necessarily as a deficit, but as a common starting point. Indeed, a number of PSMTs also acknowledged the explanatory role of proof (Hanna, 2000) and recognized proving as a venue for students to both build and communicate understanding. Coupling verification with explanation, and explicitly discussing these complementary roles of proof in preservice teacher education (Bleiler-Baxter & Pair, 2017) has the potential to expand PSMTs' view of proof and their ability to integrate proof more consistently into their future teaching.

In light of our theoretical perspective, we noted that while many of PSMTs' responses were consistent across survey items, others appeared somewhat inconsistent. For example, when asked to identify necessary components of a proof, PSMTs stated in Part I that proofs should be based on established facts or theorems and follow a logical progression, and subsequently included statements about the logical structure or use of theorems in their rationales for the acceptance (or not) of the student-generated arguments as proof. On the contrary, despite the fact that only 20% of PSMTs explicitly mentioned that a proof must be general, generality was the characteristic attended to most often when PSMTs were asked to evaluate the student-generated arguments from Part II. Indeed, in their evaluations, PSMTs demonstrated a robust understanding of proof from a mathematics perspective, and made explicit statements that examples do not constitute proof. This understanding is particularly notable in light of prior research documenting students and preservice teachers' acceptance of empirical arguments as proof (Harel & Sowder, 2007; Simon & Blume, 1996). In addition, the characteristics PSMTs attended to when evaluating arguments in Part II did not necessarily map directly onto characteristics PSMTs used when deciding which proofs they might present to students in Part III, wherein the use of multiple representations that might be accessible to a range of students was valued. Differences across survey sections highlight the importance of considering PSMTs' actions across multiple settings. From a research perspective, these findings suggest that measures of PSMTs' proof conceptions should be situated in tasks of teaching (Steele & Rogers, 2012).

Finally, we note that merely knowing that a proof must be general was not necessarily sufficient, as PSMTs had different interpretations of what constituted generality. This finding was most evident in their assessment of two "non-traditional" arguments included in the survey, one a visual argument, the other a generic example proof (Karunakaran, Freeburn, Konuk, & Arbaugh, 2014; Mason & Pimm, 1984) for a claim about divisibility. Given their accessibility to students, visual arguments and generic example proofs have the potential to 'bridge the gap' from empirical toward more deductive modes of argumentation (Karunakaran et al., 2014). Thus, we contend that increasing PSMTs' proficiency with constructing and assessing generic example proofs and visual arguments is an important step toward enhancing the role of proof in secondary mathematics classrooms. It is worth noting that the invitation to participate in this study, which resulted in a convenience sample from participants from multiple universities in the US and one university in Australia, is a limitation in this study. Although results from students across universities and countries were similar, the different requirements for the programs at the different universities could have affected PSMTs' perspectives. A future direction could involve disaggregating the data by type of program in which the students were enrolled.

There is widespread agreement that reasoning and proof should be an integral part of *all* students' mathematical experiences across content areas and throughout the grades (Stylianou et al., 2009). Meeting this goal requires that PSMTs develop a nuanced understanding of the various purposes and characteristics of proof and what might be an acceptable proof given a particular purpose. This includes the ability to distinguish among a range of valid and invalid arguments presented visually, verbally, or symbolically, as well as the ability to determine which of those arguments might be accessible to students. Moreover, PSMTs need opportunities and experiences within their teacher preparation programs that allow them to confront (and reconcile) their views of proof from mathematical and pedagogical perspectives.

References

Adler, J., & Davis, Z. (2006). Opening another black box: Researching mathematics for teaching in mathematics teacher education. *Journal for Research in Mathematics Education*, *37*, 270-296.

Australian Curriculum and Reporting Authority (ACARA) (2017). *The Australian curriculum: Mathematics.* Retrieved from https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics/key-ideas/

Bergqvist, T. (2005). How students verify conjectures: Teachers' expectations. *Journal of Mathematics Teacher Education*, *8*, 171-191.

Bleiler-Baxter, S.K., & Pair, J.D. (2017). Engaging students in roles of proof. *The Journal of Mathematical Behavior*, *47*, 16-34.

Boyle, J. D., Bleiler, S. K., Yee, S. P., & Ko, Y.Y. (2015). Transforming perceptions of proof: A four-part instructional sequence. *Mathematics Teacher Educator*, *4*(1), 32-70.

Common Core State Standards Initiative. (2010). Common Core State Standards for Mathematics. Washington DC: National Governors Association for Best Practices and the Council of Chief State School Officers.

de Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras*, *24*, 17-24.

Dickerson, D. S., & Doerr, H. M. (2014). High school mathematics teachers' perspectives on the purposes of mathematical proof in school mathematics. *Mathematics Education Research Journal*, *26*(4), 711-733.

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics*, *44*, 5-23.

Hanna, G., & de Villiers, M. (2008). ICMI Study 19: Proof and proving in mathematics education. *ZDM: International Journal on Mathematics Education*, *40*(2), 329-336.

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. in F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning,* National Council of Teachers of Mathematics (pp. 1-60).

Karunakaran, S., Freeburn, B., Konuk, N., & Arbaugh, F. (2014). Improving preservice secondary mathematics teachers' capability with generic example proofs. *Mathematics Teacher Educator*, *2*(2), 158-170.

Knuth, E. J. (2002a). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education*, 33(5), 379-405.

Knuth, E. J. (2002b). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education*, *5*(1), 61-88.

Lesseig, K. (2016). Investigating mathematical knowledge for teaching proof in professional development. *International Journal of Research in Education and Science*, *2*(2), 253-270

Mason, J., & Pimm, D. (1984) Generic examples: seeing the general in the particular. *Educational Studies in Mathematics, 15*(3), 277-289.

Putnam, R.T., & Borko, H. (2000). What do new views of knowledge and thinking have to say about research on teacher learning? *Educational Researcher*, *29*(1), 4-15.

Simon, M.A., & Blume, G.W. (1996). Justification in the mathematics classroom: A study of prospective elementary teachers. *The Journal of Mathematical Behavior*, *15*, 3-31.

Steele, M.D., & Rogers, K.C. (2012). Relationships between mathematical knowledge for teaching and teaching practice: The case of proof. *Journal of Mathematics Teacher Education*, *15*(2), 159-180.

Stylianou, D.A., Blanton, M.L., & Knuth, E.J. (2009). *Teaching and learning proof across the grades: A K−16 perspective*. New York, NY: Routledge.

Tsamir, P., Tirosh, D., Dreyfus, T., Barkai, R., & Tabach, M. (2009). Should proof be minimal? Ms T's evaluation of secondary school students' proofs. *Journal of Mathematical Behavior, 28*, 58–67.

Varghese, T. (2009). *Secondary-level student teachers' conceptions of mathematical proof.* Issues in the undergraduate mathematics preparation of school teachers: The Journal, Volume 1: Content Knowledge.

USING STRIP DIAGRAMS TO SUPPORT EXPLANATIONS FOR KEEP-CHANGE-FLIP FOR FRACTION DIVISION

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Using memorized rules and algorithms without coherence and understanding is a perennial problem for teachers and students especially in the teaching and learning of fraction operations. I present data in which prospective middle school teachers explain a commonly used rule for fraction division—keep-change-flip. I argue that using both strip diagrams and a single, quantitative definition for multiplication support prospective teachers when explaining why the rule works. The results of the study provide impetus for both mathematics teachers and mathematics teacher educators to teach with coherence across multiple mathematical topics.

Keywords: Teacher Education-Preservice, Rational Numbers, Middle School Education

Curricular documents have highlighted the crucial role for representations in the learning and teaching of mathematics (National Council of Teachers of Mathematics [NCTM], 2000) and the development of well-prepared beginning teachers (Association of Mathematics Teacher Educators, 2017). Although there are many representations used in mathematics education, I focus on representations as models of problem situations, specifically strip diagrams. The importance of using representations in educational contexts is not new (Ball, Thames, & Phelps, 2008; NCTM, 2000), and researchers' work on teachers' use of representations has helped to identify related issues. First, teachers' conceptions of representations and their role in problem solving are relegated to the periphery of mathematical activity—representations are not "real" mathematics (Stylianou, 2010). Teachers prefer to prioritize abstract, procedural rules over productive representations (Eisenhart et al., 1993). Second, teachers' content knowledge constrains their pedagogical purposes for using representations (Izsák, Tillema, & Tunç-Pekkan, 2008), and teachers' mathematical knowledge has been shown to be primarily procedural without a strong grasp of the mathematical underpinnings (Mewborn, 2003). Prospective middle school teachers, in particular, frequently use the keep-change-flip procedure (KCF; i.e., $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$) to explain fraction division but have difficulty explaining why the procedure works (Li & Kulm, 2008). This procedural knowledge of mathematics could explain the reluctance of teachers to use representations when teaching (see Eisenhart et al., 1993). One viable avenue is to provide teachers with opportunities to learn with representations which influences both teachers' knowledge and use of representations (Jacobson & Izsák, 2015). In this report, I present such an opportunity. I examine how prospective middle school teachers use strip diagrams to solve fraction division problems in a content course and how leveraging the diagram supports productive explanations for the keep-change-flip procedure.

Theoretical Framework

Definition of Representations

Researchers who have studied representation use in class (e.g., Izsák, 2003; Saxe, 2012) have generally agreed to distinguish what is being represented and what is "doing" the representing (cf. von Glasersfeld, 1987). In this study, I refer to representations as observable geometric inscriptions that can be referred or pointed to as the object of discussion (Goldin, 2002). It is this indexical and communicative nature of representations allowing students to explain their thinking and for others to engage in another's way of reasoning. When students create a display to represent their thinking, they also have a communicative aspect. In other words, they tailor their display with an audience in mind (Saxe, 2012) and thus students select salient features to highlight and point when creating and talking about representations. Additionally, I frame representations as culturally and historically rooted. A representation's cultural and historical meaning grow out of how communities have interacted with an inscription over time (Blumer, 1986). For example, a class can ascribe the meaning to the inscription "=" as "execute the arithmetic to the left" if they are continually asked to solve result-unknown problems over time.

Strip diagrams. The Common Core authors recommended strip diagrams when reasoning about ratios and rates in 6th grade (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). A strip diagram is usually drawn as a rectangle that can be partitioned into different sized parts where each part may refer to a quantity. Although researchers have identified the strip diagram as a feature of mathematics instruction in high performing countries such as Japan and Singapore, research on strip diagrams themselves is sparse especially in the United States (Murata, 2008; Ng & Lee, 2009).

Form-Function Relationships

Saxe's account (2012) of cultural forms and functions accounts for how historically-rooted artifacts change over historical time. *Forms* are socially-rooted systems of artifacts perceivable by members of the community such as the base-27 system of the Oksapmin peoples (Saxe, 2012) or a number line in a math class (Saxe, de Kirby, Le, Sitabkhan, & Kang, 2015). *Functions* are how the forms are used to achieve goals. To characterize cultural forms and their functions, the researcher must investigate the creation of a "common ground" or a taken-as-shared ways of talking and doing (Saxe et al., 2015). Saxe and colleagues identified three strands describing how individuals contribute to common ground. In this repor, I account for two of these strands:

1. *Microgenesis.* This process shows how individuals contribute to a common ground, often using a form in public, by describing how forms serve certain functions. For example, if a student wants to show how 3/2 is equivalent to 6/4, a student may create a strip diagram partitioned into three parts and partition each part into two in order to show six sub-partitions using a different color to show the relationship between the two partitions. The student is contributing to common ground by producing a particular form (strips with partitioned partitions) to describe fraction equivalency.
2. *Ontogenesis.* This process shows the continuity and discontinuity of forms to serve new functions. In some instances, if a new function is necessitated, some previously used forms may be employed (continuity) or new forms (discontinuity) may emerge to serve the new function and accomplish the goal.

Data and Analysis

Context of the Study

I analyzed four days of instruction from the second course of a sequence of two mathematics content courses for prospective middle school teachers (PSMTs) enrolled in a teacher education program. The same teacher taught both courses. The program was geared towards certification to teach mathematics in Grades 4–8. The objective of the course was to strengthen the students' mathematical understanding of middle school topics such as base-10, fractions, and ratios. The 13 PSMTs enrolled in the course were predominantly white women. Two class norms were developed in class by the time of the classes selected for analysis. First, students were expected to use a multiplicand-multiplier definition for multiplication, notated by equation $N \cdot M = P$ (Beckmann & Izsák, 2015). In this equation, N denotes the number of base units in one group (the multiplicand), M denotes the number of groups (the multiplier), and P denotes the total number of units in M groups. With this definition, the order of the factors matters e.g., 12 boxes with six donuts in each box ($12 \cdot 6$) is different from six boxes with 12 donuts in each box ($6 \cdot 12$) even if they are numerically the same number of donuts in total. The class was also expected to use the Common Core definition of fraction (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) where the fraction a/b referred to the quantity formed by a parts of size $1/b$. Lastly, students were expected to explain their thinking with drawings rather than memorized algorithms or symbol manipulation. Specifically, students were expected to use strip diagrams and double number lines.

The teacher usually began class by orienting the PSMTs to the mathematical topic of the day. She gave the class a problem to solve and the PSMTs worked at their table with two to five other PSMTs. As they worked, the teacher walked around class, supporting or pressing the PSMTs. The teacher would redirect the PSMT if they breached any of the two sociomathematical norms. PSMTs were given the option of using iPads. After a period of time, PSMTs presented the strategies in whole-class discussion. To present their strategies, they could recreate their strategies on the mounted whiteboards or project their iPad screen on one of four mounted screens. Some students used the iPad's camera to project written work. The whole-class discussion focused on students' strategies and connections between different strategies.

Data Collection and Analytical Techniques

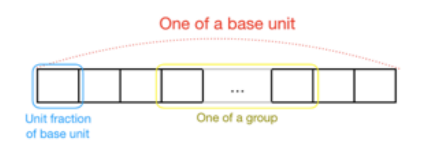
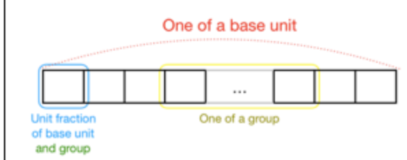
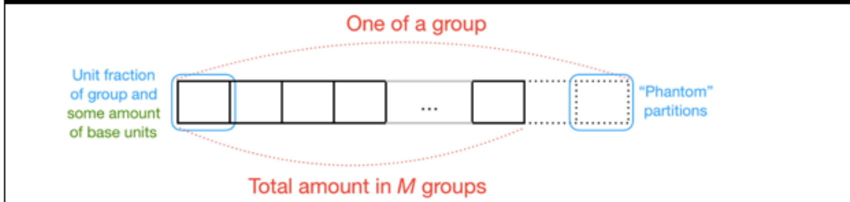
The main data corpus for this study was video and audio-recorded lessons from class. One stationary camera was set at the back of the class and captured the whole class within one frame. The other camera was also stationary during whole-class discussion but followed the teacher during small group discussions. Two microphones mounted on the ceiling captured audio during whole-class discussion while four flat microphones captured audio at each table. In post-production, all video and audio data were condensed into a single file. The two videos were synchronized and combined into a single frame. The file contained all the audio feeds such that I could select any audio and listen to one audio source.

The primary analytical techniques were modified from Saxe et al., 2015 and focused on identifying forms and functions of the representations to characterize the microgenetic process. To identify forms, I located PSMTs' inscriptions in the classroom data. I relied on both discursive and gestural indicators. As the PSMTs talked about their drawings, I noted when they physically gestured to an inscription or used pointing language such as "This is…" or "I drew…" Two grain sizes for forms emerged from this analysis. A *micro form* was a single geometric inscription as fine as a line or rectangle and a *coarse form* was a group of microforms used to address a larger goal such as an entire strip diagram with its annotations to solve a multiplication problem. To identify a function, I found pointing language referring to what a particular form represents e.g., "This is a cup" and annotations on the drawings. To address the ontogenetic strand, I searched for moments between problems wherein forms changed. I identified the difference between the question or problem that was asked before and after the change.

Results

Over the course of the lessons, the PSMTs used strip diagrams to explain their thinking. The coarse form changed when the problem type changed. In Table 1, I summarize and illustrate the coarse and micro forms used. As the lessons progressed, the function of a partition of a strip changed to accommodate multiple functions which allowed students to explain KCF.

Table 1: Coarse Forms and Functions of the Micro Forms

		Measurement Division Forms	
		First coarse form 	**Second coarse form** 
Micro forms	Whole strip	One of a base unit	One of a base unit
	Set of partitions	One of a group (N)	One of a group (N)
	One of a partition	Unit fractional amount of a base unit	Unit fractional amount of a base unit and a group
		Partitive Division Form	
			
Micro forms	Whole strip	One of a group (N)	
	Set of partition	Total amount in M groups (P)	
	One of a partition	Unit fractional amount of a group and some amount of base units	
	“Phantom” partitions	Partitions intended to complete one of a group	

Note: M, N, and P refer to quantities in the multiplication definition in Beckmann & Izsák (2015)

Measurement Division Forms

The first two lessons centered on solving measurement or how-many-groups division. The two coarse forms resembled the definition of a fraction where each strip stood for one of a base unit e.g., one cup and the partitions referred to a unit fraction of the base unit e.g., $1/n$ of a cup. The PSMTs indicated one of a group by highlighting a set number of parts (such as the different colors in Figure 1a). Using the set number of parts, they counted how many groups were in the total amount of base units. The second form resembles the first coarse form; however, the function of a partition changed to holding two referents, the base unit and the group.

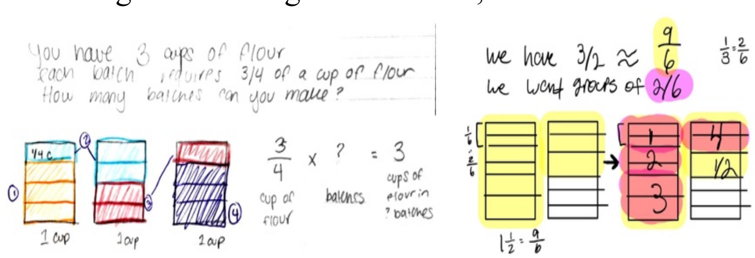


Figure 1. (a) Sophie’s Diagram for 3 ÷ 3/4 = ? (b) Jack’s Diagram for 1 ½ ÷ 1/3 = ?

In the first lesson, students created a word problem for the measurement division problem 3 ÷ 3/4 = ?. They solved the problem using strip diagrams and in some cases double number lines. All the strip diagrams produced resembled Sophie’s diagram in Figure 1a. Sophie created the word problem “You have three cups of flour. Each batch requires three-fourths cup of flour.

How many batches can you make?” During whole-class discussion, she explained, “I first drew three full cups each of those is a cup.” Based on the annotations at the bottom of her strip and her equation, Sophie created a whole strip as a cup. Because the class was annotating equations with one definition of multiplication and Sophie’s placement of cups as the multiplicand in her equation, I inferred she assigned cups as her base unit. She explained, “each of those is made of four parts, each of size one-fourth of a cup and then I colored three of them which is three fourths of a cup each makes a batch” thus a set of three partitions was one group or batch in her context. In the next lesson, PSMTs were also asked to create a measurement division problem for 1½ ÷ 1/3 = ? and solve the problem. Jack wrote the problem “You have 1½ liters of apple juice. You want to pour this apple juice into glasses, which can hold 1/3 liters each. How many glasses can you fill with your apple juice?” Similar to Sophie’s strategy, he created a whole strip indicating one cup of apple juice or one of a base unit. He shaded one and a half strip indicating the total amount he needed to count (see yellow partitions in Figure 1b). During whole-class discussion, Jack explained he anticipated the size of the partitions he wanted “[because] halves and thirds don’t mix perfectly…so I needed to put them in same sizes so the easiest one was sixths ‘coz that’s our common denominator–least common denominators.” Thus, he chose partitions that were 1/6 of the strip. He counted sets of two partitions showing one of a group or 2/6 of a liter. Finally, he commented on the left-over partition and described it simultaneously as one sixth of a liter and half of a glass. This signaled describing a partition with respect to both the group and base unit as a new function for a partition. The ontogenesis of this new function for the partition could perhaps be explained by the number choice of the problem in the first problem where the total number of partitions is divisible by the numerator of the divisor.

Partitive Division Form

The next two lessons centered on solving partitive or how-many-in-1-group division problems. The PSMTs initially worked on partitive division problems where the dividend was a whole number and the divisor was a fraction between zero and one. One coarse form emerged from solving these problems. They created whole strips referring to one of a group e.g., one serving and the partitions were a unit fraction of the group. They also referred to the partition as some amount of base units. Describing the partition with respect to both groups and base units is perhaps a function rooted in the function emerging towards the end of the measurement division lessons. Finally, students also distinguished some partitions. Jack called these partitions that “aren’t really there” as “phantom” partitions. The phantom partitions were drawn to complete one of a group because the sizes of the group that given in the problems were less than one.

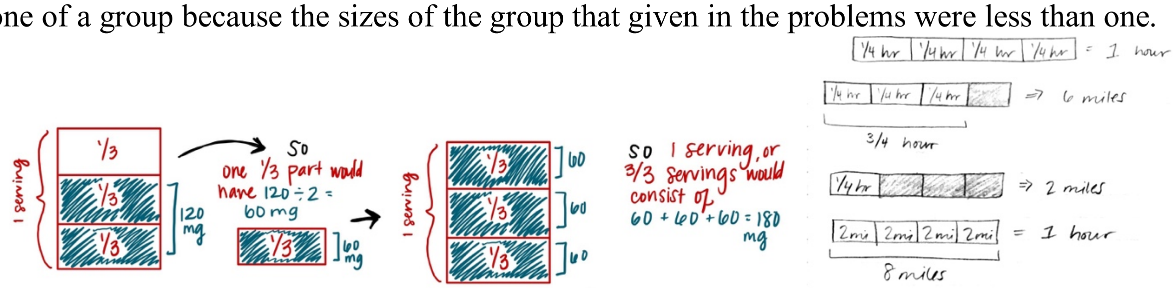


Figure 2. (a) Elizabeth’s Diagram for 120 ÷ 2/3 = ? (b) Catherine’s Diagram for 6 ÷ 3/4 = ?

First, the class worked on the problem “2/3 of a serving of noodles contains 120 mg of sodium. How much sodium is in one bowl of noodles?” Afterwards, they worked on the problem “Running at a steady pace, Anna ran 6 miles in 3/4 of an hour. At that pace, how far will Anna run in one hour?” Elizabeth and Catherine’s strip diagrams in Figure 2 are both exemplars of the coarse form PSMTs used when solving partitive division problems. In both diagrams, each whole strip denoted one of a group and assigned a subset of the partitions in a strip to denote the total amount of base units in *M* groups. In Elizabeth's first strip diagram, the set of blue partitions referred to 120mg of sodium and 2/3 serving. Catherine similarly drew her second strip with the total amount of miles in 3/4 hours. They then considered just one of these partitions and described the partition in both the quantity of the group and base unit similar to Jack's partition in the measurement division lesson. Elizabeth and Catherine also assigned equal amounts of the base unit in each partition. For Elizabeth, she annotated this in the middle of her drawing (Figure 2a) by labelling one of her partitions as both one-third of a serving and 60mg of sodium. Catherine showed this function for a partition in her third strip (Figure 2b) where she annotated one of her partitions as one-fourth of an hour and two miles. They both iterated this partition to complete the whole strip and counted the amount of base units in the whole strip. Although the functions for the whole strip and the partition shifted i.e., measurement division problems showed one of a base unit while partitive division problems showed one of a group, the function of the partition as both an amount of a group and base unit was present in both problem types.

Explaining Keep-Change-Flip

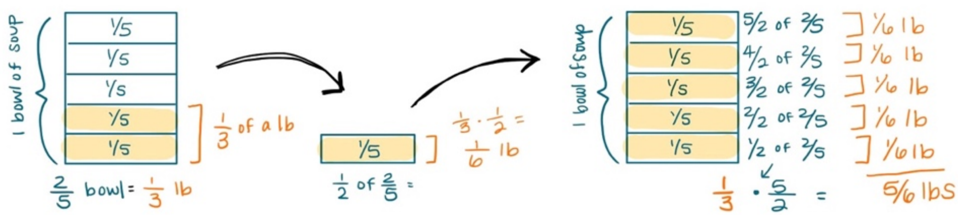


Figure 3: Elizabeth's Strip Diagram for 1/3 ÷ 2/5 = 1/3 · 5/2

PSMTs explained KCF with strip diagrams and the definition of multiplication when explaining partitive division problems when prompted by the instructor. The instructor asked the PSMTs to find ways to explain 1/3 ÷ 2/5 = 1/3 · 5/2. The PSMTs leveraged the function of a partition to describe both the base unit and the group. To explain the rule, PSMTs used a third function for a partition—describing the partition with respect to the size of the group of the total amount of base units. Consider Elizabeth's explanation for 1/3 ÷ 2/5 = 1/3 · 5/2.

Elizabeth drew the strip diagram in Figure 3 to show her thinking for 1/3 ÷ 2/5 = ? and explained her thinking in whole-class discussion. She created a partitive division word problem "A third of a pound of chicken is enough for 2/5 of a bowl of chicken soup. How many pounds of chicken is in 1 whole bowl of chicken soup?" Elizabeth's drew a strip diagram with the coarse partitive division form. First, she drew the strip on the left functioning as one of a group or one bowl of soup. She partitioned the strip into five parts and annotated her parts as 1/5 of the bowl and "colored in two of the fifths and called that 1/3 of a pound." The set of partitions referred to the size of the group, i.e., 2/5 of the bowl, but also the corresponding quantity in base units i.e., 1/3 pound of chicken. Using the function of the partition, she described one partition in two ways, as 1/5 of the bowl and 1/6 of a pound as seen in the middle of Figure 3. She iterated this part to build the whole bowl of soup and kept track of both quantities simultaneously to get 5/6 pounds of chicken in the whole bowl similar to the previous partitive division coarse forms.

To explain the equivalence 1/3 ÷ 2/5 = 1/3 · 5/2, Elizabeth described the situation considering two groups—the original group of one bowl and a new group of 2/5 of a bowl. Considering this new group, she explained the partition is also one-half of two-fifths of the bowl. This activity indicated a new function for a partition in addition to denoting a unit fractional amount of a group and base unit. She used the partition as a unit fractional amount of the size of the group of the total amount of base units in addition to one-fifth of the bowl and one-sixth of the pound. In other words, one partition refers to 1/6 pound of chicken, 1/5 of the bowl, and one-half of two-fifths of a bowl. She counted the five partitions in the whole strip and used the new function to get one bowl as five halves of two-fifths of the bowl.

Using the new group, Elizabeth created the expression $1/3 \cdot 5/2$ following the definition of multiplication used in class. In the annotation, she explained there is one third pound of chicken in one of the new group, two-fifths bowl (amount in one group, *N*) and there are five-halves of the new group in the whole bowl of soup (number of groups, *M*). In summary, Elizabeth's group changed from one bowl to two-fifths of a bowl when asked to explain KCF. Because of her new group, she added a new function to one partition. By using the class definition of multiplication, she annotated her thinking when she considered the new group to obtain the expression $1/3 \cdot 5/2$.

Two reasons could explain the ontogenesis of the new function when solving partitive division problems. First, the instructor's prompt of asking students to explain how to see $1/3 \cdot 5/2$ in their drawing seemed to initiate the creation of the function. If the PSMTs were to simply solve the partitive division problem, the third function for the partition may not have emerged. Second, the nature of partitive division problems may afford a flexible choice of groups. Elizabeth retained 1/3 in base units and flexibly chose her group as either the whole group or 2/5 of her group. In the problem Jack was solving, 1 1/2 and 1/3 both referred to base units and could potentially restrict a group to solely 1/3 of a base unit.

Discussion and Conclusion

The results of this study provide a characterization of how functions of inscriptions evolve over time. In this case, an evolving function for a partition in a strip allowed students to explain KCF. PSMTs' explanations of the algorithm were rooted in two practices—using strip diagrams and a definition of multiplication. Strip diagrams were not templates with predetermined rules and meanings. In fact, most of the forms were rooted in the activity of the class and previous uses of the strip diagram. Initially, PSMTs used the strip diagram to solve measurement division problems. By changing the number choices within the problem, a new function of the partition of a strip diagram emerged which proved to be useful in subsequent solutions. The strip diagrams remained relatively similar when solving partitive division problems. When prompted to explain KCF, the students used the partition in a new way not previously used in solving problems.

The results I report here provide future steps for both researchers and teachers. When analyzing inscriptions, researchers must attend and be explicit about the grain size of the inscription. By capturing two grain sizes, I was able to determine new uses for smaller inscriptions embedded in larger ones. Additionally, an analysis of inscription use in classrooms provides researchers with continuities and discontinuities between points in time in order to characterize teaching opportunities for new forms and functions to emerge. For teachers and teacher educators working with strip diagrams, activities wherein opportunities are provided to assign new functions to diagrams and its elements can equip students with ways of reasoning to be used in future tasks. Additionally, this report provides a case for using representations and how keeping coherent activity across multiple lessons provides students and prospective teachers with a powerful opportunity to learn mathematics.

Acknowledgments

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References

Association of Mathematics Teacher Educators. (2017). Standards for preparting teachers of mathematics. from amte.net/standards

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Beckmann, S., & Izsák, A. (2015). Two perspectives on proportional relationships: Extending complementary origins of multiplication in terms of quantities. *Journal for Research in Mathematics Education, 46*(1), 17–38.

Blumer, H. (1986). *Symbolic interactionism: Perspective and method*. Berkeley, CA: University of California Press.

Eisenhart, M., Borko, H., Underhill, R., Brown, C., Jones, D., & Agard, P. (1993). Conceptual knowledge falls through the cracks: Complexities of learning to teach mathematics for understanding. *Journal for Research in Mathematics Education, 24*(1), 8-40.

Goldin, G. A. (2002). Representation in mathematical learning and problem solving. In L. D. English (Ed.), *Handbook of international research in mathematics education, second edition* (pp. 197–218). Mahwah, NJ: Lawrence Erlbaum Associates.

Izsák, A. (2003). 'We want a statement that is always true': Criteria for good algebraic representations and the development of modeling knowledge. *Journal for Research in Mathematics Education, 34*(3), 191–227.

Izsák, A., Tillema, E., & Tunç-Pekkan, Z. (2008). Teaching and learning fraction addition on number lines. *Journal for Research in Mathematics Education, 39*(1), 33–62.

Jacobson, E., & Izsák, A. (2015). Knowledge and motivation as mediators in mathematics teaching practice: The case of drawn models for fraction arithmetic. *Journal of Mathematics Teacher Education, 18*(5), 467-488.

Li, Y., & Kulm, G. (2008). Knowledge and confidence of pre-service mathematics teachers: The case of fraction division. *ZDM, 40*(5), 833-843.

Mewborn, D. S. (2003). Teaching, teachers' knowledge, and their professional develompent. In J. Kilpatrick, W. G. Martin & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 45–42). Reston, VA: National Council of Teachers of Mathematics.

Murata, A. (2008). Mathematics teaching and learning as a mediating process: The case of tape diagrams. *Mathematical Thinking and Learning, 10*(4), 374-406.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). *Common core state standards*. Washington, D.C.: National Governors Association Center for Best Practices, Council of Chief State School Officers.

Ng, S. F., & Lee, K. (2009). The model method: Singapore children's tool for representing and solving algebraic word problems. *Journal for Research in Mathematics Education, 40*(3), 282-313.

Saxe, G. B. (2012). *Cultural development of mathematical ideas: Papua new guinea studies*. New York, NY: Cambridge University Press.

Saxe, G. B., de Kirby, K., Le, M., Sitabkhan, Y., & Kang, B. (2015). Understanding learning across lessons in classroom communities: A multi-leveled analytic approach. In A. Bikner-Ahsbahs, C. Knipping & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education: Examples of methodology and methods* (pp. 253-318). Dordrecht: Springer Netherlands.

Stylianou, D. A. (2010). Teachers' conceptions of representation in middle school mathematics. *Journal of Mathematics Teacher Education, 13*, 325–343.

von Glasersfeld, E. (1987). Preliminaries to any theory of representation. In A. A. Cuoco (Ed.), *Problems of representation in the teaching and learning of mathematics*. Hillsdale, NJ: Lawrence Erlbaum Associates.

QUANTITATIVE (AND NON-QUANTITATIVE) METHODS USED BY FUTURE TEACHERS FOR SOLVING PROBABILITY-BASED PROPORTION PROBLEMS

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We have extended two perspectives of proportional reasoning to solve problems based in probability. Four future middle grade teachers were enrolled in a mathematics content course that emphasized reasoning about multiplication with quantities. The course expected future teachers to generate and explain methods for solving proportions. Probability had not yet been discussed in the course, which gave insight into how reasoning fostered in the course was invoked in novel tasks. All four future teachers could reason about multiplication with quantities to solve probability problems and could use quantitative methods that accurately reflected how empirical probabilities approach theoretical probabilities as the number of trials increase.

Keywords: Mathematical Knowledge for Teaching, Middle School Education, Probability, Teacher Education-Preservice

Proportional relationships and ratios form an essential domain in elementary and secondary education (e.g., Kilpatrick, Swafford, & Findell, 2001; National Council of Teachers of Mathematics [NCTM], 2000) but are challenging concepts for students to learn (e.g., Lamon, 2007). Siegler et al. (2010) stressed the importance of teachers helping students conceptually understand proportion problems before utilizing rote cross-multiplication procedures. Similarly, the Common Core State Standards for Mathematics stated students should be able to "understand ratio concepts and use ratio reasoning to solve problems… by reasoning about tables of equivalent ratios, tape [strip] diagrams, double number line diagrams, or equations" (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, p. 42). Yet, studies have shown teachers have difficulties similar to their students when reasoning about proportional relationships (e.g., Orrill & Brown, 2012). Thus, it is important to understand how to better prepare teachers to reason about ratios and proportional relationships.

Additionally, standards documents (e.g., Conference Board of the Mathematical Sciences [CBMS], 2012; NCTM, 2000) emphasize the need for students to see mathematics as a coherent whole by developing connections across topics. To this end, ratios and proportional relationships can be situated within the multiplicative conceptual field (Vergnaud, 1994), which also includes multiplication, division, and fractions. Our project group has been investigating how future teachers (FTs) enrolled in content courses on multiplication develop a coherent perspective of the multiplicative conceptual field and construct viable arguments to solve tasks, especially using the multiple batches and variable parts perspectives on proportional relationships (Beckmann & Izsák, 2015). These two perspectives are explained in the theoretical framework section below.

In this study, we investigated how future middle grade teachers engaged in these two perspectives when generating and explaining methods to solve probability-based proportion problems. Due to the tendency for both students and teachers to solve missing-value proportion problems using the cross-multiplication procedure (Orrill & Brown, 2012), it is important to see if FTs can also reason quantitatively in these situations. Two potential ways FTs can reason quantitatively are through the use of the multiple batches and variable parts perspectives.

We focused on probability-based tasks for two reasons. The first is that we wanted to concentrate on a topic tied to the multiplicative conceptual field that had not yet been covered in the FTs' content courses on multiplication. By choosing a topic not covered in class, we could see if and how FTs utilized quantitative reasoning from instruction (i.e., the multiple batches and variable parts perspectives) in novel situations. Probability fit this criterion.

The second is that the variable parts perspective is rarely the focus of instruction (Beckmann & Izsák, 2015) but gives a more accurate representation of what transpires in probability tasks over a series of trials compared to the multiple batches perspective. For instance, if a spinner has five equal sectors and three of those sectors are red, the probability of the spinner landing on red is three-fifths. Yet, this does not mean as more spins are completed, three of each five consecutive spins will be red (a multiple batches approach). A more accurate representation is one based in the variable parts perspective: there are five equal-sized sectors and as the number of spins increases, each sector gets multiplicatively closer to containing the same fractional amount of the total spins (one-fifth). We mention "multiplicatively closer" because as more spins are completed, the additive difference between the number of spins in any two sectors may be growing. Therefore, as the number of spins grows, the number of spins landing on red will approach three-fifths of the total spins. On the other hand, as the number of spins increase, the fractional amount of the total spins in each of the sectors (the empirical probability) should be approaching the same number, namely the theoretical probability of landing in any sector.

Thus, there is merit in analyzing FTs' methods when solving probability-based proportion tasks. If FTs can access and utilize methods from the multiple batches or variable parts perspectives, it shows FTs can reason quantitatively in tasks that are typically solved using procedures. Further, if FTs can reason using the variable parts perspective, it shows the potential of introducing variable parts-based methods in FTs' mathematics content courses. To investigate this idea, we utilized the following research question: What methods do future middle grade teachers use when solving missing value proportion problems related to probability?

Theoretical Perspectives

In our courses with FTs, we use a quantitative definition of multiplication as a way to provide coherence among topics within the multiplicative conceptual field (Beckmann & Izsák, 2015). The definition of multiplication gives a foothold for FTs to reason about quantities in multiplicative situations and leads to four methods, two in each of the two perspectives (multiple batches and variable parts) on proportional relationships (Beckmann, Izsák, & Ölmez, 2015).

Quantitative Definition of Multiplication

Multiplication can be viewed quantitatively by considering equal-sized groups and units within those groups. We use this idea to produce the definition of multiplication in Figure 1. By consistently ordering the factors as multiplicand (N) times multiplier (M), the definition provides an organizational structure for tasks related to multiplication and allows for differentiation between multiplication, measurement (quotitive) division, and partitive division. An example is also given in Figure 1 to show how the definition of multiplication allows for this differentiation. Based on the choice of group from the situation, the definition of multiplication gives rise to four methods on proportional relationships found in the variable parts and multiple batches perspectives. We will now elaborate on these two perspectives and the four methods.

$$N \cdot M = P$$

How many base units make up one group exactly?

How many groups make up the product amount exactly?

How many base units make the product amount exactly?

Multiplication	**N • M = ?**	*If a baker can place 5 cookies on a plate and has 4 plates, how many cookies can the baker put on all of the plates?*	**5 • 4 = ?**
How-Many-Groups Division	**N • ? = P**	*If a baker can place 5 cookies on a plate, how many plates will the baker need to hold 20 cookies?*	**5 • ? = 20**
How-Many-in-1-Group Division	**? • M = P**	*If a baker has 4 plates to put cookies on and has 20 cookies total, how many cookies will the baker need to put on each plate?*	**? • 4 = 20**

Figure 1. A Quantitative Definition of Multiplication with an Example

Four Methods within the Two Perspectives on Proportional Relationships

In this section, we provide an overview of the two perspectives on proportional relationships and the four methods that arise from within these two perspectives. Due to page constraints, this overview is cursory. For a more thorough discussion of the two perspectives, see Beckmann and Izsák (2015). Similarly, for more information on the four methods, see Beckmann, Izsák, and Ölmez (2015).

The two perspectives on proportional relationships are known as *multiple batches* and *variable parts*. In the multiple batches perspective, two quantities measured by fixed units can be joined to form a "batch". A new pair of quantities, measured by the same units as above, is considered proportional to the original batch if the new pair of quantities is a multiple of the original batch. This corresponds to how-many-groups division in the definition of multiplication: the choice of a batch fixes the number of base units in the group (the multiplicand) and the number of groups (the multiplier) varies. In contrast, the variable parts perspective looks at proportionality by fixing the number of groups and varying the number of base units in the groups. This corresponds to how-many-in-1-group division in the meaning of multiplication. Beckmann and Izsák (2015) explained the variable parts perspective in the following way:

> …two quantities are said to be in the ratio A to B if for some-sized part there are A parts of the first quantity and B parts of the second quantity. In contrast to the first perspective [multiple batches], where A and B denote values based on a chosen measurement unit, here A and B specify fixed numbers of parts that can vary in size. (p. 21)

The difference between the two perspectives becomes more distinct when the four methods within these perspectives are shown. The *multiply unit-rate batch* and *multiply one batch* methods are situated within the multiple batches perspective and the *multiply one part* and *multiply whole amount* methods are situated within the variable parts perspective. Each of the four methods differs from the others by the choice of group. To clarify how these group choices affect the corresponding quantitative reasoning, we will consider the following example: *A spinner has five equal sectors with three of those sectors being red. If you spin the spinner 20 times, how many spins should land on red?* To explain how to solve this example, Figure 2 gives a math drawing and the corresponding definition of multiplication for each method. The group can be seen in blue and the number of base units in the group can be seen in red in each drawing.

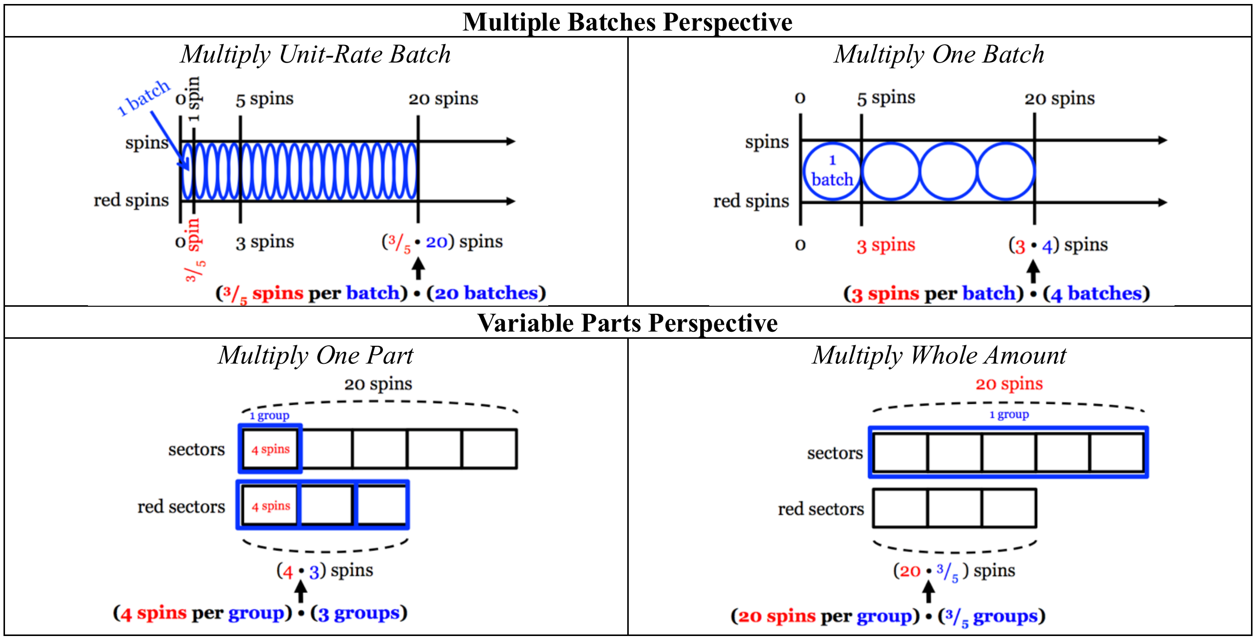

Figure 2. The Four Methods within the Two Perspectives

Data Sources

Data for this report come from a larger study at a public research university aimed at understanding how FTs in middle grade (4-8) and secondary (6-12) math education programs develop understanding of the multiplicative conceptual field. Participants in the larger study took part in six clinical interviews investigating ideas around the multiplicative conceptual field. Two cameras recorded each interview with one focused on a holistic view of the participant and interviewer and one focused on the participant's written work. The data for this report come from the sixth interview with four FTs in the middle grade program who were enrolled in the second content course focused on multiplication taught by the second author. The interview took place near the end of the Spring 2017 semester and focused on probability to see if the FTs could utilize multiplicative reasoning fostered in their content courses with yet-to-be covered topics. Each interview lasted approximately 75 minutes and the tasks used are found in Table 1.

Table 1: Tasks in Interview Involving Missing Value Proportion Problems

Task	Participant(s) Given Task	Wording of Task
1	All	If you were to spin the same spinner [on computer screen with 5 different colored sectors including red] 10,000 times or 100,000 times or 24 hundred thousand times, do you have any expectation about how many times it would land on red? Explain why or why not.
2	Nina Sophie Jack	If you were to spin the same spinner [on computer screen with 5 equal sectors, 3 purple, 2 blue] 10,000 times, 100,000 times or 240,000 times, do you have any expectations about how many times it would land on purple?
3	All	Suppose you want the spinner [5 equal sectors, 3 purple, 2 blue] to land on purple a total of 1500 times, about how many times should you expect to have to spin the spinner?
4	All	Please reconsider the spinner in Task 1. If you were to spin the spinner a bunch of times, say X times, or some specific large number of times, do you have any expectation about how many times it would land on red?

5	Sophie Molly	Please reconsider the spinner in Task 2. If you were to spin the spinner a bunch of times, say X times, or some specific large number of times, do you have any expectation about how many times it would land on red?
6a	Nina	If I wanted to get A spins to be yellow, what total number of spins would you tell me I should use? [Task is referencing a spinner with 10 equal sectors, 7 of which are purple, 3 are yellow]
6b	Sophie	Suppose I wanted to spin the spinner [referencing a spinner with 10 equal sectors, 7 of which are purple, 3 of which are yellow] enough times so that I got 2100 purples, could you give me a recommendation for how many total spins I should make?

Methods

Each interview was transcribed fully and augmented with relevant written work and hand gestures. All authors watched each interview alone and then as a group. The authors then identified all segments related to missing value proportion problems and coded these segments separately. Each author wrote notes and cognitive memos on (a) methods used, (b) concepts discussed related to the multiplicative conceptual field, (c) representations utilized, and (d) quotes that provided insights into methods/reasoning. At least two of the authors then met and discussed all authors' notes, referencing transcripts and videos as needed. Any discrepant interpretations were discussed until resolved. The first and second authors attended all meetings. These meetings led to the creation of the cumulative list of methods used by the participants seen in Table 2. The examples listed in the table do not exhaust all such instances of each method.

Table 2: Methods Used by the Participants During the Interview

Name of Method *Description*	**Example in Data**
Multiply One Part *Variable Parts Perspective* *See Figure 2*	"…we would take 1500 and divide that by 3, so that we know what each of the one-fifth parts are, because there are 3 one-fifth parts in this. And that will give us 500. So one-fifth of the total spins are 500 spins and… so then we'll multiply the 500 by 5 to get a whole, and that would be 2,500 spins." $1500 \div 3 = 500$ $500 \times 5 = 2500$ spins *–Molly, Task 3*
Multiply Whole Amount *Variable Parts Perspective* *See Figure 2*	"So if you have X spins, so if you're calling like X spins one round [annotates 'X' with 'x spins in 1 round']… then this would be one-fifth of that round [annotates '1/5' with '1/5 round']. So the question mark would be X… or not X. Question mark spins in one-fifth of a round [annotates '?' with '? Spins in 1/5 round']." $X \times \frac{1}{5} = ?$ x spins in 1 round; 1/5 round; ? spins in 1/5 round *–Molly, Task 4*
Multiply One Batch *Multiple Batches Perspective* *See Figure 2*	"Well, if you know that every 10 times you do it, it'll land on purple 7, that's like the same thing I was doing before where 10 times gives you 7 purple, how many spins gives you 2100 purple? That's multiplying by 300, so you'd have to do the same thing over here [to the 10 spins]." 10 spins = 7 P _____ 2100 P ×300 ×300 *–Sophie, Task 6b*

Multiply Unit-Rate Batch *Multiple Batches Perspective* *See Figure 2*	"If the spinner's going to land on a random spot for every spin, I think that for 1 spin, the chance is going to be three-fifths purple. And so, 10,000 spins, each one is also going to be three-fifths chance purple." $\frac{3}{5}(10,000)$ · purple *–Nina, Task 2*
Algebraic Manipulation *Writes a multiplication or division equation involving an unknown and uses numeric operations to solve for the unknown with no justification as to what the operations mean.*	Interviewer poses Task 6b, Sophie writes with no verbal justification: $X \cdot \frac{7}{10} = 2100$ $X = 2100 \times \frac{10}{7}$ $\frac{21000}{7}$ 2100 p *–Sophie, Task 6b*
Guess and Check *The participant guesses at a potential solution and uses numbers and operations in the context of the problem to check if the potential solution is correct.*	"And we want 1500 purple spins [writes '1500']. We're going to need, what, 3? We're going to need a minimum of 500 spins [writes '500']… other way around [crosses out '500', writes '4,500']. We'd need 45,000 [sic 4,500] spins. [writes '900 •3 = 2700'] Nope, that's not it [crosses out multiplication just written]…" 1500 ~~500~~ 4,500 *–Jack, Task 3*
"For Every" Language Connected with Fractions *Uses language such as "X for every Y" and states this is equivalent to either (a) "X/Y of the total", (b) writing the equation "(Total)•(X/Y)", and/or (c) writing the equation "(X/Y)•(Total)"*	"That if you spin it 10,000 times, 1 in every 5 times… or the red has the chance of being landed on 1 in every 5 times. So every 5 spins it has the chance of being landed on once, so that's the same as 10,000 times one-fifth." $10,000 \times \frac{1}{5}$ *–Sophie, Task 1*

Results

The results we present focus on the methods used by each FT when solving probability-based missing value proportion problems. Of note, all four FTs were able to engage in sound reasoning regarding the spins and spinners to solve the tasks. Additionally, all four FTs used at least one of the four methods from the two perspectives on proportional relationships during the interview, indicating they could reason about multiplication with quantities to solve probability problems. Lastly, though not always the initial method, all four FTs were able to use methods grounded in the variable parts perspective, showing this approach to proportional relationships is viable even in novel situations.

Sophie

Sophie's (all names are pseudonyms) initial method to solve four of her six interview tasks was not grounded in either of the two perspectives. Rather, Sophie used *"for every" language connected with fractions* to solve the first two tasks and *algebraic manipulation* for two other tasks (Tasks 3, 6b). When prompted for other methods to explain these tasks, Sophie justified her solutions from the multiple batches perspective, successfully explaining the *multiply one batch method* using both strip diagrams and ratio tables. Table 2 shows her typical reasoning for the *multiply one batch method*. For Sophie, using methods from the variable parts perspective only occurred when the questioning from the interviewer cued such reasoning. For instance, Sophie explained her equation, *X•1/5 = Red*, through the *multiply whole amount method* when asked to use the definition of multiplication in Task 4 and explained the *multiply one part method* only after being specifically asked to do so in Task 6b.

Jack

With the first two tasks, Jack justified his multiplication equations using *"for every" language connected with fractions*. He consistently discussed the spins landing on red as "1 out of every 5 times", an indicator of thinking about the spins from a multiple batches perspective. He approached Task 3 using *guess and check* (as seen in Table 2) but when that faltered, his language shifted from discussing the purple sectors as "3 out of every 5 times" to "3 parts of 5 that we need" and began reasoning using the *multiply one part method*. Using this method, Jack found the correct solution and provided justification with ease. Jack later revisited Task 3 and articulately explained the *multiply one part* and *multiply whole amount methods* using the definition of multiplication, drawings of spinners, and strip diagrams placed on a Cartesian coordinate system (a representation from the content course). Jack discussed his multiplication equation in Task 4 using *"for every" language connected with fractions* (again, discussing spins landing on red as "1 out of every 5 spins") but when asked about the definition of multiplication, he discussed his equation using the *multiply whole amount method*. Jack used multiple batches language throughout almost the entire interview and tended to justify his multiplication equations using such language. Yet, when these methods did not provide sufficient justification, Jack was able to discuss the two variable parts methods quite easily and with multiple representations.

Nina

Nina's use of the two perspectives was quite similar to Jack's. Nina completed the first two tasks using the *multiply unit-rate batch method* (see Table 2 for her typical explanation) and attempted Task 3 with the *multiply one batch method*. Though Nina found the solution, she was unable to justify it using her multiple batches method. After struggling for several minutes, her language shifted from discussing the purple sectors as "3 times for every 5 times" to "3 of my parts", which seemed to indicate a shift in thinking from a multiple batches perspective to a variable parts perspective. This shift allowed her to justify her answer using the *multiply one part method*. For the remainder of the interview, Nina did not use multiple batches methods again, nor did she use language that would indicate she was approaching the remaining tasks from the multiple batches perspective. Nina explained the *multiply one part method* in Task 4 when asked to use the definition of multiplication and though she used algebraic manipulation to initially solve Task 6a, she discussed the *multiply whole amount method* when asked for other methods.

Molly

Molly was the only FT to strictly use methods within the variable parts perspective throughout the interview. Molly regularly used how-many-in-one-group division to reason through the *multiply one part method* and was also facile in discussing the *multiply whole amount method*. A typical explanation of Molly's *multiply one part method* is in Table 2. When asked for differences between the two variable parts methods, Molly communicated these differences using equations, drawings of spinners, and strip diagrams. Molly needed no prompting to utilize the two methods within the variable parts perspective and used these methods to reason successfully throughout all interview tasks.

Conclusions and Implications

The results we present demonstrate that FTs taking a content course on multiplication can reason through missing value proportion problems using several methods that show conceptual understanding by reasoning with quantities. Thus, an increased focus on reasoning with quantities through a structured definition of multiplication could give FTs conceptual ways to approach problems that are frequently solved through rote procedure. Additionally, this study shows that FTs can use ideas from their content courses on multiplication to reason through novel tasks in the multiplicative conceptual field. This is promising because it indicates the FTs are beginning to see cohesiveness across topics within the multiplicative conceptual field and across mathematical ideas, which is a goal of recent standards documents (e.g., CBMS, 2012; NCTM, 2000). Though not typically the initial idea for the FTs, all four FTs could reason from the variable parts perspective and this perspective seemed to be advantageous at times compared to FTs' other methods. This result shows there is merit to giving FTs exposure to the variable parts perspective, a perspective that is typically overlooked (Beckmann & Izsák, 2015).

Future research in probability should explore the variable parts perspective. For example, a study could investigate if FTs can use the variable parts perspective to discuss the law of large numbers informally. Other studies could consider how FTs apply ideas from the multiplicative conceptual field to other mathematical areas, such as linear relationships and trigonometry. More specifically, investigating the accessibility and utilization of the variable parts perspective by FTs in these mathematical areas seems worthwhile. By investigating these topics, the field can gain a better understanding of FTs' abilities to reason quantitatively and the resources needed to support FTs in building connections between areas in the multiplicative conceptual field.

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References

Beckmann, S., & Izsák, A. (2015). Two perspectives on proportional relationships: Extending complementary origins of multiplication in terms of quantities. *Journal for Research in Mathematics Education*, 46(1), 17–38.

Beckmann, S., Izsák, A., Ölmez, I. B., (2015). From multiplication to proportional relationships. In X. Sun, B. Kaur, & J. Novotná (Eds.), *ICMI Study 23 Conference Proceedings*. Paper presented at The Twenty-Third ICMI Study: Primary Mathematics Study on Whole Numbers, University of Macau, Macau, China (pp. 518–525). Macau, China: University of Macao.

Conference Board of the Mathematical Sciences (2012). *The mathematical education of teachers II*. Providence, RI and Washington, DC: American Mathematical Society and Mathematical Association of America.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629–667). Charlotte, NC: Information Age Publishing.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author

Orrill, C. H., & Brown, R. E. (2012). Making sense of double number lines in professional development: Exploring teachers' understandings of proportional relationships. *Journal of Mathematics Teacher Education, 15*(5). 381–403. doi: 10.1007/s10857-012-9218-z

Siegler, R., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., ... & Wray, J. (2010). Developing Effective Fractions Instruction for Kindergarten through 8th Grade. IES Practice Guide. NCEE 2010-4039. *What Works Clearinghouse.*

Vergnaud, G. (1994). The multiplicative conceptual field: What and why? . In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 41–60). Albany, NY: State University of New York Press.

SECONDARY MATHEMATICS STUDENT TEACHERS' TYPES OF NOTICING WHILE TEACHING

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The purpose of this paper is to describe secondary mathematics student teachers' types of noticing while teaching. We discuss the importance of focusing on the interrelatedness of the noticing skills rather than reporting on individual skills separately. We apply the types of noticing to videos of student teachers to identify their ability to elicit and interpret student mathematical thinking in-the-moment while teaching. Results suggest that our student teachers did elicit and attend to student mathematical thinking while teaching, but how they interpreted the elicited student thinking varied. We hypothesize three reasons for why student teachers may have interpreted student mathematical thinking at a general level.

Keywords: Teacher Education–Preservice, Teacher Knowledge, Mathematical Knowledge for Teaching

Teaching is a complex activity that requires teachers to make purposeful in-the-moment decisions to attend to some activities while disregarding others. Sherin and Star (2011) noted that teachers are "bombarded with a blooming, buzzing confusion of sensory data" (p. 69) that he/she must sift through to make in-the-moment decisions to support student learning. For preservice teachers (PTs), the cacophony of sensory data can be more difficult to sift through than experienced teachers resulting in PTs becoming overwhelmed, focusing on small tasks (e.g., one group of students, one solution strategy, one students' voice) and neglecting or missing other important aspects of teaching (e.g., multiple ways of student thinking, actions of all students in the class).

Researchers have used noticing to focus teachers' attention on important aspects of teaching; however, because teachers tend to notice a variety of information in a classroom (e.g., teacher actions, student actions, classroom management, posters on the wall) guidance on what to notice, especially for PTs, is necessary (Jacobs, Lamb, Philipp, & Schappleel, 2011; Sherin & van Es, 2005; Star & Strickland, 2008; Star, Lynch, & Perova, 2011; Stockero, 2014). Jacobs, Lamb, and Philipp (2010) extended the construct of teacher noticing to —professional noticing of children's mathematical thinking, which is conceptualized "as a set of three interrelated skills: *attending* to children's strategies, *interpreting* children's understanding, and *deciding* how to respond on the bases of children's understanding" (Jacobs et al., 2010, p. 172, *emphasis added*).

Mathematics education researchers have used Jacobs et al.'s (2010) framework to learn what teachers attend to and how teachers interpret and respond to children's mathematical thinking (e.g., Schack, Fisher, & Wilhelm, 2017). As a field, we have learned more about professional noticing, specifically researchers often focus on one of the three interrelated skills (attending, interpreting, or responding) in a reflective setting, thus artificially compartmentalizing these interrelated skills. We argue in this paper that investigating *the interrelatedness* of the noticing skills is critical to understanding how teachers apply these skills when teaching. We also share how we, as researchers, utilized professional noticing to determine how our secondary mathematics student teachers elicited and acted on student mathematical thinking (SMT) in-the-moment <u>*while teaching*</u>.

In this paper, we report on the following research question: To what extent do secondary student teachers elicit SMT and interpret that thinking in-the-moment while student teaching?

Theoretical Framework

To analyze and interpret our data we frame our view of eliciting and acting on SMT through Piaget's (1955) construct of decentering and Teuscher, Leatham, and Peterson's (2017) types of noticing.

The construct of decentering provides a powerful lens for examining student-teacher interactions because it focuses on the teacher's interpretations of students' verbal and written explanations to make in-the-moment decisions while teaching. We seek to draw inferences about how student teachers' understanding of SMT relates to their actions in-the-moment of teaching. According to Piaget (1955) decentering characterizes how the actions of an observer (i.e., student teacher) attempting to understand how an individual's (i.e., student's) perspective differs from his/her own. Steffe and Thompson (2000) and Thompson (2000) distinguished two ways in which individuals interact with others, which are described by the type of model that an individual creates of other's thinking. Thompson (2000) described individuals as participating unreflectively or reflectively. Individuals who participate unreflectively were describe as creating first-order models. "The models an individual constructs to organize, comprehend and control *his or her experiences*, i.e., their own mathematical knowledge" (Steffe & Olive, 2010, p. 16 *emphasis added*). Whereas, individuals who participate reflectively were described as creating second-order models, attempting to elicit and interpret SMT, and assist students in furthering their thinking based on the students' perspectives, not their own (i.e., decentering).

Teuscher et al. (2017) extended Jacobs et al. (2010) professional noticing framework describing types of noticing. These types of noticing were based on student teachers' written journal entries completed during student teaching. Attending to SMT was divided into two categories: *general observation* or *student mathematical thinking*. Interpreting SMT was also divided into two categories: *general interpretation* or *root interpretation*. Four types of noticing specifically connect the two skills of *attending to* and *interpreting* SMT. They were: (1) general observation, no interpretation; (2) general observation, general interpretation; (3) student mathematical thinking, general interpretation; and (4) student mathematical thinking, root interpretation (see Teuscher et al., 2017 for specific definitions). We seek to apply these four types of noticing to videos of student teachers by identifying their ability to elicit and interpret SMT in-the-moment <u>*while teaching*</u> (e.g., decentering). Specifically, we are interested in identifying what SMT student teachers attend to while teaching and in what ways they interpreted the SMT that was available to them.

Methods

Pairs of student teachers where assigned a 16-week placement at a junior high or high school. The university supervisor, first author, observed the student teachers once a week and provided feedback during their student teaching experience. Data were collected on four pairs of student teachers (eight student teachers) during two different semesters. The researchers purposefully selected four pairs of student teachers to allow for variability in the data set. All eight student teachers' lessons observed by the university supervisor were videotaped. The researchers analyzed three lesson videos (beginning, middle, and end of student teaching) for each student teacher, 24 total, and coded each using the *Practice of Probing Student Thinking* Framework (Teuscher, Switzer, & Morwood, 2016) to identify instances where student teachers appeared to elicit and act on SMT. We chose probing student thinking because it allowed the researchers to identify instances of SMT and possible interpretations of the student thinking by the student teachers.

To identify a subset of instances of probing student thinking to analyze, we compared the researchers' instances to the student teachers' instances of probing student thinking and found 27 overlapping instances, which indicated that student teachers seemed aware of SMT. Using the researchers' instances from the overlapping instances, we conducted our analysis to determine how the student teachers were eliciting and acting on SMT in-the-moment while teaching. Our unit of analysis was an individual video instance of probing student thinking, defined as a complete interaction between a teacher and student(s). The researchers used the types of noticing framework (Teuscher et al., 2017) to identify the elicited SMT in the video instances and inferred the student teachers' interpretations of the SMT based on their response to the SMT in the video.

In the following section, we provide illustrative examples from our data set to describe these different types of noticing. Video instance 1 is the transcript for one probing student thinking instance that occurred near the end of a high school mathematics class. The high school students spent the class period learning about graphing quadratic inequalities and using the graphs to find the solutions to the quadratic inequalities.

Student Teacher: So, what did we notice the difference between these two answers [one answer was $x<5$ or $x>11$ and the other answer was $-5<x<1$]? The one from our previous problem and this one? What's the difference? Student 1?
Student 1: Um, it's not like infinity on it so it doesn't go on forever, it stops.
Student Teacher: Yeah, and what did we notice about the equations at the top of number nine and ten? How are they different in terms of the signs? Yeah, Student 2?
Student 2: One [function] is showing like one is less than zero and one [function] is greater than zero so that changes when it's outside or inside.
Student Teacher: And when do we know if it's going outside or inside?
Student 3: If it's greater or less than zero. Wait, what?
Student Teacher: Okay, say, what happens when it's greater than zero?
Student 3: Then the arrows on the outside go inside.
Student Teacher: Um, is it, let's see, I think in this case we also had our, both our parabolas, which way are they opening?
Students: Up.
Student Teacher: Yeah, so we want to make sure and graph and make sure they were opening up, because if it was opening down we might have a different answer.

We identified video instance 1 as a *General Observation* and *General Interpretation*. The general observation is because we could not infer the SMT. We have some idea that students are discussing whether a quadratic function is greater than zero or lesson than zero, but we were unable to determine what students meant when they said, "*that changes when it's outside or inside*" (Student 2) or "*then the arrows on the outside go inside*" (Student 3). The general interpretation was based on the student teacher's response to Students 2 and 3 because she seemed to carry the discussion on without seeking clarification of what students meant by "*that changes when it's outside or inside*" (Student 2) or "*then the arrows on the outside go inside*" (Student 3).

Video instance 2 is the transcript for another probing student thinking instance that occurred at the beginning of a middle school mathematics class. Students were learning about unit rates, constant growth rates, and linear equations. Students were shown a graph of a linear function (number of hands as a function of number of people) and asked to answer the following three questions: (1) What is the unit rate and what does it represent? (2) If there are 6 hands, how many people are present? (3) How many hands are there if there are 52 people? The transcript is from the class discussion about the first question.

Student 1: I was looking at this one right here [points to the point] this dot right here (1, 2), and since it is the lowest one that is not (0, 0) that is the unit rate because all you need to do is multiply it by two and it will keep going on forever.
Student Teacher: Multiple what by two?
Student 1: Multiply people and hands by two.
Student Teacher: People and hands? I am not sure I understand what you mean.
Student 1: One person times two is two people, well (pause). You add one person here then you add two hands. Whenever you go up one dot it goes add one person then it goes add two hands.
Student Teacher: Ok, so you are saying…
Student 2: There is a constant growth pattern.
Student Teacher: [revoicing] There is a constant growth pattern that you add one here (pointing to the person axis) and you add two here (pointing to the hands axis).
Student 1: Yes
Student Teacher: And then continue on?
Student 1: Yes
Student Teacher: Okay. Okay. Awesome, so your unit rate is what?
Student 1: One to two.
Student Teacher: [revoicing and writes on the board: 1 to 2], One to two, and what are your units?
Student 1: Units?
Student Teacher: What does this 1 represent [pointing to the 1 on the board]
Student 1: People. People. People to hands

We identified video instance 2 as Student Mathematical Thinking and Root Interpretation. The SMT was "I was looking at this one right here [points to the point] this dot right here (1, 2), and since it is the lowest one that is not (0, 0) that is the unit rate because all you need to do is multiply it by two and it will keep going on forever" (Student 1). In other words, the student is saying that you can use the point on the line closest to (0, 0) to determine the unit rate, which is the point (2, 1) to which he points. We inferred that the student teacher made a root interpretation – a more in-depth analysis of what the students might have meant by their utterance, or what that thinking means with respect to student understanding – because the teacher was unsure what Student 1 referred to when he states, "multiple it by 2." Therefore, the student teacher asks clarifying questions to which the student responds with "One person times two is two people, well (pause). You add one person here then you add two hands. Whenever you go up one dot it goes add one person then it goes add two hands" (Student 1). The student teacher seemed aware of the difficulty that middle school students have when beginning to work with multiplicative relationships. Following up on the statement, "you multiply it by 2" (Student 1), allowed Student

1's thinking to be made public. In responding, Student 1 begins to think about the multiplicative relationship between the number of hands and people. As he is explaining his thinking he realizes he is not correct and reverts to an additive relationship for which he is familiar.

Results

Table 1 displays the percentage of video instances based on the types of noticing. We found that merging the separate noticing skills captured the interrelatedness of attending and interpreting that reveals important and distinguishing aspects of how the student teachers acted on SMT. Of the 27 video instances the researchers coded 23 (85.2%) provide evidence of the student teachers eliciting SMT and 4 video instances (14.8%) provided evidence of the student teachers making a general observation of the SMT. However, of the 23 video instances that include evidence of SMT, 14 video instances (51.9%) include evidence of a root interpretation and 9 video instances (33.3%) include evidence of a general interpretation. Therefore, the results indicate that the student teachers were highly successful in eliciting SMT, but often did not generate a root interpretation of the elicited SMT.

Table 1: Student Teachers' Video Instances by Types of Noticing

Attending	Interpreting	Instance Count	Percent
Student Mathematical Thinking	Root Interpretation	14	51.9%
	General Interpretation	9	33.3%
General Observation	General Interpretation	4	14.8%
Total		27	100.0%

For the remainder of this report, we focus our analysis on the nine video instances where the student teachers elicited SMT but made a general interpretation. In these instances, there was no evidence that the student teachers generated a root interpretation of the elicited SMT. We were interested in identifying potential factors that may have led to the student teachers not generating a root interpretation. We share two transcripts that are representative of the SMT with general interpretation type.

Video instance 3 occurs at the beginning of a high school lesson. Students were working on a task to determine if different situations were fair or unfair to assist them in generating a mathematical definition of fair. The following transcript provides the conversation between the student teacher and a student.

Student: I think it is fair, but I am trying to think how it might not be fair?
Student Teacher: What is your definition of fair?
Student: It would be more fair if everyone got a prize or something, but this isn't the type of thing where everyone could get, I don't know, I'm thinking realistic.
Student Teacher: So, are you reconsidering the definition then of fair?
Student: Kind of, literally fair is everyone gets it.
Student Teacher: Gotcha, so like equality. Like everyone would get one?
Student: But this scenario is confusing me; it wouldn't be fair if they said that the tallest person gets the prize if they are short.
Student Teacher: Gotcha, so then that would be bias against the short people, right?
Student: Yeah, so this is, I don't know, this is like everyone has the same opportunity.
Student Teacher: (revoicing) Okay, so everyone has the same opportunity. So, does this one fit that definition of fair, this scenario?

Student: Yeah, everyone got a ticket and everyone had the, like, gets the opportunity to be picked, it's just not everyone gets picked.
Student Teacher: Right so we all have, so you and I might be in the drawing and we might not both get picked, but we have equal chances?
Student: Yeah
Student Teacher: Which is what you were saying? As long as you explain I am ok with whatever you come up with.

Our analysis of video instance 3 found that the student teacher attended to SMT multiple times. The student teacher asked the student if she was reconsidering her definition of fair. The student suggested that "*fair means everyone gets a prize,"* which is a common definition of fair but not the mathematical definition. Then the student seems confused because she thinks "*fair means everyone has the same opportunity*." While the student teacher seemed to identify that the student had two definitions of fair, how the student teacher acted on this SMT was revealing. The student teacher indicates to the student that it does not matter which way she thinks about it as long as she can explain her thinking. In other words, both of the student's definitions were acceptable. Throughout the lesson multiple students continued to bring up and discuss these two definitions of fair and the student teacher never distinguished the difference between the definitions. In concluding the lesson, the student teacher stated, "as long as we can explain why whatever method we use was fair then it should be fair, right?" We inferred that the student teacher's mathematical meaning for fair (Byerley &Thompson, 2017) may have resulted in her interpreting the SMT at a general level.

Video instance 4 occurs in a high school lesson where students were learning about probability. The students were given the following problem to work on in groups.

> Two students, Lee and Rory, find a box containing 100 baseball cards. To determine who should get the cards, they decide to play a game with the following rules:
> - One of the students repeatedly flips a coin.
> - When the coin lands heads up, Lee gets a point.
> - When the coin lands tails up, Rory gets a point.
> - The first student to reach 20 points wins the game and gets the baseball cards.
>
> As Lee and Rory are playing the game they are interrupted and are unable to continue. How should the 100 baseball cards be divided between the students given that the game was interrupted at the described moment? When they are interrupted Lee has 19 points and Rory has 17 points.

A student (Bill) shares with the class that Rory should get 17/36 of the cards and Lee should get 19/36 of the cards. Other students in the class respond with "*what?*" and "*where did the 36 come from?*" The following transcript captures the ensuing conversation between the student teacher and the students in the class.

Student Teacher: Okay, [student 1] has a question about where the 36 came from?
Bill: You take the 19+17 to get the 36.
Student Teacher: So how many cards do you give each person than?
Bill: Um, still trying to figure that out.
Student Teacher: (Writes $\frac{17}{36} * \frac{x}{100} =$on the board) and how many cards do they each have?
Student 2: The first one is 52
Student Teacher: (revoices) 52 for Rory

Student 3: 52 for the other one 47 for that one
Student Teacher: (writes on the board x=52)
Student 4: 52.7 and 47.2
Student Teacher: (revoices) 52.7 and 47.2 and writes them on the board.
Student 3: That should actually be 0.8.
Student Teacher: Which one?
Student 3: That one (points to the Rory column)
Student Teacher: (Erases the 52.7 and writes 52.8), Kay, student, what do you think we should do with the 0.8 and 0.2 of a card?

Our analysis of video instance 4 found that the student teacher attended to SMT multiple times. The student teacher asked the Bill to explain where the 36 came from in response to another student's question. Students provide the number of cards that they calculated Rory and Lee should receive. The student teacher seems to identify that Bill has based the number of cards that Rory and Lee should receive on their respective theoretical probabilities of winning. The student teacher accepts Bill's strategy and immediately moves to asking the class what they think they "*should do with the 0.2 and 0.8 of a card,*" assuming the rest of the students, those who initially did not understand Bill's reasoning, now understood his thinking. We inferred that the student teacher's not decentering (Piaget, 1955; Teuscher, Moore, & Carlson, 2016) may have resulted in her interpreting the other students' mathematical thinking at a general level.

Table 2 displays the three categories for the nine video instances where student teachers elicited SMT but made a general interpretation of the elicited SMT. The majority (66.7%) of video instances fell in the category of not decentering. As in video instance 4, the student teacher only seemed concerned with making sure that everyone had the correct answer, rather than helping students understand how Bill (the student) had come up with the answer and why that made sense. In two video instances (22.2%) we determined that the general interpretation was appropriate. This was because the student teacher was launching an activity where students later explored the ideas that they had shared. The goal during a launch of a task is to make student thinking public so it can be discussed; therefore, we felt that at that point in the lesson the general interpretation was appropriate. The last category was the influence of the student teacher's mathematical meaning as described in video instance 3.

Table 2: Possible Reasons Secondary Student Teachers' May Not Generate a Root Interpretation

Inferences for why student teachers make a general interpretation	Count	Percent
Not Decentering	6	66.7%
Appropriate	2	22.2%
Mathematical Meaning	1	11.1%

Implications

The purpose of this paper was to describe secondary mathematics student teachers' types of noticing while teaching. We demonstrated the importance of the interrelatedness of the attending and interpreting noticing skills because the types of noticing revealed important and distinguishing aspects of how the student teachers were acting on SMT. We found that our student teachers were highly effective with eliciting SMT in-the-moment of teaching. However, we found that our student teachers were less successful in interpreting the elicited SMT in-the-moment of teaching with a root interpretation. While there were two video instances where it was appropriate for the student teachers to interpret at the general level, most video instances were student teachers interpreting at the general level that were because the student teachers were not decentering.

While our analysis is a small subset of video instances from eight student teachers, we believe that future research can focus on identifying differences among student teachers to identify what practices PTs need to focus on during their course work that will prepare them to attend to and interpret SMT in such a way that will build on all students mathematical thinking and improve student learning in their classrooms. We recommend that teacher educators need to assist PTs in learning how to interpret SMT from the perspective of the students (e.g., decentering). While we would agree that this is a challenging skill for PTs to develop, we had student teachers who identify a root interpretation of the SMT in-the-moment while teaching, which allowed the student teacher to orchestrate a productive discussion in their mathematics classroom with all students.

References

Byerley, C., & Thompson, P. W. (2017). Secondary mathematics teachers' meanings for measure, slope, and rate of change. *The Journal of Mathematical Behavior, 48*, 168-193. doi:10.1016/j.jmathb.2017.09.003

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*, 169-202.

Jacobs, V. R., Lamb, L. L. C., Philipp, R. A., & Schappelle, B. P. (2011). Deciding how to respond on the basis of children's understandings. In M. G. Sherin, V. R. Jacobs & R. A. Philipp (Eds.), *Mathematics teacher noticing* (pp. 97-116). New York: Routledge.

Piaget, J. (1955). *The language and thought of the child*. Cleveland: Meridian Books.

Schack, E. O., Fisher, M. H., & Wilhelm, J. (2017). *Teacher Noticing: Bridging and Broadening Perspectives, Contexts, and Frameworks* J. Cai & J. A. Middleton (Eds.), *Research in Mathematics Education* doi:10.1007/978-3-319-46753-5

Sherin, B., & Star, J. R. (2011). Reflections on the study of teacher noticing. In M. G. Sherin, V. R. Jacobs & R. A. Philipp (Eds.), *Mathematics teacher noticing* (pp. 66-78). New York: Routledge.

Sherin, M. G., & van Es, E. A. (2005). Using Video to Support Teachers' Ability to Notice Classroom Interactions. *Journal of Technology and Teacher Education, 13*(3), 475-491.

Star, J. R., Lynch, K., & Perova, N. (2011). Using video to improve preservice mathematics teachers' ability to attend to classroom features: A replication study. In M. G. Sherin, V. R. Jacobs & R. A. Philipp (Eds.), *Mathematics teacher noticing* (pp. 117-133). New York, NY: Routledge.

Star, J. R., & Strickland, S. K. (2008). Learning to observe: using video to improve preservice mathematics teachers' ability to notice. *Journal of Mathematics Teacher Education, 11*:107-125. doi:10.1007/s10857-007-9063-7

Steffe, L. P., & Olive, J. (2010). *Children's Fractional Knowledge*. New York: Springer.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research Design in Mathematics and Science Education* (pp. 267-306). Hillsdale, NJ: Erlbaum.

Stockero, S. L. (2014). Transitions in prospective mathematics teacher noticing. In J. J. Lo, K. R. Leatham & L. R. Van Zoest (Eds.), *Research trends in mathematics teacher education* (pp. 239-259). Cham, Switzerland: Springer International.

Teuscher, D., Leatham, K. R., & Peterson, B. E. (2017). From a Framework to a Lens: Learning to Notice Student Mathematical Thinking. In E. Schack, M. H. Fisher & J. Wilhelm (Eds.), *Teacher Noticing: Bridging and Broadening Perspectives, Contexts, and Frameworks* (pp. 31-48). Cham, Switzerland: Springer, Cham.

Teuscher, D., Moore, K. C., & Carlson, M. P. (2016). Decentering: A construct to analyze and explain teacher actions as they relate to student thinking. *Journal of Mathematics Teacher Education, 19*: 433-456. doi.org/10.1007/s10857-015-9304-0.

Teuscher, D., Switzer, J. M., & Morwood, T. J. (2016). Unpacking the practice of probing student thinking. *Mathematics Teacher Educator, 5*(1).

Thompson, P. W. (2000). 18 Radical Constructivism: Reflections and Directions. *Radical constructivism in action: Building on the pioneering work of Ernst von Glasersfeld*, 291.

REIMAGINING DEFINITIONS OF TEACHING MATHEMATICS FOR SOCIAL JUSTICE FOR PRESERVICE SECONDARY MATHEMATICS TEACHERS

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We investigate definitions and approaches that preservice secondary mathematics teachers develop about teaching mathematics for social justice while participating in an equity-based, anti-racist professional development program. We analyzed transcripts from seminar sessions of the program where participants discussed different forms of teaching mathematics for social justice. Participants described their possible teaching with what is seen in the literature while moving beyond in formulating other possible representations of teaching mathematics for social justice. These findings suggest that it may be necessary to further theorize our understanding of teaching mathematics for social justice so that representations include everyday practices beyond representations in curriculum.

Keywords: Teacher Education, Equity and Diversity, Teacher Knowledge, Social Justice Mathematics

Mathematics education continues to attempt to address injustices in multiple ways as we look for possible solutions through teaching mathematics for social justice. Some would argue providing all students with access to rigorous mathematics is an act of social justice (Moses et al., 1989; Gregson, 2013). Critical scholars suggest we change the curriculum, using mathematics as a tool to analyze injustices in society and to propose math-based solutions (Frankenstein, 1983, Gutstein 2005). This approach has become the dominant definition of teaching mathematics for social justice (TMSJ). It allows students to develop other life skills (e.g., questioning authority) that mastering "dominant mathematics" (Gutiérrez, 2002; Valero, 2004) does not guarantee (Bartell, 2013; Felton-Koestler, 2015; Gregson, 2013; Gutstein, 2006). However, carrying out this version requires knowledges, skills, and life experiences that many teachers lack. This paper investigates how pre-service secondary mathematics teachers (PMST) are developing their definitions and approaches to what TMSJ as they become educators in the field. Our current understanding and implementation of teaching mathematics for social justice have plateaued due to neoliberal educational values (e.g., individualized learning, privatization through charter schools, increase focus on profitizing through universal curricula models), that drive mathematics education. It is safe to say that there is a need to deeper theorize social justice in mathematics education. This paper seeks to understand the range of views of TMSJ that pre-service secondary mathematics teachers embrace after being exposed to a variety of anti-racist and social justice-oriented media and individuals.

Current Understanding of Teaching Mathematics for Social Justice

Mathematics education has a persistent history where students who are Latinx, Indigenous and Black have not had access to a rigorous mathematics curriculum thereby limiting the doors and economic possibilities that they can pursue in society. Some would argue that merely providing minoritized students access to dominant mathematics and science, technology, engineering and mathematics (STEM) pipeline is a form of social justice (Brantlinger, 2013; Felton-Koestler, 2015). The Algebra Project is one prominent example of transforming algebra curricula to create an entry point for African American students (Moses et al., 1989). Through grassroots organizing, teachers, students, and parents established a pedagogy of mathematics that supported students to participate in economic and technological changes (Moses et al., 1989). Other researchers agree and have defined social justice as "empowerment, both for teachers and for their immigrant students" to succeed in dominant mathematics (Planas & Civil, 2009) and highlight the importance of students being able to navigate mathematics as a gatekeeper in the real-world (Gregson, 2013).

Definitions of social justice or critical mathematics vary depending on the populations that adopt them, the worldviews espoused, and situations at hand (Gates & Jorgenson, 2009; Stinson & Wager, 2012). For the most part, what gets counted as social justice mathematics teaching involves changing the curriculum (Bartell, 2013; Brantlinger 2007; Gregson, 2013; Gutstein, 2006) to take up examples of real-life situations (Felton, 2010) or to use mathematics to read the world and possibly write the world (Gutstein, 2006; Frankenstein, 1990; 1995). For example, Gutstein (2006) had his students survey the community on experiences with the police to calculate the likelihood that a police officer will pull over a brown or Black person, versus a white person, when driving a car in a given neighborhood, while Frankenstein (1990, 1995) had her low-income college students use mathematics to examine discrimination in costs of electrical power, home mortgage distribution, and the tax system to highlight how mathematics is sometimes used to obscure economic, political, and social issues. In fact, the inclusion of ethnomathematics lessons can be seen as a form of social justice curriculum (Borba, 1990; Powell & Frankenstein, 1997) because ethnomathematical knowledge decenters a Western and colonial view of mathematics (Gutiérrez, 2017).

Critical Perspectives to Teaching Mathematics for Social Justice

While creating access to dominant mathematics is essential for Black, Latinx, Indigenous and other minoritized students, as well as interrogating injustices that these communities may experience, some would also argue for critical representations of mathematics. Such representations move beyond using social justice mathematics as curriculum and disrupt dominant social narratives and representations of mathematics (Gutiérrez, 2017; Skovsmose, 1994). Teachers and students who participate in these types of critical representations of mathematics are participating in "tactics of resistance" (Gregson, 2013) of social and institutional inequities replicated in the classroom. Providing teachers with opportunities to envision "tactics of resistance" are limited within our current literature of TMSJ except for few. That is, students need to learn mathematics to participate in the neoliberal capitalist agenda in society while being able to critique it and "play the game/ change the game" (Gutiérrez, 2008).

Teacher Practices of Social Justice Mathematics

Beyond offering a sociopolitical context of mathematics units and lessons, it is difficult to see what other forms of social justice teachers can implement in their classrooms. Although there is a strong emphasis on changing the curriculum (e.g., adopting sociopolitical contexts for learning mathematics in particular activities or units), in fact, there is a range of practices that contribute to social justice mathematics teaching. One such form is pedagogy of questioning—a "classroom environment… co-created by students and teachers [where] students have opportunities to pose their own real, meaningful questions" (Gutstein, 2008, p. 55). Pedagogy of questioning is not necessarily curricular driven but allows students to see the importance of asking questions and challenge the notion that teacher knows all. Another practice is developing relationships with students (Planas & Civil, 2009 Gutstein, 2008) in non-stereotypical ways (Bartell, 2013) by being aware of students' realities to empower their success and participation in mathematics classrooms. Empowering students includes being able to recognize the need to address race and gender issues as institutional issues (Frankenstein, 1990; 1995). Understanding students in non-stereotypical ways are important because it pushes teachers to anticipate the types of responses that students can have in a lesson (Bartell, 2013) and support the emotions that can arise (Boylen, 2009).

While developing relationships with students and implementing pedagody of questioning are valid representations of everyday tactitcs that teachers can do on a daily basis, one has to question if they are realistic in the current mathematics classroom? Research studies that address perceptions held by teachers implementing TMSJ highlight how many can get discouraged by the difficulty of implementing such lessons (Bartell, 2013) or feel it is difficult to implement while trying to keep the rigor of higher-level mathematics (Brantlinger, 2013). The increasing demand of high stakes testing presents time constraints for implementing social justice mathematics lessons, resulting in "end of the year activities"—making it seem like it is just an add-on, or afterthought, to more crucial traditional mathematics curriculum. There are limited representations of what other practices could look like. The exception to this are representations of the pedagogy of questioning (Gutstein, 2008), creating relationships with students, and challenging one's beliefs and identities (Bartell, 2013; Boylen, 2009; Gregson, 2013 Gutstein, 2008;). These constraints limit pre-service teachers being able to become a teacher that teaches mathematics for social justice if they feel like it requires entirely or mainly changing the curriculum, something for which they may feel they have no power to do in an era of high stakes education. Furthermore, we need to ask how

Methods

Data

The data in this paper comes from an equity-based professional development program in a Midwestern University where two cohorts of undergraduate pre-service mathematics scholars (juniors and seniors) interacted. Scholars received professional development including bi-weekly seminars, mentoring sessions, a partnership with a Chicago teacher, an after-school mathematics club, and critical professional development. In seminars, scholars discussed readings/videos about issues of racism and racialization, white supremacy, gender, social justice, and mathematics; learned about mathematics in nontraditional ways; engaged with guest speakers, and interacted with alumni.

Scholars consisted of two cohorts of junior and senior scholars. Junior scholars were in the first year of the program while seniors were in their second year. Juniors consisted of six scholars, of which, three self-identify as white women, one as cis-gender Latinx man, one as Asian women and one white man. Seniors consisted of seven scholars, one self-identifies as an Asian woman, and six as white women. Two self-identify as queer. Most of the scholars originated from the suburbs of a major Midwestern City and one originated from a rural town in a Midwestern state. All scholars received a scholarship in exchange for committing to teach in a school in a community of cumulative disinvestment for four years following graduation.

We focus upon two different 3-hour seminar sessions: "Teaching Mathematics for Social Justice (part 1 and 2)." Table 1 shows the topics and participants. Before coming to the seminar, scholars wrote a letter to Gutstein, reflecting on his teaching and their stances on TMSJ, especially what was appealing or concerning for them.

Table 1

Seminar Activity Topics

Activity	Participants	Materials Covered

Day 1			
	Book Discussion Part 1	Juniors	Reading and Writing the World with Mathematics
	Study of SJM Materials	Seniors	Social Justice Math Unites created by Brian Lake
	Conversation with Scholars	All	Social Justice Math Unites created by Brian Lake
Day 2			
	Book Discussion Part 2	Juniors	Reading and Writing the World with Mathematics
	Study of SJM Materials	Seniors	Guide for Integrating Issues of Social and Economic Justice into Mathematics Classrooms and Curriculum
	Questions and Stances	All	Notes and Summaries from split sessions

Analysis of Session Audio and Materials

We placed audio recorders in multiple parts of the rooms where discussions took place. We then transcribed verbatim, focusing on the relevant conversations about social justice mathematics. We transcribed a total of 140 minutes for each of the two sessions, using codes that arose in the literature. Some of these codes include references to Social Justice Mathematics, (e.g., reading the world, writing the world, challenges, curriculum) and Classroom Norms (e.g., group work, critical thinking). Other categories such as Ethical Teacher Practices (e.g., political clarity, stance, windows/ mirrors) came from existing categories of codes that we have as a project or literature our research team has produced. During the first round of coding, we captured themes that confirmed existing trends in the literature of social justice mathematics as well as highlighted themes not captured in the literature. For example, we created the code ETP-spectrum to highlight when the scholars brought up ideas of having multiple possible stances on a sociopolitical topic represented in their classroom—a representation of teaching mathematics for social justice that Brian Lake brought up in his conversation with the scholars. During the second layer of coding, we looked for other things the scholars considered as representations of everyday social justice practices. It is important to note that the researchers never directly asked for the scholars to list their explicit definition of what they consider teaching mathematics for social justice, as we worried that they would simply tell us what they believed we wanted to hear. Instead, we looked for examples of the scholars talking about the readings or their everyday practices that reference their stances of being social justice educators who can continuously push back on dominant narratives of what happens in the classroom. We then clustered the codes into different themes that showed a representation of current definitions of teaching mathematics for social justice as well as representation of everyday practices that teachers can enact their social justice teaching stances (Emerson, Fretz, Shaw, 2011). These themes include redefining mathematics, delegating authority, and examples of teaching students to be critical participants in society. Though our scholars held definitions and stances like some found in other studies, we focus the reporting of our findings on concepts not typically or explicitly addressed in empirical studies of teachers enacting TMSJ.

Findings

TMSJ as Redefining Mathematics

One way that the scholars look at TMSJ was the way they interpreted the role of mathematics and the practices that surround mathematics. Scholars brought up multiple ways of being able to redefine mathematics. One example is making mathematics applicable across school subjects.

> *Catherine:* I think it would be more beneficial **if we actually gave them examples of how to apply it. Connect it to these types of problems that are actually happening in the**

> **world…. Like [Mathematics] wouldn't have to be a separate thing**. Schools would be better off…if we are on the same page with stuff, connect the ideas, and use what other teachers are teaching.

Catherine states that some schools and teachers treat mathematics as its own "separate thing" within the curriculum. For her, having mathematics disconnected from other subjects does not allow students to see how mathematics relates to other subjects and can be applied in solving problems in the world. TMSJ as teaching math in a holistic approach would not only benefit students but the whole school to by applying all subjects to solve problems in the world. This way of redefining mathematics would not only benefit how students learn in the classroom but also support collaboration with teachers across content areas.

For the scholars, TMSJ means redefining mathematics so that students can develop different perspectives of what is the role of mathematics in increasing opportunities for students to create and understand mathematics around them. The scholars are beginning to formulate that the role of mathematics is not only providing instrumental access to mathematics as a curriculum, but it is also teaching mathematics in a holistic approach as well as teaching students to provide a spectrum of solutions. By opening different ways of knowing mathematics and who gets to be knowers, Black, Indigenous and Latinx students can not only see themselves as having something to say but are also encouraged to develop new mathematics, a way in which students can also be authors of the world.

TMSJ as Delegating Authority

In this section our scholars talk about TMSJ as finding different ways to delegate authority in the classroom. Whether it was making decisions about curriculum or facilitating discussions, delegating authority is a way to empower students in the classroom. During the conversation of scholars one scholar brings up

> *Wendy:* I am considering **redefining the structure of the roles of students in the classroom**…how [do we] get student-led discussions, **more student-led than teacher driven**. And then have them ask the questions first. What kinds of questions are we looking at when we are examining the social justice topic?

For Wendy, redefining the structure of the roles of students in the classroom is a social justice practice in that a teacher is giving up knowing the direction of a conversation and letting students decide where it should go. In this sense, she sees students being authors of the classroom (Povey & Burton, 1999). Wendy recognizes that there are daily things a teacher can do to empower her students in the classroom that could lead to questioning other power structures and dynamics in their lives. By disrupting power structures that exist in classrooms, scholars see the connection between social justice teaching and helping students develop the skills to interact as critical participants of society.

TMSJ is capacity as critical participants in society

The scholars are not just thinking about how students can be successful in their classroom, but also what skills they can learn and carry with them that can impact students as critical participants of society. These skills include supporting perseverance; teaching students to struggle; developing opinions and being able to express them; being able to efficiently communicate given constraints of a task; and taking responsibility of their peers as a community.

Tied to students being critical thinkers, Catherine defines TMSJ as allowing students to be confident and able to communicate their opinions. she shares one of her goals is to help students develop confidence by showing students the importance of sharing their opinions.

> *Catherine:* Expression of **your beliefs** or kind of like you. The way you communicate your feelings... **you have to be able to develop an opinion, but also express it in a way that is true to yourself but respectful for other people involved with it as well**. [Students] have opinions and I feel like **being able to express them is pretty important component of social justice** and I feel like should be incorporated in the classroom as well.

Catherine recognizes that her students do have an opinion and being able to express it is important for her students to learn. Whether it is a student who is passionate about approaching a problem a certain way or an opinion in a current topic that impacts the students, Catherine acknowledges that students' opinions are not only connected to their beliefs but also to their emotions about these topics. She goes beyond to express how it is not enough for students to develop their opinions but to also "be true to yourself [and] also respectful for other people involved" in the conversation. Catherine wants students to develop skills that reflect the dignity that they would like to see beyond mathematics content. Traditional mathematics lessons tend to focus on the individual and do not necessarily focus on the well-being of others. Thus, developing an opinion that is both true to yourself as a student but also takes into consideration those around you in the conversation is a form social justice that goes beyond what can be used in mathematics classrooms and toward rehumanizing mathematics (Gutiérrez, 2018).

Pedagogical Moves for Implementing TMSJ Definitions

While some may feel like it is not an accomplishment to get pre-service teachers to talk theoretically about their definitions of social justice, we have evidence that our scholars enacted their definitions as student teachers. Some of the scholars even developed their own social justice lessons that incorporated exponential growth with issues of housing regulations. We do not analyze those social justice lessons here, as they tend to replicate the existing literature on TMSJ. Instead, we transition to examples of how these scholars are implementing pedagogical moves that would accompany their stances above.

Delegating Authority. As Wendy and Erica shared their working definitions of TMSJ as delegating authority and showing students their capacity to be critical participants of society, other scholars jumped into the conversation to share examples of everyday things they do in their student teaching classrooms.

> *Catherine:* One thing I do is I will randomly, not randomly but basically **choose a student each day and I call them my teacher assistant. It starts out with having them either pass papers out or they will help me write out the example problem on the board**, or they will help me, like, having them keep the class in check. **They get really into it and** then it gets to a point where, now, **one of my students he like loves to explain**. I am trying to look over a students work he will be like " oh I will help that student, you can go and help other students". **If you let students help with classroom management**, having one student make sure that no one is purposefully hurt, **they can help with what others may not know and that student feels special and you can put them in a situation where they feel comfortable to share with the class and build their confidence**.

Catherine shares that she delegates authority in her classroom to increase students' confidence by allowing their peers see them as an expert in class. While having students help with some task that can help her get through her daily agenda, she is also trying to communicate to the chosen student that she trusts them. Setting up a buy-in for the student then allows them to transition into building their confidence to be able to help other students, i.e., build their mathematical identities. In choosing teacher assistants, Catherine shows students that they can help her with daily tasks and also be bearers of knowledge from which other students can learn. The end goal not only helps Catherine with the daily managing in her classroom but also allows her to create impactful student leaders with positive mathematical identities.

Limitations

A limitation of our current understanding of teaching mathematics for social justice comes from theory and practice that relies on democracy (for example, Freire, 1970) which helps see mathematics as a social tool for critical consciousness (Frankenstein, 1989, Gutstein, 2005). While the current literature in social justice mathematics addresses some of the curricular dilemmas of being a social justice educator, the scholar's perceptions of TMSJ lets us with questions as to how to "play the game/ change the game" on a daily basis. Teacher educators and researchers need to consider our current definitions and understand how our current understanding of teaching mathematics for social justice has plateaued. Researchers need to look elsewhere to fields like Critical Ethnic and Gender studies to see how to examine how our current understanding of TMSJ may be reaffirming notions of anti-blackness (Martin, 2018), or not addressing modern-day civil rights of teachers and students. For example, in creating opportunities for students of color to participate in STEM fields, how might we also be asking students to simply participate in racial capitalism (Melamed, 2015), where they only count for diversity measures but not to re-imagine the field of mathematics, to show that it is not just that people need mathematics, but mathematics needs people (Gutiérrez, 2012).

Conclusion

The field of critical mathematics educators has reached the point of asking ourselves how can we better understand and theorize what it means to teach mathematics for social justice? How do we expand our current understanding of social justice beyond instrumental access and critical consciousness (Larnell, Bullock, Jett, 2016) of sociopolitical issues and a better understanding of teachers' roles in challenging systems of oppression? If we are to recognize teaching mathematics as a sliding signifier or a walking road (Cochran-Smith, 2004), how can we expand our current understanding of teaching mathematics for social justice as teachers embody their politial conocimiento (Gutiérrez, 2012)? If we were to ask scholars in Critical Ethnic and Gender studies, would they agree with our current understanding and implementation of Teaching mathematics for social justice?

If the goal of teaching mathematics is to change and dismantle what we see within the relationships that we have within society teacher educators, need to provide more representations of how community members are addressing all these topics so that they can develop Political Conocimiento(that will inform their critical stance on mathematics education. Furthermore, we need examples of studies that look at how preservice teachers who are exposed to political knowledge how they implement it in their first year and beyond. If we believe that the development of critical mathematics, include aspirations and hopes that are continuously recontextualized and reformulated and, uncertainties appear (Skovsmose, 2009), we cannot expect that teachers will be successful at teaching mathematics for social justice unless we place value on more than just curriculum.

References

Bartell, T. G. (2013). Learning to teach mathematics for social justice: Negotiating social justice and mathematical goals. *Journal for Research in Mathematics Education*, *44*(1), 129-163.

Brantlinger, A. (2013). Between politics and equations: Teaching critical mathematics in a remedial secondary classroom. *American Educational Research Journal*, *50*(5), 1050-1080.

Borba, M. C. (1990). Ethnomathematics and education. *For the learning of mathematics*, *10*(1), 39-43.
Boylan, M. (2009). Engaging with issues of emotionality in mathematics teacher education for social justice. *Journal of mathematics teacher education*, *12*(6), 427.
Emerson, R. M., Fretz, R. I., & Shaw, L. L. (2011). *Writing ethnographic fieldnotes*. University of Chicago Press.
Felton, M. D. (2010). Is math politically neutral. *Teaching Children Mathematics*, *17*(2), 60-63.
Felton, M. D., & Koestler, C. (2015). "Math is all around us and… we can use it to help us": Teacher agency in mathematics education through critical reflection. *The New Educator*, *11*(4), 260-276.
Frankenstein, M. (1983). Critical mathematics education: An application of Paulo Freire's epistemology. *Journal of Education*, 315-339.
Frankenstein, M. (1990). Incorporating Race, Gender, and Class Issues into a Critical Mathematica Literacy Curriculum. *The Journal of Negro Education*, *59*(3), 336-347.
Frankenstein, M. (1995). Equity in mathematics education: Class in the world outside the class. *New directions for equity in mathematics education*, 165-190.
Freire, P. (2018). *Pedagogy of the oppressed*. Bloomsbury Publishing USA.
Gates, P., & Jorgensen, R. (2009). Foregrounding social justice in mathematics teacher education. *Journal of Mathematics Teacher Education*, *12*(3), 161-170.
Gregson, S. A. (2013). Negotiating social justice teaching: One full-time teacher's practice viewed from the trenches. *Journal for Research in Mathematics Education*, *44*(1), 164-198.
Gutstein, E., & Peterson, B. (Eds.). (2005). *Rethinking mathematics: Teaching social justice by the numbers*. Rethinking Schools.
Gutstein, E. (2006). Reading and writing the world with mathematics: Toward a pedagogy for social justice. Taylor & Francis.
Gutiérrez, R. (2002). Enabling the practice of mathematics teachers in context: Towards a new equity research agenda. *Mathematical Thinking and Learning*. 4(2&3), 145-187
Gutiérrez, R. (2012). Embracing "Nepantla:" Rethinking knowledge and its use in teaching. *REDIMAT-Journal of Research in Mathematics Education, 1(1), 29-56*.
Gutiérrez, R. (2015). Nesting in Nepantla: The importance of maintaining tensions in our work. In Joseph, N. M., Haynes, C. & Cobb, F. (eds.), *Interrogating Whiteness and relinquishing power: White faculty's commitment to racial consciousness in STEM classrooms, (pp. 253-282)*. New York: Peter Lang.
Gutiérrez, R. (2017). Living mathematx: Towards a vision for the future. *Philosophy of Mathematics Education Journal*, 32(1), 1-34.
Gutiérrez, R. (2018). Why we need to rehumanize mathematics. In Goffney, I. and Gutiérrez, R. (eds.), *Annual Perspectives in Mathematics Education: Rehumanizing mathematics for students who are Black, Indigenous, and Latinx*. Reston, NJ: National Council of Teachers of Mathematics.
Larnell, G. V., Bullock, E. C., & Jett, C. C. (2016). Rethinking teaching and learning mathematics for social justice from a critical race perspective. *Journal of Education*, *196*(1), 19-29.
Martin, D. (2018) *Antiblackness, Citizenship, and the Desegregation of Mathematics Education.* Plenary presentation given at Critical Issues in Mathematics Education. MSRI. Berkeley, Ca.
Melamed, J. (2015). Racial capitalism. *Critical Ethnic Studies*, *1*(1), 76-85.
Moses, R., Kamii, M., Swap, S. M., & Howard, J. (1989). The algebra project: Organizing in the spirit of Ella. *Harvard Educational Review*, *59*(4), 423-444.
Planas, N., & Civil, M. (2009). Working with mathematics teachers and immigrant students: An empowerment perspective. *Journal of Mathematics Teacher Education*, *12*(6), 391-409.
Povey, Burton, (1999). 13 Learners as Authors in the Mathematics Classroom. *Learning mathematics: From hierarchies to networks*, 232.
Powell, A. B., & Frankenstein, M. (Eds.). (1997). *Ethnomathematics: Challenging Eurocentrism in mathematics education* (p. 63). Albany, NY: State University of New York Press.
Skovsmose, O. (1994). Towards a critical mathematics education. *Educational studies in mathematics*, *27*(1), 35-57.
Stinson, D. W., & Wager, A. (2012). A sojourn into the empowering uncertainties of teaching and learning mathematics for social change. *Teaching mathematics for social justice: Conversations with educators*, 3-18.
Valero, P. (2004). Socio-political perspectives on mathematics education. In *Researching the socio-political dimensions of mathematics education* (pp. 5-23). Springer, Boston, MA.

TAKING PROOF INTO SECONDARY CLASSROOMS – SUPPORTING FUTURE MATHEMATICS TEACHERS

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For reasoning and proof to become a reality in mathematics classrooms, it is important to prepare teachers who have knowledge and skills to integrate reasoning and proving in their teaching. Aiming to enhance prospective secondary teachers' (PSTs) content and pedagogical knowledge related to proof, we designed and studied a capstone course Mathematical Reasoning and Proving for Secondary Teachers. This paper describes the structure of the course and illustrates how PSTs' interacted with its different components. The PSTs first strengthened their content knowledge, then developed and taught in local schools a lesson incorporating proof components. Initial data analyses show gains in PSTs' knowledge for teaching proof and dispositions towards proving, following their participation in the course.

Keywords: Reasoning and Proof, Teacher Education-Preservice, Design-Based Research

Mathematics education researchers (e.g., Hanna & deVillers, 2012) and policy documents (e.g., NGA & CCSSO, 2010) emphasize the importance of teaching mathematics in ways that promote sense making, reasoning and proving across grade levels and mathematical topics. Yet, the reality of many mathematics classrooms rarely reflect this vision; even teachers who recognize the importance of reasoning and proof, in principle, often struggle to enact them, and tend to choose skills-oriented activities over proof-oriented ones for their own classrooms (Kotelawala, 2016). Preparing teachers who are capable of implementing such teaching practices and cultivating positive attitudes towards reasoning and proving is a critical objective of teacher preparation programs (AMTE, 2017). We used a design-based-research (DBR) approach to develop and study a novel capstone course *Mathematical Reasoning and Proving for Secondary Teachers*. Our goal was to explore how PSTs' knowledge and dispositions towards the teaching and learning of proof develop as a result of participating in the course, and to identify design principles that afford PSTs' learning.

Theoretical Framework and the Course Design

Researchers have conjectured that engaging students in reasoning and proving, that is, exploring, generalizing, conjecturing and justifying, might require a special type of teacher knowledge: Mathematical Knowledge for Teaching of Proof (MKT-P) (e.g., Lesseig, 2016). Building on their work, we theorize that MKT-P consists of four interrelated types of knowledge (Buchbinder & Cook, 2018). Teachers must have robust Subject Matter Knowledge (SMK) of (a) mathematical concepts and principles (Ball, Thames & Phelps, 2008), and (b) knowledge of the logical aspects of proof, such as proof techniques, valid and invalid arguments, the functions of proof and the role of examples and counterexamples in proving (Hanna & deVillers, 2012). Teachers also need strong Pedagogical Content Knowledge (PCK) specific to proving, such as (c) knowledge of students' proof-related conceptions and misconceptions, and (d) knowledge of pedagogical strategies for supporting students' proof activities (Buchbinder & Cook, 2018). Thus, we created opportunities for PSTs to develop these four types of knowledge.

The course comprises four three-week long modules, each module addressing one proof theme: quantified statements, conditional statements, direct proof, and indirect reasoning. These themes were identified in the literature as challenging for students and PSTs (e.g., Weber, 2010). Our first goal was to help PSTs to enhance their knowledge of the four themes of the course. Figure 1 shows the structure of the course (top) and a structure of one course module (bottom).

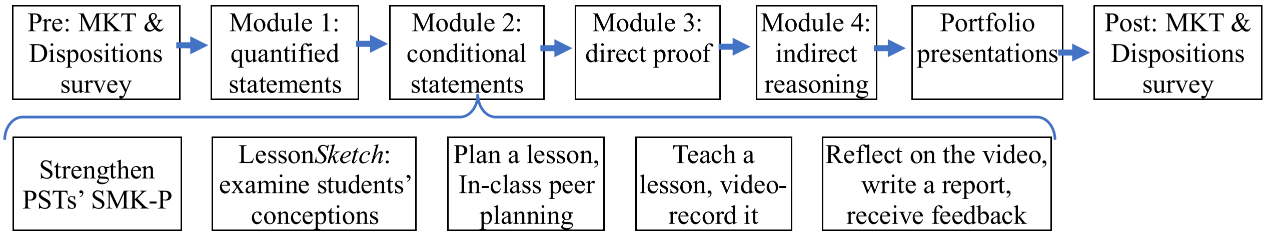

Figure 1. Design of the course *Mathematical Reasoning and Proving for Secondary Teachers*.

We designed opportunities for PSTs to enhance their proof related pedagogical knowledge, by interpreting sample student work, identifying students' conceptions of proof, and envisioning responding to students' thinking in the virtual learning platform Lesson*Sketch* (Herbst & Chazan, 2015). We also engaged PSTs in planning and implementing, in local schools, lessons that combine the proof themes with mathematical content. As PSTs enacted their lesson, they recorded it with 360° video cameras, which captured simultaneously the PSTs' teaching and the students' engagement with proof-oriented lessons. PSTs then watched and analyzed their lesson, wrote a reflection report and received feedback from the course instructor.

Methods

Participants in the first iteration of the course were 15 PSTs in their senior year (4 middle-school, and 11 high-school track; 6 males and 9 females). The PSTs had completed the majority of their extensive mathematical coursework, and two educational methods courses. Multiple measures were used to assess PSTs' learning, such as pre- and post- dispositions towards proof surveys, and an MKT-P instrument with sets of questions on the four types of MKT-P. We also collected PSTs' responses to home- and in-class assignments, video-recordings of in-class sessions, and PSTs' teaching portfolios containing four lesson plans, videos of the lessons taught, reflection reports and sample students' work. Below, we present initial results of our ongoing analysis of these data, focusing on the Conditional Statements (CS) module. The CS module comprised five activities as shown in Figure 1.

Results

The data analysis was guided by two main questions: (1) How did PSTs interact with the CS module?, and (2) What mathematical / pedagogical ideas addressed in the CS module got implemented in PSTs' lessons? In this section we present findings that respond to these questions and also describe evidence of the PSTs' learning.

PSTs' Interaction with the Conditional Statements Module

In the first activity, PSTs broke up into three groups and each group received one conditional statement, such as "A graph of an odd function, defined at zero, passes through the origin." The task was to identify its hypothesis P and conclusion Q, and use the P and Q to write statements in 11 logical forms such as: $P\ if\ Q$, $P\ only\ if\ Q$, and $If\ {\sim}P\ then\ {\sim}Q$. The PSTs were asked to differentiate between true and false statements, and to identify which statements are equivalent to $P \Rightarrow Q$. The most challenging form for PSTs to interpret was $P\ only\ if\ Q$. Some PSTs realized that it is equivalent to a contrapositive, ${\sim}Q \Longrightarrow {\sim}P$, and thus equivalent to the original statement. Other PSTs rejected this idea by arguing that P can be true "not only when Q is true." After a class discussion, all discrepancies appeared to be resolved.

The second activity utilized Lesson*Sketch*.org, a web-based platform for teacher education in which PSTs can analyze representations of classroom interactions depicted as cartoon sketches (Herbst & Chazan, 2015). The experience *Who is right?* first asked PSTs to decide whether the (false) statement: "If n is a natural number, then $n^2 + n + 17$ is prime," is true, false, sometimes true, or cannot be determined. Next, the PSTs viewed arguments of five pairs of students about the truth-value of this statement and were asked to evaluate the correctness of these arguments, identify instances of expertise and gaps in student reasoning and pose questions to advance or challenge that reasoning. The PSTs correctly identified invalid reasoning and proposed ways to acknowledge students' effort, while pointing them towards more valid modes of argumentation.

In the final activity, the PSTs analyzed current middle school and high school textbooks to identify where conditional statements appear in the secondary curriculum. PSTs also addressed students' difficulties with conditional statements and discussed ways to support student thinking.

PSTs' Implementation of a Lesson on Conditional Statements

After spending time developing lessons that involved conditional statements, PSTs implemented their lessons in the classrooms participating in the study. PSTs came up with many creative ways to integrate conditional statements in their lessons while appropriately adjusting them to the students' level. All PSTs used real-world examples, e.g., "If I do my homework, I will get good grades" to introduce the general structure of a conditional statement, and to identify their hypothesis (P) and the conclusion (Q). For example, Sam showed product advertisements, and asked students to turn them into conditional statements and analyze their structure.

The most utilized types of tasks implemented by the PSTs were True or False, and Always-Sometimes-Never, where students had to identify the hypothesis and the conclusion, determine whether the statement is true or false, and provide justifications or counterexamples. Bill had his students match hypotheses (e.g., a triangle is not equilateral) with conclusions (e.g., a triangle is isosceles), written on index cards, to produce true conditional statements. Then the students switched the order of the cards to examine how the statement and its converse compare. Overall, except for four PSTs who only minimally addressed conditional statements, the majority of PSTs successfully and creatively integrated conditional statements in their lessons.

Evidence of PSTs' Learning

To trace changes in PSTs' knowledge and dispositions following the course, we administered Dispositions Towards Proof pre- and post- surveys and a 12 items MKT-P instrument (3 question in each of the four areas of MKT-P). The analysis revealed that PSTs had relatively high initial scores on three out of four types of MKT-P. Hence instead of examining the point increase, we calculated the percentage of possible growth. E.g., although the increase in the Knowledge of Logical Aspects of Proof was only 1.15 points, it constitutes 48% of possible 2.4 points needed to obtain the maximum score. The highest gain of 66% occurred in the Knowledge of Pedagogical Practices for supporting students' learning of proof. The items measuring PSTs performance in these areas called for analyzing, interpreting and responding to students' conceptions of proof, similar to activities described above in the CS module.

The Dispositions Survey had five categories of questions: (1) notions of a proof, (2) purpose of proof, (3) confidence and comfort with proving, (4) suitability of proof in the secondary curriculum, and (5) confidence in and knowledge about teaching proof. In general, the pre- and post-test results did not show much change in categories 1, 2 and 4; a few noteworthy results occurred in categories 3 and 5. The PSTs' confidence in their ability to prove results from the school curriculum increased from 65% to 92%; the percent of PSTs recording confidence of teaching proof to students, increased from 50% to 84%. The fact that PSTs' overall growth of dispositions was relatively modest may be attributed to an initial, perhaps false, feeling of confidence due to the PSTs' prior mathematical coursework.

Discussion and Implications

We reported on the first iteration of a 3-year DBR project aimed to enhance PSTs' knowledge and dispositions for teaching proof at the secondary level. We described the overall structure of the capstone course and the theoretical underpinnings of its design. We described how PSTs interacted with the CS module through practice-based activities of analyzing students' conceptions of proof; planning and sharing lessons; classroom implementation and reflecting on it, aided by video technology. These descriptions serve as a backdrop for understanding the gains in PSTs' content and pedagogical knowledge of proof (MKT-P). The areas in which this growth was most evident are: the logical aspects of proof and pedagogical knowledge specific to proving. Although the small sample size does not allow testing statistical significance of these outcomes, we find them encouraging, since the supporting evidence goes beyond self-report.

Our data analysis is still ongoing, as we examine video of on-campus sessions and of the PSTs' teaching to create more fine-grained description of how PSTs' content and pedagogical knowledge evolved throughout the course, and to match this growth to the design principles of the course. The results of this analysis will inform future iterations of our project, and, potentially advance the field's understanding of how to support PSTs' development of MKT-P.

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References

Association of Mathematics Teacher Educators (2017). *Standards for preparing teachers of mathematics*. Available online at: amte.net/standards.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59* (5), 389 – 407.

Buchbinder, O., & Cook, A. (2018). Examining the mathematical knowledge for teaching of proving in scenarios written by pre-service teachers. In O. Buchbinder & S. Kuntze (Eds.). *Mathematics teachers engaging with representations of practice* (pp. 131-154). New York: Springer.

Hanna, G. & deVillers, M. D. (2012). *Proof and proving in mathematics education: The 19th ICMI study*. New York: Springer.

Herbst, P. and Chazan, D. (2015). Using Multimedia Scenarios Delivered Online to Study Professional Knowledge Use in Practice. *International Journal of Research and Method in Education,* 38(3), 272-287

Kotelawala, U. (2016). The status of proving among US secondary mathematics teachers. *International Journal of Science and Mathematics Education, 14*(6), 1113-1131.

Lesseig, K. (2016). Investigating mathematical knowledge for teaching proof in professional development. *International Journal of Research in Education and Science, 2*(2), 253-270.

National Governors' Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards: Mathematics.* Washington DC: Council of Chief State School Officers.

Weber, K. (2010). Mathematics majors' perceptions of conviction, validity, and proof. *Mathematical Thinking and Learning, 12*(4), 306-336.

PORTRAITURE OF ELEMENTARY PRESERVICE TEACHERS DURING A STEM CAMP EXPERIENCE

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Twenty-four preservice teachers planned, co-taught, observed peers, observed the elementary students, and participated in debriefing during a summer science, technology, engineering, and mathematics field experience. Data from lesson plans, reflections, observations of students observational noticings, and peers created a holistic view of their perspectives on teaching and learning mathematics in a diverse, authentic setting, without the challenges that often occur in traditional field-based experiences. Using the portraiture method of inquiry, the complex dynamics of this experience are captured from multiple lenses.

Keywords: Teacher Education-Preservice, Teacher Beliefs, Elementary School Education, Instructional activities and practices

Facilitating elementary preservice teachers (PSTs) cohesive understanding of teaching and learning mathematics in meaningful ways can be challenging. Experiences in mathematics field placement practice, experiences as an elementary student, and experiences in mathematics methods courses are often disconnected. One consistent complaint heard over the years is that what PSTs see in the field is often in opposition to what they are learning in their coursework (Beck & Kosnik, 2002). In addition, twelve years of classroom experiences have shaped not only their own identities, but also their beliefs about teaching and learning (Shoffher, 2008).

In this study, six PSTs observe, plan, and teach in a three-week summer Science, Technology, Engineering, Mathematics (STEM) camp. Using the narrative portraiture inquiry (Lawrence- Lightfoot, 2005) the PSTs observations, reflections, planning and teaching are analyzed to paint a portrait of the experience. These PSTs were a teaching team for 27 fourth grade students. "Teaching is not merely a cognitive or technical procedure, but a complex personal, social, and often elusive set of embedded processes and practices that concern the whole person" (Olsen, 2008, p 5). Examining the experience of PSTs from multiple perspectives supports deeper understanding of perspectives about teaching and learning mathematics.

Theoretical Framework

Research suggests that education courses linked to authentic field experiences maximize effectiveness (Weld & French, 2001). If PSTs have not experienced teaching concepts such as inquiry or project based learning, it is difficult to recreate these in mathematics classrooms or visualize what they would look like in practice (Beck & Kosnik, 2002). Teacher educators face a conundrum of how to provide opportunities for PSTs to enact and experience effective mathematics education under the guidance of mentors who also support these educational practices, when these classrooms are so rare in current K-12 settings. Finding classrooms that promote best practice, rather than ones that emulate the systemic issues of inequity and oppression that often are found in schools is a challenge.

This study is grounded in the belief that PSTs need early inquiry based field experiences with positive diverse populations, where they focus on students as mathematics learners, take teaching risks, and reflect upon the practice of teaching to meet the needs of all learners (Aguirre et al., 2013). However, it is difficult to provide these authentic placements any time in the year, but especially when coursework is in the summer. Often summer placements are in local schools that focus on remediation and drill of basic computational facts. This study centers on an alternative summer placement, which provided elementary PSTs an opportunity to reflect upon innovative mathematics teaching approaches in a positive and supportive environment. It was strategically positioned to provide PSTs experiences working with diverse populations> Instruction focused on building new knowledge and skills from the strengths and cultural competencies that each elementary student brought to the experience.

Methods, Techniques or Modes of Inquiry

Narrative portraiture (Lawrence-Lightfoot, 2005) was used to examine the perspectives and experiences of PSTs from this mathematics field placement. "Portraiture is an ethnographically oriented method of inquiry that seeks to capture and explain the ever-changing complexities of life and experience. Portraiture emerged from the desire to tell a story in such vivid detail that the event could be pictured as though it were a painting" (Lawrence-Lightfoot & Davis, 1997, p.4). In this study, a detailed, vivid portrait was created from various perspectives and then these portraits were further analyzed to develop deeper understanding of the moments. This study examined the perspectives of mathematics teaching and learning among PSTs in field placement, but the lens of each perspective was different, which created more insight and vivid portraits into the complex view of teaching and learning mathematics that preservice teachers held.

Study Context

The 23 females and one male PSTs were in their first teaching field placement in the program and came from affluent homes with many positions of privilege. Previous coursework examined lesson planning, assessment, professionalism, individual reading instruction, and culturally responsive teaching. For the semester being studied, PSTs were enrolled in mathematics, science, and reading methods coursework that were tied to the field component. The instructors of the methods courses were also the lead faculty in the summer field experience, which was a three-week Science, Technology, Engineering, and Mathematics (STEM) focused program for 97 rising third through fifth grade students. Week one focused on structures, week two focused on robotics, and week three focused on forces in motion as the unifying theme. Students explored computation through purchasing supplies in the store, which included sales. They graphed and measured experiments and explored geometric properties of shapes. Before camp began, PSTs attended mathematics methods classes to explore pedagogical concepts such as learning trajectories, assessment, technology, questioning, and project based learning within the field of mathematics. In addition, PSTs attended a 3-day workshop co-taught by the three methods faculty members. Each workshop day participants and faculty modeled and explored the content for one of the three weeks of the STEM summer experience. For example, the first day of the workshop addressed various elements they would teach, content, and projects related to structures.

This experience provided educational opportunities to elementary students from diverse backgrounds and school systems in this rural region in the southeastern United States. Sixty percent of the students were recommended for the summer experience based on financial need. A total of six systems were represented. Students from high needs schools attended alongside students from schools where resources were plentiful. Each class truly made up the diversity that public education strives to meet. The population of the camp was approximately 60% African American, 20% Asian, 15% Caucasian, and 5% other nationality representations. PSTs were able to examine their own beliefs and their practice culturally responsive teaching. PSTs were assigned a specific class for the entire summer experience. There were two third grade classrooms, one fourth grade classroom, and one fifth grade classroom. Six PSTs worked together in a classroom, but only two lead teachers per day. For the purpose of this study, the purposeful selection of a fourth-grade classroom unit was examined. Therefore, the data from the preplanning through post planning debriefing sessions for the six PSTs were the focus of this portraiture.

PSTs had five roles during the summer experience: 1) planner, 2) teacher, 3) student observer, 4) teacher observer 5) reflective, member of professional collaborative debriefing sessions. PSTs were able to plan one day and teach the next day with a teaching partner. The partner changed with each planning and teaching cycle to encourage growth and learning to work with various personalities. PSTs not teaching and planning were in the classroom observing students or observing their peer teachers. Observation instruments were used to focus on their noticings of student thinking, characteristics, and behaviors. Each day after the summer camp grade level teams would meet and debrief about the different perspectives from the day. These sessions focused on observational noticing related to teaching, students, and strategies used to make connections that informed the next day's teaching practices.

Data Sources and Evidence

PSTs reflections, lesson plans, observations of peers, field notes, and observations of elementary students were used as data sources. In order to examine data individually and for patterns about thinking of mathematics teaching and learning, there were several rounds of analysis. For each day, each data source was coded separately, as a unit, and then collectively. For example, all data (i.e. lesson plans, student observations, teacher observations, debriefing notes, and reflections) related to week one day two were examined together to note both consistencies and inconsistencies. These data sets were not only from different people, but also with different foci, but all were used to create a picture of the classroom on that day. Next data from each PSTs was analyzed as a unit to identify changes in the individual preservice teacher, as well as examining the perspective of each teacher in the various roles. Finally, the data from each role was analyzed as a unit (e.g. student observer) 15 days of student observations; 15 days of teacher observation) was examined to look for changes, patterns, and other information that helped to create a vivid image of this perspective in the classroom.

After thoroughly analyzing the data from the various perspectives, three sets of portraits were created by a team of researchers: a portrait of each individual, a portrait of each role they played in the mathematics classroom, and a portrait of how they saw mathematics teaching and learning. Researchers then analyzed the portraits for analytic themes across portraits. The themes from this analysis helped to construct a holistic visual of preservice teachers during the STEM camp experience. The portraits from the PST perspectives provide insight into their views of mathematics teaching, learning equity, and students.

Results

Findings from this study indicated that PSTs primarily noticed elements about the classroom environment and behavior, rather than focusing on the cognitive elements in the classroom. However, PSTs plans, observations and reflections were richer in focused content development as the days progressed. In addition, overall the data from each role was more focused on the perspective assigned. For example, the student observers during week 3 were more focused on students as mathematics learners, than week one which mainly focused on the teachers, despite the questions on the protocol. The portraits illustrated that each experience in the classroom provided additional insight and focus for the following experience. In addition, as PSTs gained confidence through these experiences, they were able to focus more on the mathematics and the students, rather than devoting a majority of the focus to the behavior. In addition, as time progressed, PSTs focused on strengths of individual students, rather than deficits and generalizations. It wasn't until week 2 and 3 until PSTs observing teachers and students began to notice when students could do more than they were given and that they needed more cognitive challenge. The debriefs were one space where this was highlighted by peers and discussions pivoted towards learning, which then impacted future experiences.

Scientific or scholarly significance

Examining PSTs perspectives about field experiences in diverse settings, provides insight into the connections and misconceptions they make when linking coursework, beliefs and field experiences. It provides insights into issues PSTs face and elements that are effective in deepening understanding of effective mathematics instruction. Preparing PSTs to focus on student mathematical thinking and reasoning is important in order to promote student learning (Aguirre et al., 2013). The six PSTs all looked for the positives in their students, and some recognized the role of teachers and students in the classroom dynamics, while others only focused on one of these two elements. However, the differences that each preservice teacher gained from the various perspectives provides promise into the importance of providing holistic mathematical experiences for PSTs. Studies indicate preservice teachers have the ability to develop this type of professional noticing in a teacher education program, if structures are in place (Amador et al., 2016; Fisher et al., 2014).

This study enhances research on professional noticing and teacher perspectives about mathematical teaching and learning. It provides well narratives about the mathematical teacher education identity development of PSTs in their field placements. It adds details about the nuances in the complex task of teaching and learning in elementary mathematics education.

References

Aguirre, J., Turner, E., Bartell, T. G., Kalinec-Craig, C., Foote, M. Q., & Roth McDuffie, A. (2013). Making connections in practice: How prospective elementary teachers connect to children's mathematical thinking and community funds of knowledge in mathematics instruction. *Journal of Teacher Education, 64*(2), 178–192.

Amador, J., Weston, T., Estapa, A., Kosko, K., De Araujo, Z. (2016). Animations as a transformational approximation of practice for preservice teachers to communicate professional noticing. *Journal of Technology and Teacher Education, 4*(2), 127-151.

Beck, C., & Kosnik, C. (2002). Professors and the practicum: Involvement of university faculty in preservice practicum supervision. *Journal of Teacher Education, 53*(1), 6-19.

Fisher M. Schack, E. O., Thomas, J., Jong, C., Eisenhardt, S., Tassell, J., & Yoder, M. (2014) Examining the relationship between preservice elementary teachers' attitudes toward mathematics and professional noticing capacities. In: J.J. Lo, K. Leatham & L. Van Zoest (Eds.), Research trends in mathematics teacher education: Research in mathematics education (pp. 219-237). Springer.doi: 10.1007/978-3-319-02562-9_12.

Lawrence-Lightfoot, S. (2005). Reflections on portraiture: A dialogue between art and science. *Qualitative Inquiry, 11*(1), 3-15. doi: 10.1177/1077800404270955

Lawrence-Lightfoot, S. & Davis, J. H. (1997). The art and science of portraiture. San Francisco, CA: Jossey-Bass.

Olsen, B. (2008). Introducing teacher identity and this volume. *Teacher Education Quarterly, 35*(3), 3-6.

Shoffher, M. (2008). Informal reflection in pre-service teacher education. *Reflective Practice, 9*(2), 123-134.

Weld, J., & French, D. (2001). An undergraduate science laboratory field experience for preservice science teachers. *Journal of Science Teacher Education, 12*(2), 133-142.

CRITICAL MATHEMATICS TEACHER NOTICING: USING ONLINE TECHNOLOGY TO EXPLORE HOW PRE-SERVICE TEACHERS OF COLOR CONFRONT THEIR PEERS' RACIAL POSITIONINGS OF CHILDREN

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Mathematics teacher noticing is a construct used heavily in research around mathematics teaching and mathematics teacher education. However, just as a focus on noticing draws attention to what a teacher "sees", a focus on what a teacher does NOT "see" is equally important. In this study, we utilize an online video commenting tool to analyze what elementary pre-service teachers of color notice about the language and positioning of case study students through four-weeks of problem solving interviews. This tool served as a space for teachers of color to confront the racist ideologies present in how their peers positioned students.

Keywords: Elementary School Education, Equity and Diversity, Teacher Education-Preservice, Technology

Mathematics teacher noticing has quickly become a critical aspect of mathematics teaching and teacher preparation (Schack, Fisher, & Wilhelm, 2017), centering around the idea that effective mathematics teaching extends beyond just content knowledge, pedagogical content knowledge, or even beliefs. The physical act of "noticing" or paying attention to and being reactive to the minutiae of classroom moments is key to developing any teacher's ability to empower children mathematically. Research connecting equity with teacher noticing, however, is still largely centered around how white teachers can use noticing when working with students of color, and does not recognize the way race impacts what teacher notice. In the United States, even as the teaching force grows more diverse, most students of color continue to work only with white teachers (Ingersoll, Merrill, & Stuckey, 2014), a troubling trend due to the racial and cultural blind spots of white teachers. Moreover, teachers of color who often have experiences that can reveal these blind spots are often silenced, particularly in pre-service education (Philip, Rocha, & Olivares-Pasillas, 2017).

The construct of mathematics teacher noticing is powerful, as it has given a name to a practice that all teachers engage in. So how can mathematics teacher noticing be used to delve into a mathematics teacher education that cultivates the racial and cultural knowledge necessary to empower all students? We explore this work through the following research questions: (1) When using a technology-based video commenting platform to explore children's mathematical thinking, what do pre-service teachers of color notice in contrast to white teachers? (2) What does mathematics teacher noticing look like when connected to racial awareness?

Theoretical Framework

Mathematics teachers often hold "expert blind spots", in which they find it difficult to "see" how their students struggle mathematically (Nathan & Petrosino, 2003). These blind spots are gaping holes in a teacher's vision, in which a teacher who may not have experience being positioned as mathematically inadequate cannot understand the dilemma of their struggling students. We use this construct of blind spots to refer to how teachers who come from dominant backgrounds often view their classrooms and students through their dominant culture lenses, unaware of the extent of their blind spots (Louie, 2018).

The world that is familiar to mathematics teachers, is, of course, structured by racial ideologies in ways that operate invisibly. White supremacy creates a "racial grammar" that allows for the seemingly natural reproduction of the racial order (Bonilla-Silva, 2012). This racial grammar facilitates the construction of white stories as "universal" and creates strong perceptions about how things "are" and how people of non-white backgrounds are supposed to be (Bonilla-Silva, 2012). In mathematics education, combating this racial grammar involves taking a rehumanizing stance that recognizes and positions each and every student as not only human, but someone who holds substantial mathematical knowledge connected to their cultural and racial identities (Gutiérrez, 2018). A rehumanizing stance of mathematics teacher education also values the knowledge of teachers of color, listening to the stories they tell in order to help white teachers "see" blind spots that continue to oppress and dehumanize our children of color. Teachers of color must have space to think about, discuss, and interrogate the role of race within their classroom spaces and be empowered to share these noticings. We refer to this calling out of racial grammars and other oppressive mechanisms that operate within our field as *critical mathematics teacher noticing*.

Teacher video clubs could serve as a mechanism for engaging teachers in critical dialogue with each other surrounding their practice (Van Es & Sherin, 2008). However, while video clubs facilitate teacher dialogue around particular aspects of classroom practice, video club are also sociopolitical spaces where teachers from dominant groups hold more power and "voice". One way to hinder this silencing is through technology that creates safer communicative spaces. For instance, technology that utilizes asynchronous watching and commenting allows teachers to take their time to watch a video and comment accordingly, to reflect not only on what they notice immediately, but on what they notice after repeated watching and reflection (Chao & Murray, 2013). Bringing the conversation online, then, opens up spaces for teachers to critique the deficit perspectives of fellow teachers safely, without the silencing effect of a video club.

Methodology

The data for this study came from an elementary teacher graduate degree and licensure program at a large, state university in the Midwestern United States. We focused on a diverse master's degree cohort, where six of the 28 teachers identified as persons of color. Our analysis focused on an Elementary Mathematics Teaching Methods course instructed by the first author, in which each pre-service teacher worked one-on-one with a child in their student teaching internship. The teachers completed four interviews with a case study child: a "getting to know you interview" to begin the project, then three subsequent problem-solving interviews each week (Foote et al., 2015).

We utilized GoReact, an online video commenting tool to facilitate teacher noticing during the problem-solving interviews. Teachers individually filmed their interviews with case study students and immediately uploaded the video for their peers and instructor to watch and comment on. Then, in the three subsequent class meetings, the teachers met in person to watch the videos again, read each other's comments, and engage in discussion about what they noticed. At the end of the four interviews, the teachers produced a report on what they learned from the student through the interviews and the conversations. We utilized all of this data in our analysis: the online video, the comments, and the written reports.

Findings

Our analysis focused on instances where pre-service teachers called out dismissive or deficit language they noticed their peers using, particularly if this invoked racial grammars. We found that pre-service teachers of color's comments regularly emphasized the skills and reasoning displayed by students of color and challenged conclusions of white pre-service teachers. We believe these instances reveal what we refer to as *critical mathematics teacher noticing*, which requires an awareness of the structures and functioning of racial grammars as well as the ability to interrupt them.

Ms. Simmons, a biracial, Black pre-service teacher, was paired in online discussions with Ms. Harris, a white pre-service teacher. Ms. Harris chose Amir, a Black Somali-American second-grader, as her case-study student. Ms. Harris described Amir as struggling in mathematics and posited that "it might be because he isn't paying attention and isn't focused, and sometimes it might be due to a language barrier." In Ms. Harris's first video, she posed a problem where a boy had 13 cookies and ate six, then asked how many were left. Amir wrote the equation 13-6. Ms. Harris then asked why he subtracted, to which Amir replied that the problem said he ate and that meant he should take away. Ms. Harris commented about this interaction, "I had a very hard time getting him to explain his answers no matter how long I waited, how I rephrased the question, or what questions I asked. This was the most explaining I got him to do during the whole interview."

Ms. Simmons' responded by challenging Ms. Harris' conclusion:

> I don't know what all you asked him, but I see that he understands that "you ate" means "take away" so he chose subtraction. Maybe you could have asked him if "you ate" means you get more cookies or cookies are taken away, then you could ask if take away means you add or subtract. Maybe that would've provided more of an explanation than just asking why he chose subtraction. Sometimes students don't realize all of their steps until you point them out for them.

Ms. Simmons noticed Ms. Harris' deficit perspective, implying that there was more to Amir than what Ms. Harris saw. While Ms. Harris only noticed Amir's lack of focus or skills and refusal to explain his thinking, Ms. Simmons noticed the bigger problem of how her classmate spoke to and about Amir. Ms. Simmons also expressed a belief that students are capable of more than they can put into words and that teachers play a role in developing students' mathematical confidence.

Conclusion & Discussion

The mathematics teacher noticing work has largely focused on noticing mathematical discourse or action without specifically connecting to race or culture (Jong, 2017). Our work shows how, using critical mathematics teacher noticing, we can unpack the racial blind spots pre-service teachers hold when "noticing" the mathematical thinking of their children. We posit that the online video technology space created an immediacy for teachers to interact in dynamic ways familiar to them, and therefore, grow comfortable in couching their critique in their own social identities (i.e., as a teacher of color who has to deal with continual racial microagressions from members of their cohort).

By putting the focus specifically on racial and cultural norms within mathematics education, we draw attention to the racial "blind spots" within mathematics teacher education overall. We introduce a new term to describe this unveiling of these blind spots in mathematics teachers, in which racial grammars that have been invisibilized are brought to light: critical mathematics teacher noticing.

However, this study does have limits. First, we only analyzed data from one particular cohort when they took an elementary mathematics methods course; not measuring their prior interactions with each other. Second, the first author of this paper was also the instructor of this course, introducing bias into our interpretation of the data. And third, this is a snapshot of these teachers before they actually begin their practice. Data suggest that some of these teachers will take on racist or "othering" pedagogical stances in their first years of teaching (Chao, Hale, & Cross, 2017).

Technology can be used as a democratizing tool, an agent of change that allows various views and individual interpretations of the same media. The current research examining how we prepare mathematics teachers of color is emerging, exposing how heavily our existing structures are set up to only help white teachers without threatening their fragility (Picower, 2009). Just as we encourage our mathematics teachers to engage in classroom discussion that moves beyond the easiest and most accessible mathematical strategy, we must push our teachers to engage in conversations on equity that move beyond "safe" topics of diversity and social justice to recognize and confront white supremacy. As a field, we ask teachers to orchestrate open-ended, uncomfortable mathematical conversations in the classroom. Therefore, as mathematics educators, we must also engage in uncomfortable conversations about the blind spots our teachers and ourselves hold, especially as pertaining to perpetuating white supremacy.

References

Bonilla-Silva, E. (2012). The invisible weight of whiteness: the racial grammar of everyday life in contemporary America. Ethnic and racial studies, 35(2), 173-194.

Chao, T., Hale, J. J., & Cross, S. B. (2017). Experiences using clinical interviews in mathematics methods courses to empower pre-service teachers: A conversation among three critical mathematics educators. In S. Kastberg, A. M. Tyminski, A. Lischka, & W. Sanchez (Eds.), *Building Support for Scholarly Practices in Mathematics Methods*. Charlotte, NC: Information Age Publishing.

Chao, T., & Murray, E. C. (2013). Teacher Asynchronous Noticing to Foster Students' Mathematical Thinking. In A. C. Superfine, M. Martinez, G. Larnell, T. Stoelinga, & D. B. Martin (Eds.), Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Chicago: University of Illinois at Chicago.

Foote, M. Q., Roth McDuffie, A., Aguirre, J., Turner, E. E., Drake, C. & Bartell, T. G.. (2015). Mathematics Learning Case Study Module. In C. Drake et al. (Eds.), TeachMath Learning Modules for K-8 Mathematics Methods Courses. Teachers Empowered to Advance Change in Mathematics Project. Retrieved from: www.teachmath.info

Gutiérrez, R. (2018). Rehumanizing mathematics. In I. M. Goffney & R. Gutiérrez (Eds.), Re- humanizing mathematics for Black, Indigenous, and Latinx students. Annual Perspectives in Mathematics Education. Reston, VA: National Council of Teachers of Mathematics.

Ingersoll, R., Merrill, L., & Stuckey, D. (2014). Seven trends: The transformation of the teaching force. CPRE Research Report # RR-80. Philadelphia: Consortium for Policy Research in Education. DOI: 10.12698/cpre.2014.rr80

Jong C. (2017) Extending Equitable Practices in Teacher Noticing: Commentary. In: Schack E., Fisher M., Wilhelm J. (eds) Teacher Noticing: Bridging and Broadening Perspectives, Contexts, and Frameworks. Research in Mathematics Education. Springer, Cham

Louie, N. L. (2018). Culture and ideology in mathematics teacher noticing. Educational Studies in Mathematics, 97(1), 55-69.

Nathan, M. J., & Petrosino, A. (2003). Expert Blind Spot Among Preservice Teachers. American Educational Research Journal, 40(4), 905–928. https://doi.org/10.3102/00028312040004905

Philip, T. M., Rocha, J., & Olivares-Pasillas, M. C. (2017). Supporting Teachers of Color as They Negotiate Classroom Pedagogies of Race. Teacher Education Quarterly, 44(1), 59–79.

Picower, B. (2009). The unexamined Whiteness of teaching: how White teachers maintain and enact dominant racial ideologies. Race Ethnicity and Education, 12(2), 197–215. https://doi.org/10.1080/13613320902995475

Van Es, E. A., & Sherin, M. G. (2008). Mathematics teachers' "learning to notice" in the context of a video club. Teaching and Teacher Education, 24(2), 244–276

OVERCOMING IMPLICIT BIAS: PERSPECTIVES FROM PRE-SERVICE MATHEMATICS TEACHERS

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Pre-service teacher perspectives provide important insight for teacher education and can also serve as indicators for areas of support needed in teacher preparation, especially in regards to providing training for addressing challenges in the classroom. This study examines the perspectives from pre-service mathematics teachers in an introductory mathematics education course, particularly on how challenges may be presented and addressed in the classroom based off of socially constructed implicit biases against the subject of mathematics. Participants were asked to reflect on how implicit bias may impact mathematics through two factors: 1) teacher and student challenges in the classroom and 2) addressing the concept of innumeracy. The perspectives and beliefs that these participants hold in their pre-service practice may be used to better support their clinical experiences and transition to the workforce.

Keywords: Affect, Emotion, Beliefs, and Attitudes, Teacher Beliefs, Teacher Education-Preservice

Introduction

The pre-existing perceptions and dispositions of pre-service teachers towards mathematics teaching and learning are a strong indicator of the performance and learning environment that will be established in each of these future teachers' classrooms (Franke, Kazemi, & Battey, 2007). Many factors can contribute to each pre-service teacher's development of individual perceptions and dispositions towards mathematics education, ranging from their individual backgrounds, experiences in learning mathematics, and societal influences and stereotypes on the subject area (Thompson, 1984). By examining reflections from mathematics pre-service teachers, teacher educators can establish a better understanding of how to provide professional development and teacher preparation that best supports various forms of content knowledge (curriculum-based, pedagogical, situational) needed for effective teaching (Southwell & penglase, 2005; Hill, Ball, & Schilling, 2008; Aslan-Tutak & Adams, 2015).

This study focuses on the impacts that existing social influences, which will be referred to as "implicit bias," have on pre-service mathematics teachers' perceptions of math teaching and learning. The reflections from these pre-service teachers also stem from their individual perceptions of the challenges posed by existing implicit biases towards mathematics learning. Presenting pre-service teachers with opportunities for deep self-reflection specifically on potential challenges in teaching mathematics is highly important for developing teacher accountability and better understanding of creating authentic learning opportunities in their future students (Graeber, 1999). This paper will highlight findings from this study regarding pre-service teacher perceptions on potential the implicit biases that they may encounter in their future classrooms, focusing on two main areas: 1) perceived challenges in teaching and learning mathematics, and 2) overcoming challenges presented by innumeracy.

Implicit Bias and Challenges Presented

This study is based on a framework that views perceptions and dispositions that students and pre-service teachers may have regarding mathematics teaching and learning are influenced and greatly impacted by social constructs and stereotypes (Thompson, 1984; McClain & Cobb, 2001). These pre-established societal norms about mathematics teaching and learning create challenges for teachers of all levels of expertise. For example, norms in secondary education exist regarding summer homework and readings as commonplace for English and Language Arts courses, whereas not until more recently have students and families engaged in more enrichment programs beyond the classroom for subjects such as mathematics. Additional challenges also arise from the perpetuation of negative stereotypes depicting mathematics education. For example, students (especially girls) who take a liking towards the subject area may feel outcast in their learning environments, where enjoying mathematics as regarded as "weird" by societal norms (Boaler & Greeno, 2000). Oftentimes, the statement of "I'm not a math person" may even be presented as a societal norm and something highly acceptable and perhaps even touted in commonplace (Franke et al, 2007). Through addressing these challenges outright in the classroom, teachers can set positive classroom norms for generating authentic learning opportunities that can answer the questions of "why" for learning mathematics.

Methodology

With the assumption that implicit bias in the mathematics classroom has a direct impact on teaching and learning, this study focuses on the perceptions that beginning pre-service mathematics teachers may presume prior to receiving extended mathematics education coursework and entering the teaching workforce. While this paper outlines the participants and findings of a single class, further data was collected from later cohorts as well, which would be presented in the brief research report session.

Participants and Context

This study took place in an isolated introductory mathematics education class at a public four-year university, with 11 total students. Ten of the students were mathematics majors, and one majored in statistics. All of these students were taking the introductory mathematics education course as a part of a minor program for secondary mathematics education. Each of these students had already had varying forms of clinical experiences in this teacher education minor program. It is important to note that while these students have had concurrent introductory education courses, the class in which this study was conducted was the first of their experiences with a mathematics education focus. Therefore, the responses in which these participants provided their responses regarding perceptions of implicit bias are a candid reflection of the varying levels of pedagogical content knowledge.

Data Collection and Analysis

In this study, students were asked to respond to four questions that were a part of a routine in-class survey given online through Google Forms. Identifying information (such as name, age, sex, year of study, etc.) were not recorded for this study, as this study solely examined the direct response and written perspectives of these pre-service teachers without specific association with each individual student. The intent of the survey was to generate a sampling of student responses for developing in-class discussion regarding the two factors contributing to potential implicit bias in the math classroom. Participants were allowed to provide as much or as little elaboration in their responses to the questions as they saw appropriate.

Two main factors regarding pre-service teachers' perceptions about implicit biases mathematics teaching and learning were explored in the data drawn from the survey: 1) challenges in teaching and learning mathematics (from both teacher and student perspectives) and 2) addressing the concept of innumeracy. For each of these factors, students were asked for a) their definition and understanding of each factor and b) their hypotheses and suggestions for how to approach or address each of the contributing factors.

For the first survey question, involving the participants' perceptions of challenges in teaching mathematics, keywords from participant responses were categorized as deriving from either a) limitations in a teacher's mathematics content knowledge or b) limitations in a teacher's pedagogical content knowledge. These two areas of content knowledge were chosen as the most influential indicators for effective teaching. An assumption was made that the participants may focus heavily on providing responses relating more to pedagogical content knowledge concerns due to their own identities as future mathematics educators. The second question explored participants' perspectives towards challenges faced by students in learning mathematics, and keywords from responses were categorized into two domains of either internal or external constructs. Internal constructs were defined as factors related to students' individual identity, cognition, and disposition that contribute to potential challenges in learning of mathematics. External constructs were defined as factors that were not controlled by the individual students, such as transferring of blame to a teacher or class, lack of proper curriculum or teaching, etc. The third question examined participants' perspectives towards the notion that there is a greater societal acceptance and normalization towards innumeracy (in contrast to illiteracy, which is widely disfavored by society). Participants were asked to hypothesize why this potential societal acceptance may be prevalent, and keywords from their responses were categorized into two domains: normalization of innumeracy a) developed from individual beliefs and assumptions or b) development from prior experiences or action. Factors that were classified as personal beliefs and assumptions were those such as individual perceptions on the challenge of mathematics that contributed to the formation in normalization of innumeracy. The final question asked participants to propose solutions for addressing numeracy in their own classrooms as future educators. Participant responses were categorized into two domains here as well as reflections on addressing innumeracy through a) action through change in beliefs or b) action through change in practice. In these domains, action through change in beliefs categorized suggestions from the participant responses such as generating conceptual connections for students or establishing positive environments for learning. In comparison, responses that indicated approaches for action through change in practice described keywords such as modeling, problem-based or inquiry-based learning, or implementing varied questioning in the classroom.

Findings

As predicted, participant responses greatly focused more on limitations of the pedagogical content knowledge (PCK) domain as the contributing factor to their hypotheses as to why teachers may struggle in teaching mathematics. Eight out of the eleven students noted PCK as the main contributing factor that they felt determined challenges in teaching mathematics. Responses regarding perspectives on the normalization of innumeracy likewise had a majority in favor for one of the domains than the other. Eight out of the eleven participants indicated that they perceived that the normalization of innumeracy is caused by the prior experiences or action domain. The final set of pre-service teacher perceptions showed the greatest mixture of responses. While the majority of participant responses had a distinct position or preference for one domain over the other in each survey question, there were a select few participants who did indicate keywords highlighted from their reflections for both domains.

Discussion

In addressing the first research area of pre-service teachers' perceptions of perceived challenges in teaching and learning mathematics, participant responses were highly reflective of their identities as beginning teachers. With the first question regarding challenges for teaching mathematics, the majority (eight out of eleven) participants chose to highlight pedagogical content knowledge (PCK) as the main contributing factor. As the participants mainly comprised of undergraduate students who were majoring in mathematics, an assumption exists that each of them already have a strong content background in mathematics. Therefore, this perhaps explains the lower attention and priority given to highlighting content knowledge deficits as contributing factors for creating challenges for mathematics teaching.

Although the participants were able to provide substantial responses for all four questions, there were some unclear examples of responses for these last two questions on the factor contributing to implicit bias of normalization of innumeracy. This suggests that perhaps these students did not have substantial understanding of the concept of innumeracy to adequately answer the question. They also demonstrated significantly more elaboration in their responses regarding questions that presented an actionable problem. Participants also demonstrated optimistic responses overall, particularly for the questions asking them to provide suggestions for how they might approach addressing the two factors contributing to implicit bias against mathematics teaching and learning. This further emphasizes the importance of providing opportunities for teachers (both pre-service and veteran) to engage in reflexive pedagogy where they can reflect and continue to improve on teaching.

Implications for Future Research

Future research should investigate best practices in which teachers (both pre-service teachers and veterans) can address implicit bias in their classrooms through actions such as implementing micro-affirmations for generating positive dispositions towards math as well as inclusive and reflective pedagogical methods that encourage productive mathematics learning. It is likewise important for both teachers and teacher educators to collaborate in addressing potential implicit biases that may exist in schools and school systems. Further examination of resources and other ways to support mathematics teachers in addressing these issues is also another area to explore for math teacher education research.

References

Aslan-Tutak, F., & Adams, T. L. (2015). A Study of geometry content knowledge of elementary preservice teachers. *International Electronic Journal of Elementary Education*, *7*(3), 301.

Boaler, J., & Greeno, J. G. (2000). Identity, agency, and knowing in mathematics worlds. Multiple perspectives on mathematics teaching and learning, 171-200. Chicago

Franke, M. L., Kazemi, E., & Battey, D. (2007). Mathematics teaching and classroom practice. Second handbook of research on mathematics teaching and learning, 1(1), 225-256. Chicago

Graeber, A. O. (1999). Forms of knowing mathematics: what preservice teachers should learn. Educational Studies in Mathematics, 38(1–3), 189–208.

Hill, H., Ball, D. L., & Schilling, S. G. (2008). Unpacking "pedagogical content knowledge": Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, *39*(4), 372–400.

McClain, K., & Cobb, P. (2001). An analysis of development of sociomathematical norms in one first-grade classroom. *Journal for Research in Mathematics Education*, 236-266.

Southwell, B., & Penglase, M. (2005). Mathematical knowledge of pre-service primary teachers. In H. L. Chick & J. L. Vincent (Eds.), International Group for the Psychology of Mathematics Education (vol. 4, pp. 209–216).

Thompson, A. G. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. *Educational studies in mathematics*, *15*(2), 105-127.

INDIGENIZING THE MATHEMATICS CURRICULUM WITH PRE-SERVICE TEACHERS

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In this brief research report, we examine shifts in the attitudes of pre-service teachers related to Indigenizing the mathematics curriculum through the lens of positioning theory. Written responses were examined and examples of obscuring agency, reframing of mathematics, retemporalizing and repersonalizing and more openness to rethinking attitudes emerged. By analyzing attitudes of pre-service teachers we can begin to have productive conversations.

Keywords: Indigenous Ways of Knowing, Pre-service Teacher Education, Positioning Theory.

Introduction

In recent years there has been a call to include Indigenous knowledge at all levels of schooling (BC Ministry of Education, 2015), which has proven challenging for both veteran teachers and new teachers alike. Mathematics, a discipline often associated with objectivity and abstraction, does not appear to align easily with an Indigenous approach to learning. Therefore, it is not surprising that pre-service and practicing teachers have expressed anxiety in response to British Columbia's revised curriculum, which states that mathematics teachers are to '[i]ncorporate First Peoples' worldviews and perspectives to make connections to mathematical concepts' (BC Ministry of Education, 2015). Consequently, there is a need in pre-service mathematics education to reconcile Western mathematics with Indigenous ways of knowing.

As non-Indigenous researchers, we consider teaching and learning Indigenous approaches that support holistic, personal and connected activities to be sound pedagogical practice. According to Munroe et al. (2013), 21st century learning approaches are not new ideas, but are "rooted in old ideas embedded in Indigenous knowledges" (p. 319). When the learning of mathematics includes hands-on experiences, for example, this can provide opportunities of agency, and personal relevance. In creating and in doing, mathematics becomes a discipline of practice, rather than memorizing propositional facts. The First Peoples' Principles of Learning (FPPL), created by the First Nations Education Steering Committee (FNESC) in British Columbia endeavour to reflect a respectful and holistic approach to First Peoples' values around teaching and learning. Learning is "focused on connectedness, on reciprocal relationships, and a sense of place" (FNESC, 2008). It involves generational roles and responsibilities, patience and time, an exploration of identity and recognizing the consequences of one's actions.

This research explores how a non-Indigenous instructor and teacher assistant, with a commitment to honour Indigenous values, sought to develop a respectful and culturally aware, pre-service mathematics teacher education course. This research is focused on transforming attitudes around what mathematics is and can be, while moving away from dominant ideologies and colonial practices, to more relational and respectful ways of knowing.

Theoretical Framework

We have chosen positioning theory as a way to understand the views and commitments of pre-service teachers in relation to both mathematics and to an Indigenous framing of mathematics. van Langenhove and Harré (1999) describe positioning as a stance expressed through words or actions that affects one's relation to someone else. There is an underlying moral component to the relations created during this positioning. There are three particular constructs of positioning theory: communication acts, positions and storylines. Communication acts are about the words used in an utterance, positioning is about how an interlocutor locates themselves in relation to someone else, and storyline is the underlying common narrative suggested, though not necessarily made explicit in the utterance. Each of these constructs interweave and connect with each other.

Within the teaching outlined in this study, we observed a transformation in how pre-service teachers viewed their role as teachers. Their previous experiences with mathematics learning were framed in terms of acquiring a skill set, causing them to refer to mathematics as distant and objective. However, when mathematics is more people oriented and is framed in terms of discussion, it aligns more closely with Indigenous ways of knowing. Using techniques of discourse analysis, such as voice, agency and deixis (Wagner, 2007), as well as looking at nominalization (Gerofsky, 2004) to inform positioning theory, we assess how pre-service teachers position themselves in relation to mathematics and Indigenous ways of knowing.

The goals for this research were to evaluate by the end of the course: 1) Changes in students' perceptions of the nature of mathematics; 2) Changes in students' perspectives and understandings of Indigenous ways of knowing and learning as they relate to mathematics.

Methods

Data from this study was collected from undergraduate students enrolled in a secondary mathematics methods course at a Western Canadian University in the summer of 2017. Having received some Indigenous education in previous courses, students were somewhat familiar with FPPL but had no experience connecting FPPL to mathematics. In order to ascertain student perceptions regarding the nature of mathematics and changes in perspectives of Indigenous ways of knowing, students were asked to respond in writing to the following questions at the beginning and at the end of the course: (1) What is mathematics? What does it mean to do mathematics? (2) In what ways are Indigenous ways of knowing and mathematics connected? In terms of knowing, learning and/or practicing, can you think of ways mathematics may relate to an Indigenous world view? The research team then examined all student responses to identify changes in discourse. Using a positioning theory lens, we highlight changes in student wording as a method to identify shifts in student response as well as themes that emerge.

Throughout the course, an environment was created where thinking was brought into the open to be scrutinized and re-evaluated, and notions of time, person and context were introduced into mathematical practice, rather than valuing content only. In practice, most students, are much more receptive to relational ways of engaging with mathematics because they are ‘part’ of the subject material, which aligns with Cajete (2012), who suggests ‘we’, as mathematics educators and teachers, stay away from objectification and decontextualization.

Data and Analysis

In this section we present a comparison of some of the student reflections from the beginning of the course, to those at the end. Due to the brevity of the data segments, we have combined the data and the analysis together and have categorized the data under some of the themes that emerged from our discourse analysis. All student comments are presented verbatim.

Obscuring Agency

One of the things that was noticeable was a complete lack of agency in terms of how most students described mathematics, which was indicated by the lack of an 'I' or a 'we' in their statements. However, by the end of the course, students were including not only themselves in their articulations, but others as well. In the presentation of data below (B) and (E) will represent 'beginning of course' and 'end of course' respectively.

Johanna: (B) When individuals are able to approach math problems…
(E) Mathematics is embedded in our world. It surrounds us and allows us to solve problems.

Mike: (B) Math is recognition of how patterns into an abstraction of the universe. Doing math means to discover truths.
(E) Math is a process for making sense of the patterns in the universe. To do this means to categorically recreate what we think and be able to reconstruct it for somebody else.

When a person obscures agency, they position themselves away from, or at a distance from that which they are describing (Wagner, 2007). This was also evident from the use of a common framing at the beginning of the course, that mathematics was all around them. While seemingly inclusive and holistic, this still obscures agency by not referring to a person when describing mathematics. Most students were both much more specific and present in their description of mathematics at the end of the course.

Reframing of Mathematics

Another significant change that occurred, related to how students described mathematics. The initial view that mathematics was about calculating and finding patterns had shifted to one that saw doing mathematics as involving collaborating and negotiating.

Charlotte: (B) Mathematics is a way of understanding and interpreting situations. It often involves problem solving and logical reasoning.
(E) From this course my understanding of how one does mathematics has radically changed from a more traditional, algorithmic approach to a more interactive problem based perspective where knowledge is gained through exploration, play and discussion.

Mathematics was repositioned as something that exists in activity with both people and things. This shift from understanding or interpreting something, towards ways of engaging with mathematical objects or ideas, also supports an Indigenous way of thinking that Lunney Borden (2011) calls verbification.

Retemporalizing and Repersonalizing

Jenna: (E) Math is collaborative, and every person could approach the same problem differently. We can trust that in group work, we can be inspired by other people and inspire them at the same time. Our thoughts could be broadened and completed by other people's thoughts. The discussion and collaboration may not be the most efficient way but 'learning requires time and patience'.

From this statement, it seems that group work was not simply about collaborating to find an answer, but instead spoke of inspiration, broadening perspective, and truly supporting each other in a willingness to engage with a mathematical problem and take risks while doing so.

Indigenizing the Curriculum

Although we recognize that many of the comments shared below are overgeneralized, we found that the positionality students were expressing had shifted to an openness and willingness to rethink their attitude and ways of doing mathematics.

Angel: (B) Indigenous ways of learning involves patience and time which is pertinent to mathematics education as mathematical problems are rarely solved immediately. Learning math requires dedication, practice, patience.

(E) I believe that mathematics is inherently a social endeavor, and so the process of learning and doing mathematics aligns well with Indigenous ways of knowing. I think that mathematics is a universal language that connects all of humanity.

John: (E) Learning is holistic in terms of there is a context, there is social and relational piece in it, and it is reciprocal. This is directly connected to the Indigenous belief of learning is to benefit community and the society.

These comments reflect the general attitude of many of the pre-service teachers at the end of the course and demonstrate a shift from a cultural dominant view of mathematics as a field available for study, to a more approachable discipline that engages people in supporting one another and negotiating meaning. Students have positioned themselves in a more inclusive and open space for interpretation and understanding of what it means to do mathematics.

Conclusion

This research involves attempting to change pre-service teachers' attitudes through an approach towards Indigenizing the mathematics curriculum by focusing on methods of learning mathematics and does not claim to successfully Indigenize the curriculum. Positioning theory has been used as a marker of a relationship which is relevant and informative for this course that was modeled on the relationship between people, mathematics and people, and between Indigenous ways of knowing and the pre-service teachers in this course. This research is meant to be generative of conversation, discussion and future research.

References

BC Ministry of Education. (2015). *Aboriginal worldviews and perspectives in the classroom.* Victoria, BC: Queen's Printer. Retrieved from https://www.bced.gov.bc.ca/abed/ awp_moving_forward.pdf

Cajete, G. (2012). Contemporary Indigenous education: Thoughts for American Indian education in a 21st-Century world. In Mukhopadhyaya, S. & Roth, W.-M. (Eds.), *Alternative forms of knowing (in) mathematics* (pp.33-51). Rotterdam, NLD: Sense Publishers.

First Nations Education Steering Committee. (2008). *First Peoples' Principles of Learning*. Retrieved from http://www.fnesc.ca/wp/wp-content/uploads/2015/09/PUB-LFP-POSTER-Principles-of-Learning-First-Peoples-poster-11x17.pdf.

Gerofsky, S. (2004). *A man left Albuquerque heading east: Word problems as genre in mathematics education*, New York, NY: Peter Lang.

Lunney Borden, L. (2011). The 'verbification' of mathematics: Using the grammatical structures of Mi'kmaq to support student learning. *For the Learning of Mathematics, 31*(3), 8-13.

Munroe, E.A., Lunney Borden, L., Murray, A., Toney, D., & Meader, J. (2013). Decolonizing Aboriginal education in the 21st century. *McGill Journal of Education, 48*(2), 317-337.

Wagner, D. (2007). Students' critical awareness of voice and agency in mathematics classroom discourse. *Mathematical Thinking and Learning, 9*(1), 31-50.

van Langenhove, L., & Harré, R. (1999). Introducing positioning theory. In R. Harré, & L. van Lagenhove (Eds.), *Positioning theory: Moral contexts of intentional action* (pp. 14–31). Blackwell: Oxford.

PROSPECTIVE TEACHERS' INTERPRETATIONS OF ARGUMENTATION AND CORRESPONDING TEACHING ACTIONS

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We examine connections between teacher education coursework and teaching actions. In particular, we looked at one unit of instruction addressing collective argumentation in a pedagogy course and how two student teachers supported collective argumentation in their placements during the following semester. The student teachers' support for collective argumentation aligned with their interpretations of the unit on argumentation, reflecting their slightly different emphases of the role of the teacher in facilitating student participation and focusing on the goal of the argument. Our results suggest prospective teachers' interpretations of coursework can be reflected in their practice, even in student teaching.

Keywords: Classroom Discourse, Teacher Education-Preservice

Connections between teacher education coursework and teacher actions, even in student teaching, are difficult to find, and even more difficult to document. In attempting to trace teacher learning across multiple semesters, the project described in this paper begins to document how some aspects of teacher education coursework may be visible in student teachers' practice. Rather than focusing on the specific contents of the coursework, we rely on our participants' interpretations of their coursework, as inferred from various sources, and look to see how their practices reflect these interpretations. Our focus is how teachers learn to support collective argumentation in secondary mathematics classes; this paper addresses the question: How do prospective teachers' interpretations of a unit on collective argumentation (during a pedagogy class) align with their support for collective argumentation during student teaching?

Related Literature & Framework

Argumentation and proof is an integral part of mathematics. Researchers have investigated many aspects of argumentation, including student learning through argumentation (e.g., Krummheuer, 1995), and relationships between classroom argumentation and proof (e.g., Knipping, 2008). We view argumentation as attempting to convince someone of the validity of a claim and collective argumentation as people working together to determine the validity of a claim (Conner, Singletary, Smith, Francisco, & Wagner, 2014). We follow Krummheuer (1995) in using Toulmin's (1958/2003) model of an argument, adopting the vocabulary of claim, data, warrant, qualifier, and rebuttal to describe distinct parts of an argument. We attend to who contributes these parts of arguments; distinguishing between teacher, student, and joint contributions provides insight into teacher facilitation of argumentation (Conner et al., 2014).

We view learning as situated (as discussed by Lave and Wenger, 1991). That is, we follow Peressini, Borko, Romagnano, Knuth & Willis (2004) in asserting that people learn within multiple communities, and we investigate how prospective teachers may recontextualize (as used by Ensor, 2001) their on-campus learning in their student teaching practice. For instance, prospective teachers may experience collective argumentation as they learn mathematics content, and they may learn about collective argumentation in their pedagogy courses. These experiences must be recontextualized into their own practice in student teaching.

To guide our analysis of teachers' support for collective argumentation, we use the Teacher Support for Collective Argumentation (TSCA) framework (Conner et al., 2014). This framework provides three ways teachers may support collective argumentation: directly contributing argument components, asking questions that prompt components, and responding to components of arguments using other supportive actions. Each type of support has five categories, and within the categories of questions and other supportive actions, there are multiple kinds.

Methods

We report on part of a larger project following a cohort of prospective secondary teachers through their mathematics education program, into student teaching, and beyond. We purposely selected two participants as they participated in a unit on argumentation during their course work, a practicum experience, and their student teaching experience. Cathy and William (pseudonyms) were white, cisgender and in their early 20s.

Data analysis was conducted using Transana (Woods & Fassnacht, 2010). We identified episodes of argumentation (EOA) in the student teaching data. We constructed expanded Toulmin diagrams (Conner et al., 2014) for each EOA and coded teacher support based on the TSCA framework. The Toulmin diagrams provided information on the characteristics of the arguments while the TSCA framework provided analysis of the teachers' support of the arguments. We captured these thick descriptions of the participants' argumentation in memos.

The second phase of data analysis focused on the argumentation unit during the third semester pedagogy course. During the unit, participants discussed collective argumentation and Toulmin (1958/2003) diagrams and constructed diagrams for classroom arguments. For each participant, we coded his or her reflections and discussions during the unit to describe their interpretations of argumentation. We expanded each participant's memo to examine how interpretations of argumentation aligned with argumentation practices during student teaching.

Results

Cathy's Interpretation of Argumentation

Cathy saw the role of the teacher as a facilitator who asks questions, clarifies ideas, and restates contributions. She wanted students to be the main contributors to the arguments. She also wanted to make sure students pushed each other's thinking. When discussing a class reading (Choppin, 2007), Cathy said, "Your job is not to go in and argue for the student, …you have to let them work through it and possibly have other students point out and ask, like okay why did you do this?" (In-class discussion). She believed all students were capable of contributing to the conversation as long as they were comfortable within the group. Cathy wrote the following in her reflection after orchestrating a discussion in her practicum, "I wanted each of the students in my group to feel like they contributed to the discussion…If one student hadn't shared in a while, I made sure to call on that particular student." Cathy believed her role as a teacher included having students be the ones constructing and communicating the arguments.

In her written reflection, she described an effective argument as one that involved multiple students, and she labeled as ineffective an argument involving only one student. Cathy saw effective arguments as those promoting students to share their thinking. "Why would you engage students in arguments?...So they could just like—whatever math they're doing in different ways and see there is more than one way to think about it" (In-class discussion). Overall, for Cathy, student participation in the construction of arguments was a way to provide students an opportunity to learn strategies and ways of reasoning mathematically.

Cathy's Support for Argumentation

In Cathy's classroom, students directly contributed 68 claims and data/claims (out of 75) and 41 explicit warrants (out of 47). This shows Cathy was committed to having the students be the ones constructing the arguments in the classroom. Analysis of her supportive moves reinforces this result, as she revoiced the students' contributions 64 times and expanded on their contributions minimally (7 times in two days). Specific discursive actions were taken by Cathy to promote student participation. For example, Cathy made statements like the following: "Somebody raise your hand and tell me how it's not fair. I see [St 1] and [St 2]'s hands. I need a few more, guys" and "Do y'all notice anything? I'll wait until I see a few more hands." Her desire to have students participate was also evident from her lack of contributions of argument components (claim, warrant, data). During the two days, Cathy made 16 direct contributions to arguments out of the 149 explicit argument components. The remaining 133 contributions were made by students (128 contributions) or co-constructed by students and Cathy (5 contributions).

Cathy wanted many students to participate in the construction of arguments. She thought it was important for the teacher to facilitate the students' work in argumentation, and in her student teaching, we found she asked questions that pushed students to provide their reasoning and she often restated student contributions to be sure they were understood by the class.

William's Interpretation of Argumentation.

William believed that in order to facilitate argumentation effectively, the teacher should have a clear idea about the goals of argumentation. During the argumentation unit, William said:

> I feel like a huge part of the argument is knowing where you're going with it, too. … Cause I mean it is easy to facilitate an argument and not know exactly where to go and let it, so that's when you kind of can let it spiral out of control. (In-class discussion)

William viewed argumentation as a debate, and he thought it was beneficial to let students argue with each other because "Students listen to each other more than they'll listen to us" (Interview). William wanted his students to participate in collective argumentation. However, he found that having students participate and reaching his goal to cover content was a difficult balancing act. In William's written reflection, he wrote about an argument he witnessed during his field experience between two students. He found the argument to be effective "because it required both students to approach the problem in a different way then they initially did." He expanded that the argument he witnessed was effective because "mathematical concepts are being grappled with" (Reflection). For William, an effective argument not only involves multiple approaches to a problem but also provide opportunities for students to deal with mathematical concepts behind them. In contrast, William viewed an ineffective argument as one that only dealt with simple errors about a procedure or computation.

William's Support for Argumentation

Our analysis suggested that William primarily supported the argumentation in his classroom by asking questions that requested a factual answer (63 out of 137) and requested a method (31 out of 137), which aligned with his goal-directed orientation (focusing on content). There were only a few instances in which William requested an evaluation, idea, or elaboration. In terms of contributions to the arguments, William and students contributed about the same number of components (99 and 98) and contributed a few components jointly (15). This was consistent with William's desire for students to participate in arguments. However, there was a difference in who contributed claims and warrants. Students primarily contributed the claims (64) in the arguments with some claims jointly contributed with William (7). In contrast, students contributed some warrants (29) in the arguments but most of the warrants were either provided by William (46) or were implicit (40). The contributions of students and William in the EOA align with William's interpretation of the teacher's role as facilitator in reaching the goal of the argument.

Conclusion & Implications

Cathy and William developed consistent interpretations of argumentation with respect to helping students be comfortable in participating in argumentation. Their interpretations of the teacher's role in argumentation involved similar elements, with Cathy focused on the teacher as facilitator to generate student participation and William focused on the teacher as steering students toward the final claim. Our observations of their classrooms revealed consistencies in their interpretations of the teacher's role in argumentation and how they supported argumentation in their classrooms. For instance, Cathy prioritized participation and communication for students both in her student teaching and in her interpretations of the unit. Likewise, William prioritized the goal of the argumentation in his interpretations, and this priority was evident in his focused questions and contributions of components. As prospective teachers recontextualize their learning in their student teaching placements, their interpretations of this learning may take different forms. Our results suggest prospective teachers' interpretations of coursework can be reflected in their practice, even in student teaching. However, understanding their interpretations and working toward productive interpretations of coursework is even more important as seemingly subtle differences may appear much larger in practice.

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References

Choppin, J. M. (2007). Teacher-orchestrated classroom arguments. *Mathematics Teacher, 101*(4), 306–310.

Conner, A., Singletary, L. M., Smith, R. C., Wagner, P. A., & Francisco, R. T. (2014). Teacher support for collective argumentation: A framework for examining how teachers support students' engagement in mathematical activities. *Educational Studies in Mathematics, 86*(3), 401–429.

Ensor, P. (2001). From preservice mathematics teacher education to beginning teaching: A study in recontextualizing. *Journal for Research in Mathematics Education, 32*(3), 296–320.

Knipping, C. (2008). A method for revealing structures of argumentations in classroom proving processes. ZDM: *The International Journal on Mathematics Education, 40*(3), 427–441.

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 229–269). Hillsdale, NJ: Erlbaum.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation.* Cambridge, England: Cambridge University Press.

Peressini, D., Borko, H., Romagnano, L., Knuth, E., & Willis, C. (2004), A conceptual framework for learning to teach secondary mathematics: A situative perspective. *Educational Studies in Mathematics, 56*(1), 67–96.

Toulmin, S. E. (2003). *The uses of argument* (updated ed.). New York: Cambridge University Press. Original work published 1958.

Woods, D., & Fassnacht, C. (2010). *Transana* (v2.42) [Computer software]. Madison, WI: The Board of Regents of the University of Wisconsin System. Retrieved from http://www.transana.org

REFLECTING ON THE ACT OF DEFINING

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This qualitative study sought to explore preservice elementary teachers' (PSETs) beliefs about the role definition plays in the learning of mathematics, as well as how an emphasis on the act of defining promotes the development of mathematical empathy. PSETs completed activities using geometry software, and collaboratively created definitions of special quadrilaterals. An assigned reading and written reflection of their experience followed. Data from these reflections were analyzed for themes and emergent categories. The coding instrument that resulted from iterative discussions by the researchers examined beliefs about the fluidity of definition and mathematical empathy (Araki, 2015). Findings suggest that including PSETs in the act of defining allowed for greater autonomy in authoring definitions, mathematical empathy, and the view that definitions are fluid in nature and dependent upon the audience.

Keywords: Geometry and Geometrical and Spatial Thinking; Instructional Activities and Practices; Teacher Education-Preservice

Introduction & Theoretical Frameworks

Though definitions are considered fundamental in mathematics, there is little agreement on what we believe to be a good definition (Wilson, 1990). There is disagreement in our field regarding whether a definition should be as minimal as possible; with some scholars insisting on a full reduction of extraneous properties and others honoring the role of context, allowing for more redundancy (Zaslavsky & Shir, 2005). Which properties and how many to include in a definition is somewhat arbitrary, and the value of a definition depends on the perspective of its author. That many different definitions can be written for the same concept is difficult for PSETs to understand (Linchevski, Vinner & Karsenty, 1992).

Few studies investigate student's conceptions of a mathematical definition (e.g., Zaslavsky & Shir, 2005). Using written responses and recordings of small group discussions, Zaslavsky and Shir (2005) investigated four students' conceptions of definitions for square and isosceles triangle, and how these conceptions were reflected in and developed through activities that elicit consideration of alternative ways to define a mathematical concept. They found that asking students to consider a variety of definitions is a powerful learning environment wherein personal concept definitions could be gradually refined along with conceptions of definition in general.

In this study, we are seeking to answer two research questions. 1) What are PSETs' beliefs about definitions and the act of defining; and 2) How does an emphasis on the act of defining mathematical terms promote the development of mathematical empathy?

Theoretical Framework 1: Fluidity of Definition

De Villiers (1998) studied what happens when we allow students to realize that different definitions of the same concept are possible with advantages and disadvantages for each. In addition to mathematical characteristics, we also wanted our PSETs to recognize that the choice of which definition to use in mathematics classrooms is also based on the pedagogical context. Keiser (2000) honored instructional context in her work with defining angle. We ask our PSETs to read Keiser (2000) to provide this exposure to student thinking and to support the development of concept imagery related to the act of defining.

From reading of the literature as well as our initial reading of the data, we have designed a *Fluidity of Definition* framework to capture four main themes that we wish to examine: *multiplicity*, *authorship*, *authority*, and *audience*. Multiplicity is the belief that definitions are not rigid and that many alternative definitions exist for a concept. Authorship indicates an ownership over a definition. Drawing on Tall & Vinner (1981), we view a definition as personal as it is based on concept imagery that we hold as individuals. Authority indicates the power to decide which language, style, and properties to include in a definition. Audience honors the pedagogical context and is the belief that a definition is influenced by who we intend to read and use it.

Theoretical Framework 2: Mathematical Empathy

When we take a view of mathematics as a humanistic discipline that is socially constructed and personal values influence our evaluation of results, then it is important for instruction to be participatory and for students to feel that they have agency and voice in the classroom. In order to develop a classroom culture where this is possible, we posit that teachers need to be aware of and able to comprehend the mathematical thinking of their students.

Araki (2015) refers to this skill as *mathematical empathy* defined as "the ability to comprehend another person's ideas and the true meaning or purpose behind them, seeking to utilize the other person's frame of reference" (p. 118). To elevate student thinking within instruction, mathematical empathy is required. Mathematical empathy is what allows us to play *the believing game* (Harkness, 2009) and find the truth in the mathematics that students share.

Methods

This study examines PSETs' beliefs about the role definition plays in the learning of mathematics, as well as how an emphasis on the act of defining promotes the development of mathematical empathy. We enacted this study in two sections of a course on Geometry for Teachers (PK -3) with 71 PSETs. The study was enacted similarly for both sections.

Prior to data collection, PSETs were exposed to dynamic figures constructed using interactive geometry software, and asked to collaboratively create a series of definitions for special quadrilaterals. Following these lessons, they were asked to reflect upon their experience. In order to help them frame their comments as both learners and future teachers of mathematics, we first asked them to individually read *The Role of Definition* (Keiser, 2000). The article was chosen as a catalyst for reflection on this experience because Keiser suggested that formal definitions can curtail thinking in middle grades classrooms and argued for more fluid definitions based on concept imagery (Tall & Vinner, 1981) that was relevant in classroom learning.

Following the reading, we prompted, "*After reading the article The Role of Definition, what new thoughts do you have about the conversations we had in class about defining quadrilaterals? How about using definitions with children?*" Once they had responded to the prompt on an electronic discussion board, they were given access to their classmates' reflections and were asked to participate in an online discussion of what had been shared. The qualitative data in this study comes from the initial posted reflections (n=71). We chose to focus on after-the-fact reflections because we were interested in our PSETs perceptions of the defining activity both as learners and as future educators.

To begin analysis, we read the 71 initial discussion posts and wrote memos about emerging themes from the data. Through discussions by the research team, using grounded theory (Strauss & Corbin, 1990) and top-down methods of Miles and Huberman (1994), we iteratively developed, applied and refined a coding instrument. The coding instrument that emerged was based on theory from research on *fluidity of definition* (de Villiers, 1998) and *mathematical* *empathy* (Araki, 2015). Once a set of initial codes was established, 14 discussion posts were randomly selected from the entire set and coded with this instrument by all three researchers.

Following this second phase of analysis, researchers met to discuss, compare, and come to a consensus for each discussion post. When there was low initial agreement about coding particular categories, discussions resulted in the refinement of code descriptions. In the Results section, we provide definitions and illustrations of what emerged from these discussions.

Results

More results and reflections will be shared during our presentation provided in a longer format to illustrate the different components of the frameworks and give a fuller sense of PSETs' beliefs and intentions. Here, we will present an overview of available findings, including subcategories within the two identified frameworks: *Fluidity of Definition* and *Mathematical Empathy*. These categories are not to be interpreted as mutually exclusive codes, but as different lenses through which to interpret a reflection.

Fluidity of Definition

Multiplicity. Multiplicity is the belief that definitions are not rigid and that many alternative definitions exist for a given concept. Some students reflected on their experience and found value in multiplicity, though this value took different forms. In her reflection, Rebecca expressed a belief that each child has a unique personal concept definition, and that in a classroom context, there is value in negotiating with others about those definitions. Trevor, however, believed that the personal concept definition is something that can be refined over time and experience; that even within one individual, definitions are multiple, fluid and changing. A third notion of multiplicity was described by Funda when she expressed frustration with textbooks that portray a single definition as straightforward and rigid. She sees this as restricting instruction unnecessarily and describes a freedom in expanding on the definitions in a local community.

Authorship. Authorship indicates an ownership over a definition. Funda's reflection also indicated a desire for authorship, but also a feeling of helplessness when using a textbook. Josie also saw purpose and value in being able to author definitions based on our classroom activity.

Autonomy. Autonomy indicates the power to decide which language, style, and properties are useful to include in a definition. When Sheena says "All of the definitions we decided were what made sense to us (and they were correct which is a big part of it as well)…we got to play around with our own wording and what we felt it was important to know about each different quadrilateral, which makes it much more personal to us and easier to understand."

Audience. Audience honors the pedagogical context and is the belief that a definition is influenced by who we intend to read and use it. In reflections on both the classroom activity and the assigned reading, it was clear that there was an impact on our PSETs' conception of audience as important in the act of defining. Here, definitions are not just mathematical tools to be internalized, but teaching tools that help convey meaning to others.

Mathematical Empathy

In efforts to code our data for evidence that our PSETs exhibited *mathematical empathy*, it became clear to us that there were (at least) two stages of empathetic work: *awareness* and *comprehension*.

Empathetic Awareness. Empathetic awareness lives in a general space where the speaker understands that there is multiplicity in mathematical perspectives. (Not to be confused with multiplicity in concept definitions.) Awareness can emerge as a belief that others see things differently or that students have different mathematical backgrounds, experiences, or understandings that are worthy of attention and understanding. However, not all statements that show awareness value multiplicity. Sometimes multiplicity indicates deviation from "standard" or inferior. When different means deviant, then awareness can be akin to feelings of what we have come to call *mathematical sympathy*, or the expression of feelings of pity or sorrow that we have understood another's ideas, but find them inferior or undesirable. As one student wrote, "I believe that if the students were just given a definition they would not have been able to explore what they thought it was and learn how their ideas were similar or different to the definition."

Empathetic Comprehension. Without a specific statement of understanding from another's perspective, awareness is not evidence of *comprehension*. In order for us to experience mathematical empathy, we must be able to comprehend from another perspective. This is akin to what Hufferd-Ackles, Fuson, and Sherin (2004) refer to as *revoicing* an idea shared by another. For example, Henry expressed mathematical empathy for a student, Dave, from the article, and went so far as to say that our discussion was lacking because this perspective was not available.

Conclusion

Coming to a consensus about how to define a special quadrilateral exposed PSETs to more than just the properties of quadrilaterals, but also the act of defining. Experiencing, albeit in a vicarious way, the difficulties faced by sixth-grade students in defining angle created space for mathematical empathy, including both awareness and comprehension. As Henry stated above, the story of Dave's confusion resonated with our students and provided an opportunity to see from someone else's perspective. When combined with their own fraught experience negotiating the properties of kites and trapezoids, the article about angle (Keiser, 2000) enabled PSETs to see far greater subjectivity in the discipline of mathematics and to consider, perhaps for the first time, that they, too, were both able and deserving of becoming authors of mathematical ideas.

References

Araki, M. E. (2015). *Polymathic leadership: Theoretical foundation and construct development*. (Masters dissertation, PUC-Rio).

de Villiers, M. (1998). To teach definitions in geometry or to teach to define? In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 248–255). Stellenbosch, South Africa: University of Stellenbosch.

Harkness, S. S. (2009). Social constructivism and the Believing Game: A mathematics teacher's practice and its implications. *Educational Studies in Mathematics*, *70*(3), 243-258.

Hufferd-Ackles, K., Fuson, K. C., & Sherin, M. G. (2004). Describing levels and components of a math-talk learning community. *Journal for Research in Mathematics Education, 35*(2), 81-116.

Keiser, J. M. (2000). The role of definition. *Mathematics Teaching in the Middle School*, *5*(8), 506-511.

Linchevski, L., Vinner, S., & Karsenty, R. (1992). To be or not to be minimal? Student teachers' views about definitions in geometry. In W. Geeslin, & K. Graham (Eds.), *Proceedings of the 16th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 48-55). Durham, NH: University of New Hampshire.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis: An expanded sourcebook* (2nd ed.). Thousand Oaks, CA: Sage Publications.

Strauss, A., & Corbin, J. (1990). *Basic of qualitative research: Grounded theory procedures and techniques*. Thousand Oaks, CA: Sage Publications.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151-169.

Wilson, P. S. (1990). Inconsistent ideas related to definitions and examples. *Focus on Learning Problems in Mathematics, 12*(3-4), 31-47.

Zaslavsky, O., & Shir, K. (2005). Students' conceptions of a mathematical definition. *Journal for Research in Mathematics Education, 36*(4), 317-346.

PROSPECTIVE TEACHERS' CENTRAL BELIEFS ABOUT MATHEMATICAL MISTAKES

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Standards-based documents suggest that mistakes should be viewed as more than dead ends and should be used as catalysts for discussion and learning. However, prospective teachers rarely experience the utility of mistakes, which influences their beliefs concerning mathematical mistakes and ultimately how they teach and treat mistakes in their future classrooms. The influence that beliefs have on teaching is well known, but the beliefs concerning mathematical mistakes are not. Research suggests that new experiences can change beliefs given time and reflection. This study explored the beliefs of two prospective teachers concerning mathematical mistakes and how those beliefs changed during their enrollment in a content course for teachers designed to align with NCTM recommendations.

Keywords: Preservice Teacher Education, Teacher Beliefs

Mathematics reform documents (e.g., Common Core State Standards Initiative [CCSSI], 2010; National Council of Teachers of Mathematics [NCTM], 2014) encourage teachers to teach in ways that focus on student learning and leverage students' experiences with mathematics content. Teaching in the spirit of these documents involves engaging students in meaningful tasks as well as encouraging student interaction and discussion centered on the students' work (NCTM, 2014). Many of these discussions focus on students' mistakes and provide opportunities for students to reflect on those mistakes (NCTM, 2014). However, the way reform documents describe teaching is in stark contrast to traditional ways of teaching that remain pervasive in today's mathematics classrooms (Philipp, 2007).

In mathematics, teachers' beliefs play a significant role in teaching and learning (Pajares, 1992; Philip, 2007). Their beliefs influence not only how they teach mathematics but what mathematics is, what it means to do mathematics, and how they envision the roles of the teacher and the students in a mathematics classroom (Cooney, Shealy, & Arvold, 1998; Thompson, 1992). This includes how prospective teachers (PTs) will treat mathematical mistakes in their future classrooms. If mathematical mistakes are to play the role that research and reform documents advocate, then teachers' beliefs concerning mistakes need to align with that role, or mathematics education's progress will remain compromised (NCTM, 2014).

When prospective teachers (PTs) enter their teacher preparation programs, their beliefs and belief structures are already well established (Cooney et al., 1998; Pajares, 1992). Their experiences include the PTs' years of being a student in primary and secondary mathematics classrooms (Philip, 2007; Thompson, 1992). Understanding the beliefs of PTs is the first step in ensuring alignment of PTs' beliefs concerning mistakes with mathematics education research and teaching documents (e.g., CCSSI, 2010; NCTM, 2014). Additionally, accounting for the context in which those beliefs are investigated is necessary. Therefore, the purpose of this study was to investigate PTs' beliefs concerning mathematical mistakes while considering the error climate of the classroom. This leads to the primary research question of the study: What beliefs do PTs' hold about mathematical mistakes, and how do those beliefs change, if at all, given the classroom error climate?

Theoretical Framework

Two theoretical constructs guided the study. The first of these was the *error climate* of the classroom (Steuer, Rosentritt-Brunn, & Dresel, 2013). Steuer et al. (2013) assumed that although learning from errors depended on individual characteristics of learners the classroom context was a precursor to those. Thus, classroom climate facilitates learning from errors and the potential to transcend individual attributes.

The second construct was Leatham's (2006) theoretical framework for viewing beliefs as a sensible system. The sensible system framework assumes that an individual, whether that person is able to articulate a belief, organizes beliefs into "organized systems that make sense to them" (p. 93). Leatham's (2006) framework consists of three dimensions that assist in visualizing an individual's belief. The first of which is the psychological strength of a belief. This is a description of the relative importance that a belief is to an individual that can vary from central to peripheral. The second dimension is the "quasi-logical relationship" (Leatham, 2006, p. 94) that exists between beliefs. The final dimension of the sensible system framework is the extent to which beliefs are clustered in isolation from other beliefs. This clustering of beliefs allowed for the contextualization of beliefs and explained why a belief in one context may not be as central in another. This framework provided guided the data collection and data analysis. Together, the error climate and beliefs as a sensible system framework provided a lens to examine what PTs' beliefs were concerning mathematical mistakes.

Methodology

In this study, I utilized an exploratory, multiple case design to investigate PTs' beliefs concerning mathematical mistakes and how those beliefs change, if at all. With a variety of criteria available from which to select the two cases (e.g., ACT score, mathematical content knowledge), I selected participants that displayed characteristics of different implicit theories (see Dweck & Leggett, 1988), but in the middle of the implicit theory continuum for data collection purposes. The decision to select participants based on implicit theory was made because of the role that implicit theory plays in persisting when making mathematical mistakes and that implicit theory transcends levels of mathematical ability (NCTM, 2014).

As beliefs are not able to be articulated directly and must be inferred by the researcher (Leatham, 2006; Pajares, 1992), a variety of opportunities were given to the participants to describe their beliefs concerning mathematical mistakes over the fall semester as they were enrolled in a mathematics content course designed for PTs that intend to teach at the elementary school level. The course was designed to ensure that PTs were routinely exposed to mathematical mistakes in group tasks, homework, and tests. Additionally, the participants were given numerous opportunities to reflect on the mistakes presented in class as well as their own mistakes during data collection. Data sources included error climate surveys, interview protocols, classroom observations followed by interviews, reflective journals, exit tickets, and in-class reflections. I analyzed the data chronologically using open coding. Codes were created based on distinct concepts and categories that originated in the data.

Results

Results from the error climate surveys, which were an indication of the participants' perceived error climate, differed noticeably for both participants from the beginning of the study to the end. The most notable differences for Cindy, a pseudonym, were her view of how mistakes are perceived by others in the class and how mistakes are used in a mathematics classroom. Although her view of her past mathematics classrooms indicated that a student can learn from mistakes, there was a shift in the emphasis of learning from mistakes by the end of the study. Additionally, evidence suggested that Cindy perceived that mistakes were more welcomed by both the teacher and students. In contrast, the most notable shift in Harley's perceived error climate was how she interpreted the teacher's reactions to mistakes. Evidence from the beginning of the study suggested that Harley believed that mistakes were not important for the teacher and that learning from mistakes was solely the responsibility of the learner, not the teacher or the class as a whole. By the end of the study, Harley perceived mistakes as something everyone could learn from and saw mathematical mistakes as both a personal learning and pedagogical tool that should be used often.

As the two participants' other qualitative data were analyzed, beliefs concerning mathematical mistakes emerged that were considered as central to each of them. These central beliefs transcended context and were influential in how the PTs made sense of specific mistakes as well as the repeated presence of mathematical mistakes in their classroom. A brief description of each participant's central beliefs concerning mathematical mistakes is presented below.

Cindy

From the onset of the study, Cindy believed in the value of mathematical mistakes. Throughout the course of the study, Cindy expressed how she learns from mistakes. She wrote:

> I don't like to make them at all. But even though I don't like to make them, mistakes are a part of life. You learn from mistakes, how to fix them, or make them better. You can be told how to do it, and sometimes pick up on it, but when I make a mistake, I learn why I made the mistake and how to fix it. Then, try again to figure it out. Mistakes will happen, but you need to be able to accept the mistake and learn from it. (Cindy, Journal entry, Dec. 4, 2017)

The value of mistakes for Cindy surfaced in many instances including when she could not reconcile what she learned from a mistake, stating "you can always learn something." This persisted throughout the study, but what she learned from a mistake shifted from learning how to fix the mistake to being able to go to "a deeper level" of knowledge as the study ended.

Cindy also believed that mathematical mistakes had the most service if "the majority of the class made that mistake." Void of the type of mistake (e.g., "minor" or "major") or when the mistake was made (e.g., class discussion, homework, or test), the number of people making the mistake was extremely important to her. This was by far the stickiest (Thompson, 1992) belief Cindy held throughout the study.

Finally, Cindy believed mathematical mistakes should be used as moments for learning for both students and for her as a prospective teacher. Mistakes that were presented during in-class activities, homework, and tests were done so in an effort for her to be able to anticipate what her future students might do, and more specifically what the majority of them might do. Additionally, and from Cindy's perspective, the mathematical mistakes that were highlighted and discussed in class helped her achieve a deeper of learning and to "know more than her [future] students."

Harley

Harley believed mathematical mistakes were instances where someone either made a mental mistake (e.g., adding incorrectly) or was an indication of a lack in understanding. At the beginning of the study, Harley believed that mistakes needed to be fixed either by someone pointing out the mental mistake or fixed by someone that did not have the same lack of understanding. Similar to Cindy, learning from this mistake was the responsibility of the person that made it, especially after being shown how to correct it.

Similar to Cindy, Harley also believed that mistakes had the most utility when a majority of the class was making the mistake, regardless of what the mistake was. This belief was pervasive for Harley and is best captured in her final interview when asked about when a mistake is useful and best used for class discussion:

> It depends on like how much time you have in that class and what mistakes are going to be more important, not more valuable, but more...cause they're all valuable to learning but just need to be said more than others. Like I said, if I'm the only one getting it wrong, that's just one person, but when the whole class is messing up, they're all going to fail if they don't get this. That one needs to be brought up, [and] then she can come back to me if there's time in the class. I think it's like that. (Harley, Final interview, Nov. 30, 2017)

Harley believed that mathematical mistakes were extremely valuable, and this belief strengthened as the study progressed. For Harley, mathematical mistakes could be used to ensure that you "*really* understand something," and to elicit different ways of thinking about the same problem, which is aligned with NCTM's view of the role that mistakes should play.

Discussion

Central beliefs were persistent and mediated how other beliefs are clustered, interpreted, and in some cases influenced in different contexts. Furthermore, these central beliefs were used when PTs make sense of situations that are in contention with their current beliefs, which emphasized the importance of understanding what those beliefs are. For teacher educators, understanding PTs' central beliefs concerning mathematical mistakes is crucial if there is going to be an effort to change them. Additionally, understanding central beliefs can inform teacher educator high-leverage practices to further support alignment of PTs' beliefs with literature on how mistakes should be used in the mathematics classroom and ultimately the PTs' classrooms. Findings suggest PTs notice how mathematical mistakes are used in their teacher preparation classes, especially if they are used differently compared to previous mathematics classrooms. Leveraging that environment and making the purpose of using mathematical mistakes explicit is a first step in aligning PTs' beliefs with literature recommendations.

References

Common Core State Standards Initiative. (2010). *Common core state standards for mathematics*. Washington, DC: National Governors Association Center for Best Practices & Council of Chief State School Officers.

Cooney, T. J., Shealy, B. E., & Arvold, B. (1998). Conceptualizing belief structures of preservice secondary mathematics teachers. *Journal for Research in Mathematics Education, 29*, 306-333.

Dweck, C. S., & Leggett, E. L. (1988). A social-cognitive approach to motivation and personality. *Psychological Review*, *95*, 256-273.

Leatham, K. R. (2006). Viewing mathematics teachers' beliefs as sensible systems. *Journal of Mathematics Teacher Education, 9*, 91-102.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all.* Reston, VA: Author.

Pajares, M. F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research, 62*, 307-332.

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 257-315). Charlotte, NC: Information Age Publishing.

Steuer, G., Rosentritt-Brunn G., & Dresel, M. (2013). Dealing with errors in mathematics classrooms: Structure and relevance of perceived error climate. *Contemporary Educational Psychology, 38*, 196-210.

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York, NY: MacMillan.

CO-PLANNING WITH INTERNS: ENVISIONING NEW WAYS TO SUPPORT INTERN DEVELOPMENT OF EFFECTIVE LESSON PLANNING

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We examine the potential of co-planning during the internship experience to assist interns in making the transition from being a mathematics education student to becoming a mathematics teacher. We describe six strategies to facilitate co-planning between mentor teachers and interns. We report data from interns and mentors about their comfort level with particular strategies, their use of the strategies, and the benefits and challenges of co-planning in the internship setting.

Keywords: Teacher Education-Preservice, Instructional Activities and Practices, High School Education, Middle School Education

Crucial to both a successful mentor/intern relationship and to the development of the intern as a teacher is the co-planning process. We have found that mentor teachers often struggle with how to scaffold interns' involvement in the lesson-planning process. Within our program, and in conjunction with universities across the country, we have been working to develop and study strategies to help mentors and interns plan together more effectively.

Theoretical Framework

Our work with co-teaching and co-planning during pre-service teachers' internship experiences is grounded in Lave's (1991) apprenticeship model of learning. Lave defines learning as "first and principally the identity-making life projects of participants in communities of practice" (p. 157). The work of interns is not simply to build a more refined and abstract knowledge of teaching but to work towards becoming "a respected, practicing participant" (p. 157) in the classroom. In such an endeavor the working relationship between intern and mentor teacher is a major determining factor in the intern's ability to participate productively and collaboratively in the practice of classroom teaching. The mentor teacher serves as both collaborator and mentor for the intern. We envision this mentorship as what Feiman-Nemser and Beasley (1997) termed "mentoring as assisted performance." The mentor's role is not just that of a supportive presence or critic of a performance, but as a real-time coach helping interns navigate the world of the classroom. Our strategies for engaging in co-planning provide guidance on ways in which mentors can interact to scaffold interns as they learn to plan and ways in which mentors and interns can collaborate on planning.

Project Background

This project began with a realization of how powerful the co-teaching strategies (Friend, Reising, & Cook, 1993; Bacharach, Heck, & Dahlberg, 2010) were for our interns and mentors. The co-teaching strategies provided strategies for how mentors and interns could share classroom responsibility and language for them to talk about their shared work. Despite the usefulness of the co-teaching strategies, we found that many intern-mentor pairs were not taking full advantage of co-teaching because they not effectively co-planning. Much of the literature on co-planning envisions co-planning as two teachers sitting together in the same place, at the same time, going from a general topic to a full lesson plan. Schwille (2008) provides one teacher's description: "We'd be sitting at our table and we'd have our plan books. We'd have all of the books and materials and stuff around us. We'd just be sitting at this table talking, basically." (p. 153). Just as the co-teaching strategies expanded the vision of co-teaching beyond the Team Teaching model, we sought to envision other ways that teachers could engage in co-planning.

Development of the Co-Planning Strategies

The first stage of our work was the development of a set of potential co-planning strategies. Because the success of the co-teaching strategies, we used these as inspiration for developing six co-planning strategies. We defined each approach, described how a planning session might go, and identified potential benefits and concerns for each approach. In 2015, we piloted our training with interns, mentors, and university supervisors at two universities. Based on feedback from participants, we refined the descriptions of the strategies and the training. In 2016 we used feedback from eight secondary mathematics intern/mentor pairs and 83 pairs across other grade levels and subjects to further refine our descriptions and training.

Co-Planning Strategies

These co-planning strategies are not progressive or hierarchical. However, in working with our interns, we have found that some co-planning strategies are more productively used early in the internship while others might be more appropriate later. We present them (see Table 1) in an order in which they might first be encountered in an internship setting.

Table 1: Co-Planning Strategies (adapted from Cayton, et al., 2017)

Strategy	Definition
One Plans, One Assists	Each co-teacher brings a portion of the lesson, although one clearly has the main responsibility. The team works jointly on final planning.
Partner Planning	Co-teachers take responsibility for about half of the components of the lesson plan. Then they complete the plan collaboratively.
One Reflects, One Plans	Mentor thinks aloud about the main parts of the lesson and the intern writes the plan.
One Plans, One Reacts	One co-teacher plans and the other makes suggestions for improvement.
Parallel Planning	Each member of the co-teaching team develops a lesson plan, and the two bring them together for discussion and integration.
Team Planning	Both teachers actively plan at the same time and in the same space with no clear distinction of who takes leadership.

The Current Study

In this study, we address three research questions: Which co-planning strategies are mentor teachers and interns using and are most comfortable using? What do mentor teachers and interns perceive as the benefits and challenges of co-planning? How valuable do mentor teachers and interns perceive co-planning and co-teaching to be?

Data for this study is from 11 secondary mathematics intern/mentor pairs and include pre- and post- surveys from both interns and mentors and weekly journals from interns. The interns in this study began learning about and practicing co-planning strategies during their mathematics methods course in their junior year. Interns and mentors participated in two 2-hour co-planning/co-teaching workshops and one math-specific 1-hour co-planning workshop.

Findings

Interns' usage of strategies and comfort level with strategies is summarized in Table 2.

Table 2: Usage of Co-Planning Strategies & Intern Comfort Level

Usage of Co-Planning Strategies	Interns' "Most Comfortable" Strategies	Mentors' "Most Comfortable" Strategies
One Plans, One Assists(128)	One Plans, One Assists (6 of 11)	One Plans, One Assists (4 of 9)
One Plans, One Reacts (82)	One Plans, One Reacts (6 of 11)	One Plans, One Reacts (2 of 9)
One Reflects, One Plans (51)	Team Planning (3 of 11)	Team Planning (4 of 9)
Partner Planning (41)	One Reflects, One Plans (1 of 11)	One Reflects, One Plans (0 of 9)
Team Planning (26)	Partner Planning (0 of 11)	Partner Planning (4 of 9)
Parallel Planning (5)	Parallel Planning (0 of 11)	Parallel Planning (2 of 9)

In their journals interns reported strategies used during each class period. The most used co-planning strategies were *One Plans, One Assists* (128) and *One Plans, One Reacts* (82). Three strategies: *One Reflects, One Plans* (51), *Partner Planning* (41), and *Team Planning* (26) were used moderately, while *Parallel Planning* (5) was rarely utilized. All co-planning strategies were used at least once, but only one intern used all six strategies.

The strategies interns reported as most comfortable were *One Plans, One Assists* and *One Plans, One Reacts.* No interns reported comfort with *Partner Planning* or *Parallel Planning.* Mentor teachers mentioned a broader set of strategies as comfortable with no single strategy mentioned by more than half of the mentor teachers. There was often a difference between what the interns reported and what the mentors reported as most comfortable.

We found that mentor teachers expressed more benefits than challenges on both the pre- and post-survey. There were four themes in the reported benefits: benefits for interns, benefits for mentors, benefits for high school students, and benefits for both teachers. (See Table 3).

Table 3: Benefits of Co-Planning

Benefits	Mentor Pre	Mentor Post	Intern Pre	Intern Post
For interns	6	3	8	10
For mentors	3	6	0	0
For lesson plan/students	3	3	2	1
For both teachers	2	5	3	0

Of the challenges raised by the mentors, only time for planning and allowing interns independence, were mentioned both pre and post by the same mentor. In the pre-survey, the 11 mentors identified 14 benefits and only 5 challenges. In the post-survey the nine mentors who completed the survey identified 17 benefits and 5 challenges. Interns expressed more benefits than challenges on both the pre- and post-survey. There were two main themes: benefits for interns and benefits for high school students. Compared to the mentors, interns reported more of a balance between benefits and challenges. In both the pre- and post-surveys, interns identified benefits focused primarily on themselves. In the post-survey the number of interns noting benefits for themselves increased slightly, while the number mentioning benefits for quality of lesson plans/student instruction decreased slightly.

On the post-survey interns and mentors were asked to rank "How valuable was co-teaching/co-planning in your clinical experience?" on a scale of 1 to 5. Ten interns responded; seven rated it as valuable or extremely valuable. Three were neutral. Nine mentors responded; eight rated it as valuable or extremely valuable with the ninth neutral. The high value ranking from the interns and mentors was supported by quotes from the data:

- Intern: "I really enjoyed co-planning. I would think of ideas and talk it over."
- CT: "I think that it is the best way to grow an intern and to grow as a clinical teacher. Co-Planning has to be some of my favorite times with my intern, because I learn as much as they do…"

Discussion and Implications

Participants in this study used a variety of the co-planning strategies. They felt that co-planning was beneficial for helping interns improve their planning skills and that use of the co-planning strategies helped to avoid frustration and wasted time. The strategies helped to facilitate good communication between mentors and interns. Interns reported feeling empowered to contribute to the planning process, even before they were ready to plan complete lessons.

The literature and our pre-survey results indicate concerns about the challenge of finding time for co-planning, however, one result from our study is that the co-planning strategies may help with time concerns. Many of the strategies involve a combination of face-to-face planning and individual work time so mentors and interns need less common planning time. In addition, the use of the co-planning strategies resulted in less time wasted by interns creating plans that did not meet mentors' expectations.

Although we use the co-planning strategies in co-teaching settings, they are useful in a wide variety of settings. Interns in more traditional internship models still need help learning to plan and must still share the planning with their mentor teacher. We have seen these strategies used by two or more licensed teachers and by pairs of interns. We have used these strategies ourselves as we plan our university courses or plan presentations together. While these co-planning strategies grew out of our experiences with co-teaching, they appear to be much more broadly applicable.

References

Bacharach, N., Heck, T. W., & Dahlberg, K. (2010). Changing the face of student teaching through coteaching. *Action in Teacher Education, 32*(1), 3-14. doi:10.1080/01626620.2010.10463538

Cayton, C., Grady, M., Preston, R., Sears, R., Brosnan, P., and Oloff-Lewis, J. (2017). Improving pre-service secondary mathematics clinical experiences through co-planning and co-teaching. In E. Galindo & J. Newton, (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1524-1532). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators

Feiman-Nemser, S., & Beasley, K. (1997). Mentoring as assisted performance: A case of co-planning. In V. Richardson (Ed.), *Constructivist teacher education: Building new understandings* (pp. 108–125). London, UK: Falmer Press.

Friend,M., Reising, M., & Cook, L. (1993). Co-teaching: An overview of the past, a glimpse of the present, and considerations for the future. Preventing School Failure, 37(4), 6-10.

Lave, J. (1991). Situating learning in communities of practice. *Perspectives on Socially Shared Cognition*, *2*, 63-82.
Leinhardt, G., & Greeno, J. G. (1986). The cognitive skill of teaching. *Journal of Educational Psychology, 78*(2), 75-95. doi:10.1037//0022-0663.78.2.75

Schwille, S. A. (2008). The professional practice of mentoring. *American Journal of Education*, 115(1), 139–167. doi:10.1086/590672

ASSESSING PRESERVICE SECONDARY TEACHERS' PRACTICE WITH 'STANDARDIZED STUDENT INTERVIEWS'

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This study used a standardized, practice-based assessment of 13 preservice secondary math teachers to determine how they responded to a common student algebraic error. Data included a video-recorded interview with a researcher who played the role of a ninth-grade algebra student demonstrating a common student error about squaring a binomial. Data were analyzed using mathematical task analysis guide and features of a math-talk community. Findings indicate participants were able to elicit student thinking but were not able to respond in meaningful ways that supported conceptual understanding. Symbolic manipulation, rather than connecting representations, was the primary method for addressing misconceptions.

Keywords: Algebra and Algebraic Thinking, Instructional Activities and Practices, Teacher Education-Preservice

Practice-based teacher education requires teacher educators to identify high-leverage practices (Ball & Bass, 2000; Sleep & Boerst, 2012) and deliberately design and embed in teacher education programs opportunities to develop skill with those practices. With an increased emphasis on teaching *practice* (Hiebert & Morris, 2012) comes a need to assess the developing practice of preservice teachers. This paper shares an example of a practice-based assessment for preservice secondary teachers (PSTs) focused on eliciting and responding to student thinking. Research questions address (1) what mathematical and pedagogical approaches were used, (2) the level of cognitive demand, and (3) quality of mathematical discussion established when PSTs responded to a common algebraic error.

Background

Though authentic field experiences are critical components of teacher preparation, they are problematic in several ways that indicate 'practice-based' should not be conflated with more time in the field. In K-12 classrooms, PSTs are limited to the context of a specific lesson and set of students. Classroom demands and established norms may limit PSTs opportunities to enact some practices (Campbell & Elliott, 2015). The complexity of managing a class of learners makes it difficult for a novice to isolate and target a particular practice. Competing goals of the university and K-12 school complicate which practices PSTs are encouraged to learn to enact and adopt in classrooms (Feiman-Nemser, 2001). Supporting field experiences with structured and repeated opportunities for novices to practice the work of teaching is critical to a robust practice-based teacher education program (Ball & Bass, 2000).

Using high-leverage practices to frame secondary math teacher education supports a practice-based approach and can make the complexity of teaching accessible to novices. Teacher preparation programs have designed experiences where prospective teachers can try out and develop proficiency with enacting high-leverage practices (Forzani, 2014). Coached rehearsals (Lampert et al., 2013), instructional routines (Lampert & Graziani, 2009), and collaborative teaching cycles (Hallman-Thrasher, 2017) have been shown to support preservice teachers at developing *skilled practice* (Grossman & McDonald, 2008). Analysis of video (e.g., van Es & Sherin, 2010) and creating and analyzing representations of practice (e.g., Crespo, Oslund, &

Parks, 2011) have improved preservice teachers' attention to student mathematical thinking. Only recently have researchers extended this work to the secondary level (e.g., Campbell & Elliott, 2015; Ghousseini & Herbst, 2016).

The standardized student interview used in this study is example of an *approximation of practice* (Grossman, Hammerness, & McDonald, 2009) in that it engages novices in work that is proximal to that of a practicing teacher: eliciting student thinking, recognizing a student's misconception, and responding to the misconception in ways that foster conceptual understanding. Based on 'standardized patients' used in medical training (Barrows, 1993), in the interview a researcher plays the part of student who holds a particular misconception and a preservice teacher plays the role of a teacher tasked with diagnosing the misconception. The researcher's responses, language, and actions are guided by a protocol to ensure a standard experience across PSTs. Standardized students in simulations of teaching practice have been used with preservice elementary teachers to elicit student thinking around a correct, non-standard algorithm, interpret that student's thinking, and describe that thinking to an observer (Boerst, Sleep, Shaughnessy, & Ball, 2012). This study goes a step further by assessing preservice secondary teachers' abilities to both elicit *and* respond to student thinking and to do so in the context of a common student error.

Methods

Participants were 13 preservice (grade 7-12) mathematics teachers (PSTs) enrolled in a large university's undergraduate secondary mathematics teacher education program and completing their only mathematics methods course. PSTs were given a copy of the problem "$(x+3)^2 = x^2 + 9$" along with paper, pens, graph paper, algebra tiles, and a graphing calculator. PSTs were instructed to use any of the tools available to facilitate a discussion to help the student identify the error, revise the work, and understand why the original answer was incorrect. The interview was video recorded and all written work made during the interview was collected.

Each video was coded to determine what pedagogical moves the PST made in responding to the standardized student (e.g., proposed similar problem, suggested a counterexample, asked student to explain or justify work) and what mathematical approaches were taken: numeric (comparing with squaring a constant or plugging in numbers to find a counterexample), algebraic (using properties of algebra to simplify expressions), geometric (using an area model or algebra tiles), or graphic (comparing graphs of $y = x^2 + 9$ and $y = (x + 3)^2$ or using knowledge of transformations on functions). Each video was assigned a level for cognitive demand from the mathematics task analysis guide (Stein, Smith, Henningsen, & Silver, 2000) and a level (0, 0.5, 1, 1.5) for each component of the revised math-talk framework: questioning, explaining mathematical thinking, source of ideas, and responsibility for learning (see, Hallman-Thrasher, 2011; Hufferd-Ackles, Fuson, & Sherin, 2004).

Results

Results of the study address the level of cognitive demand and quality of mathematical discussion established and the mathematical and pedagogical approaches used when PSTs responded to a common algebraic error that indicated an underlying misconception. Of the 13 PSTs, 5 elevated the cognitive demand to *procedures with connections*, 6 maintained it at *procedures without connections*, and 2 lowered it to *memorization*. The quality of the PSTs discussion as assessed by the revised math-talk framework was clumped into four categories. Three PSTs who rated the lowest level in all components of math-talk were *tellers*, consistently lowering the cognitive demand by telling the student how to correct the work or correcting the work for the student. Five PSTs were *delayed tellers* who elicited thinking before telling the student a strategy; they rated a level 1 in questioning and explaining, but level 0 in source of ideas and responsibility for learning. Three PSTs were *non-tellers;* they posed many questions and did not draw conclusions for the student, but their questioning failed to address the key mathematical concepts. They rated a level 0.5 or 1 in all math-talk components. Two PSTs were *connecters* who raised the cognitive demand by asking the student to justify work, pressing for complete explanations, and connecting to other representations or the underlying concept of distributing twice, *and* requiring the student draw conclusions or judge validity of ideas. These PSTs rated high (level 1 or 1.5) in all components of math-talk.

In terms of mathematical approaches, all 13 PSTs initially performed algebraic manipulation or guided the standardized student to do so. Many focused on interpreting symbols by asking the student, "What does squaring mean?" or, "What do parentheses mean?" Only 3 PSTs eventually used or helped the student use a different strategy or connect to a different representation (1 geometric and 2 comparing graphs). Five PSTs explicitly asked about the pneumonic device 'FOIL' and when the standardized student claimed not to know this term, 4 went on to describe how to use it to multiply two binomials. Seven others asked the standardized student how to solve a different problem and drew comparisons between squaring a binomial and multiplying two distinct binomials. Two PSTs helped the student identify the error by asking the student to notice what happened to the equation $(x + 3)^2 = x^2 + 9$ when a number was substituted for x.

The initial pedagogical move for 10 PSTs was to pose a question, whereas 2 PSTs immediately pointed out the student's error and 1 explained a correct procedure. Of those who began the interaction with questioning, 2 prompted the student towards a particular approach, "How do you expand this to look like multiplication?", and 8 elicited student thinking with some form of "How did you get the answer?" PSTs' follow-ups to their initial question varied from revoicing to clarify student explanation, pressing for justification by asking "How do you know that's the answer?", and pressing the student to further explain concepts and terms. Three PSTs who initially elicited thinking, then suggested strategies that were unrelated to the student's way of thinking. Similarly, 1 participant who initially prompted a strategy, then elicited student thinking about that strategy, and leveraged it to connect to a graphical representation.

Findings showed that there was room for improvement in PSTs' ability to maintain high cognitive demand and connect across representations in response to a student's incorrect solution. Even when PSTs raised the cognitive demand, they did not keep the responsibility for generating ideas and judging validity of approaches on the student. The PSTs persisted in suggesting new ideas for the student to execute and evaluating the correctness of the student's work. This indicates that posing questions and eliciting explanations may be more accessible to novices than shifting mathematical authority from teacher to student or supporting students in generating new ideas. Further, though most PSTs knew to start their interaction with questioning, not all questioning was mathematically productive. Eliciting student thinking was not consistently associated with high cognitive demand or high math-talk ratings, and prompting a strategy was not consistently associated with low cognitive demand and low math-talk ratings; how PSTs followed their initial inquiry mattered.

Conclusion

Though PSTs were frequently adept at asking questions, leveraging student thinking to respond in ways that build conceptual understanding was challenging. Most PSTs did not distinguish between the standardized student making an error and having no meaning for the mathematical objects. PSTs need guidance on how to interpret and what to do with student thinking once they have elicited it and they need opportunities to practice that skill. These findings may point to the need to decompose the high-leverage practice of eliciting and responding to student thinking into practices of a smaller grain-size. One such smaller practice could be the use of questioning as a formative assessment of conceptual understanding or as means of shifting authority to the student. With smaller practices identified teacher educators can develop specialized experiences that support preservice teachers in learning these practices. Findings can inform the creation of an instructional activity (Lampert et al., 2010) for responding to an incorrect student solution in ways that allow a teacher to diagnose a misconception and leverage student thinking to respond in ways that build conceptual understanding.

References

Ball, D. L., & Bass, H. (2000). Interweaving Content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics* (pp. 83–104). Westport, CT: Ablex.

Barrows, H. S. (1993). An overview of the uses of standardized patients for teaching and evaluating clinical skills. *Academic Medicine, 68*, 443–451.

Boerst, T., Sleep, L., Shaughnssy, M., & Ball, D. L. (April, 2012). Assessing Preservice Teachers' Enacted Mathematics Teaching Practice through Simulations. Working group session presented at the research pre-session of the 2012 Annual Meeting and Exposition of the National Council of Teachers of Mathematics, Fort Worth, TX.

Campbell, M. P., & Elliott, R. (2015). Designing approximations of practice and conceptualising responsive and practice-focused secondary mathematics teacher education. *Mathematics Teacher Education and Development, 17*, 146-164

Feiman-Nemser, S. (2001). From preparation to practice: Designing a continuum to strengthen and sustain teaching. *Teachers College Record, 103,* 1013–1055.

Forzani, F. M. (2014). Understanding "core practices" and "practice-based" teacher education. *Journal of Teacher Education, 65*, 357–368. doi-org.proxy.library.ohio.edu/10.1177/0022487114533800

Ghousseini, H., & Herbst, P. (2016). Pedagogies of practice and opportunities to learn about classroom mathematics discussions. *Journal of Mathematics Teacher Education, 19*, 79–103.

Grossman, P., Hammerness, K., & McDonald, M. (2009). Redefining teaching, re-imagining teacher education. *Teachers and Teaching: theory and practice, 15*, 273-289.

Grossman, P., & McDonald, M. (2008). Back to the future: Directions for research in teaching and teacher education. *American Educational Research Journal, 45*(1), 184–205

Hallman-Thrasher, A. (2011). Preservice teachers' use of mathematical discussion in the implementation of problem-solving tasks. Unpublished doctoral dissertation, University of Georgia, Athens, GA.

Hallman-Thrasher, A. (2017). Cycles of collective planning, enactment, reflection in elementary teacher education, in M. Boston & L. West (Eds.), *2017 Annual Perspectives in Mathematics* (pp. 285–306). Reston, VA: National Council of Teachers of Mathematics.

Hiebert, J., & Morris, A. K. (2012). Teaching, rather than teachers, as a path toward improving classroom instruction. *Journal of Teacher Education*, *63*, 92–102.

Hufferd-Ackles, K., Fuson, K., & Sherin, M. G. (2004). Describing levels and components of a math-talk community. *Journal for Research in Mathematics Education, 35*, 81–116.

Lampert, M., Beasley, H., Ghousseini, H., Kazemi, E., & Franke, M. (2010). Using designed instructional activities to enable novices to manage ambitious mathematics teaching. *Instructional Explanations in the Disciplines (pp.* 129–141). 10.1007/978-1-4419-0594-9_9.

Lampert, M., Franke, M. L., Kazemi, E., Ghousseini, H., Turrou, A. C., Beasley, H., Cunard, A., & Crowe, K. (2013). Keeping it complex: Using rehearsals to support novice teacher learning of ambitious teaching. *Journal of Teacher Education, 64*, 226-243.

Lampert, M., & Graziani, F. (2009). Instructional activities as a tool for teachers' and teacher educators' learning. *The Elementary School Journal, 109*, 491-509.

Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). Implementing standards-based mathematics instruction: A casebook for professional development. New York, NY: Teachers College Press.

van Es, E. A. & Sherin, M. G. (2010). The influence of video clubs on teachers thinking and practice. *Journal of Mathematics Teacher Education, 13*, 155–176.

ONLINE LEARNING EXPERIENCES AND IMPACT ON STATISTICS EDUCATION PERSPECTIVES

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In this qualitative study, we used discussion forums and lesson plans from an online course on teaching mathematics and statistics with technology in order to examine participants' statistics education perspectives. Participants' perspectives included the nature of statistics, features of a good statistical task, learning statistics, teaching statistics, and role of technology. Some expressed perspectives were also reflected in participants' lesson plans, but others were missing.

Keywords: Teacher Beliefs, Data Analysis and Statistics, Technology

Background

Although we know the importance of developing statistically literate students in a data-driven world (Kwasny, 2015), and guidelines have been endorsed by the American Statistical Association for K-12 students' learning (Franklin et al., 2007) and mathematics teacher preparation in statistics (Franklin et al., 2015), many preservice mathematics teachers are not well-prepared to teach statistics (Lovett & Lee, 2017). The purpose of this study is to examine how online materials specifically designed to provide rich experiences in teaching and learning statistics may impact the statistics education perspectives of preservice and inservice teachers.

Teachers of statistics enter a classroom with already-developed perspectives of statistics education based on their previous experiences as learners of statistics, in and out of school, or based on their belief that statistics and mathematics are the same (Gal, Ginsburg, & Schau, 1997). There is a large body of evidence that using technology to teach and learn statistics allows teachers to create data visualizations to aid in analysis, link multiple representations, and augment graphs with statistical measures (e.g., Lee et al., 2014). We hypothesized that participants' use of such technology would impact their statistics education perspectives. Teacher preparation is increasingly being offered online, with some evidence that it is effective with a supportive learning environment, higher levels of motivation of learners, contexts that encourage instructor and peer interaction, and opportunities to practice material (Noesgaard & Ørngreen, 2015). Lee, Lovett and Mojica (2017) found that experiences in an online professional development course impacted participants' perspectives about statistics teaching related to: 1) statistics being more than computations and procedures, 2) using dynamic technology, 3) using real, messy data, and 4) increasing students' levels of statistical understanding. The current study examines the question: *When teachers engage in online modules, what perspectives about teaching and learning statistics do they express, and later enact in planning to teach statistics?*

Methods

The context for this study was a mostly asynchronous required online class on teaching mathematics with technology at a large southeastern university. The course focused on teaching statistics and geometry, with weekly modules that contained articles, videos, quizzes, data investigations using technology, and forum discussions. The instructor also held synchronous class sessions approximately every other week. There were six week-long statistics modules, covering content on the nature of statistics, statistical habits of mind, levels of understanding, measures of center and variability, and inference. Participants include eight preservice (PST, 4 undergraduate, 4 graduate) and three inservice teachers (IST, graduate-level).

Data sources included posts from four discussion forums in the first two modules (of 6) on teaching statistics with technology and statistics, and nine lesson plans (some worked in pairs) on teaching a topic in statistics participants submitted after all six modules. Based on an initial reading of all (approximately 100) posts, a set of data-driven codes, which were also informed by literature, were used to code the posts. A checklist derived from these codes was then applied to lesson plans. For example, since participants discussed concerns that their own students would need time to learn technology tools; that idea became something that was coded for in lesson plans. Once lesson plans were coded, frequencies were calculated, and trends identified.

Findings

Analysis of discussion forums and lesson plans resulted in five salient areas of participants' perspectives: the nature of statistics, features of a good statistical task, learning statistics, the practice of teaching statistics, and role of technology in statistics education.

Perspectives on the Nature of Statistics

When asked to reflect on what they had learned after Week 1, over half the participants (n=6) explicitly commented on the differences between mathematics and statistics, pointing out differences such as uncertainty in answers, the level of focus on computations and procedures, the role of context and data, the role of measurement, issues of sampling, and the role of interpretation. For example, a response to the prompt included:

> …statistical thinking is very different from "standard" math thinking. "Standard" math involves a lot of procedural thinking and there is always some form of right answer. In statistics, procedure is important, but context, interpretation, and analysis is heavily involved in the process. (PST, undergraduate student)

Three participants also expressed the belief that because of these differences, statistics is, by its nature, more engaging for students than typical mathematics.

Features of a Good Statistical Task

When asked to reflect on a task they participated in, and to analyze some example tasks, their responses included descriptions of what they believe constitutes a good statistical task. The majority of participants (n=8) expressed that a good statistical task is interactive or "hands-on". For many participants, this meant students should be exploring data, looking for relationships, and creating (with technology) their own data displays. This perspective was enacted in their lesson plans, as all (n=9) planned tasks seemed to be hands-on and interactive.

Many participants (n=8) explicitly mentioned issues regarding data when assessing quality of a statistical task, with the most common sentiments being that data should be interesting to students (n=8), real (n=4), and relatively large (n=2). These perspectives were also evident in lesson plans, where all (n=9) tasks used real data, most (n=7) used data likely to be interesting to students, and most (n=5) used large data sets, often with hundreds of cases and over 10 variables.

When describing a good statistical task, most participants (n=9) referred to at least one phase of the statistical cycle (pose->collect->analyze->interpret), and five participants mentioned that a task should include all four parts. It is interesting to note that, while most lesson plans had tasks with students analyzing (n=9) and interpreting (n=8) data, only one task had students posing their own questions to investigate, and only two tasks had students collecting their own data.

Other features mentioned by multiple participants include having students share their findings with the class (n=2), including questions that require students to justify answers (n=2), and including a clear learning objective for students (n=2). These features showed up in lesson plans 7 times, 8 times, and 1 time, respectively. With each task feature, at least one participant who had expressed that perspective did not create a task that reflected that perspective.

Perspectives on Learning Statistics

Though not specifically asked about it, some participants gave their perspectives on how students learn statistics. Some participants (n=3) expressed the need for students to have time to get acquainted with data before beginning analysis. After watching a video, one PST reflected:

> [In this video], connecting students with the data before they start analyzing was emphasized. This means allowing students to know what was going on when and where the data was taken.[…] I did not use this method while I was learning statistics in high school, but I do plan on using it when teaching statistics.

Five lesson plans allotted time or specifically included questions in the task intended to help students become familiar with data before analysis began. Other participants (n=3) reflected on how technology can become a tool that can affect how a student interacts with a statistical task, or on the diversity of students' expected statistical skill levels.

Perspectives on the Practice of Teaching Statistics

Across all four discussion forums, participants expressed their perspective about the practice of teaching statistics. Three participants expressed the necessity for teachers to ask good questions during a lesson, as one IST expressed her concerns about her ability to do so:

> I am also concerned about being able to effectively direct my students with the right questions so that the task is successful. […] If I am not able to direct the task effectively we would not reach our outcome goals or be able to help the students develop along the statistical thinking chart.

Six participants stated a need for teachers to ensure their students were engaged. These participants were confident they could do so, having seen some examples in the modules. Seven lesson plans (including all six created by participants valuing engagement) included a launch phase or activities intended to get students engaged in the content of the lesson. Other perspectives included the benefit of grouping students (n=3), though all nine lesson plans included grouping students. Three participants indicated the importance of statistics in K-12, and two participants believed statistics lessons were an opportunity for cross-curricular content.

Perspectives on the Role of Technology

Perspectives on the role of technology tools generally fell into two categories: how tools can impact statistical learning, and methods for implementing tools. Six participants mentioned that quickly producing accurate graphical displays and calculations makes data easier to analyze and interpret. In addition, most (n=7) participants mentioned that being able to simultaneously see multiple linked representations would allow students to better analyze and interpret data, as well as help keep the context of data foregrounded. Two participants expressed that whether or not a tool was easy to use would have an impact on how much a student would learn while using it.

Three participants expressed concerns that statistical technology tools would be difficult for students to use. Similarly, four participants believed that students would need adequate time with the technology in order to learn how it effectively. This was also reflected in the plans, where most (n=6) lessons either included time for students to interact with the technology before the task started, or indicated students would be familiar with the tool due to their work in prior tasks. Over half (n=6) of participants stressed the importance of sufficient guidance from the teacher in order for technology to be effective learning tools. However, only two lesson plans allotted any time for such guidance to be given, anticipated any student struggles with technology, or gave any suggestions on how to address those struggles. Finally, some participants believed that whether a piece of technology was an effective statistical learning tool depended on factors such as students' age or statistical skill level, and the learning objective to be taught in the curriculum.

Discussion

Our study found the online learning environment to be effective in impacting some of the participants' perspectives on statistics education, adding to the body of literature that says that online learning is effective (Noesgaard & Ørngreen, 2015). "Effectiveness" can be measured in different ways, but for our purposes, we were solely interested in evidence of perspectives that could influence the design of lesson plans. The perspectives we observed built on those seen in previous online contexts (e.g., Lovett & Lee, 2017) but also included other perspectives such as ideas about curricular considerations and features of a good statistical task.

In general, participants were not specifically asked about their perspectives on specific topics, but were rather responding to open-ended discussion prompts or to other participants. For this reason, one cannot infer that because a participant did not express a particular perspective in discussion forums, that the participant does not hold that perspective. A prime example of this is the issue of grouping students—only two participants expressed the opinion that students should be grouped, and yet, every lesson plan included grouping students. One interesting finding is that for perspectives expressed by participants, there was often a lack of evidence of that perspective in their lesson plans. There are three clear examples of this: students posing their own statistical questions to investigate, students collecting their own data, and students needing guidance on how to use technology. It is unknown why these discrepancies existed. Perhaps they were due to the perceived constraints of the classroom (e.g. time), or perhaps participants had trouble fully assuming the role of a teacher, particularly for those with no prior teaching experience.

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References

Franklin, C. A., Bargagliotti, A. E., Case, C. A., Kader, G. D., Scheaffer, R. L., Spangler, D. A. (2015). *Statistical Education of Teachers*. Alexandra, VA: American Statistical Association.

Franklin, C., Kader, G., Mewborn, D., Moreno, J., Peck, R., Perry, M., & Schaeffer, R. (2007). *Guidelines for assessment and instruction in statistics education (GAISE) report: A Pre-K--12 curriculum framework*. Alexandria, VA: American Statistical Association.

Gal, I., Ginsburg, L., & Schau, C. (1997). Monitoring attitudes and beliefs in statistics education. In I. Gal & J. B. Garfield (Eds.), *The assessment challenge in statistics education* (pp. 37–51). The Netherlands: IOS Press.

Kwasny, M. J. (2015). Statistics in K-12: Educators, students, and us: Challenges and resources for K-12 educators and students. *CHANCE*, *28*(4), 26–29. http://doi.org/10.1080/09332480.2015.1120124

Lee, H. S., Kersaint, G., Harper, S., Jones, D. L., Driskell, S. O., Leatham, K., Angotti, R., & Adu-Gwamfi, K. (2014). Prospective teachers' use of transnumeration in solving statistical tasks with dynamic statistical software. *Statistics Education Research Journal, 13(1),* 25-54.

Lee, H. S., Lovett, J. N., & Mojica, G. M. (2017). Characterizing impacts of online professional development on teachers' beliefs and perspectives about teaching statistics. In E. Galindo, & J. Newton, (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 407-414). Indianapolis, IN: Hoosier AMTE.

Lovett, J. N., & Lee, H. S. (2017). New standards require teaching more statistics in high school: Are preservice mathematics teachers ready? *Journal of Teacher Education 68(3)*, 299-311.

Noesgaard, S. S., & Ørngreen, R. (2015). The effectiveness of e-learning: An explorative and integrative review of the definitions, methodologies and factors that promote e-Learning effectiveness. *Electronic Journal of E-Learning*, *13*(4), 278–290.

PRESERVICE SECONDARY TEACHERS' OPPORTUNITIES TO LEARN HOW TO TEACH REASONING AND PROOF

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This study investigated future secondary teachers' opportunities to learn how to teach reasoning and proof reported by instructors of required courses in three secondary mathematics teacher education programs. We examined not only what future teachers learned but also how they learned to teach reasoning and proof. We found limited opportunities and noticed variations in the amount and types of the opportunities provided by each university. We noticed that important issues such as modifying or enacting reasoning and proof tasks and using manipulative and technology were not discussed in two of the universities. We also found that instructors reported limited types of instructional activities they had used to engage future teachers in learning to teach reasoning and proof. Implications of the research were discussed at the end of this paper.

Key words: Reasoning and Proof, Teacher Education-Preservice

Reasoning and proof (R&P) play an important role in school mathematics curriculum. The field of mathematics education recommends that R&P should be integrated into students' mathematical experiences across all grades and across a breadth of content areas (Common Core State Standards Initiative, 2010; NCTM, 2000). However, many secondary students have difficulty in writing valid proofs (e.g., Knuth, Choppin, & Bieda, 2009). Studies also show that teachers' conceptions, knowledge, and prior learning experiences of R&P can have significant impact on their teaching of R&P and thus affect their students' understanding and achievement in learning R&P (Oehrtman & Lawson, 2008). However, substantial amount of literature focused on how R&P was taught and learned in secondary classrooms (e.g., Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002). There was a lack of investigation particularly on preservice secondary teachers' (PSTs) opportunities to learn how to teach R&P in their college mathematics or mathematics education courses, which is the main focus of our on-going study.

Theoretical Perspectives

Teachers' teaching practices of proofs have been studied within various curriculum contexts and educational settings (e.g., Herbst, 2002). The majority of the research has focused on factors and strategies that might aid teachers' in effectively teaching R&P. For example, Oehrtman and Lawson (2008) suggested effective proof teaching approaches, such as asking students to reflect on their procedures and reasoning patterns, identifying successful procedures and reasoning patterns, and challenging students to further reflect on their reasoning in new context. A noticeable theme of suggestions of effective teaching of R&P focuses on the teaching philosophy that emphasizes the nature of proofs rather than their external form or routine, and how the nature of proofs interacts with the instructional contexts as an organic whole. For example, Herbst (2006) suggested that, "customary, sometimes ritualistic, practices of proving in high school geometry classrooms stand in the way of engaging students in more authentic mathematical activity," and, "it seems important thus to develop customs of doing mathematical work that make students accountable for tasks that require reasoning deductively even if the product is not yet fully written as a mathematical proof" (p. 344). These suggestions guided us in the analysis of some of the initial categories of the data. We will elaborate them in our findings.

Methods

As a part of the larger NSF funded study, we examined required mathematics and mathematics education courses at three contextually different universities A (Master's degree-granting institution), B (Ph. D – granting institution with a five-year undergraduate program), and C (Ph. D – granting institution with a four-year undergraduate program). We analyzed 15 instructors' responses to the interview question "What opportunities are there to talk about activities or strategies for teaching secondary students to engage in reasoning or proof?" Interview data were analyzed using the *inductive* and *deductive* analysis methods alternately (Patton, 2002). The initial categories of data consisted of categories suggested by literature. The breath and depths of these initial categories kept evolving as the data analysis was going until we resolved all discrepancies and reached a set of categories that can capture all relevant ideas from data. After coding was done, we counted the number of instructors who mentioned a particular category of knowledge or activities at each university and reported the counts in Table 1.

Findings

As recommended by *Principles and Standards for School Mathematics* (NCTM, 2000), we considered not only *what* teachers learn but also *how* they learn to teach R&P. We report these two aspects in the following two sections.

Knowledge that may affect PSTs' future teaching of R&P

Table 1 presents opportunities for PSTs to learn to teach R&P reported by instructors as the efforts they had made to support PSTs' learning of teaching R&P. We found that these opportunities were all from mathematics education courses. All three universities reported providing PSTs opportunities to learn the importance of R&P in secondary schools, the difference between conceptions of proof in secondary school setting and college setting, and how to ask questions to support secondary students' R&P. For instance, Instructor of Secondary Math Methods 1 at University C mentioned that:

> We just talk about what can count as proofs, and I try to really encourage them to have a broader definition of proofs, what does it mean to prove something? Why do we have to write it in two columns? What happens if kids just write out a really good argument in a paragraph and they don't have all the steps in the right order, but they are able to justify it.

Instructor of Middle School Math Method at University C reported having a discussion with PSTs about asking questions:

> The idea of being able to justify anything you're saying or asking students to explain more, when I show them any kinds of videos of teaching we're focusing on the questioning. What question is she asking to draw out their understanding of what is happening? How to ask questions that get students to not just give you the answer but to explain their reasoning.

For each of the other three categories, including modifying or enacting R&P tasks, using manipulative and technology, and multiple ways to present a deduced result, we only found one instance mentioned by one university. For example, Instructor of Secondary Mathematics Methods 3 at University B mentioned discussing with his student's about "thinking about ways in which teachers can mobilize different tools from physical manipulatives to representations that they use" to promote students' understanding of R&P. In summary, we found that University A and C seemed to heavily focus on questioning and conceptions of R&P in school setting. In addition, University B also addressed important issues such as adapting proof tasks and adopting tools and technology in teaching R&P.

Table 1: Opportunities for PSTs to Learn to Teach R&P at Three Universities

Categories	A	B	C	Total
Knowledge that may affect PSTs' future teaching of R&P	**5**	**5**	**5**	**15**
Importance of R&P in secondary school	1	1	1	3
Conception of proof in school setting	2	1	1	4
How to ask questions to support secondary students' learning of R&P	1	1	3	5
Modify or enact R&P tasks	0	1	0	1
Use manipulative and technology to teach R&P	0	1	0	1
Use multiple ways to present a deduced result	1	0	0	1
Activities used to engage PSTs in learning to teach R&P	**2**	**5**	**1**	**8**
Evaluate secondary students' proof writing	1	1	0	2
Examine secondary textbooks	1	1	0	2
Watch video of teaching practices in secondary classrooms	0	0	1	1
Study research based materials	0	1	0	1
Modify existing proof tasks	0	1	0	1
Field experience with secondary students	0	1	0	1

Activities mathematics educators used to engage PSTs in learning to teach R&P

As seen in Table 1, we found interesting variations across the activities mathematics educators used to engage PSTs in learning to teach R&P. Instructors at University B used a variety of ways to engage PSTs. In contrast, instructors at University A only reported a couple of activities. Evaluating secondary students' proof writing and examining secondary textbooks were two types of activities reported by both Universities A and B. For example, Instructor of Secondary Math Methods at University A mentioned that "It may come up with looking at work students have actually produced and trying to make judgments about whether the argument that students have made in answering a problem is clear, concise, and accurate." Instructor of Secondary Math Methods 3 at University B discussed using research based materials in his teaching method course:

> We have been using the cases of R&P in secondary mathematics materials that are being developed under an NSF grant. There are case-based materials designed to increase teachers' knowledge of what is meant by R&P and how to teach R&P tasks.

The same instructor reported providing PSTs opportunities to "modify tasks from their curriculum that may not be strong in R&P to enhance the R&P potential." The only instructional activity reported by instructors at University A was to have PSTs watch and analyze videos of teaching R&P in secondary classrooms.

Discussion and Conclusions

Overall, we found limited opportunities for PSTs to learn how to teach R&P across the three teacher education programs. We also noticed variations in the amount and types of the opportunities provided by each university. University B reported the highest amount of instances with the widest range compared to the other two universities. University B reported an emphasis on discussing how to modify or enact proof tasks, which is an important component in preparing PSTs in teaching R&P due to the important role tasks play in teaching mathematics (Lin, Yang, Lee, Tabach, & Stylianides, 2012). Another reason why PSTs need opportunities to learn modifying or enacting a proof task was pointed out by Instructor of Secondary Math Methods 3 at University B. He said "since we know that there are not a wealth of really strong R&P tasks in even the better textbooks that we have in middle and high schools right now." University B also reported stressing PST' reflection on using tools and technology in teaching R&P. This aspect was not mentioned in the instructors' reports in Universities A and C. Nowadays, as technology is getting used more and more widely in K-12 classrooms to support student explorative learning, there is a need to have a discussion on how technology could possibly support or hinder secondary students' learning of R&P. In fact, many studies have reported both the positive and negative impact of the use of technology in secondary classrooms on student learning and understanding of mathematical proofs (Sinclair et al., 2016). The lack of reported instances in Universities A and C regarding the use of technology raises a question on if PSTs were well prepared to incorporate technology properly in their future teaching of R&P.

Though we aimed to report how teachers were prepared in teaching R&P, our research is not meant to be evaluative but informative. We expect that our findings will support other teacher education programs as they are reflecting on their teacher training programs in terms of promoting their future teachers' ability to teach R&P. Some of the instructional activities used by mathematics educators such as having PSTs read research based materials on R&P could be incorporated into all teaching method courses and benefit other teacher education programs.

We admit that due to the lack of classroom observation data, we were unable to capture real moments when PSTs were learning to teach R&P. However, instructors' self-reported data are mostly likely to reflect instructors' focuses of the course and thus may have considerable level of impact on PSTs' learning. In this paper, we focused on PSTs' opportunities to learn how to teach R&P as teachers. We see the possibility that opportunities for PSTs to engage in R&P as learners could also affect their future teaching practices. We will report those findings in another paper.

References

Ball, D. L., Hoyles, C., Jahnke, H. N., & Movshovitz-Hadar, N. (2002). The teaching of proof. In L. I. Tatsien (Ed.), *Proceedings of the International Congress of Mathematics* (Vol. III, pp. 907–920). Beijing: Higher Education Press. Retrieved from http://arxiv.org/abs/math/0305021

Common Core State Standards Initiative. (2010). *Common core state standards for mathematics*. Washington D.C.: National Governors Association Center for Best Practices, Council of Chief State School Officers. Retrieved from http://www.corestandards.org/the-standards

Herbst, P. G. (2002). Engaging students in proving: A double bind on the teacher. *Journal for Research in Mathematics Education*, *33*(3), 176–203.

Herbst, P. G. (2006). Teaching geometry with problems: Negotiating instructional situations and mathematical tasks. *Journal for Research in Mathematics Education*, *37*(4), 313–347.

Lin, F. L., Yang, K. L., Lee, K. H., Tabach, M., & Stylianides, G. (2012). Principles of task design for conjecturing and proving. In G. Hanna & M. de Villiers (Eds.), *Proof and proving in mathematics education* (pp. 305–325). Springer. Retrieved from http://www.springerlink.com/index/Q30715U56557272U.pdf

Knuth, E. J., Chopin, J. M., & Bieda, K. N. (2009). Middle school students' production of mathematical justifications. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Ed s.), *Teaching and learning proof across the grades: A K–16 perspective* (pp. 153–170). New York, NY: Routledge.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Oehrtman, M., & Lawson, A. E. (2008). Connecting science and mathematics: The nature of proof and disproof in science and mathematics. *International Journal of Science and Mathematics Education*, *6*(2), 377–403.

Patton, M. Q. (2002). *Qualitative research and evaluation methods*. Sage Publications, Inc.

Sinclair, N., Bussi, M. G. B., de Villiers, M., Jones, K., Kortenkamp, U., Leung, A., & Owens, K. (2016). Recent research on geometry education: an ICME-13 survey team report. *ZDM*, *48*(5), 691–719.

EXPLORING AREA AND PERIMETER: A LOOK AT THE MATHEMATICS KNOWLEDGE OF TEACHER EDUCATION CANDIDATES

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This research report explores the mathematical conceptions that teacher education candidates (TECs) bring into a teaching program, specifically the (mis)understandings about area and perimeter of rectangles. A thematic analysis was performed on the incorrect solutions in order to gain an understanding of where teacher education candidates were making mistakes. This report discusses the themes that were discovered and how some of the information provided was mathematically correct but missed the bigger connections within the questions. The information provided supports the need to consider what misconceptions are being brought to teacher education programs and how to address them within the program.

Keywords: teacher education, mathematics knowledge for teaching, measurement, misconceptions

Studies have linked mathematics knowledge for teaching to increased student achievement (e.g., Baumert et al., 2010). In Ontario, the teacher education program is a two-year degree after a bachelor's degree in another area. Since it is an after degree program, there is no way to guarantee if any mathematics courses or earlier teacher preparation courses have been taken. As such, there is a strong emphasis on ensuring that the knowledge developed during the program supports teacher development in mathematics knowledge for teaching. This study looks at the content strand of measurement, particularly area and perimeter of rectangles, in order to draw some conclusions about the understandings that are brought to a teacher education program.

Literature

Developing a deep understanding of mathematics not only influences teachers' own mathematical conceptions, but also may support better classroom teaching (Stipek, Givvin, Salmon, & MacGyvers, 2001). Teachers' mathematical conceptions, ideologies and development subsequently influence their students' mathematical development (e.g., Schommer-Atkins, Duell, & Hutter, 2005) and thus are of critical importance.

Teachers need knowledge of mathematics that is specialised for teaching (e.g., Silverman & Thompson, 2008). Knowledge of mathematical content for teaching goes beyond simply being able to solve mathematical problems. Teachers need to deeply understand concepts, see connections among mathematical ideas, have knowledge of the potential connections to mathematical pedagogy, and be able to break down all the concepts with students in order to support student learning (Ma, 1999).

As a framework for the study, the work of Ball and Bass (2000) was used to guide the exploration of the samples. Ball and Bass discuss how constructing knowledge in mathematics mirrors what mathematicians do and focus on students providing mathematical reasoning for their knowledge claims. This was used as a starting point of looking at the reasoning of the TECs in supporting their discussions around area and perimeter.

This study looks specifically at area and perimeter of rectangles. In order to put this into context, I look at where in their school the TECs had potentially encountered this information. In Ontario, students by the end of the end of grade 4 will have "determin[ed] area and perimeter relationships for rectangles" (Ontario Ministry of Education, 2005, p. 64). Perimeter is first introduced in grade 2, and concepts related to area are first introduced in grade 1 within the Ontario curriculum. According to the Common Core in the US (2010), area and perimeter are first mentioned in grade 3, and specifically look at "exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters" (Measurement).

Mode of Inquiry

The research reported in this summary is a small portion of a larger study involving teacher education candidates and their knowledge and beliefs related to teaching and learning mathematics. The data set in this study comes from a modified version of the Perceptions of mathematics survey (see Kajander, 2010). Specifically the data comes from two questions on the survey: (1) *For the rectangle below, calculate the perimeter and the area*; (2) *Is it true that as the perimeter of a rectangle increases, so does the area? Explain.* The participants in the research study were TECs from two different years of a teacher education program in Canada. The survey was given on the first day of a teacher education course related to teaching mathematics (also known as a "math methods" course). The two years were combined since there was no control over who entered into the program or their past experiences. The purpose of the study within the larger program of research was simply to explore the mathematics knowledge for teaching that participants bring to a teacher education program related to an understanding of area and perimeter of rectangles. A total of 147 TECs were included in the total data set for analysis in this research report.

Initially quantitative analysis was performed on both of the questions in order to determine the success rate for the participants with the content of the two questions. Following this, a thematic analysis (Braun & Clarke, 2006) was conducted on the 125 incorrect responses to the second question in the data set in order to determine the specific misconceptions of the TECs. Using the process outlined by Braun and Clarke (2006), the data was initially read for initial codes, and then themes were created based on the codes. The themes were then reviewed to ensure all data points were included and then named for the current research report. This research study used an "inductive approach" where the themes were tied to the data as a unit instead of existing literature (Braun & Clarke, 2006, p. 83).

Results

Quantitative Analysis

The initial results come from a quantitative analysis on the two questions in the data set. As can be seen in Table 1, the majority of the TECs in the study were able to correctly identify the area and perimeter of the 3cm by 5cm rectangle given in question 1. Interestingly, the most common incorrect answer for calculating the perimeter was 225cm (3×3×5×5).

Table 1: Results for question 1

Both area and perimeter correct	Area only correct	Perimeter only correct	Both area and perimeter incorrect
88.4% (130)	4.8% (7)	4.8% (7)	2% (3)

Note. Percentages are listed based on the percentage of total participants (N=147). The number in brackets is the total number of participants who were identified in each category.

The results for the second question were slanted in the opposite direction, as can be seen in Table 2. Less than 10% of the entire data set were able to correctly answer question 2, by indicating that just because the perimeter increased, it does not mean that the area does as well. Of that 10%

though, 2 of the participants did identify that it was not a true statement; however, were unable to explain why it was not. For the remaining portion of the research report, we look at only the TECs who gave an incorrect response to the question (n=125).

Table 2: Results for question 2

Correct with explanation	Correct with no explanation	Incorrect response	Unsure	No response/ irrelevant
8.1% (12)	1.3% (2)	85% (125)	1.3% (2)	4% (6)

Note. Percentages are listed based on the percentage of total participants (N=147). The number in brackets is the total number of participants who were identified in each category.

Thematic Analysis

In examining the quantitative data, it would be easy to conclude that the participants have limited knowledge of the question that was asked. What became clear in examining the responses for themes was the participants had strong misconceptions about the question; however, some of their responses were definitely mathematically accurate just missed the bigger concept of the question.

A total of 125 data points were used in order to conduct the thematic analysis. From this starting point, a total of 32 codes were created in order to describe each of the data points; however, some of the participants' responses included more than one code. Table 3 lists the themes that were created in order to describe all of the codes with a sample participant response to help define one instance of the category. For space, only a few of these themes will be described in more detail in this paper to add to the description.

Table 3: Themes from the thematic analysis of the data set

Theme	Sample participant response
Statement of fact	"Yes, the area increases as the perimeter of the rectangle does." Or "Larger perimeter = more area"
Relationship to multiplication / formula	"True because the area is calculated based on the length and the width."
Focus on definition of area or perimeter	"Yes this would be true because the area of a rectangle is what is inside the perimeter so therefore if the perimeter were to increase the area would as well."
Focus on increasing side lengths	"If the side lengths (perimeter) increase so would the values to find the area."
Real-life example	"Yes because the space increases. If a wall in a room increases then the space inside will as well."
Using examples to prove the incorrect conjecture	6+4+6+4=20 7+4+7+4=22 6×4=24 7×4=28
Using algebra	"Since perimeter is a linear function, while area is an essentially geometric (curved) function, no matter the values of l and w, area will tend to increase faster up to ∞."
Concepts that are not related to area/perimeter of rectangles	"This assumption isn't valid if the shape isn't symmetrical."
No response/don't know	"I don't know why."

Focus on definition of area or perimeter. In some cases, the participants looked at the definition of area or perimeter (or both) and used these to justify the conclusion that when perimeter increases that so would the area. Generally these responses did not really prove anything except that the participants knew what area and perimeter meant.

Focus on increasing side lengths. Participants whose solutions fell into this theme were focused only on the impacting that increasing either a single or both sides lengths would impact the perimeter and the area. It was notable that the statements made by these participants were mathematically correct (i.e., increasing a single side length would always increase both the area and perimeter). This, however, did not correctly answer the question since there are times when the perimeter would increase and the area would not by other manipulations.

Using examples to prove the incorrect conjecture. Examples were used by many of the participants in order to "prove" that when perimeter increases, so does the area. In some cases only a single example was included to compare to the rectangle in the previous question. The data showed up to four examples being used in order to "prove" the conclusion drawn. In one case, the participant included examples that actually proved that this was not a correct statement; however, the participant was unable to make the determination that increasing the perimeter does not always increase the area.

Conclusion and Discussion

What was interesting in the thematic analysis of the solutions was that there were elements that were mathematically correct; however, missed the bigger picture of the question. The use of examples to "prove" the statement or looking at only "increasing" side lengths showed that the TECs had applied their knowledge of area, perimeter, and multiplication correctly to the question. Some of the statements were flawed, such as indicating a direct relationship between area and perimeter of rectangles: "Perimeter and area are positively correlated –as one goes up, so does the other." As Van de Walle et al. (2014) notes, area and perimeter are a source of difficulty for elementary students, and this study points to these ideas persisting into a teacher education program. The solutions of the participants in this study provided interesting examples to discuss where the misconceptions came from and to encourage a stronger understanding of area and perimeter in rectangles during experiences within the methods course.

References

Ball, D. L., & Bass, H. (2000). Making believe: The collective construction of public mathematical knowledge in the elementary classroom. In D. C. Phillips (Ed.), *Constructivism in education: Opinions and second opinions on controversial issues: Ninety-ninth yearbook of the National Society for the Study of Education* (pp. 193-224). Chicago, IL: National Society for the Study of Education.

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A. Klusmann, U., Krauss, S., Neubrand, M., & Tsai, Y. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal*, *47*(1), 133-180.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States.* Mahwah, NJ: Erlbaum.

National Governors Association Center for Best Practices, Council of Chief State School Officers. (2010). *Common Core State Standards: Mathematics.* Washington, DC: National Governors Association Center for Best Practices, Council of Chief State School Officers.

Ontario Ministry of Education. (2005). *The Ontario curriculum grades 1-8: Mathematics* (Rev. ed.). Toronto, ON: Queen's Printer for Ontario.

Schommer-Aikins, M., Duell, O., & Hutter, R. (2005). Epistemological beliefs, mathematics problem solving beliefs, and academic performance of middle school students. *The Elementary School Journal*, *105*(3), 289-303.

Silverman, J., & Thompson, P. W. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education*, *11*, 499-511.

Stipek, D., Givvin, K., Salmon, J., & MacGyvers, V. (2001). Teachers' beliefs and practices related to mathematics instruction. *Teaching and Teacher Education*, *17*(2), 213-226.

REHEARSALS OF AMBITIOUS TEACHING IN IMMERSIVE CLASSROOM SIMULATION ACTIVITIES

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Pedagogies for engaging PTs in ambitious teaching often involve rehearsals integrated into iterative cycles and this can be difficult to imbed within large teacher preparation programs. Innovative technologies utilizing immersive classroom simulation activities (ICSA) allow PTs to rehearse instructional activities with student avatars. This study found that ICSAs provide opportunities to engage aspects of ambitious teaching such eliciting and responding to student thinking and afford PTs with unique opportunities to position students as competent. The affordances and constraints are discussed

Keywords: Teacher Education, Pre-service, Instructional activities and practices, Elementary School Education

Introduction and Purpose

As we reflect over the past forty years in mathematics education one focus of research and practice has been ambitious teaching that prioritizes the learning of all students regardless of ethnicity, race, class, or gender (Grossman, Hammerness, & McDonald, 2009). More recent research has focused on tools and structures within teacher education to successfully engage preservice teachers (PTs) in the intricacies of ambitious teaching (Thompson, Windschitl, & Braaten, 2013; Kazemi, Franke, & Lampert, 2009). Within this work pedagogies of teacher education have been examined (Lampert & Graziani, 2009; Ghousseni & Herbst, 2016) and structures such as *Cycles of Enactment and Investigation* (Lampert et al., 2013) have been designed to engage PTs in opportunities to deliberately practice specific teaching episodes and enact those episodes in classroom settings. As we look forward to the next forty years in mathematics education, it is vital we actualize this reconceptualization of teacher education at scale and utilize innovative technologies to support our work. The human resources needed to employ an iterative, practice-based process within teacher preparation are often unavailable at institutions with large programs. One innovative technology is virtual simulation software, such as that available from Mursion® (developed as TLE TeachLivE™), that provides immersive, interactive learning through practice-based teacher development (Dieker, et al., 2014). During the immersive classroom simulation activity (ICSA), pre-service teachers can engage with a classroom of five student avatars on a large computer screen. A simulation specialist operates the avatars and uses a simulation scenario to guide the interactions. This exploratory study extends research on cycles of enactment to examine 1) What opportunities are afforded with embedded prompts in rehearsals of ambitious teaching within an ICSA? and 2) How do PTs respond to embedded prompts within an ISCA?

Perspectives

Work by Grossman et al. (2009) defines three essential pedagogies of teacher preparation that enable novice teachers to learn through engagement in the teaching process. These pedagogies are the representation of practice, decompositions of practice, and approximation of practice, and all are needed to develop ambitious teaching. Lampert at al. (2013) has integrated these pedagogies of practice into an iterative cycle that utilizes rehearsal and classroom-teaching episodes, titled Cycles of Enactment and Investigation (CEI). The work presented here utilizes this framework and seeks to explore the affordances and constraints of situating approximations of practice within immersive classroom simulation activities (ICSA). Within the ICSA a scenario is created to target key aspects of instruction, scripting certain student responses to provide a vast array of opportunities with the lesson. Essentially the ICSA parallels a rehearsal within the CEI, yet does not allow the same coaching.

Methods

Larger Context

This report shares preliminary findings from a study situated within a longitudinal, comparative study designed to compare the effectiveness of ICSA and peer rehearsals in the development of PTs' ambitious teaching. PTs engage in a CEI with two number talks as the instructional activity (2 ICSA/peer rehearsals and 1 classroom enactment). The number talks are two-digit multiplication number strings. A scenario was created for the ICSA and a trained simulation specialists uses to the scenario to "act" the part of five student avatars. The PTs facilitate the number talk with the group of five student avatars (referred to as students for the remainder of the report) or with 3 peers and the teacher educator.

Participants, Data, and Data Analysis

Within this study three participants were randomly selected as case studies from the larger context. Participants were enrolled in an elementary mathematics methods course focused on grades 3-6 and had previously completed an elementary mathematics course focused on grades K-2. The data used in this study are three video-recorded ICSA sessions, one per participant. The sessions ranged from 8 minutes to 11 minutes. During the ICSA the participant lead a number talk with the problem "12x24." This problem came third in a number string.

The use of GoReact©, a video analysis tool, allowed for detailed coding of each teacher and student interaction within the video-recorded ICSA. GoReact is a secure online video analysis tool that connects comments or codes directly to the segment of video for which they address. Each video was examined multiple times and each teacher and student interaction was coded using a set of a priori codes adapted from Lampert et al. (2013) that defined elements of ambitious instruction. In addition to the a priori codes unique interactions were noted in comments and these were interpreted to develop additional codes pertaining to positioning of students as competent. The goal of analyzing the videos was to identify aspects of ambitious teaching that arose within the ISCA (exemplars and missed opportunities) and illustrate unique features of the ISCA.

Results

Analysis of the three cases resulted in exemplars of each of the elements of ambitious teaching that were strategically coded, as well as examples of missed opportunities. In the following sections the exemplars and missed opportunities are discussed in the context of the ICSA to illustrate the affordances and constraints of the ICSA.

Eliciting and Responding to Student Thinking

One of the main reasons a number talk was selected as the instructional activity is the inherit requirement to elicit students' mental strategies for solving a problem. Across the three cases the students volunteered responses, and this was a function of the classroom norms designed in the scenario. Each PT elicited strategies from three of the five students and at least one student was probed for further explanation. PTs responded to students' decomposition strategies (i.e., (20 x 12) + (4 x 12)) by probing for additional clarification about the factors. However, these follow-up questions elicited different student responses based on the open-ended or literal nature of the question. For example, one PT asked “Where did this 20 come from” for which Carlos answered “the 24” while another PT asks Carlos “Why did you break into 12 x 20 and 12 x 4?” This more open-ended question elicited a more lengthy explanation of wanting to multiply by groups of 10 to make the multiplication easier to do.

In addition to differentiating responses based on question type, the ICSA was designed for one of the student to explain that they “multiplied 12 by 2 and then added a zero to get 240” when multiplying 12 x 20. The PTs did not attend to this particular student response and did not ask for further explanation to deconstruct the mathematics

Representation of Mathematical Ideas

Overall, student strategies were recorded accurately on the board except for one instance. As the PTs represented the students’ strategies on the board, each PT recorded the decomposition strategies by placing the parts of the decomposed factor second in the set of equations they listed (ie., 12 x 20 = 240 and 12 x 4 = 48). This was due to the way the strategies were verbally shared by the students that were scripted in the scenario. This could be modified to allow for more complexity in how relationship of the factors is recorded.

There were two instances where the PT revoices the student’s strategy to ensure that she has recorded the students’ thinking. In one of these cases the PT was inaccurate in her representation of 12 x 25 = 300 -12 =288. The scenario script could address this misconception by triggering a student to ask a question about the equality of a statement, such as “Does 12 x 25 = 300 -12?” This would result in the PT having to grapple with this.

Orienting Students to One Another

Throughout the number talks, PTs did not utilize strategies, such as turn and talk, for that would allow students to orient to one another. This may have been due to the fact that certain students, Carlos and Mina, were quick to volunteer and scenarios can be modified to include longer time before raising hands. Furthermore, PTs focused on moving to the next strategy and not using moves that would ensure students were listening to one another. There was one response from a PT that directed, “Carlos please explain to your classmates how you broke apart the 24.” In doing so, Carlos turned and spoke to his classmates instead of directly to the teacher.

Building on Reasoning

The scenario was designed to have two students share strategies to solve 12x24 by using the distributive property and either decomposing 12 or 24. This design element was intended to elicit questions from the PTs to address the similarities, differences, and application of the property. Interestingly, only one PT directed attention to these two strategies and asked, “How are these ways different from each other?” When Emily responds quickly that different numbers were broken down, they PT praises Emily and solicits another strategy. This missed opportunity to make the connection to the distributive property shows an area for attention.

Students as Competent

The students expressed frustration with the problem when it was initially posed and Will shared that he did not arrive at an answer to the problem. From these student responses arose the need to code the interactions for instances of how the PTs positioned the students as competent or not. This coding revealed that PTs send mixed messages to students about mathematics and ability. At the beginning students cry out “This is crazy, we can’t do this in our heads” and ““How come these keep getting harder? I wish they used smaller numbers.” One PT responded, “You can do it. You are smart.” While she was trying to encourage students, the message she is sending them is that if you cannot do it then you are not smart. Another PT prefaces the problem with claims that it will be hard stating “This on is going to be harder…if you can’t figure it out it is ok because I had a hard time figuring it out and I had paper." This response assumes students will struggle and devalues the use of mental strategies.

When Will shares that he was not able to solve the problem, all three PTs attempt to engage Will to understand his confusion; however, they prematurely abandon their efforts. One PT tries to decipher Will's starting point when he shares, "The first thing that clicked for me was that both have twos and I thought I could do something with the two that might make it easier. But then I just got confused." In response this PT praises Will stating, "That's good that you were able to start thinking and you didn't give up when you saw the problem," and then immediately calls on another student. Will's interactions force PTs to grapple with how to address students' needs and guide them to an understanding of the problem.

Lastly, Carlos and Mina are called on first in two of the cases due to volunteering. They are also called upon for further explanation in all cases. This emerging trend highlights the positioning of certain students as competent and valued while others are expected to be passive participants in the class.

Conclusion

Analysis of recorded ICSAs indicated that certain elements of ambitious teaching were evident and afforded due to the strategic development of the scenario used within the ICSA. PTs were engaged with eliciting and responding to student thinking, representing ideas, and building on reasoning, yet they were not engaged in orienting students to one another. The missed opportunities highlight the need for additional scaffolds built into the scenario and specific feedback from teacher educators to build PTs engagement in these teaching practices. Also, the ICSA provided a unique opportunity for PTs to navigate student misconceptions and frustrations with mathematics. The authentic expressions from the students are unique to this technology and not present in peer rehearsals. This affordance within the ICSA and the standardization of the scenario make this innovative tool a plausible solution to utilizing CEI at scale. Although conducting numerous peer rehearsals may not be feasible for a teacher educator, rehearsals within an ICSA will allow for strategic scenarios to be replicated without additional work. The larger research project will provide valuable insights into the comparison of ICSA and peer rehearsals and inform the ongoing reconceptualization of practice.

References

Dieker, L. A., Rodriguez, J. A., Lignugaris/Kraft, B., Hynes, M. C., & Hughes, C. E. (2014). The potential of simulated environments in teacher education: Current and future possibilities. *Teacher Education and Special Education*, *37*(1), 21-33.

Ghousseini, H., & Herbst, P. (2016). Pedagogies of practice and opportunities to learn about classroom mathematics discussions. *Journal of Mathematics Teacher Education*, *19*(1), 79-103.

Grossman, P., Hammerness, K., & McDonald, M. (2009). Redefining teaching, re-imagining teacher education. *Teachers and Teaching: theory and practice*, *15*(2), 273-289.

Kazemi, E., Franke, M., & Lampert, M. (2009, July). Developing pedagogies in teacher education to support novice teachers' ability to enact ambitious instruction. In *Crossing divides: Proceedings of the 32nd annual conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 12-30). Adelaide, SA: MERGA.

Lampert, M., Beasley, H., Ghousseini, H., Kazemi, E., & Franke, M. (2010). Using designed instructional activities to enable novices to manage ambitious mathematics teaching. In *Instructional explanations in the disciplines* (pp. 129-141). Springer, Boston, MA.

Lampert, M., Franke, M. L., Kazemi, E., Ghousseini, H., Turrou, A. C., Beasley, H., & Crowe, K. (2013). Keeping it complex: Using rehearsals to support novice teacher learning of ambitious teaching. *Journal of Teacher Education*, *64*(3), 226-243.

Thompson, J., Windschitl, M., & Braaten, M. (2013). Developing a theory of ambitious early-career teacher practice. *American Educational Research Journal*, *50*(3), 574-615.

PRE-SERVICE TEACHERS' PERCEPTIONS OF THE USE OF REPRESENTATIONS AND SUGGESTIONS FOR STUDENTS' INCORRECT USE

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We investigated how elementary pre-service teachers (PSTs) perceive using representations in teaching mathematics and what representations they suggest to guide students' use of representations in teaching fractions. A written questionnaire was administrated to 151 PSTs and an inductive content analysis approach was used to analyze the data. Findings suggested that fraction-related topics were the PSTs' main choice for using representations, but PSTs tended to use models procedurally and predominantly depended on circular area models in guiding students who use representations incorrectly. Implications for designing mathematics methods courses in terms of effective use of representations are discussed.

Keywords: Representations, Fraction Models, Manipulatives, Preservice teachers, and Teacher Preparation

Introduction

The ability to execute a wide range of instructional practices that are deeply rooted in profound mathematical knowledge for teaching is integral to effective mathematics teaching (Hill, Ball, & Schilling, 2008). In this study, we examine PSTs' ability to analyze students' work, to choose and use appropriate mathematical representations in explanations, and to evaluate the instructional advantages and disadvantages of representations. In the hope that this study has implications for teacher preparation, which in turn will have impact on PSTs' future practices, we investigated the following research questions:

(1) How do pre-service teachers perceive using representations (e.g., manipulatives or models) in teaching mathematics?
(2) What fractional representations (e.g., manipulatives or models) do pre-service teachers suggest to guide students' incorrect use of representations in learning fractions?

Theoretical Perspectives

Knowledge of Representation

As noted, use of various representations is commonly considered necessary in mathematics instruction because of the abstract nature of mathematics and mathematical discourse. Accordingly, teachers' and students' competence in using representations has long been emphasized in mathematics education. Educators refer to this competence using varied terms such as representational flexibility, representational fluency, representational thinking, and representational competence (Greer, 2009).

While using representations is recognized as important to supporting students' sense-making process, the literature also documents that limited uses or misuses of representations, often due to teachers' limited content knowledge, can negatively affect teaching and learning (Izsák, 2008; Lee, Brown, & Orrill, 2011). However, while adequacy of teachers' content knowledge is a necessary condition for effective use of representation, that alone is not sufficient. Teachers need to be fluent in using not only their own solution representations but also pedagogical representations (Cai, 2005).

Fraction Models

Given the importance of representations in teaching and learning mathematics we further examine three major types of pedagogical representations in fractions: area, length, and set models. The area models (e.g., circular pie pieces, rectangular regions, geoboard, drawings on grids or dot paper, pattern blocks, etc.) subdivide a whole area into equal parts. The area model elucidates the part-whole concept of fractions and the meaning of the relative size of a part to the whole (van de Walle, Karp, & Bay-Williams, 2013).

Another major fraction model is the length model (e.g., fraction strips, Cuisenaire rods, number lines, etc.), in which measurements of lengths are compared rather than measurements of areas. Linear models allow students to integrate their knowledge of fractions and whole numbers in a continuous model (Siegler, Thompson, & Schneider, 2011). Some research has suggested that with the linear model it is difficult for students to develop an accurate visualization of the value of a given fraction (Behr & Post, 1992). However, another strand in the literature emphasizes the utility of linear models in that they help students understand fractions that are greater than one more easily due to their continuity, in contrast to the discrete wholes used in the area models (Lee, 2017; Siegler et al., 2011). Set models (e.g., two-color counters, objects, tallies etc.) utilize discrete collections within sets of countable objects. Literature offers mixed results and suggestions regarding the effectiveness of the set model. While many researchers recognize the importance of set models due to the applicability of students' prior knowledge of whole number strategies and the connections with many real-world uses of fractions (Behr & Post, 1992), others recommend introducing set models later than other models because it may require some understanding of ratio or the concept of fraction multiplication (van de Walle et al., 2013).

While the forms of models differ, the methods of representing a whole and its parts can be varied. Watanabe (2002) explains two distinct methods for representing fractions, depending on how the relationships between the whole and fractional parts are shown: (1) the part-whole method in which the fractional part is embedded in the whole and (2) the comparison method in which the whole and part are constructed separately. Watanabe (2002) asserts that children may benefit from more explicit treatment of the different ways that fractions can be represented.

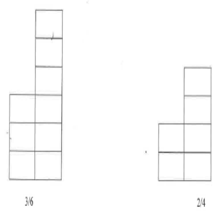
Methods

One hundred fifty one elementary PSTs at a large Southwestern university in the U.S. participated in this study. All were enrolled in an elementary mathematics content course. They had taken a course that dealt with number and operations, in which they learned about fractions and fraction operations along with manipulatives or pictorial representations such as set, length, and area models. The participating PSTs completed a written questionnaire. For this study, we focused on one open-ended question and two tasks (see Figure 1). The first and third questions below were created by the authors and the second question comes from prior research conducted by other researchers and the first author (Cross, Lee, Zeybek, & Adefope, 2015).

The written questionnaire was administered to all PSTs for one hour. To analyze the data, we used an inductive content analysis approach that included both qualitative and quantitative analyses (Grbich, 2007). First, to analyze the open-ended question, we read all PSTs' responses and identified common themes among them. Then we developed a coding scheme and used it to analyze the responses. When there were discrepancies between the two researchers' codes, we discussed them until we reached 100% agreement on the coding. Finally, to find patterns, we calculated the frequency of codes in the PSTs' responses.

To analyze the PSTs' responses to students' fractional models, we first identified the correctness of the responses and categorized the models that PSTs provided in terms of two aspects: (1) the types of models such as length model, area model, set model, and a drawing based on algorithm to find the common denominator; and (2) whether the same sized whole was used for both fractions. Then we developed a coding scheme for fractional models and used it to code all data. Finally we interpreted the data quantitatively and qualitatively.

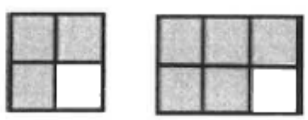
1. (1) What topics would be the most effective for elementary kids to learn mathematics with manipulatives or models? Describe the topics and explain how to teach the topics using manipulatives or models.
   (2) What do you think is the most important in teaching a math lesson to elementary kids using manipulatives or models?
2. **Task 1**: Susan is given the following question: Which is larger 2/4 or 3/6? How do you know?



Susan created the following models with the multi-link cubes to represent the fractions and help her with comparing the fractions.

(1) Do you think Susan's models will help her to compare the fractions?
(2) If you don't think Susan's models will help her, please create a model that you think would be useful for comparing 2/4 and 3/6.

3. **Task 2**: You asked a 3rd grade student, Lisa to make a drawing to compare ¾ and 5/6. She draws the following picture and claims that "3/4 and 5/6 are the same amount."



(1) Do you think Lisa's models are correct?
(2) If so, explain. If not, create a model that you think would be useful for comparing ¾ and 5/6 and explain why your model would be helpful.

Figure 1. Problems used for this study

Findings

PSTs' Perception of Using Manipulatives or Models in Teaching Mathematics

In the first part of the questionnaire, when PSTs were asked to suggest topics in which manipulatives or models can be used effectively, the most popular responses were topics involving the part-whole fraction concept, equivalent fractions, and fraction comparisons (35%), followed by whole number operations (25%), geometry (8%) and place value (7%). If we consider fraction operations (e.g., fraction addition, subtraction, multiplication, division) and other fractional numbers (e.g., decimals and percents), the topics related to fractions account for 40% of the responses. However, with regard to the topic of fractions, most PSTs pointed out the basic part-whole concept of fractions and very few (3 out of 108 responses) mentioned that manipulatives could be used to compare fractions, which is related to the focus of our study.

In addition, when the PSTs were asked what they considered to be most important in teaching mathematics with manipulatives or models, 25% of their responses indicated *understanding and reasoning*. The second most frequently mentioned aspect (16%) was providing students with *hands-on experience or opportunities to use visuals* as effective tools for solving difficult problems when they are unsure how to proceed. Almost as many responses (15%) indicated the importance of *controlled and focused management* by helping students understand how to use the manipulatives and models accurately, demonstrating their use, and establishing consistent rules for using them purposefully rather than as toys.

Identifying Correctness of Using Manipulatives for Comparing Two Fractions

Approximately two thirds of the PSTs recognized the incorrect use of models in both tasks (66% and 63% respectively), but not all of them identified valid reasons. When asked to propose models to guide students, the majority of PSTs illustrated both the invariance of the whole and equal partitioning. However, 20% of responses to Task 1 and 12% to Task 2 failed to show the invariance of the whole and/or equal partitioning in their representations. It is worthwhile to note that the most common models were area (especially circular wedges) and length/linear models (especially rectangular strips rather than a number line). Also, it was noted that even though the methods of representation used in the two tasks differ (i.e., Task 1's comparison method vs. Task 2's part-whole method), the PSTs tended to guide students using the methods familiar to themselves such as a part-whole method.

Implications

Considering that the PSTs' work in this study reflects their perceptions prior to taking a methods course, the results provide some insights into how to design a methods course. First, mathematics educators need to explicitly cover the *comparison method* as well as the *part-whole method*. Many PSTs in this study demonstrated unfamiliarity with the comparison method. Second, mathematics educators need to address *number line models* in more depth when considered that the PSTs in our study made very limited use of number line models. Third, mathematics educators need to help PSTs implement effective *pedagogical representations* as distinguished from solution representations. Finally, mathematics educators need to engage PSTs in *drawing models* by focusing on the principles of invariant wholes and equal partitioning in comparing fractions when considering that many PSTs appeared unaware of the importance of the principles.

References

Behr, M. J., & Post, T. R. (1992). Teaching rational number and decimal concepts. In T. R. Post (Ed.), *Teaching mathematics in grades K-8: Research-based methods (2nd ed.)* (pp.201–248). Boston: Allyn and Bacon.

Cai, J. (2005). U.S. and Chinese teachers' constructing, knowing, and evaluating representations to teach mathematics. *Mathematical Thinking and Learning, 7*(2), 135-–169.

Cross, D. F., Lee, M. Y., Zeybek. Z., & Adefope, O. (2015, April). Delving into the pieces: Drawing connections between different domains of teacher knowledge. Paper presented at the AERA, Chicago, Illinois, USA.

Grbich, C. (2007). *Qualitative data analysis: An introduction.* Thousand Oaks, CA: Sage

Greer, B. (2009). Representational flexibility and mathematical expertise. *ZDM, 41*(5), 697–702.

Hill, H.C., Ball, D. L., & Schilling, S.G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic specific knowledge of students. *Journal for Research in Mathematics Education*, *39*(4), 372-400.

Izsák, A. (2008). Mathematical knowledge for teaching fraction multiplication. *Cognition and Instruction, 26*(1), 95–143.

Lee, M. Y. (2017). Pre-service teachers' flexibility with referent units in solving a fraction division problem. Educational Studies in Mathematics, 96(3), 327-348. DOI: 10.1007/s10649-017-9771-6

Lee, S. J., Brown, R. E., & Orrill, C. H. (2011). Mathematics teachers' reasoning about fractions and decimals using drawn representations. *Mathematical Thinking and Learning, 13*(3), 198–220.

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology*, *62*(4), 273–296.

van de Walle, J., Karp, K.S., & Bay-Williams, J. M. (2013). *Elementary and middle school mathematics: Teaching developmentally* (8th ed.). NJ: Pearson.

Watanabe, T. (2002). Representations in teaching and learning fractions. *Teaching Children Mathematics, 8*, 457–463.

BRIDGING BELIEFS, VALUES AND PAST EXPERINCES WITH INSTRUCTIONAL PRACTICE

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Research has established the challenges that novice math teachers face as they move from teacher preparation programs into their first year of teaching (Drake, 2006; Gainsburg, 2012; Millsaps, 2000; Thompson, 1992). Specifically, math teachers are found to have problems with implementing instructional practices that promote conceptual understanding. This research places direct emphasis on understanding in detail the connections between teachers' mathematical life stories and specific instructional sense making. More specifically, how do preservice teachers inform the development and implementation of their instructional practice over the course of their credential year and into the first two years of practice? What are the factors that influence this informing and what combinations of factors attribute to their development and implementation of instructional practice?

Keywords: Teacher Education-Preservice, Teacher Beliefs, Instructional Practice

Introduction

Prospective teachers come to formal teacher preparation programs with ideas and ways of thinking that influence what they learn from courses and experiences (Ball, 1988; Silva & Roddick, 2001). During teacher preparatory programs, preservice teachers face the challenge of balancing their prior mathematical experiences, what they believe mathematics learning environments should be, with what they are taught about teaching mathematics (Gainsburg, 2012; Lampert, 1988; Thompson, 1984). Scholars have been analyzing the connection between teacher belief and practice for years (Thompson, 1992) and researchers continue to find that teacher's beliefs are reflective of the nature of the instruction the teacher provides to students. While prior research centered on the topic of preservice mathematics teachers' belief structures (Ball, 1988; Cooney, Shealy, & Arvold, 1998; Harkness, D'ambrosio & Morrone, 2007; Drake, 2006; Millsaps, 2000), little of this research places direct emphasis on understanding in detail the connections between teachers' mathematics life stories and specific instructional sense making. Furthermore, while the need to study the mental processes of teachers has increased over the past decade, research in this area has largely concentrated on what teacher candidates understand about mathematics as they enter teacher preparatory programs (Ball, 1990; Thompson, 1984, 1992). This research aims to understand why new teachers do or don't use the practices emphasized during their credential programs by examining their instructional practices in light of their prior mathematical experiences. The following questions are guiding this research: How do preservice teachers inform the development and implementation of their instructional practice over the course of their credential year and into the first two years of practice? What are the factors that influence this informing and what combinations of factors attribute to their development and implementation of instructional practice?

Theoretical Framework

Ricoeur's (1992) concept of narrative identity, which is based on one's understandings of oneself across time that are accomplished by organizing and clarifying one's experience through narrative and Sfard and Prusak's (2005) definition of identity: collections of stories individuals hear and tell about themselves, frame this study. Autobiographies, whether written or verbal, provide individuals the opportunity to see influences and experiences that have affected their view at different points in time. Narratives provide individuals space to express to not only themselves, but others, who they are now, how they came to be, and where they think their lives may be going in the future. Rather than looking for conflicts between belief and practice, this work focuses on uncovering the interplay between community, belief/identity and practice. This research will focus on the key factors provided in Table 1, which have been associated with belief, mathematical identity and instructional practice in some manner. The aspects of what builds the foundational knowledge of instructional practice for preservice and in-service educators are rooted in their K-16 mathematics apprenticeship of observation and their mathematical identity. During the first year of a teacher preparatory program additional factors are added on: cohort mentor teacher(s), cohort member(s), program coursework, program instructors, edTPA and student teaching. All of these factors inform preservice teachers' instructional practice in varying ways. Results presented in this paper focus solely on the various relationships, connections and changes related to instructional practice upon entrance into teacher preparation through the preservice year.

Table 1: Factors guiding data coding that influence instructional practice for preservice and in-service educators over time.

K-16 Experiences	Year 1 (preservice)	Year 2 (in-service & MA)
Apprenticeship of observation		
Mathematical identity		
	Mentor teachers(s)	
	Cohort member(s)	
	Program coursework	
	Program instructors	
	edTPA	
	Student teaching	
		Induction program
		School culture
		Student population
		Department
		District mandates

Research Design

This small-scale study is taking place at a university in Northern California and began fall of 2016, with data collection anticipated to conclude spring of 2020. Currently, I have data from eight participants, three males and five females. Five of the participants have completed the first year of a two-year master's in education credential program and are teaching full time as the teacher of record in secondary mathematics classrooms. The remaining three participants are in the middle of the first year of the same master's in education credential program and are completing coursework and student teaching. My intent is to follow participants for three years, beginning with the teacher preparatory program and through their second year of teaching. I collected and analyzed four main categories of data for this study: a screener survey, observations of participants during methods class, coursework and interviews.

Data Analysis

Data were analyzed in a number of ways. First, each individual interview was transcribed and then coded according to constructs identified in the literature and natural emerging themes. These codes were then applied across the multiple data sources and finally a comparison across multiple participants was completed. The following excerpts are from one participant's data that was collected during year one of the teacher preparatory program.

Ophelia is currently teaching eighth grade math and she is on track to earning her Master of Art in education spring 2018. The different data sources include preliminary thematic analysis that begin to illuminate the connection between the factors of influence on Ophelia's instructional practice.

At the start of the program, Ophelia was asked to write a mathematics autobiography, which was shortly followed by a second paper on the topic of how Ophelia thought students learn math and how she defines math. In these papers, Ophelia shares her own personal struggles with learning mathematics and how these experiences have shaped her mathematical identity.

> The only aspect I didn't like about learning math was when the only way to solve a problem was by using an equation and the teacher wouldn't provide any explanation beyond just plugging the numbers into an equation. I would feel frustrated in not understanding why that worked or where the equation came from. Luckily, not all my teachers were like that, many would find a way to give us an explanation as to why we could use a certain equation.

Ophelia's apprenticeship of observation consisted of educators that created learning environments where students were able to approach problems in a variety of ways, as well as very didactic and teacher centered environments. The frustration expressed in the previous excerpt, is tied to Ophelia's mathematical understanding and how she viewed and situated herself as a mathematics learner different types of learning environments. In a later assignment, Ophelia re-emphasizes the struggles she faced learning mathematics during her adolescence. Learning environments where knowledge production was rooted in procedural understanding and memorization over those that focused on developing conceptual understanding did not provide her the opportunities she needed to achieve the mathematical understanding and confidence she desired.

> I was a student that used memorizing as my primary tool until I discovered I couldn't do that anymore for certain topics. I then had to find new ways to learn and deepen my understanding. I do believe new knowledge is constructed based on past knowledge that we have acquired.

During the interview Ophelia shared a variety of mathematical experiences, starting when she was a little girl practicing flash cards with her sister, up through college where she completed coursework and earned a Bachelor of Science in mathematics. When asked to share a positive experience about mathematics Ophelia shared stories centered around one specific teacher. When probed further about why this was such an important memory she elaborated:

> I think it was the topic and the teacher. He made it really exciting to learn. We would do a lot of neat projects, that we would do learning on our own and then bring it back to the class to share. And I think that was the first time I started struggling with math specifically, but I also, I was ok with the struggle – my teacher made that, something you didn't necessarily

> have to get the answer as long as you started working on the process that is what he was looking for and so I think that gave me a really good foundation and I realized it was ok to be wrong and struggle with math.

Ophelia then dives even deeper and provides additional evidence of the connection between mathematical identity, apprenticeship of observation and instructional practice. The following is from one of her weekly reflections:

> I am working on taking different approaches with them [students] and finding ways to improve. I noticed how if a student gave an incorrect answer, I wouldn't respond as well as I would have liked. I am still trying to find the balance of how to do it. I have tried supporting the student to try the problem again and I have said how the student's answer was an interesting answer and opened the question back up to the class to see if anyone got anything different. Each week I find a way to push myself more. I think I will see if my [mentor] teacher would be up for changing the warm-up format to not be as traditional and rather elicit students' prior knowledge.

As Ophelia discusses and reflects upon the struggles her and her students faced, she draws upon her apprenticeship of observation when indicating the type of instruction, she wants to enact in her classroom. Her mathematical identity is tightly linked to this apprenticeship, as she was exposed to learning environments where mathematical mistakes were not only valued but encouraged and utilized. Ophelia recognizes the danger of teaching students just procedural understanding and fact memorization. Her instructional practice is heavily influenced by her prior experiences. While Ophelia is trying to apply various ideas learned during her methods coursework, at this stage of the program Ophelia's mathematical identity, and apprenticeship of observation have greater influence on her instructional practice.

Conclusion

Preservice teachers need to reflect upon their past math experiences while active in the teacher preparatory program in order to make connections between how their experiences have shaped their conceptions of math and teaching. This research on the role of prior beliefs and experiences with mathematics and teaching might produce indicators for which candidates align with the underlying principles and educational philosophies of the program they are applying for. Knowing more about the mathematical history of credential candidates could inform teacher educators about the potential hurdles their preservice teachers might face when learning how to become effective teachers that are grounded in practices that foster high levels of conceptual understandings and promote equitable access to mathematical knowledge for their students.

References

Cooney, T. J., Shealy, B. E., & Arvold, B. (1998). Conceptualizing belief structures of preservice secondary mathematics teachers. Journal for research in mathematics education, 306-333.

Drake, C. (2006). Turning points: Using teachers' mathematics life stories to understand the implementation of mathematics education reform. Journal of Mathematics Teacher Education, 9 (6), 579-608.

Gainsburg, J. (2012). Why new mathematics teachers do or don't use practices emphasized in their credential program. Journal of Mathematics Teacher Education, 15 (5), 359-379.

Millsaps, G. M. (2000). Secondary Mathematics Teachers' Mathematics Autobiographies: Definitions of Mathematics and Beliefs about Mathematics Instructional Practice. Focus on Learning Problems in Mathematics, 22 (1), 45-67.

PRE-SERVICE TEACHER DESIGN OF ACTIVITIES THROUGH COLLABORATION AND AUTHENTIC ENACTMENT

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Pre-service mathematics teachers collaborated with a Master Teacher in the authentic implementation of lesson plans in high school classrooms. The pre-service teachers made small strides in their own ability to design activities. There were indications that the pre-service teachers experienced growth in recognizing the value of collaboration and revision. There were limits to their ability to enact lesson learned about cognitive demand in their own activity design.

Keywords: Teacher Education - Preservice, Instructional Activities and Practices, Teacher Beliefs, Teacher Knowledge

Introduction

Much of the work of preservice teacher candidates is learning how to design rich activities that will engage students in inquiry-based mathematics and be accessible to all students. An extra layer in this challenge is that typically lesson plans and tasks designed in pre-service contexts, prior to student teaching, are never taught in classrooms meaning there is no feedback loop for the design beyond feedback from an instructor or peers. Our approach to this challenge was to design the CRAFTeD cycle (Meagher, Edwards & Ozgun-Koca, 2011b), inspired, in part, by Lesson Study research, to give teacher candidates the opportunity to see tasks that they designed implemented in classrooms and, therefore, promote growth in their individual task design.

The cycle emerged from our previous work (Meagher, Ozgun-Koca & Edwards, 2011a) where the decisive influence of the field placement in terms of the mentorship/exemplars students experienced became apparent.

The CRAFTeD cycle: (i) A class of preservice high school teachers design an activity; (ii) an experienced inservice teacher reviews the activity and gives feedback; (iii) the inservice teachers teach the revised activity, observed by the preservice teachers; (iv) the preservice and inservice teachers meet together to reflect on and redesign the activity.

We report the results of a small-scale intervention study designed to measure growth in teacher candidates' ability to design tasks for mathematics classrooms that require "higher level demands" (Stein & Smith, 1998) with the intervention of CRAFTeD cycles as well as to examine the influence of the CRAFTeD cycle on their growth as preservice teacher candidates.

Literature review and relationship to research

Pre-service teachers early in their program develop lesson plans - typically in isolation - that are never taught in a classroom, creating a disconnect between planning, implementation, and assessment of student learning (Allsopp et al. 2006; Meagher, Ozgun-Koca & Edwards, 2011a). While university methods instructors laud the merits of student-led inquiry, exploration, and discovery-based teaching methods, secondary mathematics teachers in too many schools favor instruction directly focused on student preparation for high-stakes, multiple choice state tests (Seeley, 2006). Developing communities of practice (Wenger, 1999) and lesson study groups (Fernandez, 2002) can help candidates and practicing teachers adopt a more research-based focus in their lesson planning and develop a shared repertoire of communal resources which transcend individual contributions. Providing candidates with opportunities to experience their lessons taught by Master Teachers in authentic classroom settings increases motivation for lesson writing. Preservice teachers demonstrate a trajectory of learning about lesson plans in a cycle of designing a lesson to be taught by a Master Teacher and reflecting critically on the implementation of that lesson (Meagher, Edwards & Ozgun-Koca, 2011b).

Methods and Methodologies

Methods and Methodologies

Participants in the study were students in the second semester of a two-semester sequence of methods classes both involving field experiences and both prior to student teaching.

Early in the semester teacher candidates designed a lesson length activity on composition of functions. Teacher candidates then participated in a CRAFTeD cycle on a different topic. At the end of the semester teacher candidates revised the original lessons on composition of functions.

For quantitative analysis both versions of the activities were scored by all members of the research team using a rubric based on Stein & Smith (1998) with a "definitive" score agreed on. For qualitative analysis the constant comparative method (Glaser & Strauss, 1967) was used to establish trends in the data.

Results

Quantitative Data:

Each component of the task was given a score: Memorization: 1pt; Procedures without Connections: 2 pts; Procedures with Connections (3 pts); Doing Mathematics (4 pts) (Stein & Smith, 1998). The scores were averaged for the entire task to get an overall score for each task.

Table 1: Researcher mean scores from Version 1 and Version 2 of the tasks

	1	2	3	4	5	6	7	8	9	10	11	12		
Vers 1	2.67	2.67	2.50	1.63	2.50	3.00	2.50	2.55	2.08	2.40	2.00	2.32	Mean	2.40
Vers 2	2.27	2.67	2.70	1.64	2.60	3.13	2.55	2.73	2.60	2.17	1.80	2.76		2.47

There was, on average, a slight improvement over the course of the semester (the class average increased from 2.40 to 2.47) toward "higher order demands" (Stein and Smith, 1998). In four of the eleven activities, the cognitive demand was reduced. This reflects a decision that some components were so open-ended as to be difficult for the students to engage with and, in revision, the cognitive demand was lowered.

Table 2: Researcher Scores vs Student Scores for Version 1 of the tasks

	1	2	3	4	5	6	7	8	9	10	11	12		
Researcher	2.67	2.67	2.50	1.63	2.50	3.00	2.50	2.55	2.08	2.40	2.00	2.32	Mean	2.40
Version 1	2.88	2.58	2.80	1.73	2.70	2.43	2.70	2.73	3.00	2.00	3.00	2.05		2.55

The numbers in the table above represent a comparison between the researchers' assessment of the average component score for the original activities and the students' own assessment of the average component score. As can be seen, the researchers assessed the components of the activities to have, on average, a slightly lower cognitive demand than in the students' assessment. This is, perhaps, to be expected with researchers who are experienced in working with pre-service teachers likely to have a more rigorous standard for cognitive demand. It is notable, however, that there is not a great deal of difference with the pre-service teachers exhibiting a good understanding of the levels of cognitive demand. This can, in large part, be ascribed to the fact that the students had to use Stein and Smith's (1998) cognitive demand framework as part of the assignment when the developed the original activities.

Qualitative Data:

CRAFTeD Cycle. At the end of the CRAFTeD cycle the teacher candidates wrote a reflective essay on the process. The following significant trends emerged in the data:

(i) a recognition of the need to give high school students an opportunity to struggle with mathematics and to discuss mathematics

Seeing a lesson that they had designed actually implemented in a class was an eye-opening and rich experience for the students. The Master Teacher with whom they collaborated stressed that student struggle is important but it was a surprise to the preservice candidates to see it in action in terms of the amount of time the Master Teacher allowed students to work on a task with which they were having difficulty and how he then handled the discussion. The preservice candidates observed the Master Teacher allow the students to struggle enough to make the subsequent discussion of the task as rich as possible. "He checked students answer and asked where they struggled with, then I heard the best conversation I have ever heard in the high school classroom." (TC 9). Another teacher candidate noted that the students, at the end of the warm up activity, "seemed to be having a difficult time connecting their observations with a kind of "main idea" but that "Once the students had a chance to discuss this out loud with [Master Teacher's] guidance the students developed a much clearer understanding. She remarked that "the class discussion is where students made the most progress." (TC 7).

(ii) an expectation that the work should be challenging and appropriate for high school students

Related to the teacher candidates' recognition of the importance of giving students an opportunity to struggle was their discomfort with that notion and the consequent initial design of tasks that were not challenging to the students either in their level or construction. The students were taken aback by the Master Teacher's rejection of their initial offerings. "A concern that [the Master Teacher] had early on in our lesson development was that our lesson was not at the high school level." (TC 3) and "[the Master Teacher] felt as though our [first idea] was too juvenile for his students" (TC 4). However, this feedback forced to both consider the level but also to consider the mind set of their approach. One teacher candidate remarked "After the presentation of our first ideas of the lessons, [the Master Teacher] made it clear to us that, when we first began this process, we were stuck in the ideas that we were taught and were not doing a great job of looking at the content as a teacher rather than a student." (TC 7).

Revision of tasks on composite functions. At the end of the semester the teacher candidates were given the task of revising their activities on composite functions and to write an essay to explain the changes they made. The following significant trends emerged in the data:

(i) appreciating continuous reflection and improvement

One of the affordances of the CRAFTeD cycle is that there is authentic feedback on the lessons that teacher candidates write. The teacher candidates are often surprised to see the different ways students interpret questions, how students may not look at a graph and see what is obvious to the teacher candidates, how the timing can be so different than what they imagined when it meets reality. This effect is heightened because of the huge amount of collaborative effort that goes into making the lessons. One result of the study is that, despite the challenge and effort the teacher candidates came to value the multiple iterations of revisions from first presenting their ideas to the Master Teacher, to negotiating within the teams about priorities for the revisions, to seeing how what they feel is a very solid lesson has many flaws when given to real students. “I learned how to listen to the reactions and feedback of students to eventually improve the lesson in any way possible. Through this project, I strengthened my ability of discussion with my fellow classmates by learning to work together to make our lesson unified and seamless. During this project, there were many hiccups and setbacks, but after each one my group came together and reflected on how we could make the lesson better.” (TC 8).

(ii) a focus in changes on fixing instructions and small details rather than elevating the cognitive demand of the task.

One of the affordances of teacher candidates seeing a lesson they designed themselves actually taught is the heightened attention paid to how the students interact with the task. One result of this was that the teacher candidates were very concerned when the students had difficulty in following instructions written on the worksheet. “When students got a hold of our lessons that we had created, they often expressed that there was a lot of writing or confusing instructions. There is no reason to confuse students with wording when the goal is mathematical.” (TC 8) and “I want students to focus on the mathematical content, and to not get caught up in instructions.” This is a valid concern but had the effect in the revisions of the lessons that a large number of the changes and justifications for changes were simply to do with wording of instructions, precision of notation etc. with a dearth of attention paid to the overall cognitive demand of the task.

Conclusion

This small-scale intervention study was designed to provide pre-service mathematics teachers with experiences in designing activities with high levels of cognitive demand and to experience some of those activities being authentically implemented in high school classrooms. The pre/post structure of one pair of activities showed the teacher candidates teachers making small strides in their activity design. There were notable limitations in their ability to craft effective open-ended questions, focus activities on key mathematical ideas and maintain cognitive demand although Stein and Smith’s (1998) proved to be a useful tool in developing their thinking about cognitive demand. The teacher candidates made larger strides in their openness to the importance of student struggle and student collaboration in mathematical work. The teacher candidates also exhibited growth in valuing collaboration and cycles of revision in lesson planning.

References

Allsopp, D. H., Kyger, M., & Lovin, L. (2006). Teaching algebraic thinking. *Designing and implementing mathematics instruction for students with diverse learning needs*, 432-488.

Fernandez, C. (2002). Learning from Japanese approaches to professional development: The case of lesson study. *Journal of Teacher Education*, 53(5), 390-405.

Glaser, B., & Strauss, A. (1967). The discovery of grounded theory. 1967. *Weidenfield & Nicolson, London*, 1-19.

Meagher, M., Edwards, M. T., Ozgun-Koca, A. (2011b). Project CRAFTeD: an adapted lesson study partnering preservice mathematics teachers with a master teacher. *In* Wiest, L. R., & Lamberg, T. *(Eds.).* Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno. PME-NA.

Meagher, M., Ozgun-Koca, A., Edwards, M. T. (2011a). Pre-service teachers experiences with advanced digital technologies: The interplay between technology in a pre-service classroom and in field placements. *Current Issues in Technology Education (CITE), 11(*4).

Seeley, C. (2006). *Teaching to the test*. Reston, VA: National Council of Teachers of Mathematics (NCTM).

Stein M. & Smith M. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics Teaching in the Middle Grades, 3(*4) 268-75.

Wenger, E. (1999). *Communities of practice. Learning, meaning and identity.* Cambridge: Cambridge University Press.

LOOKING INWARD: (RE)NEGOTIATING AND (RE)NAVIGATING MATHEMATICS, TEACHING, AND TEACHER BELIEFS

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This paper is a reflection on a co-teaching experience during the first mathematics methods course of a teacher preparation program, where a community of teachers (teacher educators and pre-service teachers) could reflect on tensions (with teacher beliefs, with practice, and with mathematics). Cognitively Guided Instruction (CGI) was a central tenet to the course material and required learnings, opening up opportunities to (re)negotiate those tensions with beliefs, practice, and mathematics. We employ poststructural theories, attending to the documents that participants produced as well as the thinking and reading happening simultaneously, using writing as a method of inquiry.

Keywords: teacher education – preservice, teacher beliefs, elementary school education

This paper is a reflection, of sorts, on a co-teaching experience during the first mathematics methods course of a teacher preparation program. This course brought together a safe space and community for *teachers*—teacher educators and pre-service teachers—to reflect on *tensions*—tensions with teacher beliefs, with practice, and with mathematics. Cognitively Guided Instruction (CGI; Carpenter, Fennema, Franke, Levi, & Empson, 2014) was a central tenet to the course material and required learnings, and that consistent focus opened up opportunities to (re)negotiate those tensions with beliefs, practice, and mathematics. As our pre-service teachers were navigating the often-overwhelming requirements of the preparation program, they were confronted by their beliefs about what mathematics is, who can do mathematics, as well as how mathematics ought to be taught and learned; meanwhile, we (as teacher educators, doctoral students, and novice scholars) were navigating the always difficult world of academia, dissertation-writing, and reading in theory. Kayla, for instance, is writing her dissertation on mathematics teacher beliefs reconceptualized as an entanglement that is not stable; and yet, she recognizes her beliefs about teaching and learning mathematics, names them, feels them, teaches them. Susan wonders how post- theories might help us navigate those tensions, or perhaps present more tensions, by providing us the space to explore them, question them, trouble them.

In the current educational context that privileges conformity and compliance by teachers (McDermott, 2013), pre-service teachers need strategies to help them exercise professional autonomy. Teachers often experience a disconnect between the identities cultivated in their university classrooms and those currently privileged in schools, which leaves them feeling like they do not have the freedom to act, feel, and think in the ways they desire (Labaree, 2010). Furthermore, teacher educators are also engaged in negotiating and renegotiating tensions between theory and practice. Consequently, forming a teaching identity becomes a complex task that involves negotiating often contradictory ideas about teachers and teaching (Britzman, 2003). Teacher education, however, often focuses on helping teachers develop a stable identity rather than negotiating the shifting identities with which teachers must contend. This study aims to reconceptualize teacher beliefs as teacher educators (re)negotiate tensions and entanglements with/in teaching frameworks, and to write stories of those tensions. We take up posthuman and poststructural theories of subjectivity and methodology to consider pre-service teachers and teacher educators as subjects (selves) whose beliefs about teaching and learning mathematics are *always-already* entangled, impossible to think as pre-existing or separate (Derrida, 1967/1974).

This study is informed by poststructural theories (e.g., Foucault, 1981/2000), the purpose of which is to describe the linkages of language, power, and identity that impact how individuals interact with and produce the social world (Davies, 2003). In adopting this perspective, we attend to the documents that participants produced (e.g., reflective writings about readings, assignments about mathematics teacher beliefs, etc.) both inside and outside of class. The goal was not to collect or understand any individual participant's thinking but to explore what it is possible to learn as teacher educators in our (re)negotiating of tensions as we connect theory with practice as well as what possibilities exist and can exist for pre-service teachers to act, think, and feel as mathematics teachers.

Imagining reflection as in-constellation (Myers, Bridges-Rhoads, & Cannon, 2017) has given us the space before to question the taken for granted practice of reflection and think about it as a relation, as reimagining, as meddling in the middle (McWilliam, 2008), all to de-stabilize the too-familiar practices we ask of pre-service teachers. In co-teaching this class and in writing this paper, we again trouble the stability and familiarity of reflection alongside teacher beliefs in mathematics teaching and learning.

Methodology

The participants in this study include 20 pre-service teachers enrolled in the first of two math methods courses during their early childhood and elementary education teacher preparation program during an undergraduate Bachelor of Science in Education degree at a large, urban university in the southeastern United States. This teacher preparation program is four semesters in duration, where pre-service teachers are enrolled in coursework as well as part-time field placements during the first three semesters, and full-time field placement (student teaching) occurs during the fourth and final semester. This first course, which takes place during their second semester of the program, was intended to focus on the primary grades and designed to introduce CGI as an instructional framework.

The following documents were produced and shared over the course of the semester: reading journals (completed weekly and outside of class), beliefs statements (written in class at the end of the semester), and course evaluations. Documents were then analyzed using a collaborative process of writing as a method of inquiry (Kvale & Brinkmann, 2009; Richardson & St. Pierre, 2005) that involved the two of us sitting together and separately to read theoretical texts and study documents while writing and talking about how they interact. In this way, we write stories about our constant (re)negotiation of teacher beliefs and instructional practices as well as theoretical paradigms and neoliberal pressures in order to illustrate how various truths about teachers and teacher educators are continuously produced as language is organized into ways of thinking, speaking, and acting. Broadly speaking, our process of writing as a method of inquiry uses writing as a way of knowing, thinking, and creating; thinking and rethinking, reading and rereading of data alongside theoretical texts, writing and rewriting as a generative and cyclical process where stories and themes are welcome to come about on the page (Jackson & Mazzei, 2012; Koro-Ljungberg, 2015; Richardson, 1997).

Here we present some of these stories of pre-service teachers negotiating their beliefs while we as teacher educators are negotiating our own. These "findings" are preliminary, not fixed or final but rather coming about in the midst of different stories and writing and thinking. We draw on class conversations, course assignments, reading journals, course evaluations, and our anecdotal notes, to begin to tell this story of tensions in beliefs, teaching, and mathematics.

The Story (Begins…)

> *Due to the way I was taught mathematics during schooling, I have never had a positive relationship with the subject. Before, I imagined math as a left-brain concept, but now, I can view it as a creative process. Math is not something to be dreaded, but teachers can and should use math to create a space where experimenting, trying, failing, and succeeding are always implied and welcome. Math is not a teacher standing in front of a chalk board, barking out orders for equation solutions, but rather, communication between students, collaboration in the classroom, and a space where creativity to solving the proposed problem is encouraged and celebrated. Over the past 15 weeks, I have had the opportunity to learn what true mathematics looks like in the elementary classroom. By learning about it in the university classroom and implementing it in the field, my entire perspective on methods of teaching mathematics and learning math has completely changed. In August, my belief was that math was taught to you. It was an art of rote memorization, and one that, quite frankly, I was no good at. However, as we learned about CGI, my perspective on the teaching and learning process regarding math completely changed.*

Beliefs were claimed to have changed in this course experience, for one reason or another, but this requires a conceptualization of beliefs as measurable or distinguishable. As the course instructors, we hope that these pre-service teachers will learn something from us, and many of them come to this course with negative and harmful past experiences in mathematics, so this new learning and outlook is subjectively good… but what happens when these pre-service teachers are confronted by all of the other aspects of teaching? What happens when it is time to enact those beliefs amidst the mandates and curricula and relationships and mathematics (and, and, and…)?

> *This type of environment can be difficult to create and even more difficult to maintain. However, I think that this system on learning is so much more beneficial to students overall than rote memorization. By refusing to "take the easy route" and just give out solutions, teachers should strive year round to make the classroom climate one that promotes the success of students and their learning processes, even if that can sometimes be difficult. But, as much as I'd love to have an environment where everything is student centered, key terms are avoided, and I allow them to find strategies rather than teach them; we must think about testing. As much as I hate it, standardized tests play a huge role in our students' academic success. And teachers are assessed based on how well our students do. I want my students to be successful and on a timed test, they do not have an ample amount of time to come up with strategies and take a deep look into every question. Therefore, I would teach tricks. However, I would only introduce tricks if the students already mastered the concept.*

The connection to practice is overwhelming. Overwhelming for us as former teachers (re)negotiating those beliefs while trying to prepare these pre-service teachers for their future classrooms (and everything that comes with them), and overwhelming for them as students in the field, getting a small dose of reality but also engaging in conversations about those negotiations, navigating the tensions and finding ways to enact with pride and confidence and responsibility. Their reading journals brought concerns to the group; questions about application and reality, ways to make teaching work for themselves and their future students.

Initial Implications

As we have begun to think with this data and witness the tensions in both our becoming researchers in mathematics education and in the students' evolving views of themselves as future educators in elementary mathematics classrooms, important questions are beginning to surface. What does/should mathematics look like? Who can do mathematics and how do we make a mathematics that is accessible to more students? (Martin, 2015; Stinson, 2013). In what ways do these new beliefs and plans for instruction change the way we think about mathematics?

> *I wholeheartedly believe that the stigma of "math is hard," "some people just aren't good at math," and "there's only one way to solve the question" are just excuses that teachers give because that's how they were taught. But if we switched those stigmas into norms- that math is easy, everyone is good at math, there are infinite ways to solve the question- then we are believing in math for what it truly can be- and believing in the students to succeed as well.*

There is opportunity and possibility in rethinking ourselves and mathematics. It changes how we do mathematics teaching. It is our hope that in further delving into and around this data, questions and moments will arise that will push us to think differently. It is in difference, not in repetition that we can make spaces for more students and pre-service teachers to find success in mathematics, and to see themselves as mathematicians.

References

Britzman, D. (2003). *Practice makes practice: A critical study of learning to teach* (2nd ed.). Albany, NY: State University of New York Press.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (2014). *Children's mathematics: Cognitively guided instruction.* Portsmouth, NH: Heinemann.

Davies, B. (2003). *Frogs and snails and feminist tales: Preschool and gender*. New Jersey: Hampton Press Incorporated.

Derrida, J. (1974). *Of grammatology*. (G. Spivak, Trans.). Baltimore: Johns Hopkins University Press. (Original work published 1967).

Foucault, M. (2000). So is it important to think? In *Power* (James D. Faubion, Ed.) (pp. 454-458). New York: The New Press. (Original work published 1981).

Jackson, A. Y., & Mazzei, L. A. (2012). *Thinking with theory in qualitative research: Viewing data across multiple perspectives*. New York: Routledge.

Koro-Ljungberg, M. (2015). *Reconceptualizing qualitative research: Methodologies without methodology*. Thousand Oaks, CA: SAGE Publications.

Kvale, S. & Brinkmann, S. (2009). *Interviews: Learning the craft of qualitative research interviewing* (2nd ed.). Los Angeles, CA: Sage.

Labaree, D. (2011). Targeting teachers. *Dissent, 58*(3), 9-14.

Martin, D. B. (2015). The collective Black and *Principles to Actions*. *Journal of Urban Mathematics Education, 8*(1), 17–23. Retrieved from http://ed-osprey.gsu.edu/ojs/index.php/JUME/article/view/270/169

McDermott, M. [UnitedOptOutNational]. (2013, July 23). Walking the labyrinth of the corporate-owned-Common-Core. [video file]. Retrieved from https://www.youtube.com/watch?t=130&v=vvUMk1ro27E

McWilliam, E. (2008). *The creative workforce: How to launch young people into high-flying futures.* Sydney, Australia: University of New South Wales Press.

Myers, K. D., Bridges-Rhoads, S., & Cannon, S. O. (2017). Reflection in constellation: Post theories, subjectivity, and teacher preparation. *Journal of Early Childhood Teacher Education, 38*(4), 322-337.

Richardson, L. (1997). *Field of play: Constructing an academic life.* New Brunswick, NJ: Rutgers University Press.

Richardson, L., & St. Pierre, E. A. (2005). Writing: A method of inquiry. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (3rd ed.) (pp. 959-978). Thousand Oaks, CA: Sage.

Stinson, D. W. (2013). Negotiating the "White male math myth": African American male students and success in school mathematics. *Journal for Research in Mathematics Education, 44*(1), 69-99.

CONSTRUCTING A FRAMEWORK ACKNOWLEDGING THE SOCIOCULTURAL, CRITICAL, SPATIAL, AND QUANTITATIVE WAYS OF KNOWING THE WORLD

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In this paper we discuss an epistemological framework rooted in a dialectic ontology of social, historical, and spatial dimensions. The framework is designed to make explicit the intersecting sociocultural, critical, quantitative, and spatial ways of knowing the world, which are inherent in mathematics education and quantitative learning environments more broadly. The intersection of these ways of knowing the world are discussed and then considered in relation to teacher education. We posit this framework could be powerful for both designing learning experiences for pre-service mathematics teachers and for making explicit to pre-service teachers a lens for making sense of their own future classroom environments to enact a more just and comprehensive mathematics curriculum.

Keywords: Teacher Education-Preservice, Teacher Knowledge

Introduction

The social and historical dimensions of reality have long been considered important when considering the teaching and learning of mathematics (Lerman, 2000), and more recently political dimensions have become highlighted (Gutiérrez, 2013), however a dimension that has been somewhat neglected is that of spatial justice. For the purpose of our research we define spatial justice as the critical examination of how geography and access impact opportunity. A recent push in that direction has been spearheaded by Dr. Laurie Rubel and her team in their work on conceptualizing *Teaching Mathematics for Spatial Justice* (Rubel, Hall-Wieckert, & Lim, 2017, 2016; Rubel, Lim, Hall-Wieckert, & Sullivan, 2016). In their work, they developed and implemented lessons involving spatial justice themes that intersected with critical pedagogy. Their work was situated in a mathematics courses for social justice in high school and undergraduate classes serving predominantly underserved students in New York City (Rubel, Lim, et al., 2016). This work shows the potential that considering spatial justice in relation to mathematics education can provide some tools and design heuristics to implement such learning opportunities in mathematics classrooms. In our work, we are drawing from the seminal work done by Rubel and company (Rubel et al., 2017; Rubel, Hall-Wieckert, et al., 2016; Rubel, Lim, et al., 2016) but are incorporating the notion of spatial justice in two fundamentally different ways. One way is by developing an epistemological framework that explicitly acknowledges sociocultural, critical, quantitative, and spatial ways of knowing the world drawing from the triple dialectic ontological perspective that Soja (2010) discusses. The second way is by investigating how such an epistemological framework could be used in constructing curriculum for pre-service mathematics teachers and also made explicit to pre-service teachers for making sense of their own future classroom environments. In this paper, we outline our epistemological framework, beginning with its ontological foundations, and then briefly discussing possible uses for it in teacher education settings.

Ontology

At the ontological level, in other words the characteristics of humans' reality, we are aligning our framework to include the commonly considered social and historic dimensions as well as the spatial (Soja, 2010). From this perspective spatial reality is not just viewed as a physical container for human social and historical realities, but instead as something that forms a triple dialectic ontological relationship with both the social and historical dimensions of reality (Soja, 2010). For example, geography or peoples' ways of knowing and interacting with their physical surroundings are shaped by their social interaction. We can also see this through the formation of political boundaries of nation states, which leads to the shaping of our spatial understanding of sociocultural interactions and the history of place. At the same time, spatial elements shape social structures. For example, rivers or mountains have often formed natural boundaries that then became social or political boundaries. This also goes beyond natural phenomenon. For example, scientists recently announced the beginning of a new geological era the Anthropocene dominated by humans reshaping of the earth (Crutzen & Stoermer, 2000). The structures that humans build both physically in terms of buildings, roads, and dams, as well as socially, such as voting boundaries, school districts, environmental laws, or norms over our interactions with the natural world have a profound effect on the spatial.

The spatial dimension also forms a dialectic relationship with the historical dimension. Natural forces work over time changing the earth, this is something that scientists have recognized over decades, but the same holds for our believed or imagined spaces and our understanding of the nature of the earth. For example, this can be seen in terms of a shift from the dominant view of the earth as flat to that of it as round and also in the methods we use to display the earth in term of cartography, and our understanding of how we can occupy space and what spaces we can occupy. We will not elaborate on the social historical dialectic, which is very well described elsewhere in mathematics education. Instead, we now shift to discussing our epistemological framework which builds off this ontological perspective.

Ways of Knowing in Quantitative Learning Environments

Our framework is based on the notion that there are many different ways of knowing or coming to know the world in which we exist. Some ways of knowing are privileged in society and by institutions whereas others are often marginalized, silenced or discredited. In the case of the ways of knowing, quantitative understanding is often privileged in dominant discourse. However, localized ways of knowing or certain cultural perspectives on the world are at times disadvantaged or marginalized particularly in the context of the mathematics classroom which is commonly seen as neutral (d'Ambrosio, 1985; Frankenstein, 1994; Gutiérrez, 2013). The framework we present here is by no means exhaustive of the ways of knowing the world. Instead what we are attempting to do is make explicit four important ways of knowing in quantitative learning environments which have been considered and investigated in various combinations by scholars in the past (d'Ambrosio, 1985; Frankenstein, 2009; Gutstein, 2006; Lerman, 2000; Rubel, Hall-Wieckert, et al., 2016; Skovsmose, 1994). Our work differs from past work in that we seek to consider and foreground all four aspects interwoven and inseparable in guiding the creation of quantitative learning experiences for pre-service students. Furthermore, we have also designed the framework to be made explicit to pre-service teachers such that the framework itself becomes part of the attunement process in making sense of quantitative problems and the world more broadly, with the hope that they will be able to transfer such ways of knowing or at least be open to different ways of knowing in other contexts. The main practices we are drawing from these ways of knowing are summarized in Table 1.

Table 1: Way of Knowing Framework

Ways of Knowing	Quantitative	Socio-cultural	Critical	Spatial
Key Practices	• Measurement • Collecting data • Modeling the world through quantification • Shaping the world through quantification • Problem solving through quantification and abstraction	• Using funds of knowledge of communities to make sense of the world • Considering the ways different communities of practice come to know the world	• Reading and writing the world • Making sense of social structures • Transforming social structures • Problematizing structural causes of (in)justice	• Considering how space and social structures shape and are shaped by one another • Understanding that spatial/social structures are historical

Implications and Future Work in Teacher Education

Rubel, Hall, and Lim (2017) discuss their Teaching Mathematics for the Spatial Justice (TMSpJ) design heuristic, applying it to two systems the lottery and alternative financial institutions. Looking back and extending the work of Rubel, Hall and Lim, we, because of our focus on working with pre-service teachers, shift from external foci such as the lottery, to looking ahead and grounding this work in spaces that remain central to the day-to-day notions of teaching. The main structure we are working to explore is that of the rural education system, and what are the opportunities students have based upon location and available resources. Teaching is a precarious profession, individuals want the best outcomes for their students but how many of the student outcomes are influenced by socio-cultural or historical decisions that challenge overall success? Soja (2010) talks about the "multiscalar view of a city", we believe that this idea is not limited to only cities (p. 32). In our study we want to challenge the notion that multiscalarity does not only occur in cities but extending this work to include issues that are found in rural settings. Although the number of influences in a city is exponential, we believe that pre-service teachers must see and validate the multitude of ways of knowing that change the opportunities for students living in areas that are rural.

Another system that is important for pre-service teachers to reflect on includes the structures related to public access and the availability of resources such as food, water, shelter, transportation etc. that ultimately become influences on opportunities for success. Children living in rural areas face similar challenges to those living in cities, but the ways of knowing do not look the same, therefore using critical pedagogies and quantitative measures a more complete understanding of equity and justice come to light.

Education and access to opportunities often determine trajectories established for students, especially in mathematics. In using the proposed framework to work with pre-service teachers, individuals will gain insight into multiple ways of knowing that will help them to better understand and support their future students. The lens in which they see and understand the world becomes inclusive, allowing their ways of knowing to look at the social, critical, and spatial challenges. When considering how the uneven development of geography impacts the opportunities of students in rural communities, it is essential to equip pre-service teachers to critically examine and validate those practices that fail to provide equal opportunity.

References

Crutzen, P. J., & Stoermer, E. F. (2000). The "Anthropocene." *Global Change Newsletter*, *41*, 17–18.

d'Ambrosio, U. (1985). Ethnomathematics and its place in the history and pedagogy of mathematics. *For the Learning of Mathematics*, 44–48.

Frankenstein, M. (1994). Understanding the politics of mathematical knowledge as an integral part of becoming critically numerate. *Radical Statistics*, *56*, 22–40.

Frankenstein, M. (2009). Developing a critical mathematical numeracy through real real-life word problems. In L. Verschaffel, B. Greer, W. Van Dooran, & S. Mukhopadhyay (Eds.), *Words and worlds: Modeling verbal descriptions of situations* (pp. 111–130). Rotterdam: Sense Publishers.

Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education*, *44*(1), 37–68.

Gutstein, E. (2006). *Reading and writing the world with mathematics*. New York, NY: Routledge.

Lerman, S. (2000). The social turn in mathematics education research. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 19–44). Westport, CT: Ablex.

Rubel, L. H., Hall-Wieckert, M., & Lim, V. Y. (2016). Teaching mathematics for spatial justice: Beyond a victory narrative. *Harvard Educational Review*, *86*(4), 556–579.

Rubel, L. H., Hall-Wieckert, M., & Lim, V. Y. (2017). Making space for place: Mapping tools and practices to teach for spatial justice. *Journal of the Learning Sciences*, *26*(4), 643–687. https://doi.org/10.1080/10508406.2017.1336440

Rubel, L. H., Lim, V. Y., Hall-Wieckert, M., & Sullivan, M. (2016). Teaching mathematics for spatial justice: an investigation of the lottery. *Cognition and Instruction*, *34*(1), 1–26. https://doi.org/10.1080/07370008.2015.1118691

Skovsmose, O. (1994). Towards a critical mathematics education. *Educational Studies in Mathematics*, *27*(1), 35–57.

Soja, E. (2010). *Seeking spatial justice*. Minneapolis, MN: University of Minnesota Press.

HALF THE BASE OR HALF THE HEIGHT? PSTS' EVALUATION OF A TRIANGLE AREA STRATEGY

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The purpose of this study was to explore how elementary pre-service teachers responded to a novel strategy for justifying the triangle area formula. We conducted interviews with 24 pre-service teachers. Findings highlighted concerns when assessing the mathematical content of the strategy, as well as differences in pedagogical responses depending on content knowledge. In many cases, pre-service teachers leveraged their understanding towards productive responses, while also grappling with generalization. Recommendations for supporting pre-service teachers in navigating the intersection between content and pedagogical knowledge are discussed.

Keywords: Mathematical Knowledge for Teaching, Measurement, Teacher Education-Preservice

Eliciting, interpreting, and responding to student thinking are essential tasks for mathematics educators, and a key component of effective mathematics instruction (NCTM, 2014). In the classroom, this practice includes seeking clues of understanding in what students say, write, or draw, and connecting to students' unexpected strategies. Given the importance of this skill, it is important to encourage development of this ability early in teacher education programs.

Successful application of this practice requires different types of knowledge, including knowledge about student thinking and how to connect to and build upon novel approaches. One area where this knowledge may be of concern is area measurement. Studies have shown that students often demonstrate procedural understanding of area, using the formula *length x width* incorrectly and without understanding (Zacharos 2006). While pre-service teachers (PSTs) have sometimes exhibited similar misunderstandings (Murphy 2012), they have also shown valuable knowledge, including understanding area as two-dimensional space and concrete measure (Baturo and Nason 1996). While perhaps not an ideal conceptual understanding of area, this knowledge represents a key starting point in mathematics education courses.

For students, utilizing formulas without understanding their development is likely to lead to incorrect usage and ineffective learning. For PSTs, lacking understanding of area may also lead to difficulties when interpreting and responding to student work. Exploring how PSTs leverage existing knowledge towards their instructional responses may help teacher educators understand how to encourage development of effective teaching practices early on. The purpose of this study was to explore how PSTs responded to a novel strategy for explaining triangle area, within the context of their existing content knowledge. The research questions for the study were: *(1) How do elementary pre-service teachers evaluate a novel strategy for justifying the triangle area formula? (2) What instructional strategies do PSTs use when responding to the novel strategy?*

Related Literature

Area measurement is a vital part of elementary school mathematics, offering connections between mathematics and the physical world, marking a shift into two-dimensional mathematics, and providing opportunities to estimate, manipulate, and visualize. However, studies have shown that students often demonstrate a more procedural understanding of area. They may use the formula *length x width* incorrectly (Zacharos 2006), or misunderstand the role of unit squares

(Kamii & Kysh, 2006). While PSTs have exhibited similar struggles (e.g. Murphy, 2012), they have also exhibited valuable knowledge about area (Baturo & Nason, 1996).

One approach for understanding how PSTs may interpret and respond to student thinking is to analyze their responses to hypothetical student work. Examining student work allows PSTs time for reflection and offers a low-risk setting for exploring response strategies. Responding to student work is a central component of teaching, and practicing this skill offers opportunities to develop effective instructional practices. Analyzing student work has helped in understanding how PSTs respond to student-invented strategies in past research (e.g. Son 2016), providing insight into how they develop responses and what may be done to support this development.

This work is further grounded in the theory of mathematical knowledge for teaching, where knowledge required for effective mathematics instruction is composed of multiple components, among them common and specialized content knowledge, knowledge of content and students, and knowledge of content and teaching (Ball et al., 2008). We were interested in exploring our PSTs' existing common and specialized content knowledge, and specifically how this knowledge informed their knowledge of students, knowledge of teaching, and instructional practices.

Methodology

Twenty-four elementary PSTs enrolled in a geometry/measurement course volunteered to participate. The participants were all female, with a range of mathematical backgrounds.

Data Sources and Collection

Data sources included both a pre-assessment and interview. The pre-assessment consisted of questions assessing both common and specialized content knowledge, as well as knowledge of content and students. Assessment tasks were adopted from prior studies (e.g. Zacharos, 2006).

The first author then conducted semi-structured interviews with all participants. In this study, we report on responses to one interview task, where participants were asked to study and respond to the diagram in Figure 1. Participants were told that the figure represented a student's attempt to explain the area formula *½ bh* via cutting and rearranging pieces of the figure. They were then asked to evaluate the strategy for mathematical accuracy, to describe their assessment of the knowledge exhibited by the student, and to offer possible instructional responses.

Pre-assessment data was collected immediately prior to the course measurement unit, with interviews were conducted in the following four weeks. While participants had been exposed to similar material prior to the interviews (both in their own education and in the course), none of the participants were previously familiar with the justification of the triangle area formula that comprised the primary data source for the present work.

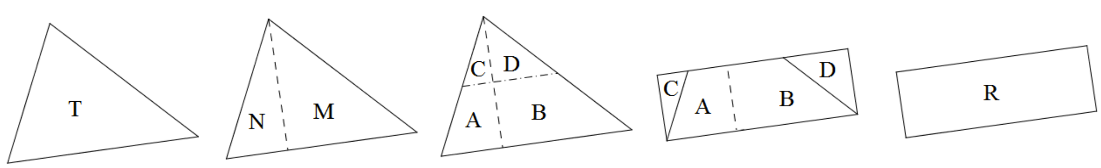

Figure 1. Interview question diagram. Novel strategy to verify area formula for a triangle.

Data Analysis

Interviews and reflective memos were qualitatively analyzed for content and instructional responses. The two authors coded interviews independently for content knowledge (see Table 1) as well as instructional response strategies (see Table 2). After initial coding, the authors met to discuss themes, resolve discrepancies between codes, and identify key findings.

Findings

PSTs' Content Concerns

The majority of PSTs (22 out of 24) grappled with evaluating the mathematical accuracy of the diagram, noting concerns regarding the fit of the pieces, connections to the triangle area formula, and generalizability to other triangles. Frequent responses are provided in Table 1.

Table 1: PSTs' Most Frequent Content-Related Responses

Description	Example
Doubt about the "fit" of pieces C and D when forming a rectangle (6 out of 24)	"You can't even know that this is correct because you don't have side lengths… just simply saying that this goes there and that goes there is not really valid" (Participant 14)
Difficulty connecting area formula for a triangle with area formula for a rectangle (10 out of 24)	"If you have one half base times height, that is not equal to base times height… You can't have the same amount of area be base times height and half of base times height at the same time" (Participant 20)
Confusion linking bisected *height* in Figure 1 with the formula indicating a bisected *base, **1/2bh*** (6 out of 24)	"I can see where the one half the height comes from, but I don't know…I get kind of confused when it comes to the base because you don't want to split the base in half because it didn't get divided and it's just not how I see it" (Participant 01)

Many PSTs' concerns were mathematically valid, engaging with pieces of specialized content knowledge and indicating critical thinking about the steps necessary to justify a formula. They sought clarity in the "fit" of the diagram, and understanding of how to connect diagram to formula. Most PSTs recognized that the strategy may not be generalizable, but struggled to fill in the details themselves. As a result, half rejected the strategy (12 out of 24) or stated the diagram was valid without verifying details (7 out of 24). Accepting without understanding or rejecting the method could lead to issues in a real teaching situation, leading either to the acceptance of incorrect strategies or the halting a productive discussion. Given that the sequence in Figure 1 is a viable strategy, our PSTs' responses indicated that struggles with understanding of the content may lead to struggles in responding to student thinking in future instructional moments.

PSTs' Instructional Responses

Our PSTs provided a variety of instructional responses, with some leveraging their content knowledge to inform their pedagogy. Frequent responses are described in Table 2.

Table 2: PSTs' Most Frequent Instructional Responses

Description	Example
Focuses on an alternate method for justifying area of a triangle (6 out of 24)	"[If] you made an exact copy [of the triangle] and flipped it on its axis… it would sort of look like a parallelogram. Then… you can do length times width or base times height and you are multiplying by point five to get half of it" (Participant 19)
Looks for additional examples or explanations (12 out of 24)	"This is very interesting! Can you show me? How do you know its correct?" (Participant 02).
Guides the student away from the strategy (9 out of 24)	"I would go over the formula again. I would bring up not to use this because it won't work every time" (Participant 12).

Twelve participants provided student-centered responses, asking the student to look for more examples or provide further explanation. Three PSTs went further and suggested the entire class investigate the strategy, so that other students could engage as well. These PSTs indicated a desire to elicit student thinking, bring about mathematical discussion, and provide room for the entire class to engage in new ideas. Although content knowledge may be concern for follow-up, these responses represented a good starting point for addressing unfamiliar strategies.

At the same time, other responses provided by PSTs were more teacher-centered. Nineteen PSTs provided procedural recommendations, such as giving area practice problems or asking the student to focus on use of the formula. Such responses indicated a higher value on procedural fluency, and frequently stemmed from prior difficulty with understanding the strategy. While testing empirical examples is a useful strategy for exploring generalization, it was unclear if the PSTs would have been able to follow up on those examples for this purpose.

Discussion and Conclusions

Our PSTs showed that it may be challenging, but certainly possible, for them to practice effective instructional practices. While struggles with content hindered some instructional responses, half of our PSTs placed the student at the center of discussion, a positive point from which to develop knowledge about teaching. They indicated a desire to learn from their students, and sought to merge the student's understanding with their own. For these PSTs, it would be helpful to discuss techniques for follow-ups to probing questions, with a focus on the specialized knowledge required to assess novel strategies. This knowledge should be explored not only to learn content, but also to see how these ideas may arise in day-to-day teaching practices.

Participants who struggled with their own justifications were less willing to explore new or unfamiliar ideas. As noted previously, eliciting and interpreting student thought is an important component of effective teaching, but may be especially difficult for those PSTs who continue to struggle with their own justifications. For these PSTs, it may be wise to focus on developing the ability to ask probing questions, eliciting students' understanding and giving themselves time for processing. Modeling how to address such challenges with probing questions, continuing dialogue, and scaffolding may help develop these key questioning skills. By integrating content knowledge with hands-on experience responding to student work, PSTs can begin realize for themselves where they may be missing understanding, and how this may impact their teaching practices in the future.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 387-407.

Baturo, A., & Nason, R. (1996). Student teachers' subject matter knowledge within the domain of area measurement. *Educational Studies in Mathematics*, *31*(3), 235-268.

Kamii, C., & Kysh, J. (2006). The difficulty of "length x width": Is a square the unit of measurement? *The Journal of Mathematical Behavior, 25*(2), 105-115.

Murphy, C. (2012). The role of subject knowledge in primary prospective teachers' approaches to teaching the topic of area. *Journal of Mathematics Teacher Education*, *15*(3), 187-206.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: NCTM.

Son, J. (2016). Preservice teachers' response and feedback type to correct and incorrect student-invented strategies for subtracting whole numbers. *The Journal of Mathematical Behavior*, *42*, 49-68.

Zacharos, K. (2006). Prevailing educational practices for area measurement and students' failure in measuring areas. *The Journal of Mathematical Behavior*, *25*(3), 224-239.

HOW DO PRESERVICE TEACHERS ELICIT THE THINKING OF A STUDENT WHO HAS MADE A MISTAKE?

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We report on a study of novice teachers' eliciting performances in a scenario in which a student has made a mistake and, if sufficiently probed, is able to recognize the mistake and revise their work. Our findings reveal the skills and capabilities of one group of preservice teachers at the start of a teacher education program.

Keywords: Teacher Education-Preservice; Instructional Activities and Practices

In mathematics teaching, a regular problem of practice is that a student does something "unexpected" when solving a mathematics problem such as using an invented approach or making a "mistake" when carrying out a known process. Interactions around "unexpected" student responses can be powerful sites of learning for both teachers and students (Hiebert et al, 1997). For example, when an answer is incorrect, the joint sense-making required to interpret student thinking goes beyond the identification and correction of mistakes or errors into the conceptual analysis of why the mistake or error was made (Borasi, 1994; Kazemi & Stipek, 2001). The information that teachers can uncover through elicitation can then guide their pedagogical response. While the importance of interacting with students around mistakes is well-documented, far less is known about the ways in which preservice teachers (PSTs) elicit student thinking when a student has made a mistake in their arithmetic process. In this study, we set out to explore the eliciting skills and capabilities of PSTs in a context in which a student has made a mistake in their arithmetic process and will recognize the mistake if asked specific questions about their reasoning. We use the term *mistake* to mean isolated and unrepresentative mis-executions of an algorithm (i.e. "careless mistakes") (Radatz, 1980). We studied the skills of PSTs at the beginning of a teacher education program in order to find out about the skills that they bring to teacher education.

The Practice of Eliciting Student Thinking

In teaching, "teachers pose questions or tasks that provoke or allow students to share their thinking about specific academic content in order to evaluate student understanding, guide instructional decisions, and surface ideas that will benefit other students" (TeachingWorks, 2011). Our decomposition of the work of eliciting student thinking includes bringing out the student's process and probing key aspects of what the student says and does to uncover the student's understanding (Shaughnessy & Boerst, 2018b). We use "skill with eliciting student thinking" to refer to the degree to which PSTs are able to engage in these areas of work.

Using a Teaching Simulation to Assess Skills with Eliciting Student Thinking

Since 2011, we have been using teaching simulations to study PSTs' skill with eliciting student thinking (Shaughnessy & Boerst, 2018a Shaughnessy, Boerst, & Farmer, 2018). In these simulations, a PST interacts with a "standardized student" (a teacher educator taking on the role of a student using a well-defined set of rules for responding) around a specific piece of written work. We design teaching simulations to have a consistent three-part format (Shaughnessy & Boerst, 2018b). First, PSTs are provided with student work on a problem and given 10 minutes to prepare for an interaction. Second, PSTs have five minutes to interact with the standardized student, eliciting and probing the "student's" thinking to understand the steps they took, why they performed particular steps, and their understanding of the key mathematical ideas involved. Third, PSTs are interviewed about their interpretations of the student's thinking.

We designed this simulation to be one in which a student makes a mistake that is evident in their original written work. This student uses an alternative algorithm for solving multi-digit subtraction problems (see Figure 1). The process involves writing the value of the minuend and subtrahend in expanded form and making any necessary trades. When used correctly, the user would then subtract the numbers place-by-place in expanded form. This student correctly applies the expanding and trading process, but mistakenly adds values by place instead of subtracting. This student has conceptual understanding of expanded form, the meanings of addition and subtraction, and when, how, and why to make trades. In this particular instance, however, the student accidentally loses sight of the fact that they are solving a subtraction problem due to the addition symbols in expanded form. During the interaction, the student will realize their mistake when pressed to: make and evaluate an estimate for the original problem, represent their process with a picture, talk about the meaning of the operation, explain why trades were made, solve another multi-digit subtraction problem, or re-solve the original problem. The teacher educators carrying out the "student" role were trained to only reveal the mistake or revise the process if pressed by the PST on one or more of these specific points.

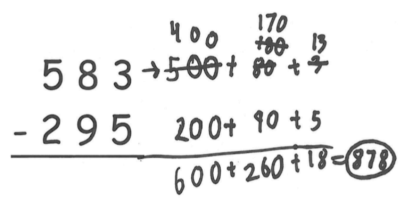

Figure 1. Student work. The student makes a mistake.

Methods

Thirty PSTs enrolled in a university-based teacher education program in the United States participated during the first week of the teacher education program. The teaching simulations were video-recorded. In this paper, we focus on three core components that we assessed: (a) eliciting the student's original process, (b) probing the student's understanding of key mathematical ideas, (c) eliciting and probing the student's mistake, including the reason for the mistake and the revised process and solution. For each of these components, we identified specific "moves" and tracked their presence or absence in each performance. We used the software package Studiocode© to parse the video into talk turns. Then, we identified "instances," which contain a question posed by a PST and the student's response to that question. Two independent coders applied all of the relevant codes to each instance. Disagreements were resolved through discussion and by referencing a code book.

Findings

Eliciting the Student's Original Process

The student's process had five steps: expanding both the minuend and subtrahend, comparing the numbers in each place, trading, adding (rather than subtracting) numbers by place, and adding the partial sums to arrive at the answer. The highest rates of eliciting occurred around the expansion (70%), the comparison of the numbers in each place (90%), and the trading steps (80%). In fact, 65% of PSTs elicited all three of these steps and 90% of the PSTs elicited two or more. However, only 53% of the PSTs elicited the step where the student made the mistake. The smallest percentage of eliciting occurred around the adding of the partial sums (10%). Because this step took place after the mistake, it may not have seemed important to PSTs who had elicited the mistake. Given that these PSTs elicited some, but not all, of the student's steps, we concluded that they brought skills relevant to eliciting a student's process that may be leveraged in the teacher education program to work towards more thorough elicitation.

Probing the Student's Understanding of the Original Process

We coded whether PSTs probed the student's understanding of six mathematical ideas underlying the process. The highest percentages of probing occurred around the operation in the problem (27%), why the student expanded (37%), and why the student traded (27%). The lowest percentages of probing occurred around the equivalence of expanded form and the "original" number (7%), the reasonableness of the answer (7%) and what trading means (0%). This suggests that limited probing of mathematical ideas occurred. However, when we looked across the set of ideas, we found that 67% of PSTs probed the student's understanding of one or more idea. Thus, we concluded that this 67% of PSTs brought skills relevant to probing the student's understanding in this scenario, but note that their probing was not focused on particular ideas.

Eliciting and Probing the Student's Mistake and Revised Process

We coded the extent to which the PSTs elicited and probed the student's mistake of adding numbers by place instead of subtracting. We found three patterns in PST' performances: eliciting that the student had made a mistake (47% of PSTs), pointing out the mistake and getting the student to agree without first eliciting the mistake from the student (20% of PSTs), and only asking questions that were not focused on the mistake (33% of PSTs). In other words, 67% of the PSTs uncovered the mistake either by eliciting it or asking the student to confirm that they made a mistake; 33% percent did not learn about the mistake.

We further examined the performances of the 47% of PSTs who elicited the mistake from the student. These 14 PSTs elicited the mistake in different ways. Eleven of the 14 asked about the operation involved in the problem and then pressed on how that operation (subtraction) related to what that the student had done (addition). Another two PSTs posed another problem for the student to solve or asked the student to redo the original problem. A third approach, used by one PST, was to ask why the student had made the trades (to get enough to subtract).

To explore what PSTs did after learning the student had made a mistake, we considered the 20 cases in which PSTs uncovered the student's mistake either by eliciting or pointing out the mistake. Three of the 20 PSTs asked questions to learn about how and/or why the mistake was made. For example, in response to the student's comment, "I think I made a mistake," one PSTs asked, "How did you make a mistake?" Five of the 20 PSTs made statements based on their own inferences about how and/or why the student made the mistake and got the student to agree. The other 12 PSTs did not ask follow-up questions about how or why the mistake was made. For example, one PST elicited the mistake, then immediately shifted focus to the revised process. Of the 20 PSTs, 15 elicited some or all of the revised process from the student. Sometimes this included asking about all of the steps in the process again, but other times it focused on redoing the step where the student had made the mistake. In 10 of the 20 cases, the student revised their final answer. These findings suggest that this group of PSTs bring skills relevant for eliciting a student's mistake but may need to learn reasons and strategies for probing how and why a mistake was made.

Discussion

Understanding the skills with teaching practices that novices bring to teacher education is key to designing experiences that are responsive to the needs of PSTs (Shaughnessy & Boerst, 2018a). This study examined the ways in which PSTs at the beginning of a teacher preparation program elicited the thinking of a student who made a mistake when using an alternative algorithm to solve a subtraction problem. Learning about the reason for a mistake is important for teachers to accurately assess and respond to student thinking. For example, a student who was genuinely unsure of the meaning of expanded form would need different instruction than a student who has made a mistake when carrying out a well-understood process. However, at the start of their teacher education program, these PSTs focused more on eliciting the revised method and/or solution than asking about why the mistake was made. The data suggest several directions for continuing research. First, the PSTs in this study were in their first week of a teacher education program. In what ways do their skills with eliciting student thinking develop over time? Second, what course activities might effectively cultivate an inclination to probe how and why students make mistakes? Third, the assessment itself involved a mistake where the student was using a "non-standard" approach to solve the problem. Anecdotally, we have reason to think that some PSTs were discounting the student's reasoning because they believed that the student should be using a different method to solve the subtraction problem. A future study could compare skills in eliciting around a mistake with a "standard" and an "alternative" algorithm.

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References

Borasi, R. (1994). Capitalizing on errors as "springboards for inquiry": A teaching experiment. *Journal for Research in Mathematics Education, 25*(2), 166-208.

Hiebert, J., Carpenter, T. A, Fennema, E., Fucson, K. A., Wearne, D., Murray, H., Olivier, A., & Human, P. (1997). *Making sense: Teaching and learning mathematics with understanding.* Portsmouth, NH: Heinemann.

Kazemi, E. & Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *The Elementary School Journal, 102*(1), 59-80.

Radatz, H. (1980). Students' errors in the mathematical learning process: A survey. *For the Learning of Mathematics, 1*(1), 16-20.

Shaughnessy, M. & Boerst, T. (2018a). Appraising the skills that preservice teachers bring to teacher education: Skill with the practice of eliciting a student's thinking. *Journal of Teacher Education. 69*(1), 40-55.

Shaughnessy, M. & Boerst, T. (2018b). Designing simulations to learn about preservice teachers' capabilities with eliciting and interpreting student thinking. In G. J. Stylianides & K. Hino (Eds.), *Research advances in the mathematical education of pre-service elementary teachers: An international perspective* (pp.125 -140). Springer.

Shaughnessy, M., Boerst, T., & Farmer, S. O. (2018). Complementary assessments of preservice teachers' skill with eliciting student thinking. *Journal of Mathematics Teacher Education.* https://doi.org/10.1007/s10857-018-9402-x.

TeachingWorks. (2011). High leverage teaching practices. Retrieved February 25, 2018, from http://www.teachingworks.org/work-of-teaching/high-leverage-practices.

FROM BATCHES TO PARTS: PROSPECTIVE TEACHERS' REPRESENTATIONS FOR PROPORTIONAL RELATIONSHIPS

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In this report, I describe how using representations facilitated a shift in reasoning about proportional relationships in a content course for prospective middle school teachers. The prospective teachers initially approached proportional relationships and created representations from a multiple batches perspective. After some support, the teachers reasoned with a variable parts perspective and modified their representations accordingly.

Keywords: Teacher Education-Preservice, Middle School Education, Number Concepts and Operations

Researchers have stressed the importance of representations in mathematical thinking (Cuoco, 2001; Janvier, 1987). Additionally, representations are a critical feature of effective mathematics teaching. In *Principles to Action,* (National Council of Teachers of Mathematics, 2014), the authors explicitly identified teaching with representations as principle for high quality mathematics teaching practice. Using and connecting mathematical representations "[engage] students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving" (p. 10). However, teachers' conceptions of representations are sometimes framed as not "real" mathematics (Stylianou, 2010) and prefer to prioritize abstract, procedural rules (Eisenhart et al., 1993). One viable avenue is to provide prospective teachers with opportunities to learn with representations which influences both teachers' knowledge and use of representations (Jacobson & Izsák, 2015). In this report, I characterize such an opportunity for prospective middle school teachers to reason about proportions using representations. Specifically, I describe two ways prospective teachers reasoned about proportions and how they were supported in transitioning from one to the other.

Theoretical Framework

Definition of Representations

In this study, I frame representations as culturally and historically rooted artifacts whose meanings grow out of how communities have interacted with an inscription over time. Saxe's account (2012) of cultural forms and functions describe how historically rooted artifacts change over historical time. Forms are socially rooted artifacts perceivable by the community such as the base-27 system of the Oksapmin peoples (Saxe, 2012) or a number line (Saxe, de Kirby, Le, Sitabkhan, & Kang, 2015). Functions are how forms are used by the community. Over time, the community may preserve historical forms but may also change them in order to serve new functions. For example, as students use a number line form in earlier grades, the form shifts to accommodate rational numbers while preserving whole numbers. In this report, I focus on strip diagrams as representations although others were found in the data (e.g., equations).

Definition of Multiplication and Proportional Relationships

The class analyzed in this study used a multiplier-multiplicand definition for multiplication notated by equation $N \cdot M = P$ (Beckmann & Izsák, 2015). In this equation, the value N denotes the number of bases units in one group (the multiplicand), the value M denotes the number of groups (the multiplier), and the value of P denotes the number of units in M groups.

Additionally, a proportional relationship is the collection of the set of values *x* and *y* satisfying the equations of the form $N \cdot x = y$ or $x \cdot M = y$. Using this definition of multiplication, Beckmann and Izsák called the first relationship a multiple batches perspective wherein the amount in one group is preserved i.e., for a certain amount of the first quantity, there are a corresponding amount of the second quantity. They name the second relationship a variable parts perspective wherein the number of groups is preserved i.e., for a certain number of groups of equal size of the first quantity, there are a corresponding number of groups of the same size of the second quantity. They also claim the variable parts perspective has mostly been overlooked in the literature but can be used to think about secondary topics such as geometric similarity and slopes of lines. The results of this report show the accessibility of the variable parts perspective and describe features of students' representations help them reason with this perspective.

Data and Analysis

I analyzed four lessons from a mathematics content course for prospective middle school teachers (PSMTs). The lessons were on proportional reasoning based on the ideas in Beckmann and Izsák (2015). The 13 PSMTs in the course were predominantly white women. In class, PSMTs were expected to use the multiplier-multiplicand definition of multiplication and explain their thinking with drawings rather than memorized algorithms or symbol pushing. I also acted as a participant-observer and sat at one of the tables asking questions to support, extend, and connect PSMTs ideas. I analyzed two introductory lessons on proportional relationships and two lessons marking the transition from using multiple batches to variable parts. I analyzed video recordings of class. One stationary camera captured the whole class and another camera followed the teacher during small group work. Each table had an iPad where PSMTs can write notes and present their drawings. The primary analytical techniques were modified from Saxe et al., 2015 and focused on identifying forms and functions of the representations. To identify a form, I characterized the geometric inscription or object the PSMT used to explain their reasoning such as a bead, strip, or partition of a strip. I identified functions by using the PSMTs verbal explanations and annotations to explain what each form represented.

Results

Initially, the PSMTs' ways of reasoning using a multiple batches perspective where a ratio represented a composite unit. With support, the PSMTs reasoned using a variable parts perspective by assigning different values to the partition of a strip and collapsing batches of strips into one. I present two students' shifts in reasoning modelling other PSMTs in the course.

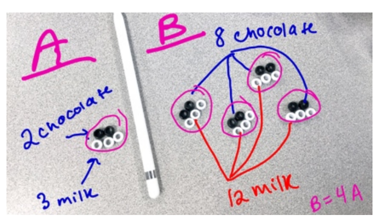

Figure 1. An Exemplar of a PSMT's Initial Arrangement

In the first lesson, the PSMTs were given a set of black and white beads and the hot cocoa problem. They were asked to determine if a hot cocoa mixture with two drops of chocolate and three drops of milk had the same flavor as a mixture with eight drops of chocolate and 12 drops of milk (see Noelting, 1980). PSMTs initially described rearranging the set of eight black beads and 12 white beads in four groups with two black beads and three white beads to show the new mixture had the same flavor (see Figure 1). This indicated a multiple batches way of reasoning i.e., $(3 + 2) \cdot x = y$ where x is the number of batches and y is the total ounces of hot cocoa.

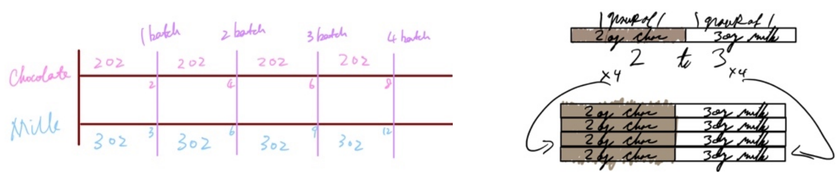

Figure 2. (a) Double Number Line and (b) Strip Diagram for the Hot Cocoa Problem

The following day, PSMTs were asked to make double number lines (DNLs) and strip diagrams (SDs) of the hot cocoa problem that captures multiple mixtures. The PSMTs found creating and explaining DNLs to be generally unproblematic. Most PSMTs created DNLs with increments of two ounces of chocolate and three ounces of milk similar to Figure 2a to show multiple mixtures of the hot cocoa. When making SDs, they created drawings resembling the DNL where strips represented a batch of two ounces of chocolate and three ounces of milk (Figure 2b). In other words, although the form of representation changed, its function remained similar in that it explicitly showed batches. At the table where I was seated, the PSMTs expressed difficulty in creating SDs that capture multiple mixtures. I pressed them to think of larger quantities of hot cocoa and how they would draw it. Molly then created the drawing in Figure 3a. During whole class discussion, she explained she initially drew one cup (the first row) of hot cocoa with five equal sized parts, two parts chocolate (blue parts) and three parts milk (pink parts). She explained that 1,000 cups of hot cocoa would also have two parts chocolate and three parts milk and stacked multiple strips to show 1,000 cups. She divided 1,000 cups into five parts to get 200 cups in one part. Describing the difference between one cup and 1,000 cups, she gestured towards the first column in her drawing, "so instead of having one-fifth in that one tiny box, [this column] is worth 200 cups." This indicated Molly was beginning to consider partitions of the strip diagram as a part with a variable amount in it i.e., either 1/5 of a cup or 200 cups.

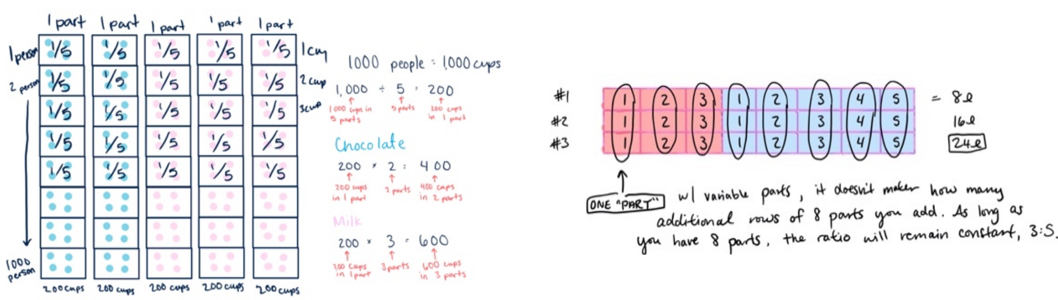

Figure 3. (a) Molly's Hot Cocoa Strip Diagram (b) Christina's Punch Strip Diagram

In the lessons transitioning from using multiple batches to variable parts, the class worked on the problem, "If you mix juice and bubbly water in a three to five ratio to make punch, how many liters of juice and how many liters of water would you need to make 24 liters of punch?" Once again, some PSMTs initially created SDs showing multiple batches. Christina's transition in reasoning during small-group discussion was similar to Molly's. Using her drawing (Figure 3b), she initially showed three batches by iterating a horizontal strip representing a batch containing three liters of juice and five liters of water. She iterated the strip three times to get 24 liters of punch. However, she was still confused about drawing a SD highlighting three and five parts instead of two parts with three liters and five liters. After talking with Molly, Christina

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.

explained she would change her drawing by collapsing multiple partitions into one partition, "I guess it would be different…I was trying to show the variable parts. It doesn't matter if [the drawing] keeps on going down, the ratio is going to be the same, so it might be clearer if this was just one big part [gestures to the first column] and not split into three because it looks like batches." Similar to Molly, Christina was flexible with the amount in a partition i.e., one partition could be one or three liters. She noted that a ratio could refer to the number of parts of same amounts i.e., $x \cdot (3 + 5) = y$ where x is the amount in a part and y is the total liters of punch.

Discussion

When initially approaching proportion problems, PSMTs used a multiple batches perspective way of reasoning. This could indicate PSMTs experience in their own education may have primarily supported a multiple batches perspective and can be leveraged to develop variable parts perspective. When prompted to think about the situation in different ways i.e., using larger numbers or focusing on the ratio as a number of parts, the PSMTs were able to successfully consider a variable parts perspective. Additionally, the partitions in strip diagrams shifted functions supporting a variable parts perspective when they "collapsed" the iterated strips and generalized that a partition could be any amount. This study demonstrates the viability of PSMTs capacity to reason with variable parts, an unnoticed perspective in the literature. Additionally, the results suggest teachers and teacher educators to support their students to reason flexibly with the mathematical functions of representations such as the strip diagram.

Acknowledgments

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References

Beckmann, S., & Izsák, A. (2015). Two perspectives on proportional relationships: Extending complementary origins of multiplication in terms of quantities. *Journal for Research in Mathematics Education, 46*(1), 17–38.

Cuoco, A. A. (Ed.). (2001). *The roles of representation in school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Eisenhart, M., Borko, H., Underhill, R., Brown, C., Jones, D., & Agard, P. (1993). Conceptual knowledge falls through the cracks: Complexities of learning to teach mathematics for understanding. *Journal for Research in Mathematics Education, 24*(1), 8-40.

Jacobson, E., & Izsák, A. (2015). Knowledge and motivation as mediators in mathematics teaching practice: The case of drawn models for fraction arithmetic. *Journal of Mathematics Teacher Education, 18*(5), 467–488.

Janvier, C. (Ed.). (1987). *Problems of representation in the teaching and learning of mathematics* Hillsdale, NJ: Lawrence Erlbaum Associates.

National Council of Teachers of Mathematics. (2014). *Principles to action: Ensuring mathematics success for all*. Reston, VA: National Council of Teachers of Mathematics.

Noelting, G. (1980). The development of proportional reasoning at the ratio concept part I - Differentiation of stages. *Educational Studies in Mathematics, 11*(2), 217–253.

Saxe, G. B. (2012). *Cultural development of mathematical ideas*: Papua New Guinea studies. New York, NY: Cambridge University Press.

Saxe, G. B., de Kirby, K., Le, M., Sitabkhan, Y., & Kang, B. (2015). Understanding learning across lessons in classroom communities: A multi-leveled analytic approach. In A. Bikner-Ahsbahs, C. Knipping & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education: Examples of methodology and methods* (pp. 253-318). Dordrecht: Springer Netherlands.

Stylianou, D. A. (2010). Teachers' conceptions of representation in middle school mathematics. *Journal of Mathematics Teacher Education, 13*, 325–343.

THE INFLUENCE OF MATHEMATICAL KNOWLEDGE FOR TEACHING AND BELIEFS ON PRESERIVCE TEACHERS' NOTICING

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A preservice teachers' (PSTs') ability to notice students' mathematical thinking can play an important role in their ability to respond to students' ideas and orchestrate productive discourse. Consequently, the topic of teacher noticing has been of interest recently, but little is known about why PSTs notice what they do. This report describes the initial analysis of a project investigating how Mathematical Knowledge for Teaching (MKT) and beliefs about the role of students' thinking in the math classroom influence elementary PSTs' noticing within the context of fractions. Regression analysis indicates that neither MKT or beliefs had a statistically significant impact on PSTs' noticing. Being in a course section that "practiced" noticing through several activities throughout the semester was the only statically significant predictor of noticing scores at the end of the semester.

Keywords: Teacher Education-Preservice, Teacher Beliefs, Mathematical Knowledge for Teaching

What teachers see or fail to see in a classroom is tightly linked to the decisions that teachers make, which makes teacher noticing critical to all aspects of teaching. Using students' thinking in instruction is essential for high quality instruction, but this can only be done if the teacher first notices students' mathematical thinking, which can be challenging, especially for novice teachers. Little is known about why teachers notice what they do, therefore leaving many unknowns about how to best improve teachers' ability to notice students' thinking and consequent capacity to make instructional decisions based on students' thinking. This study investigates how elementary PSTs' beliefs and Mathematical Knowledge for Teaching (MKT) are related to their ability to notice students' mathematical thinking.

Theoretical Framework

According to Jacobs, Lamb, and Philipp (2010), noticing of students' mathematical thinking includes: (a) attending to children's strategies while working on mathematics, (b) interpreting and making sense of children's mathematical understandings, and (c) deciding how to respond because of these interpretations. Noticing as a skill can be taught, improved, and refined (Schack et al., 2013; Schoenfeld, 2011), and it should be cultivated with PSTs. By practicing noticing with PSTs and scaffolding the complexities of noticing students' mathematical thinking, PSTs are better prepared to notice students' mathematical thinking and enact high quality math instruction in their future classrooms

Research on PSTs' noticing of student's mathematical thinking while watching videos has overwhelmingly focused on the *what* of noticing. Studies have investigated what classroom features PSTs tend to notice and in what ways their noticing of these features may change over time (Sherin & van Es, 2005; Star & Strickland, 2008). While these studies have yielded a deeper understanding of PSTs' noticing, there are still important questions left unanswered. Several authors have alluded to knowledge and beliefs impacting PSTs' noticing (Schoenfeld, 2011; Star & Strickland, 2008). However, there are not many studies that investigate the impact of beliefs or knowledge on noticing, and even fewer studies investigating these factors together.

Consequently, the relative contribution of each of these factors is unknown. Studies that have investigated beliefs and noticing have measured beliefs about mathematics in general, including PSTs' math attitudes (Fisher et al., 2014). It may be more appropriate to specifically investigate beliefs about the importance of using student thinking in instruction, as these beliefs may be more tightly connected to a teachers' noticing. Additionally, past studies that have investigated MKT in relation to noticing, cite a theoretical connection, but have been unable to find a statistically significant connection (Thomas, Jong, Fisher, & Schack, 2017). The lack of expected findings may indicate that the instruments were not well aligned.

This study is designed to build and improve on prior studies by combining attention to PSTs' knowledge, beliefs about the importance of using student thinking in classroom instruction, and their noticing. This study examines whether changes in PSTs' knowledge and beliefs correspond with changes in their noticing of students' mathematics thinking.

Methods

Participants

The study was conducted in the Fall of 2017 with 97 elementary PSTs during their first of two math methods courses. The PSTs were divided amongst four sections of the same course. Course instructors met on a weekly basis to collaboratively plan the activities and assignments each week. Three of the sections served as a comparison section to the focal section. The focal section participated in an additional two noticing activities and a video observation assignment.

Data Sources

Noticing of students' thinking about fractions. The PSTs' noticing of students' thinking is measured by a noticing activity where they watch a short video and then respond to four prompts via an online survey. The intention of the video is to simulate what a teacher would experience while listening to students collaboratively working on a task. The video showed two second graders working on a fraction subtraction problem using pattern blocks, while one of the students explained her reasoning at a conceptual level by detailing her thought process for her partner while using pattern blocks to visually represent her thinking. This clip was chosen since videos that demonstrate students' reasoning at a conceptual level result in more substantive considerations of students' mathematical thinking (Sherin, Linsenmeier, & van Es, 2009).

Following the work of Jacobs et al. (2010) and Fisher et al. (2014), the journaling prompts consist of three categories that correspond with the three components of noticing of students' mathematical thinking: attending, understanding, and responding. The prompts are: (1) *Describe what the student did in response to the problem presented.* (2) *Describe what you learned about the student's understanding of mathematics from their response.* (3) *If the student were in your classroom, what question(s) might you pose next? What would you hope to accomplish by asking these questions?*

Mathematical Knowledge for Teaching. This study uses a subset of the Learning Mathematics for Teaching (LMT) instrument to measure PSTs' MKT (Hill, Schilling & Ball, 2004). Items in the LMT address common and specialized content knowledge, asking teachers to solve mathematical problems, evaluate student solutions, and represent content to students. Since the study focuses on fractions, I created an instrument of 23 items by selecting questions that specifically require the use of fractions from both forms of the LMT rational number modules.

Beliefs Likert Scale. The survey given to PSTs' at the start and end of the semester to measure their beliefs about the role of students' thinking in math teaching combined 16 items from Beswick (2005) and 18 items from the Mathematics Beliefs Scale (Fennema, Carpenter, & Loef, 1990). Through factors analysis, the 34-item scale was pared down to 18 items ($\alpha = .83$)

that consist of five factors: what teachers should tell students, the role of student thinking in math teaching, the role of productive struggle, the importance of understanding students' errors, and the role of student justification in the math classroom. The mean of the items within each factor was used as each PSTs' score for that factor.

Analysis

PSTs' noticing was analyzed according to the three components of noticing. All coding was binary, with a 1 indicating that the response connected to the mathematical thinking of the student in the video and a 0 indicating that the PST did not connect their answer to the student's thinking. The responding score considered both the question posed by the PSTs and their reasoning for asking that question. The score from the three components were summed to create an overall noticing score.

The measures were administered at the beginning and end of the semester. Using SPSS, I performed linear regression analysis to determine how the change in predictors (LMT, beliefs, and section) related to PSTs' noticing score at the end of the semester. Given the results of the regression analysis, an independent samples T-Test was conducted to investigate how the focal and comparison sections differed.

Results

The linear regression model contains the following predictors: the change in LMT scores, the change in each of the five factors, noticing score at the start of the semester, and if the PST was in the focal section or not. Of all predictors, the only statistically significant factor is if the PST was in the focal section or not ($p = .011$). In order to amplify if there was any predictive power of the predictor variable to the final noticing score the regression was run six times with the following three predictors: if the PST was in the focal section or not, noticing score at the start of the semester (as a control), and the predictor of interest. The results of these regressions are in Table 1, demonstrating that none of the factors are statistically significant. In each individual model run in Table 1, the focal section predictor continues to be significant (with a $p < .005$), while the noticing score from the start of the semester is never significant.

Table 1: Regression Models Run with Focal Section as Control

Model	Variable	Std. Beta	t	Sig.
1	Change in LMT Score	.05	.503	.616
2	Change in Mean: *What Teachers Should Tell Students*	.109	1.052	.296
3	Change in Mean: *Role of Student Thinking in Math Teaching*	-.047	-.474	.637
4	Change in Mean: *Role of Productive Struggle*	.112	1.105	.272
5	Change in Mean: *Importance of Understanding Students' Errors*	.016	.158	.875
6	Change in Mean: *Role of Student Justification in Math Classrooms*	.002	.015	.988

An independent samples T-Test was run with all predictors to investigate how the two groups compared. Table 2 shows the only significant result when equal variances are assumed between the two groups is for the noticing score at the end of the semester and the change in noticing score over the semester. In regard to the PSTs' MKT and beliefs, the two groups are statistically similar.

Table 2: T-Test Results with Focal and Comparison Sections

Predictor	Focal Mean	Comp. Mean	Focal Std. Dev.	Comp. Std. Dev.	Focal Std. Error Mean	Comp. Std. Error Mean
Change in Noticing**	1.26	.4	.964	1.175	.201	.138
Post Noticing***	1.91	1.15	.848	.892	.177	.104
*Note: ** p < .01, ***p < .001*						

Discussion

The results from the regression analysis demonstrate that the only statistically significant predictor of the PSTs' noticing score at the end of the semester was if they were in the focal section or not. The predictors for knowledge and beliefs did not yield statically significant results. This does not mean that knowledge and beliefs had no impact on noticing, but the statistical analyses did not indicate a strong enough impact to yield statically significant results. The focal and comparison sections differed in two ways: the PSTs in the focal section participated in a noticing activity three additional times throughout the semester. They watched and discussed two additional video clips (different from the clip shown at the start and end of the semester) and conducted a video observation. All of these activities contained the same question prompts as those used to measure PSTs' noticing at the start and end of the semester. The results indicate that activities within the methods course that used real classroom video to "practice" noticing students' mathematical thinking impacted PSTs' noticing in a significant way.

References

Beswick, K. (2005). The beliefs/practice connection in broadly defined contexts. *Mathematics Education Research Journal*, *17*, 39-68.

Fennema, E., Carpenter, T. P., & Loef, M. (1990). *Mathematics beliefs scales*. Madison, WI: University of Wisconsin, Madison.

Fisher, M.H., Schack, E.O., Thomas, J., Jong, C., Eisenhardt, S., Tassell, J., & Yoder, M. (2014). Examining the relationship between preservice elementary teachers' attitudes toward mathematics and professional noticing capacities. In J. Lo, K.R. Leatham & L.R. Van Zoest (Eds.), *Research Trends in Mathematics Teacher Education* (pp. 219-237). Switzerland: Springer International Publishing.

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *The Elementary School Journal, 105*, 11- 30.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*, 169-202.

Schack, E. O., Fisher, M. H., Thomas, J. N., Eisenhardt, S., Tassell, J., & Yoder, M. (2013). Prospective elementary school teachers' professional noticing of children's early numeracy. *Journal of Mathematics Teacher Education, 16*, 379–397.

Schoenfeld, A.H. (2011). Noticing matters. A lot. Now what? In M.G. Sherin, V.R. Jacobs & R.A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 134-151). New York, NY: Routledge.

Sherin, M.G., & van Es, E.A. (2005). Using video to support teachers' ability to notice classroom interactions. *Journal of Technology and Teacher Education*, *13*, 475-491.

Star, J.R., & Strickland, S.K. (2008). Learning to observe: Using video to improve preservice mathematics teachers' ability to notice. *Journal of Mathematics Teacher Education*, *11*, 107-125.

Thomas, J., Jong, C., Fisher, M. H, & Schack, E. O. (2017). Noticing and knowledge: Exploring theoretical connections between professional noticing and mathematical knowledge for teaching. *The Mathematics Educator, 26,* 3-25.

PROSPECTIVE TEACHERS' USE OF CHIP MODEL

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Ten elementary and middle school prospective teachers (PTs) participated in clinical interviews where they modeled integer addition and subtraction number sentences with two-colored chips. The PTs constructed various models using the two-colored chips that both matched and did not match the number sentences presented to them. Although the prospective teachers sometimes created models that did not match the integer addition and subtraction number sentences, some recognized these inconsistencies. The results highlight spaces of PTs' accomplishments and struggles with using two-colored chips for certain integer number sentences. Implications of this study support facilitating PTs' construction of models and leveraging their thinking and uses of models in instruction.

Keywords: Teacher Education-Preservice, Number Concepts and Operations, Cognition

Understanding the ways that prospective teachers (PTs) reason is important so that we can leverage discourse and instructional experiences in teacher education. PTs use various manipulatives, such as the two-colored chips, as they engage in activities around the teaching and learning of integers. Yet, we know little about the ways PTs construct models for integer operations with two-colored chips. This research report highlights the results of an investigation that sought to make sense of the following research question:

> In what ways to elementary and middle school PTs use two-color chips to model integer addition and subtraction?

Conceptual Framework

The minus symbol has multiple meanings (i.e. unary, binary, opposite) that are often confounded (Bofferding, 2014; Vlassis, 2004, 2008). In fact, students often operate with the negative integers by simply omitting or "ignoring" the minus symbol and adding it later (e.g., Ayres, 2000; Bell, O'Brien, Shiu, 1980). Confounding the meaning of the symbol or omitting it all together will interfere with the ways that integer addition and subtraction is modeled with the chips.

Subtraction can be interpreted in two different ways, with take-away or distance (Selter, Prediger, Nührenbörger & Hußmann, 2012). The take-away interpretation of subtraction aligns best with chip models, as discrete objects are used and may be removed; and distance interpretations align with other models, such as number lines. The use of a subtraction model does not automatically insure success; knowing how the model is connected to the mathematics is what is needed to develop conceptual understanding (e.g., Kamii, Lewis, & Kirkland, 2001).

In this study, we focus on a subset of models (Van Den Heuvel-Panhuizen, 2003), *manipulatives*, and specifically the use of a two-colored chips. The use of two-colored chips is often referenced as the chip model, but these manipulatives can be used in differing ways (e.g., Murray, in press). Thus, there are many chip models, and we explore the various different models that PTs may construct in this study.

Using two-colored chips to model integer addition and subtraction has both affordances and constraints, or places where the model breaks down (Murray, in press). No model is vigourous enough to support all problem types (Vig, Murray, & Star, 2014). Integer number sentences like 3 – 5 = ▢, for example, presents a constraint in using two-colored chips. If one starts with 3 chips of one color, representing the positive 3, and tries to remove five chips of the same color, this does not work. Consequently, the model needs to be adapted—such as adding in "zero pairs" to the model (i.e., representing 1 + -1 with two different colored chips).

Consistency refers to how things like contextual situations match number sentences (Wessman-Enzinger, in press). Similarly, consistency can also refer to how a model matches a number sentence. The use of discrete objects for 2 + 3 may not necessarily match 2 – -3. Attention to PTs' consistency as they use two-colored chips to model integer addition and subtraction will highlight potential affordances and hindrances of models (Vig et al., 2014).

Methods

Ten elementary and middle school PTs volunteered to participate in structured, task-based interviews (Goldin, 2000). The PTs participated in the study as freshmen, during their first mathematics content course for teaching elementary and middle school mathematics, prior to using two-colored chips in their university coursework.

The PTs worked in pairs within the interviews to better elicit natural discourse about tasks. Although in pairs, each PT received the number sentence on a single sheet of paper, with colored markers, and two-colored chips (i.e., two-sided red and yellow chips). The following number sentences were given to the PTs: -5 + 9 = ▢, 1 – 5 = ▢, 7 + -2 = ▢, -5 – 4 = ▢, 2 – -3 = ▢, -8 – -3 = ▢, -5 – -7 = ▢. The PTs solved each integer addition or subtraction number sentence any way they wanted, modeled the number sentence with a number line, and then modeled the number sentence with two-colored chips. This research brief focuses on the ways PTs used the two-colored chip models for integer addition and subtraction.

Results

The results are described in two parts: (a) ways that PTs used two-colored chip models that are *consistent* with the number sentences and (b) ways that PTs used two-colored chip models that are *inconsistent* with the number sentences.

Consistent Uses of Two-Colored Chips

Zero pairs. Half of the PTs consistently used zero pairs with the two-colored chips for the number sentences -5 + 9 = ▢ and 7 + -2 = ▢ in this study. Brooke and Sayoni, a pair of PTs, matched up two sets of chips. Brooke used 9 red chips and 5 yellow chips for -5 + 9 = ▢; Sayoni used 9 yellow chips and 5 red chips for -5 + 9 = ▢. Although they represented the -5 flexibly with either red or yellow chips, they paired the chips up similarly. Brooke stated, "I know these cancel out," pointing to the pairs of chips. And, Sayoni highlighted the solution, "I have these 4 left" after the canceled pairs.

Comparing two sets. PTs compared two sets of chips with -5 – -7 = ▢ only twice. Kaye compared two sets of chips for -5 – -7 = ▢. She constructed two rows, placing five red chips to represent -5 in one row and seven red chips to represents -7 in another row. Kaye reflected:

> I do not know how to show this in a way that universally makes sense … Since you are subtracting, this equal zero somehow (pulls away five pairs of red/red chips)...They are not negative, they are positive (points at the remaining two red chips), that's the part that I am stuck on.

As Kaye compared the two sets, she confounded zero pairs (1 + -1 = 0) with her reflection on how -1 – -1 = 0, which presented a challenge in determining how the solution she knew (-5 – -7 = 2) matched the remaining two red chips in her model. Although she confounded zero pairs, her comparison aligned to the number sentence.

Taking away from one set. The only consistent use of chip model with a take-away interpreation of subtraction included taking chips away from one set of chips for the number sentence -8 – -3 = ▢. Rochelle started with eight red chips and took away three red chips:

> You have the 8 negative chips. You are subtracting the negative three, so you are taking away three of the negatives (pulls away 3 red chips). And that leaves you with negative five.

Inconsistent Uses of Two-Colored Chips

Zero pairs. Each of the PTs used zero pairs, combining two different colored chips to represent additive inverses (i.e., 1 + -1 = 0). However, in doing so the PTs often changed the number sentences (e.g., changing 1 – 5 = ▢ to 1 + -5 = ▢). Crystal, for example, modeled 1 – 5 = ▢ with one yellow chip representing +1 and five red chips representing -5. She combined the chips, paired a red and yellow chip together, pulled the pair off to the side, and pointed to the four remaining red chips as the solution. Crystal, although she constructed this model for 1 – 5 = ▢, recognized that this model did not fit the original addition number sentence well: "one minus five is hard to do with the chip model because you are taking away from a smaller number."

Joining. Joining two sets of chips was a common strategy when PTs solved the number sentences -5 – 4 = ▢ and 2 – -3 = ▢. However, the PTs often changed these number sentences before modelling the chips with joining. In particular, the PTs changed -5 – 4 = ▢ to -5 + -4 = ▢ and 2 – -3 = ▢ to 2 + 3 = ▢. Kacee, for instance, modeled 2 – -3 = ▢ by using 2 red chips and 3 red chips. She joined these two chips, illustrating a solution of 5. Kacee, without prompting, shared: "If I did not already know it was five, I think it would be a lot harder to figure out what I am supposed to do with them [reference to the chips]." Although she recognized that that she did not think this was the best model to use with the chips, she did not provide an alternative model.

Taking away from one set. Starting with a set of chips of a singular color, the PTs modeled integer number sentences by taking away chips. For 1 – 5 = ▢, Jakob started with five red chips and took away one red chip—illustrating 5 – 1 = ▢ rather than 1 – 5 = ▢. Jakob referenced this take-away strategy while recognizing that his model did not incorporate subtraction. Even so, he did not offer an alternative model.

Flipping chips. PTs occasionally flipped chips as they used them. At times, this method occured as the PTs employed another strategy, and other times it stood alone. Kaye, flipped chips as a stand alone method for 2 – -3 = ▢: "If you were to add these, it would be negative. But since you are subtracting it, they're all like (and flips chips over to yellow)." Kaye started with 2 yellow chips and 3 red chips. She flipped the red chips to yellow. In some ways this matches 2 – -3 = ▢ in the sense that, her model matched the relationship between subtraction and adding the opposite (e.g., 2 – -3 = 2 + 3).

Concluding Remarks

Although the PTs' focus on procedures interfered with their ability to use models consistently (e.g., changing 2 – -3 = ▢ to 2 + 3 = ▢), the PTs often expressed unprompted recognition that their model did not match the number sentences. These types of observations have potential to be spaces where mathematics teacher educators can build on PTs' thinking, facilitating discussion about models and issues of consistency.

It is important for PTs to be consistent with number sentences and models; they need to connect the mathematics with the model (Kamii et al., 2001). Even so, when PTs provide an inconsistent model, mathematics teacher educators can leverage this as a way to encourage discussion. Given the challenges for children transitioning to models from whole number to integers, it seems likely these may be spaces that PTs will need to have discussions in their own classrooms in the future.

Acknowledgements

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References

Ayres, P. (2000). An analysis of bracket expansion errors. In T. Nakahara, & M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 25–32). Hiroshima, Japan: Nishiki Print Co., Ltd.

Bell, A., O'Brien, D., & Shiu, C. (1980). Designing teaching in light of research of understanding. In R. Karplus (Ed.), *Proceedings of the 4th Conference of the International Conference for the Psychology of Mathematics Education* (pp. 119–125). Berkeley, California: University of California.

Bofferding, L. (2014). Negative integer understanding: Characterizing first graders' mental models. *Journal for Research in Mathematics Education, 45*(2), 194–245.

Kamii, C., Lewis, B. A., & Kirkland, L. (2001). Manipulatives: When are they useful? *Journal of Mathematical Behavior, 20*, 21–31.

Murray, E. (in press). Using models and representaitons: Exploring the chip model for integer subtraction. In L. Bofferding, & N. M. Wessman-Enzinger (Eds.), *Exploring the Integer Addition and Subtraction Landscape: Perspectives on Integer Thinking*. Springer.

Selter, C., Prediger, S., Nührenbörger, M., & Hußmann, S. (2012). Taking away and determining the difference—a longitudinal perspective on two models of subtraction and the inverse relation to addition. *Educational Studies in Mathematics, 79,* 389–408. doi:10.1007/s10649-011-9305-6

Van Den Heuvel-Panhuizen, M. (2003). The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics, 54*, 9–35.

Vig, R., Murray, E., & Star, J. R. (2014). Model breaking points conceptualized. *Educational Psychology Review, 26*(1), 73–90.

Vlassis, J. (2004). Making sense of the minus sign or becoming flexible in 'negativity.' *Learning and Instruction, 14*(5), 469–484. doi:10.1016/j.learninstruc.2004.06.012

Vlassis, J. (2008). The role of mathematical symbols in the development of number conceptualization: The case of the minus sign. *Philosophical Psychology, 21*(4), 555–570.

Wessman-Enzinger, N. M., & Mooney, E. S. (2014). Making sense of integers through story-telling. *Mathematics Teaching in the Middle School, 20*(4), 203–205.

Wessman-Enzinger, N. M. (in press). Mathematically equivalent, but not contextually equivalent. *Mathematics Teaching in the Middle School.*

DEEPENING K-8 PRESERVICE TEACHERS' UNDERSTANDING OF MATHEMATICS AND MATHEMATICAL FEEDBACK VIA LETTER-WRITING

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Letter-writing activities, in which pre-service teachers exchange written correspondence with K-12 students, have a rich history in mathematics education. The purpose of this research was to explore the affordances of letter-writing exchanges in the development of preservice K-8 teachers' understanding of mathematics and mathematical feedback processes. Analyzing formative feedback regarding a mathematical task at the level of the task, the processes used to analyze the task, and the self-regulatory mechanisms involved in these processes yielded interesting results. Among these were that exposure to different ways of thinking about a task were sufficient to elicit new thinking in participants, that a metacognitive awareness of the feedback cycle could prompt change in the manner that feedback was perceived, and that improving mathematical understanding is an incremental process that is not easily measured.

Keywords: Algebra and Algebraic Thinking, Instructional Activities and Practices, Mathematical Knowledge for Teaching, Teacher Education-Preservice,

Introduction and Purpose

Letter-writing activities, in which pre-service teachers (PSTs) exchange correspondence with K-12 students, have a rich and diverse history in mathematics education. In their earliest iterations, these projects served as diagnostic tools and pedagogical aids, providing PSTs the opportunity to examine K-12 students' mathematical thinking and evaluate their skills and attitudes towards the subject (Fennell, 1991; Phillips & Crespo, 1996). Later letter writing-projects shifted the focus to examine PSTs' learning, and provided a firm theoretical framing for these activities within the realm of the knowledge and beliefs needed for teaching mathematics (Crespo, 1998, 2003). More recent projects further developed this framework by considering the type of problems and cognitive demand of the tasks PSTs explored with the students (Crespo, 2003; Norton & Kastberg, 2012) and the nature of the feedback processes involved in these exchanges (Kastberg, Lischka, & Hillman, 2016). It is from this fertile heritage that we implemented a letter-writing project in which PSTs from four U.S. universities exchanged correspondence with one another around a shared, high-cognitive-demand task in order to evaluate the development of their mathematical understanding of the task and their ability to provide productive feedback.

The purpose of this research was to explore the affordances of this letter-writing exchange in the development of the PSTs' understanding of mathematics and mathematical feedback processes. Project participants across the four universities engaged in a shared mathematical task focused on algebraic reasoning and exchanged solutions with a partner from another university in the form of written letters. Upon receiving feedback from their partners, participants revised their written solutions, composed new letters, and engaged in a second feedback cycle. The artifacts of these exchanges, including participants' written descriptions of their formative thinking, feedback, responses to feedback, reflections on the feedback cycle, and summative analysis of the initial task supported the researchers in examining the relationships between participants' understanding of the mathematical task and the feedback cycle.

Research Perspective

Earlier letter-writing research provided four types of practical and theoretical perspective for our project. First, the research suggested a variety of pedagogical benefits for PSTs that justified the substantial time they committed to participate in the project. These benefits included practice with the assessment of mathematical thinking (Fennell, 1991), engagement in the feedback cycle (Kastberg, Lischka, & Hillman, 2016), and enjoyment of the activity as it offered them opportunities to test out their mathematical ideas, exposed them to new perspectives and ways of thinking regarding mathematics, and challenged their notions of the nature of productive mathematical exchanges (Crespo, 2003). Second, the research base highlighted shifts in PSTs' metacognitive thinking that could occur with appropriate implementation of the project. These shifts included a transition from directed, unproductive questions and commentary to responses that would examine and challenge their partners' unique ways of thinking based upon analysis of the work presented for the task (Crespo, 2003; Norton & Kastberg, 2012). Third, prior studies elaborated on the potential pitfalls of projects of this type and informed the design of our project. As forewarned in the literature, we encountered instances of participants blaming their peer's lack of effort for their own inability to engage in the feedback process, redirecting the peer's thinking to align with their own rather than building on the ideas presented in the writing, and a failure to offer actionable feedback regarding incorrect thinking for the task (Kastberg, Lischka, & Hillman, 2016; Norton & Kastberg, 2012). From these precedents, we were able to address these concerns both implicitly, in the form of prompts offered for our participants' written reflections regarding the feedback process, and explicitly via classroom instruction. These interventions helped to mitigate the negative impact of these factors to some degree. Finally, and most significantly, past research directed our attention to the importance of the nature of the formative feedback offered in letter-writing exchanges (Kastberg, Lischka, & Hillman, 2016) and the participant's reflections on these processes (Norton & Kastberg, 2012).

This focus on formative feedback and reflection established the theoretical framing for the current study. Hattie and Timperley's (2007) model of feedback as a means of reducing "discrepancies between current understandings and performance and a goal" (p. 86) comprised the central definition for evaluation of our participants' work, and the four levels of focus for feedback established in this model guided the initial thematic analysis of our participants' feedback and reflections. These levels include feedback focused on the project's mathematical task, the processes used to understand and record one's thinking about the task, the self-regulatory processes needed to engage with the task successfully, and evaluations of the individual completing the task. We worked under the assumption that feedback along the first three of these levels would offer opportunities for increased mathematical understanding of the task by shifting participant's attention to the productive aspects of their work (Kluger & DeNisi, 1996) while the final level would provide affective and emotional reinforcement for engaging with the task.

Methodology

The research was conducted across four universities with a mathematics teacher educator (MTE) at each institution supervising the process; the 187 participants in the study were K-8 preservice teachers in mathematics education courses. There were two classes of juniors from

University A (n = 56), a class of juniors and seniors from University B (n=16), three classes, primarily constituted of freshman, from University C (n = 81), and one class juniors and seniors at University D (n=34). All participants were given the same task: discover and explicitly describe patterns across the stages of a growing tile pattern (Foreman & Bennett, 1996).

Participants from different universities were paired with one another. For example, participant 1 from University A was paired with participant 2 from University B, with these participants completing the first draft of the task at the same time in the form of a letter to another unknown participant. These letters were collected by each MTE and shared with the other MTEs in a google drive folder so all four had access to participant responses. The MTEs then provided participant 1's letter to participant 2 and participant 2's letter to participant 1. The participants read each other's solution and provided feedback to help shape the mathematical precision and accuracy of the presented solution. Participants were provided with a rubric to focus their feedback on specific components of the solution. This feedback was shared with the original participant, following a similar exchange process facilitated by the MTEs to maintain anonymity among the participants, who had time to implement feedback and make revisions to their original solution. Feedback from the peer was elicited a second time, and participants had another chance to make changes in light of the feedback before submitting a final analysis of the task to their respective MTE. Throughout the exchanges, and again after the final task analysis, participants were asked to reflect on the processes of giving and receiving feedback in relation to the task. Critical cases were selected from each participating university, and thematically coded via the framework introduced earlier.

Results

These findings include preliminary results from student work at University B. The majority of participants at University B had rather thorough and mathematically appropriate initial drafts, and their work improved nominally throughout the process. A few participants produced initial drafts that needed much improvement, but after analyzing their first and final work along with the feedback, either there was limited improvement or no improvement could be linked directly to the feedback provided. While such results seem to indicate that the peer feedback process had limited impact, a closer look at the data sheds light on more nuanced ways the exchange contributed to participants' understanding. The exchanges of three participants are described below to highlight some of the impact of the letter writing activities.

Participant A's initial work was strong and the feedback she received was focused on aspects of both the task and the process used to solve it. Additionally, the correspondence introduced a new way of thinking about the pattern that considered the change in the number of tiles between each stage. Participant A reflected on this approach and made a connection to the first derivative of the explicit formula they had originally developed, even though this way of thinking was not provided in the exchange. This case demonstrated that exposure to different ways of thinking can promote a deepened reflection regarding the mathematics of the task.

Participant B's initial work involved a recursive pattern, which proved to be an ineffective approach to analyzing the task. However, the feedback she received was superficial and process based, without providing any concrete suggestions for improvement or description of the problem. Participant B, who had been seeking task-based feedback, realized before writing her second draft that the recursive nature of her approach was unproductive and chose a completely different path for the second draft. Although Participant B's reflection indicated that the feedback received in this peer exchanges was ineffectual in improving her mathematical understanding, she noted that the iterative nature of the feedback processes prompted her to think about the nature of feedback itself. More specifically, her reflections on the process indicated that she had identified characteristics of helpful feedback that would benefit her in the future.

Participant C's initial work consisted of a systematic and quantitative approach, but the relationships observed were not made in terms of the stage number of the pattern. Consequently, the participant was unable to write an explicit formula. The feedback she received targeted all three levels of the framework: task, process, and self-regulation, and was highly detailed. This prompted Participant C to see new features of the pattern that were better aligned with an explicit representation, however she was still unable to write a formula which captured these relationships. Nonetheless, she noted how the feedback was helpful in allowing her to describe this new way of thinking. Additionally, she commented on her need to state her thinking in a different way, perhaps one that would be "simpler for students to follow." Although the feedback process appeared to help this participant in understanding the nature of the task, only incremental gains were seen in her final task analysis. This incremental growth was not easily measured and could be entirely masked by examining her mathematical results in isolation.

Discussion

One benefit of peer feedback exchanged between participants from different universities is the honest nature of the feedback from an anonymous peer with similar professional goals. Without the pressure of direct relationships, participants who understood the need for it were better able to provide critical feedback. Additionally, participants from another university would often provide different perspectives on the task that spurred new ways of thinking. Our initial results identified two areas in which participants benefit from this intervention: 1) deepening their mathematical knowledge by looking at strategies they may not have initially considered, and 2) self-reported improvement in their self-efficacy regarding the task, their motivation to engage with the task, and their confidence in their results. Additionally, many participants commented on the need for increased precision when explaining their thinking or giving feedback, as both could be interpreted in unintended ways. Future analysis will seek to generalize the results noted here across the full sample and to identify other critical cases of the affordances of letter-writing exchanges for the development of preservice K-8 mathematics teachers' understanding of mathematics and mathematical feedback processes.

References

Crespo, S. (1998). *Math penpals as a context for learning to teach: A study of preservice teachers' learning* (Doctoral dissertation). Retrieved from https://open.library.ubc.ca/cIRcle/collections/831/items/1.0054969

Crespo, S. (2003). Learning to pose mathematical problems: Exploring changes in preservice teachers' practices. *Educational Studies in Mathematics, 52*, 243-270.

Fennell, F. (1991). Diagnostic teaching, writing, and mathematics. *Focus on Learning Problems in Mathematics, 13*(3), 39-50.

Foreman, L. & Bennett A. (1996). *Visual mathematics, course II.* Salem, OR: The Math Learning Center.

Hattie, J., & Timperley, H. (2007). The power of feedback. *Review of Educational Research*, *77*(1), 81-112.

Kastberg, S., Lischka A., Hillman, S. (2016). Exploring prospective teachers' written feedback on mathematics tasks. In M. Wood, E. Turner, M. Civil, & J. Eli (Eds.) *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp 783-790). Tucson, AZ: The University of Arizona.

Kluger, A. & DeNisi, A. (1996). The effects of feedback interventions on performance: A historical review, a metaanalysis, and a preliminary feedback intervention theory. *Psychological Bulletin 119*, 254-284.

Norton, A., & Kastberg, S. (2012). Learning to pose cognitively demanding tasks through letter writing. *Journal of Mathematics Teacher Education, 15*, 109-130.

Phillips, E., & Crespo, S. (1996). Developing written communication in mathematics through math penpal letters. *For the Learning of Mathematics, 16*(1), 15-22.

EXCHANGING DEFICIT FOR AFFORDANCE: DEVELOPING PRESERVICE TEACHERS' CAPACITY TO SEE EMERGENT BILINGUAL STUDENTS' RESOURCES

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We present a Scholarship of Teaching and Learning (SoTL) research project study aimed at helping pre-service teachers to grasp the importance of their emergent bilingual students' bilingualism and cultural backgrounds as resources to serve them in apprehending mathematics instruction and explaining their mathematics to others.

Keywords: Teacher Education-Preservice, Teacher Knowledge, Equity and Diversity, Instructional Activities and Practices

Suppose that a teacher asked his elementary-aged students the simple question, "What is ten plus four?" How should the teacher evaluate the response of a student who claimed that "10 + 4 = 2"? There seems to be an error, and we might expect teachers to make the same assessment of the students' work at face value; the student was incorrect and needs remediation. But might she be correct? It is more difficult to see evidence of correctness. How might our evaluation change if, when asked about this solution, the student had explained that she had found this solution by looking at the clock? Withholding judgment and listening to the student leads us to perceive advanced mathematical reasoning (employing modular arithmetic) at play, where superficial assessment would fail to capture the understandings as well as resources that she had drawn upon to solve it.

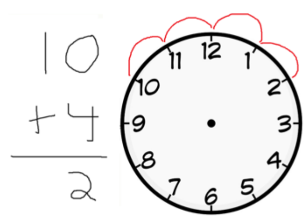

Figure 1. Ten plus four is two (mod 12)

This instance serves as an analogy for mathematics education research that has focused on issues of cultural and linguistic complexity: mathematics teachers and researchers may have been more prone to see inaccuracies than accuracies, and deficits than affordances, when considering students in the context of linguistic and cultural diversity. The research has sufficiently described educational achievement gaps with respect to language minority students (see Gutiérrez, 2008; and Moschkovich, 2002, for discussions), but only more recently has been moving away from investigating issues of equity in education through deficit lenses toward seeing students through affordance lenses, which movement may be made more difficult because of habitually negative evaluative mindsets in mathematics teaching. This paper reports brief findings from a Scholarship of Teaching and Learning (SoTL; Bishop-Clark & Dietz-Uhler, 2012) research project aimed at understanding how PSTs can be helped to perceive emergent bilingual students' learning of mathematics on the line between deficit and affordance perspectives, and in enabling them to see "difference not deficit" (Lewis, 2014) in emergent bilingual students.

Diversity Growth in US Classrooms and Theoretical Framework

Teachers in US schools are seeing an enrichment in racial and ethnic diversity in their classrooms (*Musu-Gillette et al., 2016*) and students' first languages are increasingly found to be other than English. For instance, in the last decade the percentage of students participating in English Language Learner (ELL) programs has steadily increased from 9.1 to 9.4 percent of all students (NCES, 2016) – a total of about 4.6 million students in the US. (The terms ELLs, English Learners (ELs), and, more recently, emergent bilinguals (EBs) have variously been used with respect to students whose first and often stronger languages are other than English. Although varying in connotations, they are used here synonymously.) In this paper we study an effort to help PSTs to be "well-prepared beginning teachers of mathematics", as the recent AMTE Standards for Preparing Teachers of Mathematics (2017) define them, by improving their ability to "recognize the difference between access to and advancement in mathematics learning and work to provide access and advancement for every student" (p. 21), especially in culturally and linguistically diverse environments. This study contributes to the recent work by others who have attended to the need of preparing PSTs for teaching diverse students (see for example, Gallivan, 2017; White et al., 2016; and Meskill, 2005), and also to the complexity of accounting for EBs in measures of educator effectiveness (Jones et al., 2013; Turkan et al., 2014).

The present study used Wilson's (2016) model of *pedagogical content knowledge for teaching mathematics to ELLs* (PCK-MELL) as a theoretical framework for the type of knowledge that we aimed to develop in our PSTs. This model posits knowledge for teaching mathematics to EBs as a subset of pedagogical content knowledge related to knowledge of students and knowledge of teaching within Ball, Thames, and Phelps's (2008) *mathematical knowledge for teaching* (MKT) model. We chose this framework, which was used in studies concerned with mathematics teachers' knowledge in relation to issues of linguistic diversity (Sorto et al., 2018), because it addresses not only the obstacles that emergent bilinguals can face in the classroom, but also the resources that such students draw upon as a result of their emerging bilingualism and non-dominant cultural backgrounds.

Method and Participants

The research method used for this SoTL study was an instructional intervention which we call the Teaching Emergent Bilinguals (TEB) Project that was based on the PCK-MELL framework. The intervention employed *challenge-based instruction* (CBI; Crown et al., 2012) by positioning PSTs as test-writers tasked with developing multiple-choice teacher licensure exam questions that could be used to ascertain whether aspiring teachers possessed a sufficient understanding of resources that emergent bilinguals bring to the learning of mathematics. A research question that guides this analysis is the following: within the constraints of the multiple-choice test-writing task, what resources do PSTs identify that EBs draw upon in specific mathematics classroom work and how do they operationalize these resources? The 52 participants in this study came from a large Hispanic Serving Institution (HSI) in the southwestern United States and greater than 90% of participants in this study were Latino/as, mostly female (>90%), all of whom were enrolled in mathematics courses as part of degree plans leading to teacher certification. PSTs in this study worked in groups of 2 or 3 students (25 groups in all), across two different class sections, and submitted their items and accompanying explanation documents via email to the course instructor.

Findings

As indicated above, the TEB project required PSTs to reason about the experiences of emergent bilingual students in the classroom, what linguistic and cultural (or other) resources they draw upon in English-taught mathematics classrooms, and to operationalize these resources in the form of multiple-choice teacher licensure exam items. Analysis of their items involved carefully reading their items, as well as supplemental explanations they had provided, and then forming categories of resources identified by students across all of their items and explanations. In this brief report, we present summary findings and also one example test item, selected because it exemplifies both one of the resources identified by PSTs as well as the way in which many students in this sample applied their own past experience to the task. Resources identified in PSTs' test items included the following:

- Bilingualism –general communicative ability in both languages with full or partially fluency
- Specific Spanish-English cognates: words with similar appearances and meanings in both languages
- Prior mathematical knowledge: familiarity with the mathematical topic or object
- Bilingual teachers
- Parents, friends, relatives that understand English
- Pictures and other visual displays as substitutes for words

An example item is the following.

Marilyn and Karina didn't know what to wear the following day to school. They both had the same number of possible combinations.

Pants/Shorts	Shirts
Blue Pants	Pink
Blue Pants	Red
Blue Pants	Yellow
Black Pants	Pink
Black Pants	Red
Black Pants	Yellow
Khaki Shorts	Pink
Khaki Shorts	Red
Khaki Shorts	Yellow

Marilyn: "Que me voy a poner?"

Karina: "Pos you can wear pantalones con una shirt de cualquier color que quieras."

Marilyn: "Pero esta haciendo calor afuera."

Karina: "Bueno you can also think of the posabilidad of wearing chores."

Marilyn: "Bueno, I have 2 pairs of pantalones, 1 short and 3 different colored shirts."

Karina: "Okay, entonces tenemos que hacer una table to show our possibilities."

Count the objects and use pictures to find the possible combinations.

Make a chart with all the possible combinations.

3 shirts
yellow, pink, red

2 pants
black, blue

What helped the EL's to figure out how to solve the problem?

A) Mathematical Terms

B) Their use of Bilingualism and peer work

C) Talking to the teacher about the problems they encountered

D) Looking for information in their math books.

Figure 2. An item indicating PSTs' observation of bilingualism as a resource to EBs

In this item the darker inserted box should represent the problem upon which the emergent bilingual students are working and the table should represent their written mathematical work.

Discussion

Figure 2 shows how one group of PSTs operationalized bilingualism as a resource for EBs. However, participants frequently experienced a considerable challenge in the task of identifying resources that emergent bilinguals use in the classroom, often asking us, "What are we supposed to do for the Resources part? We just can't think anything." On one hand, writing test items about teaching mathematics and not merely about mathematics was a challenging task for them. But when asked about obstacles that EBs might face, we observed that they could more readily generate lists of things that might cause difficulty for EBs than of EBs' resources. One explanation may be that, similar to the analogy that begins this paper, PSTs may be more prone to envision disability rather than capability when thinking about EB students. This paper has presented brief findings suggesting that the TEB intervention may promote PSTs' perceptions of the affordances of the bilingualism of their EB students in their future mathematics classrooms.

References

Association of Mathematics Teacher Educators. (2017). *Standards for Preparing Teachers of Mathematics.* Available online at amte.net/standards.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407.

Bishop-Clark, C., Dietz-Uhler, B. (2012). Engaging in the scholarship of teaching and learning: A guide to the process, and how to develop a project from start to finish. Sterling, Virginia: Stylus Publishing, LLC.

Crown, S. W., Fuentes, A. A., & Freeman, R. A. (2012). Pedagogy for pedagogy: Using a Wiki to promote the adoption, development, and implementation of challenge-based instruction in STEM education. *Proceedings of the ASEE Annual Conference* (Session 2012-5328). Washington, DC: ASEE.

Gallivan, H. R. (2017). Supporting prospective middle school teachers' learning to revise a high level mathematics task to be culturally relevant. *Mathematics Teacher Educator, 5*(2), 94–121.

Gutiérrez, R. (2008). A "gap-gazing" fetish in mathematics education? Problematizing research on the achievement gap. *Journal for Research in Mathematics Education, 39*(4), pp. 357–364.

Jones, N. D., Buzick, H. M., and Turkan, S. (2013). Including students with disabilities and English learners in measures of educator effectiveness. *Educational Researcher, 42*(4), 234–241.

Lewis, K. E. (2014). Difference not deficit: Reconceptualizing mathematical learning disabilities. *Journal for Research in Mathematics Education, 45*(3), 351–396.

Meskill, C. (2005). Infusing English language learner issues throughout professional educator curricula: The training all teachers project. *Teachers College Record, 107*(4), 739–756.

Moschkovich, J. N. (2002). A situated and sociocultural perspective on bilingual mathematics learners. *Mathematical Thinking and Learning*, 4(2 & 3), 189–212.

Musu-Gillette, L., Robinson, J., McFarland, J., KewalRamani, A., Zhang, A., & Wilkinson-Flicker, S. (2016). *Status and trends in the education of racial and ethnic groups 2016 (NCES 2016-007).* U.S. Department of Education, National Center for Education Statistics.

National Center for Education Statistics (NCES). (2016). *Number and percentage of public school students participating in English language learner (ELL) programs, by state: Selected years, fall 2004 through fall 2014. (Table 204.20).* U.S. Department of Education, National Center for Education Statistics.

Sorto, M.A., Wilson, A., & White, A. (2018). Teacher knowledge and teaching practices in linguistically diverse classrooms. In J. Moschkovich, D. Wagner, A. Bose, J. Rodrigues, & M. Schütte (Eds.) Language and communication in mathematics education, international perspectives, (pp. 12). Dordrecht: Springer.

Turkan, S., de Oliveira, L., Lee, O., & Phelps, G. (2014). Proposing the knowledge base for teaching academic content to English language learners: Disciplinary linguistic knowledge. *Teachers College Record, 116*(3), 1–30.

White, D. Y., DuCloux, K. K., Carreras-Jusino, Á. M., González, D. A., & Keels, K. (2016). Preparing preservice teachers for diverse mathematics classrooms through a cultural awareness unit. *Mathematics Teacher Educator*, *4*(2), 164–187.

Wilson, A. T. (2016). Knowledge for equitable mathematics teaching: the case of Latino ELLs. *The Mathematics Enthusiast. 13*(1&2), 111–129.

PROMOTING KNOWLEDGE INTEGRATION IN TEACHER PREPARATION

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In this study, we investigated the opportunities secondary pre-service teachers (PSTs) had to integrate their knowledge of mathematics, learners, and pedagogy in a methods and content course taught in tandem. In particular, we describe and report on the instructional moves associated with this potential knowledge integration. Implications for pre-service teacher education and directions for further research will be provided.

Keywords: Teacher Knowledge, Teacher Education-Preservice

Secondary mathematics teachers must develop a thorough and connected understanding of mathematics (M), pedagogy (P), and learners (L) in order to plan effective lessons, implement mathematical tasks, and interpret student thinking. Unfortunately, pre-service secondary mathematics teachers (PSTs) may not have the opportunity to connect their knowledge of mathematics, pedagogy, and learners. The traditional structure of teacher preparation programs often delivers pedagogy and content in different departments on campus, leaving it up to PSTs to integrate their fragmented knowledge (Ball, 2000).

To address the challenge of promoting connections between PSTs' knowledge of content, pedagogy, and learners, we investigated the opportunities for knowledge integration in a methods and a content course for secondary pre-service teachers (PSTs) taught in tandem. This paper will describe some preliminary findings for the following research questions: What instructional moves appeared to promote knowledge integration in a methods and content course for secondary PSTs?

Theoretical Foundation

For this study, we define *knowledge integration* as the coordinated use of multiple knowledge types in order to reflect on, or make, instructional decisions. Bishop and Whitefield (1972) proposed that teacher decisions are made using a framework or schema. The main operation of a schema is to store knowledge through a network of connected pieces of knowledge called "elements" (Marshall, 1995, p. 43). The more connections that exist within a schema, the more useful the schema will be. Hence, the process of knowledge integration leads to the development of a more robust schema, resulting in the potential for more informed instructional decisions. The knowledge types used in this study are defined as follows:

Knowledge of Mathematics (M): knowledge regarding the mathematical concept(s) related to the content under investigation. Includes the connections and relationships among ideas, the way(s) and mean(s) of justifying and proving these ideas, and conversations focused on mathematics and reasoning about the mathematical topic.

Knowledge of Pedagogy (P): knowledge regarding the tasks, curriculum, instructional goal(s), or questions used to further the lesson. Includes comments centered on the lesson implementation or decision-making regarding the flow of the lesson.

Knowledge of Learners (L): knowledge regarding student thinking. Includes observed as well as anticipated student thinking, conversations about student characteristics, habits, or misunderstandings.

Our definitions of knowledge of mathematics, learners, and pedagogy are related to other models of teacher knowledge, such as mathematical knowledge for teaching (MKT) (Hill, Ball, & Shilling, 2008) or pedagogical content knowledge (Shulman, 1987). However, the definitions in this study are broader in scope, as we are focused more on the *interactions* among these knowledge sets (i.e. knowledge integration). Our study seeks to uncover instructional actions that promote knowledge integration among PSTs.

Methods

To answer our research question, field notes were collected for every session of a senior-level secondary methods and capstone content course at a teacher preparation program in the Midwest. The content and methods courses were taught in tandem, with the same set of students. The field notes were initially read to identify individual episodes—sections of the transcript centered on a single idea. Following the identification of episodes, the above definitions were then applied to code all episodes. If multiple forms of knowledge were used during a single episode, we defined this to be an example of potential knowledge integration. For instance, if an episode contained both M and L statements, we identified this as an opportunity for PSTs to integrate their knowledge of M and L. We note that evidence of multiple knowledge types does not ensure knowledge integration, only that the potential of knowledge integration was present. Our next step in analysis was to investigate the instructional moves that appeared to promote this potential knowledge integration. We applied open coding to develop categories for these different instructional moves. Discrepancies in coding were discussed by the research team, and definitions and categories adjusted, until all disagreements were resolved.

Findings

From the field notes, we identified 405 different episodes in the content course and 273 episodes in the methods course. Table 1 displays the number of episodes that contained evidence of the different knowledge combinations. For instance, the 26 in the last column for the content course indicates there were 26 episodes in the content course where the knowledge of mathematics, learners, and pedagogy were exhibited.

Table 1: Combinations of Knowledge Used in Episodes From Both Courses

Course	M	L	P	M-L	M-P	L-P	M-L-P
Content	265	4	18	41	32	19	26
Methods	4	10	92	1	17	135	14

Table 2 provides the definitions for several of the categories of instructional moves identified through our open coding process. This list includes the most common instructional strategies used in episodes with the potential for knowledge integration.

Table 2: Definitions of Categories

Reflecting on Instruction: The teacher educator reflects, has the PSTs reflect, or PSTs organically reflect on the instruction in a course. The conversation is focused on the pedagogical moves, or their justification, associated with this observed teaching experience.
Reflecting on a Shared Learning Experience: PSTs reflect on how a set of tasks or activities affected them from the perspective of the learner. The teacher educator or the PSTs can initiate this reflection.

Development of a Pedagogical Product: The teacher educator asks the PSTs to develop or modify a pedagogical product. (e.g., excerpt of lesson plan, assessment, task, lesson objective, classroom norms or rules), or PSTs discuss the process of developing a pedagogical product.
Evaluation/Critique of Pedagogical Products: The teacher educator has PSTs evaluate/critique a provided pedagogical product (e.g., excerpt of lesson plan, assessment, task, etc.). PSTs do not create the pedagogical product being evaluated.
Focus on High School Student Thinking: The teacher educator considers, has the PSTs to consider, or PSTs organically consider how a ***high school student*** would think about a particular mathematics topic.
Discussion of a Daily Topic: The teacher educator has the PSTs discuss a reading or daily topic in a course.

Table 3 provides the distribution of instructional moves for each knowledge combination in both the content and methods course. For example, the 14 in the first row indicates there were 14 episodes in the content course where reflecting on instruction resulted in the coordinated use of mathematics and pedagogy.

Table 3: Distribution of Instructional Moves

	Content				Methods			
Category	ML	MP	LP	MLP	ML	MP	LP	MLP
Reflecting on Instruction	0	14	3	6	0	1	45	2
Reflecting on a Shared Learning Experience	10	0	0	2	0	1	4	4
Development of a Pedagogical Product	0	2	2	9	0	7	11	4
Evaluation/Critique of Pedagogical Products	0	0	1	1	0	2	7	3
Focus on High School Student Thinking	12	1	2	6	0	0	0	2
Discussion of a Daily Topic	2	0	5	0	1	0	55	0

We note that the instructional moves identified above may not be the only moves that promote integration, these were the instructional moves used by these particular instructors that promoted potential integration.

The development of a pedagogical product appeared to stimulate the widest variety of knowledge integration, independent of the course. Moreover, episodes within this category possessed the largest frequency of knowledge integration involving all three knowledge types. When a teacher develops a pedagogical product, there is a need to draw upon mathematics, and to consider the effect on learners, when making these important decisions. It is interesting to note that we had three categories related to pedagogical products—development, enactment, and evaluation/critique. Of these three categories the development of a pedagogical product resulted in the most knowledge integration, enactment of a pedagogical product resulted in the least.

Before conducting this research, one of the main tools employed in the methods course was to provide PSTs with readings to be discussed during the next class. These discussions focused on how PSTs believed the ideas in the articles related to their teaching and future students. We identified 55 episodes in the methods course where potential LP integration took place as a result of discussing a daily topic. Unfortunately, episodes that include integration with M were not present, with the exception of a single ML episode. This finding has led to the realization that we need to provide opportunities for our PSTs to apply these pedagogical ideas to specific mathematical contexts, which could be done using instructional moves such as development of a pedagogical product. This finding aligns with Steele and Hillen's (2012) design principles for a content-focused methods course.

Reflecting on instruction promoted knowledge integration in both the content and methods course. In the content course the main type of potential integration was MP and in the methods course it was LP. Taking the time to reflect on and process experiences obtained while observing others teach (the instructor in the content course and clinical teachers in the methods course) appeared to help PSTs integrate their knowledge.

Discussion and Implications

Examining one's own practice is exciting and terrifying at the same time. We often believe that the activities and tools used in teaching methods and content courses are tried and true and lead to expert teachers leaving our universities to share mathematics with the youth of the world. Analyzing our teacher preparation practice through the lens of knowledge integration has led to some surprising results that will have an impact on our program in years to come. So far, the data in our study point to a mixed review of effectiveness in promoting knowledge integration. On the one hand, having students create pedagogical objects (e.g., lesson plans, tasks, effective questions, etc.) seems to be quite effective at promoting knowledge integration. On the other hand, having students read and respond to articles, without additional activities to apply the concepts, seems to do little to promote knowledge integration in PSTs. As we continue to analyze the data we will continue to refine our list of effective teaching actions that secondary mathematics teacher educators can use to promote knowledge integration. Our next step will be to further incorporate these strategies into our program, and hopefully programs at other universities, to continue to push towards our goal of promoting knowledge integration.

References

Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning. *Journal of Teacher Education, 51*(3), 241-247.

Bishop, A. J. & Whitfield, R. C. (1972). *Situations in teaching*. London, UK: McGraw-Hill.

Hill, H., & Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372-400.

Marshall, S. P. (1995). *Schemas in problem solving*. New York, NY: Cambridge University Press.

Shulman, L. S. (1987). *Knowledge and teaching: Foundations of the new reform*. Harvard Educational Review, 57(1), 1-22.

Steele, M.D., & Hillen, A.F. (2012). Content-focused methods courses: Integrating pedagogy and mathematical content that is central to the 7-12 curriculum. *Mathematics Teacher Educator*, (1), 52-69.

PRESERVICE TEACHERS' USE AND CONNECTIONS OF REPRESENTATIONS OF QUADRATIC FUNCTION IN SOLVING AND PLANNING TO TEACH

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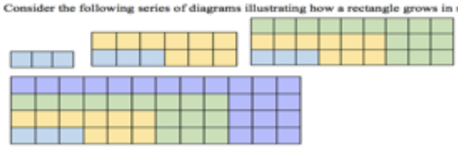
This research was conducted in response to an enduring challenge of supporting preservice teachers (PT) to learn about teaching mathematics from a conceptually oriented approach. We situate this work as an inquiry into how PTs use representations in learning and planning to teach. In prior work, numerical, graphical, tabular and verbal representations strengthened students' sense-making through creating and flexibly moving among representations (Doorman et al., 2012; Lobato, et al., 2012). Less is known about how such understandings influence teachers' plans for instruction. The research questions guiding our inquiry are: How do PT use and connect representations to reason about rate of change of quadratic functions?; and What trace is observed from PT representational activity to their planning for teaching? Participants were five high school PT in the final semester of their program. We conducted a 60-min. audio-recorded task-based interview with each PT. We curated four quadratic function tasks set in a rate of change context (e.g., see Figure 1). We asked PT to pick one task to solve, and to solve it in more than one way. We prompted PT to think aloud and clarify meanings (e.g. "Did the graph help you to answer the question?"). Both researchers debriefed after each interview on what we noticed and wondered. We share initial results from analyses of each interview.

Consider the following series of diagrams illustrating how a rectangle grows in several iterations.



This remote-controlled car is continually speeding up. The table shows the total distance the car has traveled after different amounts of time.

How far will the car have traveled after 14 minutes? Is the car accelerating? What is the acceleration?

Total Time (minutes)	**Total Distance Traveled** (feet)
0	0
2	2
4	8
6	18
8	32

Figure 1. Rate of change of quadratic function tasks (from Ellis (2011); Lobato et al. (2012)).

Initial analysis revealed PT were explicit about the nature of the growth in the dependent quantity (e.g., length or speed), and independent quantity (e.g, area or time) and were able to identify constant rate of rate of change in tables, but less explicit in graphical representations. We observed little evidence of PT using representations to connect to meanings of quadratic function as modeling constant rate of rate of change in linked quantities (as advocated by Ellis, 2011). We also found a strong trace from PT use of representations for their own problem solving to their planning for instruction. For example, Claire said "So if my students are like me, I'm a definitely need to see colors … 'cause it was hard to see the groups of three with just all this gray". Next steps will focus on elaborating an empirically based theoretical framework to understand observed trace from PT own problem solving to their plans for instruction.

References

Doorman, M., Drijvers, P., Gravemeijer, K., Boon, P., & Reed, H. (2012). Tool use and the development of the function concept: From repeated calculations to functional thinking. *International Journal of Science and Mathematics Education*, 10(6), 1243–1267.

Ellis, A.B. (2011). Generalizing promoting actions: How classroom collaborations can support students' mathematical generalizations. *Journal for Research in Mathematics Education*, 42(4), 308 – 345.

Lobato, J., Hohensee, C., Rhodehamel, B., & Diamond, J. (2012). Using student reasoning to inform the development of conceptual learning goals: The case of quadratic functions. *Mathematical Thinking and Learning*, 14(2), 85-119.

PRACTICING TEACHERS PERCEPTIONS OF CONNECTIONS TO KNOWLEDGE AND SKILLS LEARNED IN THEIR TEACHER PREPARATION PROGRAM

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Keywords: Teacher Education-Preservice, Teacher Knowledge

Research on the preparation of mathematics teachers has provided insight into issues such as specialized content knowledge for teaching (Hill, Schilling, & Ball, 2004), understanding of ways in which students learn (Ball & Forzani, 2011) and social aspects of classroom culture (Gutiérrez, 2013). AMTE has integrated this research into their publication *The Standards for Preparing Mathematics Teachers* (AMTE, 2017). The publication of this document serves as an impetus for teacher educators to reflect on the ways in which their own teacher preparation programs can grow to better serve their students.

Previous research has examined the effectiveness of teacher preparation programs through comparisons of outcomes such as student test scores and professional licensure exams (Boyd et.al., 2009). More recent efforts have focused on examining the connections between the knowledge gained by pre-service teachers in their preparation program and the application of this knowledge after becoming practicing teachers (Morris & Hiebert, 2017). Results of their work suggests that mathematics teacher educators must carefully choose the "small sets of knowledge and skills that matter most for beginning teachers" (Morris & Hiebert, p.555).

The purpose of this study was to better understand the knowledge and skills related to teaching of mathematics in-service teachers perceive as important for beginning teachers and what knowledge from their pre-service preparation they were able to apply in their instruction. Data was collected from practicing elementary and middle grades teachers (n=101) who received their undergraduate preparation within the past eight years from a university in the Northeastern United States. Participants completed a questionnaire that included Likert and short answer questions. Results suggested that there were specific content (e.g., fractions and geometry) and learning experiences (exploring multiple approaches to solving problems) that participants associated with supporting their current teaching practice. However, responses also suggested a pattern in which participants perceived a lack of preparation (knowledge of curriculums). Participants provided suggestions on learning experiences related to knowledge and skills needed to teach mathematics effectively that would benefit future pre-service teachers.

References

Association of Mathematics Teacher Educators. (2017). Standards for Preparing Teachers of Mathematics. Available online at amte.net/standards.

Ball, D. L., & Forzani, F. M. (2011). Building a Common Core for Learning to Teach: And Connecting Professional Learning to Practice. *American Educator*, *35*(2), 17.

Boyd, D. J., Grossman, P. L., Lankford, H., Loeb, S., & Wyckoff, J. (2009). Teacher preparation and student achievement. *Educational Evaluation and Policy Analysis*, *31*(4), 416-440.

Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education*, *44*(1), 37-68.

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *The elementary school journal*, *105*(1), 11-30.

Morris, A. K., & Hiebert, J. (2017). Effects of Teacher Preparation Courses: Do Graduates Use What They Learned to Plan Mathematics Lessons?. *American Educational Research Journal*, *54*(3), 524-567.

ELEMENTARY PRESERVICE TEACHERS BELIEFS RELATED TO MATHEMATICS

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Introduction

The purpose of this study is to understand the relationship between mathematics related affective measures, including professional mathematics identity, mathematics identity, mathematics mindset, mathematics anxiety, teaching self-efficacy, and beliefs about teaching and learning mathematics. In addition, this study explored the impact of two different field experience implementation techniques to determine whether either resulted in a change in beliefs over the course of a semester. The following research questions were used to guide this study: 1) are elementary preservice teachers' self-perceptions and beliefs about mathematics and teaching correlated and 2) do elementary preservice teachers self-perceptions and beliefs about mathematics and teaching differ based on different field experience models as part of their mathematics methods course?

Methods

Data were collected using a pre- and post-survey of elementary preservice teachers (EPT) in the fall of 2017 at one university in the Mid-west region of the United States. A total of 70 EPTs in four different sections of an intermediate mathematics methods course participated in the study. In addition to demographic information, the survey included items to measure professional (teacher) mathematics identity, mathematics identity, mathematics mindset, mathematics anxiety and teacher self-efficacy. Additionally, EPTs were asked to complete an online self-assessment to determine where EPTs' beliefs fell in terms of a student-centered versus teacher-centered educational philosophy.

Results

Using Pearson's correlation test, each of the mathematics constructs were found to be significantly correlated with each other except teacher self-efficacy with mathematics identity and mathematics anxiety. In order to address research question 2, a series of t-tests were conducted with the post-survey data, finding that no significant differences existed between the immersive and traditional field-based groups of EPTs.

Discussion

Results from this study indicate that many of the mathematics-related constructs are significantly correlated with one another. However, some of the variables are more highly correlated than others, such as professional mathematics identity with mathematics identity as well as professional mathematics identity and mathematics identity with mathematics anxiety. Conversely, teaching self-efficacy was either weakly correlated or not correlated with the constructs measured. Additionally, mathematics teaching philosophy was only weakly correlated with mathematics mindset. When addressing the second research question, no significant differences were found between the groups. If the goal is to assist students in developing more positive beliefs related to mathematics and teaching mathematics, the immersive versus traditional program investigated in this study did not result in significant change. While much is gained by participating in methods courses and field experiences, longer exposure to these, or similar experiences, may be needed to significantly affect beliefs and self-perceptions.

ELEMENTARY PRESERVICE TEACHERS WHOLE-CLASS INSTRUCTIONAL DECISION MAKING

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The purpose of this research study is to understand how PST's analysis of individual student written work relates to PSTs' whole class instructional decision making (Jacobs et al., 2010). For this study, PSTs at four different institutions in four different states, participated in a multi-digit multiplication module designed with a goal of developing PSTs' skill of making whole class instructional decisions based on analysis of a set of individual students' written work taken as representative of a whole class. The research question for this study is: How does PSTs' professional noticing of individual student work samples relate to their whole class instructional decision making?

Keywords: Professional Noticing, Student Work Analysis, Instructional Decision Making

Methodology

Participants were 68 elementary PSTs enrolled in a mathematics methods or content course at four different institutions in the U.S.; class sizes ranged from 10 to 30 students. PSTs were either enrolled in a four-year elementary education program or in a post-baccalaureate elementary credential program. In each course, participants were taught a professional noticing module focused on children's solution strategies for multiplication. For this study, we report on a portion of the module in which PSTs professionally noticed five individual student work samples and synthesized their noticing to make whole class instructional decisions.

PST responses were scored using a 4-point scale analytical rubric that was developed by the researchers and reliability was verified using the *Flesis Kappa* statistical measure (Landis & Koch, 1977). We then determined the Kendall's tau coefficient to measure the ordinal association between the two mean scores.

Preliminary Results and Implications

We are in the midst of scoring all PST noticing rubrics, thus our analysis is of a preliminary nature. Current results show that the more robust the PSTs' professional noticing of individual student work, more robust their whole class instructional next steps. Since making instructional decisions based on students' mathematical thinking is a difficult skill to develop, providing PSTs with opportunities to analyze sets of student work samples that can be considered representative of a whole class, and thus support all students, is recommended.

References

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*, 169-202.

Landis, J. R., & Koch, G. G. (1977). The measurement of observer agreement for categorical data. *biometrics*, 159-174.

A RESEARCH EXPERIENCES FOR UNDERGRADUATES PROGRAM: PREPARING FUTURE RESEARCHERS IN STEM EDUCATION

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Introduction and Research Questions

Our Research Experiences for Undergraduates (REU) project has exposed pre-service K-12 teachers to timely problems involving STEM teaching and learning through original research conducted with eight STEM Education faculty mentors. At this time, 23 undergraduate Fellows have fully completed the academic-year program. Previous research has shown that undergraduate research programs can allow students to make a more educated decision regarding the pursuit of a graduate degree (Willis, Krueger, & Kendrick, 2013). Our REU project's main objective for pre-service teachers was to instill an appreciation of the interplay between society, education, and research. This study aimed to answer the following questions: (1) To what extent can undergraduate STEM Fellows' knowledge and confidence in STEM Education research change through participation in an intensive nine-month research program? and (2) What research products and career trajectories will result from REU Fellows' participation in our REU program?

Design, Analysis, and Findings

This research employed a mixed methods approach (Creswell & Plano Clark, 2007). Quantitative data included administering a pre/post survey (Kardash, 2000) concerning undergraduate research to determine Fellows' research expectations, familiarity with research literature, and ability to conduct statistical analyses. Qualitative data was comprised of Fellows' pre-, mid-, and post-evaluation interviews and their final research papers and presentations. Using a two-tailed t-test to analyze the results of the survey, we discovered significant growth in the Fellows' ($n = 22$) confidence levels in studying, conducting, and analyzing research ($t(21) = -4.488, p < .001$). This significant growth was also echoed in their interviews with the program evaluator. Further, our findings showed Fellows entered the program with confidence in analyzing literature, but still needed improvement in the areas of designing their own research projects and performing statistical analyses of their data. Fellows believed this program would greatly increase their abilities to conduct research and write manuscripts for publication. At the end of their nine-month research experiences, Fellows stated they felt they had gained expertise in coding and analyzing data, conducting clinical interviews, using technology, writing, and presenting, but most frequently noted their increase of interpersonal skills and collaborating with peers. Additionally, most of the Fellows still planned to work as a teacher at the K-12 level, but the experience had helped them gain deep pedagogical knowledge and strategies that could be integrated into their future teaching.

References

Creswell, J. W., & Plano Clark, V. L. (2007). Designing and conducting mixed methods research. Thousand Oaks, CA: Sage.

Kardash, C. (2000). Evaluation of an undergraduate research experience: Perceptions of undergraduate Fellows and their faculty mentors. Journal of Educational Psychology, 92(1), 191-201.

Willis, D. A., Krueger, P. S., and Kendrick, A. (2013). The influence of a research experiences for undergraduates program on student perceptions and desire to attend graduate school. *Journal of STEM Education (14)*2, 21-28.

ELEMENTARY PRE-SERVICE TEACHERS' USE OF RESOURCES TO MAKE SENSE OF CURRICULAR PRESENTATIONS OF PROOF

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Keywords: Reasoning and Proof, Teacher Education – Preservice, Curriculum

Because of the important role that proof plays in mathematical learning and sense-making (Yackel & Hanna, 2003), mathematical reform documents have called for proof to be central to all levels of schooling (e.g. NGA & CCSSO, 2010). Considering this at the elementary level, what teachers know about proof and how they have experience the process of proving will affect how they promote (or don't promote) proof in their classrooms (Stylianides & Ball, 2008). As such the nature of learning opportunities around proof and proving for elementary pre-service teachers (EPSTs) are important to consider. Some of the only such opportunities take place in mathematics content courses for EPSTs and the associated textbooks. In these texts EPSTs are dominantly given the opportunity to examine others' completed proofs, but there are many instances where informal or exploratory resources are connected to a completed proof (McCrory & Stylianides, 2014), which can offer increased insight into why the proof is true (Weber, 2005).

My research is guided by the questions: 1) *How do EPSTs use resources from in-text explorations of the proving process as they make sense of written proofs?* and 2) *To what extent are there connections between how EPSTs interpret particular resources as they read proofs and how they use those resources as they prove?* This poster will present findings from analysis grounded in data from interviews and written work of EPSTs as they engaged in making sense of two curricular presentations of proof.

EPSTs' proof understandings have been an "enduring challenge" in the field, yet few studies examine the process of proving and understanding proofs rather than simply the product. To better understand this process, my finding report on the types of in-text resources (e.g. definitions, diagrams, written descriptions of proof approach) EPSTs utilize as they make sense of written proofs and how EPSTs connect exploratory resources to written proofs. My findings also address parallels between the processes of reading proofs and proving. Through these parallels, this work has implications to better understand EPSTs' current learning opportunities around proof and enrich them through informing curriculum work and teaching experiments.

References

McCrory, R. and Stylianides, A. J. (2014). Reasoning-and-proving in mathematics textbooks for prospective elementary teachers. *International Journal of Educational Research, 64*, 119-131.

National Governors Association Center for Best Practices, & Council of Chief State School Officers, (2010). *Common core state standards mathematics*. Washington, DC: National Governors Association Center for Best Practices/Council of Chief State School Officers.

Stylianides, A. J. and Ball, D. L. (2008). Understanding and describing mathematical knowledge for teaching: Knowledge about proof for engaging students in the activity of proving. *Journal of Mathematics Teacher Education, 11*, 307-332.

Weber, K. (2005). Problem-solving, proving, and learning: The relationship between problem-solving processes and learning opportunities in the activity of proof construction. *Journal of Mathematical Behavior, 24*(3-4), 351-360.

Yackel, E., & Hanna, G. (2003). Reasoning and proof. In J. Kilpatrick, W. G. Martin, &D. Schifter (Eds.), *A research companion to ''Principles and standards for school mathematics''* (pp. 227–236). Reston, VA: National Council of Teachers of Mathematics.

PRESERVICE TEACHERS' RESPONSIVENESS TO STUDENT MATHEMATICAL THINKING IN REHEARSALS OF INSTRUCTIONAL PRACTICES

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Rationale

A current trend in mathematics teacher education is to support preservice teachers (PTs) to develop complex mathematics teaching practice by identifying core practices and by creating opportunities for PTs to enact, to reflect, and to investigate those practices (Grossman, Hammerness, & McDonald, 2009). Research-supported core practices for mathematics teaching include *posing purposeful questions* and *eliciting and using evidence of student thinking* (NCTM, 2014). An approach to developing PTs' practices is to rehearse core practices within interactions that approximate classroom instruction. It is important to attend to the extent to which PTs coordinate these practices. We investigate the question: How do PTs respond to student thinking while rehearsing the practice of posing purposeful questions?

Method

We analyzed video of 12 secondary mathematics PTs' engagement in three rehearsals of the practice of posing purposeful questions. In Rehearsal 1, PTs worked in pairs in interaction with one simulated student. In Rehearsal 2, PTs worked individually to interact with a pair of simulated students. In Rehearsal 3, PTs worked individually with a small group of peers. Using the Teacher Response Coding Scheme (TRC, Peterson et al., 2017), we coded each PT response to student thinking for the Actor who is invited to consider an instance, the Move represented by the response, the extent to which students' actions are explicitly referenced, the extent to which student ideas are central to the response, and the extent to which the response connects to a related mathematical point (Peterson et al., 2017). We searched for patterns among the nature of the communication moves, for the extent to which the moves incorporated student thinking, and for whether the moves yielded explicit statements of student thinking.

Results

Our ongoing analysis suggests that posing questions that elicited evidence of student thinking and that built on student thinking were challenging for PTs. Student statements were not always explicit, and PTs' questioning did not necessarily help to draw out clarifying information. Further, relationships between the teacher's response and the student's thinking were often implicit or could not be easily inferred based on the teacher statement. An implication is that the development of questioning as a routine must be further supported by helping PTs to attend to the quality of the information that is elicited and to the clarity and directness of their responses.

References

Grossman, P., Hammerness, K., & McDonald, M. (2009). Redefining teaching, re-imagining teacher education. *Teachers and teaching: Theory and practice, 15*(2), 273–289.

National Council of Teachers of Mathematics. (2014). *Principles to Actions: Ensuring mathematics success for all.* Reston, VA: Author.

Peterson, B. E., Van Zoest, L. R., Rougée, A. O. T., Freeburn, B., Stockero, S. L., & Leatham, K. R. (2017). Beyond the "move": A scheme for coding teachers' responses to student mathematical thinking. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.), *Proceedings of the 41st annual meeting of the International Group for the Psychology of Mathematics Education*, *Vol. 4* (pp. 17-24). Singapore: PME.

INSTRUCTIONAL STRATGIES USED TO TEACH REASONING AND PROOF IN SECONDARY TEACHER EDUCATION PROGRAMS

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Despite the importance of proof, a considerable body of research suggests that many secondary school students face challenges in constructing or understanding proofs (e.g., Knuth, Choppin, & Bieda, 2009). Some pre-service secondary school teachers are confused about the process of constructing, evaluating, or understanding mathematical proofs (Knuth, 2002). Teacher preparation program accreditation agencies such as the Conference Board of the Mathematical Sciences (2012) recommend that preservice teachers engage in different types of proof related reasoning. This raises the question: "What opportunities do pre-service secondary teachers have to develop their knowledge of reasoning and proof?" Yackel & Cobb (1996) suggests the importance of the way in which mathematical ideas and process are conveyed during the instruction. Thus, in this study, we are interested in "What teaching strategies do instructors use to support pre-service secondary teachers' learning of mathematical ideas and processes related to proof?"

The study was conducted at secondary mathematics teacher preparation programs in three contextually different universities. Data include instructor interviews of five required mathematics or mathematics education courses at each university. Each interview was audio-recorded and transcribed. Three researchers iteratively coded the data and resolved discrepancy.

We developed a framework to capture instructional strategies. Our framework includes proof specific teaching strategies and general teaching strategies. We found variations in terms of the strategies used among three universities. We also found that a majority of instances of proof specifics strategies involve engaging students. However, half of the instances of general teaching strategies were instructors' sole efforts without engaging students. We also noticed that more instructional strategies, both proof specific teaching strategies and general teaching strategies, were reported in mathematics courses than in mathematics education courses.

This work has implications for the current learning opportunities for future teachers. It offers insights on detailed differences between teacher education programs. It informs changes that need to happen in teacher education programs.

References

Conference Board of the Mathematical Sciences [CBMS]. (2012). *The Mathematical Education of Teachers II*. Providence RI and Washington DC: American Mathematical Society and Mathematical Association of America.

Knuth, E. J. (2002). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education*, *33*, 379–405.

Knuth, E. J., Chopin, J. M., & Bieda, K. N. (2009). Middle school students' production of mathematical justifications. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Ed s.), *Teaching and learning proof across the grades: A K–16 perspective* (pp. 153–170). New York, NY: Routledge.

Yackel, E.. & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27*(4), 458-477.

TRANSFORMING LEARNING THEORY: A LENS TO LOOK AT MATHEMATICS COURSES FOR PREPARING FUTURE TEACHERS

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Research has shown that prospective teachers (PTs) enter their mathematics content courses with procedural understandings of mathematics (e.g. Thanheiser et al., 2014). However, in their work as teachers, they will be required to know and understand more than just *how* to solve mathematics problems (AMTE, 2017). Teachers should possess a “robust knowledge of the mathematical and statistical concepts that underlie what they encounter in teaching.”(AMTE, 2017, p. 6).

Additionally, PTs often believe that they already know the elementary mathematics that they will need to teach (Thanheiser, 2018). Therefore, it is the job of mathematics teacher educators (MTEs) to help PTs see the need for, and develop the specialized mathematics content knowledge that they will need to use in their work as teachers. We suggest that transformative learning theory (TLT; Mezirow, 1991) provides a model through which MTEs can help PTs move from shallow procedural understandings to conceptual understanding with deeper procedural understandings as a base.

We present a 4-step cycle based on TLT: Disorienting Dilemma; Critical Reflection; Rational Dialogue; Action. A disorienting dilemma involves a task where PTs’ preconceived understandings are challenged or where the procedures that they already know are not enough to solve the problem. In the critical reflection phase, PTs are asked to work through the dilemma independently while reflecting on their previous assumptions. The rational dialogue phase focuses on PTs justifying and explaining their thinking to peers in order to reach an equilibrium between their prior assumptions and the disorientation presented by the task. The fourth and final step, the action phase is the most important because it involves making connections between the PTs’ procedural fluency and the conceptual understanding, helping PTs to see how the procedures they learned (their prior assumptions) are related to the knowledge needed to teach mathematics (their transformed understanding).

References

Association of Mathematics Teacher Educators. (2017). *Standards for Preparing Teachers of Mathematics*. Available online at amte.net/standards.

Mezirow, J. (1991). *Transformative dimensions of adult learning*. San Francisco, CA: Jossey-Bass.

Thanheiser, E. (2018). The effects of preservice elementary teachers’ accurate self-assessments in the context of whole number. *Journal for Research in Mathematics Education, 49*(1), 39-56.

Thanheiser, E., Browning, C., Edson, A.J., Lo, J., Whitacre, I., Olanoff, D., & Morton, C. (2014). Mathematical content knowledge for teaching elementary mathematics: What do we know, what do we not know, and where do we go? *The Mathematics Enthusiast, 11* (2), 433-448.

RIGHTS OF THE LEARNER AND ROUGH DRAFT THINKING: TWO COMMITMENTS FOR HUMANIZING MATHEMATICS TEACHER EDUCATION

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The purpose of this paper is to illustrate two commitments to humanizing mathematics education in the context of teacher education: Rights of the Learner (RotL) and Rough Draft Thinking (RDT). Based on the vision of Olga Torres, a mathematics teacher and teacher educator, the Rights of the Learner (RotL) situate students' ideas as valuable contributions to the process of learning mathematics. Students should have the right to: be confused; claim a mistake; speak, listen and be heard; and write, do and represent what makes sense to them (Kalinec, 2017). The RotL align with divergent formative assessment (Pryor & Crossouard, 2008), which encourages teachers to learn *what* students know rather than only *if* they know something, which encourages differing approaches and leverages the collective strengths of a group as they solve a task (Lotan, 2003). Rough Draft Thinking (RDT) is the process of sharing unfinished, in-progress ideas and remain open to revising those ideas. This commitment is inspired by the concept of "exploratory talk," forwarded by Douglas Barnes, which is using talk to "work on understanding" or talking to learn (Barnes, 2008, p. 3). Rough draft talk sounds like false starts, expressions of uncertainty, and incomplete or imperfect sentences (Jansen, Cooper, Vascellaro, & Wandless, 2016). Articulating one's own thinking as a draft that can be revised honors the continual process of making sense of a new idea. During rough draft thinking, learners are focused on sorting out their thoughts rather than performing for an audience (Barnes, 2008). Promoting RDT can be seen as a part of promoting mastery goals (oriented to engage in school in order to learn and understand) over performance goals (oriented to engage in school in order to perform well or to avoid performing poorly) (Ames, 1992). This poster theorizes about how pre-service teachers can learn about RotL and RDT in order to resituate their perspectives on teaching and learning mathematics. By leveraging asset-based mathematics practices that foreground what students already know and can do (and thereby resist a deficit perspective of students' thinking is something that always needs to be immediately corrected or changed), the authors define the theoretical underpinnings of the RotL and RDT before giving examples in practice. The poster includes examples from practice in teacher education.

References

Ames, C. (1992). Classrooms: Goals, structures, and student motivation. *Journal of Educational Psychology,* 84(3), 261 – 71.

Barnes, D. (2008). Exploratory talk for learning. In N. Mercer & S. Hodgkinson (Eds.), *Exploring talk in school* (pp. 1 – 15). London, UK: Sage Publications.

Jansen, A., Cooper, B., Vascellaro, S., & Wandless, P. (2016). Rough-draft talk in mathematics classrooms. Mathematics Teaching in the Middle School, 22(5), 304–307. doi:10.5951/mathteacmiddscho.22.5.0304

Kalinec-Craig, C.A. (2017). The Rights of the Learner: A sociocultural framework for promoting equity through formative assessment in mathematics classrooms. *Democracy and Education Journal*. 25(2), Article 5.

Lotan, R. A. (2003). Group-worthy tasks. *Educational Leadership, 60*(6), 72-75.

Pryor, J., & Crossouard, B. (2008). A socio-cultural theorisation of formative assessment. *Oxford Review of Education,* 34(1), 1–20.

THE DEVELOPMENT OF DISCOURSE FACILITATION AMONG PRESERVICE ELEMENTARY TEACHERS: A LONGITUDINAL INVESTIGATION

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The recently released Standards for Preparing Teachers of Mathematics (AMTE, 2017) point to the need to *develop* preservice teachers (PSTs) in discourse facilitation. However, facilitating high-quality discourse in a way that impacts learning is difficult (Michaels & O'Connor, 2015), particularly for PSTs (Neergard & Smith, 2012). Therefore, understanding *how* they develop in their facilitation of mathematical discourse, the very focus of this study, seems particularly important for the work of mathematics teacher educators. Specifically, this study investigated how PSTs in an undergraduate elementary teacher preparation program (TPP) developed in facilitation of discourse (herein referred to as "math talk") over four semesters.

Four participants were purposefully selected from a group of 16 who were studied in depth during their TPP and first year of teaching. The four participants were selected because they demonstrated the most "success" (Brinkerhoff, 2003), in comparison to the larger group, in facilitating math talk during the first year of teaching, as measured by an observational protocol (Walkowiak, Berry, Meyer, Rimm-Kaufman, & Ottmar, 2014). We used the Mathematics Discourse Matrix (Sztajn, Heck, & Malzahn, 2013) to code each five-minute segment of four previously video-recorded lessons, one per semester, as to which discourse type on the matrix was primarily represented. We created graphical representations of the codes to make sense of development during the TPP.

Findings indicate two of the participants showed progress over time in how they were facilitating math talk while the other two participants' lessons remained relatively similar. This study can spark discussion in our field about how we measure, make sense, and generate developmental trajectories for mathematics pedagogy. Furthermore, the findings point to the importance of understanding factors that influence variation in teacher development.

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References

Association of Mathematics Teacher Educators (2017). *Standards for preparing teachers of mathematics.* Available online at amte.net/standards.

Brinkerhoff, R. O. (2005). The success case method: A strategic evaluation approach to increasing the value and effect of training. *Advances in Developing Human Resources*, *7*(1), 86-101.

Michaels, S., & O'Connor, C. (2015). Conceptualizing talk moves as tools: Professional development approaches for academically productive discussion. In L. B. Resnick, C. Asterhan, & S. N. Clarke (Eds.), *Socializing intelligence through talk and dialogue*. Washington DC: American Educational Research Association.

Neergaard, L., & Smith, T. (2012). *Comparing beginning teachers' instructional quality growth on subject-specific and global measures.* Paper presented at SREE Conference, Washington, DC.

Sztajn, P., Heck, D., & Malzahn, K. (2013). *Project AIM: Year three annual report.* Raleigh, NC: North Carolina State University (Chapel Hill, NC: Horizon Research, Inc).

Walkowiak, T. A., Berry, R. Q., Meyer, J. P., Rimm-Kaufman, S. E., & Ottmar, E. R. (2014). Introducing an observational measure of standards-based mathematics teaching practices: Evidence of validity and score reliability. *Educational Studies in Mathematics, 85*(1), 109-128.

QUALITY RESEARCH OF MATH WORD PROBLEM SOLVING INTERVENTION FOR ELEMENTARY STUDENTS WITH LEARNING DISABILITIES

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Theoretical Perspective

Special educators have formulated guidelines, Council of Exceptional Children's new set, for identifying evidence-based practices in special education (Cook et al. 2015). However, recently there is limited research that evaluated the body of work using Council of Exceptional Children (CEC) quality indicators. The purpose of this review is to identify the quality of previous investigations of empirical single subject design studies on mathematics word problems intervention of elementary students with learning disabilities. The very recently studies of effective intervention represent an attempt to improving the skills of mathematics performance.

Research Questions and Design

This review will address the following research question: what is the quality of the research on math word problem solving interventions for students with learning disability?

Data Collection Techniques and Analysis

This literature review was conducted for studies which provided intervention. Online databases were scanned to locate math word problem solving intervention studies for elementary students with math learning disabilities. Relevant studies were reviewed based on title, key words, abstract and method to check if the studies involved word problem solving. The Council of Exceptional Children's (CEC, 2014) new set of standards were used for identifying evidence-based practices in special education. The articles were evaluated by CEC quality indicator for single-subject study by reviewing method, result and discussion sections of the articles.

Summary of Findings

The finding indicated that not all studies met CEC quality indicator for single-subject study. Researchers need to continue to apply the quality indicator into their studies. The studies involved interventions that require special training, but the authors didn't describe any training of interventionists and didn't note that no training were required. Future researchers could use CEC quality indicator to evaluate their studies and publish more high quality studies that meet all the quality indicators.

References

Cook, B. G., Buysse, V., Klingner, J., landrum, T. J., McWilliam, R. A., Tankersley, M., & Test, D. W. (2015). CEC's Standards for Classifying the Evidence Base of Practices in Special Education. *Remedial and Special Education*. 36(4)220-234.

Council for Exceptional Children. (2014). Council for exceptional children standards for evidence-based practices in special education. Retrieved from http://www.cec.sped.org/~/media/Files/Standards/Evidence%20based%20Practices%20and%20Practice/EBP%20FINAL.pdf

PRESERVICE MATH TEACHERS' PERCEPTIONS OF PRODUCTIVE STRUGGLE

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The importance of student struggle in learning is well documented in the literature (e.g., Piaget, 1960; Hiebert & Grouws, 2007). In mathematics education, more students are being encouraged to engage in productive struggle to learn mathematics at a deeper, more conceptual level (NCTM, 2014; Warshauer, 2015). Hiebert and Grouws classify productive struggle as the effort by students "to make sense of mathematics, to figure something out that is not immediately apparent" (p. 387). The goal of this exploratory study was to identify and characterize the nature of prospective teachers' struggles as they engaged in a non-routine mathematical task in a mathematics content course. The researchers sought to determine 1) how prospective elementary, middle, and secondary teachers engage in productive struggle as they complete non-routine mathematical tasks and 2) how prospective teachers perceive and characterize their struggle during non-routine mathematical tasks.

Thirty-two prospective elementary, middle, and secondary teachers from three different mathematics content courses engaged in a non-routine mathematical task. Researchers observed each class and students completed exit tickets where they discussed the struggles encountered while completing the task. The exit tickets and observation notes from each of the three content courses were coded using Warshauer's (2015) list of four types of student struggles. We will describe the method used to code the data where four additional themes emerged from what Warshauer's team identified. In addition, we will share the analysis of exit tickets and researchers' observation notes that revealed that prospective teachers 1) attempted to describe their own struggles while they focused on getting the correct answer and making sense of their work; 2) identified positive and negative types of struggles; and 3) the responses in the elementary course were notably different than those from the other two courses. The poster will also highlight examples of each type of struggle that were identified as well as a discussion of the results and the possible implications on teaching prospective mathematics teachers and future research.

References

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students'learning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 371–404). Charlotte, NC: Information Age.

National Council for Teachers of Mathematics (NCTM). (2014). *Principles to action: Ensuring mathematical success for all.* Reston, VA: Author.

Piaget, J. (1960). *The Psychology of Intelligence.* Totowa, NJ: Littlefield Adams & Co.

Warshauer, H. K. (2015). Productive struggle in middle school mathematics classrooms. *Journal for Mathematics Teacher Education, 18*, 375-400.

USING SELF-DIRECTED LEARNING TO PROMOTE MATHEMATICAL ENGAGEMENT AMONG PRESERVICE TEACHERS

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Many Pre-Service Teachers (PSTs) enter professional development programs without "meaningful understandings of mathematical content, appreciation of what mathematics entails, or confidence in their own ability to do and to learn mathematics,"(Ball, 1990, p. 5). However, many students experience college learning in a similarly disengaged way (Blum, 2017). It has been suggested that meaningful learning occurs when students have an opportunity to engage with the topic in a way that includes space for pausing in mastery and enjoyment (Handa, 2003).

Self-directed learning (SDL) is a pedagogical "process in which individuals take the initiative" in elements of their learning, from diagnosing learning needs to deciding how to demonstrate mastery (Knowles, 1975, p. 18). This poster presents a conceptual argument for using SDL to promote student engagement in mathematics teacher education, supported by the experience of two teacher educators who have used SDL in their classrooms.

Many studies show that SDL increases learner engagement. SDL in marketing and business courses led to increased curiosity and interest in the studied areas (Boyer, Edmondson, Artis, & Fleming, 2013). College students who report studying a topic/subject for autonomous reasons (like having an interest for the topic) showed more motivation and perseverance toward, and performed better on, tasks than their extrinsically-motivated peers (Black & Deci, 2000). Self-directed learning among in-service teachers on the job tends to be more emotionally impactful than formal learning on the learners (affecting their identities as teachers); formal learning is often "seen as a bureaucratic requirement" (McNally, Blake, & Reid, 2009, p. 259).

This is in line with our own experiences using SDL in our respective college classrooms. Our evidence of engagement includes a record of the personalized projects students have constructed tailoring math to their own interests and teacher journals documenting observed instances of student engagement.

References

Ball, D. L. (1990). Breaking with experience in learning to teach mathematics: The role of a preservice methods course. For the Learning of Mathematics, 10(2), 10–16.

Black, A. E., & Deci, E. L. (2000). The effects of instructors' autonomy support and students' autonomous motivation on learning organic chemistry: A self-determination theory perspective. Science Education, 84(6), 740.

Blum, S. D. (2017). I love learning I hate school: An anthropology of college. Cornell University Press.

Boyer, S. L., Edmondson, D. R., Artis, A. B., & Fleming, D. (2013). Self-directed learning: A tool for lifelong learning. Journal of Marketing Education, 36(1), 20–32.

Handa, Y. (2003). A phenomenological exploration of mathematical engagement: Approaching an old metaphor anew. For the Learning of Mathematics, 23(1), 22–29.

Knowles, M. S. (1975). Self-directed learning: A guide for learners and teachers. Chicago, IL: Follett Publishing Company.

McNally, J., Blake, A., & Reid, A. (2009). The informal learning of new teachers in school. Journal of Workplace Learning, 21(4), 322–333.Aguirre, J.M., Mayfield-Ingram, & Martin, D.B. (2013) *The Impact of Identity in K-8 Mathematics: Rethinking Equity-based Practices*. Reston, VA: National Council of Teachers of Mathematics.

USING THE ARRAY MODEL TO DEVELOP PROSPECTIVE TEACHERS' UNDERSTANDING OF MULTIPLICATION AND ITS PROPERTIES

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Research (e.g., Smith, 2017) suggests that it is important to focus on models of multiplication beyond equal groups. From our experience teaching mathematics content courses for prospective elementary teachers (PTs), we have found that our students sometimes struggle with understanding the array or area model of multiplication.

In order to address these issues, we designed a sequence of mathematical tasks whose overarching goal is to help PTs develop a better understanding of the array model of multiplication. We designed these tasks by engaging in a previously developed task design cycle (Tobias et al., 2014) that focuses on modifying a task designed for children into a task for PTs.

The initial task begins with a 29x23 rectangular array. We ask PTs to come up with three ways to determine the number of 1x1 unit squares in the array by breaking the picture into regions. Subsequent tasks focus on using base-10 blocks to model the array and exploring how the model relates to the partial products algorithm for multiplication. Our goals for these tasks are for PTs to connect the number of 1x1 unit squares in the array with the product 29 x 23. Specifically, our goals for the tasks are for PTs to 1) recognize that a product can be found by summing, or combining, partial products, and that this procedure can be modeled by decomposing a rectangular array into different regions and 2) use a rectangular array model to make sense of the distributive property of multiplication over addition as a driving force behind the partial products (by place value) and the standard algorithms.

Preliminary data analysis indicates that we were somewhat successful in achieving our goals. However, many PTs still focused only on the total number of squares in an array, rather than on the components that made up the multiplication problem. In our poster, we will share the task sequence and examples of student work. We will also discuss how we have begun to modify the tasks in order to help us better achieve our goals.

References

Tobias, J.M., Olanoff, D., Hillen, A., Welder, R., Feldman, Z., & Thanheiser, E. (2014). Research-based modifications of elementary school tasks for use in teacher preparation. In Karen Karp (Ed.) *Annual perspectives in mathematics education 2014: Using research to improve instruction.* Reston, VA: NCTM.

Smith, J.P. (2017). Conceptual deficits in curricular introductions to multiplication. In *Proceedings of the 39th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* Indianapolis, IN.

ELICITING PERSONIFICATION TO STUDY PRE-SERVICE TEACHERS BELIEFS

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The recently released Association of Mathematics Teacher Educations Standards for Teacher Preparation claims that “All teachers, including well-prepared beginners, must hold positive dispositions about mathematics and mathematics learning, such as the notions that mathematics can and must be understood, and that each and every student can develop mathematical proficiency” (p 2.7, AMTE, 2017). A goal for mathematics classes for PSTs should be to try and align the students’ beliefs that align with those positive dispositions about mathematics and mathematics learning. Studies (e.g., Philippou & Christou, 1998) have shown that teachers tend to come into these courses with negative dispositions about mathematics. In this study, we used the method of eliciting personification (Zazkis, 2015) to see what is revealed about PSTs beliefs that other methods (e.g., surveys, journaling) fail to capture.

Sixty-eight sophomore PSTs participated in the study using an assignment to elicit their personification by describing a character called Math and writing a dialogue between them and their math character. The assignment was given during the beginning of the semester. After completing the course, the students were asked to describe the character Math from the course and compare this Math to the one they described at the beginning of the semester. Additionally, they were asked to read their first assignment and respond to their previous assignment by creating a dialogue between their current selves and themselves as the start of the class.

The assignments were analyzed by the authors using a thematic analysis (Braun & Clarke, 2006) approach to open-code and develop themes related to beliefs about mathematics. At the beginning of the semester the PSTs that engaged in the task viewed mathematics as an activity that occurred only schools. They first met math when they entered school and all their described interactions, both positive and negative, occurred in a school setting or by engaging in tasks related to schooling (e.g., homework). They described a volatile relationship where math was alternating between a positive relationship and one that was out to hurt them. In their end of class write-ups their focus was on a more approachable mathematics that wants to help the students understand. Instead of blaming themselves for not working hard enough, they instead realized that they could ask questions of others to help understand the content and that math was kind and wanted the best for them. Going forward, research is needed to focus on how to leverage activities such as this to help build positive identities and dispositions towards mathematics.

References

Association of Mathematics Teacher Educators. (2017). Standards for Preparing Teachers of Mathematics. Available online at amte.net/standards.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative research in psychology*, 3(2), 77-101.

Phillippou, G. N., & Christou, C. (1998). The effects of a preparatory mathematics program in changing prospective teachers’ attitudes towards mathematics. Educational Studies in Mathematics, 35, 189-206

Zazkis, D. (2015). Monsters, lovers and former friends: Exploring relationships with mathematics via personification. For the Learning of Mathematics, 35(1), 33-38.

WHAT DOES IT MEAN TO BE GOOD AT MATH: TOOLS FOR DISCOVERY

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Looking back into the recent past, research has shown that emphasizing the importance of effort for learning mathematics can have positive effects on students' achievement, engagement, and sense of competence (Dweck, 2010; Middleton & Jansen, 2011). To adopt this focus, teachers must embrace the belief that success (or being "good at math") is attributable to effort (i.e., growth mindset) rather than inherent ability (i.e., fixed mindset). To support this goal, teacher educators must address related issues with prospective teachers in their teacher preparation programs. As a result of their own experiences as learners of mathematics, many prospective elementary teachers (PETs) have fixed mindsets and negative ideas about their own mathematical abilities and what it means to be successful in mathematics (e.g. Stoehr, 2017).

Looking ahead to the future of this issue, we developed and piloted a suite of instruments designed to elicit PETs' beliefs related to success in mathematics. This suite of tools included three complementary elements: (a) a personification of mathematics task (adapted from Zazkis, 2015), (b) a drawing protocol imaging someone who is "good at math," and (c) a semantic differential. In the personification task, PETs write a description of mathematics as a person, draw a picture of mathematics, and script a conversation they would have with mathematics. For the drawing protocol, PETs reflect on the characteristics of someone who is "good at math" and create an image of this person. Finally, the semantic differential presents 20 sets of paired words (e.g., fast/accurate or ease/effort) and asks participants to make a selection along a continuum as to which of these words more appropriately describes someone who is "good at math."

As a trio, these instruments provide a rich description of PETs' views of what it means to be "good at math," and allow us to consider these ideas through a conceptual framework examining attitudes towards mathematics along three interconnected dimensions: *vision of mathematics, perceived competence, and emotional disposition* (Di Martino & Zan, 2010). One's vision of mathematics includes beliefs about what it means to be successful in mathematics, while perceived competence positions one's own competence within this constructed reality. These factors are regulated by emotional dispositions that account for the positive and negative affective characteristics that PETs express in relation to mathematics. Our suite of instruments and preliminary data from our pilot study provide snapshots of PETs' dispositions related to teaching and learning mathematics, which we hope can inform mathematics teacher educators who support these emerging teachers.

References

Di Martino, P., & Zan, R. (2010). 'Me and maths': Towards a definition of attitude grounded on students' narratives. *Journal of Mathematics Teacher Education,* 13(1), 27-48.

Dweck, C. S. (2010). Mind-sets and equitable education. *Principal Leadership, 10*(5), 26-29.

Middleton, J. A., & Jansen, A. (2011). Motivation matters and interest counts: Fostering engagement in mathematics. Reston, VA: National Council of Teachers of Mathematics.

Stoehr, K. J. (2017). Building the wall brick by brick: One prospective teacher's experiences with mathematics anxiety. *Journal of Mathematics Teacher Education*, *20*(2), 119-139.

Zazkis, D. (2015). Monsters, lovers and former friends: Exploring relationships with mathematics via personification. *For the Learning of Mathematics, 35*(1), 33-38.

PROSPECTIVE ELEMENTARY TEACHERS' NOTICING: INTRODUCING A TRAJECTORY OF CHILDREN'S MULTIPLICATIVE THINKING IN A CONTENT COURSE

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Keywords: Learning Trajectories; Teacher Education – Preservice; Teacher Knowledge

Effective teaching involves understanding children's mathematical thinking, interpreting their thinking to assess current understanding, and deciding where children need to go next in order to advance their learning (Heritage, 2010). Researchers refer to this set of skills as the professional noticing of children's mathematical thinking (Jacobs, Lamb & Philipp, 2010). We argue prospective teachers (PTs) need to learn to effectively notice children's mathematical thinking and further, need multiple opportunities in their preparation program to develop their skill at noticing. It is important thus to orient PTs to the work of teaching early in their program, long before they take a methods course. In this study, we designed and enacted an intervention to support PTs development of noticing children's thinking in a first semester content course and examined the following research questions: (1) How does the OGAP framework support PTs' noticing of children's multiplicative thinking; and (2) How does the OGAP framework support PTs' development of knowledge of children's multiplicative thinking?

Drawing from research on learning trajectory-based professional development, we designed an intervention using the Ongoing Assessment Project (OGAP) framework (Petit, 2013) that scaffolded PTs' noticing according to a trajectory of children's multiplicative reasoning. We designed a series of four, 50-minute class sessions which utilized number talks and video clips to introduce PTs to the five levels of Children's Multiplication Strategies (Petit, 2013): early-additive, additive, early-transitional, transitional, and multiplicative.

This poster reports on some of the results of a pre-post survey created by the authors which was used to measure PTs ability to attend to and interpret four samples of children's multiplicative strategies in a variety of ways. Four multiplication problems and associated sample of student work were used as a context for PTs to answer a set of identical questions. One question asked PTs to anticipate children's multiplicative thinking for the given task, a second asked PTs to describe the provided sample of student work, and a third asked PTs to posit the next step in a progression of children's multiplicative thinking. We utilized the OGAP framework as a lens analyze PTs responses to these questions.

The results shared indicate that PTs made improvements in their noticing and knowledge of children's multiplicative thinking from pre- to post-survey. For example, PTs' responses included specific language from the OGAP framework to describe children's thinking; as exemplified by Toni's response on the post-survey, "A child might solve this problem using the transitional strategy, displaying an area model that considers both dimensions but not every individual unit".

References

Heritage, M. (2010). *Formative assessment: Making it happen in the classroom*. Corwin Press.
Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. Journal for research in mathematics education, 169-202.
Petit, M., Hulbert, E., & Laird, R. (2013). *OGAP Multiplicative Reasoning Framework*. Moretown, Vermont.

PRESERVICE ELEMENTARY TEACHERS USE OF JOURNALS AS A MEANS OF REFLECTING ON PROFESSIONAL VISIONS OF MATHEMATICS INSTRUCTION

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Keywords: Teacher Education-Preservice; Teacher Beliefs

Before entering their teacher preparation program, future preservice teachers develop preconceived notions of what good teaching looks like based upon their time as students of mathematics (Masingila & Doerr, 2002). However, once they become a preservice teacher of mathematics, these ideals, or visions to which they strive (Hammerness, 2003), often do not align with the standards-based practices of their teacher preparation program. Nevertheless, teacher preparation programs can help to refine preservice teachers' visions of instruction. One way in which teacher preparation programs can help preservice teachers to refine their visions is through writing prompts (Duffy, 2002). The hope is that through reflection, preservice teachers are able "to develop a clearer sense of their purposes for teaching and of their commitment to the profession" (Hammerness, 2003, p. 55). Therefore, two main questions guided this study:

1. How do preservice elementary teachers' visions of mathematics instruction evolve throughout an elementary mathematics methods course?
2. What impacts their vision of mathematics instruction throughout the methods course

Participants & Data Collection

Thirty-two preservice elementary teachers volunteered to participate in the study that took place during their second mathematics methods course, during their senior year. Throughout the course, participants were exposed to a variety of tasks, readings, and discussions to enrich their content and pedagogical knowledge. In addition to the course assignments, the participants were asked to complete six journals throughout the course which specifically asked them to reflect on their visions of mathematics instruction.

Results & Discussion

Preliminary results of this study show the impact of Jo Boaler's *Mathematical Mindsets* text used throughout the mathematics methods course. Many participants described their vision as changed after reading and discussing growth versus fixed mindsets during the course. They also attributed their experience in the course, where the instructor modeled a growth mindset and made the environment safe for sharing mistakes, to their changing vision. This change shows the importance of teacher educators modeling reform-oriented practices in methods courses and how specific readings and discussions attribute to the change of preservice teachers visions of effective mathematics teaching.

References

Duffy, G. G. (2002). Visioning and the development of outstanding teachers. *Reading Research and Instruction*, *41*(4), 331–344.

Hammerness, K. (2003). Learning to hope, or hoping to learn? The role of vision in the early professional lives of teachers. *Journal of Teacher Education*, *54*(1), 43–56.

Masingila, J. O., & Doerr, H. M. (2002). Understanding preservice teachers' emerging practices through their analyses of a multimedia case study of practice. *Journal of Mathematics Teacher Education*, *5*, 235–263.

Chapter 9

Statistics and Probability

Research Reports

Brief Research Reports

Posters

MIDDLE AND SECONDARY TEACHERS' INFORMAL INFERENTIAL REASONING

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This study examined middle and secondary mathematics teachers' knowledge structures and informal inferential reasoning (IIR). Using task-based clinical interviews (Goldin, 1997) and cross-case analysis, nine teachers responded to four LOCUS assessment tasks (Jacobbe, 2016). Responses were used to construct knowledge structure maps for measures of center, spread, and shape (Groth & Bergner, 2013). Teachers' IIR was analyzed for the appropriateness of responses (Means & Voss, 1996) and key components of IIR were identified. Teachers with more connected knowledge structures and fewer undesirable knowledge elements exhibited more acceptable forms of IIR. Although teachers engaged in the inference and data components of IIR (Makar & Rubin, 2009), they rarely referenced uncertainty. Implications for teacher education and future research are discussed.

Keywords: Data Analysis & Statistics, Teacher Knowledge, Cognition

Globally, there has been an increased recognition of the need for statistical literacy. This is particularly evident in the U.S. from the increased number of statistics standards included in the Common Core State Standards for Mathematics (CCSSM) and from other standards reform documents (e.g., Franklin et al., 2007). This push is due, in part, to consistent findings that tertiary students struggle to improve in their statistical reasoning abilities throughout introductory courses (e.g., delMas, Garfield, Ooms, & Chance, 2007). Not surprising, the majority of K-12 mathematics teachers in the U.S. feel unprepared to teach statistics content, despite completing at least one college level statistics course (Banilower et al., 2013) and despite the consistent calls for mathematics teacher education reform over the past twenty years (Conference Board of the Mathematical Sciences, 2001, 2012; Franklin et al., 2015). More importantly, standards reform documents, such as the CCSSM, require teachers to engage their students in informal ways of reasoning, which is not typically found in tertiary courses (Garfield, DelMas, & Zieffler, 2012), thereby leaving many teachers unable to draw on experience to support such teaching. Several studies have found great success in engaging students in informal inferential reasoning (IIR) as a way to promote students' statistical literacy across middle and secondary grades (see for example, Makar & Ben-Zvi, 2011, special issue in *Mathematical Thinking and Learning*; Pratt & Ainley, 2008, special issue in *Statistics Education Research Journal*). However, there has not been an associated level of research into teachers' engagement in IIR, leading to recent claims that research connecting teachers' content knowledge and IIR is a "critical area for future research" (Langrall, Makar, Nilsson, & Shaughnessy, 2017, p. 517). This study examined teachers' knowledge structures for measures of center, spread, and shape as it related to their engagement in tasks designed to elicit IIR. The goal was to understand the ways in which teachers' knowledge structures may be constructed, and how they support the ways teachers engage in IIR, in the context of task-based clinical interviews. This work adds to research base on teachers' IIR—a relatively thin area compared to research on students.

Theoretical Framework

This study draws on the work of Zieffler and colleagues (2008), who defined informal inferential reasoning (IIR) as "the way in which students use their statistical knowledge to make arguments to support inferences about unknown populations based on observed samples" (p. 44). Their framework for IIR describes three essential components—making claims about populations from samples, utilizing prior knowledge, and using evidence-based arguments to support claims about populations (Zieffler et al., 2008, p. 45). We further expanded this framework, drawing upon Rossman (2008), to include inferences made about causality between variables in addition to claims about populations from samples.

In considering the role of knowledge in this IIR framework, we took the stance of Franklin and colleagues (2007) who assumed that knowledge and informal reasoning develop alongside one another. Furthermore, in Zieffler and colleagues' literature review, they found that informal reasoning did not improve with "maturation, education, or life experience" (2008, p. 44). Therefore, we theorized that IIR occurs at the intersection of statistical knowledge and informal reasoning, and neither knowledge nor reasoning are required for the development of the other.

Research Questions

This study addressed the following two research questions: What knowledge structures do middle level and secondary mathematics teachers have regarding center, spread, and shape of distributions (RQ1)? How do teachers' knowledge structures support IIR (RQ2)?

Method

Setting and Participants

A stratified purposeful sample (Patton, 2002) of nine practicing middle and secondary mathematics teachers each participated in two task-based clinical interviews. Teachers were required to have taught statistics content that included data analysis explicitly using measures of center, spread, and shape of distributions. Teachers were also chosen in order to obtain four strata: 1) statistics taught as a unit within middle level mathematics ($N = 3$), 2) statistics taught as a unit within secondary mathematics ($N = 2$), 3) Non-Advanced Placement (AP) Statistics ($N = 2$), and AP Statistics ($N = 2$). There was wide variability in teachers' backgrounds that was not concentrated in any one strata, all teachers had Master's degrees in education-related fields, and all had completed at least one tertiary course in statistics.

Data and Analysis

Task-based clinical interviews were conducted with each teacher, using two tasks for each of two 60–90 minute video recorded interviews. The four tasks were selected from released items from the *Levels of Conceptual Understanding of Statistics* (LOCUS) assessment (Jacobbe, 2016) in order to align with Huey and Jackson's IIR task framework (2015) to maximize the potential for observing IIR engagement. Due to space limitations, the tasks are not included, but they were: New Year's Day Race, Tomatoes and Fertilizer, Extended School Day, and Jumping Distances. Teachers' written and verbal responses aided in answering both research questions. Interview protocols were developed, following the suggestions of Ginsburg (1981) that questions should require reflection, determine the seriousness of responses, confirm that participants understood the question, and evaluate the strength of belief of responses by challenging them. Moreover, we followed suggestions from Goldin (1997) that follow-up questions be non-directive and that the protocol anticipate as many contingencies as possible. Contingencies were more completely anticipated by using LOCUS tasks because a range of student responses are included along with each released task. Due to space limitations, instruments and tasks will not be shown here but will be provided during the presentation.

Data analysis first involved transcribing each interview and coding teachers' responses for constructed knowledge elements related to center, spread, and shape of distributions (RQ1).

Connections between knowledge elements were then also hypothesized through open-coding across all tasks. Node-link diagrams were used in order to visually represent each teacher's knowledge structure. We followed a similar method to that of Groth and Bergner (2013) that involved mapping Partially-Correct Constructions for each participant. Thus, teachers' responses were further classified as either *desirable* or *undesirable*. A *desirable* constructed knowledge element is one that has been identified in the literature as having the potential for supporting the development of other knowledge elements or more complex depictions of that knowledge element. An *undesirable* element is one that may support the development of inconsistent or disconnected knowledge elements, or not allow for a *desirable* element to be constructed from it. *Desirable* elements were visually represented by blue rectangles, and *undesirable* elements with yellow rectangles with rounded vertices. These value laden terms should not be misconstrued as a teacher's lack of expertise because an *undesirable* element does not necessarily imply that it is inherently incorrect. Observed connections between knowledge elements were depicted with a double-headed arrow. After all knowledge maps had been constructed, a within-case analysis was carried out to confirm each knowledge structure, and then a cross-case analysis was carried out to identify common themes across structures to categorize types of structures.

To address RQ2, we drew upon the work of Means and Voss (1996) to first identify teachers' arguments—their claims and reasons for them—and if they were supported by *acceptable* or *unacceptable* evidence. Next, we employed Makar and Rubin's informal statistical inference framework (2009) to identify which of the three components, evidence of IIR, were observed in teachers' arguments—generalization beyond the data, data as evidence, and probabilistic language. A cross-case analysis (Creswell, 2013) was carried out to identify categories of types of IIR. Lastly, types of IIR were compared with types of knowledge structures to characterize how teachers' knowledge structures may support their IIR.

Findings

Knowledge Structures for Center, Spread and Shape

After a cross-case analysis, 3 basic types of knowledge structures were identified (see Figure 1): *desirable-connected* ($N = 3$), *undesirable-connected* ($N = 4$), and *undesirable-disconnected* ($N = 2$). *Desirable-connected* structures included almost no *undesirable* knowledge elements and knowledge elements were highly connected, with connections observed both *within* and *between* knowledge types (center, spread, shape), as can be seen in the case of Rosalynn's knowledge structure as a case of a *desirable-connected* structure in Figure 1. This knowledge structure category was highly consistent across teachers—salient features were observed across all cases. On the other end of the spectrum, *undesirable-disconnected* knowledge structures contained multiple *undesirable* knowledge elements that were observed to be largely disconnected, with no connections observed between center and shape knowledge types. These characteristics were consistent across the two cases of *undesirable-disconnected* structures, as can be seen in the case of Amalia's knowledge structure in Figure 1. Knowledge structures described as *undesirable-connected* also contained multiple *undesirable* knowledge elements, but knowledge elements were highly connected—thus integrating *undesirable* knowledge elements into the overall structure, as can be seen in Ellie's knowledge structure in Figure 1. Knowledge structures in this category were more varied than the other types. For instance, not all structures included direct connections *between* all knowledge types (center, spread, shape), but all included direct connections *between* at least two of the three types. Moreover, the majority of knowledge elements were connected to at least one other knowledge element—reflecting the highly integrated nature of teachers' knowledge. It is of note that in all cases of both *undesirable-*

connected and *undesirable-disconnected* structures, at least one direct path connecting two elements of center, spread, and shape included at least one *undesirable* knowledge element.

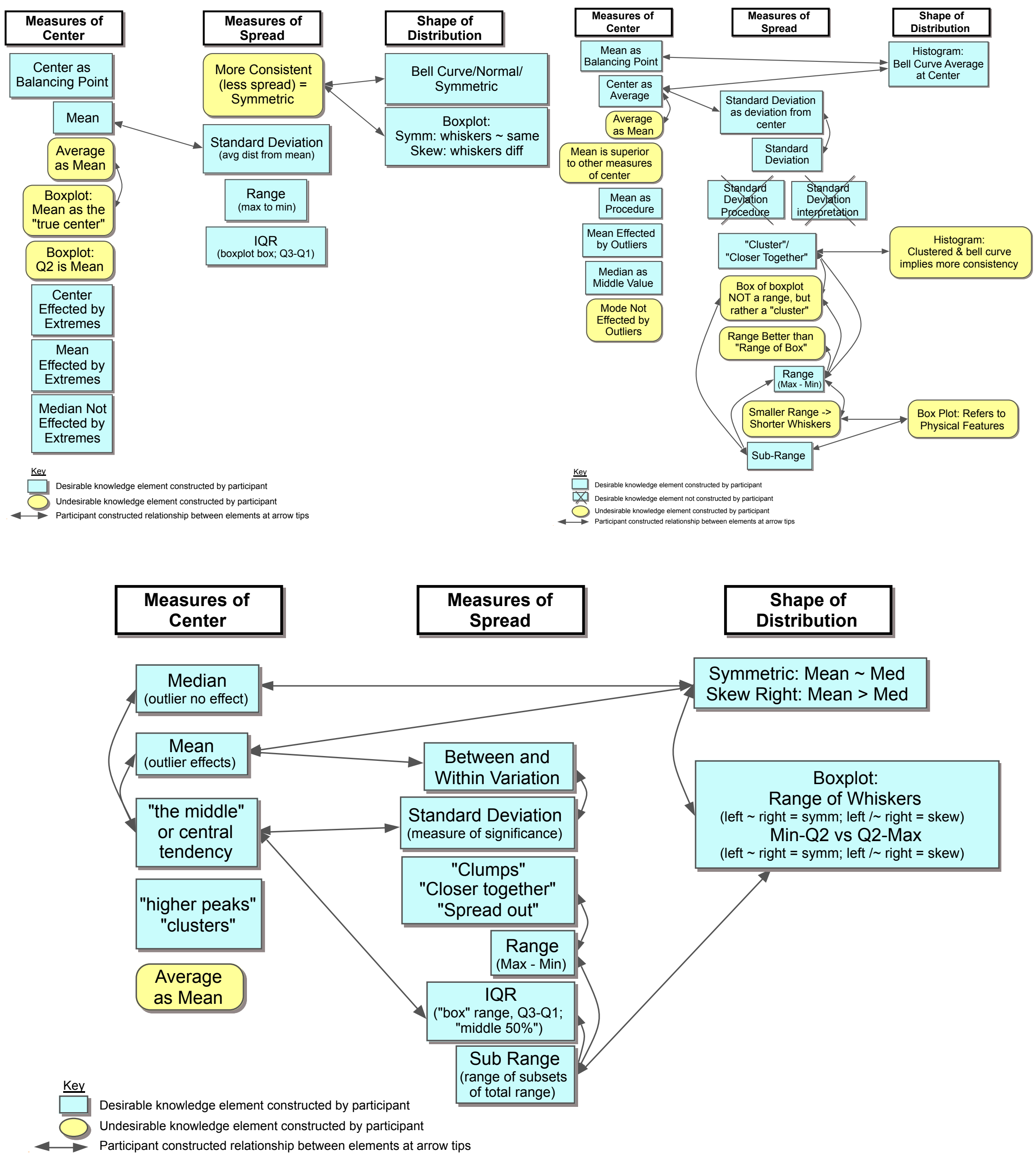

Figure 1. Representative cases of the three types of knowledge structures

Knowledge Structures and Support for IIR

Tasks that did not require engagement in IIR explicitly (e.g., asking only about characteristics of samples) produced different findings than those that did. In particular, teachers more frequently exhibited *acceptable* supports in their arguments when reasoning in these non-

IIR contexts than in those tasks that demanded IIR (see Table 1). For example, five teachers were observed to reason using *acceptable* supports (i.e., drawing on *desirable* knowledge elements) for at least 75% of their responses (note that teachers were encouraged to offer multiple responses per task per the interview protocol) and two of those (Rosalynn and Tim) reasoned in this way for 100% of their responses. In contrast, within IIR contexts, none of the teachers were observed to reason using *acceptable* supports for at least 75% of their responses. Although teachers' reasoning in IIR contexts largely made use of *unacceptable* supports in their arguments, when the task only required drawing upon center, spread or shape, 8 of the 9 teachers were able to offer at least one response coded as *acceptable*. However, 5 of these 8 teachers simultaneously offered responses coded as *unacceptable*, indicating wider variation in responses for IIR contexts than non-IIR contexts.

Table 1: Comparing Reasoning Types and Knowledge Structures

Non-IIR Reasoning	Teacher	Knowledge Structure	IIR Reasoning
Mostly Unacceptable	Kathy	Undesirable-Disconnected	Mostly Unacceptable IIR
Mixed	Amalia	Undesirable-Disconnected	Mostly Unacceptable IIR
Mixed	Ellie	Undesirable-Connected	Mostly Unacceptable IIR
Mixed	Ruby	Undesirable-Connected	Mostly Unacceptable IIR
Mostly Acceptable	Michaela	Undesirable-Connected	More Unacceptable IIR
Mostly Acceptable	Harrison	Undesirable-Connected	More Unacceptable IIR
Mostly Acceptable	Mike	Desirable-Connected	More Unacceptable IIR
Mostly Acceptable	Rosalynn	Desirable-Connected	Mostly Acceptable IIR
Mostly Acceptable	Tim	Desirable-Connected	Mostly Acceptable IIR

Tasks that required IIR were also coded according to the three components of informal statistical inference described by Makar and Rubin (2009). Two teachers utilized all three components (generalization/causation, data as evidence, probabilistic language) for one response each and 8 of the 9 teachers incorporated both a generalization and data as evidence when their reasoning was coded as *acceptable*. The use of probabilistic language was exceedingly rare, observed in only the two cases where all three components were included, indicating that teachers largely did not consider the deterministic nature of their inferential statements.

Comparing reasoning types to knowledge types, the greatest observable pattern is that teachers with mostly *desirable* elements that were highly integrated also reasoned in *acceptable* forms for the majority of their responses, while teachers with many *undesirable* elements that were largely *disconnected* tended to reason in *unacceptable* forms for the majority of their responses. A second salient pattern is that those who reasoned in more *acceptable* forms across both contexts (non-IIR and IIR) tended to have more connected knowledge structures. Moreover, the relationship between reasoning types and knowledge structures appeared most evident at opposite ends of the spectrum. To illustrate this, on the Jumping Distances task, participants were asked to compare center, spread and shape of the distances a sample of students jumped by examining a pair of boxplots—one boxplot representing jumping distances for a subset who was provided with a target to jump towards, the other not having a target to jump towards (see Figure 2). The following responses come from Amalia and Rosalynn as they respond to this item:

> *Amalia*: The no target looks to be more symmetric. This one [target plot] is close to be symmetric, but it is slightly skewed. […] The whiskers make me think [the no target plot is] not skewed because they look almost the same length. Whereas this one [target plot] is a little longer on the left [gesturing to both whiskers of target plot] […] But yeah, I was just looking

at the whiskers, and just the fact that the target group, there is a bigger difference between the largest and the shortest jump, versus here [no target], there's not as much of a difference. (lines 10 and 21)

Rosalynn: The shape of the distribution […] Since you do have a longer whisker on the left, you're looking at a slight left skew for the target group versus a more symmetric distribution for the no target group. […] The left part [gesturing to distance from minimum to median] is more spread out for this one [target] than the right part. (lines 6 and 9)

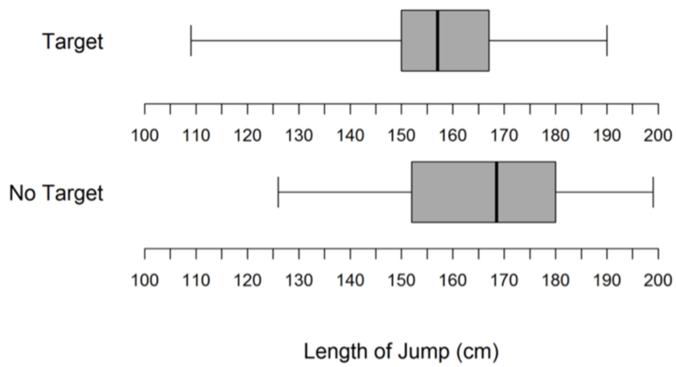

Figure 2. Boxplots provided on Jumping Distances task.

Within these excerpts, you can see that Amalia first begins to describe the distribution's shape using *desirable* knowledge elements—referring to the respective lengths of the whiskers for each plot to hypothesize about the shape. However, she then creates a second argument that draws on an *undesirable* knowledge element—that the smaller range of the no target plot implies more symmetry. Thus, without making contradictory claims, she simultaneously argued in ways coded as both *acceptable* and *unacceptable*. This perspective of spread as being connected to shape in an *undesirable* way led to her *unacceptable* form of reasoning. Amalia's knowledge structure was categorized *undesirable-disconnected* (see Figure 1), and her reasoning was impacted by the integration of *undesirable* elements, thus resulting in more responses coded as *unacceptable*. In contrast, Rosalynn's responses remained in comparison to whether things appeared more evenly spread to the left and right of center, drawing on *desirable* knowledge elements for each response. Therefore, Rosalynn's knowledge structure, categorized as *desirable-connected* (see Figure 1), supported her reasoning in ways that led to more *acceptable* forms of reasoning. Teachers whose knowledge was more integrated yet contained multiple *undesirable* elements, were observed offering both *acceptable* and *unacceptable* forms of reasoning simultaneously—leading to the characterization of having a more mixed form of reasoning.

Discussion

A major finding of this study is that middle and secondary teachers' knowledge structures for center, spread and shape of distributions fell into three categories—*desirable-connected*, *undesirable-connected*, and *undesirable-disconnected*. Looking across these categories, 7 of the 9 teachers were found to have highly interconnected knowledge structures that all made connections among the three knowledge types (center, spread, shape), and some made connections across all three. Moreover, despite many teachers having *undesirable* elements integrated into their knowledge structures, most were observed to have at least one connection between *desirable* elements of center and spread. Although there is evidence that teachers' knowledge structures were highly interconnected, teachers struggled to integrate their knowledge of center, spread and shape. For instance, teachers in this study were largely observed to recognize that connections existed between knowledge element types (less so between center and shape), yet they did not utilize these connections and draw upon multiple knowledge types simultaneously, as prior research has found for students and pre-service teachers (Doerr & Jacob, 2011; Groth & Bergner, 2006; Noll & Shaughnessy, 2012). Moreover, these studies also found that hierarchically higher levels of reasoning required the integration of multiple knowledge elements simultaneously—such as describing a measure of spread in relation to a measure of center—thus, supporting the notion that *desirable-connected* knowledge structures are associated with more *acceptable* forms of reasoning.

Although the theoretical framework for this study does not assume that there is any particular prior knowledge necessary for engaging in IIR—and that knowledge and informal reasoning work together during engagement in IIR—the teachers who engaged in IIR in *mostly acceptable* forms were those with *desirable-connected* knowledge structures. This finding aligns with Makar and colleagues' (2011) claim that statistical knowledge is an important support for IIR. Moreover, as found by Huey (2011) and Watson (2003), teachers in this study tended to focus only on measures of center unless explicitly prompted otherwise.

These results promote the need for statistics teacher education to include explicit opportunities, embedded within tasks, for teachers and preservice teachers to grapple with connections both *within* knowledge types and *between* them. For instance, some teachers in this study believed that outliers were either *always* or *never* ignorable, that the term average *always* implied the arithmetic mean, or that the mean was *always* or *never* a better measure than the median. Situations that confront teachers with these dichotomies will allow them to consider a more nuanced understanding and more flexible knowledge structure. Moreover, teachers in this study largely drew on ideas of center to support inferences, even when tasks did not restrict them in this way. Consequently, tasks designed to engage teachers in IIR should also encourage attention to all of center, spread, and shape—not isolating one type.

More research is needed in these areas in order to explore the kinds of task features that are more likely to encourage integration of all three knowledge types, thus supporting teachers in strengthening their knowledge structures and reasoning. One possible fruitful area of research is to provoke teachers into weighing the evidence of their inferential statements through requiring the construction of multiple inferential statements and responding to multiple inferential statements to explore their validity. This will allow engagement in the statistics practice described in the *Statistical Education of Teachers* report as being able "to compare the plausibility of alternative conclusions and distinguish correct statistical reasoning from that which is flawed" (Franklin et al., 2015, p. 14).

A final needed area of research is in explicitly engaging teachers in, and drawing attention to, the use of probabilistic language. A profound lack of attention to this component of IIR was observed in this study. Unlike mathematics, where deterministic statements are the norm, inferences in statistics are predictions and therefore come with a degree of uncertainty.

References

Banilower, E. R., Smith, P. S., Weiss, I. R., Malzahn, K. A., Campbell, K. M., & Weis, A. M. (2013). *Report of the 2012 national survey of science and mathematics education*. Chapel Hill, NC: Horizon Research, Inc.

Conference Board of the Mathematical Sciences. (2001). *The mathematical education of teachers*. Washington, DC: American Mathematical Society and Mathematical Association of America.

Conference Board of the Mathematical Sciences. (2012). *The mathematical education of teachers II*. Providence, RI and Washington, DC: American Mathematical Society and Mathematical Association of America.

Creswell, J. W. (2013). *Qualitative inquiry and research design: Choosing among five approaches* (3rd ed.). SAGE.
delMas, R., Garfield, J., Ooms, A., & Chance, B. (2007). Assessing students' conceptual understanding after a first course in statistics. *Statistics Education Research Journal*, *6*(2), 28–58.
Doerr, H. M., & Jacob, B. (2011). Investigating secondary teachers' statistical understandings. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education* (pp. 776–786). University of Rzeszów, Poland.
Franklin, C., Bargagliotti, A. E., Case, C. A., Kader, G., Scheaffer, R., & Spangler, D. A. (2015). *The Statistical Education of Teachers*. Retrieved from http://www.amstat.org/education/SET/SET.pdf
Franklin, C., Kader, G., Mewborn, D., Moreno, J., Peck, R., Perry, M., & Scheaffer, R. (2007). *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report. American Statistical Association*. Alexandria, VA. Retrieved from http://www.amstat.org/Education/gaise/GAISEPreK-12_Full.pdf
Garfield, J., DelMas, R., & Zieffler, A. (2012). Developing statistical modelers and thinkers in an introductory, tertiary-level statistics course. *ZDM - The International Journal on Mathematics Education*, *44*(7), 883–898.
Ginsburg, H. (1981). The clinical interview in psychological research on mathematical thinking: Aims, rationales, techniques. *For the Learning of Mathematics, 4*(3), 1–11.
Goldin, G. A. (1997). Chapter 4: Observing mathematical problem solving through task-based interviews. *Journal for Research in Mathematics Education. Monograph*, *9*(Qualitative Research Methods in Mathematics Education), 40-62.
Groth, R. E., & Bergner, J. A. (2006). Preservice elementary teachers' conceptual and procedural knowledge of mean, median, and mode. *Mathematical Thinking and Learning*, *8*(1), 37–63.
Groth, R. E., & Bergner, J. A. (2013). Mapping the structure of knowledge for teaching nominal categorical data analysis. *Educational Studies in Mathematics*, *83*(2), 247–265. https://doi.org/10.1007/s10649-012-9452-4
Huey, M. E. (2011). *Characterizing middle and secondary preservice teachers' change in inferential reasoning*. University of Missouri, Columbia, MO.
Huey, M. E., & Jackson, C. D. (2015). A framework for analyzing informal inferential reasoning tasks in middle school textbooks. In T. G. Bartell, K. N. Bieda, R. T. Putnam, K. Bradfield, & H. Dominguez (Eds.), *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 431–438). East Lansing, MI: Michigan State University.
Jacobbe, T. (2016). Levels of Conceptual Understanding in Statistics (LOCUS). Retrieved April 18, 2016, from https://locus.statisticseducation.org/
Langrall, C. W., Makar, K., Nilsson, P., & Shaughnessy, J. M. (2017). The teaching and learning of probability and statistics: An integrated perspective. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 490–525). Reston, VA: National Council of Teachers of Mathematics, Inc.
Makar, K., Bakker, A., & Ben-Zvi, D. (2011). The reasoning behind informal statistical inference. *Mathematical Thinking and Learning*, *13*(1–2), 152–173.
Makar, K., & Ben-Zvi, D. (2011). The role of context in developing reasoning about informal statistical inference. *Mathematical Thinking and Learning*, *13*(1–2), 1–4.
Makar, K., & Rubin, A. (2009). A framework for thinking about informal statistical inference. *Statistics Education Research Journal*, *8*(1), 82–105.
Means, M. L., & Voss, J. F. (1996). Who reasons well? Two studies of informal reasoning among children of different grade, ability, and knowledge levels. *Cognition and Instruction*, *14*(2), 139–178.
Noll, J., & Shaughnessy, J. M. (2012). Aspects of students' reasoning about variation in empirical sampling distributions. *Journal for Research in Mathematics Education*, *43*(5), 509–556.
Patton, M. Q. (2002). *Qualitative Research and Evaluation Methods* (3rd ed.). Thousand Oaks, California: Sage Publications Inc.
Pratt, D., & Ainley, J. (2008). Introducing the special issue on informal inferential reasoning. *Statistics Education Research Journal*, *7*(2), 3–4.
Rossman, A. (2008). Reasoning about informal statistical inference: One statistician's view. *Statistics Education Research Journal*, *7*(2), 5–19.
Watson, J., & Callingham, R. (2003). Statistical literacy: A complex hierarchical construct. *Statistics Education Research Journal*, *2*(2), 3–46.
Zieffler, A., Garfield, J., delMas, R., & Reading, C. (2008). A framework to support research on informal inferential reasoning. *Statistics Education Research Journal*, *7*(2), 40–58.

AN ANALYSIS OF HIGH SCHOOL STUDENTS TAKING STATISTICS USING THE HSLS:09 DATASET

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In this study I report results of an quantitative analysis using a large scale public dataset (HSLS:09) to investigate which students are taking statistics courses in high school to begin to understand the access students have to opportunities to learn statistics concepts and practices in high schools in the United States. The main result of this study is that predominantly the top academically performing high school students are earning credit for taking statistics. This is concerning as all students should have experiences learning concepts and practices from statistics to be prepared to engage in and play active roles in today's data centric societies. In line with the conference theme of looking forward, implications for future research and policy are discussed.

Keywords: Data Analysis and Statistics, Equity and Diversity, High School Education

Background and Statement of Problem

Society today is drenched in data (Steen, 2001). Individuals are surrounded by data aimed at influencing their decisions about what school policies to implement, politician to vote for, or medicine to use. Statistics, the science of data, is becoming an increasingly important discipline for people to be familiar with the key concepts and practices of, because of society's reliance on data (Ben-Zvi & Garfield, 2008). As Franklin et al. (2007) states, "every high school graduate should be able to use sound statistical reasoning to intelligently cope with the requirements of citizenship" (p.1). For students to develop sound statistical reasoning it is important that a goal of public K-12 education is to provide students with opportunities to have rich learning experiences with concepts and practices of statistics if they are going to be critical citizens in today's data-driven societies.

Teaching statistics in the K-12 setting is firmly rooted in the mathematics curriculum and access and opportunities to learn are crucial equity issues to consider in mathematics education (Gutiérrez, 2009; Schmidt & McKnight, 2012). In the United States, students' opportunities to learn mathematics have been found to vary greatly due to the decentralized nature of education in the U.S. (Schmidt & McKnight, 2012). The implementation of the Common Core State Standards for Mathematics (CCSSM; National Governor's Association Center for Best Practices (NGA Center) & Council of Chief State School Officers (CCSSO), 2010) was meant to serve as a unifying force to reduce some of the variation in the mathematics content covered across states in the U.S. However, not all states have signed on to adopt the standards, and other states have begun to modify the standards to be implemented in an attempt to distance themselves from some of the controversy and politics around the standards (Orrill, 2016). In the CCSSM, data analysis and statistics have gained emphasis in grades 6-12 compared to most previous state standards. However, there has also been the loss of much of the statistics and probability content at the K-5 level, which could have serious ramifications in the future (Lubienski, 2015).

It is important to point out that standards are not the only factor influencing curriculum as classroom teachers and local contexts have a direct influence on the enacted curriculum students experience in the classroom (Remillard & Heck, 2014). However, this is a unique issue in the case of statistics because although the instruction of statistics is firmly rooted in the mathematics curriculum at the K-12 level, statistics is a distinct discipline with concepts and practices that are non-mathematical (Cobb & Moore, 1997; Groth, 2013). This can cause problems because many K-12 mathematics teachers have often had little to no prior coursework in statistics (Franklin et al., 2015; Shaughnessy, 2007), which could have very serious repercussions for the opportunities students have to learn statistics in school. Since the enactment of mathematics curriculum varies greatly from classroom to classroom based on a number of factors (Remillard & Heck, 2014; Schmidt & McKnight, 2012), it is very difficult to study what opportunities students have to learn statistics concepts and practices on any kind of large scale.

In this study, I used the National Center for Educational Statistics' (NCES) High School Longitudinal Study of 2009 (HSLS:09) public access dataset to begin to investigate which students are earning statistics credit in high school in an effort to better understand high school students access to opportunities to learn statistics concepts and practices. More specifically, I investigated the following research questions:

1. What relationship is there between demographic characteristics (i.e. sex, race, and SES) of students who earned at least one credit of statistics and those that did not?
2. What relationship is there between the academic performance of students who earned at least one credit of statistics and those that did not?
3. What relationship is there between the beliefs/attitudes of students who earned at least one credit of statistics and those that did not?

Conceptual Framework

This study is framed in a larger equity framework, specifically investigating issues around achievement and access, which are dimensions of what Rochelle Gutiérrez (2009) refers to as the dominant axis of equity. Gutiérrez (2002) states a basic definition of equity as being the, "erasure of the ability to predict students' mathematics achievement and participation based solely on characteristics such as race, class, ethnicity, sex, beliefs and creeds, and proficiency in the dominant language" (p.153). Drawing upon this definition this study is focused on the characteristics of race, class, sex, and beliefs and whether or not there is a relationship between students' statistics course taking and such characteristics, in an effort to begin to investigate the equity in student's participation in opportunities to learn statistics concepts and practices. To consider the issue of achievement in this study, students' mathematics achievement in terms of algebraic reasoning as well as their achievement in terms of their GPAs was considered in relation to whether or not they earned at least one credit in statistics or not.

Mode of Inquiry

Data source

The public access dataset from the HSLS:09 was the data source for this study. The goal of the HSLS:09 is to provide data to "better understand the impact of earlier educational experiences (starting at 9th grade) on high school performance and the impact of these experiences on the transitions that students make from high school to adult roles" (Ingels et al., 2015, p. 6). As such, the HSLS:09 was designed to gather data on a sample that is nationally representative of students entering 9th grade in 2009 (n=23503). One of the goals of the HSLS:09 is to provide data to investigate, "the nature of the paths into and out of STEM (science, technology, engineering, and mathematics) curricula" (Ingels et al., 2015, p. 6). At this point data is available for the base year (Fall 2009), first follow-up (Spring 2012), post-secondary status update (Summer/Fall 2013), and high school transcript report (2013-2014).

Methods

For this study, quantitative methods were employed to investigate variables related to students' demographics, mathematics course taking, academic performance, and beliefs/attitudes towards math and science. Students' mathematics course taking was considered in terms of whether or not they earned at least one credit in statistics/probability in high school and what was the highest level mathematics course students took in high school. As a note, AP Statistics course taking was not included in this analysis, because that data is only available in the restricted use data file. The HSLS:09 has a number of different academic performance variables, which include students' performance on a mathematics assessment designed to assess students' algebraic reasoning. In this analysis, I chose to use students' theta score for their performance on the mathematics assessment, which provides a norm-referenced measurement of achievement, for the base year assessment and the first follow-up assessment. Students' mathematics, STEM, and all academic course GPAs were also used as performance variables. Finally students' identity, self-efficacy, utility of and interest beliefs/attitudes towards mathematics and science, which were measured as normalized scale variables during the base year and the first follow-up, were considered. A complete listing of the variables and their description that were used in the analysis can be found in Table 1.

An initial exploratory data analysis (Tukey, 1977) was conducted and important descriptive statistics are reported in the results. An inferential analysis of the data was then done to address the research questions to investigate the relationship between students' demographics, academic performance and beliefs/attitudes and whether they earned credit for taking a statistics course of not. The inferential statistics used included two-sample t-tests for scale variables, which included variables for students' academic performance, beliefs/attitudes towards math and science, and SES, grouped based on whether or not they had at least one credit of statistics. In other words, the specific relationship that was investigated was whether or not there were differences in the variables considered, between the two groups. For categorical demographic variables, chi-square tests were used to determine if there were associations between the variables and the two groups considered. Standardized residuals were also employed in the case of statistically significant results of the chi-squared tests to get a better idea of which categories were significantly different from what was expected. Design effects normalized analytic weights were used for all inferential statistics. Effects sizes are reported for the quantitative variables by converting the t-test statistics to r and then using the commonly used cutoff's of r=.1 for small, r=.3 for medium, and r=.5 for large effect sizes (Field, 2013). Effect sizes for the categorical variable χ^2-test statistics are reported using Cramer V and the same cutoffs of .1, .3, and .5.

The method of focusing on transcript data to investigate students' course taking is not new. The transcript studies conducted periodically as part of the National Assessment of Educational Progress (NAEP) have been a source of such information over the past few decades. Based on such information, it was recently reported that, statistics/probability course taking in high school increased from 1% of high school graduates in 1990 to 11% in 2009 (NCES, 2016). Unfortunately, the NAEP transcript data is also limited in the information it collects on students, as it is focused mostly on transcript data of course taking and is only linked to students' performance on the NAEP assessment in 12th grade in some cases. The HSLS:09 public access dataset includes significantly more variables related to students' course taking, academic performance, demographics, and beliefs/attitudes towards mathematics and science, making it useful to investigate factors related to who the students are that are taking statistics courses.

Table 1: List of variable names and the descriptions from the HSLS:09 codebook (Ingels et al., 2015) for the variables analyzed in this study.

Variable Name	Description of Variable
X3T1CREDSTAT	Indicates at least one Carnegie unit in Statistics/Probability, does not include AP Statistics
X3THIMATH	Highest Mathematics course
X3TWHENALG1	Indicates the grade level the student took Algebra I.
X1TXMTH & X2TXMTH	The mathematics theta score represents the student's ability level on a continuous scale. The theta score provides a norm-referenced measurement of achievement, that is, an estimate of achievement relative to the population (fall 2009 9th graders) as a whole. It provides information on status compared to peers.
X1MTHID & X2MTHID	This variable is a scale of the sample member's math identity. Sample members who tend to agree with the statements "You see yourself as a math person" or "Others see me as a math person" will have higher values.
X1MTHUTI & X2MTHUTI	This variable is a scale of the sample member's perception of the utility of mathematics; higher values represent perceptions of greater mathematics utility.
X1MTHEFF & X2MTHEFF	This variable is a scale of the sample member's math self-efficacy; higher values represent higher math self-efficacy.
X1MTHINT	This variable is a scale of the sample member's interest in his or her base-year math course; higher values represent greater interest in the base-year math course.
X1SCIID & X2SCIID	This variable is a scale of the sample member's science identity. Sample members who tend to agree with the statements "You see yourself as a science person" or "Others see me as a science person" will have higher values for X2SCIID.
X1SCIUTI & X2SCIUTI	This variable is a scale of the sample member's perception of the utility of science; higher values represent perceptions of greater science utility.
X1SCIEFF & X1SCIEFF	This variable is a scale of the sample member's science self-efficacy; higher X2SCIEFF values represent higher science self-efficacy.
X1SCIINT	This variable is a scale of the sample member's interest in his or her base-year science course; higher values represent greater interest in the base-year science course.
X3TGPAMAT	GPA in Mathematics.
X3TGPASTEM	GPA in STEM courses.
X3TGPAACAD	GPA in Academic courses.
X1SEX	Student's sex
X1RACE	Student's race/ethnicity-composite
X1SES	Socio-economic status composite for base year
X2SES	Socio-economic status composite for first follow-up year

Results

Of the sample of students, there was not transcript data available for 1575 (6.7%) of the students. 19748 (84%) students did not earn any credit in statistics/probability, and 2180 (9.3%) students earned at least one credit in statistics/probability, which in looking back is slightly lower than the 11% of students reported in 2009 NAEP transcript study (NCES, 2016).

Demographic Characteristics

In considering the relationship between students' demographic characteristics and whether or not they earned at least one credit in statistics, the characteristics of sex, race, and SES were investigated. In comparing sex to earning statistics credit, there was no significant association (χ^2=0.538, df=1, p=0.463). The association between race and earning statistics credit was statistically significant (χ^2=17.923, df=7, p=0.012). However, the effect size was small (Cramer V=.083). In looking at the standardized residuals of the observed and expected cell counts the only significant residual (those $\geq\pm1.96$) was for Asian students earning a credit in statistics with z=2.8, meaning there were significantly more Asian students who earned at least one credit in statistics than expected. A further caution is that three of the expected counts where less than five, though that only constitutes 18.8% of the cells, which is below the 20% cutoff that is generally considered acceptable (Field, 2013). Finally, in considering the relationship between SES and earning statistics credit there were statistically significant differences between the groups in both the base and first follow-up years, the results of which can be seen in Table 2. The group differences in SES also had small effect size.

Table 2: Difference in scale variables between students who did not earn any credits in statistics/probability versus those who earned at least one credit in statistics/probability.

	≥1 Stat Cred Mean (SD)	No Statistics Mean (SD)	t	df	Mean Diff (SE)	95% CI of Diff	Effect Size (r)
BY Math theta score	.563(.893)	-.089(.937)	-10.51***	2581	-.652(.062)	[-.773, -.530]	.203
FY Math theta score	1.295(1.102)	.538(1.123)	-10.28***	2581	-.758(.074)	[-.902, -.613]	.198
BY Math Identity	.350(.970)	-.014(1.002)	-5.46***	2548	-.364(.067)	[-.495, -.233]	.108
FY Math Identity	.331(1.047)	-.003(.997)	-5.02***	2510	-.334(.066)	[-.464, -.203]	.100
BY Math Utility	.018(.981)	-.001(1.001)	-.270	2277	-.019(.070)	[-.155, .118]	.006
FY Math Utility	.176(.979)	.018(.991)	-2.40*	2503	-.158(.066)	[-.287, -.029]	.048
BY Math SE	.225(.919)	-.007(1.012)	-3.31**	2269	-.231(.070)	[-.368, -.094]	.069
FY Math SE	.180(1.014)	.004(.996)	-2.64**	2477	-.176(.067)	[-.307, -.045]	.053
Interest F09 math	.277(.992)	-.014(.993)	-4.16***	2227	-.290(.070)	[-.427, -.154]	.088
BY Science Identity	.245(.978)	.006(.993)	-3.62***	2542	-.239(.066)	[-.369, -.109]	.072
FY Science Identity	.248(1.007)	-.008(.997)	-3.87***	2495	-.257(.066)	[-.387, -.126]	.077
BY Science Utility	.071(1.014)	.004(.989)	-.95	2088	-.067(.071)	[-.207, .072]	.021
FY Science Utility	.015(.071)	-.001(.069)	-3.44**	2487	-.016(.005)	[-.025, -.007]	.069
BY Science SE	.217(.991)	-.006(.985)	-3.15**	2079	-.223(.071)	[-.362, -.084]	.069
FY Science SE	.209(.968)	.004(1.005)	-3.05**	2437	-.205(.067)	[-.337, -.073]	.062
Interest F09 science	.127(.996)	-.014(1.009)	-1.93	2040	-.141(.073)	[-.285, .002]	.043
GPA STEM courses	2.889(.722)	2.297(.930)	-9.76***	2576	-.592(.061)	[-.711, -.473]	.189
GPA all courses	3.023(.684)	2.456(.889)	-9.79***	2576	-.567(.058)	[-.680, -.453]	.189
GPA Math	2.811(.810)	2.218(.974)	-9.32***	2576	-.594(.064)	[-.719, -.469]	.181
BY SES Composite	.281(.780)	-.109(.745)	-7.83***	2581	-.390(.050)	[-.487, -.292]	.152
FY SES Composite	.306(.717)	-.0750(.715)	-7.85***	2394	-.381(.049)	[-.476, -.286]	.158

Note. Equal variance assumed for independent t tests. *p<.05, **p<.01, ***p<.001. BY=Base Year, FY=First Follow-up Year.

Beliefs/Attitudes

Looking at the belief/attitude scale score variables (see Table 2); during the base year students who earned at least one credit in statistics/probability had significantly stronger mathematics and science identities, mathematics and science self-efficacy beliefs, and interest in their 9th grade mathematics course than students who did not. This pattern continued for the identity and self-efficacy variables in the first follow-up of the study. There was no difference in the mathematics or the science utility scale scores during the base year of the study. However, at the time of the first follow-up, students who earned at least one credit in statistics/probability had significantly higher mathematics and science utility scale scores than students who did not. It seems that over the course of three years of high school education the population of students who earned at least one credit in statistics/probability by the end of high school, were students who began to view mathematics as more useful on average than students who did not, whose average scale score changed minimally from the base year to the first follow-up year. The same cannot be said in the case of science where the scale score for both populations decreased on average from the base year to the first follow-up year. It is important to acknowledge that there is no way of knowing when students earned at least one credit in statistics, so no inferences should be made that temporal shifts in beliefs and attitudes might be influenced by statistics course taking. It is also important to note that even though a number of the belief/attitude variables differed significantly between the groups, none of the differences had more than small effect sizes, and quite a few were below the r=.1 cutoff for small effect sizes.

Academic Performance

In looking at the mathematics course taking of those students who earned at least one credit in statistics/probability, a striking trend emerges that students taking statistics are largely the top mathematics students. Nearly half (n=1053) of the students earning statistics/probability credit in high school completed Algebra I in 8th grade, just over a quarter took statistics as their most advanced mathematics course in high school, and in fact over a quarter of students took calculus in high school (see Table 3). This trend is also supported by the mathematics performance of students who earned a statistics/probability credit versus those who did not (see Table 2). Students who earned at least one credit in statistics/probability had significantly higher theta scores on the algebraic reasoning assessment on average in both the base year and the first follow-up than students who did not earn a statistics/probability credit. This same trend can be seen in the GPA of students' mathematics courses taken in high school, and in their GPA for STEM courses, and all academic courses (see Table 2). Furthermore, the differences had small to moderate effect sizes. These results seem to indicate that the students who earned at least one credit in statistics/probability may include more than just top performing mathematics students, but perhaps even the top performing students in general.

Table 3: Frequency of students' highest level mathematics course taken for students who earned at least one credit in statistics/probability

X3 Highest level mathematics course taken/pipeline	Frequency of students who earned at least one credit in statistics/probability
No Math	0
Basic math	0
Other math	0
Pre-algebra	0
Algebra I	0
Geometry	0
Algebra II	0
Trigonometry	0
Other advanced math	0
Probability and statistics	697
Other AP/IB math	260
Precalculus	596
Calculus	144
AP/IB Calculus	483
Total	2180

Discussion and Scholarly Significance

Considering the results as a whole the two most significant factors that differed between the group of students who earned at least one credit in statistics and the group of students that did not were academic performance and SES. Though there are other factors that differed with statistical significance between the two groups, they were all of small or less effect sizes and given the large sample size should be considered with caution as only small differences are needed for statistical significance, which may not be significant in a practical sense. One possible reason, for predominantly top academically performing students earning at least one credit in statistics is that they have more time in their schedule to take additional mathematics courses like statistics.

Most states require at least three years of mathematics, which generally includes an Algebra I, Geometry, Algebra II trajectory or a more integrated Math I, Math II, Math III trajectory where the various strands of mathematics are taught together over three years. That means that students who take the typical 9th grade mathematics course of Algebra I or Math I in 8th grade have more time to take statistics as an elective course in their junior or senior year.

The cohort of students in the HSLS:09 were in high school during the very beginning of the influence of the CCSSM (NGA Center & CCSSO, 2010), which does recommend that statistics concepts be included in all students' high school mathematics education. However, the enacted curriculum that students experience is generally carried out by the classroom teacher who mediates the influence that official curriculum, such as the CCSSM, have on the enacted curriculum (Remillard & Heck, 2014) and as I discussed earlier, statistics has a unique position in this regard as many mathematics teachers have often had little to no prior experience with learning statistics (Shaughnessy, 2007). Though the results here do not show that all students do not have access to opportunities to learn statistics, they do point out that there may, at the very least, be an issue in how statistics courses are offered or advertised to students, or perhaps in students' perceptions of the usefulness of statistics, or who should take it. The results do show that the majority of students are not earning any credits in statistics/probability, which means if they are not experiencing statistics in their other mathematics courses, they are not having any experiences with statistics in high school. It is important that more research is done around students' opportunities to learn statistics at the K-12 level especially in the context of the enacted curriculum that students experience and how students are provided access to opportunities to learn statistics at the school level.

A limitation in this analysis is that using the HSLS:09 data it is only possible to determine which students earned at least one credit of statistics/probability in high school. It does not provide information on specific course performance, when the student took the course, or identify those students who were enrolled in a statistics/probability course but did not earn credit, which limits the results reported. Furthermore, it is not possible to determine the quality of the students' instruction or whether or not they had any opportunities to learn statistics in any of their other courses. However, given the current dearth of empirical research looking at the teaching or learning of statistics on any kind of large scale, these results help to give some idea of patterns in statistics course taking, which have implications for further research and for policy. Another limitation is not having transcript data on who earned credit for AP Statistics. However, given that AP courses are advanced courses the inclusion of such data would have likely only amplified the results as the students often taking AP courses are generally high performing and of a higher SES. However, it is still important that future analyses include such data.

Related to the equity framework used in this study, the results reported here are significant in that if the goal of education is democratic equality, preparing students to participate as critical citizen's in society where statistical reasoning is crucial, than there is a serious issue in that it appears that predominantly only the top performing students in high school are earning credits in statistics/probability. It is promising that students' sex was found to have no association with their taking of statistics. However, race does appear to have a weak association and there are significant differences in the academic performance and SES of those who earned credit in statistics and those who did not, which points to inequity in students' access to statistics. Going back to Gutiérrez's (2009) equity framework, it is also important to point out that using this data, equity can only be considered through the dominant axis, which means there is still the issue of the critical axis of equity, namely identity and power, to consider in future work. Though the variables of math and science identity were considered in the HSLS:09, they are different than the identity construct that Gutiérrez (2009) discusses. The identity variables in the HSLS:09 consider an individual's identity relative to its alignment with the discipline, whereas Gutiérrez (2009) is looking at identity from the perspective of the individual and how they see themselves in relation to the discipline and curriculum. It is crucial, that if we are to achieve the promise of equal educational opportunity, that all students should have experiences learning statistics concepts and practices, and that they have experiences in seeing themselves in the curriculum and considering how to read and write the world with statistics (Weiland, 2017).

References

Ben-Zvi, D., & Garfield, J. (2008). Introducing the emerging discipline of statistics education. *School Science and Mathematics*, *108*(8), 355–361.

Cobb, G. W., & Moore, D. S. (1997). Mathematics, statistics, and teaching. *The American Mathematical Monthly*, *104*(9), 801–823.

Field, A. (2013). *Discovering statistics using IBM SPSS Statistics* (4th ed.). London: Sage Publications Ltd.

Franklin, C., Bargagliotti, A., Case, C., Kader, G., Scheaffer, R., & Spangler, D. (2015). *Statistical Education of Teachers*. American Statistical Association. Retrieved from http://www.amstat.org/education/SET/SET.pdf

Franklin, C., Kader, G., Mewborn, D., Moreno, J., Peck, R., Perry, M., & Scheaffer, R. (2007). *Guidelines for assessment and instruction in statistics education (GAISE) report: a pre-K--12 curriculum framework*. Alexandria, VA: American Statistical Association.

Groth, R. E. (2013). Characterizing key developmental understandings and pedagogically powerful ideas within a statistical knowledge for teaching framework. *Mathematical Thinking and Learning*, *15*(2), 121–145.

Gutiérrez, R. (2009). Framing equity: Helping students "play the game" and "change the game." *Teaching for Excellence and Equity in Mathematics*, *1*(1), 5–7.

Gutiérrez, R. (2002). Enabling the practice of mathematics teachers in context: Toward a new equity research agenda. *Mathematical Thinking and Learning*, *4*(2–3), 145–187.

Ingels, S. J., Pratt, D. J., Herget, D. R., Bryan, M., Fritch, L. B., Ottem, R., … Christopher, E. M. (2015). *High school longitudinal study of 2009 (HSLS:09) 2013 update and high school transcript: Data file documentation*. Washington, DC: National Center of Educational Statistics.

Lubienski, S. T. (2015, November 4). What the drop in NAEP math scores tells us – about Common Core and NAEP. *Washington Post*. Washington, DC.

National Center of Educational Statistics. (2016). *The condition of education: High school coursetaking*. Retrieved from http://nces.ed.gov/programs/coe/indicator_cod.asp

National Governors Association Center for Best Practices [NGA Center], & Council of Chief State School Officers [CCSSO]. (2010). *Common core state standards for mathematics*. Washington DC: Authors. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Orrill, C. H. (2016). The process is just messy: A historical perspective on adoption of innovations. *The Mathematics Educator*, *25*(2), 71–94.

Remillard, J., & Heck, D. J. (2014). Conceptualizing the curriculum enactment process in mathematics education. *ZDM*, *46*(5), 705–718.

Schmidt, W. H., & McKnight, C. C. (2012). *Inequality for all: The challenge of unequal opportunity in American schools*. New York, NY: Teachers College Press.

Schmidt, W. H., Wang, H. C., & McKnight, C. C. (2005). Curriculum coherence: An examination of US mathematics and science content standards from an international perspective. *Journal of Curriculum Studies*, *37*(5), 525–559.

Shaughnessy, M. (2007). Research on statistics learning and reasoning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 957–1009). Charlotte, NC: Information Age Publishing.

Steen, L. (Ed.). (2001). *Mathematics and democracy: The case for quantitative literacy*. United States: The National Council on Education and the Disciplines.

Tarr, J. E., Chávez, Ó., Reys, R. E., & Reys, B. J. (2006). From the written to the enacted curricula: The intermediary role of middle school mathematics teachers in shaping students' opportunity to learn. *School Science and Mathematics*, *106*(4), 191–201.

Weiland, T. (2017). Problematizing statistical literacy: An intersection of critical and statistical literacies. *Educational Studies in Mathematics*, 96(1), 33-47.

PROBABILITY AND INDEPENDENCE: A COMPARISON OF UNDERGRADUATES

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Probability and independence are difficult concepts, as they require the coordination of multiple ideas. This qualitative research study used clinical interviews to understand how three undergraduate students conceptualize probability and probabilistic independence within the theoretical framework of APOS theory. One student's reasoning was consistent with a process conception, one student with at least an object conception, and one with a schema conception of probability and independence. Differences in students' thinking are analyzed, with a specific focus on intuition and simultaneously occurring events.

Keywords: Probability; Post-secondary Education

Review of Literature

"Students have difficulty just sorting out the mathematics of whether events are statistically dependent or independent in probability problems" (Shaughnessy, 2003, p. 221). The sentiment of this quotation resounds throughout the literature on probabilistic independence, and it has been well documented for students at varying ages. Considering the definition of independence, for two events A and B, $P(A) = P(A|B)$, there is a clear relationship between conditional probability and independence which students must come to understand. In recognition of the need for students to coordinate these concepts, Tarr and Jones (1997) constructed a framework for understanding how middle school students conceptualize independence and conditional probability. When students' understanding of these two concepts were analyzed within the four levels of this framework, it was determined that students' levels of reasoning regarding conditional probability and independence tend to be the same; this finding advances the apparent connection between conditional probability and independence, and furthermore, implies that students may construct these concepts in tandem.

Undergraduate students' struggles with probabilistic independence have been well documented. Kelly and Zwiers (1988) delineated three common misconceptions undergraduates have with independence: determining whether events are (in)dependent, understanding that dependence does not imply cause, and understanding that (in)dependence is not reliant upon time. Plaxco's (2011) findings support this third struggle. In interviews with undergraduates, Plaxco found that all students alluded to a temporal conception of independence and two of the three students included an element of time in their definition of independence. Students' inabilities to separate an element of time from their understanding of independence is clearly problematic, as the conditioning event may occur at the same time as or after the second event. In response to some of the difficulties students experience in conceptualizing independence, Keeler and Steinhorst (2001) call for the improvement of instruction in undergraduate probability courses, and specifically call instructors to capitalize on students' intuitions. They posit that an inquiry-based learning environment facilitates students' understanding of probability by building upon students' intuitions. Abrahamson (2014) also encourages probability instruction that "guide[s] students to appropriate the cultural resource as a means of supporting and empowering their tacit inference" (p. 250). His findings indicate that students' tacit inferences are a powerful instructional tool that instructors can leverage by linking it to formal mathematics. These researchers build upon the work of Fischbein (1987), who defines students' intuitions as primary or secondary. Primary intuitions are those students develop prior to instruction, whereas secondary intuitions are developed through formal mathematical experience. Secondary intuitions are not right and primary intuitions wrong; rather, secondary intuitions replace primary intuitions in situations where primary intuitions fail. Thus, for both Keeler and Steinhorst, and for Abrahamson, it seems that helping students construct secondary intuitions to complement primary intuitions regarding independence may be a powerful instructional approach.

On the other hand, Ollerton (2015) found that although undergraduate students have an intuitive understanding of independence, they struggle to conceptualize independence in a mathematically appropriate manner. Thus, it remains to be seen to what extent students' intuitions of probabilistic independence are beneficial in a mathematical context. Therefore, the purpose of this qualitative research study is to construct a more comprehensive understanding of how undergraduate students conceptualize both conditional probability and probabilistic independence, including how students use intuition to understand probabilistic independence.

Theoretical Framework

APOS theory (Dubinsky, 1991) utilizes reflective abstraction as the mechanism for learning mathematics, and specifically delineates the process by which individuals mentally construct mathematical schemas by progressing through action (A), process (P), and object (O) conceptions of mathematical concepts. These progressions ultimately formulate a schema (S), which is a "person's own cognitive framework which connects in some way all of the ideas that the individual either consciously or subconsciously views as related to the piece of mathematics" (Mathews & Clark, 2003, p. 3).

An *action* conception is the most rudimentary, in that mathematical tasks are completely external (Mathews & Clark, 2003); therefore, it is necessary for the student to carry out the actions of solving. Once an individual has interiorized the mathematical actions, they are able to act with a *process* conception (Dubinsky, 1991), enabling the individual to perform processes internally and to "reflect on, describe, or even reverse the steps of transformation" (Mathews & Clark, 2003, p. 2). As one's mental constructions become more powerful, processes can be encapsulated into *objects* (Arnon et al., 2014), which are static entities in and of themselves to which actions can be applied (Arnon et al., 2014). Ultimately, a *schema* conception, or a collection of thematized mathematical objects (Mathews & Clark, 2003), is the most powerful, and allows for the coordination of objects either within or between mathematical concepts.

Probability and Independence

The following is a preliminary genetic decomposition (Arnon et al., 2014) used to facilitate the analysis of students' reasoning. Within APOS, students operating with an action conception of probability are likely to calculate probabilities by applying a formula, or by relying upon physical manipulatives or a drawn sample space. As a process, students are able to anticipate the result of a probability problem without physical manipulations, and can therefore compare and reverse probabilities without actions. Once probability is encapsulated into an object, students are able to conceptualize compound and conditional probabilities because simple probability is now a static entity that can be combined (compound) and nested (conditional). As a schema, students can coordinate encapsulated objects within probability, including coordinating multiple representations of compound and conditional probability.

Probabilistic independence is a component of probability, and thus, its construction will be linked with that of probability (Tarr & Jones, 1997). With an action conception of probability and independence, determining independence will be reliant upon a formula with no justification of its use. Anticipation of the results of probability, indicative of a process conception, may allow students to develop an intuition for independence, without explicit calculations. As probability is encapsulated into an object, students begin to conceptualize conditional probability, which facilitates reasoning with regard to the definition of probabilistic independence. Finally, as a schema, multiple representations of probability and independence can be coordinated; this includes contingency tables, formulas, and definitions, to name a few.

Methods and Analysis

The participants in this qualitative study include three undergraduate students at a large, research university in the southeastern United States. The participants were recruited because they were at least 18 years of age and had taken a probability course. All participants were individually interviewed once, for approximately 40 minutes, and were assigned a pseudonym. Interviews were video recorded for the purposes of retrospective analysis. During the interview, each student completed nine questions meant to elicit their understandings of conditional probability and probabilistic independence. Questions six and seven are adapted from Manage and Scariano (2010; Appendix A) and question nine is from the research of Tversky and Kahneman (1980; Appendix A). Additional questions are described in the results section.

The data were analyzed using APOS theory. Accordingly, students' responses to each task were compared to the preliminary genetic decomposition described above. Note that it is possible for responses of students who have constructed more sophisticated conceptions of a mathematical concept to align with the reasoning of a less sophisticated conception. However, students with a less sophisticated conception cannot act in a manner consistent with a more sophisticated conception. For example, a student with an object conception of independence may write out all of the steps to completing a task because they believe they are supposed to show work. To an observer, this may seem to indicate independence is external, or an action, for the student; however, it is not a counter-indication of a process or an object conception. Conversely, a student with only an action conception cannot act in a manner consistent with an object. Thus, student responses throughout the interview were taken as a whole, and the most sophisticated conception of probability and independence observed was attributed to the student.

Results

John

The first participant, John, was a sophomore electrical engineering major who had completed a probability course in the previous semester. When asked to find the probability of rolling a six on a six-sided die given the result is even, John answered two related, albeit incorrect, probability problems: What is the probability of rolling a six? And, what is the probability of rolling an even? With some prompting he realized "Ohhh, oh, oh. *Given* that it's even." On its own, this could be interpreted as a misunderstanding, however, John treated conditional probability problems as two related problems, instead of as one problem with a conditioning event, three out of four times throughout the interview. In question nine he went as far as to say, "So this is, like, two parts." I interpret these responses as John's inability to recognize the need to adjust the sample space of the problem as a result of the conditioning event; and therefore interpret this an indicator that John has not encapsulated probability into an object because he does not nest two probabilities.

In the instances when John did solve conditional probability problems correctly, he relied heavily on a formula. On one question in particular, he applied a formula for conditional probability and when asked to justify its use, said, "so you find (laughs)… I don't know how to explain this… it's just…" An explanation never followed. Later, in response to question nine, John tried to find a relationship between the formulas he had inappropriately applied and the contingency table provided in the problem. He explained:

> I'm thinking this is definitely true (points to his written formula), but I'm thinking maybe some of those (pointing to numbers in the contingency table) could actually give me the answer. … doesn't (as he checks one number in the contingency table against his formula), doesn't (checks a second number), doesn't (checks a third). No, it doesn't.

John's conclusion is that the numbers in the contingency table were unrelated to the formula. This speaks to his inability to conceive of the relationship between the two events within the task, which is a counter-indication of constructing an object conception of probability and independence because he does not justify the nesting of two probabilities. This is also a counter-indication of a schema conception because he does not relate multiple representations of conditional probability.

Although John relied heavily on formulas to reason about conditional probability, when reasoning about independence, he used formulas to justify his responses, not the other way around. In a question about the probability of being dealt two different sets of cards, he indicated that all cards dealt were independent of one another, and later used calculations to explain this. John went as far as to say, "just by intuition, they're the same," meaning the two hands are equally likely to be dealt. What John termed intuition, I consider to be his anticipation of the result, and therefore, evidence of at least a process conception of probability and independence. An area in which John continued to struggle with respect to independence was when two events were occurring simultaneously. In question seven, events C and D occur on a single roll of one die. John intuited these to be dependent events (incorrect), however, in justifying this response with a formula, he found them to be independent (correct). This was the only question regarding independence on which John's intuition led him astray. This was also the only question in which one event could not be interpreted to occur before the other event. Ultimately, John explained, "I believe in the formula. The results of the formula, they can't be false. … [but] if you just read it, not thinking about the formula, it would seem like they're kind of dependent." In this situation John struggled to discern independence from dependence as a seeming result of temporal reasoning; this limitation indicates that John has not coordinated conditional probability with independence in situations involving simultaneous events.

I attribute to John a process conception of probability and independence; that is to say, he has not yet encapsulated probability into an object upon which he can perform actions. John was able to apply formulas for probabilistic independence as a justification for his reasoning, not as a substitution for his reasoning. This is evidence of a process conception. However, John did not consistently adjust the sample space to account for the nesting of probabilities in conditional probability problems, which is a counter-indication of him having constructed an object conception. Furthermore, he did not demonstrate an understanding of the relationships between contingency tables and conditional probability, nor could he coordinate conditional probability with independence when events occurred simultaneously. These limitations are counter-indications of John having constructed a schema conception of probability and independence.

Dan

The second participant, Dan, was also a sophomore electrical engineering major, and was in the final three weeks of his probability course. In all conditional probability questions, Dan immediately recognized the need to adjust the sample space to account for the conditioning event. When asked to find the probability of rolling a six given the result is even, Dan explained, "If it's even it's going to be one-third because there's one possibility and there are three times it can be even – two, four, six. You want it to be six, so the number is one over three." This cogent justification implies that for Dan the probabilities of rolling a six and rolling an even are static objects that he acts upon by nesting them; this indicates at least an object conception of probability and independence.

He also coordinated his understanding of conditional probability with contingency tables. In question nine, for instance, he visually demonstrated this apparent coordination when he circled one column on the contingency table and said, "So it's gonna be this right there." He proceeded to verbalize the relationship between the table and the formula for conditional probability. This is evidence that Dan has constructed a schema conception of probability because of the coordination between multiple representations within the mathematical concept.

With regard to probabilistic independence, Dan relied upon his intuitions to anticipate the results of events within a set without performing calculations. In response to question six, he said, "I mean, there's only three possible outcomes. I'm just thinking about it … so that's the same for all of these … they can't be independent." When asked if he was comparing the probabilities of each event occurring, Dan repeated that he was "just thinking about it." This is evidence of his use of intuition, rather than mental calculations, which reinforces the indication of at least a process conception of probability and independence.

Dan's anticipation of probabilistic independence failed him, however, when responding to question seven. His intuition led him to believe that the simultaneous events occurring on a single die should be dependent, but upon calculation, he determined them to be independent. Reflecting on his calculations, he said, "It doesn't make sense. I'm just trying to picture this as a single event. It's not like you're saying this happened after the other event. … I just feel like it shouldn't be independent because they're happening at the same time." Dan spent several minutes trying to think of and explain other situations of independence as a means of eliminating his cognitive conflict, but was unsuccessful; he remained perturbed by the idea of independent events resulting from the roll of a single die. This is evidence of Dan not having coordinated objects of independence and probability in the case of simultaneous events, which is a limitation of his conception, and a counter-indication of him having thematized these objects into a schema. Considering probability tasks alone, it is possible to attribute to Dan a schema conception. However, when considering in tandem his reasoning with regard to probabilistic independence, it seems inappropriate to attribute to him the same conception. Based on this evidence, it is appropriate to attribute to Dan at least an object conception of probability and independence.

Aaron

The third student, Aaron, was a sophomore mathematics and economics major; he had completed AP Statistics in high school and received college credit for the course. Aaron calculated conditional probabilities correctly on all appropriate tasks, and did so largely mentally. In each of these questions he adjusted the sample space correctly in response to the conditioning event. In question nine, for instance, he explained that the sample space was changing from 1,000 taxis to 290 taxis by saying, "You add up how many times he actually says blue, which is the 290, and then of those times it's actually blue only 120. … That's (points to 120/290) the probability that the cab is blue given that he said blue." This excerpt exemplifies Aaron's ability to operate on one probability nested within another. When asked to explain his response to question nine, Aaron also constructed a tree diagram, and stated it was better organized than the table. He also clarified, "This is just what the table was telling me." As a whole, this is evidence that Aaron can coordinate formulas, contingency tables, and tree diagrams in reasoning about conditional probability.

In response to an independence task in which a set of people's behaviors and socio-economic statuses were given in a contingency table, Aaron calculated P(A|B) in determining whether event A depends on event B. When asked a follow-up probe (is event B dependent upon event A), he said "Maybe that's something where it's always if it goes one way [A depends on B] then it goes the other [B depends on A]. No, that isn't right." When asked how he arrived at this conclusion, he did not describe performing any calculations; instead, he described reasoning that P(A|B) is not equivalent to P(B|A), which is evidence of using a mental structure to anticipate the results of conditional probability. Aaron's ability to fluently solve conditional probability problems, to compare and reverse the conditioning events, and to reason using multiple representations, are all indicators of at least an object conception of probability. Moreover, his ability to compare and reverse conditioning events indicates he is likely operating on probabilities as static entities in their own right; for this reason, it seems likely that Aaron has constructed a schema conception of probability.

To further examine the schema attribution, it is important to consider his means of operating with regard to probabilistic independence. Aaron began the interview by explaining that probabilistic independence "means that the knowledge of one of them [events] happening shouldn't affect the likelihood of the other one happening." His metaphor of "dependence as knowing" was recurrent; I interpret his use of this metaphor as a mental structure that Aaron engaged to anticipate the independence of events. He used this metaphor again in question six: "you know that if you win, you don't lose and so the information of winning would help you figure out whether or not you lose. … they are dependent." Here, his metaphor of dependence as knowing includes the idea that "information" about winning provides knowledge, and therefore, implies dependence. Aaron's metaphor of dependence as knowing facilitates the coordination of independence with conditional probability, as demonstrated on question seven:

> If you know you rolled C [1 or 3], it doesn't actually help me know how likely it is that you rolled D [1 or 4], it just lets me know that it's more likely that you rolled a one… that means these would be independent… And then it goes the other way. If I knew about event D I wouldn't know any more about event C happening.

Aaron reasons about independence by explaining that knowing event C occurred doesn't give him more knowledge about whether event D occurred. Aaron does not struggle with the idea of events C and D occurring simultaneously because he coordinates conditional probability with probabilistic independence by applying his metaphor; this is evidence of a schema conception.

Discussion

John, Dan, and Aaron demonstrated different levels of reasoning; John's being consistent with a process, Dan's with at least an object, and Aaron's with a schema. A major difference between John's and Dan's conceptions is that Dan had encapsulated probability into an object upon which he could act, making it possible for him to appropriately interpret conditional probability. Dan understands the conditioning event to require an adjustment of the sample space, and justifies the use of probability formulas in his reasoning. John could not nest probabilities. Fischbein and Gazit (1984) found that difficulty reconstructing sample space in conditional probability is widespread, and this was experienced by John. Fischbein and Gazit's findings may be explained through the lens of APOS, as it appears that students must objectify probability to appropriately reconstruct sample space. This is an area for future research.

The main difference between Dan’s and Aaron’s reasoning was regarding independence. Aaron was able to coordinate probabilistic events as static entities and act upon these events to determine independence regardless of a temporal connection. The task on which this difference was the most apparent was task seven, in which two dice are rolled simultaneously. On this task, Dan’s object conception did not allow him to coordinate multiple objects (probabilities) occurring simultaneously, whereas Aaron’s schema conception did. It is possible that Dan was relying on primary intuitions (Fischbein, 1987) in this situation, whereas Aaron had constructed secondary intuitions that allowed him to intuit results in the situation of simultaneously occurring events. However, the role of simultaneously occurring events in students’ constructions of independence requires further empirical consideration; specifically, why this difficulty limits students’ coordinations of probability and independence, and how a perturbation, such as Dan’s, can be exploited in instruction to engender the construction of more sophisticated conceptions. Abrahamson (2014) indicates that probability instruction should link students’ intuitions to analytic reasoning and empirical activities with regard to probability. Perhaps by helping students construct these links, they will be engendered to construct secondary intuitions to supplement their primary intuitions; without secondary intuitions, it seems as though students may be restricted to thinking about independence as reliant upon time.

Interestingly, Aaron had the most sophisticated conception of probability and independence among this group of students, and was the only student to rely on a metaphor for determining probabilistic independence. Sfard (1994) argues that “reification is, in fact, the birth of a metaphor which brings a mathematical object into existence and thereby deepens our understanding” (p. 54). With this understanding, it stands to reason that Aaron’s metaphor of dependence as knowing was born out of his thematization of probability and independence into a schema, thereby allowing him to more meaningfully intuit independence. Aaron’s use of metaphor was an unexpected result in this research study, and it remains to be seen whether reasoning with metaphor with regard to probabilistic independence advantages students’ reasoning in some way over the intuitive reasoning engaged by the other two students.

Conclusions

As is frequently noted in the literature, students’ abilities to determine the independence of probabilistic events is problematic. Although these undergraduates were all STEM majors with formal instruction in probability and independence, each demonstrated a different conception. With a process conception, John was significantly limited; moreover, his intuitions regarding independence were primary (Fischbein, 1987). Similarly, Dan relied on primary intuitions in the situation of simultaneous events. Although Dan had constructed at least an object conception of probability, he had not constructed secondary intuitions regarding independence involving simultaneous events. John’s and Dan’s intuitions advantaged their reasoning in different ways, demonstrating that not all intuitions are equally beneficial. Future research should examine the conception of probability and independence that supports students’ constructions of secondary intuitions, specifically with regard to temporal reasoning.

The preliminary genetic decomposition utilized in this research is preliminary in that it requires refinement, and the varying reasoning of these students can direct these revisions. First, intuition was included as an indicator for the construction of a mental structure for independence, and thus, a process. While this was appropriate for John, Dan’s intuitions were more sophisticated. As a result, intuition needs to be more thoroughly examined within the APOS framework. Furthermore, these results indicate that even primary intuitions can be beneficial to students’ reasoning, which supports leveraging students’ intuitions in instruction. By capitalizing on primary intuitions, perhaps researchers can identify the mental constructs that support the construction of secondary intuitions and schemas, and instructors can begin to engender these constructions in their classrooms.

References

Abrahamson, D. (2014). Rethinking probability education: Perceptual judgment as epistemic resource. In E. J. Chernoff & B. Sriraman (Eds.), *Probabilistic thinking: Presenting plural perspectives* (pp. 239–260). New York, NY: Springer.

Arnon, I., Cottrill, J., Dubinsky, E., Oktac, A., Fuentes, S. R., Trigueros, M., & Weller, K. (2014). Genetic decomposition. In *APOS theory: A framework for research and curriculum development in mathematics education* (pp. 27–55). New York, NY: Springer.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In Advanced mathematical thinking (pp. 95–126). Netherlands: Springer.

Fischbein, E. (1987). *Intuition in science and mathematics: An educational approach*. New York, NY: Kluwer.

Fischbein, E., & Gazit, A. (1984). Does the teaching of probability improve probabilistic intuitions?: An exploratory research study. *Educational Studies in Mathematics, 15*, 1–24.

Keeler, C., & Steinhorst, K. (2001). A new approach to learning probability in the first statistics course. *Journal of Statistics Education, 9*(3). Retrieved from www.amstat.org/publications/JSE/v9n3/keeler.html

Kelly, I. W., & Zwiers, F. W. (1988). Mutually exclusive and independence: Unraveling basic misconceptions in probability theory. In R. Davidson & J. Swift (Eds.), *Proceedings of the Second International Conference on Teaching Statistics* (pp. 96–100). Victoria, BC: University of Victoria.

Manage, A., & Scariano, S. (2010). A classroom note on: student misconceptions regarding probabilistic independence vs. mutual exclusivity. *Mathematics & Computer Education, 44*(1), 14–21.

Mathews, D. & Clark, J. (2003). Successful Students' Conceptions of Mean, Standard Deviation and the Central Limit Theorem. Unpublished paper. Retrieved October 20, 2007, from http://www1.hollins.edu/faculty/clarkjm/stats1.pdf

Ollerton, R. L. (2015). A unifying framework for teaching probability event types. *International Journal of Mathematics Education in Science and Technology, 46*(5), 790–794.

Plaxco, D. (2011). The temporal conception: Student difficulties defining probabilistic independence. In L. R. Wiest & T. Lamberg, (Eds.) *Proceedings of the 33rd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 276-283). Reno, NV: University of Nevada, Reno.

Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics, 14*(1), 44–55.

Shaughnessy, J.M. (2003). Research on students' understanding of probability. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 216–225). Reston, VA: NCTM.

Tarr, J. E., & Jones, G. A. (1997). A framework for assessing middle school students' thinking in conditional probability and independence. *Mathematics Education Research Journal, 9*(1), 39–59.

Tversky, A., & Kahneman, D. (1980). Causal schemas in judgment under uncertainty. In M. Fischbein (Ed.), *Progress in social psychology* (vol. 1, pp. 50–72). Hillsdale, NJ: Lawrence Erlbaum.

Appendix A

6) Assume a competitive game can end for team A in a win (P(win) = 0.4), loss (P(loss) = 0.5), or tie (P(tie) = 0.1). Are these events pairwise independent or dependent?

7) Roll a single, fair, four-sided die once and observe its upper face. Define two events: "C: Rolling either a 1 or a 3" and "D: Rolling either a 1 or a 4." Are these events independent or dependent?

9) A cab was involved in a hit and run accident at night. There are two cab companies that operate in the city, a Blue Cab company and a Green Cab company. It is known that 85% of the cabs in the city are Green and 15% are Blue. A witness at the scene identified the cab involved in the accident as a Blue cab. The witness was tested under similar visibility conditions and made the correct color identification in 80% of the trial instances. What is the probability that the cab involved in the accident was a Blue Cab rather than a Green one?

**note that a contingency table was also provided to participants for question 9.*

POSITIVE INTERDEPENDENCE THROUGH DATA MODELING

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This paper describes work to integrate statistics and data modeling deeply into weekly routines of classroom instruction, and to use data modeling practices to support positive interdependence at small-group and whole-class levels. This research pursued a local problem of practice identified by a collaborating teacher: how to promote her students' individual and collective agency in respect to their achievement. Our hypothesis was that designing, creating, interpreting, and revisiting anonymized data representations of aggregated assessment information could offer a context for classroom groups to formulate, guide, and monitor various efforts to improve. Here, we report preliminary findings from work to foster relations among data, representations, conceptual structures, and collective agency in these classes. We also discuss implications of these early findings for researchers in data science education.

Keywords: Data Modeling, Positive Interdependence, Classroom Assessment, Representations

How can students in mathematics classes develop a culture of positive interdependence? Although research has shown that shared action and responsibility in classrooms can be powerful in supporting students to learn (Deutsch, 1962; Johnson & Johnson, 1989), we also know that mathematics classes create individualism and competition (Johnson & Johnson, 1999). Here we explore how learners' creation, negotiation, and use of aggregate representations of *classroom assessment data* can drive effective individual and collective action to improve their learning. This follows a line of work that leverages student-generated mathematical objects in aggregate representations to support participation in discourse and reasoning within the classroom community (Wilensky & Stroup, 1999; Abrahamson & Wilensky, 2004; Brady, White, Davis & Hegedus, 2013). We seek to build more visible relations between the social structures in the classroom that support or undermine positive interdependence and the conceptual structures that underpin the data generated by assessment tools during regular classroom routines. Relations among students' collective and individual perceptions as learners are deeply connected to the ways their learning is measured and represented, and how these data are used for interpretation. Yet we know that there are significant conceptual barriers for students when reasoning about data in aggregate. For example, students need deliberate support to move from case-based thinking about data to seeing aggregate "collective" data shape (Petrosino, Lehrer & Schauble, 2003). With this research in mind, we sought to capitalize on these strategies as resources for collective learning. In this paper we describe initial findings about design conjectures guiding our work, and identify opportunities that we are discovering and refining in ongoing work to map the *mathematical structures* behind assessment data and *social structures* in the class onto one another to promote interpretive discussion (cf., Stroup, Ares, & Hurford, 2005). Given the importance in data science of flexibly connecting information and interpretations about individuals and aggregates, we believe such tools are particularly powerful here.

Study Design and Data Collection

The data for this analysis have been generated in an ongoing year-long design-based research study (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) conducted in collaboration with a 6th grade teacher in a racially and economically diverse urban public school in the Southeastern US. This teacher had been participating in a larger professional development project to improve middle-grades statistics instruction, and this study emerged to address a problem of practice that she identified: supporting her students in making sense of weekly quiz data. This teacher has been teaching for less than 5 years with a master's degree in education, and she is highly motivated with a strong commitment to supporting her students. Our collaboration has led us to focus on designing tools and routines to help students (a) see their grades as products of a coherent system; and (b) better navigate data aggregations at multiple levels: responses to quiz questions, individual quiz scores, individual report card scores, classroom aggregates of quiz scores, and classroom aggregates of quiz item responses.

We collected data from each of this teacher's three classes. The school has labeled two of the classes as *regular* 6th grade math classes and one as *advanced*. Many of the students in the classes had historically experienced little success in school mathematics, but the teacher is committed to supporting them to experience success in her class. The school's assessment routine requires students to evaluate assessment scores weekly, to create learning plans to improve, and then to "re-take" comparable assessments.

Over the Fall, we integrated ourselves into the classroom community, with the goal of becoming familiar resources for the students and getting to know the culture and rhythms of the three classes. We participated in class on Mondays as students examined quiz and discussed the instructional "map" for the week. After becoming integrated into regular classroom activity, we began collecting written student artifacts and video records of classroom conversations about assessment scores on each Monday. We recorded conversations with one stationary camera at the back of the room. We qualitatively analyzed classroom videos of conversations with an eye toward students using data representations to make inferences about their learning, or about the mechanisms that created the assessment data. We also looked for ways students coordinated different representations of data aggregations with social formations of collective responsibility.

Results

This work is ongoing design research. As such, some of our most valuable research products are the evolving *conjectures* we are iteratively testing and refining (c.f. Sandoval, 2014) and the *tools and routines* that we are developing, implementing, and refining as we build a theory of the activity we are working to support. Here we describe some of these conjectures and tools.
Our early work in this classroom highlighted that most students were unfamiliar with the mathematics underlying their grades (e.g. averaging), and in the face of an inscrutable process of converting quiz and other grades to a report-card grade, many substituted morally-charged explanations for these grades. These explanations often implied self-critiques, such as not having studying hard enough or not doing their homework well. However, these critiques did not appear to be concretely actionable, and learners were unable to take small steps to improve.

Conjectures: Challenges in Reasoning about Aggregate Data and Collective Activity

Our aim in supporting reasoning about classroom assessment data has been to foster concrete actions that build (and build *on*) connections between individuals and aggregates in both the mathematical and social spheres. Below is a list of current conjectures guiding our design: (1) Connecting individual and aggregate levels, and related changes in those levels, would be challenging in both mathematical and social dimensions, and creating representations that support students to *link* these domains would support fruitful analogous reasoning across the mathematical and social. (2) Exhortations to make visible changes to aggregate quiz data would spur reasoning about concrete social actions that could be taken at the whole-class level. (3) Seeing changes in aggregate data representations could support a social interpretation as feedback about success in collective action, and data representations would increasingly become tools to motivate, guide, and refine social action over time. (4) Shared experiences of success would promote a sense of individual and collective agency and responsibility, extending beyond the narrow bounds of quizzes to the broad enterprise of classroom learning.

Tools: Supporting Varieties of Reflection and Analysis

In supporting students reasoning about actions to change performance on quizzes we have developed two broad forms of data representation. The first is closely related to a standard "dot" plot: it built upon the students' early efforts to order and structure data captured on square post-it notes and arranged on the class whiteboard during interpretive discussion. This tool has been useful for promoting students' ability to read data displays, reason about distributions and clumping, and argue for "summative" measures to answer the question "Did we improve?" For instance, proposals for evidence of improvement included: Fewer 0s; More 100s; More people above the pass/fail cut score of 70; More people going up than down, and so forth. In addition, the use of *Tinkerplots* provided opportunities to trace changes in data across linked representations. For example, students traced those that scored "0" in the quiz to observe their score on the retake as potential site for evidence about collective improvement. On the other hand, large numbers of absences complicated not only claims about individual improvement but also inferences and claims at the class level, a subtle and group-worthy data modeling topic.

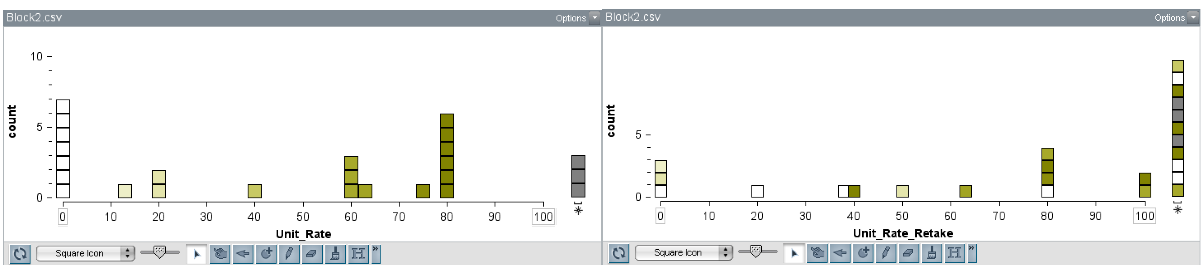

Figure 1. Post-It Inspired Dot Plot displays for a quiz (left) and a retake (right).

However, there have also been limitations to the discussions supported by the dot plot representations. Although we wanted to foster discussions about concrete actions that the group could take to improve, students often left these conversation with morally charged interpretations that were similar to our initial observations. Thus, we developed a second tool, supporting students in disaggregating quiz scores into item level data, and then aggregating that item level data to the class level in a non-standard representation that we have named DIGI/WIGI ("**D**id **I** **G**et **I**t/**W**hy **I** **G**ot (or didn't get) **I**t"). Students record their score on each (dichotomous) item on the quiz, and then describe the mathematical knowledge they used (for correct answers) or would need to have (for incorrect ones). Students have used DIGI aggregations (see Fig 2) to explore counts (which detected data-entry errors), assess relative item difficulty, and identify mathematical gaps for the class to work on together. Our use of this form suggests it can help to shift the focus of conversations from only *student quiz scores* to include comparative group-level performance on *items*. We have seen preliminary evidence that this shift may support more meaningful and concrete interpretations of assessment data.

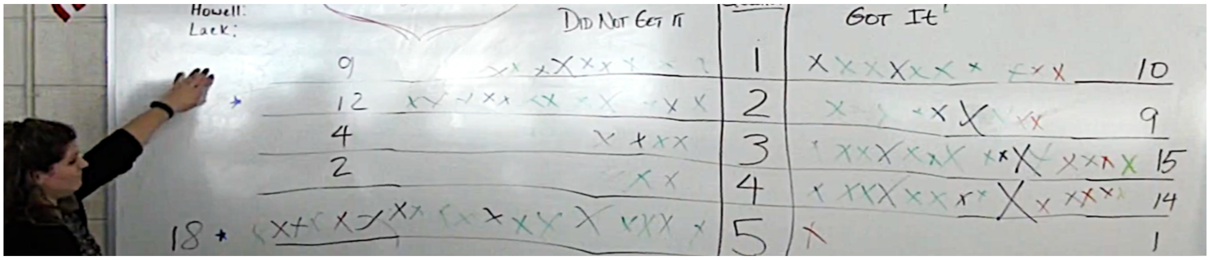

Figure 2. Aggregation of DIGI data for a 5-question quiz.

Discussion

Data science education can help us understand how to apprentice learners in creatively making sense of themselves and phenomena in the wide, information-saturated world (e.g. Lehrer & Schauble, 2002; Taylor & Hall, 2013). Nonetheless, it is important for this emerging field *also* to attend to quotidian activities and data systems that significantly impact students' identities, learning opportunities, and lived experiences. In this paper, we illustrate the rich learning opportunities afforded by mundane but periodic, small-scale classroom assessment data. The meanings students generate from data are constructed in relation to local goals, routines, and conceptual systems. We are exploring how learners' individual and collective relations to classroom assessment data can support them in flexibly constructing meaning with representations. Reciprocally, we are exploring how building interpretations about aggregate data representations can drive individual and collective action to effect improvements detectable by those representations. Although weekly quizzes may seem prosaic, we are finding evidence that students can become empowered to operate more critically, intentionally, and effectively as individuals and groups in this familiar but consequential context.

References

Abrahamson, D., & Wilensky, U. (2004, July). SAMPLER: Collaborative interactive computer-based statistics learning environment. In *Proceedings of the 10th International Congress on Mathematical Education* (pp. 4-11).

Brady, C., White, T., Davis, S., & Hegedus, S. (2013). SimCalc and the networked classroom. In The SimCalc Vision and Contributions (pp. 99-121). Springer, Dordrecht.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9-13.

Deutsch, M. (1962). Cooperation and trust: Some theoretical notes. In M. R. Jones (Ed.), *Nebraska symposium on motivation* (pp. 275–319). Lincoln, Nebraska: U Nebraska Press.

Johnson, D. W., & Johnson, R. T. (1989). *Cooperation and competition: Theory and research.* Edina, MN: Interaction Book Company.

Johnson, D. W., & Johnson, R. (1999). *Learning together and alone: Cooperative, competitive, and individualistic learning* (5th Ed.). Boston: Allyn & Bacon.

Lehrer, R., & Schauble, L. (2002). *Investigating real data in the classroom: Expanding children's understanding of math and science*. New York, NY: Teachers College Press.

Petrosino, A. J., Lehrer, R., & Schauble, L. (2003). Structuring error and experimental variation as distribution in the fourth grade. *Mathematical Thinking and Learning*, *5*(2-3), 131-156.

Sandoval, W. (2014). Conjecture mapping: An approach to systematic educational design research. *Journal of the learning sciences*, *23*(1), 18-36.

Stroup, W., Ares, N., & Hurford, A. (2005). A dialectic analysis of generativity: Issues of network-supported design in mathematics and science. *Mathematical Thinking and Learning, 7*(3), 181-206.

Taylor, K. H., & Hall, R. (2013). Counter-mapping the neighborhood on bicycles: Mobilizing youth to reimagine the city. *Technology, Knowledge and Learning*, *18*(1-2), 65-93.

Wilensky, U., & Stroup, W. (1999). Learning through participatory simulations: Network-based design for systems learning in classrooms. *Proceedings of Computer Supported Collaborative Learning (CSCL'99)*. Stanford, CA.

CONSTRUCTING INFORMAL VOCABULARY: A MIDDLE-SCHOOLER'S PATH TO PROBABILISTIC REASONING

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Our research investigates the probabilistic reasoning of middle-school students making sense of sample space, distribution, and long-run likelihood. Eight students participating in a 3-week summer class on probability participated in games and activities that focused on concepts from the Common Core State Standards for grades 6-8. These activities utilized both concrete and simulated environments. Through engagement with the games and activities, we captured seeds of probabilistic reasoning that emerged with one student in particular, Evan, and took note of the vocabulary he and the class created to make sense of seemingly paradoxical situations. Implications for instruction on probability are briefly discussed.

Keywords: Probability, Interactive Simulations, Distribution, Sample Space

The Common Core State Standards have called for instructional emphases in probability and statistics in the wake of a data-driven society (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA & CCSSO], 2010). However, probability has proved notoriously difficult to teach, due in part to its varying colloquial uses as both something subjectively felt and something objectively calculated (Batanero, Chernoff, Engle, Lee, & Sánchez, 2016). Conflation is furthered through the term "randomness" due to nuanced distinctions in its use in statistics, mathematics, and everyday life (e.g., random sampling, random variable, randomly occurring) (Jones, 2006). The integration of probability in the middle-grades standards puts a considerable burden on teachers who report low self-efficacy for teaching probability and statistics (Chernoff & Zazkis, 2011; Lovett & Lee, 2017).

This report examines middle-school students' experiences grappling with probabilistic concepts, highlighting how one student's reasoning emerged. We discuss the role of interactive simulations in his development of probabilistic reasoning and the class' use of vocabulary to name these nuanced distinctions of probability and randomness. Our results also offer insight for instruction and the importance of student construction of vocabulary.

Conceptual Framework

This report investigates what role students' informal vocabulary played in developing greater understanding of probabilistic concepts. We use probabilistic reasoning as a lens for recognizing how these spaces connect. Savard (2014) states that probabilistic reasoning involves considering both variability and randomness. Variability is a consideration of what is possible by considering an outcome in terms of a distribution and a sense of how those results should manifest over the long-run. Randomness involves conceiving probability as non-deterministic, meaning we cannot ascribe certainty to outcomes. We view students' grappling with variability and randomness as the basis for probabilistic reasoning and an avenue for making sense of unfamiliar situations.

We emphasize a constructivist perspective on students' development of probabilistic reasoning that seeks to identify the ways students orient themselves to probabilistic situations and draw on their own experiences to make sense of these phenomena. The literature notes the seemingly paradoxical nature of many probabilistic situations (e.g., Batanero et al., 2016). For that reason, we are wary to assess students' progression of learning on this topic in terms of adopting and articulating rule-based responses. Rather, we keep in mind considerations offered by Chernoff and Zazkis (2011), who define a desirable pedagogical approach as "one that uses the learner's ideas as a starting point" (p. 19). Our methodology and analysis kept this point central as we sought to identify what students were bringing to this topic rather than their take-up of certain instructional ideas. We ask: How do middle-school students' use of vocabulary help them reason probabilistically about sample space, distribution, and long-run likelihood?

Methodology

Setting

This work stems from a research project studying students and teachers' use of PhET interactive simulations (sims) in middle-school mathematics (https://phet.colorado.edu/). The original intent of this summer class was to test drive elements from a two-week probability sequence that the research team had created for 7th grade involving the PhET sim "Plinko Probability." An instructional sequence was created prior to the summer class, with revisions being made after each lesson to adjust for students' progress and pressing questions. Eight students who were rising 6-8th graders participated for at least half of the classes. The students were gender and ethnically diverse. Each class was co-taught by us (the authors), with each class taking place at a local library branch and lasting 90 minutes.

Our findings focus on Evan, a rising 8th grader who reported "disliking" mathematics on an entrance questionnaire. We highlight Evan because he was present for more classes than any other student (11 out of 12) and elicited noteworthy probabilistic reasoning that did not often lead to correct conclusions. However, these episodes of reasoning were conceptually complex and led to important discussions that led to the creation of more refined vocabulary.

Data Collection and Analysis

Data sources include video recordings from each of the 12 classes, student work, including activity sheets and posters, student journals, and pre- and post-assessments. This preliminary report identifies examples of Evan's probabilistic reasoning embedded in brief narratives that help explain activity elements that facilitated his reasoning. Because of the instructional setting, we were able to implement activities and learning opportunities that responded to students' reasoning from day to day, allowing us to test explanations we were developing. Across the instructional sequence, students' vocabulary emerged as central to facilitating opportunities for probabilistic reasoning. We report two sequences from the class, highlighting how Evan's reasoning developed and made sense of class vocabulary when encountering new ideas.

Findings

Sloppy and Unsloppy Sample Spaces

After two introductory days of ice-breakers, administrative details, and a fraction review, students spent the rest of the first week immersed in the probability of spinners. When reasoning about the probability of landing on a certain space with an equally spaced spinner, students were all comfortable using a *how-many-out-of-how-many* approach (e.g., there is a 3/8 probability of landing on a space with a star since 3 spaces out of 8 have stars). When the size of the spaces on the spinner were no longer equal, students initially tried to adapt the how-many-out-of-how-many approach before recognizing that it was different. Students decided to describe this as a "sloppy" sample space, which they defined as a sample space with unequally likely outcomes. Figure 1 presents the poster that Evan's group made to describe this situation. This designation of sample spaces was an important step in distinguishing how variability can be apportioned differently across sample spaces—a stepping stone into thinking about distributions.

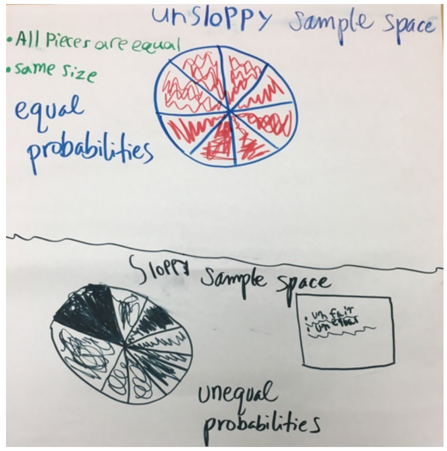

Figure 1. Student Poster for "Sloppy" Sample Space

Bountiful and Scarce Data

When reasoning about the sample space of rolling one die, Evan and others quickly described this sample space as unsloppy, with each of the options 1-6 being equally likely. However, when we discussed the probabilities associated with the sum of rolling two dice, students were initially divided. Jada felt it would be unsloppy: "I think they'd still be equal, even if you have two of them, because the possible numbers would still be the same." Evan claimed that the sum of two dice was a sloppy sample space: "To get a 5, you can get there a few different ways, but there's only one way to get a 2." Next, each group rolled a pair of dice 75 times, agreeing as a group that the sum of two dice appeared sloppy. Evan responded, "we didn't even get a two…in order to be unsloppy, you need to get an even amount of each number." The teacher then probed this idea: "If it wasn't exactly even, what if I got like six 2's, five 3's, five 4's, eight 5's, four 6's, like if it was just a little bit bumpy, would that be okay?" Evan said no, adding that "Unsloppy's perfect."

We followed up on this question of perfectly even by asking the class what they thought would happen if we threw just one die 75 times and recorded results. Evan changed his mind, now believing this might also be sloppy. After recording our results, the distribution was quite bumpy, with considerably more 5's than any other value. Students posed several ideas, claiming that the dice were either rigged, unreliable, or actually more likely to land on certain sides due to the distribution of dots. Evan viewed this as evidence for a sloppy sample space. We gathered 75 more rolls and continued to find bumpy results, further reinforcing students' notions that the sample space was quite sloppy. When the teacher reminded students that they had all agreed the theoretical probability was 1/6, Evan posed the idea that we could think of there being a theoretical sample space and an experimental sample space that could be considered together.

We want to highlight Evan's reasoning here because at one level, it could be easily dismissed as incorrect, but at another, it is quite complex and resonant with Bayesian approaches (Chib, Clyde, Woodworth, & Zaslavsky, 2003). Evan was cognizant of a theoretical (prior) distribution, while also being responsive to experimental results. The next day, we wrestled with how to balance these two notions. We began with a presentation about sloppy and unsloppy, but we introduced a new idea that emerged: having few results and tons of results (i.e., an experimental and theoretical distribution). Using online dice simulations, we revealed a more complete picture of rolling one and two dice many times. Evan used "bountiful" to describe when lots of data was collected, with "scarce" describing when only a few results were collected. This vocabulary helped Evan and others consider underlying variability in tandem with randomness as a factor.

Discussion

Our findings highlight two layers of vocabulary that, together, helped students make sense of rather complex notions involved in reasoning about probability. These vocabulary layers assisted students as we ventured into the last segment of the course, long-run likelihoods. Using both a coin-flipping game (involving making left and right turns based on the flip) and Plinko Probability, students were able to connect their vocabulary from the dice activities with new situations to conceive of both an underlying theoretical distribution and the randomness of experimental outcomes. Figure 2 displays how students were able to make sense of balls bouncing down the plinko peg-board based on theoretical probabilities, but could also explain why only running a handful of balls down the board resulted in clumpy results using the bountiful/scarce language. This combination of vocabulary layers led to productive probabilistic reasoning by attending to both distribution and randomness in the data. There are instructional implications from these findings, including the importance of leaving room for student ideas and names, even when these names are not officially in the curriculum. We hope to explore these ideas further by implementing an updated instructional sequence in a 7th grade classroom.

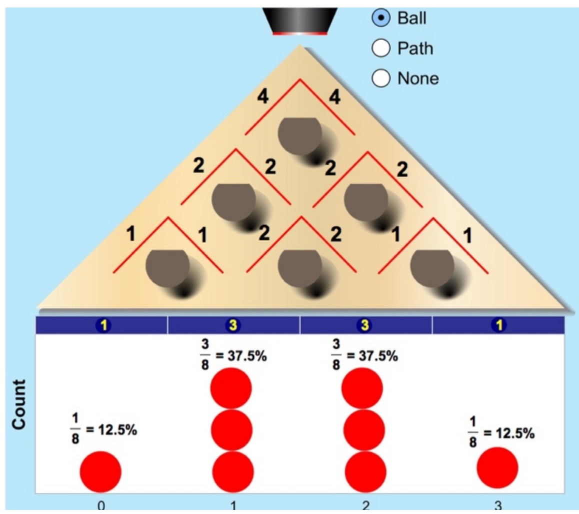

Figure 2. Reconciling Distribution and Randomness through Path Likelihoods

References

Batanero C., Chernoff E.J., Engel J., Lee H.S., Sánchez E. (2016) Research on Teaching and Learning Probability. In: *Research on Teaching and Learning Probability*. ICME-13 Topical Surveys. Cham, Switzerland: Springer.

Chernoff, E. J., & Zazkis, R. (2011). From personal to conventional probabilities: from sample set to sample space. *Educational Studies in Mathematics*, *77*(1), 15-33.

Chib, S., Clyde, M., Woodworth, G., & Zaslavsky, A. (2003). *Subjective and objective Bayesian statistics: Principles, models, and applications* (Vol. 590). Hoboken, NJ: John Wiley & Sons.

Jones, G. A. (Ed.). (2006). *Exploring probability in school: Challenges for teaching and learning* (Vol. 40). Springer Science & Business Media.

Lovett, J. N., & Lee, H. S. (2017). New standards require teaching more statistics: Are preservice secondary mathematics teachers ready?. *Journal of Teacher Education*, *68*(3), 299-311.

National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). Common core state standards. Washington, DC: Authors.

Savard, A. (2014). Developing probabilistic thinking: What about people's conceptions?. In Chernoff E. & Sriraman B. (Eds.) *Probabilistic Thinking* (pp. 283-298). Dordrecht, The Netherlands: Springer.

INTERPRETATION LEVELS OF FREQUENCY GRAPHS BY KINDERGARTEN TEACHERS IN TRAINING

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We hereby present an exploration about graphs interpretation by students who are becoming kindergarten teachers. We also pretend to identify the notions which have about representative value from a data set. In this study there were 91 future kindergarten teachers participating to whom some information represented through a frequency graph was presented for them in order to answer a set of questions. The analysis shows the levels of interpretation of the participants, it is perceived that most of the teachers gather explicit information from the graph. However, they present some difficulties to integrate and combine data in order to make appropriate interpretations. The results are a reference in order to design intervention proposals which can help in the future teachers in training and the ones in current service as well as to reduce the breaches in the statistics learning in kindergarten.

Keywords: Data Analysis and Statistics, Teacher Knowledge.

Introduction

The interpretation of statistic graphs is an everyday activity which people do. Nowadays, most of the information is represented through graphs or tables so that when interpreting them, people can make decisions or just satisfy their curiosity about any topic of their interest. This has led into the necessity to train the future teachers so that they have the abilities and knowledge necessary about statistics and so, in that way they train the future citizens. In Mexico, the study of representation and interpretation of graphs is seen from the kindergarten level (see SEP, 2011, p. 57). This objective is part of the knowledge that a kindergarten teacher in training must acquire during his/her initial training (see SEP, 2012, pp. 20 and 22). Due to this, in this current research, we have the objective to identify the interpretation level of the future kindergarten teachers about frequency graphs as well as to explore their notions in relation with the representative value from a set of data.

There are several studies which give relevant information about graphs interpretation by the basic education teacher. González, Espinel and Ailey (2011), Jacobbe and Horton (2010), Alacaci, Lewis, Brien and Jiang (2011) and Sanoja and Ortíz (2013) report that the difficulties which the teacher face when interpreting the graphs, sometimes, are the result of the lack of understanding of basic concepts in statistics (e.g. scale, origin, axis, variable, dependency, independency, scattering, among others). From this, the interest of some researchers has focused on the mathematical knowledge for interpreting graphs (e.g. Alacaci et al., 2011). For Ball, Lubienski and Mewborn (2001) the teacher must build a solid knowledge during his/her initial training so that he/she has the basis to teach mathematics, thus statistics. On the other side, Llinares (2013) considers that the future teacher has to re-learn the mathematics he/she already knows in order to build a specialized knowledge which lets him/her state and give meaning to the mathematics to be taught in the classroom.

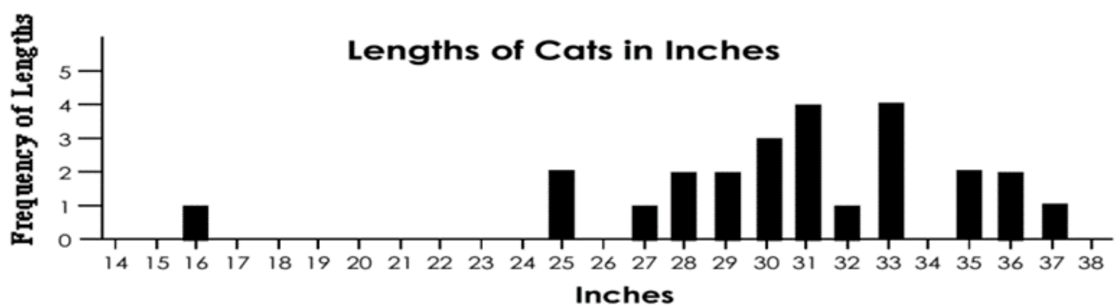
Conceptual Framework

For the interpretation of statistical graphs, Curcio (1989) defines three levels: 1) *Reading the data*, copy or reproduce the information or the facts which are explicitly expressed in the graph without going any further; 2) *Reading between the data*, combine and integrate data from the graph in order to obtain information, for example, by making comparisons or basic algorithms; 3) *Reading beyond the data*, predict or infer non-directly expressed information in the graph. In terms of Ball, Thames and Phelps (2008), the first two levels make reference to *basic knowledge (Common Content knowledge)* to extract information or data from statistic graphs, and the last level is related to the *specialized knowledge* which involves statistic reasoning.

Method

In this study, 91 future kindergarten teachers in training from a Mexican public school participated. Their ages were from 18 to 21 years old. A questionnaire about four problems, which included frequency graphs and a set of questions related with these graphs, was implemented for each participant. It is important to mention that the problems were taken from scientific literature. In the current document, only the results of one of the problems are reported (see Figure 1). The analysis of the gathered information is of the descriptive type and it was made through coding (Birks & Mills 2011). For this, firstly, words or groups of words or phrases were identified which derived from the answers of the participants and then, they were categorized based on the levels of Curcio (1989).

A group of students has been investigating information about their pets Several students have cats. They decided to collect some information about each of the cats. One set of data they collected was the length of the cats measuring from the tips of the cats' noses to the tips of their tails. Here is a bar graph showing the information they found:



1. How many cats are 30 inches long from nose to tail? How can you tell?
2. How many cats are there in all? How can you tell?
3. If you added up the lengths of the three shortest cats, what would the total of those lengths be? How can you tell?
4. What is the typical length of a cat from nose to tail? Explain your answer.

Figure 1. Problem of *lengths of cats*, taken from Bright and Friel (1998).

Results

The levels of interpretation on which the participants, taking the answers to the problems as a reference, are exposed as follows.

1. How many cats are 30 inches long from nose to tail? How can you tell? The answer to this question is the lowest level, *Reading the data*, and it was expected that the participants located the value of the variable corresponding to 30 inches and its frequency. This level was achieved by 90% of the participants. One example of the answers given was the following: "There are three cats with a length of 30 inches. Because in the bars graph it is shown that in the frequency of the inch number 30 is 3." The rest of the participants made at least one mistake when reading the frequency in the Y axis of the graph. They based their answer on the idea that each bar represented a cat so they gave confusing answers or did not answer the question.

2. How many cats are there in all? How can tell? In order to answer this question, the level *Reading between the data* was expected because it only required basic knowledge such as identifying the frequency in each bar and add them up. Only 67% of the participants achieved to give a correct answer and get into this level of interpretation. This is reflected on the answer which one of the participants gave: “There are 25 cats, because it is the total sum of the frequencies.” The answer to this question showed the difficulty for some of the teachers in training to read among the data. Such difficulty is related to making mistakes when counting or adding up the frequencies of the cats, interpreting each bar as a cat, not knowing how to determine that total as well as to give confusing answers or not answering at all.
3. If you added up the lengths of the three shortest cats, what would the total of those lengths be? How can you tell? The answer involves the level *Reading between the data*, due to locating the minor values of the variables is enough. These were 16 and 25 inches, and their frequencies 1 and 2 respectively. Besides, they had to add those values up ($16 + 25 + 25 = 66$). The future teachers associated the idea of small cat with the bars with the lowest frequency. 54% of them added up three bars of the lowest frequency. In some cases, they were three consecutive bars and in others, three bars chosen by any other criteria which is not clear. An example of this is the following: “$16 + 27 + 32 = 75$. Because the smallest cats have a measure of 16, 27 and 32 inches. Then 16 + 27 are 43 + 32 are 75.” 25% answered correctly (66 inches). Other participants made mistakes adding up, gave confusing answers or stated not being able to determine the required value. In some cases, they just added up the values of the variable which corresponded to the cats with the smallest length without considering the frequency of the variable.
4. What is the typical length of a cat from nose to tail? Explain your answer. The answer makes reference to *Reading beyond the data*. The notion which the future teachers handle of typical value matches with the one of average as mode (61.5%), as they took the mode to interpret a distribution and this is reflected when considering just as “the most” and not as a representative value (Mokros & Russell, 1995). An example of those answers is: “31 – 33 inches because it is the number which is repeated most of the times.” Other teachers did not answer. They considered that the typical value matched with the frequency which was repeated the most (2), they mentioned the average, considered the median or did not show a clear answer.

Conclusions

The teachers in training are able to reading the data without difficulty but when requiring them to reading between the data, they show difficulty to infer or interpret the graph. In relation with the answers of the second and third questions, the participants backed up on their basic knowledge such as the algorithm of adding up or multiplying (this was a mistake), they made it evident in a lower number of answers corresponding to combine and integrate the data because it was difficult for them. As in the study of Bright and Friel (1998), where the participants were children from 11 to 14 years old, the difficulties from the teachers in training are recurrent such as interpreting each bar of the graph as the representation of a cat, or that the small cats were represented by the bars with lower frequency. As it was mentioned in the introduction, some authors consider that this type of difficulties appear because teachers do not have clear some of the basic concepts in statistics. The future teachers show confusion when determining the frequencies in the bar graphs and operating with them. Besides this, the interpretation of the variable together with its frequency can be a non-achieved task by any of them. Concerning the approximations of the representative value, from a set of data, the average as mode is the one used the most, maybe 8% of the participants is close to the notion of average as a media point, as they consider the median to justify their answers. The results also show that a great percentage of teachers in training for kindergarten level have basic and specialized knowledge that allows them to literally read the graph or infer information from it. However, the need to reinforce this knowledge during their training is evident. Due to the exigency of the society and educational reforms which we live nowadays in Mexico, it is expected that the future teacher has wide knowledge (Jaworski, 2008), These results show that research focused on how the kindergarten teacher presents in the classroom the use and integration of the graphs is required and so to study the idea of arithmetic media as a representative value from a set of data.

References

Alacaci, C., Lewis, S., Brien, G. E. O., & Jiang, Z. (2011). Pre-service elementary teachers´ understandings of graphs. *Eurasia Journal of Mathematics, Science & Technology Education*, *7*(1), 3–14. doi: https://doi.org/10.12973/ejmste/75171

Ball, D. L., Lubienski, S. T., & Mewborn, D. S. (2001). Research on teaching mathematics: the unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (pp. 433-456). New York: Macmillan.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: what makes it special? *Journal of Teacher Education, 59*(5), 389-407. doi: 10.1177/0022487108324554

Birks, M., & Mills, J. (2011). *Grounded theory: A practical guide.* Londres: Sage Publications.

Bright, G. W., & Friel, S. N. (1998). Graphical representations: Helping students interpret data. In S. P. Lajoie (Ed.), *Reflections on statistics: Agendas for learning, teaching, and assessment in K–12* (pp. 63–88). Mahwah, NJ: Erlbaum.

Curcio, F. R. (1989). *Developing graph comprehension*. Reston, VA: NCTM.

González, T., Espinel, C., & Ainley, J. (2011). Teachers' graphical competence. In C. Batanero, G. Burrill, & C. Reading (Eds.), *Teaching Statistics in School Mathematics. Challenges for Teaching and Teacher Education. A Joint ICMI and IASE Study* (pp. 187–197). New York: Springer.

Jacobbe, T., & Horton, R. M. (2010). Elementary school teachers' comprehension of data displays. *Statistics Education Research Journal*, *9*(1), 27–45. Available from http://www.stat.auckland.ac.nz/serj

Jaworski, B. (2008). Mathematics teacher educator learning and development. In B. Jaworski, & T. Wood (Eds.), *Handbook of Mathematics Teacher Education* (Vol. 4, pp. 1-13). Rotterdam, The Netherlands: Sense.

Llinares, S. (2014). Conocimiento de matemáticas y Tareas en la formación de maestros. *Cuadernos de Investigación y Formación en Educación Matemática,* (12). 205-220. Available from https://revistas.ucr.ac.cr/index.php/cifem/article/view/18921/19035

Mokros, J., & Russell, S.J. (1995). Children's concepts of average and representativeness. *Journal for Research in Mathematics Education, 26*, 20-39. doi: 10.2307/749226

Sanoja, J., & Ortiz Buitrago, J. (2013). Conocimiento de contenido estadístico de los maestros. In J. M. Contreras, G. R. Cañadas, M. M. Gea, & P. Arteaga (Eds.), *Actas de las Jornadas Virtuales en Didáctica de la Estadística, Probabilidad y Combinatoria* (pp. 157-164). Granada, Departamento de Didáctica de la Matemática de la Universidad de Granada.

Secretaría de Educación Pública (SEP). (2011*). Programa de estudios. Guía de la educadora*. México: SEP.

Secretaría de Educación Pública (2012). *Programa del curso. Procesamiento de Información Estadística*. México: SEP

WORKING ON THE EDGE OF MATHEMATICS, STATISTICS, AND BIOLOGY: BIOLOGY UNDERGRADUATE STUDENTS' GRAPHS CONSTRUCTIONS

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Historically, mathematics, statistics, and biology are courses taught in isolation. Yet, there are some tasks that invite students to combine these different types of knowledge. In this study, 10 undergraduate biology students were asked to make a graph from a given set of data embedded in a biology context. Graphs were analyzed to identify features such as variables plotted, number of graphs created, and type of graph used to represent the information (e.g., scatter plot, line graph, or bar graph), and whether the graph showed an understanding of the purpose and features of the scenario. Participants made a variety of graphs. Graphing data was a task that invited them to combine knowledge from different disciplines. Looking forward to the field of mathematics education, graphing could be one activity that invites our community to interact with other disciplines, embrace our common practices and diverse ways of knowing.

Keywords: Graphing, Biology, Statistics, Undergraduate, Interdisciplinary

In mathematics classrooms, we usually ask students to make a graph to represent linear, quadratic, or exponential patterns. In these tasks, students usually identify the role of two given variables (i.e., independent and dependent variable), determine a couple of coordinates to then construct a graph. Meanwhile in statistics classrooms, students use graphs such as a line graph, bar graph, or scatterplot as a way to analyze patterns in the data. If students are given a set of data, it usually contains more than one variable and students have to explore and determine what relationship could be represented from the given set of data (Wild & Pfannkuch, 1999). While it is known that mathematics and statistics tasks have real-world contexts in common, in mathematics this could be optional, while in statistics, the context is an essential element of the situation (Scheaffer, 2006). In other science fields, such as biology, one skill that students are expected to develop is the construction of graphs as part of the scientific process where they can use them to summarize data, which comprises evidence to support the validity of a hypothesis. One practice that is common across mathematics, statistics, and biology is understanding information that graphs convey in order to create an effective representation of data with which the graph creator and others can interpret and draw conclusions. For example, in biology, students might be asked to design experiments to evaluate the effectiveness of a treatment. In this context, students need to integrate different types of knowledge to create an appropriate graph of their data to discern patterns. Knowledge about the biological system under study, so that they understand the context, limitations of measurement and study design, and properties of the measured variables. Knowledge about mathematics, to operate symbols and use a mathematical language to communicate with others. Knowledge about statistics, to explore data, acknowledge variation, and select an appropriate graph that allows them to communicate patterns in the data.

In the construction and reading of graphs, adults and K-12 learners struggle at the time of conveying information and making interpretations from them. For example, Chick (2004) found that secondary school students' graph choices were not always suitable and they did not used strategies that could have made their graphs clear. Working with a similar population, D'Ambrosio, Kastberg, McDermott, and Saada (2004) found that students had difficulties interpreting graphs. Specifically, students wrestled integrating the context and the mathematics involved in graphs. Studying graphs created by adults, Tufte (1983) identified graphs published in magazines or periodicals that promoted misleading information for readers. In science fields such as biology, Angra and Gardner (2017) explored graph constructions of professors, undergraduate and graduate biology students. The authors found that in comparison to professors and graduate students, undergraduate biology students struggled deciding which data to select and represent, what type of graph to use, and aligning their graph construction decisions with the question and/or hypothesis under consideration.

The difficulties learners face making and reading graphs could be seen as a lack of mathematics or statistics content preparation. However, we contend that it is not competence with the knowledge and skill in any one of the relevant disciplines, but that in order to successfully analyze, represent data, and communicate its patterns in biology contexts; students need a combination of all of these types of knowledge. We acknowledge that this task is challenging given that students are seldom given explicit instruction preparing them to combine knowledge from different disciplines. Therefore, in this study the authors built from Williams' et al. (2016) concept of interdisciplinarity to explore graph construction as a task that involves working on the edge of three different disciplines: statistics, mathematics, and biology. Particularly, the authors explored the types of graphs biology undergraduate students constructed from a given set of data and discuss the mathematics, statistics, and biology knowledge used in their graphs' construction.

Methods and Analysis

These data were collected as part of another study (Angra & Gardner, 2016, 2017) in which biology undergraduates, graduate students, and biology professors were engaged in a pen-and-paper think-aloud graphing task. The data presented here are from a subset of the full data corpus and are comprised of data from 10 undergraduate biology students' first graphs construction and their descriptions. There were two isomorphic biology scenarios assigned to the participants: two treatments, five time points, and three replicates within the context of plant growth and cell culture experiments. Participants were asked to graph any aspect of a given set of data. Figure 1 shows data from the plant growth context.

Plant leaves scenario
Imagine you are a botanist. You are particularly intrigued by how the amount of water influences plant growth. In order to answer your question, you set up an experiment that measures the growth of a particular type of plant at two different water amounts. You collect your data and display it in a chart shown below.

	Number of Leaves					
Time (Hours)	15ml of water/day			5ml of water/day		
	Plant 1	Plant 2	Plant 3	Plant 1	Plant 2	Plant 3
0	0	1	0	0	0	0
30	2	1	3	0	1	1
60	3	3	5	2	1	2
90	4	3	5	2	1	3
120	6	5	7	3	1	4

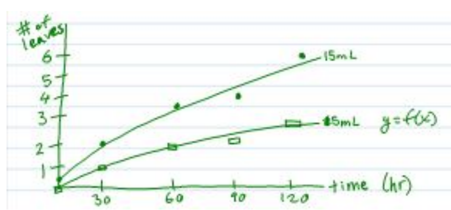
Figure 1. Scenario and data given to the participants.

In a first level of analysis, participants' graphs were examined independently of their think-aloud. The graphs were coded according to different characteristics. Variables plotted, number of graphs created, type of graph used to represent the information (e.g., scatter plot, line graph, or bar graph), and whether the graph is *aligned* with the purpose and features of the study in the scenario (i.e. comparing growth over time in two treatment groups and the use of replicates within each treatment). In a second level of analysis, commonalities and differences among participants' graphs were identified and organized along with participants' verbal descriptions of their graphs during the think-aloud. The authors used Konold, Higgins, Russell, and Khalil's (2015) framework as a way to describe their interpretations of participants' views of data: aggregate, classifiers, case values, and pointers.

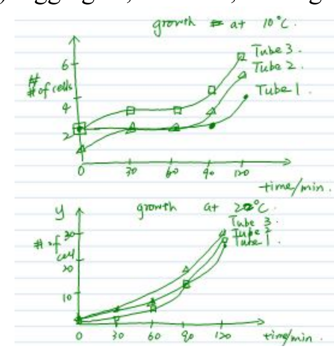
Results and Discussion

Four out of 10 participants drew a graph that showed a comparison of the two treatments (see Figure 2a)). In these cases, time was considered the independent variable, and the average of the results per plant, as the dependent variable. Three out of 10 participants graphed the results of each plant, or tube, per treatment using two separated graphs (see Figure 2b) and c)). In Figure 2b) the participant was assigned a context related to cells culture and she drew two graphs. One graph representing the number of cells in the tubes exposed to the 10 degrees Celsius treatment, and the other graph for the 22 degrees Celsius treatment. Finally, three out of 10 participants represented the results of each plant under the two treatments. This approach generated three different graphs, one per plant (see Figure 2d)).

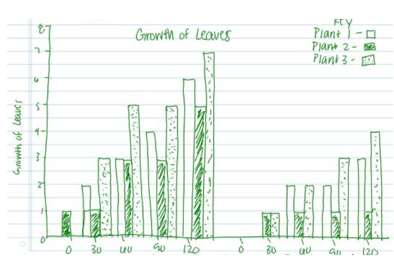
a) Aggregate (reduced) no variation, aligned



b) Aggregate, raw data, misaligned



c) Aggregate, raw data, aligned



d) Case values, misunderstanding replicates, aligned

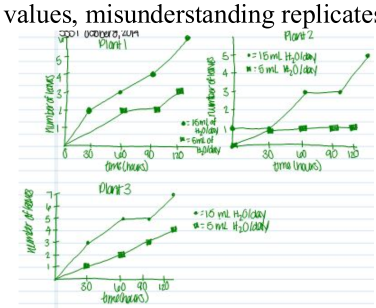


Figure 2. Examples of participants' graphs using Konold's et al. (2015) language.

When researchers looked at the participants' background information, they learned that those with research experience created one graph that communicated a comparison of the two treatments. However, none of them explicitly acknowledged variability in their graph construction (e.g., using error bars). In relation to the type of graphs, six out of 10 participants used line graphs, 2 out of ten scatter graphs, and the rest bar graphs. The majority of the participants used time as the independent variable in the x-axis, and number of leaves, or cell, as the dependent variable. Participants' reasoning for using time as the independent variable was based on understanding time as a continuous variable, which for them it meant to be placed in the x-axis.

Students created a variety of graphs to represent the same set of data. However, the majority of them considered time as the independent variable. As shown in other studies (e.g., Chick, 2004; D'Ambrosio et al., 2004) representing a given set of data in a graph was not a simple task for participants. Creating a clear graph that conveyed information about the results of the experiment was something that four out of ten participants successfully did (see Figure 2a)). Yet, statisticians would argue that those participants' graphs with their descriptions lacked of explicitly acknowledging variability.

The authors consider that those participants who created one graph that showed a comparison of the two treatments are examples of students combining biology, statistics, and mathematics knowledge. The need for a graph originated from the context of the task: exploring the results of an experiment where plants or cells where exposed to two different treatments. To compare both treatments, students needed to see data as aggregate (Konold et al., 2015), which four students interpreted as plotting time versus the average (reduced data) of the results of each plant or cell. This procedure, or way of thinking, could be associated to mathematics or statistics depending on the student's academic background. Finally, the type of task used in this study seems to encourage students to combine different types of knowledge learned in college coursework. However, authors cannot claim with certainty what prompted participants to use a combination of mathematics, statistics, and biology knowledge. It seems that having research experience could be an antecedent to be considered for future studies. The authors also consider that teaching graphing to undergraduate students using interdisciplinary lenses is another way to advance their graphing skills further.

References

Angra, A., & Gardner, S. M. (2016). Development of a framework for graph choice and construction. *Advances in Physiology Education*, *40*(1), 123-128.

Angra, A., & Gardner, S. M. (2017). Reflecting on Graphs: Attributes of Graph Choice and Construction Practices in Biology. *CBE-Life Sciences Education*, *16*(3), ar53.

Chick, H. (2004). Tools for transnumeration: Early stages in the art of data representation. In Putt, I., Faragher, R., & McLean, M. (Eds.), *Mathematics Education for the Third Millennium: Towards 2010. Proceedings of the Twenty-seventh Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 167-174). Sydney, Australia: MERGA.

D'Ambrosio, B., Kastberg, S. E., McDermott, G., & Saada, N. (2004). Beyond reading graphs: student reasoning with data'. In Kloosterman, P. & Lester, F. (Eds.), *Results and interpretations of the 1990-2000 mathematics assessments of the National Assessment of Educational Progress* (pp. 363-381). Reston, VA: NCTM.

Konold, C., Higgins, T., Russell, S. J., & Khalil, K. (2015). Data seen through different lenses. *Educational Studies in Mathematics*, *88*(3), 305-325.

Scheaffer, R. L. (2006). Statistics and mathematics: On making a happy marriage. In Burrill, G., & Portia C. E. (Eds.), *Thinking and reasoning with data and chance, 68th. Yearbook,* (pp. 309-321). Reston, VA: NCTM.

Tufte, E. (2001). *The visual display of quantitative information*. Connecticut: Graphics Press.

Watson, J. M., & Moritz, J. B. (1998). The beginning of statistical inference: Comparing two data sets. *Educational Studies in Mathematics*, *37*(2), 145-168.

Wild, C. J., & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry. *International Statistical Review*, *67*(3), 223-248.

Williams, J., Roth, M., Swanson, D., Doig, B., Groves, S., Omuvwie, M.,… Borromeo F. R., & Mousoulides, N. (2016). *Interdisciplinary mathematics education: A state of the art.* Springer Open. https://doi.org/10.1007/978-3-319-42267-1_1

UNDERSTANDING PROBABILITY LITERACY OF HIGH SCHOOL STUDENTS

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Probability and statistics have taken prominence in the era of big data. Thus, there is a need to ensure that high-school students have a thorough grounding in probability concepts or probability literacy. Probability literacy is a collection of knowledge, skills, attitudes, and habits of the mind about probability. Gal (2005) placed all aspects of this into two categories with subtopics under each. Those two categories are knowledge elements and dispositional element. The underlying concerns of this study are the big ideas and figuring probabilities under the knowledge element. Multiple reliable tasks will be used to centered around randomness, chance, independence, and sample space. The result of understanding probability literacy of high school students will help understand the teaching needs of students in schools. Currently, there is little research that examines the conceptions that high school students have about randomness, chance, independence, and sample space (Jones, Langrall, & Mooney, 2007).

The purpose of the study is to understand the probability literacy of high-school students as stated by Gal's (2005) framework. Namely: what is the probability literacy of high-school students? In particular, what do high school students understand about the following probability concepts of chance, randomness, independence and sample space?

Fischbein, Nello, and Marion (1991) reported students' responses from a study of probability where students correctly answered probability questions with incorrect reasoning. One such response was, "the probability is the same because the result cannot be predicted" (Fischbein, Nello, & Marion, 1991, p. 538). The student's response was to a task involving the flipping of two coins. The student correctly answered the question, but the reasoning points to the fact the student has little understanding of probability. Hence the need for more thorough understanding of students' misconceptions or probability is needed to better inform the instruction to prevent such misconceptions from propagating. The information that this study hopes to uncover will provide researchers and practitioners with insight about probability literacy among high school students.

The research will be conducted using taped semi-structured oral interviews and parallel student written interviews as adapted from Rubel (2007) where she was seeking to uncover students' justifications in addition to correctness. She found inconsistencies in student responses between the two forms that might lead to insight to students' probability literacy. A pilot study will be conducted during the summer of 2018 to inform the main study that will take place during the spring of 2019.

References

Gal, I. (2005). Towards "probability literacy" for all citizens: Building blocks and instructional dilemmas. In *Exploring probability in school* (pp. 39–63). Springer.

Fiscbein, E., Nello, M. S., & Marino, M. S. (1991). Factors affecting probabilistic judgements in children and adolescents. *Educational Studies in Mathematics*, *22*(6), 523–549.

Jones, G. A., Langrall, C. W., & Mooney, E. S. (2007). Research in probability: Responding to classroom realities. *Second handbook of research on mathematics teaching and learning, 2*, 909-955.

Rubel, L. H. (2007). Middle school and high school students' probabilistic reasoning on coin tasks. *Journal for Research in Mathematics Education*, 531–556.

MIDDLE AND SECONDARY TEACHERS' PCK WITHIN IIR CONTEXTS

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Over the past decade, research on pedagogical content knowledge (PCK) for statistics has been identified as a critical area of need, particularly the need to study teachers' PCK in the context of informal inferential reasoning (IIR) (Langrall et al., 2017). A majority of the work in statistics education, for both PCK and IIR, has focused on *students* across PK-16. Within the narrow body of literature on middle level and secondary mathematics teachers' PCK, although some teachers have been found to develop strong PCK (Peters, 2013), many struggle to develop it, despite strong content backgrounds (Callingham et al., 2011). This study utilized Groth's (2013) statistical knowledge for teaching framework—focusing on the PCK components of teachers' knowledge of content and students, and knowledge of content and teaching—and operationalized it through the descriptions of four hierarchical PCK levels expressed by Callingham and Watson (2011), set within the context of responding to tasks designed to illicit IIR. Due to space, this study focuses only on teachers' PCK and does not attempt to analyze their IIR engagement.

Data come from a stratified purposeful sample of 9 middle and secondary mathematics teachers as they responded to a set of 4 LOCUS (Jacobbe, 2016) tasks during two 60-90 minute task-based interviews. After completing the tasks on their own, the following two questions were asked, as in Watson et al (2008), to aid in identifying teachers' statistics PCK: 1) What are some appropriate and inappropriate student responses? 2) How might you respond to a student who offers an inappropriate response? The interview protocol included follow-up questions based on predicted teacher responses, as well as example student responses to offer, when necessary. Results indicate that when teachers were responding to explicit IIR items, as opposed to those not requiring reasoning about inferences (e.g., reasoning only about center, spread, and shape), they tended to fall within the lowest level of PCK. However, when responding to items not requiring IIR, their PCK could be characterized at higher levels of PCK and there was less consistency with content background knowledge. Additional findings will be shared and implications for future research and teacher education will be offered on the poster.

References

Callingham, R., & Watson, J. (2011). Measuring levels of statistical pedagogical content knowledge. In C. Batanero, G. Burill, & C. Reading (Eds.), *Teaching statistics in school mathematics-Challenges for teaching and teacher education. A Joint ICME/IASE Study* (pp. 283–293). The 18th ICMI Study. New York, NY: Springer Science + Business Media.

Groth, R. E. (2013). Characterizing key developmental understandings and pedagogically powerful ideas within a statistical knowledge for teaching framework. *Mathematical Thinking and Learning*, *15*(2), 121–145.

Jacobbe, T. (2016). Levels of Conceptual Understanding in Statistics (LOCUS). Retrieved April 18, 2016, from https://locus.statisticseducation.org/

Langrall, C. W., Makar, K., Nilsson, P., & Shaughnessy, J. M. (2017). The teaching and learning of probability and statistics: An integrated perspective. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 490–525). Reston, VA: National Council of Teachers of Mathematics, Inc.

Peters, S. A. (2013). Developing understanding of statistical variation: Secondary statistics teachers' perceptions and recollections of learning factors. *Journal of Mathematics Teacher Education*, *17*(6), 539–582.

HIGH SCHOOL STATISTICS TEACHERS' DEVELOPMENT OF KEY UNDERSTANDINGS OF HYPOTHESIS TESTING THROUGH SIMULATION

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Keywords: Data Analysis and Statistics, Mathematical Knowledge for Teaching, Technology

In general, students and teachers can often learn the associated computations and procedures of a hypothesis test but struggle with the reasoning and basic concepts of this procedure (Harradine et al., 2011). Even with intense remediation efforts, it is often difficult to convey an understanding of these abstract ideas (Thompson et al., 2007). Using simulations for hypothesis testing is a growing trend in statistics education due to the belief that simulations help make the abstract ideas behind hypothesis testing become more concrete and understandable (Erickson, 2006). Cobb (2007) described the simulation process as "The three R's: randomize, repeat, reject. Randomize data production; repeat by simulation to see what's typical; reject any model that puts your data in its tail" (Cobb, 2007, p. 12). In other words, using simulated sampling distributions to reject or fail to reject the null hypothesis.

With this trend in statistics education in mind, I conducted an explanatory case study to answer the following question, "How does engaging in the process of using simulations to conduct a hypothesis test influence a high school statistics teacher's understanding of this topic?" Before engaging in the simulation tasks, the participant could correctly define terms and explain some of the steps that should occur for hypothesis testing. However, when attempting to answer questions or provide explanations that required a more in-depth understanding, she was unable to do so. For example, when she did not have the steps in front of her, she stated the you should reject the null hypothesis when the *p*-value is large. She believed a large *p*-value meant that your data would likely be replicated again, so your null hypothesis is probably true.

My poster will describe the lesson plan design used for the simulation tasks and describe how the participant interacted with the tasks. Additionally, the results of how the tasks influenced the teacher's content knowledge for hypothesis testing will be shared.
The tasks allowed the teacher to visualize the sampling distribution and correct many of her misconceptions. After the tasks, she correctly described when to reject the null hypothesis and was able to articulate how the alpha-level was related to a Type I error. Additionally, the lesson plan design helped her to understand the logic of hypothesis testing. The poster will highlight important elements to incorporate into professional development to help teachers foster a deeper understanding of hypothesis testing using technology to conduct simulations for inference.

References

Cobb, G. W. (2007). The introductory statistics course: a Ptolemaic curriculum? *Technology Innovations in Statistics Education*, *1*.

Harradine, A., Batanero, C., & Rossman, A. (2011). Students and teachers' knowledge of sampling and inference. In C. Batanero, G. Burrill, & C. Reading (Eds.), *Teaching statistics in school mathematics-challenges for teaching and teacher education* (pp. 235-246). The Netherlands: Springer.

Thompson, P. W., Liu, Y., & Saldanha, L. A. (2007). Intricacies of statistical inference and teachers' understandings of them. *Thinking with Data*, 207-231.

A FRAMEWORK FOR ANALYSIS OF AP STATISTICS TEXTBOOKS FOR INFERENTIAL REASONING

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The textbook has served as the centerpiece of learning for almost every academic discipline, although the relationship between the course and its textbook is complex. As a fundamental resource, Advanced Placement (AP) Statistics' textbooks have the potential to shape the way the instructors teach and the students learn (Zieffler, Isaak, & Garfield, 2013). While a growing body of research has sought better ways to improve the teaching of inferential reasoning in AP Statistics course, there is a lack of research on particularly focusing on the textbooks' analysis for inferential reasoning (Zieffler et al., 2013). Inferential reasoning is not only an ultimate learning goal of the AP Statistics course, both its early introduction and implicit instruction is also recommended by the Guidelines for Assessment and Instruction in Statistics Education (GAISE) report (Franklin et al., 2007). Recent research efforts have focused on its implicit learning goal as "informal" inferential reasoning where students drawing conclusions about populations based on data before the procedures of formal inference are introduced (Zieffler et al., 2013).

In this poster, we will describe a framework that we have developed to answer the following: What is the nature and extent of inferential statistics concepts contained in AP Statistics textbooks?

We have developed a two-dimensional, four by five matrix structured framework. One dimension is the four phases of the statistical investigative cycle, which is defined by GAISE as *posing a question, collecting data, analyzing data* and *interpreting results*. The other dimension is comprised of the five key aspects of inferential reasoning proposed by Lee (2017). They are the five columns of this framework as *Context, computations and graphs, variability, uncertainty* and *beyond data at hand*. Each cell of this matrix will have evaluation criteria. For instance, in the cell of the intersection of the row of *posing a question* and the column of *context*, students' interest will be the evaluation criteria. The two dimensions of our framework embody the key recommendations for developing informal inferential reasoning and capture the common attributes, which pave the way to inferential reasoning. This framework acknowledges a continuum within each dimension that reveals implicit and explicit pedagogical choices made by textbook authors and publishers that directly influence students' opportunities to learn.

Our framework analyzes the expositions and tasks in AP Statistics textbooks, which are recommended by College Board. Under the lenses of our framework, we will present, the preliminary findings of the analysis of two AP Statistics' textbooks,

References

Franklin, C., Kader, G., Mewborn, D., Moreno, J., Peck, R., Perry, M., & Scheaffer, R. (2007). Guidelines for assessment and instruction in statistics education (GAISE) report. *American Statistical Association.*

Lee, H. S. (2017). What is inferential reasoning? In *Teaching Statistics Through Inferential Reasoning MOOC for Educators,* Friday Institute for Educational Innovation: NC State University, Raleigh, NC.

Zieffler, A., Isaak, R., & Garfield, J. (2013), The Course as Textbook: A Symbiotic Relationship in the Introductory Statistics Class, *Technology Innovations in Statistics Education,* 7(3).

ESTUDIO DE CASO DE UNA ENSEÑANZA DE LA ESTADÍSTICA BASADA EN LENGUAJE R: ESQUEMAS DE ACCIÓN Y DISCURSO

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Palabras clave: Actividades y Prácticas de Enseñanza, Discurso del Aula, Educación Post-Secundaria, Análisis de Datos y Estadística

Con el objetivo de estudiar la evolución de los procesos de enseñanza de la estadística cuando se integran recursos tecnológicos digitales, analizamos los esquemas de acción y utilización (Gueudet & Trouche, 2012) de una profesora de estadística universitaria, quien integra en sus clases el lenguaje de programación R. Más específicamente, esta profesora diseñó su curso a través de una serie de actividades basadas en R, con el objetivo de promover el entendimiento de conceptos de estadística, por parte de sus alumnos. Se observó la práctica de esta profesora en dos cursos avanzados de nivel universitario, durante un periodo de tres semanas; dichas observaciones fueron complementadas con datos de una entrevista semi-estructurada, recursos materiales (las hojas de trabajo de las actividades en R, evaluaciones, etc.) y otros recursos relacionados con su práctica docente. Los resultados fueron analizados cualitativamente usando el Enfoque Documental (Gueudet & Trouche, 2012) para inferir la génesis documental de esta profesora relativa al uso que hace de los recursos. La génesis documental se refiere al proceso mediante el cual, a través de la selección, modificación y recombinación de recursos, el profesor desarrolla un sistema de recursos junto con un esquema de utilización y/o acción que guía la actividad de enseñanza situada.

Para este póster, nos centramos en los esquemas de utilización y acción de los recursos del lenguaje que surgieron al analizar las relaciones e interacciones entre los recursos digitales y no digitales. Se identificó cómo esta profesora usaba diferentes tipos de lenguajes –oral, corporal, estadístico y de programación en R— como esquemas de acción. En el discurso de la profesora, hay una continua traducción entre el lenguaje estadístico, el lenguaje de programación y los lenguajes oral y corporal, por medio del cual se expresan las hipótesis, las relaciones observadas y las interpretaciones que se hacen a partir de la exploración de los datos contenidos en los problemas planteados en las actividades. De esta manera, los lenguajes oral, corporal y estadístico se combinan y articulan con el lenguaje de programación R, constituyéndose así en 'recursos de lenguaje', y sus usos son esquemas de acción de la profesora que promueven la comprensión de conceptos y de relaciones entre éstos, a la vez que ayudan a desarrollar la capacidad de los alumnos para conducir análisis estadísticos.

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Referencia

Gueudet, G., & Trouche, L. (2012). Teachers Work with Resources: Documentational Geneses and Profesional Geneses. En Gueudet, G., Pepin, B., & Trouche, L. (Eds.). *From Text to "Lived" Resources: Mathematics Curriculum Materials and Teacher Development* (pp. 23-41). Netherlands: Springer.

A STATISTICS TEACHING CASE STUDY BASED ON THE R LANGUAGE: SCHEMES OF ACTION AND DISCOURSE

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In order to study the evolution of teaching processes for statistics when integrating digital technological resources, we have analyzed the schemes of action and utilization (Gueudet & Trouche, 2012) of a university statistics teacher, who conducts her lessons using the R programming language. More specifically, this teacher designed her course through a series of R-based tasks, with the aim of promoting a more conceptual understanding in her students. We observed this teacher in two advanced university courses over the course of three weeks; those observations were complemented with data from semi-structured interview, material resources (the R tasks worksheets, assessments, etc.) and other documents related to her teaching practice. The results were analyzed qualitatively using the Documentational Approach (Gueudet & Trouche, 2012) in order to infer this teacher's documentational geneses related to the use of resources, The documentational genesis refers to the process by which, through selection, modification and recombination of resources, a teacher develops a system of resources together with schemes of utilization and/or action that guide the situated teaching activity.

For this poster, we focus on the schemes of utilization and action of language resources that emerged when analyzing the relationships and interactions between digital and non-digital resources. We identified that the teacher used different types of language –verbal, body, statistical and R programming ones— as schemes of action. In the teacher's discourse, there was a continuous back-and-forth and translation between the statistical language, the programming language, and the oral and body languages, through which she described and expressed the hypothesis, observed relationships and interpretations derived from the explorations of the data contained in the problems presented in the R-based tasks. In this way, the verbal, body and statistical languages are combined and articulated with the R programming language to shape the 'language resource', and its use constitutes schemes of action that help students understand and identify relationships between concepts, as well as develop abilities to carry out a statistical analyses.

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Reference

Gueudet, G., & Trouche, L. (2012). Teachers Work with Resources: Documentational Geneses and Professional Geneses. In G. Gueudet, B. Pepin & L. Trouche (Eds.). *From Text to "Lived" Resources: Mathematics Curriculum Materials and Teacher Development* (pp. 23-41). Netherlands: Springer.

Chapter 10

Student Learning and Related Factors

Research Reports

Brief Research Reports

Posters

SUPPORTING SECONDARY STUDENTS' PERSEVERANCE FOR SOLVING CHALLENGING MATHEMATICS TASKS

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Perseverance, or initiating and sustaining productive struggle in the face of obstacles, promotes making sense of mathematics. Yet, engaging in struggle can be grueling and is avoided for some students. I investigate the effect of scaffolding mathematics tasks on student perseverance. The results show how prompting secondary students to conceptualize a mathematical situation prior to problem-solving can encourage re-initiating and re-sustaining mathematically productive effort upon reaching an impasse. For learning mathematics with understanding, these findings suggest specific methods by which student perseverance in problem-solving can be supported.

Keywords: Instructional Activities and Practices, Problem-Solving, High School Education

Perseverance, or initiating and sustaining in-the-moment productive struggle in the face of one or more mathematical obstacles, setbacks, or discouragements, is a productive process by which understandings are developed. The idea of withstanding uncertainty and persevering past obstacles has long been recognized as a key process by which meanings can be formed (Dewey, 1910; Polya, 1971). These ideas have been echoed for learning mathematics because students develop their understandings through productive struggle, or as they grapple with mathematical ideas that are within reach, but not yet well formed (Hiebert & Grouws, 2007). Additionally, reconciling moments of impasse, or times of significant uncertainty or discouragement, is critical for mathematics learning; the processes of struggle to approach, reach, and make continued progress despite a perceived impasse puts forth cognitive demands upon the learner that are conducive for development of conceptual ideas (VenLehn et al., 2003). As such, encouraging student perseverance has been made explicit as a mathematical and pedagogical demand for reform to help improve teaching and learning in mathematics education (NCTM, 2014).

Supporting Student Perseverance with Mathematics Tasks

Despite widespread educational support around the notion of struggling with challenging mathematics, both students and teachers can be reluctant to engage in and offer opportunities for perseverance (DiNapoli & Marzocchi, 2017). Consequently, several recent research efforts have sought to make explicit classroom practices that support or constrain student perseverance with challenging mathematics. Kapur (2009) found that providing consistent opportunities for students to persevere with unfamiliar mathematical tasks encouraged more variability in problem-solving strategies and greater learning gains, compared to providing consistent opportunities to engage with more procedural mathematics. All students, however, were unlikely to make additional attempts at solving a problem after reaching an impasse – an area of concern. Bass and Ball (2015) explored the nature of perseverance by implementing classroom tasks with familiar entry points yet a complex structure (i.e., low-floor/high-ceiling tasks). The researchers observed children leveraging these opportunities to persevere in their efforts despite challenge and seemingly make mathematical progress. The authors urge for future research to document the student perspective around such perseverance and if it was indeed productive. Warshauer's (2014) exploration of productive struggle found that teachers used probing questions and encouragement as effective methods to nurture perseverance at times of immense student struggle, but had great difficultly providing consistent support for all classroom students for logistical reasons. Noteworthy outcomes of these studies are ideas for how perseverance can be operationalized, and suggestions for future research to carefully study perseverance from the student point of view to better understand how it can be supported.

Additionally, in a recent journal series on nurturing perseverant problem solvers, several teaching practices were identified and described as advantageous. These included encouraging independent student thinking by restricting teacher reassurance feedback during problem-solving (Bieda & Huhn, 2017), scaffolding students' experiences with tasks through assessing questions, advancing questions, and judicious telling (Freeburn & Arbaugh, 2017), and scaffolding student engagement with mathematics through establishing a culture of guiding, exploratory self-questioning (Kress, 2017). These studies offer insight into effective teacher moves for supporting student perseverance with mathematical tasks, but also call attention to logistical concerns about teachers providing targeted support for each and every student (Warshauer, 2014).

One method to bypass these logistical concerns is to embed similar scaffold supports into the mathematical tasks themselves to better insure each student has an opportunity to engage with them. Anghileri (2006) explains different levels of scaffolds that can be applied to tasks to help students problem-solve. Most relevant to supporting student perseverance are embedded conceptual thinking scaffolds, which provide opportunities for students to conceptualize the situation by making connections from their prior mathematical knowledge to the task at hand and mapping out their own strategies for problem-solving. Such conceptualization scaffolds provide a structure for thinking and acting that can organize a problem-solving plan coming entirely from a student's own ideas. Moreover, these scaffolds align with Polya's (1971) stages of problem-solving through which learners approach a task. While stages 3 and 4 help describe the actions of perseverance, it is stages 1 and 2 that can theoretically support those actions by encouraging students to conceptualize the mathematical situation at hand (see Figure 1).

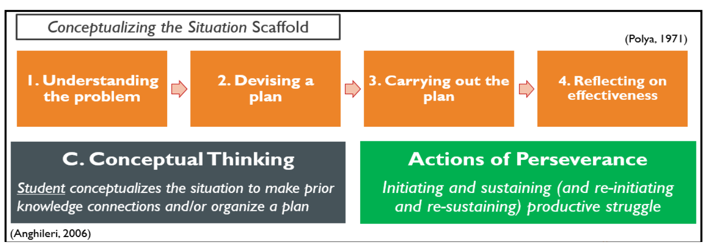

Figure 1. Scaffolding Perseverance in Problem-Solving

There is ample work that shows how conceptualization scaffolds encourage initial engagement with challenging mathematical tasks: when students have opportunities to make connections to what they already know and record all of their ideas and plans prior to the actual execution of problem-solving strategies, they are better suited to initiate and sustain their engagement with the task (e.g., Hmelo-Silver & Barrows, 2006). However, it is less clear how conceptualization scaffolds affect student perseverance upon reaching a perceived impasse – a moment that could be leveraged for key conceptual learning gains (VenLehn et al., 2003). If scaffolds are to support student perseverance during problem-solving, they must not only support initiating and sustaining productive struggle at the outset, but also support re-initiating and re-sustaining productive struggle after a student encounters a significant setback and is uncertain about how to continue. Therefore, via the student point of view, this study addressed how initially engaging with conceptualization scaffolds embedded at the start of low-floor/high-ceiling mathematics tasks supported perseverance, especially after a perceived impasse.

Methods

The participants for this study were 10 ninth-grade students from one suburban-area high school algebra class in a Mid-Atlantic state. These participants were purposely chosen to have demonstrated, via pretest, the prerequisite knowledge necessary to initially engage with each mathematical task included in the study.

To collect data, each participant was observed engaging with five mathematical tasks, one per week. These tasks were rated as analogous, expert-level tasks by the Mathematics Assessment Project (MAP) because of their low-floor/high-ceiling structure, two objectives, and required generalization a mathematical situation. I also solicited independent mathematics education experts to rate the difficulty of these tasks; no differences were reported. Additionally, each participant was given an opportunity to reflect on task difficulty after their participation and reported no differences. Three tasks were randomly chosen to be scaffolded, and two tasks were randomly chosen to be non-scaffolded. The conceptualization scaffold embedded into the scaffolded tasks was "*Before you start, what mathematical ideas or steps do you think might be important for solving this problem? Write down your ideas in detail*." Each participant worked on these set of five tasks in a random order. For context and to help follow the results in this paper, Cross Totals (a scaffolded task) asked students to generalize rules about how to arrange the integers 1-9 in a symmetric cross such that equal horizontal and vertical sums would be possible or not possible. Triangular Frameworks (a non-scaffolded task) asked students to generalize rules about how to build different triangles using the triangle inequality theorem if the longest side was even or odd in length.

This study prioritized the student point of view of their perseverance, so I incorporated many opportunities for participants to make explicit their in-the-moment perspectives during problem-solving. For each task and participant, I conducted think-aloud interviews while they worked on a task and video-reflection interviews immediately after they finished working. Additionally, once a participant had engaged with all five tasks (and thus all five think-aloud interviews and video-reflection interviews), I conducted an exit interview to give each participant an opportunity to comment on their overall experience working on the five tasks. In all, I conducted 11 interviews with each participant, or 110 interviews in total for this study.

To analyze the data, I developed the Three-Phase Perseverance Framework (3PP) (see Table 1) that I used to operationalize the construct in this context. It was designed to reflect perspectives of concept, problem-solving actions, self-regulation, and making and recognizing mathematical progress. The 3PP considered first if the task at hand warranted perseverance for a participant (the Entrance Phase), considered next the ways in which a participant initiated and sustained productive struggle (the Initial Attempt Phase), and considered last the ways in which a participants re-initiated and re-sustained productive struggle, if they reached an impasse as a result of their initial attempt (the Additional Attempt Phase). A participant was determined to have reached a perceived impasse if they affirmed they were substantially stuck and unsure how to continue (VenLehn et al., 2003). Mathematical productivity was determined based on the extent to which the participant perceived themselves as better understanding the mathematical situation as a result of their efforts. To substantially capture the student point of view during engagement with tasks, coding decisions – or whether or not certain engagement constituted evidence of perseverance – depended on student cues from all interviews.

Table 1: Three-Phase Perseverance Framework (3PP)

Entrance Phase	
Clarity	Objectives were understood

Initial Obstacle	Solution pathway not immediately apparent
Initial Attempt Phase	
Initial Effort	Engaged with task
Sustained Effort	Used problem-solving heuristics to explore task
Outcome of Effort	Made mathematical progress
Additional Attempt Phase (after perceived impasse)	
Initial Effort	Engaged with task
Sustained Effort	Used problem-solving heuristics to explore task
Outcome of Effort	Made mathematical progress

I used a point-based analysis with the 3PP to help inform deeper investigation of the ways in which participants persevered on scaffolded and non-scaffolded tasks. Each participants' experiences with each task were analyzed using the framework, and each component in the Initial Attempt and Additional Attempt Phases were coded as 1 or 0, as affirming evidence or otherwise, respectively. Coding decisions were based off of interview quotes. For instance, participants earned a 1 for an Outcome of Effort component only if they affirmed perceiving mathematical progress; such decisions were not based on my own perceptions. Since each task had two objectives and six components per objective, there were 12 framework components to consider, per participant, per task. Thus, 3PP scores ranged from 12 to 0, depicting optimal to minimal demonstrated perseverance in this context, respectively. Participants did not need to completely solve the task to earn a 12, they just had to exhibit perseverance in all 3PP components. Once 3PP scores were determined for all participants' experiences with all tasks, I conducted matched-paired *t*-tests to compare means of 3PP scores between work on scaffolded and non-scaffolded tasks. I also inductively coded interviews to uncover from the participant perspective why they persevered differently on scaffolded tasks compared to on non-scaffolded tasks. For trustworthiness, I enlisted help from two independent coders to analyze participant perseverance and their reasons for doing so. Our inter-rater reliability was 93%.

Results

Participants demonstrated higher quality perseverance when working on scaffolded tasks compared to on non-scaffolded tasks, in general, as evidenced by significantly greater mean total perseverance scores ($M_S = 8.73$, $SD_S = 2.07$, $M_{NS} = 6.00$, $SD_{NS} = 2.60$; $t(9) = 6.816$, $p < .001$) (see Table 2), as well as by participant reports. Differences in participants' perseverance in the Additional Attempt Phase of the 3PP drove this general finding. Participants demonstrated higher quality perseverance after encountering a perceived impasse while working on scaffolded tasks compared to on non-scaffolded tasks. This was evident by significantly greater mean Additional Attempt Phase perseverance scores ($M_S = 3.67$, $SD_S = 2.50$, $M_{NS} = 1.60$, $SD_{NS} = 2.58$; $t(9) = 4.083$, $p = .003$) (see Table 2), as well as by participant reports. All participants affirmed in the Entrance Phase that they understood the objectives, but did not know how to solve each task. Also, all participants reported a perceived impasse as a result of their engagement with each task.

Table 2: Three-Phase Perseverance Scores

Total Perseverance Scores (Maximum of 12 Points)		
Task Type	**Mean**	***n***
Scaffold	8.73	30 (3 tasks per 10 participants)
Non-Scaffold	6.00	20 (2 tasks per 10 participants)
Difference	2.73*** (*p* < 0.001)	

Additional Attempt Perseverance Scores (Maximum of 6 Points)		
Task Type	**Mean**	***n***
Scaffold	3.67	30 (3 tasks per 10 participants)
Non-Scaffold	1.60	20 (2 tasks per 10 participants)
Difference	2.07** ($p < 0.01$)	

The most prevalent difference between participants' perseverance on scaffolded and non-scaffolded tasks was whether and how they re-initiated and re-sustained a productive additional attempt at solving the problem. This means that while working on scaffolded tasks, participants often continued to productively struggle toward a solution after reaching a perceived impasse. This was not often the case after reaching an impasse while working on non-scaffolded tasks. When considering the specific Additional Attempt Phase components of re-initiating, re-sustaining, and outcome of effort, participants working on scaffolded tasks demonstrated more aspects of perseverance compared to their work on non-scaffolded tasks (see Table 3).

Table 3: Perseverance Frequencies in Additional Attempt Phase

Component	On Scaffolded Tasks	On Non-Scaffolded Tasks
Re-Initiated Effort	80% (24 out of 30)	35% (7 out of 20)
Re-Sustained Effort	70% (21 out of 30)	35% (7 out of 20)
Productive Outcome of Effort	57% (17 out of 30)	20% (4 out of 20)

While talking about their engagement with the five problem-solving tasks in the study, all participants reported a general positive effect of their preliminary conceptualizing work prompted by the scaffolds on their mathematical engagement. Related, none of the participants engaged in any noticeable preliminary conceptualizing work on tasks that did not prompt it (i.e., the non-scaffolded tasks). Most of the participants explicitly mentioned in their interviews that the reason they did not engage in such preliminary work on non-scaffolded tasks was because the task did not specifically ask them to do so, even though they recognized such work was helpful to them. Several more specific themes emerged from the analysis of all interviews that helped explain why participants found it easier to persevere on scaffolded tasks compared to on non-scaffolded tasks, especially after reaching an impasse (e.g., the scaffolds helped participants re-visit a different idea, re-conceptualize the situation after a mistake, stay organized, and establish momentum to stay interested and engaged).

Illustrative Case: James's Experiences with a Scaffolded and Non-Scaffolded Task

To illustrate how perseverance was better supported in scaffolded tasks compared to non-scaffolded tasks, especially by helping participants make a quality additional attempt, consider participant James's experiences with the Cross Totals task and Triangular Frameworks task. For James, Cross Totals was his fourth overall task and his third and final scaffolded task. Triangular Frameworks was his fifth and final task, the second of two non-scaffolded tasks. James passed through the 3PP Entrance Phase on both tasks by affirming he understood the objectives, but that had "no idea what to do." James earned 3PP scores of 6 in his Initial and Additional Attempt Phases, resulting in a maximum overall score of 12 for his work on Cross Totals. On Triangular Frameworks, James earned 3PP scores of 6 and 0 in his Initial and Additional Attempt Phases, respectively, resulting in an overall score of 6, with the most notable difference being no additional attempt at solving after reaching an impasse on Triangular Frameworks.

On Cross Totals, James began his work with the scaffold prompt, brainstorming about the mathematics present, including ideas about the parameters of possible and impossible cross totals and that the middle number would be included in both horizontal and vertical sums. After recording his conceptualization ideas, James started his initial attempt at solving the problem stating he would try to "find some possible ones first, and those might connect to impossible ones." He initiated and sustained his effort by guessing and checking different arrangements of integers along two lines representative of the lines of the cross, with one integer in the middle of both lines, and finding their sums. James noticed he was not having success with his plan, and affirmed he was at an impasse when he said "I'm stuck. This is harder than I thought." During his video-reflection, James revealed "I felt stressful here because I never learned this and there was not any way I would know how to do it. I definitely panicked." He did, however, believe he made some mathematical progress toward both objectives by figuring out "that you couldn't just throw numbers in [the cross], you had to think about big numbers and small numbers."

After James revealed he was at an impasse during his think-aloud, he paused and admitted "I don't know what to do now." Frustrated, he started looking around at his papers in front of him and eventually pointed to his list of mathematical ideas under the scaffold prompt and said, "Well I haven't used [the middle number] yet really. Something might be special about the middle number." During his video-reflection, he explained his point of view during these moments, "I had kind of forgotten that I wrote all this stuff down. So I saw again that middle number idea. It was kind of like my life-preserver…it kept me from giving up."

When James revisited his original conceptualization of this problem, prompted by the embedded scaffold, he was propelled forward into making an additional attempt at solving. He thought-aloud about his plan to re-initiate his effort by studying the middle number in possible and impossible cross totals. He began re-sustaining his effort by changing his point of view, a different problem-solving heuristic, and examined the given example. He noticed that "this one has 9 in the middle to get a 27, they are balancing the big and small numbers... something about the evens and odds." James wrote down his observations around the provided example and went on to explore his own examples (see Figure 2) and solve the problem by concluding and defending that "all solutions are odd, like the middle can't be even to do a cross total."

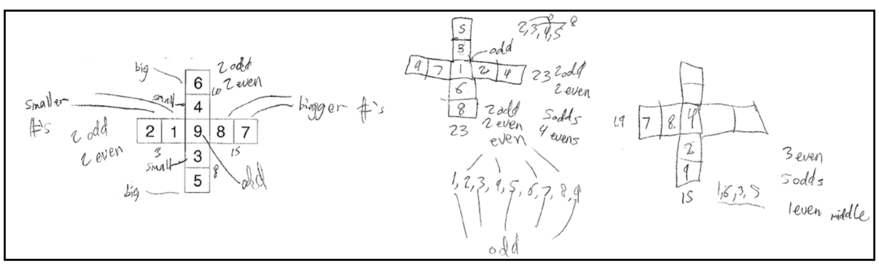

Figure 2. James's Work in Additional Attempt Phase on Cross Totals

On Triangular Frameworks, a non-scaffolded task, James did not record his initial conceptualization of the mathematics in the task. He started his initial attempt toward both objectives by saying "I'll try to find some other evens and odds that work out." He initiated and sustained his effort by guessing and checking different triangles, the same initial heuristic he used in Cross Totals the week before. James examined the given example of six frameworks made with a longest side of 7m, and made mathematical progress when he built four frameworks with a longest side of 6m, writing "6-5-4, 6-5-3, 6-5-2, 6-4-3" on his paper. James concluded, "It looks like it's one less for odd or two less for even, so the rules might be that." Before James wrote his rules, he said, "Wait, let me look at something." Clearly troubled, James went on to write "5-4-3" and "5-4-2" on his paper. Then, seemingly ignoring what he had just done, he wrote his rules about the situation (rules that were incorrect), that if the longest side, *c*, is odd he can make *c*-1 frameworks, and if *c* is even he can make *c*-2 frameworks. Finally, he said "I'm done" and stopped working without making an additional attempt.

While reviewing the video of the aforementioned moments during his first attempt at solving Triangular Frameworks, James admitted that he was, in fact, at an impasse during the latter stages of his first attempt. He said, "I was looking at if 5 was the longest side and my rule didn't work! There should have been four of them but there were only two, 5-4-3 and 5-4-2…so I panicked and pretended like it worked." When asked why he pretended, he said, "I panicked when it didn't work. I didn't even know where to start to fix it. So I really wanted to stop." James' experiences with Cross Totals and Triangular Frameworks are illustrative of the ways in which participants persevered while working on scaffolded tasks compared to on non-scaffolded tasks. Similarly to his engagement on Cross Totals, James leveraged his initial conceptualization work while working on the two other scaffolded tasks to help make a quality additional attempt at solving despite encountering frustrating impasses. On the other hand, similar to his work on Triangular Frameworks, James did not make an additional attempt at solving the other non-scaffolded task; he cited overwhelming stress and feeling disorganized after encountering a setback as the primary reason he did not persevere. During his exit interview, James shared his perspective on how initially attending to conceptualizing the mathematical situation had a positive effect on his engagement for all scaffolded tasks:

> You have to have a plan of what you can try to do…Like the ones that made me write down what I thought first. That was really good. I used that a lot because sometimes you forget where you're going in a problem, it's like chaos – like on [Triangular Frameworks]. But on other ones, like [Cross Totals] I got stuck but had a way out of it. It seems easier with that because you don't have to think of a way out when you're mad or stuck or something.

For James, and for almost all participants in this study, responding first to the conceptualization scaffold served as a "life-preserver" of sorts later, when participants were "panicked" and most tempted to give up. In these moments, the conceptual thinking recorded after engaging with the scaffold prompt acted as an organizational toolbox from which to draw a fresh mathematical idea, or a new connection between ideas, to use to help re-engage with the task upon impasse and to continue to productively struggle to make sense of the mathematical situation. Participants were persevering in problem-solving cyclically, with each additional attempt as a new opportunity to productively struggle supported by their own conceptual ideas (see Figure 3). Without recording their conceptual thinking on non-scaffolded tasks, participants felt lost and frustrated after a setback and often gave up without making an additional attempt at solving.

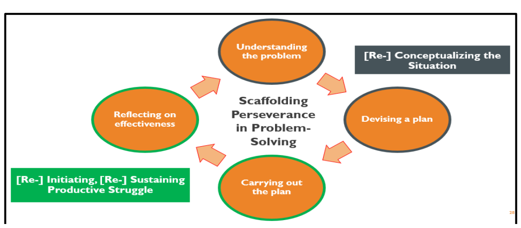

Figure 3. Rethinking Scaffolding Perseverance in Problem-Solving

Discussion and Conclusion

Prior research on supporting student perseverance with mathematical tasks was limited to a focus on the low-floor/high-ceiling structure of the task itself (e.g., Bass & Ball, 2015), teacher moves that nurtured independent student thinking (e.g., Freeburn & Arbaugh, 2017), and using conceptualization scaffolds to encourage initial engagement (e.g., Hmelo-Silver & Barrows, 2006), but did not examine in detail the student perspective around moments when perseverance is necessary, especially upon perceived moments of impasse. Addressing classroom logistical concerns by embedding scaffolds directly into low-floor/high-ceiling tasks (Anghileri, 2006), this study extends past research by investigating from the student point of view how conceptualization scaffolds help students persevere re-initiate and re-sustain productive struggle after reaching an impasse. In sum, the results showed positive effects of prompting students to record their initial conceptualization of a mathematical situation prior to problem-solving, as evidenced by more and higher-quality perseverance on scaffolded tasks compared to on non-scaffolded tasks. Arguably the most notable effect occurred when students reached an impasse while problem-solving, because often they were able to leverage their initial conceptualization to overcome that obstacle and continue to make progress. This implies such scaffolding can help students persevere past impasses and continue to make progress, if provided the opportunity to explore mathematics as a discipline of activity (Schoenfeld & Sloane, 2016). Further, greater efforts are needed to establish attending to conceptual thinking as a normative and welcomed school practice during all problem-solving, not just when prompted (Warshauer, 2014).

Future work should seek to further validate the impact of conceptualization scaffolds on student perseverance after impasse. Especially important is collecting evidence to refute the alternative explanation that differences in task difficultly could explain these data reported here. Although I made several efforts to ensure each task was similarly difficult (vetted by MAP, analyzed by experts, rated by participants), a follow-up study could randomize the assignment of scaffolds to tasks for each participant to better control for this potential bias.

References

Anghileri, J. (2006). Scaffolding practices that enhance mathematics learning. *Journal of Mathematics Teacher Education, 9*(1), 33-52.

Bass, H. & Ball, D. L. (2015). *Beyond "you can do it!": Developing mathematical perseverance in elementary school*. http://www.spencer.org/collected-papers-april-2015

Bieda, K. N. & Huhn, C. (2017). Investigating problem-solving perseverance using lesson study. *Mathematics Teacher, 111*(3), 207-212.

Dewey, J. (1910). *How we think.* Boston: Heath.

DiNapoli, J. & Marzocchi, A. S. (2017). Productive struggle: What we can learn from working with pre-service teachers. *The ComMuniCator, 41*(4), 10-13.

Freeburn, B, & Arbaugh, F. (2017). Supporting productive struggle with communication moves. *Mathematics Teacher, 111*(3), 176-181.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. *Second Handbook of Research on Mathematics Teaching and Learning*, *1*, 371-404.

Hmelo-Silver, C. E., & Barrows, H. S. (2006). Goals and strategies of a problem-based learning facilitator. Interdisciplinary *Journal of Problem-Based Learning, 1*(1), 21-39.

Kapur, M. (2009). Productive failure in mathematical problem solving. *Instructional Science, 38*(6), 523-550.

Kress, N. E. (2017). 6 essential questions for problem solving. *Mathematics Teacher, 111*(3), 190-196.

NCTM. (2014). Principles to actions: Ensuring mathematical success for all. Reston, VA: Author.

Pólya, G. (1971). *How to solve it* (2nd ed.). Princeton: Princeton University Press.

Schoenfeld, A. H., & Sloane, A. H. (Eds.). (2016). *Mathematical thinking and problem solving*. Routledge.

VanLehn, K., Siler, S., Murray, C., Yamauchi, T., & Baggett, W. (2003). Why do only some events cause learning during human tutoring? *Cognition & Instruction, 21*(3), 209–249.

Warshauer, H. K. (2015). Productive struggle in middle school mathematics classrooms. *Journal of Mathematics Teacher Education*, *18*(4), 375-400.

ACHIEVING, LABORING, AND SURVIVING: POSITIONING THROUGH MATERIAL AND IDEATIONAL IDENTITY RESOURCES IN STUDENT AUTOBIOGRAPHY

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This study examined the identity resources used and the subject positions taken by secondary students in autobiographical narratives of their experiences with mathematics (mathographies). Data consisted of mathographies from 54 students in two differently tracked math courses at a large, diverse public high school in Brooklyn, NY. The study asked 1) What material resources do students use to position themselves as different kinds of mathematics students? What typology of mathematics identities is revealed? And, 2) What ideational resources do students use to position themselves as different kinds of mathematics students? Findings revealed three predominant mathematics subject positions: celebrated high achiever, earnest laborer, and trauma survivor. Ideational identity resources included discourses of mathematics, of America and American schooling, and of race, language, and immigration.

Keywords: Equity and Diversity, Identity, High School Education

Mathematics is a high status, high power discipline–one that confers special status on those with recognized facility in the content and practice (Apple, 1993). The notion that there exists such a thing as a 'math person,' and assumptions about who gets to be one, are connected to issues of power. Who is seen as smart, who has a pathway to college, and who has access to an upwardly mobile career are all connected through the status of mathematics as a gatekeeper in American society (Martin, Gholson & Leonard, 2010). Even with increasingly abundant research that conceptualizes learning as a process of inclusion, participation and belonging (e.g. Wenger, 1998), questions that examine the conditions of possibility for belonging are still left largely unattended in mathematics education research.

Students, teachers and schools are faced daily with the high-stakes negotiation of who is recognized as a 'math person,' and students from historically marginalized groups in particular have especially complicated access to this high-status marker of success. Studies of the educational landscape of the United States often use demographic data to document stratification at the levels of both access and achievement for young people in mathematics. These studies repeatedly demonstrate inequitable outcomes (National Center for Education Statistics, 2013), as well as inequitable opportunities, (Storage, Horn, Cimpian & Leslie, 2016) along the lines of race, ethnicity, language, gender, (dis)ability and socio-economic status. Even as some research has shifted from a focus on an outcome-oriented "achievement gap," to an input-oriented "opportunity gap," these studies still tell an incomplete story. Adding an otherwise absent dimension to this work, recent scholarship demonstrates ways in which schooling, and even mathematics learning itself, can include "*racialized forms of experience* [emphasis original]—that is, … experiences where race and the meanings constructed around race become highly salient" (Martin, 2006, p. 198). Beyond its well-understood status as gatekeeper, mathematics education has the potential to reproduce racialized experiences present in other aspects of social, political and economic life.

In one example of racializing experiences found in mathematics learning spaces, Shah & Leonardo (2016) share the story of a South Asian American student, Akshay, who performs at a high level in his AP Calculus course only to find his effort dismissed by a classmate who comments that he simply does well because he is Indian. In the same school, a Black student, James, is treated with surprise and even disbelief when his classmates see that he is enrolled in an advanced math course. As these examples suggest, mathematics education may actually contribute in complex ways to *minoritization* – the formation of a marked minority status due to racial, ethnic, linguistic, immigration or another social status that diverges from an historical or ideological norm (Omi & Winant, 2002).

Contrary to the popular belief that mathematics is culture-neutral or culture-free, research in the last decade has begun to illuminate how mathematics teaching and learning may play a role in the process of minoritization (e.g. Shah & Leonardo, 2016). Minoritizing experiences of mathematics are particularly problematic because they function to "other" certain students in both their successes and their failures, reproducing exclusive notions of who gets to participate or belong within the discipline of mathematics. Throughout this paper, the word "minoritized" references a broad group of young people for whom their status as a minority is often constructed in direct contradiction to their numerical presence in their school, neighborhood, or even city. This contradiction makes clear the need to understand how this status is produced and reproduced, and in particular what role mathematics education plays in this process.

With these concerns in mind, mathematics education research has begun to attend to the identities of minoritized youth in mathematics, both by asking about the math learning experiences of specific social groups, and by asking about how mathematics identities are shaped by being part of a social identity group (Berry, Thunder & McClain, 2011). Studies of identity, often understood as including both the ways a student defines themself as well as how they are perceived by others, have the ability to bring together questions about learning, participation and inclusion. However, even as identity has become widely accepted as a crucial analytic lens for understanding learning (Gee, 200; Wenger, 1998), the ways that people understand and analyze identity in mathematics education vary significantly (Darragh, 2016; Langer-Osuna & Esmonde, 2016) and we still know little about the interaction between, and co-construction of, social and academic identities in mathematics. Furthermore, we know almost nothing about the *identity resources*–ideas, materials and relationships (Nasir & Cooks, 2009)–that students draw upon or how they use them in order to make sense of themselves in the world of mathematics. In order to explore this gap, the research questions guiding the study were:

1. What material resources do students use to position themselves as different kinds of mathematics students? What typology of mathematics identities is revealed?
2. What ideational resources do students use to position themselves as different kinds of mathematics students?

Methodology

This study took a new approach to understanding the identity work of young people. By looking at the discourses that students draw upon, and analyzing how students use these discourses as identity resources, this work provides insight into the conditions of possibility that shape and constrain student identities in mathematics. By examining *mathographies*–narrative autobiographies of student experiences with mathematics–written by Brooklyn high school students in response to a prompt from their math teacher, this work centers student voices as they describe and comment on their own experiences. Discourse analysis was used to illuminate the sets of language and ideas that students drew upon as identity resources and to identify common subject positions that students took as they positioned themselves as different types of math students.

Context and Data

The 54 mathographies examined in this study were written by students enrolled in a Regents Algebra 2 and Trigonometry course (n=29) and students enrolled in a non-Regents Geometry course (n=25) at a large, racially and linguistically diverse public high school in Brooklyn, NY. In their mathographies, students mentioned over 20 language varieties and 15 different ethnic, racial, or national origin groups. Students in both courses were in the 10th or 11th grades. The mathographies ranged from one to six paragraphs and 200-1043 words. By examining the writing of students at the same school, and in the same grade band, but in two differently tracked courses, the sample attempted to account for a broad representation of experiences of success and perceived ability with regards to mathematics while holding other factors such as age, school site and teacher constant.

Mathography writing was an instructional activity assigned at the beginning of a new semester by one mathematics teacher to two classes of students. The activity was framed as both a “getting to know you” task and an opportunity for student reflection on previous learning experiences. Students had one week to complete and submit their written work to their teacher. The same writing prompt was used in both classes.

The questions in the writing prompt included inquiry into personal details about family and language, as well as questions about experiences with mathematics such as what does math at home look like, when have you learned the most math, and have you ever had a ‘math moment’? Students were also invited to share hopes and dreams for the future and something a math teacher might not know about them if they only knew them in math class. Students were prompted to respond in narrative form to any subset of the questions, provided it told a story of them and mathematics. They did not have to respond to all questions in the prompt.

Coding and Analysis

This study made use of discourse analysis (Wortham & Reyes, 2015), examining student language in their mathographies in order to map the ways in which their language use both served to situate themselves within existing discourses and also at times worked to challenge those discourses. Discourse analysis allowed the researcher to track the linkages between micro-interactional moments and social structures at the institutional and societal levels. Insight into how the micro and macro work together is essential in the treatment of identity and mathematics education. As students negotiate identity they do so drawing not only on classroom-based interaction but also on their full repertoire of experiences as members of the social, cultural, and political world.

Discourse analysis entails identification of patterns both within and across “speech events” (Wortham & Reyes, 2016). In this study, the speech event unit was one mathography (n=54) and the speech act unit was a paragraph (as demarcated by students using conventional line breaks or indentation) (n=233). Coding schema were developed through open coding, and then focused coding was used to further nuance the coding schema (Emerson, Fretz & Shah, 2011). Analysis was conducted through the use of analytic memos. Strategic searches for code co-occurrence and patterns across the memos were conducted using both Excel and Dedoose, a qualitative coding software.

Coding was conducted in three phases. The first phase of coding and analysis focused on research question 1, *What material resources do students use to position themselves as different kinds of mathematics students*? *What typology of mathematics identities is revealed?* First, I identified speech acts (paragraphs) in which students made claims about themselves as mathematics learners or recounted others making claims about them as particular types of mathematics students (n=148) and wrote an analytic memo for each. Patterns were noted in the identity claims being made by students. The memos were compared and cross referenced such that it was possible to propose a typology of mathematics student identity positions that accounted for all of the instances. These speech acts were then also coded for material evidence and analysis identified the material evidence most commonly associated with each identity position from the typology.

The second phase of coding and analysis focused on research question 2, *What ideational resources do students use to position themselves as different kinds of mathematics students?* This phase started by re-examining the speech acts that included student positioning and the material resources that students used to position themselves in order to identify the discourses of mathematics that students were drawing on in these moments. Another round of analytic memos were written for each positioning act. These memos asked for each positioning act, how does the student frame mathematics in the world. Specifically, memos addressed "what is math?", "mathematics for what?" and "mathematics for whom?" for each positioning act.

In order to expand from discourses of mathematics to other potential discourses at play, a third phase of coding identified speech acts in which students positioned their families, languages, culture or nation of origin in relation to their own experiences of mathematics (n=85). A final round of analytic memos were written examining each of these speech acts. Two additional discourse categories were identified: discourses of America and American schooling, and discourses of race and immigration, along with core sub-components of each discourse.

Findings

Three Math Student Subject Positions and the Material Evidence for Each

In response to research question 1, I identified three mathematics subject positions: *celebrated high-achiever*, *earnest laborer*, and *trauma survivor*, explained below. These subject positions can also be thought of as available identities that a mathematics student might assume, achieve, or have ascribed to them. In their mathographies, as students positioned themselves as particular types of math students, or recounted having been positioned by someone else, each student also provided evidence to substantiate the position. The evidence they provided reveals the *material identity resources* that students found available to them for making sense of themselves as different kinds mathematics learners. In the 54 mathographies, 24 students positioned themselves at one point as celebrated high achievers – those with status based on the visibility of their successes in math. Twenty-seven (27) students positioned themselves at one point as earnest laborers – those whose status relies on effort, and 21 students positioned themselves at one point as trauma survivors – those with a significant negative experience that has shaped their current orientation toward math. Note that the sum is greater than 54 (the number of individual mathographies) because many students made use of more than one position over the course of their mathography. A summary of the typology with the related material evidence and one example from each position is provided in Table 1.

Table 1: Typology of positional mathematics identities with material identity resources

Position	Material Resources	Example
Celebrated High Achiever (n=24)	Course grades Test scores Regents Exams Course enrollment Class rank Awards Competition Visibility (positive) Being first, being fast	In school I was doing math like it was nothing. I was top in my class. But every year I'll have a competition with someone to get top place. Always someone that smarter than me. So I'll go home and try to be ahead, going to tutoring and asking my dad, freshman year I was a 95% average student in math had the best teacher, Ms. Alexander. That year math was my focus, I past every test with 95-100. That made the teacher and my parent proud. (MS23G)
Earnest Laborer (n=27)	Course grades Test scores Regents Exams Too much content, too fast Attending tutoring Homework completion Learning steps and procedures Practice and repetition	I'm really not that great in math but I somewhat like mathematics because every time I solve a math problem it makes me feel like I'm intelligent, I'm like everybody else, it's a challenge sometimes that comes with my best effort. Every time I use math in my daily life it greatly benefits me, that being said it is something I use every day and almost at any time especially when it's dealt with money. It's always been difficult for me to learn math, I sometimes feel as if math is like my kryptonite. I know I can't get every question right but at least I know I tried my best even if I don't know what I'm doing. (MS09T)
Trauma Survivor (n=21)	Course grades Test scores Regents Exams Course enrollment Competition Public humiliation Being last, being slow Visibility (negative)	My most negative math moment would be long dividing in the fourth grade. I absolutely hated it, and most of my peers already knew how to do it. Which made me more embarrassed to know that I was the only one in the class that didn't know how to do it. Another negative would be in the 6th grade where I basically got 60s on all my math tests because I didn't get what he was teaching as well as how I even got a 65. I was at the lowest level of dumb in math. (MS26T)

Three Discourses as Ideational Resources

Findings showed that students used cultural and institutional discourses as ideational resources as they positioned themselves, their families, languages and cultures or nations in relation to mathematics. The discourses they used throughout were *discourses of mathematics*, *discourses of American schooling and meritocracy*, and *discourses of race, language, and immigration* in the United States. An overview of the three discourses that students drew upon as identity resources, and the core components of each, as used by students is found in Table 2.

Dominant Discourses of Mathematics. Findings show that even as students relied on varied material resources to position themselves as different kinds of mathematics students, the material resources they used point to a set of widely shared dominant discourses of mathematics itself. Those discourses were that 1) mathematics ability is innate and fixed, and 2) mathematical competence is comparative, exclusive, and measurable through time constrained performances, usually of procedures. These discourses are visible throughout the examples shared in Table 1.

For example, one Algebra 2/Trig student wrote, "I'm really not that great in math" (MS09T), reflecting the perceived fixed-ness of her mathematics ability status. A Geometry student marks her competence through indicators of ease and comparison to her classmates, followed by a description of a competition to prove her status: "In school I was doing math like it was nothing. I was top in my class" (MS23G).

Discourses of American Schooling and Meritocracy. In their mathographies, students also drew on discourses of schooling and the American social context more broadly, layering these with the dominant discourses of mathematics. The two core components of discourses of America and American schooling that students drew upon were 1) Innate ability in tension with development psychology and lived experience and 2) American meritocracy as advancement through legitimate systems that reward both innate talent and work ethic. Whereas discourses of mathematics elevate ideas about innate talent, which are corroborated in discourses of American meritocracy, discourses of American schooling and meritocracy offer an alternative but well-established narrative that elevates the importance of lived experience as well as the merits of hard work. Students who positioned themselves as earnest laborers or trauma survivors wrote about challenges faced and overcome through personal effort and perseverance that shaped the student's relationship with mathematics.

Discourses of Race, Language and Immigration. American liberal ideals that value both individual ambition and competition as well as egalitarianism are complicated by the pressures of assimilation for minoritized racial and ethnic groups and immigrants in the United States. Both standing out and fitting in are complicated for minoritized students. Furthermore, discourses of racial inferiority, both genetically and in terms of educational attitudes, sit as a backdrop, in particular for minoritized students, as they attempt to position themselves with worth and dignity in front of their current math teacher. Throughout their mathographies minoritized students consistently explained experiences of visibility – both positive and negative – as attributable to some form of racial, ethnic, linguistic or cultural experience of difference. They describe experiences of confronting ideologies of racial and linguistic inferiority and they draw on discourses of cultural deficiency themselves. One student wrote,

> The most negative experience happened in second grade. My teacher at that time didn't try to teach me but also made fun of me. I didn't know that much English at time and some of the math problems were in words. It was really difficult to me. My teacher gave up on me but she didn't try at all. She insisted that I should go get my ears checked. She also asked if I'm a mute (MS16T).

At the same time, many more students position themselves and their families as model minorities, citing appropriate attitudes toward education and orientations of aspiration and investment that make them stand out in their success, or, if not in their success, at least in their work ethic. One student attributes her own success to her immigrant parents, positioning their attitude toward mathematics as the appropriate one:

> Having parents that find math one of the most important subjects in school shaped me to be how I am in math class today. I've always found math to be an easy subject to wrap my head around; getting 4's on my state tests. … Whenever my parents talk about currency I love to convert the dollars to taka (Bangladesh currency) with a pen a paper instead of just searching it up on google. It's nice when I get the calculations correct (MS03T).

Table 2: Discourses used as ideational identity resources in student narratives

Dominant Discourses of Mathematics	Discourses of American Schooling and Meritocracy	Discourses of Race, Language, and Immigration
Mathematics ability as innate and fixed	Innate ability in tension with psychology, development, and lived experience	Ideologies of racial and linguistic intellectual inferiority
Mathematics competence as comparative, exclusive, procedural, and time-bound	American meritocracy, work ethic, and advancement through investment in legitimate systems	Discourses of deprivation and deficit alongside discourses of investment and models of success

Discussion

This study contributes to existing theory on the development of mathematical identities for young people by proposing new ways of understanding the identity work in which young people engage. The voices of students as heard through their mathographies reveal the resonance of discourses of mathematics, of schooling and American society, and of race and immigration in the United States. These discourses both shape and constrain the possibilities for how young people understand themselves and present themselves to the world. At the same time, these discourses act as available tools that students necessarily use when shaping ideas about themselves, and bidding for particular representations of themselves in the world.

Ideas about what math is significantly shape and constrain how students see themselves as different types of mathematics students and learners. The material evidence that students used to instantiate their various mathematics student positions demonstrates the limited nature of the notion of mathematics most readily available to students. Discussion of grades, tests, speed, awards, homework, attendance records, and comparisons to other students abound as students position themselves as mathematics students of a certain type. This reveals the extent to which student notions of mathematics are constrained by the dominant discourse of mathematics as a set of skills and procedures, measured comparatively and through time-bound mechanisms. Students position themselves with self-worth and merit in front of their math teacher within the constraints of this dominant discourse of mathematics. Their reliance on the positions of celebrated high-achiever, earnest laborer and trauma survivor likely reveals more about the ways that dominant discourses in mathematics shape the identities available to students than it does about the students themselves as participants in or learners of mathematics.

Discourses of American ambition and American meritocracy complicate and constrain how minoritized students see themselves and present themselves because the meanings behind visibility and achievement have different stakes when they are racialized. Discourses of American society both resonate with and diverge from the dominant discourses in mathematics. As is true in mathematics, American ideals celebrate ambition, standing out and high achievement, especially as compared to others. At the same time, American ideals of hard work, labor and perseverance act as an alternative axis of merit available to students for whom traditional high achiever positions are inaccessible. In their mathographies, students drew widely on the American ideals of hard work and perseverance, presenting their own efforts as valuable and as deserving of recognition, both when they told stories of high achievement, and when their efforts were called upon to stand in the place of high performance.

That said, standing out and fitting in are inscribed with different meanings when the students doing so are racialized or minoritized as existing outside of the dominant norm. Fitting in, when it means assimilation, can be high stakes. Where students associated family or language with negative visibility they often employed deficit discourses to themselves or their home lives. At the same time, where minoritized students experienced positive visibility it was often framed as standing out as compared to other minoritized students. Rather than being celebrated as an exceptional student (full stop), the celebration was of being a model *minority* student. As seen in their mathographies, young people work consonantly to find ways to understand and position themselves as worthy mathematics students, often contesting the centrality of innate ability. However, if the other most readily available discursive tools are the also-racialized American myths of hard work and meritocracy, student options remain severely constrained.

References

Apple, M. W. (1993). What post-modernists forget: Cultural capital and official knowledge. *Curriculum Studies*, *1*(3), 301–316.

Berry, R. Q., Thunder, K., & McClain, O. L. (2011). Counter narratives: Examining the mathematics and racial identities of Black boys who are successful with school mathematics. *Journal of African American Males in Education*, 2(1), 10–23. Retrieved from http://journalofafricanamericanmales.com/

Darragh, L. (2016). Identity research in mathematics education. *Educational Studies in Mathematics*, 1–15. http://doi.org/10.1007/s10649-016-9696-5

Emerson, R. M., Fretz, R. I., & Shaw, L. L. (2011). *Writing ethnographic fieldnotes*. University of Chicago Press.

Gee, J. P. (2000). Identity as an analytic lens for research in education. *Review of Research in Education*, *25*, 99–125.

Langer-Osuna, J. M., & Esmonde, I. (2016). Insights and advances on research on identity in mathematics education. In *First Compendium for Research in Mathematics Education* (pp. 1–36).

Martin, D. B. (2006). Mathematics learning and participation as racialized forms of experience: African American parents speak on the struggle for mathematics literacy. *Mathematical Thinking & Learning: An International Journal*, *8*(3), 287–307. http://doi.org/10.

Martin, D. B., Gholson, M. L., & Leonard, J. (2010). Mathematics as gatekeeper: Power and privilege in the production of knowledge. *Journal of Urban Mathematics Education*, *3*(2), 12-24.

National Center for Education Statistics. (2013). *The Nation's Report Card: A First Look: 2013 Mathematics and Reading* (NCES 2014-451). Institute of Education Sciences, U.S. Department of Education, Washington, DC.

Nasir, N. S., & Cooks, J. (2009). Becoming a hurdler: How learning settings afford identities. *Anthropology and Education Quarterly*, *40*(1), 41–61. http://doi.org/10.1111/j.1548-1492.2009.01027.x.41

Omi, M., & Winant, H. (2002). Racial formation. In P. Essed & D. T. Goldberg (Eds.), *Race Critical Theories: Text and Context* (pp. 123–145). Blackwell.

Shah, N., & Leonardo, Z. (2016). Learning discourses of race and mathematics in classroom interaction. In I. Esmonde & A. Booker (Eds.), *Power and privilege in the learning sciences: Critical and sociocultural theories of learning* (pp. 50–69). New York and London: Routledge.

Storage, D., Horne, Z., Cimpian, A., & Leslie, S. (2016). The frequency of "brilliant" and "genius" in teaching evaluations predicts the representation of women and African Americans across fields. *PLOS ONE*, 1–18. http://doi.org/10.1371/journal.pone.0150194

Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. Cambridge University Press.

Wortham, S., & Reyes, A. (2015). *Discourse analysis beyond the speech event*. Routledge.

SOCIAL JUSTICE DRIVEN STEM: ACHIEVING EQUITY GOALS THROUGH INTEGRATED MATHEMATICS EDUCATION

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This study investigated how integrating social justice issues, STEM practices, and mathematics may support equity in mathematics education. We analyzed video of four lessons focused on inverse trigonometry and disability rights from a STEM project-based geometry class. Using an established observation protocol, we identified themes related to access and participation in coherent and cognitively demanding mathematics, student voice, and opportunities to develop positive mathematics identity. Findings provide insights into project and lesson structures that support balancing mathematics and social justice goals across STEM projects and point to additional considerations of equity not fully captured by the existing observation protocol.

Keywords: Equity and Diversity; Geometry; High School Education

Calls for integrated STEM education in K-12 schools have increased in recent years, and strategic STEM education plans often identify project-based learning (PBL) as one approach to integrated STEM education (e.g., Chief Education Office, 2016; STEM Leadership Council, 2018). These STEM education plans argue that full integration of STEM education in K-12 learning is necessary to ensure all students, particularly those from economically disadvantaged in STEM fields. Simultaneously, efforts to use mathematics to critically investigate social justice issues increasingly offers another approach to integrating mathematics and other learning goals (e.g., Esmonde, 2014; Keith McNeil & Fairley, 2016). Some mathematics education scholars argue that teachers face a moral and ethical imperative to transform mathematics classrooms into spaces for the development of critical social awareness and social transformation (Stinson, 2014). Both of these approaches to integrating mathematics with other content areas and topics overlap in their goals to make mathematics learning more accessible and equitable for those students who have been systematically and historically marginalized in mathematics.

Envisioning integrated models for mathematics education may hold promise for addressing longstanding inequities in mathematics. This study investigates the intersection of STEM PBL and social justice in mathematics classrooms with students from economically disadvantaged backgrounds and minoritized racial/ethnic groups. In integrated mathematics lessons, however, balancing mathematics learning goals with other goals (e.g., social justice, STEM) presents significant challenges for teachers (Bartell, 2013), and little is known about whether or not students actually experience these integrated lessons as more equitable (Harper, in press). Thus, the possibility exists that attempts to integrate social justice goals and STEM might further exacerbate equity issues. We aim to investigate the potential for integrating social justice and STEM goals into mathematics education by asking: does (and if so, how does) the integration of social justice mathematics with STEM PBL support goals for equity in mathematics education?

Theoretical Framework

Historically, research towards equity in mathematics education has prioritized achievement and access. *Achievement* relates to tangible outcomes on measures such as standardized tests or course-taking patterns, and *access* relates to tangible resources such as high-quality mathematics teaching and rigorous, coherent mathematics curricula (Gutiérrez, 2012). Achievement and access reflect dominant dimensions of equity in mathematics education because of the widespread, historical attention these aspects have received and because of the acceptance of their importance in mainstream mathematics education research. In addition to these dominant dimensions of equity, we include a focus on under-recognized and under-explored dimensions of equity, identity and power, in this study (Gutiérrez, 2012). Students develop their mathematics *identities* based on dispositions and beliefs about their ability to learn, do, and use mathematics (Martin, 2006). Because school mathematics traditionally marginalizes certain ways of knowing, students must negotiate their personal (racial, ethnic, gender, etc.) identities as they develop their mathematics identities, but students of Color and students who are economically disadvantaged rarely have opportunities to see themselves as competent mathematics learners, doers, and users (Gutiérrez, 2012; Martin, 2006). Alternative ways of knowing in mathematics challenge traditional *power* structures in mathematics classrooms, and a more complex conception of equity attends to issues of social transformation such as whose voice is heard and what mathematics reveals about social justice issues (Gutiérrez, 2012). Identity and power represent critical dimensions of equity in mathematics education because attention to identity and power reflects a more recent sociopolitical turn in mathematics education (Gutiérrez, 2013). We adopt this framework for equity, which attends to both dominant and critical dimensions, and aim to describe what these dimensions look like in classroom interactions among teachers and students. Namely, we sought to answer the following questions: (1) How does access to mathematics content and the emphasis on student identity and power vary across a social justice-oriented, STEM project in a mathematics classroom?; and (2) What dimensions of equity within a social justice-oriented, STEM project-based classroom are not captured by current frameworks for observing mathematics classrooms?

Research Design and Methods

This study drew on a subset of data from a larger study that examined how students took up, negotiated, shifted, or resisted an innovative approach to coupling equity-minded mathematics instruction with STEM PBL across an academic year. The study took place at a STEM-themed magnet school, in a low-income area of a small Midwestern city, whose mission emphasizes technology-driven (1:1 student-to-computer ratio) PBL. The study occurred in one ninth grade geometry classroom in which the teacher integrated various equity-oriented instructional approaches, including integrating social justice and mathematics topics, into STEM PBL in mathematics. Of the 16 consented research participants, six are young men (4 white, 1 Black, 1 Latino), and ten are young women (9 Black; 1 Asian American). The teacher participant is a White woman who was in her fourth year of teaching. She and the first author collaborated on social justice mathematics for three years, and the teacher sought various other professional development opportunities (e.g., technology, equitable collaboration).

The research design was rooted in ethnographic and critical traditions of educational research (Anderson-Levitt, 2006; Skovsmose & Borba, 2004). The first author observed 93 total class sessions as a participant-observer. Data included video and audio recordings, field notes, and photographs from seven major projects and two mini-projects (all teacher-designed), as well as interviews assessments with students This study examined only video data from a mini-project in late April focused on inverse trigonometry and disability rights (hereafter, the mini-project).

Mini-Project Overview

Across four days, students worked in pairs to determine whether or not ramps at the school were compliant with regulations set by the *Americans with Disabilities Act* (1990). The primary mathematics goal focused on learning to use inverse trigonometric functions to find an unknown angle when two or more side lengths of a triangle are known. The primary social justice goals focused on understanding the meaning and need for the *Americans with Disabilities Act* (ADA) and reflecting on whether persons with disabilities had fair access to and within the school. Students utilized technology and engaged with some aspects of design work, but the integration of STEM practices was more limited in the mini-project than in the major projects. The design aspects of the project involved collecting measurement data from physical ramps at the school and creating scaled drawings of ramps. Although students had an option to use dynamic mathematics software to model ramps, all students elected to draw by hand. Technology use mostly involved word processing, calculators, and researching ADA history and regulations. As the final mini-project artifact, students wrote letters to the principal about their investigation, making recommendations for school improvements using recently acquired funding. Although there were distinct goals for mathematics learning, social justice learning, and STEM practices, various activities throughout the mini-project and the final letter required integration across these goals and practices.

Data Analysis

Because achieving equity goals in mathematics education depends upon particular relationships between teacher and student behaviors, we sought to operationalize access, identity, and power based on observations of the entire class – teacher and students – throughout the mini-project. We did not consider achievement because pre- and post-assessment data reflected progress over the entire year, not over individual projects. We used the Teaching for Robust Understanding of Mathematics (TRU Math) Rubric (The Algebra Teaching Study and Mathematics Assessment Project, 2014) to describe how access to mathematics content and the emphasis on student identity and power varied across the mini-project. We chose this rubric because it was designed to capture overall activities and interactions (i.e., not distinguishing between teacher and student behaviors) along five dimensions, which the rubric developers claim comprehensively describe powerful mathematics classrooms (Schoenfeld, 2014). The identification of these five dimensions as key aspects of mathematically powerful classrooms (i.e., "classrooms that produce students who do well on tests of mathematical content and problem solving" (Schoenfeld, 2014, p. 406)) was based on cumulative research in mathematics education suggesting that access to coherent and cognitively challenging mathematics, an emphasis on mathematical sense making, and equal participation are necessary aspects for students' mathematics success. These dimensions of powerful mathematics classrooms, and thus the TRU Math Rubric, map onto ways equity in mathematics education has historically been theorized (Gutiérrez, 2012), making this rubric a particularly good starting point for considering whether social justice, STEM, and mathematics integration achieved equity goals. More specifically, we used the TRU Math Rubric to operationalize *access* as the extent to which participation in coherent and cognitively demanding mathematics was available to students and *identity* and *power* as the extent to which student voice, thinking, or contributions drove the mathematics and gave students an opportunity to develop a positive mathematics identity. Table 1 gives an overview of the five dimensions included in the TRU Math Rubric and shows how we mapped each dimension to the theoretical constructs of access, identity and power.

Table 1: A Map of Dimensions of Equity to the Dimensions of the TRU Math Rubric

Access	**Mathematics**	*The extent to which the mathematics discussed in the observed lesson is focused and coherent, and to which connections between procedures, concepts and context are addressed*

(dominant dimensions)	**Cognitive Demand**	*The extent to which classroom interactions create and maintain an environment of productive intellectual challenge that is conducive to students' mathematical development*
	Access to Mathematical Content	*The extent to which classroom activity structures invite and support the active engagement of all of the students in the classroom with the core mathematics being addressed*
Identity & Power (critical dimensions)	**Agency, Authority, & Mathematical Identity**	*The extent to which students have opportunities to conjecture, explain, make arguments, and build on one another's ideas in ways that contribute to students' development of agency, authority, and identities as doers of mathematics*
	Uses of Assessment	*The extent to which the teacher solicits student thinking and instruction responds to those ideas, by building on productive beginnings or addressing emerging misunderstandings*

Each author independently analyzed video recordings using the TRU Math rubric to: (1) identify segments by participation structure (whole class, group work, or individual work); (2) divide segments by participation structure into five minute or less segments; and (3) rate (1 – low level, 2 – mid level, 3 – high level, 8 – not enough information, and 9 – not applicable) the activities of each segment along the five dimensions described in Table 1 by using unique rubrics corresponding to each participation structure (for full rubrics see Schoenfeld, A. H., Floden, R. E., & the Algebra Teaching Study and Mathematics Assessment Project, 2014). Each recording was approximately one hour in length and was split into fourteen segments on average. We used spreadsheets to record the participation structure, duration, TRU Math rating for each dimension, and brief memos about observations on each segment. We ensured reliability by meeting and comparing our ratings and memos for each segment and resolving differences through discussion and rewatching that segment together until we reached consensus.

Findings

Across the mini-project, we observed three classroom participation structures: whole class activities (36.49% of total time across four days), group work (28.3%), and individual work (31.37%). Note that 3.85% of total class time was identified as not applicable and is shown in gray in Figures 1 and 2. Whole class activities included launch of activities (18.62% of total time across four days), teacher exposition (6.94%), and whole class discussion (23.15%). Table 2 summarizes mini-project activities and their durations.

Table 2: Summary of Mini-Project Activities and Their Duration

	Activity Summary / Participation Structures	Time (min)	% of Lesson
Day 1	Warm-up by solving equations using inverse operations Launch, Individual work, Discussion	11.75	23.8%
	Introduction to using inverse trigonometric functions to solve for unknown angles in right triangles Launch, Exposition	5	10.2%
	Inverse trigonometry task, adapted from the CPM (2013) *Core Connections: Geometry*, Lesson 5.1.3, involving hypothetical wheelchair ramps Launch, Group work	32.5	66.0%

Day 2	Connecting math practices necessary for project to evaluation rubric	32.95	58.9%
	Launch, Individual work, Discussion		
	Measuring base, height, and/or hypotenuse of selected ramp in school	22.95	41.1%
	Launch, Group work, Exposition, (N/A)		
Day 3	Revising measurements, producing scaled drawings, calculating angles of measured ramps	46.4	83.0%
	Launch, Group work, (N/A), Exposition		
	Writing letters to summarize findings and make recommendations	9.5	17.0%
	Individual work, Exposition		
Day 4	Reflection on evaluation from previous project	5.9	11.6%
	Launch, Individual work, Exposition		
	Writing letters to summarize findings and make recommendations	33.2	65.4%
	Launch, Individual work, Exposition		
	Discussion of meaning/importance of ADA law and equity concerns raised through this investigation	11.7	23.0%
	Launch, Discussion		

Access to coherent and cognitively demanding mathematics content

Figure 1 shows the average TRU Math Rubric scores for three dimensions for each participation structure across the four days of the mini-project. Duration of participation structures are drawn to scale, with whole class activities in blue (light blue for launch, dark blue for exposition, and blue for discussion); group work in green; and individual work in orange.

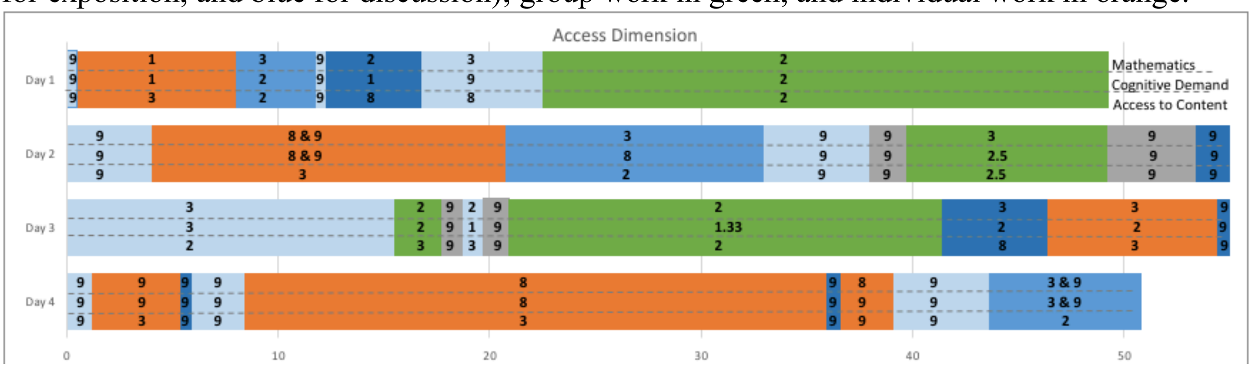

Figure 1. Access Dimension. Average TRU Math Rubric Scores for Mathematics, Cognitive Demand, and Access to Mathematics Content by Participation Structure

Taken together, the three dimensions shown in Figure 1 provide insights into how participation in coherent and cognitively demanding mathematics (i.e., access), as well as STEM practices and social justice content, varied by participation structure and day. Average scores (calculated by averaging scores across segments for the duration of each participation structure) for the "access to content" dimension suggest that, overall, broad (i.e., equitable) participation was achieved (score of 3). In some cases, uneven participation was observed, but the teacher made efforts to achieve equitable participation (score of 2). The average scores for the "mathematics" and "cognitive demand" dimensions provide information about the nature of that participation, respectively, the coherence/meaningfulness of mathematics and the level of cognitive challenge involved in participation. Although overall average scores of 2-3 were common for these dimensions, we noticed some variation by participation structure and day.

During group work on Day 1 and 3, the mathematical content students engaged with was largely skills-oriented (solving for an unknown angle using inverse trigonometry) with some connections to the meaning of inverse (in Day 1) and to the context of the ramps (Day 1 & 3) ("mathematics" score of 2), but the teacher largely drove these connections ("cognitive demand" score of 2). On Day 2, however, students had an opportunity to engage with explanation of mathematics practices/content as they collaborated to measure a ramp within the school ("mathematics" score of 3). The ramp students selected could not be measured in a straightforward way because it was two combined ramps (i.e., involved a switchback) and was adjacent to stairs. As students worked to decide how to define and measure the height and base of each ramp, they (rather than the teacher) took on the cognitive challenge involved with explaining and justifying their mathematical thinking ("cognitive demand" score of 2.5).

Unlike in group work, whole class and individual activities often focused on coherence and connections in mathematics ("mathematics" score of 3), but the teacher was the primary driver making these connections ("cognitive demand" score of 2). Exceptions included the launch on Day 3 in which students' were challenged to make sense of their ramps measurements by connecting to mathematical ideas and the context and the final whole class discussion (Day 4) in which students undertook the cognitive work of making connections between the mathematics and the social justice issue ("cognitive demand" scores of 3). Further, scoring the "mathematics" and "cognitive demand" for whole class and individual activities presented challenges (indicated by scores of 9: not applicable and 8: not enough information). In some cases, these dimensions were not applicable because the activities did not involve mathematics content. For example, launches and teacher exposition commonly scored 9 because these activities focused on logistics (e.g., how to complete an activity) without any attention to mathematics content. In other cases, however, the challenge of using the rubric to rate segments arose because of the interdisciplinary and integrated nature of activities. For example, rating these dimensions during individual letter writing was challenging because it was impossible to discern students' mathematical thinking from their thinking about writing and the social justice issue. Further, topics of discussion were not always explicitly connected to mathematics (e.g., whole class discussion on Day 4).

Emphasis on student identity and power

Figure 2 shows the average TRU Math Rubric scores for two dimensions for each participation structure across the four days of the mini-project. Duration of participation structures are drawn to scale, with whole class activities in blue (light blue for launch, dark blue for exposition, and blue for discussion); group work in green; and individual work in orange.

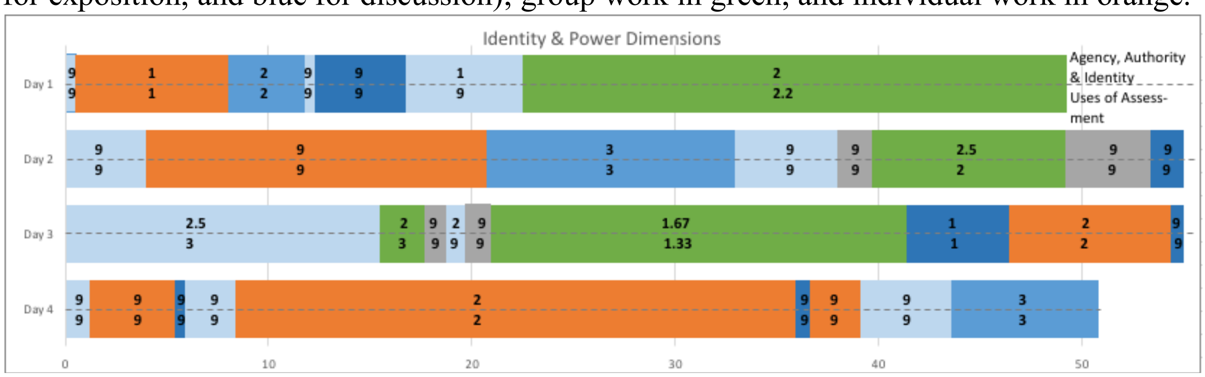

Figure 2. Identity and Power Dimensions. Average TRU Math Rubric Scores for Agency, Authority, and Identity and Uses of Assessment by Participation Structure

Taken together, the dimensions in Figure 2 provide insight into the extent to which student voice, thinking, or contributions drove the mathematics (i.e., power) and gave students an opportunity to develop a positive mathematics identity. Overall, during group and individual work, at least one student had an opportunity to talk about mathematical content, but the teacher drove conversations and determined mathematical correctness. Students were not supported to build on each other's thinking ("agency, etc." score of 2), and the teacher did not necessarily build on student ideas (e.g., lead students in the "right" direction) ("assessment" score of 2). In whole class activities on Day 2, 3, and 4, however, students had an opportunity to explain their thinking, with other students building on those ideas, the teacher ascribing ownership to those ideas ("agency, etc." score of 3) and instruction building on those ideas ("assessment" score of 3). These instances corresponded to high levels of mathematical coherence/connections and cognitive demand (scores of 3) but uneven participation ("access" score of 2; Figure 1).

Discussion and Conclusion

With increasing recognition of mathematics' important relationships to other disciplines, attempts to integrate other topics (e.g., social justice, STEM) into mathematics education will necessarily require that teachers spend instructional time focused on topics that are not explicitly mathematical. In efforts to integrate social justice goals and mathematics goals, concerns about balancing the focus on mathematics and other goals are not new (Bartell, 2013). A major concern is that time taken away from a direct focus on mathematics may exacerbate inequities in mathematics (Harper, in press). This study provides insights into how particular project structures might support balancing mathematics goals with social justice and STEM goals.

Although aspects of the mini-project focused on skill-oriented mathematics, non-mathematical topics, or project logistics, the activities that scored poorly on measures of coherent and cognitively demanding mathematics seemed to have served an important purpose in the overall scope of the mini-project. The work that students did on Day 1 and the first half of Day 2, combined with the integration of STEM practices in the second half of Day 2, likely prepared students to engage in coherent, cognitively demanding mathematics as they measured ramps and made sense of those measurements by connecting to mathematical ideas (from Day 1) and the context at the beginning of Day 3. Although subsequent mathematical work was largely procedural and letter writing was not always focused explicitly on mathematics, scores of participation were highest when students worked in pairs or individually on their letters to integrate mathematics, social justice, and STEM learning from across the mini-project. In other words, *all* students engaged with the intended mathematics, social justice, and STEM ideas, at least to some degree. Further, the final whole class discussion provided some insight into the integrated and equitable learning that occurred through this letter writing process. Although participation was somewhat uneven in this final discussion, the project culminated with high levels of coherent, cognitively demanding mathematics and student voice and authority with meaningful connections to the social justice issue.

Finally, this study provides insights into additional considerations that are likely important for achieving equity goals through integrated mathematics education. Namely, models for powerful mathematics classrooms, and the observation protocols designed to capture dimensions of those classrooms, do not account for the ways that non-mathematical aspects of social justice-oriented STEM PBL support access to coherent and cognitively demanding mathematics (as discussed previously) and how those non-mathematical aspects may foster additional aspects of power and identity. The framework used in the current study only captured power as student voice and authority and focused only on mathematics identity. In this study, the level of engagement with and passion for discussing the social justice issues during the whole class discussion on Day 4 are not reflected in the current analysis. Because students negotiate their personal (racial, ethnic, gender, etc.) identities as they develop their mathematics identities (Martin, 2006), attempts to frame equity in observations of mathematics teaching and learning must also incorporate a way to identify opportunities for students to bring other salient identities into the mathematics space through integrated mathematics education. For social-justice, STEM PBL specifically, this might involve adding an additional dimension that identifies the extent to which students use mathematics and/or STEM practices as analytical tools to understand and transform social justice issues (i.e., power) and how students see those issues as important and relevant to them (i.e., personal identity). Considering the level of engagement with disability rights that students came to recognize as relevant and important through this mini-project, non-mathematical discussions present an important opportunity to understand how students see their other identities as salient in mathematics when that mathematics is integrated with social justice and STEM. Looking ahead towards integrated mathematics education demands that we re-imagine what dimensions are necessary for powerful mathematics classrooms so that those classrooms support broader equity goals.

References

Anderson-Levitt, K. M. (2006). Ethnography. In J. L. Green, G. Camilli, P. B. Elmore, A. Skukauskaite, & E. Grace (Eds.), *Handbook of Complimentary Methods in Educational Research* (pp. 279–295). Muhwah, NJ: Lawrence Erlbaum Associates.

Bartell, T. G. (2013). Learning to teach matheamtics for social justice: Negotiating social justice and mathematical goals. *Journal for Research in Mathematics Education*, *44*, 129–163.

Chief Education Office (2016). *STEM education plan: Driving individual, community, and state prosperity*. Salem, OR: Chief Education Office. Retrieved from http://education.oregon.gov/wp-content/uploads/2016/11/STEM-Education-Plan-Final_CEdO_Nov_2016.pdf.

CPM Educational Program (2013). *Core Connections Geometry.* Sacromento, CA: CPM.

Esmonde, I. (2014). "Nobody's rich and nobody's poor...It sounds good, but it's actually not": Affluent students learning mathematics and social justice. *Journal of the Learning Sciences*, *23*, 348–391.

Gutiérrez, R. (2012). Context matters: How should we conceptualize equity in mathematics education? In B. A. Herbel-Eisenmann, J. Choppin, D. Wagner, & D. Pimm (Eds.), *Equity in discourse for mathematics education: Theories, practices, and policies* (pp. 28–50). New York: Springer.

Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Mathematics Research Education*, *44*, 37–68.

Harper, F. K. (In press). A meta-analysis of teaching mathematics for social justice in action: Promises and problems of practice. *Journal for Research in Mathematics Education*.

Keith McNeil, C., & Fairley, C. (2016). Developing sociopolitical consciousness through Lorraine Hansberry's A Raisin in the Sun: An interdisciplinary project. *Teaching for Excellence and Equity in Mathematics*, *7*, 22–35.

Martin, D. B. (2006). Mathematics learning and participation as racialized forms of experience: African American parents speak on the struggle for mathematics literacy. *Mathematical Thinking and Learning*, *8*, 197–229.

Schoenfeld, A. H. (2014). What makes for powerful classrooms, and how can we support teachers in creating them? A story of research and practice, productively intertwined. *Educational Researcher*, *43*, 404-412.

Schoenfeld, A. H., Floden, R. E., & the Algebra Teaching Study and Mathematics Assessment Project. (2014). An introduction to the TRU Math document suite. Berkeley, CA & E. Lansing, MI: Graduate School of Education, University of California, Berkeley & College of Education, Michigan State University. Retrieved from http://map.mathshell.org/trumath/tru_math_rubric_alpha_20140731.pdf.

Skovsmose, O., & Borba, M. (2004). Research methodology and critical mathematics education. In P. Valero & R. Zevenbergen (Eds.), *Researching the socio-political dimensions of mathematics education: Issues of power in theory and methodology* (pp. 207–226). Dordrecht, The Netherlands: Kluwer Academic Publishers.

STEM Leadership Council. (2018). *STEM strategic Plan: An integrated K-12 STEM proposal for Tennessee*. Nashville, TN: Division of College, Career, and Technical Education, TN Department of Education. Retrived from: https://www.tn.gov/content/dam/tn/education/ccte/ccte_stem_strategic_plan.pdf.

Stinson, D. W. (2014). Teaching mathematics for social justice: An ethical and moral imperative? *Journal of Urban Mathematics Education*, *7*(2), 1–5.

TACIT, TRICK, OR "TEACH": WHAT IS GINA'S MATHEMATICAL REALITY?

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We present study findings that depict the natural and fractional number knowledge of one third grade student with learning disabilities (LDs) in seven experimental sessions. We utilize qualitative analysis methods to illustrate how this student evidenced her knowledge of natural number and fractions through her interactions with varied learning situations. We argue the child reverted to pseudo-empirical abstractions and tricks to make sense of the situations as opposed her own reasoning. Providing children interventions that promote procedures and actions may be preventing children from adapting their thinking structures.

Natural and rational number understandings are among the most relentless areas of difficulty in school mathematics, especially for children who have varying exceptionalities, such as learning disabilities (LDs) (Mazzocco & Devlin, 2008). Difficulties in understanding natural numbers and fractions impact these students from the early elementary years through their adult life (e.g., Lewis, 2014; Mazzocco, Myers, Lewis, Hanich, & Murphy, 2013) and effect problem solving, computational procedures, and development of reasoning and sense making.

In this paper, we present findings from a design experiment that depicts the natural and fractional number knowledge of one third grade student with learning disabilities (LDs) in seven experimental sessions. We utilize qualitative analysis methods to illustrate how this student evidenced her knowledge of natural number and fractions through her interactions with varied learning situations. Through presenting this data, we raise questions about the child's reality and what the child's apparent knowing and learning was relying upon. The research questions are (1) What is one child identified with a LD's mathematical reality specific to her number and fractional knowledge as modeled across varying task types and (2) What persistent difficulties did the child experience? What seemed to be the function of these difficulties?

Theoretical Framework: The Mathematical Reality of Children with LDs

Historical perspectives regarding the mathematical realities of children with LDs arose from a medical depiction of LD as either innate, neurological impairment (Lewandowsky & Steelman, 1908) or an acquired 'disabling' of calculation centers in the brain despite normal language and speaking skills (Henshan, 1928). Over time, children not affected by injury and thought to have otherwise average or above average intelligence showed similar 'impairments' in mathematics performance (U.S. Office of Education, 1968). Researchers began to define ways to address the perceived impairments in mathematical knowledge (e.g., Hudson & Miller, 2006). In recent years, specific factors (e.g., working memory, processing, spatial reasoning, retrieve basic facts, identifying and/or compare number magnitudes and symbols) were tested alongside instruction (or before and after instruction) in a predictive manner to both explain and remediate the child's mathematical reality (e.g., Compton, Fuchs, Fuchs, Lambert, & Hamlett, 2012; Jordan, 2007; Mazzocco & Devlin, 2008; Murphy, Mazzocco, Hanich, & Early, 2007; Vukovic, 2012). By and large, the field continues to define LDs in children's mathematical realities as innate, neurological differences even today.

Yet, this definition of LD is incomplete and at times misleading for those who do not equate *biological difference* as a *disabler* or a *remediation* with *learning*. We argue that biological variations in the brain are far more dissimilar than they are similar (Compton et al., 2012) and that regardless of variations from a norm, these children still hold knowledge (DiSessa, 1988). If this is true, then we also have to accept that even if children labeled LD may evidence knowing or engage with situations in unexpected ways that their reasoning cannot be conceptualized as “impaired” or even “different”. It must be conceptualized as *their knowing*: a complex, unique organism truly known only to the child. In this way, we draw from Piaget (1972/1980), who viewed *learning as adaptation*, one that any child makes to her internal cognitive schemes, as she navigates her environment. In this way, Piaget (1972) links adaptation with instruction.

Adaptations children make in their reasoning, or not, become negotiated by *the environment and the larger system*. These negotiations *do* begin in the child’s mind, yet the child’s mind and the goal-driven activity that produces learning is variant and system dependent. Depending on the system, a child with “LDs” might very well be considered disabled in one classroom or learning environment and not another (McDermott, 1993). In a similar way, this child could be disabled in one kind of mathematical instruction (Lambert, 2014) but not in another. These factors can work to enable or disable aspects of the child’s unique knowing and learning.

Number Sequences and Composite Units

Within the child’s mind lies the potential construction of a meaningful mathematical reality. Steffe and Cobb (1988) describe four distinct types of number sequences children may evidence and adapt to understand other mathematics: (1) Initial Number Sequence, (2) Tacitly-Nested Number Sequence, (3) Explicitly-Nested Number Sequence, and (4) Generalized Number Sequence. Each number sequence can be related to children’s composite unit coordination.

Initial Number Sequence. Steffe (1992) posits that children who “count on” think with an *Initial Number Sequence (INS).* A child who uses an INS is characterized by her counting of single units and then segmenting of a numerical sequence. Through this segmenting, the child unitizes numerical sequences and interiorizes the rules of patterns (e.g., develop one composite unit and count on from this composite unit to a second composite unit, 4…5-6-7-8).

Tacitly-Nested Number Sequence. Once a child has developed and coordinated composite units through an INS, she can use composite units both as a unit in which to count on from and one to keep track of when counting. This is a *Tacitly-Nested Number Sequence (TNS).* The activity, while similar to an INS, supports the child to hold a start and stop value AND count on additional items from the stop value (e.g., three more items were added to the total of 8…9-10-11). Essentially, a child who thinks with a TNS take the result of her counting as a composite unit in which to use for a new problem. Ulrich (2015) explains this child is aware of her counting acts as both (a) a segment of a numerical sequence (i.e., the difference between 4 and 11 is 5, 6, 7, 8, 9, 10, 11) and (b) the numerical sequence created through this segmenting (i.e., 5 is 1, 6 is 2, 7 is 3, 8 is 4, 9, is 5, 10 is 6, 11 is 7). The awareness of one number sequence contained inside another, or *double-counting,* is an indication of TNS, as is a skip count to solve early multiplicative kinds of problems, such as how many 3’s are contained in 39.

Ulrich and Wilkins (2017) further distinguish between two types of TNS, early TNS (eTNS) and advanced TNS (aTNS). The two main distinctions between these subsets of TNS are described as the degree of abstraction a child relies on when understanding one composite unit while developing a second composite unit. For instance, when asked to find the parts of a whole in a fraction task (with a continuous model), a child who thinks with an eTNS will engage in *equi-segmenting* where she develops a composite unit that she needs to adjust in relation to the whole (Steffe, 2002; Ulrich & Wilkins, 2017). A child who thinks with an aTNS anticipates using a composite unit prior to the activity in a task. The common aspect of each way of reasoning is the propensity to be aware of one number sequence in relation to another number sequence.

Explicitly-Nested Number Sequence. Children who use composite units and the whole simultaneously disembed parts from wholes and develop iterable units of one. These children are thinking with an *Explicitly-Nested Number Sequence* (ENS) (Olive, 1999; Ulrich & Wilkins, 2017). An ENS supports children to multiplicatively understand a single unit and a composite unit without disrupting the whole because they see the summation of the parts and the whole as interchangeable (2017). Olive explains that each nested unit needs to be understood abstractly, allowing two levels of units as a singular abstract composite unit. This nesting of abstract units provides children number structures that are necessary for multiplicative reasoning. For example, a child at an aTNS stage versus an ENS stage would solve the following task in very different ways: 1 + 1 + 1 + 1 + 1. A child at an aTNS stage would need to consider each 1 as its own unit in which to understand the whole (e.g., 1+1=2, 1+2=3, 1+3=4; 1+4=5). A child at an ENS stage would be able to consider the five 1s and the whole 5 simultaneously.

Other sequences: Doubles and halves. Doubling and halving are described as representing initial forms of partitioning and iterating development that children use when transitioning from additive structures towards multiplicative structures. Research studies focused on doubling and halving are minimal in mathematics education (i.e., Confrey, 1994; Kieran, 1994; Empson & Turner, 2006; Steffe, 2002). Findings from these studies suggest that children do not need multiplicative structures to initialize engagement with halving because they can develop doubling operations through symmetry and halving operations through activity such as considering parts of parts (Author, Confrey, 1994).

Kieran (1994) first noted that young children were aware of multiplicative patterns through paper folding; Empson and Turner (2006) further explored this development with elementary school age students. Empson and Turner found that engagement with empirical material resulted in four types of development (non-recursive activity, emergent recursive activity, recursive activity, and functional multiplication). Through this development, Empson and Turner noted that young children began by not being able to relate their folds to the number of parts and to use this solution to solve problems (non-recursive activity) and that first-grade students did not transition beyond this activity. After transitioning from non-recursive activity, students could build towards early recursive activity (emergent recursive) and develop recursive activity (recursive activity) and multiplicative structures where solutions were anticipated (2006). These stages in students' development suggest another means in which to make sense of how students transition from additive, natural number structure structures, towards other mathematics, such as multiplicative or fractional knowledge.

Methods

"Gina"

"Gina" (age = 10 years) attended elementary school in the Northwestern United States. She was identified by her school system as having a learning disability that affected her mathematics performance. Gina's performance on the mandated standardized state measure of math performance was at a low level. Her reading scores were at average levels. Gina received additional support in mathematics concepts and operations that included explicitly modeled strategies and procedures for operations. Gina's individual learning differences included significant difficulties with working memory.

Initial Interview and Design Experiment

Data collection was collected in two semi-structured clinical interviews (Ginsburg, 1997) and seven sessions of a design experiment (Cobb, Confrey, DiSessa, Lehrer, & Schauble, 2003). All sessions were used in data analysis. Sessions took place during school hours and were in addition to the child's regular math class time. The first author was the interviewer and the researcher-teacher and attended all sessions. The second author collaborated throughout the retrospective analysis of sessions. The first author utilized explanations and indicators of Gina's possible number (Olive, 2001), multiplicative (Hackenberg, 2013; Tzur et al., 2013), and fraction (Olive, 2001; Steffe & Olive, 2010; Tzur, 1999, 2000, 2007) conceptions to build a theoretical model of Gina's initial ways of operating. Then, we worked with Gina individually in sessions that lasted about 40 minutes. Although sessions were planned for 40-min time increments, not all sessions lasted 40 min (range of time = 30 to 50 min; average time per session ≈ 40 min). Researchers collected three sources of data: video recordings, written work, and field notes.

Tasks to promote Gina's number knowledge. Consulting the available research and resources (e.g., Tzur & Lambert, 2011), we designed several learning situations to support Gina to adapt her schemes for number toward abstract composite units. Our goal was to promote Gina to "know" that fluent addition strategies use number relationships and the structure of the number system- that numbers can be decomposed and added on in parts, not just by ones. One platform task involved rolling a dice and moving a marker across a linear number game board (Tzur & Lambert, 2011). The object of the game was for the child to tell "How Far from the Start" she was after combining two addends. Another platform task involved counters and covers. In some cases, counters were quantified by the child; the researcher then added to or subtracted from the total and asked about the new total. Constraints on the platform tasks (i.e., covering of an addend or both addends, larger numbers, moving the location of the unknown questions to promote anticipation) were planned such that Gina would hold a start value in her head, count on using the structure of numbers through ten as parts and wholes mentally to add two numbers- an early ENS with small composites. A third task involved small and large units presented as bakery pans. Gina was asked to convert back and forth between the larger (e.g., 3) and the smaller (i.e., 1) to "fill orders".

Fraction knowledge. We also included a second set of learning situations to assess and support Gina's unit fraction concept (Tzur, 2007). Unit fraction concepts can be initially accessed with a tacit number sequence. One platform task was to equally share one item among two people and then share among three and four people. Constraints (no folding for 3 sharers; make the size in fewest number of attempts for 4+ sharers) were designed such that Gina would use iteration, or repeating, of a unit she estimated to be the correct length across the whole and make adjustments opposite the number of pieces.

Data Analysis

Ongoing analysis of critical events (Powell, Francisco, & Maher, 2003) in the child's thinking and learning were noted and discussed before and after each session. The focus was on generating (and documenting) initial hypotheses as to what conceptions could underlie the child's apparent problem-solving strategies during these critical events. These hypotheses led to planning the following teaching episode.

Next, the researchers engaged in a blend of retrospective analyses (Cobb et al., 2003) and fine-grained analysis (Siegler, 2007) to consider Gina's adaptations and interactions with the learning situations. To begin, we gathered the video data for all sessions and transcribed these sessions verbatim. We also examined corresponding written student work and anecdotal notes taken during these sessions. Then, we closely inspected the data for evidence of Gina's understandings and/or shifts in understandings. That is, we conducted a line by line examination of the transcribed data with a side by side of video data to consider how Gina's thinking progressed, or not, across the sessions. We then chunked the data into smaller, more meaningful parts. For each identified segment, we examined very closely each child's spoken and gestured actions along with her utterances and representations, consistently and systematically searching for confirming or disconfirming evidence (Strauss & Corbin, 1994) to ensure credibility. This process allowed us to elaborate on the model of Gina's mathematics as she interacted with the learning environment over time (Steffe, 1995) to gain a clearer picture of adaptations made to her thinking.

Tacit, Trick, OR "Teach": What is Gina's Mathematical Reality?

Gina's Initial Mathematics: Clinical Interview and Session One

In the clinical interview, Gina solved several tasks involving whole number operations the revealed a seemingly tacit nature of her thinking. For instance, she skip-counted by threes to 18 yet had to resort to the use of ones after 15 (e.g., "3, 6, 9, 12, 15…16-17-**18**; 19-20-**21**…."). When asked to solve 8 + __ = 14 in a contextualized story problem, Gina stated, "bottom number bigger, break apart a ten", first counting back with her fingers from 14, keeping a double count, and wrote a number sentence (algorithm). Interestingly, she writes "108" as the answer. When asked if that matched what she did with her fingers, Gina stated that "you have to put a ten there". When asked different ways to make 17, Gina stated there were not many. When given a start number of six, Gina counted up from five to 11 with her fingers, then counted up another six verbally to 17. She did not put the five and six together as a solution. We were unsure of Gina's number sequence.

We also asked questions around Gina' fraction knowledge. When asked to say for which of two cookies she would get a bigger share (one is cut into two parts vertically; the other diagonally), Gina stated that they were the same because "they each are a split of one cookie". In another problem, Gina worked to share a thin rectangular paper called a French fry (Tzur & Hunt, 2014) between two friends. Gina suggested partitioning the bar "into halves", partitioning the bar in the center. When asked if there was a way to know the parts are equal, the child made three parts on each side of the partition line and seemed to explain "equal" as the same number on each side. When asked to share the fry among three people and constrained not to use cubes or fold, the child drew three relatively equal parts that did not take up the entire rectangle. To address the space not used by her three parts, Gina suggested we cut off the rest of the rectangle.

To confirm the participatory nature of Gina's tacit number knowledge, we began Session 1 with the "How Far from the Start Are You?" game. *Excerpt a* begins with Gina's response to "How Far from the Start" she was after rolling 9 then 11 (the first nine spaces were covered with a paper and a "9" was written on the cover):

Excerpt a: Move nine spaces, then move 11 spaces. How far from start are you?

R: Are you thinking about how to make nine?
G: Yeah let me show you [counts up five, places a finger, then counts another four and places a finger, then counts up one and places a finger].
R: Hmm. So, you did five- 1-2-3-4-5 – and then you figured out that the other part of nine was four [child nods]. Did you figure that out in your head before you did anything?
G: Yeah, I was about to do three plus one is four [rolls dice again and gets 11]. I'm trying to figure out how to get to the 11; I haven't done that one before [child uses game board to count ten more by ones, then another for 11 spaces].

R: So, we went nine [grabs cover and writes “9”; covers first nine spaces] and then we went 11. How far from the start are we?
G: [*pauses for 6 seconds; the grabs pencil and begins to write algorithm*]. R: Can you figure it out without writing it down?
G: [*pauses for 9 seconds*] 20. R: How’d you know it was 20?
G: Whenever I do math problems, it’s like a paper and pen to the wall showing the math problem and the numbers magically appear. It’s just my brain.
R: Tell me what you were thinking in your brain. G: [writes down 9 + 11 = 20 vertically]
R: Tell me what that means.
G: This [points to 9 + 11 that she wrote, write 9 + 1 = 10] is how much this adds up. It’s sort of like we’re grouping.
R: So, you took the nine and the one from the 11 and you did ten?
G: [Smiles] Or - you could do 9 [lifts fingers one at a time]…10-11-12-13-14-15-16-17-18-19-20.

In the first session, Gina evidences what could be viewed as tacit knowledge of number through a count on. That is, she seems to start from the nine and count up 11 to stop at 20. Yet, because she already solved the problem visually by “showing the math problem on the wall”, we asked her another problem involving 2 + __ = 8. Gina’s thinking in this problem told a different story.

Namely, Gina began to count up from five and stopped. She then stated that four and four is eight, raised two fingers, began to count up and stopped at seven. The child then began to verbally recite number facts and, after several attempts, eventually stated that two and six would make eight. We also used the fraction tasks from the first session, yet this time we asked Gina to test a share size to see if it was the correct size for one of three equal shares. To do so, Gina repeated the share three times to show a length, yet called the part “one-half”. She then described her activity as “two times”, further explaining the activity as 2 + 1. Looking across the tasks, Gina’s activity suggested that she was operating with a participatory tacit number sequence. Yet, procedural residue made it difficult to say this with certainty.

Sessions Two and Six

In session two, Gina continues to display what we called tacit knowledge in both number and in fractional knowledge. Excerpt b begins with Gina’s response to a missing addends task where she was shown six counters, then the counters were covered. The researcher then said, “I placed more counters under the cover; altogether there are 13. How many counters in my hand?”:

Excerpt b: 6 counters covered, 7 in the researcher’s hand
G: [mumbling]. Six… [pauses 12 seconds] Trick question. R: How so?
G: Well, I already know that six plus four is ten. The trick I was taught was to add another three [shows three fingers to R]. So, it would have been six plus…like…13 [shugs]. Because there’s possibly no way to get to 13 because we’ve already got six [shows six fingers, stares at her fingers. Then begins to raise two more fingers]. Eight…. [pauses 7 seconds] it’s 12. I have to get to….
R: Well I had six and I added some more and now I have 13. You were trying to think about… G: Is this some sort of trick question or what [smiles]?
R: [smiles at child] Let’s back up to something you were saying. Did you say that you had six, and then you put four more onto that and got to ten?
G: [nods]
R: Then how many more did you say to 13? G: Three more.
R: So [shows three on one hand and hour on the other]?
G: 30…34. No, seven. Seven?
R: Do you think I put seven under there?
G: Yes [lifts covers and counts to check, smiles].

In the second session, Gina continues to show a kind of tacit knowledge of number through a count on. That is, she seems to start from the six and counts up to 10. Interestingly, she then seems to no longer rely on her own knowledge. Instead, she reverts to a “trick” to get from 10 to 13. Good evidence for this claims rests in the notion that Gina could not put the four [6… 7-8-9- 10] with the 3 to arrive at a missing addend of seven. Instead, Gina sees a “trick question”.

Similar activity seemed to continually show and then fade in Gina’s reasoning throughout the remainder of the sessions, and became exacerbated when the location of the session was changed from the small environment used in the first four sessions to the room where Gina was used to coming for small group math intervention time during the school day. In the seventh session where Gina’s response to task where she was shown 4 red units (each red unit measured 2 white units). The researcher asked Gina to think about how many white units would be needed to equal the same length. Gina aligned the white units long ways, reasoning that three white units aligned with the red unit as opposed to two and muttered, “3, 6, 9, 12, 15, 18” and smiled. Excerpt c begins with a request from the teacher to consider the total amount of white units if two more red units were added. Gina answered incorrectly and was asked to explain:

Excerpt c: Add two more red units (3s)- how many total white units (1s)?

G: Because this and that [taps table]...it’s like a multistep...so...what we do in my classroom. It’s like, um, it’s supposed to be hard. It used to be hard for me but...it’s easy [writes down the word multistep on the paper]. So that would be considered multistep. Six red and you need two more red. And you already have this over here down [references red units she modeled earlier]. And so, six plus two equals eight [points to where she wrote "multistep"]. It’s a multistep. Then 18 [writes 18] plus two, I mean eight.
R: Why eight?
G: [looks over at other teacher in room] Because six plus two equals eight.
R: That is true. How many white units equal one red unit? G: Three.
R: How many reds are you adding? G: Ah ha! 11! This all equals 11! R: Tell me how.
G: Because this right here... [draws a circle in the air over her earlier representation of the red and white units]. Because this [swipes where she wrote "multistep"] and three equals ... 11.

This excerpt provides clear evidence that what may have disabled Gina was not her innate, cognitive differences but instead prior “tricks” and teaching “strategies” that prohibited her from negotiating her own reasoning in the learning situation. Moreover, the mere presence of the teacher in the room seemed to change Gina’s goals away from reasoning and sense making and toward displaying learning “strategies”. While these strategies were arguable now part of her reality, we question whether this thinking was really “hers” in that she could not call upon it to explain and justify her words. We will expound upon further examples to illustrate this point further in session.

Discussion

We investigated how one student, Gina, evidenced her tacit use of composite units and early fractional reasoning. We begin to describe the difficulties this child experienced - namely, the use of tricks and previously taught strategies to engage with the mathematics as opposed her own reasoning. Toward this end, we add a unique contribution to the literature. Namely, we found that when Gina engaged in natural number tasks, she did appear to be relying on tacit knowledge, at least in part. However, when engaging with other tasks, she reverted to pseudo-empirical abstractions and tricks to make sense of the situations. Gina seemed to take up procedures that prevented her from reflecting on her actions upon units such that she might adapt her thinking structures. For instance, in session 2, Gina segmented the number sequence between six and 13 into four and three. However, we posit that these actions were not organized in such a way that Gina could make sense of the potentially powerful way of operations (i.e., a through ten thinking, or a breaking of seven into four and three). Instead, strategies and tricks seemed to be taught arbitrarily to Gina to afford her success in counting and performance in adding and subtracting.

In other sessions, Gina seemed to have a goal of mimicking her intervention teacher. This is evidence that learning is contributed to by both children and teachers and are negotiated by factors such as interactions, school curriculum, classroom culture, and so forth. For example, a teacher's reaction to a child with an exceptionality who has supplied a solution that they did not expect or time taken to answer may be, "That's not right" followed by explicitly taught teacher thinking. These responses likely reinforce the child as possessing the "problem." Yet, another response (e.g., "Say more") might change the context, opening up space for the child to access the mathematics from his or her own conceptions and develop practices that align with academic success. The interactional context of, "Did they say what I was expecting?", becomes replaced by, "How are they thinking about it?". The problem, removed from the child, becomes explorable as the narrative of the child as disabled begins to unravel.

We argue that instruction for Gina would have better served her mathematical needs if she had been given more opportunities for adapting *her own thinking* constructed within her own mathematical reality. When well-intentioned educators provide children interventions that promote procedures and actions, not only are they not serving their children's mathematics learning needs, they may be preventing them from engaging in learning situations that support the children to adapt their thinking structures and advance their learning. In this sense, from working with Gina, we have lingering questions regarding her development. For instance, over the course of the sessions, Gina evidence an affinity towards doubles and symmetry both within natural number development and early fractional reasoning. Other studies (e.g., Empson & Turner, 2006) have found students rely on doubles and halves in varying tasks. Although not fully reported here, Gina seems to rely heavily on these units across learning situations and across natural and fractional number reasoning. Could this have been a pathway forward which Gina could utilize to overcome her reliance on tricks and make sense her own thinking? More research is necessary to determine real causes of students' difficulties with number alongside research that conforms models of teaching that facilitate these children's understanding, reasoning, and sense-making.

References

Empson, S. B., & Turner, E. (2006). The emergence of multiplicative thinking in children's solutions to paper folding tasks. *The Journal of Mathematical Behavior*, *25*(1), 46-56.

Ginsburg, H. (1997). Entering the child's mind: The clinical interview in psychological research and practice. Cambridge University Press.

Hudson, P., Miller, S. P., & Butler, F. (2006). Adapting and merging explicit instruction within reform based mathematics classrooms. *American Secondary Education*, 19-32.

Hunt, J.H. & Tzur, R. (2017). Where is difference? Processes of mathematical remediation through a constructivist lens. *Journal of Mathematical Behavior*, 48, 62-76.

Mazzocco, M. M., & Devlin, K. T. (2008). Parts and 'holes': Gaps in rational number sense among children with vs without mathematical learning disabilities. *Developmental science*, *11*(5), 681-691.

Tzur, R. & Hunt, J.H. (2015). Iteration: Unit fraction knowledge and the French fry tasks. *Teaching Children Mathematics*, 22(3), 148-157.

OBSERVING CHANGE IN STUDENTS' ATTITUDES TOWARDS MATHEMATICS: CONTRASTING QUANTITATIVE AND QUALITATIVE APPROACHES

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A student's attitude towards mathematics affects how they learn and perform in mathematics. What exactly is meant by attitude and how this interacts with mathematics education is a current debate in the mathematics education research community. Regardless, practitioners often acknowledge a consideration of improving students' attitudes towards mathematics in their course design. This creates an impetus to study attitudes towards mathematics in a way that lends itself to observing changes over a course in mathematics. The current study draws on two approaches to observing and measuring attitudes towards mathematics in an effort to contrast disparate approaches and deepen an investigation of students' changes in attitudes. Results indicate there is no superior approach; that the multi-dimensional nature of attitude defies a succinct description, but methods exist to allow us to get a handle on this construct, nonetheless.

Keywords: Attitudes towards mathematics, undergraduate mathematics education

Introduction and Literature Review

Education practitioners often lament their students' attitudes towards mathematics, and this is employed as an explanation of their poor performance (Di Martino and Zan, 2010). But what is meant by attitude? How might this construct be operationalized, and with a clear articulation of "attitude" in hand, how might students' attitudes be transformed through education and these transformations be observed? Despite decades of research on affect in mathematics education, these questions are only now being addressed in any substantial way (Pepin and Roesken-Winter, 2015; Goldin, et al., 2016).

The most current perspective on attitude is that it is a multi-dimensional construct, with affective, social, emotional, temporal, and other components (Hannula, 2012). Attitude can co-emerge with the setting, and depends on social and individual aspects. In brief – attitude's messy.

Three-dimensional Model for Attitudes (TMA)

In an effort to improve parsimony of the attitude construct, Di Martino and Zan (2010) introduce the Three-dimensional Model of Attitudes (TMA) framework, which conceives of attitudes towards mathematics having i) Emotional, ii) Vision of Mathematics, and iii) Perceived Competence components. These three form stable dimensions of the attitude construct, being based on ~1,600 student essays describing their relationships with mathematics, but each dimension could be further refined in a multitude of ways. Vision of Mathematics could be split into instrumental and relational views following Skemp (1976), for example, or it could be understood in terms of Sfard's structural/relational model (1991). Di Martino and Zan (2010) provide a way to parse the otherwise difficult to grasp attitude construct.

However, theirs is only one such way; indeed, the origins of attitudinal work in mathematics education are in the creation and validation of Likert-type scales; see, for example, (Fenema and Sherman, 1976). Di Martino and Zan (2010) level a valid criticism against much of these origins, in particular against the "measurement era" of creating Likert-type instruments. In particular, Di Martino and Zan (2010) characterize much of the literature as defining "attitude" a posteriori through the instruments used to measure it, creating a researcher's tautology: we chose our research foci because the outcomes of our research indicate they are important and problematic.

It is necessary, then, to have a relatively clear articulation of “attitude” before rigorous research on it can proceed.

Other criticisms towards instruments intended to measure attitudes are presented in Di Martino and Zan (2010). Among these are, i) items run the risk of being chosen in a way that may not be relevant to the survey/instrument responders – responders may be tasked only to give opinions on the items and their underlying structures that are of value to the researchers; ii) how are the scores for grading the items determined? This question is especially relevant to Likert-type instruments, where researchers sometimes take the, totally erroneous, approach of assigning numbers to the Likert responses, calculate subsequent means, for example, and use these for comparison; iii) the act of measuring with a pre-defined scale necessarily compresses or projects the multi-dimensionality of attitude onto smaller dimensions, and thereby loses nuance. They mention further that the enterprise of measuring attitude assumes it can be measured (read: quantified) and can therefore be related to other quantifiable variables present in education. These are valid and powerful criticisms and highlight to this author a need to investigate further the interactions between various measurement approaches present in the literature and, for example, the TMA framework proposed by Di Martino and Zan (2010) to further explicate the attitude construct.

Mathematics Attitudes and Perceptions Survey

Not all attitudinal survey instruments have been created in the same way: at least one, the Mathematics Attitudes and Perceptions Survey (MAPS; Code, et al., 2016) is grounded in empirical data, with the items and scales originating from students' comments about learning in STEM fields, and used not as an objective measure of students' attitudes, but rather as a measure of students' attitudes relative to mathematicians'.

The MAPS items were initially adapted from the Colorado Learning Attitudes About Science Survey (CLASS; Adams, et al., 2006), which were in turn emergent from student interviews. Items were subsequently refined, added or dropped based on student and faculty interviews. The survey then was completed by 3,411 students in differential, integral, and multi-variable calculus, and introduction to proof. An exploratory factor analysis was performed on a subset of this data, with a subsequent confirmatory factor analysis on another. From this cyclical development process, seven categories emerged: growth mindset, a view of the relevance of mathematics to the real world (shortened: real world), confidence, interest, persistence, a drive to make sense of mathematical answers (shortened: sense making), and a view on the nature of mathematical answers (shortened: answers). Though these descriptors are largely self-explanatory, the reader is referred to (Code, et al., 2016) for further elaboration of the categories.

Importantly, data was also gathered from mathematicians, giving the final MAPS items an expert consensus rating. A student's response to each item is scored relative to the corresponding expert consensus: +1 if in the same “direction” as the mathematician – that is, agree or disagree – and -1 if in the opposite direction. A score of 0 is given for neutral responses.

The intention with this approach was to construct an instrument that could observe movement of students' expert-like views of mathematics. This approach avoids the arbitrariness of other instruments (Fennema and Sherman, 1976) and assesses the students' attitudes, or aspects thereof, relative to the mathematics community's prevailing attitudes. This facilitates a way of understanding how an education in mathematics might bring students into the culture of mathematics, or push them away.

Previous implementations of the MAPS survey have revealed interesting results that corroborate results in the wider STEM literature (Code, et al., 2016): aspects of attitude do correlate with academic achievement, though the directionality of this relationship is unclear; the higher the course in the mathematics sequence, the more expert-like dispositions of the students, though this may be a self-selection effect; and the typical mathematics course experience tends to push students away from more expert-like attitudes towards the field. Disheartening as this, this appears to be the norm; post-secondary education selects those most oriented towards the field, and seldom develops such orientation.

Coordinating approaches to observing attitudes

Given the proliferation of methods and ways of observing, measuring, and understanding attitudes towards mathematics, it seems a worthwhile endeavour to employ multiple methods on a particular group of students in a particular context to explore how each measure might contribute to an overall picture of the students' attitudes towards mathematics. Both the TMA and MAPS have strengths – the TMA provides a way for the students' voices to be heard, and the MAPS lends itself to large-scale educational institutions – and weaknesses – the TMA is applied only to what the student articulates, and the quantitative aspect of MAPS can be construed as feigning "rigor", with educators desiring metrics. It seems that both ought to lead to a more complete picture of students' attitudes towards mathematics.

This study takes this approach with two such methods: the MAPS instrument and a TMA analysis of students' writing about experiences with mathematics. In particular, we attend to the questions: i) is the same attitudinal construct identified by students' free-form writing about their experiences with mathematics identified with a TMA analysis of their writing as that that is identified by their responses to the MAPS instrument? ii) Are the two approaches complementary or supplementary? iii) how might both be used to asses *changes* in attitudes exhibited by students over a mathematics course?

Methods

The data for this study comes from students enrolled in a 5-week Summer pre-university preparation program at San José State University (SJSU). Students in this program were admitted to SJSU, but did not pass an entry-level mathematics test. As a consequence, they were required to enroll in developmental courses during their first year of university. In an effort to get a lead on these courses before the start of the semesters, those students of greatest financial and academic need were invited to attend the Summer program, which is intended to smooth the students' transitions to university and improve their overall chances of success.

The Summer program consisted primarily of courses in elementary mathematics and English, but also included a series of sessions conducted by the university's counseling services that targeted students' attitudes towards mathematics. Specifically, the sessions focused on mindfulness, fostering a positive attitude, self-esteem and confidence in relation to performance, academic skills, stereotype threat, and relaxation. The inclusion of these counseling sessions was intended to target and improve the developmental students' attitudes towards mathematics, as developmental students are known, in general, to have less favourable attitudes towards mathematics than their non-developmental counterparts and that this significantly hinders their progression through university (Maciejewski and Tortora, *under review*). The content of the Summer program is not the focus of the current paper. Rather, I seek to observe changes in students' attitudes towards mathematics as revealed in their writing about mathematics and their responses to a survey instrument on attitudes and dispositions towards mathematics.

Specifically, students in the Summer program were invited at the start and end of the program to write a short response to the prompt:

Tell us about a personal experience you've had with math. Try to write at least 200 words.

This prompt was chosen to be as open as possible and to not narrow responses to be specifically about attitude or approaches to mathematics, etc. The intention here is for the student to recall a memory of their own interactions with mathematics; such memories are known to have associated emotional content, which is often articulated (Maciejewski, 2017).

Prior to writing their response to the above prompt, the students responded to the MAPS instrument. Both the MAPS and the essay responses were completed online, during a class time.

Start-of-program essays ($N = 134$) were matched with end-of-program essays ($N = 134$) and complete MAPS responses (start: $N = 123$; end: $N = 123$) to form the dataset ($N = 116$ start/end matched essay pairs and MAPS responses) for this study. Each essay was scored by the author according to the TMA framework of Di Martino and Zan (2010) following the descriptors in Table 1. As an example of this scoring, consider the following essay.

> I usually prepare for a test by doing a practice test with sample questions. However, I could never get a good grade one a test because when the test comes, my mind freezes. The problem feels completely different and more difficult. Even though sometimes the difference in the problem was just a few numbers. I try to get through the problem by thinking hard about the practice test and writing all the formulas down.

This essay was assigned a "-" in the Emotional category for the language around the problem feeling "different and more difficult"; an "i" in the Vision category for the view of mathematics as formulas, and a "l" in Competence, because of their admission of not being able to attain a good grade.

Table 1: TMA dimensions and accompanying scores, in parentheses.

TMA Dimension	Possible Score		
Emotional Disposition	N/A	Positive (+)	Negative (-)
Vision of Mathematics	N/A	Relational (r)	Instrumental (i)
Perceived Competence	N/A	High (h)	Low (l)

After each essay was scored, the aggregate scores were assessed using χ^2 and z tests to test for statistically-significant differences between start- and end-of-term essay response categorizations. As will be discussed, there is not a singular best way to analyse the aggregate score data. However, the dichotomous parsing of the TMA categories employed here is sufficient to observe appreciable differences in start and end of term essay responses.

The MAPS responses were scored by the author according to the instructions in the MAPS literature (Code, et al., 2016). The start and end of term MAPS results were compared on each category using Kruskal-Wallace tests.

Both TMA and MAPS data were then compared with standard logistic regression methods. This was intended to articulate the relationships between the MAPS and TMA categories .

Results

The results presented below reveal some aspects of the participating students' attitudes towards mathematics changing over the course of the Summer program. However, that is not the focus of the results; the reader is encouraged to attend to i) how the start and end of term TMA data are compared – the TMA framework has not been used to assess changes in students' attitudes over the duration of a course or program, as far as this author is aware; ii) the contrasting MAPS results; iii) relationships between MAPS and TMA categories present in the data.

In all, the underlying theme of the results takes the form of a question, that deserves being presented as a result in and of itself:

How best to determine if a change of attitudes occurred?

TMA

As reported elsewhere (Maciejewski, *under review*), there was a statistically-significant change in the TMA scores, taken as a proportion of positive/relational/high to total number of non-N/A scores ($p < 0.01$). A summary of the raw counts for each TMA category is in Table 2. The main observation here is that a TMA analysis of the student essays reveals a change in their attitudes towards mathematics.

Table 2: Results of a TMA analysis of the student essays.

	Pos./rel./high		Neg./ins./low		N/A	
	Start	End	Start	End	Start	End
Emotional	30	44	49	41	37	31
Vision	10	12	63	53	43	51
Competence	32	53	76	56	8	7

Changes in individual students' attitudes towards mathematics were observed by comparing their start and end of term essays. However, as is presented in (Maciejewski, *under review*), these are not necessarily *improvements*, which would be a normative assessment of the student writing. Indeed, the writing was often sufficiently rich to defy such a normative assessment.

MAPS

A comparison of the start and end-of-program MAPS scores indicate only one category with a statistically-significant change: Interest decreased ($p = 0.03$). On the surface, this seems to be a null, or negative, result. However, results from previous studies conducted with the MAPS instrument typically indicate a decrease in *all* MAPS categories over the duration of a program or course (Code, et al., 2016). The maxim, *if you can do no good, at least do no harm*, seems apropos.

Interactions Between MAPS and TMA scales

In an effort to understand better the seemingly disparate results from the TMA essay analysis and the MAPS results, I seek statistical relationships between the two. That is, this subsection of the results answers the question, do high or low MAPS category scores correspond to either of the values of the TMA category variables?

To this end, a logistic regression analysis is performed between each of the, non-zero, TMA variables and the MAPS variables; the N/A values in the TMA categories are not included in this analysis, to reflect the practice of excluding neutral responses in the MAPS analysis.

The results of the logistic regression analysis are summarized in Table 3. This analysis reveals that there are significant interactions between each TMA variable and multiple MAPS categories.

Table 3: Logistic regression model results for MAPS and TMA data. Only significant ($p < 0.05$) results are reported. Odds ratios are reported with χ^2 and p-values in parentheses.

	TMA Dimension		
MAPS Category	Emotion	Vision	Competence
Growth Mindset			
Real World	10.45 ($\chi^2(1) = 12.48$, $p < 0.01$)		
Confidence	10.81 ($\chi^2(1) = 7.46$, $p < 0.01$)	14.61 ($\chi^2(1) = 11.11$, $p < 0.01$)	15.76 ($\chi^2(1) = 19.76$, $p < 0.01$)
Interest	10.04 ($\chi^2(1) = 12.5$, $p < 0.01$)	4.88 ($\chi^2(1) = 4.93$, $p = 0.03$)	3.28 ($\chi^2(1) = 4.77$, $p = 0.03$)
Persistence			
Sense Making			
Answers			5.66 ($\chi^2(1) = 6.35$, $p = 0.01$)

Many of these relationships ought to be expected – confidence and interest cut across all TMA categories, for example. It is a reasonable expectation that these two do lead to more positive emotions towards mathematics and are related in some way to competence, though the analysis here cannot discern the direction of the relationship; perhaps improved competence leads to both greater confidence and interest. Persistence not being related to any of the TMA categories is also a plausible result – people persist for any number of reasons, whether they are competent or like it or not.

Some of the relationships are not expected, both in terms of relationships present and missing. For example, it is not clear why the Real World category ought to be related to emotions in mathematics – perhaps an appreciation of the pervasiveness of mathematics in one's life results in greater comfort or otherwise positive emotions. The lack of a statistically-significant relationship between the MAPS Answers and the TMA Vision of Mathematics categories is also unexpected, as these ought to be two different perspectives on the same underlying phenomena. However, the p-value ($p = 0.08$) in the logistic regression analysis between these variables was not far from the arbitrary significance cut-off of 0.05. The students in this study had a fairly uniform instrumental vision of mathematics, and the dichotomous parsing of the Vision category may have overlooked some of the nuance present in their true views of the nature of mathematics.

Discussion

This work concerned observing change in students' attitudes towards mathematics through two different methods and analyses. The first, an application of the TMA framework (Di Martino and Zan, 2010) to student essay responses, revealed significant positive changes in student attitudes towards mathematics over the duration of the program. The second, a start and end-of-program administration of the MAPS instrument (Code, et al., 2016), revealed only one significant change: student interest decreased. How might we understand these seemingly disparate results?

Returning to the research questions posed near the start of this paper, I address (iii) first. Changes in students' attitudes towards mathematics are difficult to identify and describe. The MAPS categories, by their quantitative nature, give the illusion of readily being able to describe change. However, changes do not always occur in terms of moving towards or away to experts. It is possible for a student to have a "horizontal" shift in their attitudes – they may develop a love for solving rich modeling problems and increasingly despise computation, the net effect of which is that the student is no closer to the experts in that aspect of attitude towards mathematics. The MAPS is also restricted to revealing those aspects of the students' attitudes that are asked about by the MAPS items. Though the MAPS categories tend to be identified in the research

As revealed through logistic regression analyses, there are significant interactions between the TMA essay analyses and MAPS results. This lends support to the following hypothesis, phrased in terms of a mathematical analogy.

A student's attitude towards mathematics is a multi/high-dimensional object. As articulated in the literature (Goldin, et al., 2016; Pepin and Roesken-Winter, 2015; Hannula, 2012), *attitude* consists of psychological, social, emotional, and temporal states. Any attempt to observe, describe, and otherwise measure the attitude object necessarily results in a reduction of dimension. A student writing about their relationship with mathematics reveals only some features of attitude, and a subsequent TMA analysis slices through those features. A MAPS instrument implementation reveals sections of a students' attitude object, that they may or may not have written about, and attempts to quantify these, metrizing the sections along the MAPS categories/dimensions. The MAPS sections may, as in the data reported here and revealed through the predictive variables in the logistic regression, intersect with the TMA slices, or be parallel.

The key with this mathematical analogy is that a high-dimensional object is difficult to comprehend by humans embedded in a spatially three-dimensional world. This difficulty is not an impossibility, since we have ways of bringing high-dimensional objects into three-dimensions, but this dimensional reduction necessarily results in a loss of information. In our efforts to understand that which is not readily graspable, we lose features of the object. Bringing the object into our world in one way may reveal the smooth curves of one side, and leave the sharp corners and divots of another hidden.

Taking this view of attitude towards mathematics as a high dimensional object avoids privileging of one method to understand that attitude over another. That is, a qualitative analysis of a student's writing about their relationship with mathematics is neither better nor worse than the student's responses to a Likert-based questionnaire on attitudes towards mathematics, or vice-versa. Both approaches reveal only restricted features of attitudes, and subsequent analyses further reduce those features. Both taken together can reveal more features of overall attitude than either one alone.

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References

Adams, W.K., Perkins, K.K., Podolefsky, N.S., Dubson, M., Finkelstein, N.D., and Wieman, C.E. (2006). New instrument for measuring student beliefs about physics and learning physics: the Colorado Learning Attitudes about Science Survey. *Physical Review Special Topics – Physics Education Research, 2*, 010101.

Code, W., Merchant, S., Maciejewski, W., Thomas, M., & Lo, J. (2016). The mathematics attitudes and perceptions survey: an instrument to assess expert-like views and dispositions among undergraduate mathematics. *International Journal of Mathematical Education in Science and Technology, 47(6),* pp 917-937.

Di Martino, P., and Zan, R. (2010). 'Me and maths': towards a deffinition of attitude grounded on students' narratives. *Journal of Mathematics Teacher Education, 13*, 27-48.

Fennema, E., and Sherman, J. (1976). Fennema-Sherman Attitude Scales: instruments designed to measure attitudes toward the learning of mathematics by females and males. *Journal of Research in Mathematics Education, 7*, 324-326.

Goldin, G.A., Hannula, M.S., Heyd-Meztuyanim, E., Jansen, A., Kaasila, R., Lutovac, S., Di Martino, P., Morselli, F., Middleton, J.A., Pantziara, M., & Zhang, Q. (2016). *Attitudes, Beliefs, Motivation and Identity in Mathematics Education: An Overview of the Field and Future Directions*. Hamburg, Germany: Springer Open.

Hannula, M. (2012). Exploring new dimensions of mathematics-related affect: embodied and social theories. *Research in Mathenatics Education, 14(2)*, 137-161.

Maciejewski, W. (*under review*). Changes in attitudes revealed through students' writing about mathematics.

Maciejewski, W. (2017). Mathematical knowledge as memories of mathematics. In B. Kaur, W.K. Ho, T.L. Toh, & B.H. Choy (Eds.) *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education, Vol. 3*, (p. 209-216). Singapore: PME.

Maciejewski, W., and Tortora, C. (*under review*). Developmental students' attitudes towards mathematics: deviating from the norm?

Pepin, B., and Roeskin-Winter, B. (2015). *From beliefs to dynamic affect systems in mathematics education.* Cham: Springer.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1-36.

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20-26.

MATHEMATICS AND LANGUAGE INTEGRATION IN A CLIL CLASSROOM: AN ANALYSIS OF GENRE

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Teachers in mathematics classrooms where the language of instruction differs from students' first language face competing demands regarding mathematics and language teaching. In this study, I conducted an analysis of genre of teaching episodes to examine how a teacher in a Spanish immersion third-grade classroom in the US responded to those demands. Findings indicate that the genres on which the teacher drew are associated with mathematics and with language classrooms, in addition to general genres that can be found in either classroom. I argue that the teacher's awareness of school demands and of students' needs influenced how genre-switching unfolded. I discuss how genre analysis adds to conceptualisations of mathematics and language as integrated in CLIL classrooms.

Keywords: Equity and Diversity, Instructional activities and practices

Mathematics teachers face competing demands in classrooms where the language of instruction differs from students' first language.[1] Teachers have the double responsibility of supporting students' development of mathematics and the additional language (Nikula, Dalton-Puffer, Llinares, & Lorenzo, 2016). These teachers confront the decision of how to split class time between these two competing demands. An alternative is to approach mathematics and the additional language as integrated. Integration is the premise of Content and Language Integrated Learning, CLIL, frameworks (Nikula et al., 2016). In CLIL classrooms, the language of instruction differs from students' first language[1]. Rather than regarding the role of the teacher as complying with competing content and language demands, CLIL teachers integrate the two.

In this study, I draw on the concept of genre to consider the role that societal forces play in mathematics CLIL classroom interactions. I follow Bakhtin's (1986) notion of genre that considers both linguistic and social influences on interactions. I propose a view of classroom episodes as representing teaching genres that are particular to mathematics or to language classrooms. I ask the following research question: How does a teacher in a CLIL classroom integrate mathematics and foreign language to respond to competing demands? I argue that a dialogic dynamic among language, content, students' needs and school demands drove the teacher's efforts toward integration. Both individual and school orientations toward mathematics and the foreign language influenced how integration unfolded.

This paper relates to the theme of the conference of celebrating 40 years of PME-NA. The paper focuses on one of the enduring challenges in mathematics education research that the conference addresses: supporting *all* students through equitable teaching informed by bodies of knowledge that the conference has supported. Specifically, the paper sheds light on the role

[1] Following Johnson & Johnson's (1999) definitions, I refer to **first language** as the language to which a person is exposed from birth and in which the person is raised. A **second language** refers to a language other than the first language that is widely used in the person's community (for example, English is a second language for Spanish speakers living in the US). A **foreign language** is a language other than the first language and that is not widely used in a person's community. An **additional language** encompasses all languages other than a person's first language.

teachers play in mediating competing demands in mathematics classrooms where the language of instruction is different from students' native language.

Theoretical Framework

Mathematics and Language

I draw on a perspective that regards mathematics and language as integrated and inseparable. Instead of delimiting where mathematics ends and where language begins, this perspective defines the two as an entanglement. Rather than a final representation of an already existing idea, language is part of developing, communicating and refining mathematical ideas (Barwell & Pimm, 2016). Accordingly, mathematics and the additional language develop in tandem. Mathematical activity involves continuous learning and relearning the additional language as it emerges in interaction (Brown, 2002). Additionally, using particular languages influences mathematical activity. For example, certain languages express particular mathematical concepts in a more transparent way than other languages (Martínez, 2018). Teachers may draw on the language that more transparently represents a mathematical idea if such language is valued at school. The intersection between a language and mathematical activity presents particular complexities, including the social orientations about what languages to use in academic settings.

CLIL in Mathematics

I draw on definitions of CLIL as an umbrella term that refers to classrooms where students learn both content and an additional language (Nikula et al., 2016). CLIL classrooms have both content and language goals and expectations. The focus of CLIL, however, is on integration: "[CLIL's] distinctiveness lies in an integrated approach, where both language and content are conceptualised on a continuum without an implied preference for either" (Coyle, 2007, p. 545). Because of this attention to integration, CLIL is consistent with this study's purpose of exploring how a teacher integrates language and content to respond to competing demands.

In addition to the focus on integration, CLIL highlights that contextual forces influence classroom interactions. The purpose of CLIL pedagogies is not exclusively to bring a set of pre-existing grammar structures to the mathematics classroom. Instead, CLIL draws attention to how expectations about language, mathematics and schooling affect pedagogical decisions: "CLIL teaching [is] a site where diverse institutional, educational, personal and pedagogical scripts and purposes intersect" (Nikula et al., 2016, p. 27).

Genre Analysis

I follow a Bakhtinian definition of genre as a socially discernible type of language use (Bakhtin, 1986). Rather than being classifications that specialist linguists propose, genres are sets of cultural conventions (Gerofsky, 1999). That is, genres are what a collective believes them to be and they are influenced by the ideologies, expectations and histories of a specific culture (Gerofsky, 1999). This simultaneous attention to language use and social influences is consistent with this study's purpose of exploring a mathematics CLIL classroom in light of institutional and societal expectations and demands. I build on research on genre that is consistent with a perspective of mathematics and language as integrated. These studies attend to both linguistic features and social orientations and demands that influence language use. For example, when studying lectures as a genre in mathematics education, Gerofsky (1999) drew attention to linguistic features such as the non-inclusive we/us, ambiguous use of nouns, and saturation of imperative forms. Simultaneously, he regarded social expectations for the interaction as constitutive of the genre, including who is supposed to be silent and who is expected to speak.

For the purpose of this study of exploring how a CLIL teacher integrated mathematics and language to respond to competing demands, I focus on genres as teaching episodes associated with language or mathematics classrooms, as well as general genres associated with multiple classrooms. Particular types of language use, and teachers' and students' actions can be thought of as typical of language teaching or mathematics teaching. Different genres respond to specific institutional and social demands, with some genres motivated by demands about language acquisition and others motivated by demands about mathematics learning. Drawing on different genres throughout a lesson and a unit, teachers respond to competing demands: "The focus would be on the different intended and perceived purposes embedded in different genres, and on a mastery and enjoyment of genre-switching" (Gerofsky, 1999). This approach acknowledges social influences in recognising and using language at the level of classroom interactions.

Methodology

Sites and Participants

This elementary school was located in the Midwest region of the United States in a mostly White, middle class, English monolingual community. The school offered a Spanish language immersion program. Following a language enrichment full immersion model, all instruction was in Spanish, except for one hour taught in English per day. Mathematics was taught in Spanish. The classroom was a third grade with nine female students (one Latina, one African American, seven White) and 14 male students (two African American, 12 White). All students spoke English at home. All 23 students were able to comfortably initiate and sustain a conversation in Spanish. The teacher, señora Abad (all names are pseudonyms) was a US-born Latina who considered both English and Spanish her native languages. Her educational background was in Spanish bilingual education. She had been a teacher for three years.

Data Generation

I used ethnographic methods (participant observation, interviews and field notes) to generate data on how the teacher integrated language and mathematics and how such integration related to competing demands external to the classroom. Consistent with this study's purpose of illustrating how the teacher drew on different genres rather than claiming this is an exhaustive list, I focus on one geometry unit. I chose this unit because it took place toward the end of the academic year when patterns of interaction were stable. I video recorded three lessons per week which resulted in nearly 11 hours of video for 12 lessons. I transcribed all video recorded lessons. Throughout the unit, I conducted unstructured interviews where the teacher and I discussed preliminary findings, confirming or redirecting my analyses. I documented references to forces external to the classroom that influenced the lesson, including the teacher's references to the pacing guidelines, communications with parents, the textbook, and assessment. I transcribed interviews.

Data Analysis

Analysis focused on video data and interview data supported interpretations. The unit of analysis were teaching episodes which I defined as class moments beginning with a change in topic, task or teaching method. The 12 lessons in this unit resulted in 85 teaching episodes. Consistent with the discussion of genre above, during a first stage of analysis I examined both linguistic features and social conventions in each episode. Drawing on Gerofsky's (1999) tools to analyse genre, I annotated the following dimensions in each teaching episode: who influenced the direction of the conversation, modes of communication (written, spoken, gestures), types of questions asked and who asked them, resources being used (for example, textbooks and manipulatives), who addressed who (teacher-student, teacher-whole class, student-student), mathematical content, mathematical practices (for example, giving an answer and describing a problem solving strategy), languages being used, and the level of attention to language accuracy.

During a second stage of analysis, I looked for patterns across transcript annotations. I grouped similar episodes and characterised each group. I conducted this process three times, refining classifications and interpretations, resulting in 11 teaching genres. During the third stage of analysis, I classified each genre as associated with language or with mathematics teaching, or as general. This classification corresponds to the focus of the episode, the teacher intentions as expressed in interviews, and my interpretation of whether the observed episode was recognizable as typical of mathematics or language classrooms or both.

Findings

Analysis revealed that the teacher drew intermittently on clearly discernible mathematics and language teaching genres to respond to demands on each of the two areas. Moreover, some of the genres seem to be associated with both mathematics and language classrooms. Table 1 synthesises the distribution of teaching episodes in each genre and genre category.

Table 1. Number of teaching episodes by genre.

Genre Category	Genre	Number of episodes
General	Whole class discussion	21
	Small group task	18
	Task launching	11
	Teacher explanation	8
	Student presentation	4
Language teaching	Info gaps	6
	Artifact description	4
	Role playing	2
Mathematics teaching	Textbook work	6
	Number talk	3
	Test	2
Total number of episodes		85

Due to space restrictions, I focus on the most frequent genre in the language and the mathematics categories. In the transcripts below, text in brackets describes actions.

Mathematics Episodes: Textbook Work

In the textbook work genre, students worked individually or in pairs solving problems from the textbook. One example of this genre took place during the seventh lesson of the unit. Students were working in pair on a page that asked them to find three rectangles with a perimeter of 12 centimetres. Afterwards, students needed to fill out a table with the lengths of the adjacent sides and the area of each rectangle. The teacher told students they could get geoboard dot paper if they wanted to draw their rectangles. A few minutes into the task, Señora Abad approached two students, Lisa and John. They were working in silence and looking at each other's textbook. John had drawn a 2x2 square and Lisa had written 2 and 2 in the first row of the chart and 2 and 4 in the second row. Señora Abad asked the students what the perimeter of their square was. Putting her finger in the middle of the square, Lisa answered that the perimeter was four.

Table 2. Example of the textbook work genre.

Speaker	Original Utterance	Transcribed Utterance

Señora Abad	El perímetro es afuera.	Perimeter is outside.
Lisa	[Mira el dibujo en silencio.]	[Looks at the drawing quietly.]
John	[Mira a la señora Abad] ¿Estamos sumando o multiplicando?	[Looking at señora Abad] Are we adding or multiplying?
Señora Abad	¿Qué es perímetro? [Mira a Lisa y a John.]	What is perimeter? [Looks at Lisa and John].
Lisa	Afuera.	Outside.
Señora Abad	Afuera. Entonces, ¿qué hago para buscar el perímetro de esta figura?	Outside. So, what do I do to look for this shape's perimeter?
Lisa	Sumar.	Add.
Señora Abad	Mmhm. ¿Qué sumo? [Mira a los dos.]	Mmhm. What do I add? [Looks at both.]
Lisa	Dos mas dos mas dos mas dos.	Two plus two plus two plus two.
Señora Abad	Y eso es…	And that is…
Lisa	¿Ocho?	Eight?
John	[Borra los números 2 y 2 que había escrito en su cuadro.]	[Deletes numbers 2 and 2 that he had written in his table.]
Señora Abad	Ocho. Pero yo necesito rectángulos con un perímetro de doce. Tú me dices que dos y dos te da un perímetro de doce.	Eight. But I need rectangles with a perimeter of twelve. You say from two and two you get a perimeter of twelve.
Lisa	No.	No.
Señora Abad	OK. Puedes usar tu cuadrícula de puntos [señala al papel de John.] Aquí en esta [señalando a la segunda fila del cuadro en el libro] tú me estás diciendo que un lado tiene 2 y otro 4 en un rectángulo, ¿si? Si esto es 4, esto también va a ser 4 [Dibuja un rectángulo y señala dos lados opuestos]. Si esto es 2, esto también va a ser dos [señala los otros dos lados opuestos]. ¿Cuál es el perímetro de esto?	OK. You can use your dot grid [points at John's dot paper.] Here in this one [pointing at the second row in the table in the textbook] you are telling me that a side is 2 and another one is 4, right? If this is four, this has to be 4 too [Draws a rectangle and points at opposite sides]. If this is 2, this is going to be 2 too [points at the other opposite sides]. What is the perimeter?
Lisa	Doce.	Twelve.
Señora Abad	¡Ajá! Entonces ese, está bien ese. Ahora tienes que buscar dos rectángulos mas que tengan un perímetro de doce.	Aha! So, this one, this one is correct. Now you have to look for two more rectangles that have a perimeter of twelve.

Several characteristics of the textbook work genre are present in this excerpt. Both teacher and students influenced the direction in which the conversation went. The teacher elicited students' thinking and definitions of concepts (i.e "¿Qué es perímetro?"—*what is perimeter?*). Students' questions had to do with the correctness of their answers, what they had to do, or specific definitions or procedures (i.e "¿Estamos sumando o multiplicando?"—*Are we adding or multiplying?*). Lisa presented her answers as questions, indicating the exploratory nature of her work and acknowledging that there was one correct answer that señora Abad could confirm. Modes of communication included spoken language, mathematical symbols and drawings.

Despite the interactive nature of the excerpt above, episodes in the textbook work genre did not necessarily involve student talk. Students could decide to work silently, as was the case at the beginning of the episode described here. Conversation frequently started when the teacher approached each pair. Although both the teacher and the students contributed to this conversation, students had the option of remaining silent or limiting their spoken utterances. For example, John only spoke once in the excerpt above and most of Lisa's turns consisted of one word. On the contrary, señora Abad's turns are longer and consist of full sentences. The focus of episodes in the textbook work genre was on mathematical ideas. These explorations, however, did not revolve around the generation of ideas but rather on recollection of definitions and strategies previously discussed. In the example above, the teacher did not elicit explanations or justifications. Instead, she gave definitions (i.e. "perímetro es afuera"—*perimeter is outside*) or reminders and students used phrases or isolated words to give answers.

In an interview, the teacher mentioned she used the textbook genre to review mathematical ideas, focusing on accurate application of procedures to get correct answers. The teacher used this genre after students had generated ideas and made sense of specific concepts. According to the teacher, she frequently drew on this genre as formative assessment that informed subsequent lessons, as well as a review before tests. In this example, the class had already discussed perimeter and area, and strategies to calculate each. The teacher mentioned that from previous lessons she had concluded the class needed to work on differentiating area and perimeter, which was one of her goals in this episode.

Language Episodes: Info Gap

In the info gap genre, students worked in pairs. Each student had some of the information necessary to solve a problem. The teacher gave a slip of paper to each student in a pair and students could not see each other's slip. One slip was designated "Student A" and the other one "Student B." First, each student needed to figure out what the information in the slip was by completing a language driven task. Then, students requested the missing information from each other. Although the information discussed was mathematical, the focus was on accurate production of the Spanish language.

One example of the info gap genre took place during the eighth lesson of the unit. In pairs, one student got a slip labelled Student A with information about a shape. This student needed to unscramble the words in each sentence. Student B had incomplete sentences about the same shape. This student needed to formulate questions to request the missing information. Señora Abad approached two students, Malik and Daniel.

Table 4. Example of the info gap genre.

Speaker	Original Utterance	Original Utterance
Señora Abad	¿Qué te hace falta aquí? [señalando la palabra "cuadrilatero."]	What's missing here? [pointing at the word "cuadrilatero."]
Malik	Oh! [Agrega tilde en "cuadrilátero."]	Oh! [Adds accent mark in "cuadrilátero."]
Señora Abad	Muy bien. [Murmura mientras señala cada frase.] Los lados cortos suman… La figura tiene… Hmm… ¿Cómo se dice esto?	Very good. [Whispering as she points at each sentence.] Both sides add up… The shape has… Hmm… How do you say this?
Malik	Oh! La figura tienes 4 ángulos rectos. Wait, no! Es correcto.	Oh! The shape have four right angles. Wait, no! It's correct.
Señora Abad	Hmm… ¿Recuerdas la historia de los caminos? Los caminos eran…	Hmm… Do you remember the story about the roads? The roads were…
Malik	Um largos y… Oh! [Borra "rectos ángulos" y escribe "ángulos rectos."]	Um long and … Oh! [Erases "rectos ángulos" and writes "ángulos rectos."]
Señora Abad	¡Muy bien! ¡Ángulos rectos! Ahora trabaja con Daniel que ya terminó.	Very good! Right angles! Now, work with Daniel; he is done.
Daniel	Ahora yo te pregunto.	I ask you now.
Malik	Mmhm.	Mmhm.
Daniel	¿Qué figura es?	What is the shape?
Malik	La figura es un cuadrilátero.	The shape is a quadrilateral.
Daniel	[Escribe en su papel] ¿Cuánto suman los lados adyacentes?	[Writes on his paper] What is the sum of the adjacent sides?
Malik	Los adyacentes lados suman nueve centímetros [Sonríe.]	The sum of the sides adjacent is nine centimetres[Smiles.]
		[Starts to write] What?! No!

Daniel	[Comienza a escribir] ¿Qué?! ¡No!	[Laughs.]
	[Risas.]	
Daniel	Malik, ¿no sabes decir eso?	Malik, don't you know how to say this?
Malik	[Sonríe] Es los lados adyacentes suman nueve centímetros.	[Smiles] It's the sum of the adjacent sides is nine centimetres.
Daniel	[Escribe.]	[Writes down.]

This example illustrates characteristics of the info gap genre. For example, students worked independently some of the time, with the teacher intervening only when she saw inaccuracies in language use. When the teacher intervened, she explicitly pointed at errors and asked the student to correct. That was the case when señora Abad asked Malik what was missing in the word he had spelled as "cuadrilatero" and when she asked him how to write a specific sentence ("ángulos rectos"). The questions señora Abad asked were not related to the mathematical meaning of the ideas in the slips. The teacher asked questions that drew Malik's attention to language inaccuracies ("¿qué te hace falta aquí?" and "cómo se dice esto"). Specifically, señora Abad asked "¿recuerdas la historia de los caminos?" referring to a short story the class had read about long roads and straight roads (caminos largos y caminos rectos) to focus on the noun-adjective word order in Spanish. Instead of inquiring about students' mathematical sense making of the ideas, the teacher focused on accurate spoken and written language production. That was the case when Malik playfully changed the standard "lados adyacentes" for "adyacentes lados" and how Daniel realised there was an error. Additionally, interactions in this genre were guided. Students engaged in the task writing or taking turns to ask and answer questions. Both students in the pair needed to communicate actively in order to complete the task.

In an interview, the teacher mentioned she used the info gap genre to focus students' production of mathematics discourse. Moreover, the class used the product of tasks in this genre as input for problem-solving tasks focused on mathematical ideas. Señora Abad commented that she drew on this genre when she thought it was necessary for the class to review the mathematical meaning and accurate use of specific terminology and phrases. The teacher mentioned that parents were frequently familiar with these tasks as associated with language learning and that they appreciated this focus on language. Therefore, she sometimes sent the individual part of these tasks as homework.

Discussion

This study suggests that the teacher navigated the competing demands by flexibly drawing on mathematics and language specific genres. The tasks, types of interactions, linguistic features, resources used, and purposes of episodes were similar within a genre and were typical to either mathematics or language teaching. This dynamic genre-switching resonates with Gerofsky's (1999) conceptualisation of genre-switching as a pedagogical tool to reach and support diverse students. Through genre-switching, the teacher foregrounded one of the two areas at different moments of the unit. In doing so, the teacher responded to mathematics and language related demands. This finding sheds light on how the teacher used genre-switching as a mechanism to respond to competing demands in a CLIL classroom specifically.

This study illustrates an analytical framework that does not disrupt the mathematics and language integration. By drawing on genre analysis of teaching episodes as associated with mathematics or language teaching, this study adds to previous research on CLIL mathematics classrooms that has mainly focused on students' language acquisition (Nikula et al., 2016). Rather than differentiating whether each teaching episode dealt with language or with mathematics, this genre analysis maintained the assumption that the two are inseparable. It was the types of tasks, interactions and teacher moves that were susceptible of being characterised as typical of a mathematics classroom or a language classroom. Analysis of teaching episodes focused on the interplay between mathematics and language during mathematical activity. During episodes in all genre categories the teacher provided opportunities to use and develop the language while simultaneously developing mathematical ideas.

This study attended to school influences on classroom interactions that the teacher perceived. Summative assessment, school pacing guidelines and parents' perceptions of language tasks informed how the teacher selected and sequenced specific genres. Characteristics of the school also informed the interplay between mathematics and language in this classroom. The teacher was a Latina, Spanish native speaker with formation in bilingual education. Moreover, she was aware of parents' interest in their children language proficiency. The school guidelines required the teacher to use Spanish at all times, resorting to English only to clarify specific words. Students had been in this language immersion classroom for nearly three years and were able to understand and maintain conversations in Spanish. Accordingly, in this classroom, Spanish was valued and positioned as useful at all times. In other contexts where language minority students learn mathematics in the community's dominant language, teachers frequently lower expectations, water down the mathematical curriculum or pull students out of mathematics classes until they develop what the school considers sufficient language proficiency (Setati & Moschkovich, 2013). In this classroom, rather than seeing students' language acquisition as a barrier, school, teacher and student characteristics supported the teacher's pursuit of mathematically challenging tasks in Spanish.

References

Bakhtin, M. M. (1986). *Speech genres and other late essays* (McGee, V. trans). Austin, TX: University of Texas Press.

Barwell, R., & Pimm, D. (2016). Bakhtin and some dilemmas of mathematics-language. In M. Phakeng & S. Lerman (Eds.), *Mathematics Education in a Context of Inequity, Poverty and Language Diversity* (pp. 7–18). Cham, Switzerland: Springer International Publishing.

Brisk, M. E. (2011). Bilingual education. In E. Hinkel (Ed.), *Handbook of research in second language teaching and*

Brown, T. (2002). *Mathematics education and language: Interpreting hermeneutics and post*-structuralism. Dordrecht, Netherlands: Kluwer Academic Publishers.

Coyle, D. (2007). Content and language integrated learning: Towards a connected research agenda for CLIL pedagogies. *International Journal of Bilingual Education and Bilingualism*, *10*(5), 543–562.

Gerofsky, S. (1999). Genre analysis as a way of understanding pedagogy in mathematics education. *For the Learning of Mathematics*, *19*(3), 36–46.

Johnson, K., & Johnson, H. (1999). *Encyclopedic dictionary of applied linguistics: A handbook for language teaching*. Oxford, UK: Blackwell Publishers.

Martínez, J. M. (2018). Language as resource: Language immersion mathematics teachers' perspectives and practices. In R. Hunter, M. Civil, B. Herbel-Eisenmann, N. Planas, & D. Wagner (Eds.), *Mathematical discourse that breaks barriers and creates space for marginalized learners* (pp. 85-100). Rotterdam, The Netherlands: SENSE Publications.

Nikula, T., Dalton-Puffer, C., Llinares, A., & Lorenzo, F. (2016). More than content and language: The complexity of integration in CLIL and bilingual education. In T. Nikula, E. Dafouz, P. Moore, & U. Smit (Eds.), *Conceptualising integration in CLIL and multilingual education* (pp. 1–28). Bristol, UK: Multilingual Matters.

Setati, M., & Moschkovich, J. (2013). Mathematics education and language diversity: A dialogue across settings. *Journal for Research in Mathematics Education*, *44*(1), 119–128.

THE INTERPLAY OF FRUSTRATION AND JOY: ELEMENTARY STUDENTS' PRODUCTIVE STRUGGLE WHEN ENGAGED IN UNSOLVED PROBLEMS

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In this study, I investigated Grade 4 and 5 students' emotions while they engaged in the exploration of unsolved mathematics problems, including parts of the Graceful Tree Conjecture. Ten students from an after-school program in the Midwest participated in seven task-based interviews. The students exhibited a variety of emotions throughout the study, with frustration and joy displayed most frequently. I found the interplay of these emotions, joy and frustration, to describe the productive struggle that the students experienced while working on parts of the unsolved problems. A descriptive case of two students, Bernice and Alia, are included to describe how the interplay of frustration and joy characterize productive struggle.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Elementary School Education; Problem Solving

The purpose of this paper is to document what productive struggle look like. To do this, I examined the emotions ten students displayed while they were engaged in problem-solving with unsolved mathematics problems. The use of the unsolved problems permitted the students the opportunity to display multiple emotions and engage in struggle. Hiebert and Grouws (2007) defined *struggle* as when "students expend effort to make sense of mathematics, to figure something out that is not immediately apparent" (p. 387). Allowing students, the opportunity to struggle is beneficial (e.g., Hiebert & Grouws, 2007; Kapur, 2010; Reinhart, 2000). When students have spent time learning mathematics while engaged in productive struggle, they significantly outperform similar ability students who were not given the opportunity to struggle on a task (Kapur, 2010). According to Kapur (2010), these same students who have engaged in productive struggle are also able to engaged and transfer their knowledge to work on challenging mathematics concepts they have yet to explore. While past research has documented that productive struggle is beneficial, there is no research that described what productive struggle actually looks like as children are engaged in mathematical problem solving (Warshauer, 2015; Zeybek, 2016).

When students are engaged in problem solving, they experience both positive and negative emotions (Goldin, 2000a; Hannula, 2015). However, the bulk of the previous research on emotions is limited to surveys and does not document students' emotions while they are engaged in problem solving (Hannula, 2015). In order to look forward in mathematics education, it is critical to document the different emotions students experience while engaged in problem solving and to describe what it looks like when students are experiencing productive struggle.

Research Questions

The following questions guided my research study:

1. What are the emotions students displayed when they were engaged in the exploration of unsolved problems?
2. How are the emotions of frustration and joy related to the struggle students experience while they are engaged in the exploration of unsolved mathematics problems?

Theoretical Framework

Positioning theory has been used in mathematics education as a way to analyze social interactions (e.g., Turner, Dominquez, Maldonado, & Empson, 2013; Wood, 2013; Yamakawa, Forman, & Ansell, 2009). Daher (2015) and Evans, Morgan, and Tsataroni (2006) have identified that students' emotions are linked to their positioning. In this study, positioning theory (van Langenhove & Harré, 1999) was employed as an overarching theoretical framework for how students positioned themselves through their interactions with other students and the mathematics. Their positions were displayed through their dispositions or more specifically, their emotions.

Positioning theory is "the study of local moral orders as ever-shifting patterns of mutual and contestable rights and obligations of speaking and acting" (van Langenhove & Harré, 1999, p. 1). It is "the discursive construction of personal stories that make a person's actions intelligible and relatively determinate as social acts and within which the members of the conversation have specific locations" (van Langenhove & Harré, 1999, p. 16). People can position themselves or be positioned by others in many different ways, such as being good at mathematics or bad at mathematics. A person is positioned based on actions and conversations. The conversations create storylines. The storylines can document students emotions at the moment they are displayed through their dispositions.

Methods

In this qualitative study, I investigated ten Grade 4 and 5 students (Alia, Amanda, Becca, Bernice, Edward, Hector, Iris, Joella, Karly, and Trevor; all of the names have been changed) while they engaged in the exploration of unsolved problems. The study took place at an after-school program at a community center in the Midwestern United States. Students participated in seven semi-structured, task-based interviews (Goldin, 2000b), which I will refer to as problem-solving sessions. These seven problem-solving sessions took place over three weeks and each lasted between 35 and 45 minutes. For the first six Problem-Solving Sessions, students worked on the Graceful Tree Conjecture. During Session 7, students were introduced to Collatz Conjecture.

Graceful Tree Conjecture

During the first six problem-solving session students worked on the Graceful Tree Conjecture. The Graceful Tree Conjecture is an unsolved conjecture from graph theory that is accessible to young children. The problem has students exploring different types of tree graphs, graphs that are connected (one piece) with no cycles (see Figure 1). This means a tree graph is acyclic, which means if you follow a path from node to node along the edges, you will never cycle back to the same node without repeating an edge. This means that tree graphs always have one more edge than node.

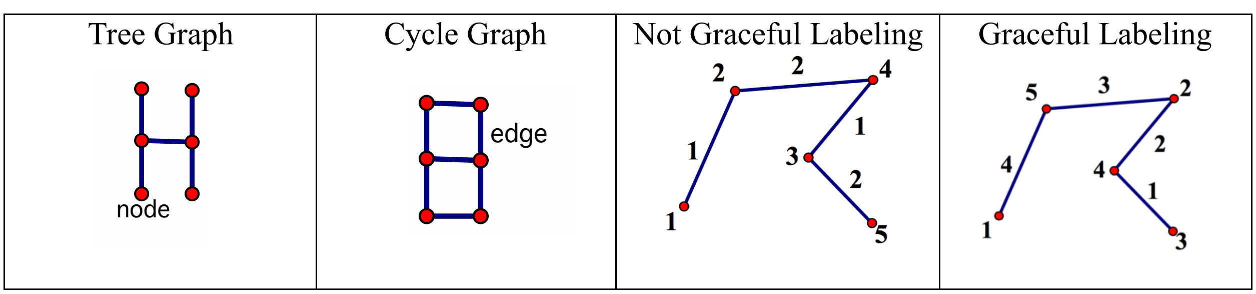

Figure 1. Concepts of the Graceful Tree Conjecture.

Tree graphs are assigned numbers to the nodes that induce labeling of the edges. For a tree graph of order n, every node is labeled distinctly from 1 through n and the edges are labeled with the absolute value of the difference of the labels on their endpoints. For a tree graph to be labeled gracefully, the edges need to be labeled distinctly 1 through n-1 (see Figure 1).

Overview of Problem-Solving Sessions

During the first six sessions, students explore different categories of tree graphs in increasing sophistication (see Figure 2). I challenged the students to not only create a graceful labeling for each graph but to find a pattern or justification to show that any tree graph in the category could be labeled gracefully. While exploring the different tree graphs, students were given a page that contained four distinct tree graphs in a given category, enlarged copies of each of the graphs, and numbered circle and square chips (see Figure 3). This allowed students to try multiple solutions without having to erase.

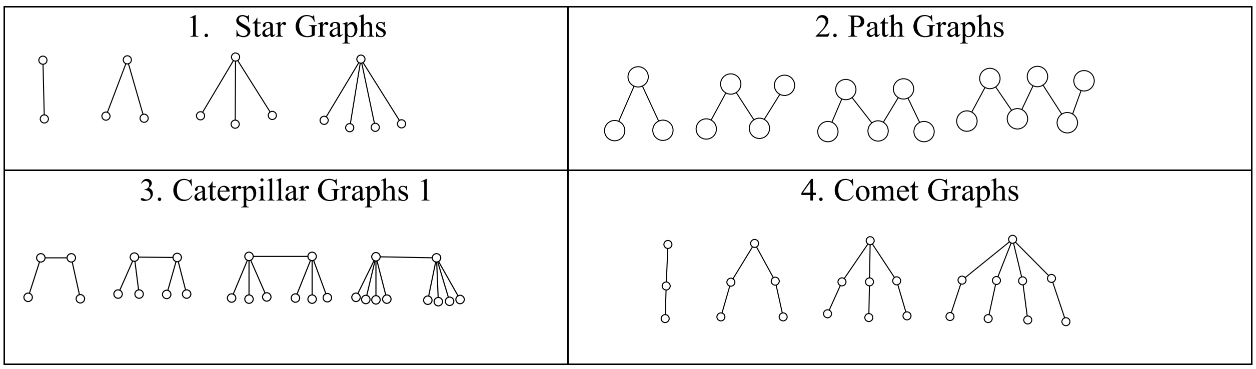

Figure 2. Tree Graphs in increasing sophistication.

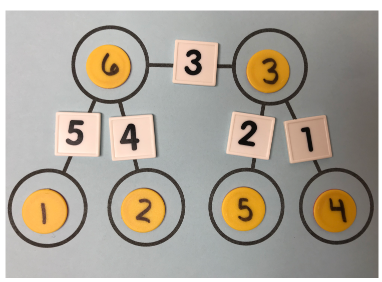

Figure 3. Enlarged Caterpillar Graph and circle and square numbered chips.

During Session 1, students were introduced to tree graphs, learned what a graceful labeling was, and explore a five node tree graph. Throughout Session 2, students worked to find a graceful labeling for Star Graphs. In Session 3, students finished Star Graphs and began to work on labeling Path Graphs. During Session 4, students continued to work on labeling Path Graphs. In Session 5, students finished labeling Path Graphs and began to work on Caterpillar Graphs. In Session 6, students worked on labeling Caterpillar Graphs. In Session 7, Collatz Conjecture was introduced but after twenty minutes of work the participants requested to continue working on the Graceful Tree Conjecture. The students then began to work on gracefully labeling Comet Graphs.

Analysis

All seven problem-solving session were video recorded and student work was collected. All video was transcribed including non-verbal actions, such as facial expressions or arms raised in the air. Next, I constructed an analytic framework based on the ideas from Else-Quest, Hyde, and Hejmadi (2008) to characterize the emotions students displayed while they were engaged with the mathematics. During my first round of analysis, I modified Else-Quest et al.'s framework by combining emotions such as joy, pleasure, and pride because I was not able to distinguish a difference between these emotions without interviewing students to know how they were specifically feeling (see Table 1 for the emotion framework).

Table 1: Emotion Analytic Framework

Emotion/Code	Definition	Example
Anger/Disgust	Irritability, temper, rage	I hate you.
Tension	Worry, Tautness, rigidity	I don't want to work with my sister.
Sadness	Anguish, grief, misery	I am really sad.
Boredom/Apathy	Fatigue, lack of interest, indifference	I don't want to work on this today. I am tired.
Affection/Caring	Friendliness, warmth, encouragement	I can help you with this one.
Humor	Banter, joking, comedy	That looks like a tree.
Contempt	Snobbery, ridicule, mimicking	Bad job for you. Stop being scary.
Joy/Pleasure	Delight, excitement, happiness	I got it! Graceful!
Distress/Frustration	Complaining, impatience, upset, disappointment	I don't get it. I can't do it.

After creating my analytic framework, I examined all of transcripts from the problem-solving sessions and used the framework to document anytime an emotion was displayed by a student. If the student displayed several statements of frustration in a row, each individual statement was documented as its own sign of frustration. I did this process twice to check for consistency, similar interpretation, and to make sure nothing was missed. This process was completed using the qualitative software, Transana (Woods & Fassnacht, 2016). Next, I created reports through the use of Transana to account for each emotion that was displayed by a student.

Results

During analysis, I explored the emotions students displayed when they were engaged with the exploration of unsolved problems (see Table 2). Through my Transana reports, I found that the two most common emotions students displayed were joy and frustration.

Table 2: Emotions Displayed by Session

Emotions	1	2	3	4	5	6	7	Total
Anger/Disgust	0	0	0	0	0	0	0	0
Tension	0	0	0	0	0	0	0	0
Sadness	0	0	0	0	0	0	0	0
Boredom/Apathy	1	0	1	2	0	0	0	4
Affection/Caring	2	5	1	0	3	7	0	18

Humor	3	9	0	3	1	2	3	21
Contempt	4	13	6	3	3	15	5	49
Joy/Pleasure	23	32	21	14	25	44	18	177
Distress/Frustration	22	26	56	21	23	38	37	223

Frustration

Documented a total of 223 times, frustration was displayed the most throughout the seven problem-solving sessions. Bernice, Trevor, and Joella displayed the emotion of frustration most commonly during the sessions (see Table 3). These emotions were most typically displayed while the students were working on finding a graceful labeling of a graph. Comments such as, "I don't get this," "Oh my gosh," "This is hard," and "It won't work" were common statements students made that indicated the emotion of frustration. They also had actions like slamming their hand on the table or making moaning sounds.

Joy

Joy was the second most common emotion displayed. It was documented 177 times. Becca, Alia, and Bernice displayed the emotion of joy most frequently throughout the seven sessions (see Table 3). Joy was displayed most frequently when students found a graceful labeling for a graph. Comments marked as joy included statements such as, "Got it!" "I am done," "I did it, I finished it!" "Finished," and Graceful!" Many times these comments were also made with students throwing their arms in the air in celebration or with clapping.

Productive Struggle

Because of the large number of times students displayed joy and frustration, I wanted to examine frustration and joy in more detail. I broke down frustration and joy by student and session (see Table 3). When I did this, I found that every student displayed both frustration and joy multiple times throughout the problem-solving sessions. Upon a closer look at when the students were displaying the frustration and joy, I found that students would display multiple signs of frustration followed by a moment of joy. Most of the time the joy was displayed when a student found a graceful labeling of a graph. I will share two students' stories throughout a session to document the interplay of frustration and joy.

Table 3: Frustration and Joy By Student and Session

	Alia		Amanda		Becca		Bernice		Edward		Hector		Iris		Joella		Karly		Trevor	
Session #	F	J	F	J	F	J	F	J	F	J	F	J	F	J	F	J	F	J	F	J
1	2	4	3	1	-	-	6	6	2	5	0	1	0	0	3	3	1	0	5	3
2	0	12	1	2	3	3	8	0	2	2	-	-	0	3	3	2	-	-	9	8
3	8	5	1	2	5	2	15	2	1	2	4	0	1	3	10	3	5	2	6	0
4	-	-	1	1	7	1	9	5	1	4	-	-	1	1	2	1	-	-	0	1
5	3	9	0	1	0	5	5	4	1	1	2	0	2	2	5	2	0	1	5	0
6	4	5	2	7	2	10	6	6	1	3	7	4	1	2	11	1	1	2	3	4
7	2	6	1	0	9	3	9	5	1	0	-	-	0	2	1	1	-	-	14	1
Total	19	41	9	14	26	24	58	28	9	17	13	5	5	13	35	13	7	5	42	17

Note. – means a student was absent for the problem-solving session. F stands for frustration and J stands for joy.

Bernice. Bernice was a Grade 4 student. Her story was chosen to be documented because she was vocal in her feelings and displayed the most frustration and the second most amount of joy.

During the Session 6, Bernice began with attempting to gracefully label the fourth graph in the Caterpillar class. This graph contained 10 nodes (previously she had only done graphs that contained eight nodes). Bernice began her attempts at labeling the graph. Several minutes into working she said, "Oh, dang it. I need a nine," showing frustration. She continued to work and changed her frustration into excitement with the statement, "I am doing it, I am doing it." Only two minutes later, while she was still trying to label the graph gracefully she said, "One, two. What happened? Oh no, I missed one." (A labeling of a node.) Again shifting from joy back to frustration. This frustration continued as she worked and the following conversation took place with myself, the teacher.

Bernice: Seven minus three equals four (has her yellow page almost filled). Oh my god, Ms. Teacher, look how close I was. But this last one doesn't make sense.

Teacher: Well are you looking at your sheet for a pattern?

Bernice: No.

Then Bernice continued to struggle while displaying signs of frustration with the following statements: "This makes, oh my god this doesn't make sense" and "It does not make sense."

Several minutes later, Bernice completed a graceful label and shouted with her arms in the air, "I did it. I am done. I need a pen. Ms. Teacher, I did it." At this time her frustration oscillated to joy (see Figure 4 of her work on the first set of Caterpillar graphs).

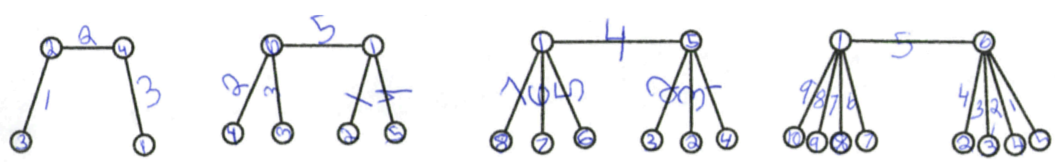

Figure 4. Bernice's work on Caterpillar Graphs.

Alia. Alia was a Grade 4 student. She had a very positive and cheery demeanor. She also displayed her productive struggle through an oscillation of frustration and joy. Her story was chosen be documented because she displayed the most total accounts of joy throughout the sessions.

During Session 6, Alia was working on gracefully labeling Caterpillar graphs. After a few minutes of working, Alia said, "I am lost" and erased her page, displaying frustration. After another thirty seconds, Alia stated aloud, "Okay, I got it" and continued to work for several more minutes. Alia then noticed that she was missing an eight on one of her nodes and shouted, "No! I need an eight." This showed her frustration. Alia continued to work on finding a graceful labeling and just twenty-seven seconds following her last shout of frustration, she clapped her hands and then threw her arms in the air and shouted, "I did it! Woo!" changing her frustration to joy.

Alia began to work on labeling her next caterpillar graph gracefully. Two minutes into working, Alia took her hand and slammed it on the table and stated, "Oh, I messed up." Only twenty-four seconds later, Alia clapped her hands together and said, "I was right. I was right." This again showed her oscillation from frustration to joy. Alia continued to work. After six more minutes of work, Alia shouted with her arms in the air, "Oh, I am good at this" and continued to work. Several seconds later, Alia said, "No" displaying frustration and a solution that was not a graceful label. After six more minutes of working, Alia shouted, "Yes" with her arms in the air. After recording her solution, Alia turned to the instructor and said, "Okay, I am ready for the next one." Alia continued working throughout the rest of the time with the similar pattern of joy and frustration. She displayed frustration when she was struggling to find a solution followed by joy and excitement when she made progress on her work.

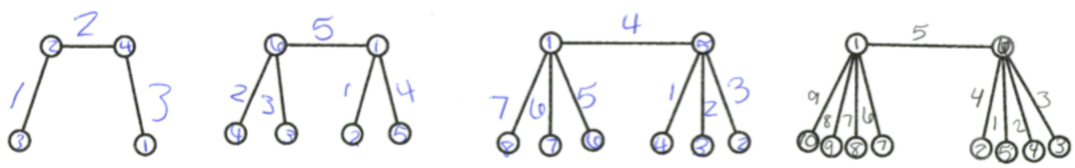

Figure 5. Alia's work on Caterpillar Graphs

Similar to Bernice, Alia also displayed a range of emotions and oscillated between being frustrated and then displaying signs of joy or pride when she produced a graceful labeling. I found this oscillation between joy and frustration a sign of productive struggle. This struggle was similar to the productive struggle Alia, Bernice, and the other students demonstrated during all the problem-solving sessions.

Discussion and Conclusions

The most common emotions that students displayed were frustration and joy. These results are similar to Else-Quest et al. (2008) who found the most common negative emotion students displayed were frustration and distress and the most common positive emotions students displayed were joy and pride. Students in this study also showed emotions of contempt, sadness, boredom, humor, and acts of caring.

Past research has documented different ways to help students during productive struggle (Warshauer, 2015; Zeybek, 2016), and that productive struggle is beneficial (Kapur, 2010). But, there is limited research in what productive struggle actually looks like as children are problem solving (Warshauer, 2015; Zeybek, 2016). Through this research study, I have documented that while students are engaged in productive struggle, they display the emotions of frustration and joy. Students display frustration while struggling through a portion of a problem. Frustration was displayed multiple times throughout their work. Once a student found some success or made progress towards the solution, they enacted joy. This process was repeated throughout the mathematical problem solving.

Overall, I believe a significant finding of this study pertains to the oscillation of joy and frustration which I have interpreted as the construct of productive struggle. All of the students in this study were able to persist through the struggle and were able to find joy and pride in their work. Unsolved mathematics problems are not typically part of elementary school education but I found that these types of problems allowed students the opportunity to engaged in productive struggle and experience mathematics more similar to how a mathematician might experience mathematics.

References

Daher, W. (2015). Disursive positioning and emotions in modeling activities. *International Journal of Mathematical Education in Science and Technology, 46*(8), 11149–1164. doi:10.1080/0020739X.2015.1031836

Else-Quest, N. M., Hyde, J. S., & Hejmadi, A. (2008). Mother and child emotions during mathematics homework. *Mathematical Thinking and Learning, 10*(1), 5–35. doi:10.108/10986060701818644.

Evans, J., Morgan, C., & Tsatsaroni, A. (2006). Discursive positioning and emotion in school mathematics practices. *Educational Studies in Mathematics, 63*(2), 209–226. doi:10.1007/s10649-006-9029-1

Goldin, G. A. (2000a). Affective pathways and representation in mathematical problem solving. *Mathematical Thinking and Learning, 2*(3), 209–219. doi:10.1207/S15327833MTL0203_3

Goldin, G. A. (2000b). A scientific perspective on structures, task-based interviews in mathematics education research. In A. E. Kelley & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517–545). Mahwah, NJ: Erlbaum.

Hannula, M. S. (2015). Emotions in problem solving. In S. J. Cho (Ed.), *Selected regular lectures from the 12th international congress on mathematical education* (pp. 269–288). Seoul, Korea: Springer. doi:10.1007/978-3-319-17187-6_16

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 1, pp. 371–404). Charlotte, NC: Information Age Publishing.

Kapur, M. (2010). Productive failure in mathematical problem solving. *Instructional Science, 38*(6), 523–550. doi:10.1007/s11251-009-9093-x

Reinhart, S. C. (2000). Never say anything a kid can say! *Mathematics Teaching in the Middle School, 5*(8), 478–483.

Turner, E., Dominquez, H., Maldonado, L., & Empson, S. (2013). English learners' participation in mathematical discussion: Shifting positionings and dynamic identities. *Journal for Research in Mathematics Education, 44*(1), 199–234. doi:10.5951/jresematheduc.44.1.0199

van Langenhove, L., & Harré, R. (1999). Introducing positioning theory. In R. Harré & L. van Langenhove (Eds.), *Positioning theory* (pp. 14–31). Oxford, United Kingdom: Blackwell.

Warshauer, H. K. (2015). Productive struggle in middle school mathematics classrooms. *Journal of Mathematics Teacher Education, 18*(4), 375–400. doi:10.1007/s10857-014-9286-3

Wood, M. B. (2013). Mathematical micro-identities: Moment-to-moment positioning and learing in a fourth-grade classroom. *Journal of Research in Mathematics Education, 44*(5), 775–808. doi:10.5951/jresematheduc.44.5.0775

Woods, D. K., & Fassnacht, C. (2016). *Transana v3.02*. Madison, Wi: The Board of Regents of the University of Wisconsin System, http://www.transana.org.

Yamakawa, Y., Forman, E., Ansell, E. (2009). The role of positioning in constructing an identity in a 3rd grade mathematics classroom. In K. Kumpulainen, C. E. Hmelo-Silver, & M. César (Eds.), *Investigating classroom interaction: Methodologies in action* (pp. 179–202). Rotterdam, the Netherlands: Sense.

Zeybek, Z. (2016). Productive struggle in a geometry class. *International Journal of Research in Education and Science, 2*(2), 396–415. doi:10.21890/ijres.86961

MATHEMATICS ANXIETY AND MEDITATION

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Mathematics anxiety debilitates many students, reducing academic success and quantitative literacy. Although various treatments have been researched, there has been no successful intervention for this phenomenon. Meditation has been recognized as helping individuals reduce other forms of anxiety. Research in this field is limited due to the internal nature of meditation. In this study, a device called a Muse Headband addressed some of these limitations. Results showed that meditation can potentially help students reduce their mathematics anxiety.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Technology

Mathematics anxiety, "a feeling of tension and anxiety that interferes with the manipulation of numbers and the solving of mathematical problems in … ordinary life or academic situations" (Richardson & Suinn, 1972), challenges many students. This phenomenon, although related, is distinct from general or test anxiety because of the association with mathematics. Researchers disagree on the prevalence of mathematics anxiety. However, conservative estimates indicate that this problem disproportionately affects females (Dowker, Sarkar, & Looi, 2016).

The consequences for students who suffer from mathematics anxiety can be significant. These students can have lower grades (Richardson & Suinn, 1972), be less fluent in mathematics (Ashcraft & Faust, 1994), avoid enrolling in mathematics courses (Stubblefield, 2006), and prefer procedural mathematics courses that are not conceptually driven (Norwood, 1994). These factors lead to students being less quantitatively literate, affecting prospects for employment.

Interventions to help alleviate or prevent mathematics anxiety vary, although there is no comprehensive cure (Dowker et al., 2016). Such interventions include having a student reappraise the student's belief in the negative consequences of their mathematics anxiety. Another intervention has the student write out their worries and fears. Focused breathing exercises have also been shown to help math-anxious students better perform on a series of arithmetic tasks (Brunyé et al., 2013). Overall, interventions to alleviate mathematics anxiety focus on calming the mind, directing it away from worrisome thoughts.

Considering mathematics anxiety as a specific type of general anxiety, successful interventions for general anxiety may inspire ways to help students with mathematics anxiety. Meditation is one particular intervention. Mindful meditative practices, where an individual directs attention to their breathing or focuses on their body while in a relaxed state has shown to potentially help an individual's cognition (Zeidan, Johnson, Diamond, David, & Goolkasian, 2010) and mitigate their anxiety (Khusid & Vythilingam, 2016). Mindfulness and the associated calmness, clarity, and stability in the mind can allow an individual to "respond" rather than "react" to potentially anxiety-producing situations more effectively (Miller et al., 1995, p. 197).

Meditation and related activities have been used in mathematics classrooms to support student learning. Students have demonstrated increased speed and accuracy in computations after practicing tai chi and yoga (Field, Diego, & Hernandez-Reif, 2010). Furthermore, mindfulness impacted general anxiety, but did not impact performance on lower-stakes mathematics homework assignments (Bellinger, DeCaro, & Ralston, 2015). By helping students explore meditative states and mindfulness, can we positively impact students' mathematics anxiety?

Researching mindfulness and meditation is difficult due to the internal nature of focused meditation (Davidson & Kaszniak, 2015). There is no way to measure to what extent a participant is mentally focused or actively meditating. Furthermore, due to the internal and personal nature of meditation, it is difficult for a novice to know when they are in a meditative state and how they affect their meditative experience. How can a person know whether they are truly meditating? These challenges can be overcome with an electroencephalogram (EEG) to provide data and feedback. An EEG measures an individual's brainwaves and determines the degree to which an individual is in a meditative state. EEGs are cumbersome and expensive. Researchers need a portable, affordable way to measure students' mental state; students need a way to receive feedback to understand and control their state of mind.

The Muse Headband: Measuring States of Mind

A Muse Headband is a portable EEG, a lightweight headband that measures brainwaves, and interprets and records the extent to which a user is in a meditative state. Users wear the headband which is paired with a mobile application (app) while listening to audio prompts during the session. A Muse session occurs in two phases, a calibration phase and a meditation phase.

In the calibration phase, the user is prompted to close their eyes and let their mind flow naturally, allowing wandering or distraction. A baseline wherein the individual's general state of mind is then recorded. Since an individual's mental state varies, calibration must occur at the start of each session. After calibration, the meditation phase of the Muse session begins. The user sets the length of the session and is prompted to focus on their breathing, while trying to calm and focus the mind. Background sounds, such as water lapping or ambient music, are played. The data collected in this phase are compared with data collected during calibration to determine the extent to which the user's mental activity differs from the calibration, measuring the extent to which the user can focus his or her mind. Depending on how the mental activity during the meditative session differs from the calibration session each moment is labeled as active, neutral, or calm. The user's goal is to maximize how long they were calm or were able to maintain a difference in brain activity between the calibration and meditation phases of the Muse session.

Using a Muse Headband, students can learn how to focus their mind and maintain a meditative state. Consequently, many challenges posed by researching meditation can be eliminated since actual brain activity is recorded and both users and researchers can know whether they are in a calm focused state. As a result, it is possible to determine the extent to which meditation can help students alleviate their mathematics anxiety.

Methods

In this study, students at a small university in the northeastern United States participated in meditation sessions using a Muse Headband to reduce their mathematics anxiety. This project sought to answer the question: *Does using a Muse Meditation Headband significantly reduce students' mathematics anxiety?* This study was conducted during the Fall 2017 semester.

Two algebra classes, an experimental and control group, participated in the study. The course covered factoring polynomials, graphing and solving quadratics, and manipulating rational expressions. Both classes were taught by the same instructor and had identical assignments, assessments, and lessons. Nine students in the control class and 13 students in the experimental class consented to participate. At the beginning and end of the semester, all students were given the Mathematics Anxiety Rating Scale – Short Version (MARS-S) (Suinn & Winston, 2003), a scale which measures mathematical test anxiety and numerical (calculating) anxiety. The experimental class meditated with the Muse Headband for 3-5 minutes at the beginning of each class, including exam days. Lessons were modified to account for this time. On the last day of data collection, students meditated for 10 minutes prior to taking the MARS-S post-test.

Results

Complete data from seven students in the experimental group and nine students in the control were collected and analyzed. These data included pre- and post-test MARS scores in both classes and data from at least 10 meditation sessions in the experimental class. A total of 10 males and 6 females participated. A repeated measures ANOVA was conducted to address the research question. This analysis was conducted twice; the first analysis included a student in the control group, the second did not. This student experienced a death in his immediate family early in the semester. At the end of the semester, prior to completing the MARS-S posttest, he was again absent due to related family obligations. This student's MARS-S score decreased by 16 points, a decreased unmatched by any participant in either class. It is possible that this decrease in the MARS-S score was related to this personal trauma. The student may have been more focused on their personal loss than on academic success. This student's unfortunate experience requires that we consider these scores an outlier and analyze the data with and without these data.

Results of these repeated measures ANOVA are mixed. If the outlier is not included in the analysis, results indicated a significant decrease in students' anxiety scores from the pre-test to the post-test for students in the experimental class, $F(1, 13) = 6.712$, $p = 0.022$. If the outlying student in the control group is included in the analysis, then results did not indicate a significant decrease in students' anxiety, $F(1, 12) = 3.342$, $p = 0.089$. The source of the interaction effect was the difference between the experimental and control groups at post-test. A pairwise comparison of post-test scores is not statistically significant, $p = 0.094$. Overall there is a plausible decrease in anxiety scores that warrants further investigation. See Tables 1 and 2.

Table 1: MARS-S Scores, Excluding Outlier

	Pre-Test (M ± SD)	Post-Test (M ± SD)
Control Group (n=8)	71.75 ± 16.49	78.75 ± 12.36
Experimental Group (n=7)	69.57 ± 22.28	64.29 ± 18.4

Table 2: MARS-S Scores, Including Outlier

	Pre-Test (M ± SD)	Post-Test (M ± SD)
Control Group (n=9)	73.44 ± 16.24	77.89 ± 11.85
Experimental Group (n=7)	69.57 ± 22.28	64.29 ± 18.4

Limitations

The small sample size in this study cannot be overlooked. It would be premature to draw conclusions stating that meditation contributes to differences in students' mathematics anxiety. The sample size also makes it difficult to discern whether a students' anxiety level decreased due to having the opportunity to meditate or just from having a moment to mentally pause and quietly reflect at the start of class. Finally, the sample size makes it difficult to determine what aspects of this experience contributed to potential changes in students' mathematics anxiety.

Although the researcher sought out identical classes, the two classes did not have identical student bodies. The control class consisted mostly of first-year students who placed directly into the class. Students in the experimental class were mostly second-year college students and were retaking this course or had taken and failed the prerequisite developmental course.

Discussion

This study demonstrates a potential for meditative practices to reduce mathematics anxiety. Future research studies with a more robust sample size could allow for a more thoughtful investigation into the connections between meditation and alleviating mathematics anxiety, including whether changes in MARS-S scores could be attributed to long-term effects that resulted from meditating over the course of the semester, or from the meditation session which occurred immediately preceding the student completing the MARS-S.

Finally, as mathematics educators and teachers, we should recognize how mathematics anxiety affects students' experiences and progress in our classrooms. This project has once again demonstrated that mathematics anxiety is a common phenomenon and needs to be addressed. By focusing on affective factors such as mathematics anxiety, a phenomena that has continuously challenged mathematics education researchers, we would be taking another step towards re-humanizing mathematics and mathematics education (Gutierrez, 2018). Reducing mathematics anxiety has the potential to address the needs of all students.

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References

Bellinger, D. B., DeCaro, M. S., & Ralston, P. A. S. (2015). Mindfulness, anxiety, and high-stakes mathematics performance in the laboratory and classroom. *Consciousness and Cognition*, *37*, 123–132.

Brunyé, T. T., Mahoney, C. R., Giles, G. E., Rapp, D. N., Taylor, H. A., & Kanarek, R. B. (2013). Learning to relax: Evaluating four brief interventions for overcoming the negative emotions accompanying math anxiety. *Learning and Individual Differences*, *27*, 1–7. https://doi.org/10.1016/j.lindif.2013.06.008

Davidson, R. J., & Kaszniak, A. W. (2015). Conceptual and methodological issues in research on mindfulness and meditation. *American Psychologist*, *70*(7), 581–592. https://doi.org/10.1037/a0039512

Dowker, A., Sarkar, A., & Looi, C. Y. (2016). Mathematics Anxiety: What Have We Learned in 60 Years? *Frontiers in Psychology*, *7*. https://doi.org/10.3389/fpsyg.2016.00508

Field, T., Diego, M., & Hernandez-Reif, M. (2010). Tai chi/yoga effects on anxiety, heartrate, EEG and math computations. *Complementary Therapies in Clinical Practice*, *16*(4), 235–238.

Gutiérrez, R. (2018). Rehumanizing mathematics. In I. M. Goffney & R. Gutiérrez (Eds.), *Re-humanizing mathematics for Black, Indigenous, and Latinx students. Annual Perspectives in Mathematics Education.* Reston, VA: National Council of Teachers of Mathematics.

Khusid, M. A., & Vythilingam, M. (2016). The Emerging Role of Mindfulness Meditation as Effective Self-Management Strategy, Part 1: Clinical Implications for Depression, Post-Traumatic Stress Disorder, and Anxiety. *Military Medicine*, *181*(9), 961–968. https://doi.org/10.7205/MILMED-D-14-00677

Nielsen, I. L., & Moore, K. A. (2003). Psychometric data on the mathematics self-efficacy scale. *Educational and Psychological Measurement*, *63*(1), 128–138.

Norwood, K. (1994). The effect of instructional approach on mathematics anxiety and achievement. *School Science and Mathematics*, *94*(5), 248–254.

Richardson, F. C., & Suinn, R. M. (1972). The Mathematics Anxiety Rating Scale: Psychometric data. *Journal of Counseling Psychology*, *19*(6), 551–554. https://doi.org/10.1037/h0033456

Suinn, R. M., & Winston, E. H. (2003). The mathematics anxiety rating scale, a brief version: Psychometric data. *Psychological Reports*, *92*(1), 167–173. https://doi.org/10.2466/pr0.2003.92.1.167

Zeidan, F., Johnson, S. K., Diamond, B. J., David, Z., & Goolkasian, P. (2010). Mindfulness meditation improves cognition: Evidence of brief mental training. *Consciousness and Cognition*, *19*(2), 597–605. https://doi.org/10.1016/j.concog.2010.03.014

THE UNDERGRADUATE MATHEMATICS LEARNING CENTER: PERSPECTIVES OF PEER TUTORS

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Virtually all calculus-offering post-secondary institutions in the United States provide mathematics tutoring for undergraduates, but these tutoring interactions are not well studied. This study observed tutoring in a mathematics learning center (MLC) and interviewed four tutors about their role within the MLC and the role of the MLC in mathematics learning at the institution. The tutors all felt that the MLC was a unique space both socially and academically that provided a qualitatively different learning opportunity for students. Formal observations of interactions within the MLC support this view. Suggestions are made for increased research of MLCs and tutors providing distinct mathematics learning opportunities, with the intention of improving student outcomes through extending research-based instructional practices to the learning space provided by an MLC.

Keywords: Affect, emotions, beliefs, and attitudes; Informal education; Post-secondary education; Instructional activities and practices

According to the recent MAA National Study of College Calculus (Bressoud, Mesa, & Rasmussen, 2015), a study that collected data from departments and individuals in a stratified random sample of over 200 non-profit colleges and universities across the United States, almost all calculus-offering institutions in the United States offer mathematics tutoring on campus. Furthermore, 40% of Calculus I students surveyed reported that they had utilized the campus-provided tutoring center for mathematics at some point and the study showed that merely the existence of a tutoring center improved student attitudes toward mathematics (Bressoud et al., 2015). Clearly, mathematics tutors and tutoring centers impact or have the potential to impact post-secondary mathematics students in positive ways. Yet at the same time, there is very little extant research into undergraduate mathematics tutoring and mathematics learning centers (MLCs) (Mills, Tallman, & Rickard, 2017).

One study by Solomon, Croft, and Lawson (2010) considered MLCs at two UK universities through the theoretical lens of figured worlds (Holland, Lachicotte, Skinner, & Cain, 1998). Figured worlds provides a socio-cultural perspective of identity formation through ongoing authoring and positioning work in social interactions. Their primary finding was that the MLC was viewed as a distinct academic and social space by mathematics students, and that these distinctions allowed for different types of mathematical identity work versus the interactions that took place in a classroom or within the offices of their professors or teaching assistants. The present study sought to extend our understanding of the MLC and undergraduate peer tutors by considering an MLC as a similarly potentially unique social space where mathematics learning and teaching takes place in ways outside of traditional, classroom-based paradigms. In contrast to Solomon, et al (2010), this study had a focus on the perspective of tutors rather than students, and took place in the US where tutors and tutoring centers may have different, though equally distinct and perhaps even more ill-defined, academic and social connotations. The study sought to address the research question: how do undergraduate mathematics peer tutors describe the figured world of a mathematics learning center and their role within it? The figured world framework allows for a description of the social space that accounts for broader culture, specific interactions, and the beliefs and attitudes of individuals. There is currently a lack of research around undergraduate mathematics peer tutoring in the US (Mills, Tallman, & Rickard, 2017) so rather than an empirical study this work was exploratory to describe an MLC and thus give context to inform continuing and future research.

Setting

The study took place in the MLC of a large, public university in the Southwestern United States, which offers drop-in tutoring for all mathematics courses, emphasizing the Precalculus through Calculus II sequence. The university also has an appointment-based writing center and various on-campus organizations offer supplemental instruction sections and other venues for tutoring, including mathematics tutoring. The MLC is located in the campus library and employs 20-30 undergraduates for drop-in tutoring over 46 hours during the week. Between two and five tutors may be working at a time, depending on the time of day and other factors like upcoming exams. Observed tutoring interactions lasted an average of 10-20 minutes with an individual student, although depending on tutoring demand, tutors sometimes remained longer or returned again to the same student for another interaction.

Methods & Analysis

Four undergraduate peer tutors were recorded going about their normal tutoring duties once a week over several weeks. Following each recorded observation the tutors joined the researcher in watching the footage of their tutoring interactions and explaining their decision-making and thought process to the researcher in the form of a stimulated recall interview (Dempsey, 2010). During the stimulated recall interview the participant was asked to "think aloud" about their tutoring actions and explain to the researcher their decision-making processes. The research also asked clarifying questions about specific enactments. A total of 15 tutoring shifts were video recorded and 71 distinct tutoring interactions were discussed in stimulated recall interviews. Each participant also participated in a in a final semi-structured interview with the researcher where they were asked to more globally describe their perspective on the MLC and their role(s) within it. The table below summarizes key characteristics of the participating tutors.

Table 1. Case study participants' demographic information.

Case Study Participants*	Year	Major	Ethnicity	Time Tutoring in the MLC
Danielle	Junior	Mathematics	White/Caucasian	1-2 years
Eric	Senior	Mathematics	Chinese-Filipino	1-2 years
Jake	Sophomore	Engineering	White/Caucasian	<1 year
Lily	Junior	Mathematics & Physics	Cambodian	1-2 years

**Gender-preserving pseudonyms chosen by the participants*

Stimulated recall interviews and final interviews were transcribed. Episodes and potentially relevant actions or utterances were logged for observed tutoring shifts which allowed for easy reference when analyzing stimulated recall interviews, but observations were not completely transcribed. Grounded theory was used to develop a coding scheme (Corbin & Strauss, 1990). Interactions were coded as a unit with field notes, log notes from the researcher watching the recording of the interaction, and stimulated recall transcripts were considered together.

Results

Aligning with the work of Solomon, et al (2010) tutors reported that the MLC provided something more than a central location or extended hours – that the space and the help provided there were different than that provided by professors and TAs. From the perspective of figured worlds, these tutors explained the MLC as a figured world where mathematical struggle is valued, and where the role of a tutor within the MLC to be that of a more advanced peer. Their role included normalizing student struggles, creating a non-judgmental environment, and explaining things in ways that were accessible to students. As one tutor put it:

> "The intimacy level and the time provides a different environment if that makes sense. Not only do [professors], sometimes, not only would they know what's going on better but they might try to reach it faster and not be as elaborate in their explanations. And again because the tutor might not be on the same page as the lecture it might take some more bungling and some more muddling and some more sharing of experiences and tools." – Eric

Eric saw this muddling as a good thing, validating student struggles and reinforcing that the material isn't meant to always be easy. He saw himself in the position of a near-peer offering help to a fellow student. Lily shared a similar view, seeing in fact her lack of extensive expertise in more advanced mathematics and her having recently taken the courses herself as potentially beneficial in providing clear explanations and making students feel welcome:

> "[W]e have this math learning center here so that students can come ask for help with people who are around their age, also going through the same ideas, the same process of going to school, trying to learn, they're struggling also, in a way. But that they've gone through these classes and they know what it's like. Because, I mean, professors if they're not the newest professors it's been quite a while since they've taken a class on Calc II… Meanwhile we're students and we're like 'oh, we just took this two years ago' or maybe even a year ago," – Lily

Jake and Danielle echoed that it was important for there to be a nonjudgmental space, a space were students would feel comfortable asking questions. The existence of such a place, Jake believed, allowed students to persist rather than giving up:

> "They can ask a question about their class and not worry about their professor being like 'Did you read the syllabus?' And so I think it makes it more comfortable for them and yet it helps them learn. I think that obviously more people would be failing if we didn't have a place like this because we have a place to come ask questions rather than just giving up on it." – Jake

> "Well I'm not going to judge them… I think that by coming to the MLC talking to someone who doesn't know you at all and who isn't grading your stuff, there's not a lot of judgement that is there." – Danielle

The tutors studied here see their work as being distinct from that of a professor or TA, and the MLC as providing a nonjudgmental space that encourages a certain form of help-seeking and provides a valuable and unique service for mathematics students.

Observations of participant's tutoring and their explanations in stimulated recall interviews suggest that these beliefs about their roles and the figured world of the MLC influence tutoring interactions. For example, participants were observed to sit whenever possible while tutoring, and when a chair was not available to bend or kneel rather than remaining upright. When asked about this behavior, multiple participants expressed that being eye-level was important so that the student never felt that they were being talked down to.

Conclusion

The perspectives of the tutors in this study highlight the ways in which a peer tutor and a MLC are likely to be distinct in their social and academic environments from classrooms and classroom-based instructors. In particular, the tutors interviewed described themselves as being less intimidating and perhaps more helpful to students as near-peers than the more intimidating, non-peer role of a professor or TA. This informed their role as tutors to include behaviors like intentional affirmation of the realities of struggling as a mathematics student, and positioning themselves in non-authoritative ways. While the current study is considering a single mathematics learning center and a subset of tutors, it is my hope that further studies into MLCs and the tutors who work in them will continue to be conducted as we learn more about the role that these individuals and spaces play in improving student outcomes in mathematics.

References

Bressoud, D. M., Mesa, V., & Rasmussen, C. L. (Eds.). (2015). Insights and recommendations from the MAA national study of college calculus. MAA Press.

Corbin, J., & Strauss, A. (1990). *Grounded theory research: Procedures, canons and evaluative criteria*. Qualitative Sociology, 19(6), 418–427.

Dempsey, N. P. (2010). Stimulated recall interviews in ethnography. *Qualitative Sociology*, 33(3), 349–367. http://doi.org/10.1007/s11133-010-9157-x

Holland, D. C., Lachicotte, W., Skinner, D., & Cain, C. (1998). *Identity and Agency in Cultural Worlds*. Cambridge, MA: Harvard University Press.

Mills, M., Tallman, M., & Rickard, B. (2017). Research opportunities for RUME researchers in the context of mathematics resource centers. Presented at the annual meeting of the Special Interest Group of the Mathematical Association of American on Research in Undergraduate Mathematics Education, San Diego.

Solomon, Y., Croft, T., & Lawson, D. (2010). Safety in numbers: Mathematics support centres and their derivatives as social learning spaces. *Studies in Higher Education*, 35(4), 421–431. http://doi.org/10.1080/03075070903078712

MATHEMATICAL IDENTITIES: NARRATIVES AND DISCOURSES OF FEMALE STUDENTS IN AN 8TH GRADE CLASSROOM

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This research in progress examines two 8th grade female students' socially constructed mathematical identities. Using ethnographic and discourse analytic methods, the study examined how these evolving mathematical identities intersect with gender identities. Preliminary results indicate female students in this study actively disrupt stereotypes of femininity and of women in mathematics. This study offers a fine-grain analysis of how female students identify with mathematics at a social and cultural level, and it provides insight on how these mathematical identities can develop and change over time.

Keywords: Classroom Discourse, Gender, Diversity and Equity, Middle School Education

Introduction

Between May of 2009 and May of 2015 employment in science, technology, engineering and mathematics (STEM) occupations grew by 10.5% in the United States (Fayer, Lacey & Watson, 2017). The STEM job market is the fastest growing job sector in the United States, and mathematics and computer occupations are predicted to have the most growth in the coming years, over 25% between 2014 and 2024 (Fayer, Lacey & Watson, 2017). Yet, women are still vastly underrepresented within STEM occupations, particularly mathematics and computer science, with women occupying only 25.5% of these occupations in 2016 (U.S. Department of Labor, Bureau of Labor Statistics, 2017). There are many questions that arise as to why this under-representation of women in the STEM labor force persists, and what factors may be contributing to this under-representation. An area of literature that has yet to be explored in detail is the social and cultural construction of mathematical identities in the classroom (Bishop, 2012), in particular female students, and how they identify themselves in relation to mathematics while also navigating the complexities of social life in schools. Mathematics coursework is a foundation for STEM majors and careers, and this academic area is part of students' educational trajectories from their first days in school (Shapiro & Sax, 2011). Middle and high school mathematics experiences are particularly foundational for students interested in pursuing STEM careers (Shapiro & Sax, 2011). How students identify with mathematics is an essential component of students' mathematics experiences (Bishop, 2012). Thus, this study in progress proposes to examine students' socially constructed mathematical identity and how this intersects with gender over the course of 8th grade.

Background

A prominent way of defining identity for educational research purposes, is that identity is constructed at a social and cultural level and realized through social interactions (Anderson, 2007; Boaler, 2002; Bucholtz & Hall, 2005; Davies & Harre, 1990; Duff, 2002; Mendick, 2005; Nasir & de Royston, 2013; Solomon, Lawson & Croft, 2011). Since this study takes a social and cultural lens of classroom and student interactions, *identity* is defined as socially constructed and negotiated in specific contexts and informed by personal history and narratives (Anderson, 2007; Boaler, 2002; Bucholtz & Hall, 2005; Davies & Harre, 1990; Duff, 2002; Mendick, 2005; Nasir & de Royston, 2013; Solomon, Lawson & Croft, 2011). For the purposes of this research, the socially constructed conception of identity is a persuasive definition based on theoretical grounding of social constructivism, in which meanings that we take from the world are inherently dependent upon social relationships which we are a part of, and our sense of self must be generated through social processes (Gergen, 2009). Now following in the broader definition of *identity*, *mathematical identity* is located, which for this study is defined as a person's socially constructed and negotiated identity in the mathematics context, which is also informed by a personal narrative of one's mathematical self both inside and outside of the classroom (Anderson, 2007; Bishop, 2012; Boaler, 2002; Cobb, Gresalfi, & Hodge, 2009).

For the purposes of this study, gender is conceptualized as a broader category than merely sex, male vs. female, and instead as a socially constructed phenomenon of interest that is consequential in mathematical identity formation (Damarin & Erchick, 2010). When a student is identified by others in a social context as female, and/or if the student self-identifies as female, this holds social consequence, especially in mathematics where mathematical ability is often gendered as a male characteristic (Mendick, 2005). This study holds significance in understanding how gender plays a role in students' construction and performance of their mathematical identities in classroom spaces during a formative time in their schooling, which is their 8th grade year of mathematics, one year prior to the transition to high school.

Research Design and Methods

This study provides results from the first year of a two-year ethnographic and discourse analytic study of students' mathematical identities over the course of 8th grade and 9th grade, as students have to navigate a major transition in their academic life. This study highlights the importance of two female students' construction of their mathematical identities and narratives through two interviews, and performance of their mathematical identities in day to day classroom interactions during their 8th grade year. This study also examines how these two female students' mathematical identities intersect with social constructions of gender. In conducting the research for this project, both an ethnographic and discourse analytic perspective were taken (Blommaert & Jie, 2010; Bloome, et. al, 2005; Green & Bloome, 2004; Heath & Street; 2008).

The site for this research project collection is an 8th grade mathematics classroom at a middle school in a large suburban district in the Mid-Western United States. The two female focal students, Cayla and Paris, were selected based on their well-articulated personal narratives that they gave in their first interviews, and their self-awareness of how mathematics relates to their lives. To protect the participants within the study, all names are pseudonyms.

The data collected for this study were two interviews with each student, one conducted in the fall and the other conducted in the spring, of their 8th grade year of school. The interviews were conducted as semi-structured interviews, and each interview was audio recorded and later transcribed. The other source of data for this study were 50 classroom observations throughout the school year, field notes and video recordings of each classroom observation. Selections from the classroom observations were transcribed, which illustrate the mathematical identities of the two students and also instances where gender becomes salient within the two students' classroom discourse were transcribed and analyzed. The transcripts of the interviews were analyzed from an ethnographic perspective using discourse analytic techniques (Bloome, et. al., 2005).

Findings

The in the interviews, first focal student, Cayla, expressed a dislike of mathematics prior to her 8th grade year, but now enjoys the algebra concepts that they are learning. She is positioned by the teacher and other students as being one of the top performers in the class. She positions herself as a hard-worker who now enjoys mathematics, but does not label herself as smart. Through classroom observations, there has been a shift in her social engagement over the course of the year from being excessively social with classmates, to the point of distraction, to being more focused and often times choosing to separate herself from other students who may be potential distractions.

In the interviews, the second focal student, Paris, expresses her relationship with mathematics as closely tied to her struggle managing Attention Deficit Hyperactive Disorder (ADHD) symptoms and school. She says that her mathematics experience is very dependent on how responsive the teacher is to her needs as a student with ADHD. Paris is positioned by her teacher and peers as a middle performer within the class. Paris positions herself as a meticulous worker, who mainly associates with a single friend in the class. In class she has proclaimed that she has a love-hate relationship with mathematics, and sees herself as being stuck with mathematics through school, but enjoys mathematics applied in other contexts like science.

Data show Cayla and Paris display mathematical identities that have shifted over time. They are also actively engaged in producing identities that disrupt traditional notions of femininity, and make conversational moves to assert that even though they are young women, this does not mean they have to be constrained to feminine constructs. To illustrate this type of move, the following is an excerpt from class where Cayla disrupts traditional feminine notions of a classmate as they work on a project creating a poster that illustrates an example of a linear equation, with an emphasis showing slope and y-intercept:

Richie: This is easy. (Pointing to the assignment instructions.)
Cayla: I can't draw.
Richie: You literally just have to do that (points at instructions for the assignment.) Just write little flowers and show growth.
Cayla: Just because I'm a girl doesn't mean I'm feminine and have to draw flowers on my paper.
Richie: Yeah, it does.
Kyle: No, it doesn't.
Cayla: No, I don't even like flowers.
Richie: Then draw a bicycle.
Cayla: What do bicycles have to do with slope and y-intercept?

Cayla is asserting that her conception of what it means to be female does not have to align with traditional conceptions of femininity, as well as at the end of the conversation shifting the focus of the conversation back to the mathematics assignment, indicating focus as a mathematics student. Another illustration of this type of move occurred when

Conclusion

The implications of this study are important for better understanding the female experience in male-dominated areas of STEM, and why there is a persistent underrepresentation of women in STEM majors and careers (U.S. Department of Labor, Bureau of Labor Statistics, 2017). If the experiences of Cayla and Paris are indicative of the broader experience of young women, then it can be seen how much work is required in navigating mathematics as a young woman. Paris views mathematics as a necessary step towards her personal goals, but nothing that she wishes to pursue further. Cayla's narrative is similar, despite feeling an affinity towards the mathematics content that she has learned in 8th grade, she does not proclaim excitement or a strong desire to continue mathematics coursework beyond what is required of her. It is important to continue to explore young women's experiences in mathematics in middle and high school related to their identity development, to better understand how these experiences influence choices to pursue STEM majors and careers which require advanced coursework in mathematics.

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References

Anderson, R. (2007). Being a mathematics learner: Four faces of identity. *The Mathematics Educator*, *17*(1), 7-14.

Bishop, J. P. (2012). "She's always been the smart one. I've always been the dumb one": Identities in the mathematics classroom. *Journal for Research in Mathematics Education*, *43*(1), 34-74.

Black, L., Williams, J., Hernandez-Martinez, P., Davis, P., Pampaka, M., & Wake, G. (2010). Developing a 'leading identity': The relationship between students' mathematical identities and their career and higher education aspirations. *Educational Studies in* Mathematics, *73*(1), 55-72.

Blommaert, J., & Jie, D. (2010). *Ethnographic fieldword: A beginner's guide*. Bristol, UK: Multilingual Matters.

Bloome, D., Carter, S., Christian, B., Otto, S., & Shuart-Faris, N. (2005). *Discourse analysis and the study of classroom language and literacy events.* Mahwah, NJ: Erlbaum.

Boaler, J. (2002). The development of disciplinary relationships: Knowledge, practice and identity in mathematics classrooms. *For the Learning of Mathematics*, *22*(1), 42-47.

Bucholtz, M., & Hall, K. (2005). Identity and interaction: A sociocultural linguistic approach. *Discourse Studies*, *7*(4-5), 585-614.

Cobb, P., Gresalfi, M., & Hodge, L. L. (2009). An interpretive scheme for analyzing the identities that students develop in mathematics classrooms. *Journal for Research in Mathematics Education*, *40*(1), 40-68.

Damarin, S., & Erchick, D. B. (2010). Toward clarifying the meanings of" gender" in mathematics education research. *Journal for Research in Mathematics Education*, *41*(4), 310-323.

Darragh, L. (2016). Identity research in mathematics education. *Educational Studies in Mathematics*, *93*(1), 19-33.

Davies, B., & Harre, R. (1990). Positioning: The discursive production of selves. *Journal for the Theory of Social Behavior*, *20*(1), 43-63.

Duff, P. A. (2002). The discursive co-construction of knowledge, identity, and difference: An ethnography of communication in the high school mainstream. *Applied Linguistics*, *23*(3), 289-322.

Fayer, S., Lacey, A., & Watson, A. (2017). BLS *Spotlight on Statistics: STEM occupations: Past, present, and future*. Washington, D.C.: U.S. Department of Labor, Bureau of Labor Statistics.

Gergen, K. (2009). *An invitation to social constructionism.* (2nd ed.). Thousand Oaks: Sage.

Green, J., & Bloome, D. (2004). Ethnography and ethnographers of and in education: A situated perspective. In J. Flood, S. B. Heath, & D. Lapp (Eds.), *Handbook of research on teaching literacy through the communicative and visual arts* (pp. 181-202). New York: Simon & Schuster Macmillan.

Heath, S. & Street, B. (2008). *On ethnography: Approaches to language and literacy research.* New York: Teachers College Press.

Mendick, H. (2005). A beautiful myth? The gendering of being/doing 'good at maths'. *Gender and Education*, *17*(2), 203-219.

Nasir, N. S., & de Royston, M. M. (2013). Power, identity, and mathematical practices outside and inside school. *Journal for Research in Mathematics Education*, *44*(1), 264-287.

Shapiro, C. A., & Sax, L. J. (2011). Major selection and persistence for women in STEM. *New Directions for Institutional Research*. *2011*(152), 5-18.

Solomon, Y., Lawson, D., & Croft, T. (2011). Dealing with 'fragile identities': Resistance and refiguring women mathematics students. *Gender and Education*, *23*(5), 565-583.

U.S. Department of Labor, Bureau of Labor Statistics. (2017). Labor force statistics from the current population survey. Retrieved from https://www.bls.gov/cps/cpsaat11.htm.

PROVIDING STUDENTS CORRECT VS. INCORRECT CALCULUS EXAMPLES

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This study seeks to determine differences in learning when calculus learners analyze correct or incorrect work samples and to investigate students' perceptions of the effect of analyzing work samples on their learning of mathematics. Results suggest that when students analyze incorrect work samples, they are less likely to repeat the errors they have seen. Results also suggest that students perceive correct work samples as more beneficial to learning than incorrect work samples. However, both correct and incorrect work samples were found to challenge students' thinking and promote student independence. These findings have implications for teacher practice and curriculum development.

Keywords: Metacognition, Instructional Activities and Practices, Advanced Mathematical Thinking

Although experienced teachers may predict common student misconceptions, this knowledge is fruitless unless it can inform instruction. While many believe that any exposure to errors will overwhelm or confuse students and thus avoid addressing common errors altogether, others recommend having students analyze others' work, including incorrect work, in hopes of challenging student thinking and revealing common misconceptions (Kramarski & Zoldan, 2008). Borasi (1994) suggests using errors as springboards for inquiry and found examination of errors to improve learning and attitudes about math. This article tackles this instructional dilemma by investigating learning associated with analyzing correct and incorrect student work samples and by investigating students' perceptions of how this analysis impacted their learning: (1) Are there differences in learning outcomes (exam scores, problem-solving, avoiding errors) associated with students' analysis of correct or incorrect examples? (2) Are there differences in how students perceive the benefits or hinderances of analyzing correct and incorrect examples? (3) How do students describe analyzing work samples and its influence on learning?

Literature Review

Errors can encourage communication of ideas, justification of solutions, questioning, reflection, critical thinking, inquiry, and flexibility in reasoning (Kramarski, 2004). However, it is unclear what the role of students' errors and misconceptions should be within ambitious teaching practices. Participants' experiences were viewed as intertwinings of thoughts, observations, and understandings, consistent with Marton's (1995) phenomenographic approach. This study uses the framework of IMPROVE (Mevarech & Kramarski, 1997), a metacognitive approach which encourages students to ask themselves questions and includes four factors: (a) comprehension, (b) connections, (c) strategies, and (d) reflection. We assumed that using this framework to analyze others' work would also encourage students to ask questions when they are solving problems themselves. This study was influenced by Knowles' (2011) self-directed learning, allowing learners to take initiative in learning, assuming learners' experiences and reflections are valid learning resources, and assuming learners are somewhat intrinsically motivated to learn due to curiosity and perceptions of relevance.

Materials and Methodology

Learning modules (LMs), asked students to analyze work samples by presenting them online. Students saw the same math problem, but some students saw a correct solution, and others saw an incorrect solution. Completion was compulsory and conducted outside of class meetings.

Participants. Our data was collected from 181 enrollees in 10 sections of Basic Calculus at a large university in the United States, and nine of those were interviewed the following semester. The sections met together for 50-minute lectures twice per week and separately for 75-minute recitations (led by graduate students) once per week. The gender distribution of participants was 68.51% female and 31.49% male. Each recitation section had two approximately equal-sized randomly assigned groups within it, one analyzing correct and one analyzing incorrect work samples. Interviewees were selected by their positive response to an electronic invitation.

Data Collection. From relevant problems on exams, scores and error rates were collected. Also, students' perceptions were measured by survey; each LM ended with a survey asking students to rate the LMs' helpfulness on a five-point scale. The survey's alpha coefficient was .96, suggesting high internal consistency. Semi-structured 30-minute interviews (n=9) were conducted. After a pilot study, the pre-test, the sampling techniques, the exams, the interventions, and the interview protocol were improved based on expert validator critique.

Data Analysis. (1) The pre-test and final exam scores were analyzed, using ANCOVA, to reveal if work-sample correctness was associated with students' overall achievement. (2) Responses to certain pre-test and exam questions, similar to those in the LMs, were scored by rubrics. Prior knowledge was accounted for by pre-test questions, and ANCOVA was conducted to analyze possible association with ability to solve similar problems. (3) To reveal differences in error rates, all pre-test and exam problems which provided opportunities to make those errors were analyzed through ANCOVA. (4) Each LM ended with a survey asking students to rate the LMs' helpfulness in learning. Independent samples *t*-test tests were conducted to compare students' ratings of correct work samples to incorrect work samples. (5) Semi-structured interviews (n=9) were transcribed and analyzed using inductive analysis. Domains emerged by finding semantic relationships in the data. Then, weak domains were abandoned, some were combined, and some emerged as salient. Further analysis was conducted within and between them to identify themes.

Interventions. The incorrect-work LMs featured common errors. Questions were formed that would elicit students' reflection, metacognition, and advanced mathematical thinking and in a manner that would offer approximately the same level of challenge to students in both groups.

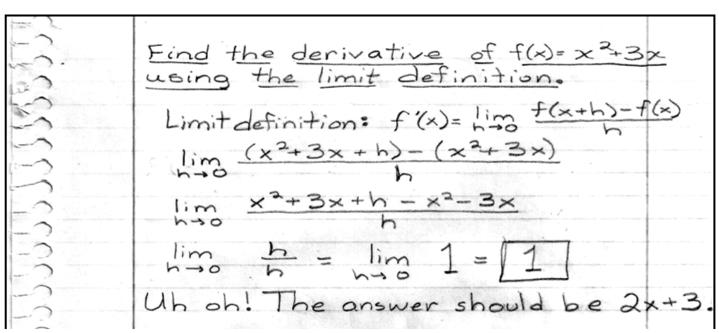

Figure 1. An example of an incorrect example showing a common error

Results

Final Exam Scores and Problem Solving

After controlling for pre-test scores, the ANCOVA revealed no significant difference in the two groups' final exam scores; $F(1, 169) = 1.24$, $p = .267$. When adjusted for pre-test scores, the mean final exam score for analyzers of incorrect work was 82.79, while the mean for correct work was 80.78. When analyzing only problems similar to those in the LMs, the difference between exam-question scores for analysts of correct and incorrect work was not statistically significant after controlling for pre-test scores; $F(1, 169) = 0.82$, $p = .367$. Exam-problem scores for analysts of incorrect work had an adjusted mean of 79.92, while the scores for analysts of correct work had an adjusted mean of 78.35.

Making Errors

Error rates differed significantly, after controlling for pre-test error rates; $F(1, 1048) = 4.80$, $p = .029$. Participants who analyzed incorrect work made these errors 3.7% of the time, while the participants who analyzed correct work made them 5.5% of the time. Table 1 shows ANCOVA results and adjusted error rates (e(c)= error rate for students who analyzed correct work, e(i) = error rate for students who analyzed incorrect work) for the two groups.

Table 1: Making Errors, Results of ANCOVAs

Incorrect Cancellation	$F(1, 159) = 1.13$, $p = .289$	e(c)=2.5% e(i)=3.3%
f(x)+h instead of f(x+h)	$F(1, 150) = 1.98$, $p = .162$	e(c)=5.3% e(i)=1.8%
Omission of negative solution	$F(1, 163) = 1.17$, $p = .280$	e(c)=8.2% e(i)=6.2%
Confusion of exponent rules	$F(1, 126) = 0.14$, $p = .705$	e(c)=8.5% e(i)=7.0%
Not using quotient rule	$F(1, 154) = 2.12$, $p = .147$	e(c)=5.0% e(i)=2.2%
Confusion of f(x) & derivatives	$F(1, 143) = 2.58$, $p = .110$	e(c)=7.9% e(i)=4.3%

Students' Perceptions about Effectiveness of Interventions

Those who examined correct work consistently rated the LMs as having greater association than those who examined incorrect work. Independent samples *t*-test indicated that overall, ratings by those who examined correct work were significantly higher than those who examined incorrect work. Also, tests were conducted to compare the perceptions of helpfulness to each aspect of learning, and significant differences were discovered within all aspects except for attitudes.

Interview Analysis Results

After reading the transcripts and considering that the overarching goal of this study, the frames of analysis were selected to be (a) understanding underlying concepts, (b) using rules and procedures, (c) using strategies, (d) communicating about math, and (e) attitudes. Domains were identified using semantic relationships. To conduct an analysis within these domains, Marton's perspective of *capability* and *constraint* (1995, p. 171) was considered, and domains were viewed as either depictions of capability, which positively associate with learning, or depictions of constraints, which inhibit learning. The major findings from this study are delineated below and mirror the 5 strands of mathematical proficiency (NRC, 2001):

- Both correct and incorrect LMs made students think about concepts underlying procedures (**Conceptual understanding**).
- Explaining math was also a task that promoted use of procedures (**Procedural Fluency)**.
- The correct LMs helped students form strategies to begin problems. Explaining connections between steps emerged as an aid to the use of specific strategies (**Strategic Competency)**.

- When asked how the LMs influenced the students' abilities to talk about math to others, several mentioned the LMs "made them think" about the mathematics. (**Adaptive Reasoning).**
- Related to confidence, incorrect work samples seemed to ease anxiety by illustrating others make mistakes and mistakes are fixable (**Productive Disposition)**.

In addition, we observed two overarching themes from analyses of semantic relationships between domains—independent learning and student engagement. Students felt as if they had interacted (or were more engaged), as opposed to hearing a lecture, which led to more independent learning as students were not just "fed" the material. The open-ended nature of the analyses, both of correct and incorrect work, influenced some aspects of learning, such as improving attitudes and flexible use of strategies.

Discussion

This study reveals that if students analyze incorrect work, they may be less likely to make the errors. However, ironically, students perceive correct work samples as more helpful to learning. With these juxtaposed findings, we see that students are not always aware of what is most effective in their own learning. They may experience challenge and interpret it as less helpful than a low-risk lecture. Teachers should not avoid using both correct and incorrect work samples when teaching and developing tasks because, as evidenced in this study, (a) both are effective in promoting problem-solving skills and achievement, (b) seeing common errors in work samples of moderate difficulty-level discourages students from making those errors, and (c) using both may prevent predictability that can constrain attitudes and student thinking. Teacher educators should make preservice and in-service teachers aware of these results in order to counter the fears that students will replicate errors. Future research could investigate whether results will differ with error type, or could include a group analyzing both correct and incorrect work.

References

Borasi, R. (1994). Capitalizing on errors as "springboards for inquiry": A teaching experiment. *Journal for Research in Mathematics Education, 25*(2), 166-208.

Knowles, M. S., Swanson, R. A. & Holton, E. F., III. (2011). *The adult learner: The definitive classic in adult education and human resource development* (7th ed.). London: Elsevier.

Kramarski, B. (2004). Making sense of graphs: Does metacognitive instruction make a difference on students' mathematical conceptions and alternative conceptions? *Learning and Instruction, 14*(6), 593-619.

Kramarski, B., & Zoldan, S. (2008). Using errors as springboards for enhancing mathematical reasoning with three metacognitive approaches. *The Journal of Educational Research,102*(2), 137-151.

Marton, F. (1995). Cognosco ergo sum. Reflections on reflections. *Nordisk Pedagogik, 15*(3), 165-180.

Mevarech, Z. R., & Kramarski, B. (1997). IMPROVE: A multidimensional method for teaching mathematics in heterogeneous classrooms. *American Educational Research Journal, 34*(2)*,* 365-394.

National Research Council. (2001). *Adding it up: Helping children learn mathematics.* J. Kilpatrick, J. Swafford, & B. Findell (Eds.). Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.

Son, J. W., & Moseley, L. M. (2012, February 9) *How pre-service teachers respond to student-invented strategies.* PowerPoint presented at the 16th Annual Conference of Association of Mathematics Teacher Educators, Fort Worth, TX.

DEVELOPMENT OF MATHEMATICS COMPETENCE, IDENTITY, AND SENSE OF BELONGING TO A COMMUNITY OF MATHEMATICS LEARNERS

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In this paper, we consider the ways in which out-of-school educational programs extend support to young high-achieving African-American students with an interest in STEM. We present a qualitative case study of three female African-American students enrolled in a mathematics summer camp intended for high school students interested in higher-level mathematics. We report on how these students found the program impacted their own growth and development in several specific domains.

Keywords: Equity and Diversity, Gender, High School Education, Informal Education

There is growing evidence that out-of-school educational programs contribute to young people's interest, understanding, and participation in STEM (National Research Council, 2015). In this paper we consider the ways in which out-of-school educational programs extend support to young high-achieving African-American high school students with an interest in STEM. In particular, we focus on how these students describe the support provided to them from their own perspective. The research questions guiding this study are: How do young African-Americans attending an out-of-school mathematics program describe the support provided to them with respect to the development of their (a) mathematical competence, (b) mathematical identity, and (c) sense of belonging to a community of mathematics learners?

Literature Review

Gasman and colleagues (2009) present a comprehensive review of the literature on students of color in STEM. They note that STEM education has traditionally been characterized by a survival-of-the-fittest approach in which "lack of attainment in STEM fields for underrepresented minorities is attributable to students' background characteristics, including socioeconomic status, parental education, family support, preparation in high school, and even genetics." (p. 66) Strayhorn (2011) measured the effect of summer bridge programs (SBP) on preparation for college in four areas: academic self-efficacy, sense of belonging, and academic and social skills. He found that students participating in SBP with high self-efficacy earned greater first semester GPA's than those with low self-efficacy, and that academic achievement prior to college was the most powerful predictor of success in the first semester of college.

Namakshi (2016) conducted a qualitative case study in which the impacts of an informal mathematics program on four former female students were examined. Namakshi found that the math camp had a crucial impact on the participants' future higher-education/career trajectory by increasing their social capital and positively influencing their mathematical identity.

Methods

Seven African-American students were enrolled in the math camp in this study. Each of the seven students had been provided a scholarship to enroll in the program through a private foundation that provides assistance to young African-Americans. Three of the seven students were chosen for this study based on their transcripts, letters of recommendation, and application essays submitted during the application process to the math camp. The students selected were chosen to represent a wide range of previous academic achievement.

Context of the study

The math camp in this study is a 6-week multi-level residential program for 60-68 talented high school students located in the southwest part of the United States. The participants were from 18 different states. The first-year participant class was 25% African-American. Students are immersed in studying mathematics at a high level while learning the process of doing research. Students learn to explore problems deeply, compute examples, make conjectures and give careful, rigorous mathematical proofs. In their first year, students study number theory with an associated Mathematica programming lab. In subsequent summers, the students take more advanced courses including Abstract Algebra, Combinatorics and Analysis, and work in groups on a research project guided by a faculty mentor.

All of the students work in nightly study groups of 4 students, with each study group having a camp counselor. The counselors are former campers who oversee daily problem sessions and provide feedback on the problem sets for student work. They are role models who give the campers help and guidance throughout the summer.

Participants

The first participant, Abigail, was a rising sophomore in high school and exhibited a strong love for mathematics in her application essay. She had been attending math enrichment programs since the 3rd grade and enrolled in a college-level calculus course in the 7th grade. Abigail had independently studied subjects such as linear algebra. Abigail's primary interests in coming to the math camp were to develop her knowledge of mathematics so that she could better understand how to apply mathematics to her interests in physics and computer science.

Brianna was a rising junior in high school. In her application essay, she stated that she wanted to "be the best that I can be in all things math and science." At her school, Brianna won a competition in which she successfully remembered the most digits of the number Pi. Brianna has an interest in going to medical school to study oncology or genetic exploration. To accomplish this goal, Brianna intends to apply to several elite universities.

Candice was a rising junior in high school. As a child, Candice was strongly encouraged to join STEM focused clubs in her elementary school. She had received a perfect score on the math portion of the PSAT in her sophomore year of high school. Candice was described by her teacher as a critical thinker and problem solver with a willingness to help others. Candice's interests in coming to the camp came from her desire to conduct meaningful research in a STEM field that could make a significant impact in the lives of others.

Data Collection

In addition to the transcripts, letters of recommendation, and application essays, this study also drew upon several other sources of data. While at the math camp, the students completed weekly electronic journals that served as reflections of their experiences from that week. Observations of the nightly study groups were conducted, and field notes were taken. After each observation, a one-page reflection report was written that focused on the interactions and general flow of events that occurred. During the fifth week, semi-structured interviews were conducted with the three participants and their counselors. Electronic post evaluations were collected from the students. These reflections provided the students a chance to critically reflect on their overall experience in the camp and opened a door to discuss any problems or concerns directly with the Director.

Data Analysis

We defined mathematical competence following Boaler and Greeno (2000), who defined knowing and understanding as "… [the ability to] engage in sense-making and problem-solving using and developing mathematical representations, concepts, and methods as resources… [through participation] in the practices of mathematical discourse and thinking" (p. 172). To conceptualize mathematical identity, we defined identity as a set of stories about a person (Sfard & Prusak, 2005). In this way, the mathematical identity of the participants could be revealed through the stories they told about themselves within their application essays, journals, and interviews.

The application essays, journal entries, and interviews were read over to gain familiarity with the data. The codes were then compared to ensure reliability. The coding consisted of identifying evidence of change in mathematical competence, identity, and sense of belonging as a result of attending the camp. This analysis was followed by a cross-case synthesis (Yin, 2017). In the following section, we present a synthesis of the results

Preliminary Findings and Results

Mathematical Competence

Each of the three students entered math camp with a strong record of achievement in mathematics as attested by their transcripts, applications, and teacher recommendations. They all reported improvement in problem solving techniques and proof writing skills. Abigail, in addition, felt that during camp she improved in her ability to attack problems and explain the solution to others once they were solved. Candice also found how, "effective communication in mathematics was an important skill to have and led to a better understanding of math." Brianna referred to changes in her views of learning, and how "true understanding stems from being able to take apart these core concepts …. learning is about always asking why, questioning everything …asking questions is never a sign of weakness or ignorance." Candice attributes camp to her change in attitude towards math. Rather than take formulas as true statements, she now wants to understand why the formula works. She also felt that she was overcoming her fear or failure because she had learned how important failure is in learning to write proofs. While Brianna and Candice developed a sense of confidence and comfort through challenges, they acknowledged there are "still some topics that are confusing for me." By the end of camp, Abigail reported how she would like to test her mathematical competence by entering math competitions and, "hopefully qualify for math prizes." The students remain focused on the goals that they had set prior to attending camp. They reported that they felt an increase in their mathematical ability and competence in Number Theory, writing proofs, and in doing mathematics in general.

Mathematical Identity

The mathematical identity of the participants could be seen to evolve through their perceptions of what mathematics means to them. When prompted to describe her views and her personal growth and development over the summer, Brianna responded that "Being here at [the camp] has given me an all new meaning to math. It has taught me that math doesn't have to be a part of just another agonizing cyclic process in school that has been defined as 'learning.'" In addition, she explains that she felt no limitations when it comes to learning. Brianna and Abigail talk about their persistence despite struggles, stating that "my optimism keeps me going along with me keeping my eye on the prize and knowing that the struggle won't last very long and the destination is in sight." Both of Abigail and Candice also provided commentary on how they had developed a greater appreciation for the utility of mathematics. Abigail stated in one journal that "[the camp] has shown me that math can be important all across the STEM field." Candice wrote about her fear of failure and how she learned to "embrace failure and learn from my mistakes." Thus, the mathematical identity of the participants can be seen to have been changed by positively impacting their feelings, attitudes, and beliefs.

Sense of Belonging

The participants described their sense of belonging to a community of mathematics learners through their interactions with other students. Each of the participants placed value on the opportunity to work on challenging mathematics together with likeminded individuals. Both of Abigail and Brianna described themselves as being motivators for the study groups to interact with each other. Abigail explicitly identified herself as the leader of her study group during her interview. At times, the participants expressed doubt about their ability to keep up with other students. Candice mentioned in her final evaluation that, "After talking with the other students at [the camp] on the first day, I felt underprepared when I entered the first Number Theory class…" However, Candice explained in the same entry that she felt she was at the same level as her peers in the beginning because the number theory class started from the basics.

Conclusion

In this paper, we have reported on several ways in which the participants of this study described the support provided to them during their time at a summer math camp. The participants recounted their experiences in the camp as it related to the positive development of their mathematical competence, mathematical identity, and sense of belonging to a community of mathematics learners. The impact of these out-of-school experiences can be critical to reaching out to and preparing African-American students (and all students) for careers in STEM. Further work is needed to better understand the supports provided, and how these supports each played a role in supporting the participants growth and development.

References

Boaler, J., & Greeno, J. G. (2000). Identity, agency, and knowing in mathematics worlds. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 171-200). Westport, CT: Ablex Publishing.

Gasman, M., Perna, L. W., Yoon, S., Drezner, N. D., Lundy-Wagner, V., Bose, E., & Gary, S. (2009). The path to graduate school in science and engineering for underrepresented students of color. In M. F. Howard-Hamilton, C. L. Morelon-Wuainoo, S. D. Johnson, R. Winkle-Wagner, & L. Santiague (Eds.), *Standing on the Outside Looking In:Underrepresnted Students' Experiences in Advanced Degree Programs* (pp. 63-81). Sterling, VA: Stylus.

Namakshi, N. (2016). *Creating a pathway to STEM: Role of an informal mathematics program.* (PhD Dissertation), Texas State University,

National Research Council. (2015). *Identifying and supporting productive STEM programs in out-of-school settings*. Washington, DC: National Academies Press.

Sfard, A., & Prusak, A. (2005). Telling identities: In search of an analytic tool for investigating learning as a culturally shaped activity. *Educational Researcher, 34*(4), 14-22.

Strayhorn, T. L. (2011). Bridging the pipeline: Increasing underrepresented students' preparation for college through a summer bridge program. *American Behavioral Scientist, 55*(2), 142-159.

Yin, R. K. (2017). *Case study research and applications: Design and methods* (6th ed.). Thousand Oaks, CA: Sage publications.

INQUIRY AND INEQUITY IN THE UNDERGRADUATE MATHEMATICS CLASSROOM

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To investigate performance differences between undergraduate mathematics students whose instructors implemented an inquiry-oriented curriculum and those whose instructors did not, we analyzed 513 content assessments. The data was disaggregated by gender to examine how inquiry-oriented instruction relates to the pattern of underrepresentation of women in mathematics. While we detected no gender performance-gap in a national comparison sample, we did find one in the inquiry-oriented group. Notably, this gap came from male inquiry-oriented students outperforming male students from the comparison group, while female inquiry-oriented students performed the same as female students from the comparison group.

Keywords: Post-Secondary Education, Gender, Equity and Diversity, Assessment and Evaluation

Active learning in undergraduate STEM classes has been linked to improved student success and learning (see Freeman et al.'s (2014) meta-analysis of 225 studies). Additionally, in undergraduate mathematics specifically, Laursen, Hassi, Kogan, and Weston's (2014) work suggests that Inquiry-Based Learning (IBL) courses – which promote collaborative learning environments – 'level the playing field' for male and female students. While these findings are powerful, the mechanisms connecting active learning approaches to more equitable student outcomes are not well understood. As Singer and colleagues argued in the National Research Council's (2012) report, the identification of critical features of instructional approaches is crucial for exploring the ways those approaches may impact particular groups of students. Thus, in this study we present a narrow investigation into the relationships between one carefully defined instructional approach, namely inquiry-oriented instruction (IOI) (Kuster, Johnson, Keene, & Andrews-Larson, 2018), and student performance on a content assessment.

Our work draws on data collected in an NSF-funded project, Teaching Inquiry-oriented Mathematics: Establishing Supports (TIMES), that aims to explore what is needed to support instructors learning to teach in inquiry-oriented ways. Of specific interest here is one of three sets of materials utilized in this project, namely the Inquiry-Oriented Abstract Algebra (IOAA) curriculum (Larsen, Johnson, & Weber, 2013). We analyzed 513 completed content assessments to investigate performance differences between students whose instructors implemented the IOAA curriculum (with professional development support from the TIMES project), and those whose instructors did not. Specifically we address the following two research questions: 1) What is the relationship between inquiry-oriented instruction, as manifested by the TIMES program, and student performance on a content assessment?, 2) Is the pattern consistent across genders? Given Freeman et. al's (2014) findings, we expect to see the TIMES students outperforming Non-TIMES students. Additionally, given the similarities between inquiry-oriented instruction and IBL, we expect to see Laursen et al.'s (2014) findings replicated, with men and women performing more similarly in the IOI setting than in non-IOI settings.

Literature Review and Theoretical Perspective

In IOI, the intent is for the students "to regard the knowledge they acquire as their own private knowledge, knowledge for which they themselves are responsible" (Gravemeijer, 1999, p. 158). Thus, through inquiry and collaborative group work, the objective is to reposition students as central to the process of constructing and reinventing important mathematical ideas. Some literature suggests that classrooms emphasizing collaborative work, problem solving, and communication may be supportive for women (Du & Kolmos, 2009; Springer, Stanne, & Donovan, 1999). However, research on the nature of social interactions in collaborative decision-making and facilitated discussions offers additional insights into the negative ways in which students may experience the mathematics classroom as a gendered and racialized space. For example, when groups are tasked with arriving at a decision, women in predominantly male groups speak less and are interrupted more than men (Karpowitz, Mendelberg, and Shaker, 2012). Further, within the same classroom, students often receive different opportunities to participate in whole-class math discussions in ways that follow patterns of gender, race, and class (Black, 2004; Walshaw & Anthony, 2008).

Most closely related to our research context, i.e., mathematics education at the undergraduate level, Laursen et al. (2014) found that "in non-IBL [inquiry-based learning] courses, women reported gaining less mastery than did men, but these differences vanished in IBL courses" (p. 415), suggesting that instructional approaches involving collaborative problem solving may 'level the playing field' for male and female students. Their IBL context is commensurate with, but distinct from, our inquiry-oriented context. Both IOI and IBL promote collaborative group work, prioritize student engagement with authentic mathematical activity, and decentralize classroom authority. However, IOI places special emphasis on reinvention and inquiry – by both the students and the teacher (see Kuster et al. (2017) for a description of IOI).

Thus, due to the undergraduate mathematics setting and similarities between IBL and IOI, we had reason to believe that we would see Laursen et al.'s (2014) findings replicated in our study. However, our study had some key differences. Specifically, our analysis examines student content assessments (not affective reports of learning gains) and takes place within the context of a long-term, multi-faceted professional development model that emphasized a narrowly defined instructional approach.

Methods

There were 12 IOAA TIMES Fellows. These fellows received three forms of professional development support: curricular support materials, summer workshops, and online working groups that met every week as the Fellows were implementing the IOAA curricular materials.

The TIMES Fellows asked their students to complete the Group Theory Content Assessment (GTCA) (Melhuish, 2015). This assessment, developed to measure conceptual understanding of key concepts in group theory, included: binary operations and their properties, group structures, element properties, and functions. In order to produce a robust validity argument, the GTCA began with open-ended tasks, which were converted to a multiple-choice format based on student responses. As a result of the development process, we have a large, nationally representative 'control' group from which we can draw comparisons with our IOI students.

From the 12 TIMES Fellows, there were a total of 174 students, 80% of whom completed the GTCA. For our "Not-TIMES" group, we analyzed data from 374 students from 33 institutions. For these students, we make no claim about the form of their instructional experiences. Thus, they are a control for the TIMES project, but not necessarily for IOI. However, nationally the proportion of teachers using "non-lecture" instructional approaches in abstract algebra is less than 10% (Fukawa-Connelly, Johnson, & Keller, 2016). Between the treatment (TIMES, *n* = 139) and control (Not-TIMES, *n* = 374) groups, we have a total of 513 student content assessments. In total there were 269 males, 237 females, and 9 who otherwise identify or declined to identify their gender. The percentage that identified as female was not significantly different ($p = .413$) between Not-TIMES and TIMES (47.59% and 42.45%, respectively).

The primary analysis in this research was an investigation of performance on the GTCA and group comparisons therein. Summary statistics were computed for the overall sum score based on the 15-item subset that was consistent across data collection iterations. Independent sample *t*-tests were used to compare mean performance by gender within instructional treatments and overall; effect sizes were computed where indicated.

Results and Discussion

When tabulating the collective assessment results of all students on the GTCA, we report the average performance to be a score of 6.33 items correct out of 15, with a standard deviation of 2.95. We found that the TIMES students slightly outperform the Not-TIMES students by about half a unit (6.65 vs. 6.21), but this difference is not statistically significant ($t = -1.516$, df = 511, $p = .130$). When considering performance differences by gender, we see that males generally outperform females (when aggregating TIMES and Not-TIMES students) by about one correct answer (6.75 vs. 5.79). This difference, while statistically significant ($t = 3.703$, $p < .001$), is of little practical importance as judging by the small effect size ($d = .33$). Thus, we view this as a fairly minimal performance gap. However, when disaggregating the data by instructor classification, this trend is inconsistent.

Gender comparisons within each group revealed no significant difference in the Not-TIMES group (males outperform females by about half-point on average, $p = .095$). In the TIMES group however, males outperformed females by over 2 points on average. This difference was appreciable both in terms of statistical significance ($p < .001$) and in practical importance as judging by the large effect size ($d = .7257$).

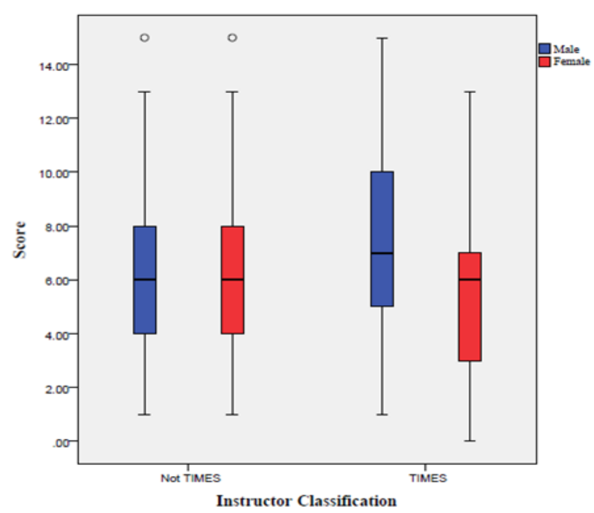

Figure 1. Gender Comparisons of Students on GTCA

In summary, males outperform females in both groups (though not significantly in the Not-TIMES group), but when considering the differences within gender across categories of instruction, we see that the performance gap widens in the TIMES group because the average male score registers a statistically significant improvement of 1.1 units (t = 2.69, p = 0.008) while the average female score declines (albeit insignificantly) by .58 units (t = -1.37, p = .174).

We see the detection of a gender performance-gap within the IOI setting as an unfortunate finding. We are not insinuating that the TIMES project, or the implementation of IOI in general, is detrimental to female student achievement; in fact, both male and female TIMES students performed as well or better than the national comparison sample. We take the improved scores for male students as evidence that IOI has the potential to improve student learning. However, the question remains as to why female students did not share the same degree of improvement. It is our intention to use our findings to inform a critical examination of the effect of our interventions on the gendered experiences of our students and call on others to do the same.

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References

Black, L. (2004). Differential participation in whole-class discussions and the construction of marginalised identities. *Journal of Educational Enquiry*, *5*(1), 34–54.

Du, X., & Kolmos, A. (2009). Increasing the diversity of engineering education–a gender analysis in a PBL context. *European Journal of Engineering Education*, *34*(5), 425-437.

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences*, *111*(23), 8410-8415.

Fukawa-Connelly, T., Johnson, E., & Keller, R. (2016). Can math education research improve the teaching of abstract algebra? *Notices of the AMS 63(3).*

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, *1*(2), 155-177.

Karpowitz, C. F., Mendelberg, T., & Shaker, L. (2012). Gender inequality in deliberative participation. *American Political Science Review*, *106*(3), 533–547.

Kuster, G., Johnson, E., Keene, K., & Andrews-Larson, C. (2018). Inquiry-oriented instruction: A conceptualization of the instructional principles. *PRIMUS*, *28*(1), 13-30.

Larsen, S., Johnson, E., Weber, K. (Eds). (2013*).* The Teaching Abstract Algebra for Understanding Project: Designing and Scaling up a Curriculum Innovation. *Journal of Mathematical Behavior, 32(4)*

Laursen, S. L., Hassi, M. L., Kogan, M., & Weston, T. J. (2014). Benefits for women and men of inquiry-based learning in college mathematics: A multi-institution study. *Journal for Research in Mathematics Education*, *45*(4), 406–418.

Melhuish, K. M. (2015). *The Design and Validation of a Group Theory Concept Inventory.* Unpublished doctoral dissertation. Portland State University, Portland, OR.

National Research Council. (2012). *Discipline-based education research: Understanding and improving learning in undergraduate science and engineering.* National Academies Press. Washington, DC.

Springer, L., Stanne, M. E., & Donovan, S. S. (1999). Effects of small-group learning on undergraduates in science, mathematics, engineering, and technology: A meta-analysis. *Review of Educational Research*, *69*(1), 21-51.

Walshaw, M., & Anthony, G. (2008). The teacher's role in classroom discourse: A review of recent research into mathematics classrooms. *Review of Educational Research*, *78*(3), 516– 551

LATINX BILINGUALS' PERSEVERANCE ON A MATHEMATICAL TASK

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This paper examines the problem-solving practices of Latinx bilingual students with a challenging mathematics task. Studying the collaborative efforts of two groups of Latinx bilinguals, our findings suggest these students can spontaneously and dialogically leverage communicative resources to help persevere with in-the-moment obstacles, but only when provided freedom to explore mathematics via their dynamic bilingualism.

Latinx students are one of the fastest growing populations in the U.S. (NCES, 2016), yet dehumanizing school practices continue to marginalize this group's linguistic, social, and cultural capital (Langer-Osuna et al., 2016). All students must have access to study rigorous mathematics, and thus be supported in their perseverance with such challenges (NCTM, 2014). This notion of perseverance exists at specific times during problem-solving when productive struggle, or grappling with mathematical ideas not yet well formed, is necessary to develop conceptual knowledge (Hiebert & Grouws, 2007). Despite efforts to support perseverance by emphasizing collaborative communication, Latinx bilinguals' mathematical engagement is often viewed from deficit perspective by ignoring their linguistic and cultural repertoire and instead privileging the dominant school language (Garcia, 2017; Morales Jr., 2004). This naturalistic study aims to describe the translanguaging and perseverance practices of Latinx bilingual students during problem-solving with challenging mathematics.

Translanguaging and Persevering to Learn Mathematics

We draw on translanguaging to reconceptualize bilingualism as a liberating and empowering communicative practice, and a resource capable of transforming learning that goes beyond the transition to a dominant school language. Translanguaging is an interrelated communicative practice that make up bilinguals' linguistic repertoire. Garcia (2017) posits, "…speakers use their languaging, bodies, multimodal resources, tools and artifacts in dynamically entangled, interconnected and coordinated ways to make meaning" (p. 258). A translanguaging lens allows us to better understand holistically how bilingual students use their linguistic, multimodal, and mathematical repertoire to make meaning as they persevere with challenging ideas. Perseverance is a cycle of initiating and sustaining in-the-moment productive struggle in the face of one or more obstacles, setbacks, or discouragements (DiNapoli, 2018). Because productive struggle is helpful in developing mathematical meaning, analysis of perseverance should consider the ways in which students first wrestle with an uncertain mathematical situation, and, if necessary, how they amend their first effort to continue to make mathematical progress. Yet, for Latinx bilingual students, questions remain about how perseverance might manifest working across two languages and how teachers might take a translanguaging stance (see Garcia, 2017) to provide opportunities to leverage communicative and linguistic repertoires to build understandings.

Methods

We observed two groups of 12th-grade Latinx bilingual students from different classes with the same monolingual, English-speaking teacher (Ms. Patrick) working on a multi-day task necessitating perseverance. The task referenced *Alice in Wonderland,* in which Alice's height is doubled or halved by consuming certain potions. All but one student (Ines, see below) were born

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.

in the U.S., yet all students still use home language (Spanish) as part of their linguistic repertoire. We employed a perseverance analytical framework (see DiNapoli, 2018) that identified how productive struggle was initiated and sustained from the outset with a challenging task (Initial Attempt) and how it was re-initiated and re-sustained in a new way (Additional Attempt(s)) if a first effort did not lead to a solution. We consider how students engaged in translanguaging practice and persevered at specific moments during their problem-solving.

Findings

Here we share detailed findings from Group A, who exemplified translanguaging and perseverance practices while conceptually exploring exponential growth. We then briefly share, due to space limitations, apparent themes from Group B, who demonstrated little to no evidence of such practices. Our observations show how student spontaneous translanguaging can help support perseverance with challenging mathematics, but only when students had an opportunity to productively struggle without teacher intervention. Consider the following vignettes from Group A's three-day collaboration around the task. Group A had demonstrated evidence of understanding the goal of the problem on Day 1, but did not know how to solve it. We share findings from Day 2, as Group A makes their Initial Attempt at solving.

JESSICA: ¿Qué era la primera, se hace así? Okay, dice: Alice changes when she eats the cake, assume that her height doubles for each ounce she eats, so that means if she eats one ounce, that means that she grows twice, dos ¿qué? Double, no double, two. (*What was the first one, do you do it like this? Okay, it says: Alice changes when she eats the cake, assume that her height doubles for each ounce she eats, so that means if she eats one ounce, that means that she grows twice, two what? Double, no double two*.)
JESSICA: See, so when two is four, and then three is six, and four is eight, y así, y así vamos hacer la graph. Going like that para arriba. (Gesturing) You get it? (*See, so when two is four, and then three is six, and four is eight, like this, and this is how we are going to make the graph. Going like that, up.* (Gesturing) *You get it?*)
ELENA: Um hmm. Pero, how do we times it? (*But how do we times it?*)
JESSICA: Porque mira, two, times two. Well no. (*Because look, two, times two. Well no*.)
ELENA: But that's what you were telling me yesterday y yo pensé que no. Okay so we. (*But that's what you were telling me yesterday and I didn't think so. Okay so we*.)
JESSICA: Double it by, nomás double the number of ounces…[Elena interrupts] (*Double it by, just double the number of ounces…*)
ELENA: Two times two, y luego four times two, y luego six times two, is that your idea? (*Two times two, and then four times two, and then six times two, is that your idea?*)
JESSICA: Más o menos como sumando el mismo número. (*Like adding the same number*.)
CARINA: Pero es lo mismo de sumando si lo multiplicas por dos. (*But it is the same as adding if you multiply by two.*)
INES: [Referencing table from year one] Lo que parece es como hicimos un in/out table y ya lo sacamos. (*It looks like we just did an in/out table and that's it*.)
CARINA: Yeah. In times two equal out.

The students made their Initial Attempt at solving by coordinating their meaning making actions and deploying their linguistic repertoires. After reading, the students began to interpret the task and the nature of doubling. In dialogically entangled ways, they expressed linguistically across Spanish and English what doubling meant to them (e.g. double, two, dos, grows twice, sumando el mismo numero, multiplicas por dos, times it). They also explored and questioned how to represent their ideas symbolically, which demonstrates their productive struggle regarding understanding the nature of doubling. Drawing on such linguistic and mathematical resources is further evidence of spending diligent effort during their Initial Attempt to make sense of the function. They did not, however, realize immediately that they were multiplying the number of ounces of cake by two, instead of *Alice's* height. Their equation correctly spanned the table of values, yet these representations did not model an exponential function.

Not completely agreeing with the other students' mathematical representation, Ines' own metacognitive awareness of the mistake during their Initial Attempt helped collectively move the discussion in a direction that considered *Alice's* initial height. The entire group collaborated around this new idea and persevered together as a learning community. At the end of Day 2, Ines recalled her prior experience with the *Alice* problem from her sophomore year and raised the issue of starting with an initial height. This helped the others rethink about the mathematics, cross out their table (Figure 1, left), and begin on Day 3 to think about starting a new one.

> *INES*: …empezamos de cuatro pies. Si toma si come un pedacito son ocho, si come un pedacito son dieciseis, el tercer pedazo dieciseis y dieciseis. Treinta y dos ¿no? (*…we start at four feet. If she drinks, if she eats one piece it becomes eight, if she eats one piece it becomes sixteen, the third piece, sixteen and sixteen, thirty two, no?*)
> *JESSICA*: Pero, ¿cómo sacastes eso? (*But how did you get that?*)
> *INES*: Porque si empezamos con cuatro pies, como yo les digo, si come un pedacito y sale, aumenta de altura de doble. (*Because, if we start at four feet, like I'm telling you, if she eats one piece and it comes out to, her height grows double*.)
> *JESSICA*: Ohh, her height doubles.
> *ELENA*: You know it's the same thing mira. Dos, you multiply one times two is two, two times four is eight, y si pones two times two is four, four times four is sixteen. (*You know it's the same thing look. Two, you multiply one times two is two, two times four is eight, and if you put two times two is four, four times four is sixteen*.)
> *CARINA*: In squared times 2 is equal to your out.

At the start of Day 3, Ines tried to help the group understand that *Alice's* initial height is necessary to compute subsequent heights. The task was written in English, yet Ines leveraged her home language to re-voice the problem. She modeled mathematically the concept of doubling for her group members in Spanish and also gestured to demonstrate how *Alice's* height doubles for each ounce of cake she eats. This helped Jessica have her "a-ha" moment realizing that she needed to double *Alice's* height not the number of ounces of cake.

This translanguaging exchange exemplified further perseverance by exploring what it means to double and illustrated how Group A became aware about the flaws in their first attempt to make sense of the function. What makes this so significant is the students' commitment to collaboratively draw on linguistic (including their home language) and mathematical resources fluidly as they continue to try to overcome the conceptual obstacle of doubling. The combination of these interactions affords Jessica her liberating moment of understanding, which leads these students to amend their plan and adapt their thinking to continue to persevere.

Following this discussion, the students created a new table of values (Figure 1, center) that correctly modeled *Alice's* exponential growth. This new approach demonstrated the end of their Initial Attempt and the start of their Additional Attempt by the group amending their plan and making a second attempt to make sense of the *Alice* function. Unfortunately, the equation was not correct and did not span all of their entries. Again, this is an important opportunity for the group to recognize more mistakes and continue to persevere. Near the end of Day 3, the students used a graphing calculator to make a table of values for $y = 2x^2$ (Figure 1, right) and discovered that it did not match their current table. Consequently, they were ready to continue to explore ways to change their equation and make another Additional Attempt, but Ms. Patrick interjected to go over the answers. Although Group A did not yet find the proper equation to span their IN/OUT table in this episode, they were agents of their own translanguaging repertoire and leveraged this practice to help persevere and think more deeply about exponential functions.

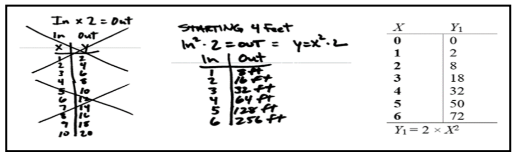

Figure 1. Progression of Student Tables (recreated by authors)

Group B did not have similar learning experiences as Group A, likely because Group B was not given the chance to persevere with the *Alice* task themselves. Instead, Ms. Patrick directed students to a correct procedure from the outset and students rehearsed these routines. As such, these students had no opportunity to use translanguaging practice to help overcome conceptual obstacles, and instead simply waited for Ms. Patrick to instruct them about what to do next.

Discussion and Conclusion

What emerges from the data are different trajectories of mathematics teaching and learning, revealing how the unintended consequences of instructional choices can detrimentally affect Latinx students' perseverance. Our observations show how the level of involvement and proceduralized guidance by the teacher can impact opportunities for student perseverance and conceptual development. When students had freedom to struggle, this facilitated an evolving mathematical understanding of exponential functions. Students leveraged their translanguaging repertoire to help persevere past mathematical obstacles and make progress toward a solution. Without freedom to struggle, other students only used their bilingualism to monitor routine steps, not navigate meaning. This suggests Latinx bilinguals can use bilingualism to their advantage when engaging with challenging mathematics, but only when given the opportunity to grapple with uncertain mathematical relationships. However, we cannot expect students to always spontaneously engage in translanguaging practice. Latinx bilinguals need explicit classroom support systems to encourage such engagement. Future research must clarify how teachers can create space to leverage their students' bilingualism for forming mathematical meanings.

References

DiNapoli, J. (2018). *Supporting secondary students' perseverance for solving challenging mathematics tasks.* Proceedings of the 40th annual meeting of PME-NA, Greenville, SC.

García, O. (2017). Translanguaging in schools: Subiendo y bajando, bajando y subiendo as afterword. *Journal of Language, Identity & Education, 16*(4), 256-263. doi:10.1080/15348458.2017.1329657

Hiebert, J. & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. *Second Handbook of Research on Mathematics Teaching and Learning*, *1*, 371-404.

Langer-Osuna, J. M., Moschkovich, J., Norén, E., Powell, A. B., & Vazquez, S. (2016). Student agency and counter-naratives in diverse multilingual mathematics classrooms: Challenging deficit perspectives. In R. Barwell et al. (Eds.), *Mathematics Education and Language Diversity* (pp. 163-173): Springer.

Morales Jr., H. (2004). *A Naturalistic Study of Mathematical Meaning-Making by High School Latino Students.* Unpublished Dissertation, University of Illinois at Chicago.

National Council for Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: Author.

NCES. (2016). *The condition of education.* National Center for Educational Statistics. http://nces.ed.gov/pubsearch

EFFECTS OF AN INTENSIVE REMEDIAL MATH COURSE ON ENGINEERING STUDENTS' MATH ANXIETY AND MATH SELF-EFFICACY

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This article explores the relationship between an intensive remedial math course for freshmen engineering majors and students' math anxiety (MA) and math self-efficacy (MSE) levels. MA and MSE have been shown to influence students' pursuit and completion of a math-related major, making these important factors to consider for these students in their first semester. Data were collected from 565 students using items from the existing Mathematics Anxiety Rating Scale and the Mathematics Self-Efficacy Survey. This study uses Analysis of Covariance (ANCOVA) to determine the effects of the remedial course on students' MA and MSE while controlling for confounding variables such as student's gender and type of high school. Results showed there was a significant increase in students' MA and MSE levels after the intensive course. Implications for math instructors designing this type of remedial course are discussed.

Keywords: Affect, Emotion, Beliefs, and Attitudes, Post-Secondary Education, Measurement

Introduction

There is a current need to match the projected demands for new engineering professionals in some countries like the U.S., and one way to reach this goal is keeping more engineering students in their undergraduate programs. Engineering majors currently have high attrition rates (Geisinger & Raman, 2013), and students' math preparation and attitudes toward math-related activities have been shown to be significant factors in students' decision to pursue and complete an engineering major (Hackett, 1985). This research focuses on two important factors influencing students' perceptions about math-related activities: math anxiety (MA) and math self-efficacy (MSE). Once enrolled, engineering students may perceive math courses as barriers, and struggling to complete these courses could make them more likely to quit or change majors (Suresh, 2006).

Richardson & Suinn (1972) defined **math anxiety** as "feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic settings" (p. 551). Math anxiety is not just a dislike of mathematics, but a set of feelings that affect performance in math-related activities that may lead to avoidance of math courses. Math anxiety is typically linked to perceptions of low math ability, prior unsuccessful experiences, and lack of studying and poor test preparation skills. High MA levels has shown to be a significant predictor of poor math performance and is also negatively correlated with the decision of pursuing a math-related major (Suinn & Winston, 2003). Women have shown to be more likely to report MA feelings when they perform math activities, and these negative feelings normally affect them in their decision to continue taking math courses (Sherman & Fennema, 1977).

According to Bandura (1986), **self-efficacy** refers to "people's judgments of their capabilities to organize and execute courses of action required to attain designated types of performances" (p. 391). Self-efficacy beliefs are task-specific, and one's skills in a particular task may influence only the individual performance on that task. Students with a high MSE have shown to be more interested in pursuing a math-related major like engineering (Hackett, 1985; Lent et al, 1991). On the other hand, low MSE levels have been shown to have an influence on students' decisions to avoid math-related activities that may lead them to have feelings of stress and anxiety. Math self-efficacy has been shown to be lower for students leaving STEM majors, and this factor was more significant for students leaving college during their first semesters (Eris et al., 2010).

If MA and MSE are not established and measured as independent factors, developing a conclusion about their effects on students' behaviors becomes a challenging task. However, it is important to recognize that MSE is related to MA, and that these two factors are normally influenced by similar past experiences (Hackett & Betz, 1989). Although students' intentions to engage in math-related activities are related to their MA levels, this effect has shown to lose significance when MSE is also examined (Meece et al., 1990). If engineering educators could develop interventions focused on improving students' MSE and ameliorate students' MA, then students may be better prepared to complete the math courses required by their engineering major. This led to the following research question: What is the effect of an intensive math intervention on the MA and MSE of first-year engineering students?

Methods

Participants were selected from a Mexican university that were engineering majors. To be accepted in this university, students must complete a four-week summer math course designed to standardize their math knowledge with a grade of 70% or better. Students meet 5 times a week to have a fast pace review of basic pre-college math, and they are evaluated once a week.

All freshmen students taking the math course for the Fall 2016 semester ($N=565$) completed a survey on the first and last days of the course before an exam (diagnostic and final). The survey presented 20 MA Likert-type items with scores from 1 "not anxious at all" to 5 "very anxious", and 36 MSE items where students rated their level of confidence performing math-related activities on a scale ranging from "no confidence at all" (0) to "complete confidence" (10). The MA items were selected from the 30-item Mathematics Anxiety Rating Scale (MARS 30-item) (Suinn & Winston, 2003), based on relevance to the context in the Mexican university. For the MSE items, this research used 36 of the 52 Mathematics Self-Efficacy Survey (MSES) items (Betz & Hackett, 1983). Only the math problem-solving (18) and the everyday math tasks (18) MSES items were considered relevant for this population. All survey items were translated from English to Spanish. Both the MARS 30-item and the MSES were selected for this study due to its high reliability shown in prior studies, with coefficient alpha values ranging from 0.84 to 0.96 (Brown & Burnham, 2012).

An average of the items was calculated as a total MA score (between 1 and 5) and a total MSE score (between 1 and 10), and independent scores for each of the constructs were also calculated. In order to determine if the math intervention had an effect on students' MA and MSE, a repeated measures Analysis of Covariance (ANCOVA) was conducted. This statistical model tests for a significant treatment effect among repeated observations while controlling for confounding variables. A mixed model was developed for each construct (total math anxiety, math tests anxiety, math activities anxiety, total math SE, math activities SE, and math problems SE). Fixed terms were included for treatment (pre/post), covariates (gender, type of high school, origin) and the interaction effects. A within-subjects random factor was also added to account for the repeated measures of the students. Finally, a correlation analysis was conducted to determine the relationship between MA and MSE before and after the math intervention.

Results

The results for the fixed effect F-tests of the treatment terms for MA and MSE are given in Tables 1 and 2 below. Covariate adjusted means for each construct for before and after the intervention are also given.

Table 1: Treatment Effect and Adjusted Means for Math Anxiety

Construct	F- Ratio	p- value	Pre- Mean	Post- Mean
Total Math Anxiety	14.2030	0.0002*	2.4153	2.7463
Math Test Anxiety	12.8993	0.0005*	2.8717	3.2434
Math Activities Anxiety	7.2271	0.0081*	1.9579	2.2493

Table 2: Treatment Effect and Adjusted Means for Math Self-Efficacy

Construct	F- Ratio	p- value	Pre- Mean	Post- Mean
Total Math Self-Efficacy	31.0563	<.0001*	7.1439	8.0029
Math Activities Self-Efficacy	1.2462	0.2663	8.4603	8.2966
Math Problems Self-Efficacy	65.7355	<.0001*	5.8262	7.7051

The ANCOVA results revealed a significant difference between pre and post mean scores for total MA and total MSE (see Tables 1 & 2). Each of the different constructs, math tests anxiety, math activities anxiety, and math problems self-efficacy, showed a significant difference with higher adjusted mean post scores. The only construct with no significant difference in the mean scores was the math activities self-efficacy. Also, there was a significant difference in math test anxiety scores between male ($M=2.87$) and female students ($M=3.25$) ($p=0.0052$). For all constructs, no interaction terms were significant, meaning that the treatment effect was consistent regardless of students' gender, type of high school, and origin. Next, the correlation analysis revealed a significant negative correlation between total MA ($M=2.416$, $SD=0.658$) and total MSE ($M=7.33$, $SD=1.24$) before the intervention ($r = -0.47$, $p < 0.0001$). There is also a significant negative correlation between total MA ($M=2.63$, $SD=0.74$) and total MSE ($M=8.16$, $SD=1.08$) after the intervention ($r =-0.54$, $p < 0.0001$). Even though this correlation is stronger for the post scores, it is not significantly different than the pre scores ($p= 0.1008$).

Discussion and Conclusion

The increase of students' MA and MSE after the course suggests that intensive remedial courses could help students improve their math knowledge, but there may be secondary effects on students' feelings about their math abilities. The increase of MA levels is a clear disadvantage for this type of course, making students more likely to decide to avoid math-related activities, thinking that they will feel stressed and perform poorly (Betz & Hackett, 1983). On the other hand, the MSE increment may have a positive impact. Higher MSE feelings could help to keep students' interest in developing their math abilities and desire to complete their engineering major despite facing some struggles during their first college math courses (Eris et al., 2010).

Developing a deeper understanding about the possible effects of this type of remedial math course could help math educators to better design these courses. The focus of these courses should be on enhancing students' math knowledge and building better MSE, but there must be an emphasis on creating an environment that could help students to develop their math abilities without generating anxiety feelings during the process. For this specific course, the stress generated by the need of obtaining a passing grade in the weekly tests may have contributed to creating a stressful environment for the students. Math and engineering educators should consider the stress and anxiety generated by frequent assessments, especially for students who may need more time to practice enough to feel confident to be evaluated. Female engineering students showed to be more affected by the math test anxiety than their male peers, but these MA levels were not higher in the total MA assessments. Despite literature suggesting that females are more likely to experience feelings of anxiety while performing math-related activities (Hackett & Betz, 1989), the research findings from this study that focuses on engineering students show no significant difference between males and females for total MA. These results suggest that this could be because female engineering students might be better prepared to deal with math courses in college than other female students in different majors.

This research found that MA and MSE have a negative correlation, suggesting that MSE beliefs are normally higher when MA levels are low. Despite finding a negative correlation, both factors showed a significant increment in the post course measurements. The MA increment stresses the importance of considering strategies that are directly focused to ameliorate negative feelings when students are learning math. Even when students start improving their math knowledge and self-efficacy by engaging with math-related activities, if the learning environment makes them feel stressed and anxious, then these students may be less likely to get interested in seeking more math activities aiming to further develop their math knowledge.

Math courses should keep the goal of improving students' math abilities, but there must be a greater awareness about the negative repercussions that students may have if they learn math in stressful environments. Groups like females have shown to be more likely to feel stressed during math tests, and this perception could result in higher MA feelings that may push students away from math courses and activities. If math courses increase students' feelings of anxiety, then these students could start doubting their abilities to finish their engineering major due to the challenges that completing all the required math courses could present. Math instructors should be prepared to support students and create a class environment that promotes their learning and MSE while developing the necessary tools to manage their anxiety while performing math.

References

Bandura, A. (1986). Social foundations of thought and action: A social cognitive theory. Englewood Cliffs, NJ: Prentice Hall.

Betz, N. E., & Hackett, G. (1983). The relationship of mathematics self-efficacy expectations to the selection of science-based college majors. Journal of Vocational Behavior.

Brown, S., & Burnham, J. (2012). Engineering Student's Mathematics Self-Efficacy Development in a Freshmen Engineering Mathematics Course. International Journal of Engineering Education, 28(1), 113–129.

Eris, O., Chachra, D., Chen, H. L., Sheppard, S., Ludlow, L., & Rosca, C. (2010). Outcomes of a Longitudinal Administration of the Persistence in Engineering Survey. Journal of Engineering Education, 99(4), 371.

Geisinger, B. N., & Raman, D. R. (2013). Why They Leave : Understanding Student Attrition from Engineering Majors. International Journal of Engineering Education, 29(4), 914–925.

Hackett, G. (1985). Role of mathematics self-efficacy in the choice of math-related majors of college women and men: A path analysis. Journal of Counseling Psychology, 32(1), 47–56.

Hackett, G., & Betz, N. E. (1989). An Exploration of the Mathematics Self-Efficacy/Mathematics Performance Correspondence. Journal for Research in Mathematics Education, 20(3), 261–273.

Lent, R., Lopez, F., & Bieschke, K. (1991). Mathematics self- efficacy sources and relation to science-based career choice. Journal of Counseling Psychology, 38(4), 424–430.

Meece, L., Wigfield, A., & Eccles, S. (1990). Predictors of math anxiety and its influence on young adolescents' course enrollment intentions and performance in math. Journal of Educational Psychology, 82(1), 60–70.

Richardson, F. C., & Suinn, R. M. (1972). The Mathematics Anxiety Rating Scale : Psychometric Data. Journal of Counseling Psychology, 19(6), 551–554.

Sherman, J., & Fennema, E. (1977). The Study of Mathematics By High School Girls and Boys: Related Variables. American Educational Research Journal, 14(2), 159–168.

Suinn, R., & Winston, E. (2003). The mathematics anxiety rating scale, a brief version: Psychometric data. Psychological Reports, 92, 167–173.

Suresh, R. (2006). the Relationship Between Barrier Courses and Persistence in Engineering. Journal of College Student Retention: Research, Theory and Practice, 8(2), 215–239.

LATENT CLASSES OF MIDDLE GRADES STUDENTS ON MATH ANXIETY

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The purpose of this study was to identify differences in middle grades students' math anxiety levels based on 952 Turkish 8th grade students' responses to the Math Anxiety Scale (Erol, 1989) from nine schools located in different regions of Turkey. A mixture partial credit model analysis revealed three distinct latent classes based on homogeneities in response patterns. Latent class membership indicated that students in Class 1 had the lowest anxiety towards mathematics lessons, mathematics in daily life, and test and evaluation, and highest self-confidence about mathematics as opposed to students in Class 3 with the highest anxiety in these characteristics and with the lowest self-confidence about mathematics. Characterizing differences between members of each latent class extends recent advances in measuring math anxiety.

Keywords: Affect, Emotion, Beliefs, and Attitudes, Middle School Education, Research Methods

One main challenge in current educational settings is to identify affective constructs, such as anger, anxiety, and depression, and to provide intervention as early as possible for improving students' learning. Math anxiety is one of these constructs that lead to confusion about manipulation of numbers and solving problems related to mathematics. (Richardson & Suinn, 1972). In addition to physiological reactions such as sweaty palms and being sick, a student with high math anxiety level tend to have less self-confidence about mathematics (Luo, Wang, & Luo, 2009), and select career paths that to do not involve mathematics (Ashcraft & Moore, 2009).

To identify math anxiety levels of students, past research has used exploratory and confirmatory factor analysis for measuring the dimensions of math anxiety (e.g., Baloğlu & Zelhart, 2007) and structural equation modeling of the relationship between math anxiety and variables such as mathematics achievement (e.g., Harari, Vukovic, & Bailey, 2013). However, these studies were based on total scores obtained from summing all ratings from each student to items of a math anxiety scale, and using total scores has the potential of missing fine-grained information on characteristics of each student for math anxiety levels. Given the importance of math anxiety on learning and career choice, there is an urgent need for using methods that rely on individual item scores instead of total scores to detect students with different math anxiety levels and to identify those differences on math anxiety. One such method is the use of mixture item response theory (IRT) models. Based on past research (e.g., Rost, 1990), mixture IRT models can be useful in identifying latent classes of individuals on affective constructs such as differences in personality traits including depression (Hong & Min, 2007). The only study, to my knowledge, that has examined the utility of using mixture IRT models in detecting latent classes of 6th and 7th grade students on math anxiety is Ölmez & Cohen's (2017) study and they found two latent classes with students in Class 1's being less anxious than those in Class 2 towards mathematics lessons, mathematics in daily life, and test and evaluation, students in Class 2's having less self-confidence about mathematics. Hence, the purpose of this study was to identify differences in 8th grade students' math anxiety levels based on their responses to the Math Anxiety Scale (MANX; Erol, 1989). 8th grade is a critical grade level for dealing with math anxiety as early as possible because math anxiety has been known to peak during the secondary grades (Hembree, 1990). The following research questions were addressed in this study:

1. How many distinct latent classes of 8th grade students do exist in the population?
2. What does the existence of these latent classes (if any) imply about the different response patterns of math anxiety that exist in this population?

Theoretical Framework

The theoretical framework for this study follows the mixture Rasch model (MRM; Rost, 1990) that assumes the existence of distinct latent classes in the population. While the MRM specifies a separate item difficulty for each latent class, it determines a probability of being a member of a particular latent class for each individual. The polytomous form of the MRM based on scoring items with more than two categories such as never, sometimes, usually, and always, is called a partial credit model (PCM; Masters, 1982). In the mixture form of this model, the mixture partial credit model (MixPCM), the relationship between the probability of selecting a particular response category and the latent trait (i.e., math anxiety) changes across latent classes. The differences in response patterns to each item of a scale indicate homogeneities in characteristics of members of each latent class. Based on the differences in response patterns, the MixPCM could assign two individuals with same total scores to different latent classes.

Methods

The sample consisted of 952 Turkish 8th grade students from nine schools located in different regions of Turkey. Students' age range was between 14 and 16 years (female=478, male=474). A written consent form was obtained from one of the parents of each student before the study. The *Math Anxiety Scale*, developed by Erol (1989), is a four-point Likert type scale written in Turkish with options for each item ranging from "never" to "always." Higher scores demonstrate a higher math anxiety level. An internal consistency reliability estimate of .95 was obtained in this study. In a sample of 754 middle school and high school students, Erktin, Dönmez, and Özel (2006) identified four factors, which were test and evaluation anxiety, apprehension of math lessons, use of mathematics in daily life, and self-confidence for mathematics.

To analyze the data with the MixPCM, WINMIRA (von Davier, 2001) program was used. First, different numbers of latent classes were estimated in separate models to determine the relative fit of each model. Second, three indices for each model were compared to select the best fitting model: the Akaike information criterion (AIC), the Bayesian information criterion (BIC), and consistent AIC (CAIC). The model with the smallest BIC value was selected as the best fitting model (Li, Cohen, Kim, & Cho, 2009). Next, the characteristics of each latent class were analyzed by focusing on places where members of one latent class considered items to be easier or harder to endorse than other latent classes.

Results

Unidimensionality of the Scale

An exploratory factor analysis using maximum likelihood estimation indicated eigenvalues of the first three factors as 14.3, 2.6, and 1.9, with the total variance explained by the first factor as 31.7%. This indicated an evidence for essential unidimensionality of the Math Anxiety Scale based on Reckase's (1979) criterion.

Number of Latent Classes

Table 1 presents the indices for model selection. Minimum values for BIC as 66738.28, and CAIC as 67148.28 suggested a three-class solution in the data with 383 students (40.2%) in Class 1, 354 students (37.2%) in Class 2, and 215 students (22.6%) in Class 3.

Table 1. Model fit indices of the mixture Rasch model

Model	AIC	BIC	CAIC
One class	67128.86	67748.38	67884.38
Two classes	65687.50	66931.12	67204.12
Three classes	64870.58	**66738.28**	**67148.28**
Four classes	64500.46	66992.25	67539.25
Five classes	**64180.87**	67296.73	67980.73

Note. AIC = Akaike information criterion; BIC = Bayesian information criterion; CAIC = Consistent Akaike information criterion; the smallest information criterion index is bold.

Item thresholds indicate the extent of propensity when members of each latent class endorse each item. While thresholds lower on the scale such as -3 or -2 indicate a greater propensity of examinees to endorse that response category, thresholds higher on the scale such as 2 or 3 show that examinees have a greater propensity to endorse a higher response category.

Analysis of Item Location Parameters and Raw Responses

Item location parameter is the mean of all item threshold parameters for the particular item. While higher numbers on the logit scale demonstrate lower propensities of endorsement, lower numbers refer to higher propensities of endorsement. Thus, item location parameters are helpful for identifying differences in response patterns across latent classes. Moreover, I also compared item response distributions between the three latent classes by examining the percentages of raw responses in each latent class. This analysis led to three main results: (1) Students in Class 1 had the lowest anxiety towards mathematics lessons, mathematics in daily life, and test and evaluation, and highest self-confidence about mathematics than students in Class 2 and in Class 3; (2) students in Class 3 had the highest anxiety in terms of these characteristics, and lowest self-confidence about mathematics; and (3) students in Class 2 had medium anxiety and self-confidence regarding all these characteristics.

Items reflecting anxiety related to mathematics lessons (i.e., Items 7, 16, 36) were most difficult to endorse for Class 1 and easiest to endorse for Class 3. For example, for Item 36, "I struggle with listening to the teacher in the math class" the proportions selecting the options of "never" and "always" were 83% and 1%, respectively, in Class 1; 39% and 6%, respectively, in Class 2; and 29% and 35%, respectively, in Class 3. Furthermore, items that asked students to rate their ideas about mathematics exams and evaluation anxiety (i.e., Items 2, 3, 14, 25, 30, 41) were hardest to endorse for students in Class 1 and easiest to endorse for students in Class 3. For Item 25, "I feel uneasy a week before the math exam" (Class 1, Class 2, and Class 3 item locations were 0.75, 0.21, and 0.18), the proportions selecting the option "never" were 86%, 33%, and 44% for students in Class 1, Class 2, and Class 3, respectively. And, the proportions selecting the option "always" were 0% for Class 1, 8% for Class 2, and 31% for Class 3.

Items focusing on anxiety about the use of mathematics in daily life (i.e., Items 9, 29, 38) were hardest to endorse for students in Class 1 and easiest to endorse for students in Class 3. For Item 38, "I feel bothered by the necessity of making calculations by solving mathematical problems in my daily life even if they are simple", 80% of the students in Class 1, 38% of the students in Class 2, and 30% of the students in Class 3 selected the option "never."

Finally, items involving self-confidence for mathematics (i.e., Items 23, 27, 43) were hardest to endorse for students in Class 1 and easiest to endorse for students in Class 3. For Item 23, "I cannot dare to ask the points I do not get in the math class", the proportion selecting "always" was 1% in Class 1, 13% in Class 2, and 37% in Class 3.

Discussion

In this study, I examined the differences in middle grades students' math anxiety levels by identifying their latent classes using the MixPCM on a math anxiety scale. The first research question indicated three distinct latent classes with different patterns of math anxiety. The second research question showed that Class 1 consisted of students who were less anxious towards their mathematics lessons, mathematics in daily life, and test and evaluation, and who were more self-confident about mathematics than students in Class 2 and Class 3. On the other hand, Class 3 consisted of students who were more anxious regarding these characteristics, and who were less self-confident about mathematics than those in Class 1 and Class 2. The results of this study confirm the findings reported by Ölmez & Cohen (2017) based on 244 Turkish 6th and 7th grade students' responses to the same math anxiety scale. The only difference between the two studies appears to be the number of latent classes in each study (2 latent classes versus 3 latent classes). This might either stem from the fact that the sample sizes are different in both studies (N = 244 versus N = 952) or due to the focuses on different grade levels (6th and 7th grades versus 8th grade). The present study revealed that relying on individual item scores instead of total scores can provide fine-grained information about the differences in students' math anxiety levels. By using the MixPCM analysis, teachers can identify students in their classrooms with different patterns of math anxiety and make interventions accordingly based on the needs of each student. Future research should focus on conducting similar studies with other popular math anxiety scales to demonstrate a more complete understanding of math anxiety in middle grades.

References

Ashcraft, M. H., & Moore, A. M. (2009). Mathematics anxiety and the affective drop in performance. *Journal of Psychoeducational Assessment, 27*(3), 197–205.

Baloğlu, M. & Zelhart, P. F. (2007). Psychometric properties of the revised mathematics anxiety rating scale. *The Psychological Record, 57*, 593–611.

Erktin, E., Dönmez, G., & Özel, S. (2006). Psychometric characteristics of the mathematics anxiety scale. *Education and Science, 31*(140), 26–33.

Erol, E. (1989). *Prevalence and correlates of math anxiety in Turkish high school students*. Unpublished master thesis, Bogazici University.

Harari, R. R., Vukovic, R. K., & Bailey, S. P. (2013). Mathematics anxiety in young children: An exploratory study. *The Journal of Experimental Education, 81*(4), 538–555.

Hembree, R. (1990). The nature, effects, and relief of mathematics anxiety. *Journal for Research in Mathematics Education, 21*, 33–46.

Hong, S., & Min, S. (2007). Mixed Rasch modeling of the Self-Rating Depression Scale: Incorporating Latent Class and Rasch Rating Scale models. *Educational and Psychological Measurement, 67*(2), 280–299.

Li, F., Cohen, A. S., Kim, S. H., & Cho, S.-J. (2009). Model selection methods for dichotomous mixture IRT models. *Applied Psychological Measurement, 33*(5), 353–373.

Luo, X., Wang, F., & Luo, Z. (2009). Investigation and analysis of mathematics anxiety in middle school students. *Journal of Mathematics Education, 2*(2), 12–19.

Masters, G. N. (1982). A Rasch model for partial credit scoring. *Psychometrika, 47*, 1491–74.

Ölmez, İ. B., & Cohen, A. S. (2017). Detecting math anxiety with a Mixture Partial Credit Model. In Galindo, E., & Newton, J. (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1399–1406). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Reckase, M. D. (1979). Unifactor latent trait models applied to multifactor tests: Results and implications. *Journal of Educational Statistics, 4*, 207–230.

Richardson F. C., & Suinn, R. M. (1972). The mathematics anxiety rating scale: Psychometric data. *Journal of Counseling Psychology, 19*(6), 551–554.

Rost, J. (1990). Rasch models in latent classes: An integration of two approaches to item analysis. *Applied Psychological Measurement, 14*, 271–282.

Von Davier, M. (2001). WINMIRA [Computer Software]. St. Paul, MN: Assessment Systems Corporation.

LATINA/O STEM MAJORS' PERSPECTIVES OF EXPERIENCES SUPPORTING THEIR MATHEMATICAL SUCCESS

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This study uses an identity lens and qualitative methods to examine eight Latina/o STEM majors' perceptions of experiences that supported their mathematical success throughout their lives. Using NVivo software, an iterative coding scheme is applied to analyze interview data. Drawing on Leyva's (2016a) three-tiered analytical framework, critical race theory, and Latina/o critical theory, preliminary cross-case analysis reveals that participants' mathematical success is related to complex experiences negotiated within institutional, interpersonal, and ideological dimensions. Such experiences are related to their racial identity constructions, gender identity constructions, the co-construction of their mathematical identities with other salient identities, and experiences negotiated in multiple sociopolitical contextual layers.

Keywords: Equity and Diversity, Post Secondary-Education, Gender

Purpose and Background

This research project is a response to Leyva's (2016a) call "for more nuanced conceptualizations of gender in future mathematics education research that examine its social construction at different intersections of identity (e.g., gender/race) as well as at different levels of influence (e.g., institutional, interpersonal, ideological)" (p. 190). Drawing on Leyva's (2016a) three-tiered analytical framework, this research uses an identity lens and qualitative methods to examine eight Latina/o STEM majors' perceptions of experiences that supported their math success at the institutional, interpersonal, and ideological levels. The research questions are: (1) How do participants define experiences that support their math success at the institutional, interpersonal, and ideological levels? and (2) How do participants negotiate *what it means to be an underrepresented student* in relation to experiences that support their math success at these three levels? Understanding the complexities, avenues, and transformations that support Latina/o students' math success will further expand knowledge about how to provide students with math supports that are identity-affirming and equitable.

Literature Review

Emerging math education research has advocated for conducting more holistic explorations of underrepresented students' participation, learning, and experiences in mathematics, including the roles of sociopolitical, sociocultural, and psychological mechanisms (e.g., Gutiérrez, 2010). Of the small number of math education studies that have conducted in-depth explorations of underrepresented students' perspectives of their math experiences, few have investigated the lives of *mathematically successful* underrepresented students to understand their perceptions of experiences that have supported this success (exceptions include Leyva, 2016a, 2016b). Further, much of this work has been conducted on African American students and not Latina/o students (exceptions include Leyva, 2016a, 2016b). We argue that capturing aspects of Latina/o students' experiential knowledge they perceive as linked to their math success can assist the math education community in better supporting both underrepresented and represented students in attaining math success. Such information will shed light on how to create equitable math learning environments, how to empower students as math learners, and how to provide opportunities that support students' math success throughout the educational pipeline.

Mathematics education equity scholarship situated within broader sociopolitical perspectives (e.g., Gutiérrez, 2010; Martin, 2009) offers theoretical approaches for uncovering the mechanisms underlying underrepresented students' math participation, learning, and experiences that contribute to their success. Sociopolitical scholarship supports using critical race theory (CRT) and Latina/o critical theory (LatCrit) to examine marginalized students' math experiences (Gutiérrez, 2010; Martin, 2009). CRT and LatCrit frameworks encourage examining marginalized students' experiences through counter-storytelling methods while emphasizing the significance of race and racism and how they intersect with other forms of oppression (Solórzano & Yosso, 2002). Leyva (2016a) drew on both CRT and LatCrit to apply his analytic framework, which consisted of institutional, interpersonal, and ideological dimensions, to explore Latina/o students' math success. Such work expands knowledge about how Latina/os construct positive mathematical identities while taking into consideration the intersectional nature of race and gender.

This study addresses a gap in the math education literature by employing CRT and LatCrit and applying Leyva's framework (2016a) to examine how Latina/o STEM majors succeed in mathematics. Understanding the math experiences of this particular student population is critical because they can reflect on their elementary through undergraduate math experiences and reveal the roles of, for instance, sociohistorical forces and constructs, contextual influences, agency, and resilience. Understanding their experiences will broaden the math education community's understanding of how to devise math supports drawn from the experiences of underrepresented students who have succeeded in mathematics. We will be able to understand critical factors that supported them in persisting and succeeding in math throughout their lives, obstacles they had to overcome in their math trajectories to succeed, the various sociopolitical contexts in which they managed factors that supported their success in math, and how their identity constructions (e.g., math, racial, gender, class) contributed to their mathematical success.

Methods

This project uses qualitative methods (Yin, 2009) to explore participants' experiences related to their mathematical success and how such experiences are related to their lives as underrepresented STEM students. The data analyzed for this study is part of a larger study on underrepresented STEM students conducted at a diverse, Hispanic-serving urban university located in Chicago, Illinois during 2013-2015. This larger study involved recruiting African American, Latina/o, Native American, low-income, and/or first generation college students. Recruitment tactics involved sending emails to students in 200 and 300 level undergraduate STEM courses and posting flyers near classrooms where STEM classes were held. All Latina/o STEM majors that responded to our recruitment tactics and completed the entire data collection process were selected for this particular study. STEM majors included students that were actively pursuing a STEM major or had recently graduated with a STEM major at the time of the study. Table 1 describes participant demographic information for the eight Latina/o participants.

Table 1: Participant Demographic Information

Students	Racial/Ethnic Background	Gender	Birth Country	University Classification
Amelia	Latina/o	Male	Mexico	Junior
Antonio	Latina/o; Honduran	Male	Honduras	Graduate student
Elias	Latina/o; Columbian-American	Male	US	Junior

Caleb	Latina/o; Mexican	Male	US	Senior
Diego	Latina/o; Columbian	Male	Columbia	Senior
Juan	Latina/o; Mexican-American	Female	US	Senior
Maria	Latina/o; Mexican	Female	US	Junior
Ricardo	Latina/o; Mexican; Puerto Rican	Male	US	Senior

In-depth interviews were the primary data source; questionnaires were secondary data sources. The in-depth interview consisted of open-ended questions that aimed to capture a holistic understanding of participants' perspectives of their life-long math experiences. As categories were identified, additional interviews were used to investigate the emergent categories. Data collection continued until saturation of main categories was reached. All interviews were audiotaped, a subset of interviews were videotaped, and all interviews were transcribed. Using NVivo software, an iterative coding scheme will continue to be applied to analyze interview data. Interview codes will continue to be developed and compared until theoretical saturation is reached (until no new codes or categories emerge). Finally, a theoretical model that explains the experiences participants described as supporting their mathematical success will be developed.

Preliminary Results

Table 2 displays preliminary findings for this study. Preliminary cross-case analysis resulted in the following main themes regarding experiences that supported the participants' math success: (a) access to advanced math, math curriculum and pedagogy, and STEM support resources in the institutional dimension; (b) experiences involving teacher, family, and peer interactions and relationships in the interpersonal dimension, and (c) powerful experiences related to self-histories, critical roles of math transitions, complex aspects of math identities, and the intersectional nature of math identities and other complex identities in the ideological dimension. Such experiences that supported participants' math success were complexly related to racial identity constructions, gender identity constructions, the co-construction of their math identities with other salient identities, and experiences negotiated in multiple sociopolitical contextual layers. For example, emergent connections between experiences that supported their math success and their racial identity constructions included: resisting racialized experiences, functioning as cultural math role models, and engaging with cultural academic role models. Participants' counter-stories reveal how inequities, sociohistorical forces, sociopolitical constructs, contextual influences, agency, and resilience are intimately related to their math success.

Table 2: Preliminary Themes and Subthemes

INSTITUTIONAL DIMENSION	INTERPERSONAL DIMENSION	IDEOLOGICAL DIMENSION
Access to Advanced Math • Advanced math course tracking • Math classes with older students • Summer math classes **Math Curriculum & Pedagogy** • Non-traditional aspects of the math curriculum	**Teacher-Participant Interactions & Relationships** • Math classroom management approaches • Nurturing personalities • Passionate personalities	**Powerful Experiences Related to Self-Histories** • Experiences that motivated them to pursue their major • Overcoming life challenges **Critical Roles of Mathematical Transitions** • Elementary to high-school shifts in perspective • High-school to college shifts in perspective

• Effective math pedagogical approaches • Connections between math curriculum and broader academic/ professional/ personal goals **STEM Support Resources** • Math tutoring • STEM-related professional development programs/activities • STEM-related mentorship/ advisement	**Family-Participant Interactions & Relationships** • Family valuing/ supporting educational pursuits • Comparisons to family members **Peer-Participant Interactions & Relationships** • Cooperative learning inside & outside of classroom spaces • Comparisons to peers	• Community college to four year college shifts in perspective **Complex Aspects of Mathematical Identities** • Awareness of mathematical strengths • High mathematical expectations • Recognition of the importance of mathematics in their lives • Recognition of the important role of work ethic/curiosity/creativity in learning mathematics • Agency/resilience as math learners **Intersectional Nature of Mathematical Identities and Other Complex Identities** • Role of racial identities • Role of gender identities • Role of socioeconomic statuses • Role of the intersectional nature of complex identities

Conclusion

Emerging math education equity research indicates that we cannot fully understand students' math achievement, participation, and learning without taking into account their identity development as math learners. Such research also supports that the math education community must embrace the sociopolitical nature of underrepresented students' math participation, learning, and experiences because this more expansive theoretical lens can provide additional knowledge regarding *how* and *why* they attain mathematical success. This research project addresses a gap in existing math education research by raising important considerations regarding the significance of understanding what it means to be Latina/o with respect to math participation, learning, and success.

References

Gutiérrez, R. (2010). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education*, *41*, 1-32.

Leyva, L. A. (2016a). *" Mapping the margins [in mathematics]": Examining the gendered and racialized intersectionality of mathematics experiences among African American and Latina/o undergraduate students in a large, predominantly white university*. Rutgers The State University of New Jersey-New Brunswick.

Leyva, L. A. (2016b). An intersectional analysis of Latin@ college women's counter-stories in mathematics. *Journal of Urban Mathematics Education*, *9*(2).

Martin, D. (2009). Researching race in mathematics education. *Teachers College Record*, *111*(2), 295−338.

Solórzano, D., & Yosso, T. (2002). Critical race methodology: Counterstorytelling as an analytical framework for education research. *Qualitative Inquiry*, *8*(1), 23−44.

Yin, R. K. (2009). *Case study research: Design and methods*. Thousand Oakes, CA: Sage.

STUDENTS' EXPERIENCES IN AN UNDERGRADUATE MATHEMATICS CLASS: CASE STUDIES FROM A LEARNER-CENTERED PRECALCULUS CLASS

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In this case study, we report the results from qualitative research set in a large-size undergraduate mathematics classroom. We provide evidence that students perceived that their participation in tasks where students work together to solve real life math problems involving precalculus concepts was helpful to their learning. Students expressed positive attitudes with regards to the collaborative efforts in this math classroom, often different from a traditional math course. Additionally, they expressed gratitude towards learning math in a way that was different than traditional mathematics courses, which we interpret to mean a more conceptual versus procedural-type of a mathematical learning.

Keywords: Post-Secondary Mathematics; Affect, Emotions, Beliefs, and Attitudes; Instructional Activities and Practices

There is great interest in improving undergraduate STEM courses across the country (P-CAST, 2012). More specifically, there is significant ongoing work on how to improve undergraduate mathematics courses by introducing the use of learner-centered teaching strategies, particularly what is often called active learning. This educational research is important for mathematics educators and policy makers in order to learn whether new instructional approaches support positive student affect and learning in STEM fields. Freeman and colleagues (2014) published significant results of a meta-analysis that addressed articles about using active learning in STEM and student outcomes. Performance outcomes, attitudes, persistence, and confidence in doing math has been shown to be higher when incorporating these strategies into the classroom (e.g. Freeman et al, 2014; Laursen, Hassi, Kogan, & Weston 2014).

There has not been significant work done to study the results of introducing new practices into service-level mathematics courses, such as Precalculus, that have been taught traditionally with procedural content or significant work done in large lecture classes. Being able to show how these strategies can be introduced and the results of using them in these large classes is an important contribution to mathematics (and more generally STEM) education.

In this case study, we interviewed several students from a large-scale Precalculus course where learner-centered strategies were used. These students expressed their attitudes and beliefs about different aspects of the course. We specifically chose to listen to students' experiences from their own perspective because too often, their perspectives are not acknowledged. This research is important because it sheds light on how new strategies that are implemented in these large-scale mathematics course affect students' overall experiences. The research question addressed in this paper is: What are students' perceptions of their experience in a large-scale Precalculus course where learner-centered strategies are employed?

Literature Review

With the existence of a great variety of learner-centered instructional strategies, we propose that there are many ways to vary methods of teaching in a large lecture classroom beyond direct instruction (Kennedy, 1998). Personal response systems, also knowns as Clickers, is one tool that can enhance students' learning when done properly (e.g., Boyle & Nicol, 2003; Vajravelu &

Muhs, 2016). Vajravelu and Muhs (2016) found that when courses are intentionally designed with the structure and opportunity for students to become active learners in conjunction with their classmates through clickers and team-based activities, students will work together to solve mathematical problems and engage in discussion in large lecture classrooms.

In addition to using learner-centered strategies, student attitudes and self-efficacy are important areas for further study. Núñez-Peña and colleagues (2013) found that poor success rates in math courses are related to negative attitudes towards math. Additionally, both student attitudes and self-efficacy play an integral part in perceived usefulness of math, in choosing careers involving math, and in opting to take more math courses beyond those required for their degree (e.g., Núñez-Peña, Suárez-Pellicioni, & Bono, 2013; Wilkins & Ma, 2003). Wilkins and Ma (2003) report on the importance of social influence, challenging curricula, and engaging activities as ways to help students see the importance of math in their own lives. The choices instructors make on the activities presented in class can make a difference in how students view math in general (Wilkins & Ma, 2003). These activities should be both engaging and include problem-solving. Yusof and Tall (1999) found that "when students are doing problem-solving, their attitudes change in the direction desired by the mathematics lecturers" (p. 77).

Methods

This case study was designed to study the introduction of evidenced-based instruction to explore students' experiences as they relate to various instructional components. For this study, outcomes are defined to mean achievement, attitudes, persistence, interest, and self-efficacy. This was part of a larger study, in which student-centered strategies were analyzed holistically.

This research was carried out at a large (~50,000 students), public university located in southern United States. There were 94 students enrolled in a 4-credit hour Precalculus course that met for 1.5 hours, 4 days a week during the summer of 2016. The students sat inside of an auditorium-style lecture hall, large enough to seat 300 students. Most class meetings consisted of traditional lecture delivered by the instructor with daily Clicker questions. The clicker questions were given to students frequently throughout the daily lecture and were based on the material being taught that day. These included a mixture between procedural and conceptual questions.

Along with normal classroom hours, students were required to participate in a computerized portion of the course called Smart Lab for 2 hours per week, which was facilitated by the course's 5 graduate teaching assistants (TAs). During these hours, students received extra help, were encouraged to complete math assignments, and completed examinations for the course.

Several times throughout the course, students also participated in Team Activities that lasted an entire class. Students worked in teams of 3-4 to solve application problems. An example of a team activity is when students used trigonometry and a sextant to estimate the height of a building. The five TAs were available during the days in which Team Activities were completed.

In the first week of class, students were asked to complete an online survey. This 43-item pre-survey addressed attitudes, perceived self-efficacy, and interest in math. Towards the end of the course, the students were given a post-survey which consisted of the same items. Finally, during the last week of the course, students were also asked to participate in a semi-structured interview where they answered questions about their attitudes and experiences towards different parts of the course, i.e., Team Activities, Clickers, and Smart Lab. Although many students were included in data collection, only the 5 students who completed the pre-survey, post-survey, and interview were included in this study. The participants included: Magdelina, Kadeem, Nancy, Teo, and Rajiv (pseudonyms). Their experiences are reported next.

Results

Overall, the participants had relatively positive experiences in the course where evidence-based strategies were introduced. In this section, we report on these experiences.

Team Activities

With regards to Team Activities, participants' experiences appeared fairly mixed. One of the main positive comments made by the participants pertained to the social aspect of the Team Activities. In particular, students enjoyed and felt like they benefitted from working with others. For example, Nancy commented, "It's nice working with other people because sometimes you don't get it and they get it." Another participant, Kadeem, also stated, "I think they contributed to your learning because if you, if you're trying to solve a problem, umm, and you don't really know what to do or there's a problem, you have teammates there that probably know something you don't know. Or something you forgot."

Although the participants had mixed feelings about the Team Activities, their negative feelings consisted of comments such as: they are conceptual, they don't correlate to the tests, they are difficult, they are too long, etc. For example, Magdelina commented "we didn't have enough time to finish most of them", while another participant, Nancy, stated "I just feel like team activities are really hard or really long and it takes too long to finish". These concerns, although conveyed as negative from the students, could be interpreted as positive from a researcher standpoint. Thus, these mixed feelings may not tell the whole story.

Clicker Questions

With regards to the Clicker questions, in general, the participants found them to be helpful for learning, they allowed productive collaboration among their peers, and they were presented in a more conceptual way that tied into their learning. For example, Kadeem stated "I, umm, I liked those because it's like, it's more interactive. And usually the problems he would put up there were not the typical homework problems." Another participant, Teo, also stated,"It kind of umm, helped the progression, because as he was going through the class, it kind of helped you okay, at like different points you could say 'Yeah, okay. I understand it up to here. Or I understand this concept up until right here'." Of the students interviewed, only one student stated that Clicker questions were not helpful to his learning.

Smart Lab

Similar to the Clicker questions, Smart Lab was also perceived as mostly positive. The main comments addressed by the participants with regard to the Smart Lab were that it provided a space to specifically focus on math homework and that the presence of the TAs was helpful. For example, Kadeem, mentioned:

> Whenever you have an issue with a certain problem, [the TAs] are right there to help you. And say if you went to the lecture and didn't understand anything, and that happened once or twice. Umm, the tutors are right there, and they can explain it for you.

Similarly, Magdelina expressed, "That's just very helpful if you don't understand something, you have, umm… either a tutor or then you can click on like an option that guided you step by step on how to do a problem." With regards to providing a space to work on assignments, Teo stated, "it gives you a time, umm… a set time where you can just focus on your math stuff and that if you need some help, you can, you know […] there's someone there to help you."

Surveys

Overall, the results from the surveys of these 5 participants were positive. We saw an increase in average score across all items from the pre-survey to the post-survey. Among our participants, the largest increases from pre- to post-survey occurred on the following items:

"Knowing math will help me in my life", "Mathematics is important for a good career", "I am not nervous when I am in math class", and "Mathematics is easy for me to understand". This leads us to believe that including learner-centered instructional strategies can have an effect on students' perceived experiences in a large-scale pre-calculus course, with those focused more on mathematics in the real-world showing the largest gains.

Conclusion

In this paper, we chose to discuss one portion of our research. Currently, most research focuses on either student outcomes or teachers' moves, and gives little-to-no voice to the students themselves. Therefore, we found it important to give those students a voice in this study and listen to their perspective to help inform our analysis.

We found that it is possible to bring learner-centered instructional strategies to a large-sized mathematics undergraduate class. The students in this study expressed mostly positive attitudes towards these student-centered learning strategies, in addition to demonstrating a similar change in attitude from a pre- to post-survey. To implement these strategies successfully, we suggest starting with a few activities and then slowly adding more activities to the course. We also found the TAs to be an important addition to the course. With a large class size of about 100 students, five TAs were utilized to help the Team Activities run more smoothly. We suggest including several TAs for these student-centered learning strategies to be more of a success.

In conclusion, we propose trying to incorporate the types of student-centered learning strategies as described in this paper into undergraduate mathematics courses. Educators need to be mindful of the pace at which these strategies be introduced in these courses and the number of TAs needed to make the course as successful as possible. With thoughtfulness and more research, these student-centered learning strategies could be the next step towards improving student attitudes and self-efficacy in large-scale, undergraduate mathematics courses.

References

Boyle, J. T., & Nicol, D. J. (2003). Using classroom communication systems to support interaction and discussion in large class settings. *Research in Learning Technology*, *11*(3), 43–57.

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences*, *111*(23), 8410-8415.

Kennedy, M. M. (1998). Education reform and subject matter knowledge. *Journal of Research in Science Teaching*, *35*(3), 249-263.

Laursen, S. L., Hassi, M.-L., Kogan, M., & Weston, T. J. (2014). Benefits for women and men of inquiry-based learning in college mathematics: A multi-institution study. *Journal for Research in Mathematics Education*, *45*(4), 406–418.

Núñez-Peña, M.I., Suárez-Pellicioni, M., & Bono, R. (2013). Effects of math anxiety on student success in higher education. *International Journal of Educational Research, 58*, 36-43.

Vajravelu, K., & Muhs, T. (2016). Integration of digital technology and innovative strategies for learning and teaching large classes: A calculus case study. *International Journal of Research In Education and Science (IJRES)*, *2*(2), 379–395. Retrieved from http://files.eric.ed.gov/fulltext/EJ1105125.pdf

Wilkins, J. L. & Ma , X. (2003) Modeling Change in Student Attitude Toward and Beliefs About Mathematics, The Journal of Educational Research, 97:1, 52-63.

Yusof, Y., & Tall, D. (1999). Changing attitudes to university mathematics through problem solving. *Educational Studies in Mathematics, 37*, 67-82.

DOES TEACHER FEEDBACK ON HOMEWORK IMPROVE STUDENT MATHEMATICS ACHIEVEMENT BETTER THAN ONLINE COMPUTER FEEDBACK?

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Despite the benefits of homework and teacher feedback on students' mathematics achievement, relatively limited work has been done on analyzing the effect of feedback on homework. This study examined the effects of four types of feedback on eighth grade students' achievement in mathematics: no feedback, effort-based teacher feedback on paper-and-pencil-based homework, ability-based teacher feedback on paper-and-pencil-based homework, and online programmed feedback on computer-based homework. A total of 59 students from an inner-city school district in New York participated in the study. A repeated measures experimental design was employed to isolate/minimize the between group variability. Analysis of results revealed that although effort-based feedback appeared to be a variable of interest, there was statistically insignificant relationship between the effect of different type of feedback and academic achievement.

Keywords: Teacher feedback, Effort feedback, Online feedback, Student achievement

Introduction

Homework and feedback have been identified as significantly related to academic achievement. The fundamental reason for assigning homework is meant to extend and supplement in-class activities (Cooper, Robinson & Patall, 2006). Most studies examining the relationship between academic achievement and homework found a positive and statistically significant correlation between the two variables (Cooper & Valentine, 2001; Cooper et al., 2006; Doorn, Janseen, & O'Brien, 2010). Although feedback can lead to large positive gains on student performance, relatively limited work has been done on analyzing the effect of feedback type on homework and its relationship with mathematics achievement.

The purpose of this study is to investigate the effect of teacher feedback on paper-and-pencil-based homework compared to computer feedback on online homework. Although an extensive amount of research has been conducted on teacher feedback on student homework, relatively limited work has been done in analyzing the effect of feedback type on homework and its relationship with academic achievement. In particular, despite the availability and popularity of online homework, the research on the impact of online assignments is limited in the current literature. Dweck (2006) highlights the importance of a growth mindset over a fixed mindset and the need for effort-based feedback/praise over ability-based feedback in motivating students to accept more challenging tasks. However, little is known about the effects of teachers' different feedback types (e.g., ability versus effort feedback) on students' homework behaviors and motivation in the paper-and-pencil-based homework setting compared to computer-based homework setting. This study hypothesizes that not all feedback types have a positive relationship to achievement. This study focuses on two distinct areas – 1) homework and teacher feedback and 2) the intersection of homework type (i.e., paper-and-pencil versus online) and feedback type (i.e., effort-based feedback, ability-based feedback and computer-generated feedback). The study analyzes the effect of no feedback, effort-based feedback, ability-based feedback on paper-and-pencil based homework, and computer-programmed immediate feedback on online homework and their relationship with the improvement of middle school students' achievement in mathematics. This study also explores differences in students' attitudes towards mathematics depending on feedback type. The research questions that guide this study include: (1) Is there any difference in students' mathematics achievement depending on feedback type (i.e., no feedback, effort-based feedback, ability-based feedback using paper-and-pencil based homework)? and (2) Is there any difference in students' mathematics achievement depending upon homework type (i.e., paper-and-pencil versus computer based)?

Theoretical Framework

This study was guided by expectancy-value theory (Eccles 1994; Wigfield, Tonks & Eccles, 2004) to explain and understand the cognitive and behavioral aspects of students' reasoning regarding homework. In addition, this study builds on mindset theory as proposed by Dweck (2006) for feedback type. First, according to the expectancy-value theory, student achievement and choices related to achievement are determined by two factors: expectancy for success, or how confident the students are in their ability to complete the task on hand, and subjective task values, which is the perceived value of the task for the student. Expectancy-value theory is used in this study to understand and evaluate the factors that motivate a student to choose a particular behavior while completing homework assignments. In addition, this study builds on Dweck's (2006) mindset theory to guide the feedback aspect. Mindset theory offers a correlation between a person's self-belief or mindset and their motivations and achievements. If students believe that they are capable of improving their knowledge or intelligence, they have adopted a growth mindset, whereas if they are under the assumption that intelligence is a precise and pre-established concept then they are following a fixed mindset. Two main types of feedback or praise have been reported in the literature: effort-based praise ("Good try"; "You've been working hard"), where teachers provide feedback focusing on the process of students' work and task performance and highlight the possibilities for learning to motivate students, and ability-based praise ("Well done; you are really smart"; "Gee, you are a good student"), where teachers provide feedback focusing on students' innate attributes or ability rather than the quality of their work (Son, 2016). The study compared the effects of four methods of feedback on student achievement: no feedback on paper-and-pencil based homework, effort-based teacher feedback on paper-and-pencil-based homework, ability-based teacher feedback on paper-and-pencil-based homework, and online programmed answer based feedback on computer-based homework.

Methods

Repeated measures experimental design, a powerful tool to control for existing subject differences between groups (Howitt & Cramer, 2011), was employed to explore the correlation between homework, feedback, and academic achievement, in which each student is part of each treatment condition. The participants for this study was comprised of two 8th grade classes from an inner-city junior high school located in New York State; 59 students in total with ages ranging from 13 to 14 years. 53% participants were female and 47% were males. In the repeated measures design, all participants took part in all possible treatment conditions, and they were assigned to each treatment condition randomly.

All participants took a pre-test and filled out a Homework Feedback Survey to collect information on students' reported practices and perceptions regarding homework, feedback from teachers, quality of homework, value of homework, and expectancy beliefs. Participants were then randomly divided into four groups A, B, C, and D comprised of about 15 students each coming from two different classes. The four treatments were: assigned homework returned with no feedback, assigned homework returned with teacher effort-based feedback, assigned homework returned with teacher ability- based feedback, and online-based homework with IXL-generated immediate feedback.

To answer the first research question that explored whether different types of homework feedback improved students' mathematical achievement, one-way repeated measures ANOVA was used to test that the difference between means were statistically significantly for feedback type. To answer the second research that investigated the effect of teacher provided feedback on students' attitudes towards mathematics in the light of different type of feedback, a two factor repeated measures ANOVA was conducted. Feedback type was treated as factor one with three levels (no feedback, effort-based feedback and ability-based feedback), and attitude was factor two with three levels (homework effort, expectancy, and value) as well.

Summary of the Findings

Effect of Feedback Type on Achievement

The first research question asks whether different types of homework feedback improve students' mathematical achievement. In order to answer the first research question, the effect of four different types of feedback, provided on students' mathematics homework and its relationship with mathematics achievement as measured by gain scores (difference between post-test and the pre-test scores) were explored. Table 1 shows the mean and standard deviations of pre-test scores and post-test scores, along with gain scores by each feedback type provided on students' homework.

Table 1: Descriptive Statistics by Feedback Groups

Feedback Type	Pre-Test		Post-Test		Gain Scores	N
	Mean	SD	Mean	SD		
No Feedback	49.82	20.16	53.38	18.16	3.47	59
Effort-based Feedback	48.05	19.25	54.58	21.02	6.53	59
Ability-based Feedback	48.14	16.40	52.37	18.92	4.24	59
IXL-based Feedback	48.98	18.80	52.29	20.26	3.31	59

Regardless of the feedback type, students showed improved scores in the post-test. In addition, the gain scores show the difference between mean pre-test scores and the mean post-test scores were the highest for participants provided with effort-based feedback; they showed an increase of little more than 6.5 percent points. The gain in post-test scores shown by ability-based feedback was next at little more than 4 points, whereas the gain made by IXL based immediate feedback and no feedback were comparable at about 3.5 points. We found that the difference between gain scores among the four feedback conditions were not statistically significant, $F(2.71, 157.15) = .38, p = .75 > .05$. Although, observed pairwise comparisons from Table 2 show positive differences in favor of effort-based feedback, post-hoc test using Bonferroni correction revealed that none of these differences were statistically significant. Although there are differences in gain scores among the four feedback types, this study was not able to find a statistically significant relationship between teacher feedback and students' mathematics achievement.

Students Attitudes and Feedback

The second research question asked whether there was any difference in student's attitudes towards mathematics depending on feedback type. To answer this research question, Homework Feedback Survey, which consisted of six constructs comprising of 26 items, administered to 236 participants was used. To analyze the relationship between feedback and attitude, a two-way repeated measures ANOVA was employed, the two within-group factors were attitude with three levels (i.e., expectancy belief, homework effort, and value belief) and feedback also with three levels (i.e., no feedback, effort-based feedback, and ability-based feedback) and gain scores by feedback type as the dependent variable. We found that the main effect of attitude is statistically significant $F(1.63,96.64) = 30.47$, $p = .001 < .05$. This finding indicates that students' attitudes are positively associated with their performance in mathematics. However, the interaction effect of feedback type and attitude is not statistically significant, $F(3.39,196.70) = 1.57$, $p = .19 > .05$. Thus, although students' attitudes are positively related to mathematics achievement, this relationship is not moderated by different type of feedback employed in this study.

Discussion and Implications

The findings of this study suggest that the effect of feedback type on mathematics homework did not have a positive relation to academic achievement. The study could not find statistically significant difference between the gains made by students who were provided different types of feedback on mathematics homework, though nominal difference in favor of effort-based feedback were higher, especially for females. The study also did not find a statistically significant interaction between feedback and gender, suggesting the above findings are equally applicable to both male and female students. In addition, although attitude was strongly related to mathematics achievement, this relationship was not moderated by the three feedback types employed in this study. However, our study showed that students who were provided effort-based feedback perceived teachers' feedback to be more useful as compared to ability-based feedback and no-feedback groups, and they attached a higher value to it. Nevertheless, this usefulness did not translate into statistically significant higher achievement levels. This study provides implications for future researchers and educators to gain an inside access to a low performing classroom and observe how feedback and homework are related to the academic performance of students in mathematics.

References

Cooper, H. (1989). Synthesis of research on homework. *Educational Leadership, 47*(3), 85-91.

Cooper, H., Robinson, J. C., & Patall, E. A. (2006). Does homework improve academic achievement? *A synthesis of research, 1987-2003. Review of Educational Research, 76*(1), 1-62.

Doorn, D. J., Janssen, S., & O'Brien, M. (2010). Student attitudes and approaches to online homework. *International Journal for the Scholarship of Teaching and Learning, 4*(1), 1-20.

Dweck, C. S. (2006). *Self-theories: Their role in motivation, personality, and development.* Philadelphia, PA: Psychology Press.

Eccles, J. S., & Wigfield, A. (1995). In the mind of the actor: The structure of adolescents' achievement task values and expectancy-related beliefs. *Personality and Social Psychology Bulletin, 21*(3), 215-225.

Eccles, J. S. (1994). Understanding women's educational and occupational choices: Applying the Eccles et al. model of achievement-related choices. *Psychology of Women Quarterly, 18*(4), 585-609.

Howitt, D., & Cramer, D. (2011). *Introduction to research methods in psychology*. Harlow, England: Pearson.

Son, J. (2016). Preservice teachers' response and feedback type to correct and incorrect student-invented strategies for subtracting whole numbers, *Journal of Mathematical Behavior*, *42*, 49-68.

Wigfield, A., Tonks, S., & Eccles, J. S. (2004). Expectancy value theory in cross-cultural prospective. In McInerney, D. M., & Etten, S. V. (Eds.), *Big theories revisited (*pp. 165-198). Greenwich, Connecticut: Information Age Publication.

EXPLORING THE PHENOMENON OF PEDAGOGICAL EMPATHY

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Interactions between teachers and students are an essential part of learning in educational settings. In this study, I explore the phenomenon of pedagogical empathy, which is conceptualized as empathy that influences teaching practices. Specifically, I examine how mathematics graduate teaching assistants (GTAs) might express pedagogical empathy when providing feedback to student questions. Data was collected through interviews of current pre-calculus instructors in which participants were shown samples of student work and asked to respond to questions about that work. Preliminary analysis has revealed varying abilities of GTAs to express and attend to student emotion.

Keywords: Affect, Emotions, Post-Secondary Education

Learning mathematics is an emotional experience for students (Hannula, 2002). Mathematics education researchers have studied student affect and teacher affect independently within the context of learning and teaching mathematics. However, little attention has been given to the relationship between teachers and student affect (Philipp, 2007). At the post-secondary level, both emotional reactions and interpersonal relationships have been shown to influence what is learned in the classroom (Lowman, 1994).

Graduate teaching assistants (GTAs) play a large role in the instruction of lower level undergraduate mathematics courses (Speer, Gutmann, & Murphy, 2005). As a result, GTAs have opportunities to interact with a variety of students on a day-to-day basis and develop interpersonal relationships with them. These interactions likely influence GTAs' identities as teachers and shape their teaching philosophies (Kung, 2010). However, little is known about GTAs' teaching experiences and only recently has the mathematics education community begun to study their development as teachers and potential future faculty members (Kung, 2010; Speer et al., 2005).

This study seeks to add to the growing body of information about mathematics GTAs' pedagogical beliefs and teaching practices by investigating the awareness of GTAs to student feelings using a qualitative research design. In doing this, I aim to develop a better understanding of a new theoretical construct, *pedagogical empathy*. Building upon a definition used by Tettegah and Anderson (2007) in their study of empathic dispositions of pre-service teachers, I define pedagogical empathy as "the ability to express concern and to take the perspective of a student" (p. 50) and *the influence that this ability has on teaching decisions*. With this in mind, the purpose of this study is to explore how GTAs might express pedagogical empathy when providing feedback to students. To examine this phenomenon, I pose the following three research questions:

1. Can the feedback that GTAs provide to student questions be characterized as attending to possible student emotions?
2. Given sample written work on a typical pre-calculus problem, what feelings might GTAs attribute to students?
3. How might GTAs take student feelings or emotions into account when answering student questions?

Theoretical Framing

The process of learning is complex and involves both cognitive and affective factors. In particular, emotions have an effect on student learning and teachers play a significant role in influencing those emotions (Mortiboys, 2012). To better understand the relationship between emotions and learning, Hannula (2002) developed a framework to analyze a student's attitude towards mathematics using the psychology of emotions as a foundation. This framework separates attitudes into four evaluative processes: 1) The emotions the student experiences during mathematics related activities; 2) the emotions that the student automatically associates with the concept 'mathematics'; 3) evaluations of situations that the student expects to follow as a consequence of doing mathematics; and 4) the value of mathematics-related goals in the student's global goal structure (Hannula, 2002, p. 26).

With respect to this study, I focus on the first part of this framework, which attends to the emotions that students experience while working on math problems. Whereas the framework was analyzed from the perspective of a student, I aim to explore how the framework might be viewed from a teacher's perspective and how the responses of a teacher might take into account the initial process of the framework when interacting with students. For this study, it is also important to distinguish between feelings and emotions. Hansen (2005) defines feelings as conscious perceptions used to describe emotions. Because feelings are perceivable and can be articulated by the individuals who experience them, I use this term for discussing student displays of emotion.

Methods

The participants in this study were 14 mathematics GTAs at a large Midwestern university. Each GTA had at least one semester of experience as the instructor of record for a pre-calculus class, and none were international GTAs. Data was collected during the 2016-2017 school year through structured interviews with the GTAs. Participants selected their own pseudonyms, which are used below. During the interview, participants were asked to solve a typical pre-calculus problem in order to familiarize them with the problem. After working through the problem, the participants were then shown five different samples of student work for the problem and asked to respond to questions about the work. The samples of student work were fictitious examples of actual student work based on the author's experiences as a pre-calculus instructor. Student questions about the work were presented through audio recordings intended to simulate an actual student asking the question. Examples of questions asked in the audio recordings include, "Am I doing this right?" and "How do you know what is x and what is y?" Finally, at the end of the interview the participants were asked to reflect on how they thought each student might have felt when working on the problem and if they would respond differently to any of the students after thinking about what they might be feeling. Participants were provided with a list of feeling words to use a reference during this part of the interview. After data was collected, select interviews were transcribed and analyzed using open coding.

Preliminary Findings

By conducting this study, I aimed to determine whether feedback to student questions could be characterized as attending to emotion. Through open coding, I found that many of the responses to interview questions focused on helping students develop either procedural or conceptual mathematical knowledge and did not attend to student emotion. However, several participants mentioned potential student feelings during the interviews, even before being prompted by the interviewer to describe how each student might have felt after viewing their work. In a few notable cases, GTAs were unable to articulate possible student feelings using descriptive feeling words. These responses were categorized under the code “non-feeling” and were common among only a few participants. In addition to the mathematical responses and emotional responses, GTAs also gave reflective responses during the interview. These reflective responses often involved taking the perspective of a student or drawing upon prior personal experiences, which may be an indicator of expressing pedagogical empathy. Table 1 shows the primary codes that emerged from the interviews along with representative excerpts from 4 different participants. These codes help to categorize GTA responses to student questions.

Table 1: Characteristics of GTA Responses

Code	Description	Interview Excerpt
Procedural Mathematical Knowledge	Responses involving procedures, algorithms, rules, or formulas that do not directly attend to conceptual mathematical ideas	*Nicole:* If I plug in 25,000 students, do I get out the 13,000?
Conceptual Mathematical Knowledge	Responses connected to underlying mathematical concepts or content including discussion of abstract ideas or relationships	*Floyd:* What sort of function are we trying to come up with here?...Think of the function as a machine.
Student Feelings or Emotions	Anything about what a student might be feeling or anything related to underlying emotions that students might be experiencing	*Paul:* It’s less that they don’t know the math and more sort of fear or being uncomfortable with story problems.
Non-feelings	Use of words that are unrelated to emotions to describe what a student might be feeling when specifically prompted	*Phillip:* They probably feel medium.
Student-centered Reflection	Evidence of reflection centered on student thinking or past experiences with students	*Paul:* Even though my students see me make mistakes, they still want my pat on the back.
Instructor-centered Reflection	Evidence of self-reflection that is centered on the participant rather than the students, including personal beliefs	*Phillip:* I feel like in that situation, I would feel stupid.

Further analysis is ongoing to identify common feelings that GTAs might attribute to particular students and whether some feelings were more prevalent than others. Several feelings were mentioned in the interviews including fear, nervousness, confusion, anger, annoyance, uncertainty, insecurity, frustration, anxiety, vulnerability and confidence. Some participants also made an effort to distinguish between different levels of student emotion. For example, Floyd tried to distinguish between vulnerability and insecurity when discussing how one student might have felt while working on the problem:

> Floyd: Maybe vulnerable is a good way to describe that. It’s not so much insecure as much as they’re afraid to make a mistake I think – like a different level of insecurity than just not being sure of what they’ve done.

From the preliminary analysis, it is evident that the GTAs who participated in this study were aware of the potential for student feelings to arise when working on a math problem, but varied in their abilities to express those feelings. For example, James found it difficult to attribute emotions to students:

James: I guess I have a hard time ascribing emotion to people as they're working on math problems. That's not something I really consider too much.

Other GTAs, however, were able to articulate student feelings and discussed how taking student feelings into account was something that they would usually do when responding to student questions. In response to the question, "Would you respond differently to any of the students after thinking about what they might be feeling?" both Nicole and Aubrey replied similarly:

Nicole: I don't think so. I mean I think that's something that I do think about when a student asks me a question, like where they are in terms of not only like mathematically, but also like emotionally.

Aubrey: Probably not. I try to think on the spot about how they're feeling and look at people's faces...You get to know your students, and I try to pay attention to how they're feeling.

Discussion

The feedback that GTAs provided in response to student questions not only addressed the mathematical content required to solve the problem, but also attended to potential student feelings. In addition, a variety of possible feelings were brought up organically by several GTAs throughout the interviews. This observation draws attention to the fact that GTAs are trying to account for student emotion when helping students work through problems. However, further analysis is still needed to explore how pedagogical empathy might be manifested in teaching practices like providing feedback to students and how different experiences might help the development of pedagogical empathy in mathematics teachers.

References

Hannula, M.S. (2002). Attitude towards mathematics: Emotions expectations and values. Educational Studies in Mathematics, 49(1), 25-46.

Hansen, F. (2005) Distinguishing between feelings and emotions in understanding communication effects. Journal of Business Research, 58(10), 1426-1436.

Kung, D. (2010). Teaching assistants learning how students think. Research in collegiate mathematics education VII, 7, 143.

Lowman, J. (1994). What constitutes masterful teaching. In K.A. Feldman & M.B. Paulsen (Eds.), Teaching and learning in the college classroom. (pp. 213-225). Needham Heights, MD: Simon & Schuster.

Mortiboys, A. (2012). Teaching with emotional intelligence: A step-by-step guide for higher and further education professionals. Routledge.

Philipp, R.A. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester (Ed.). Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics. (pp. 319-369). Charlotte, NC: Information Age Pub.

Speer, N., Gutmann, T., & Murphy, T. J. (2005). Mathematics teaching assistant preparation and development. College Teaching, 53(2), 75-80.

Tettegah, S., & Anderson, C. J. (2007). Pre-service teachers' empathy and cognitions: Statistical analysis of text data by graphical models. Contemporary Educational Psychology, 32(1), 48-82.

SELF-EFFICACY BELIEFS OF ADULTS WHO DISLIKE MATH: CHALLENGING HOW WE MEASURE THE SOURCES

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According to Bandura (1977, 1997), four sources of information inform people's self-efficacy beliefs: mastery experiences, vicarious experiences, verbal persuasion, and physiological and emotional states. To date, most research on the sources of mathematics self-efficacy has been quantitative in nature. The purpose of this study was to qualitatively examine the sources of self-efficacy for a small group of young adults who dislike mathematics to uncover the more covert aspects of the sources of their mathematics self-efficacy. In this paper, I share evidence from this study that reveal flaws in the methodology used in prominent quantitative studies examining the sources of mathematics self-efficacy.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Mathematics Education; Self-efficacy

In the late 1970s, Bandura (1977) introduced a framework for understanding how people's self-beliefs affect how they act in given situations, especially when the circumstances are challenging. Bandura (1977, 1997) defined one such self-belief, *self-efficacy*, as the belief one has about his or her own capabilities to organize and execute the courses of action needed to attain certain goals. Since the 1970s, self-efficacy has received considerable attention in educational research and has been shown to affect students' level of achievement, academic motivation, persistence, competence, and feelings in various domains (e.g., Hackett & Betz, 1989; Pajares & Valiante, 1997).

With such striking correlations, it would be important to better understand the sources, or the factors that influence self-efficacy beliefs. Bandura (1977, 1997) hypothesized people form self-efficacy beliefs based on four sources of information, the most influential of which is one's interpretations of their performance outcomes, or *mastery experiences*. Also influential are *vicarious experiences,* where one relies on social comparison with and outcomes achieved by a model as a source of information about his or her capabilities (Bandura, 1997). *Verbal persuasion* can also influence self-efficacy beliefs, and is hypothesized to have a more limited effect on self-efficacy than the previous two, as outcomes are only described, not directly encountered (Bandura, 1997). Lastly, fatigue, anxiety, and other *physiological and emotional states* are often interpreted as indicators of capability (Bandura, 1997).

Much of the research on the sources of mathematics self-efficacy has been quantitative and has focused on secondary and college students. Here I attend to some inherent problems in measurements used by three particular oft-cited quantitative studies (Klassen, 2004; Lent, Lopez, & Bieschke, 1991; Matsui, Matsui, & Ohnishi, 1990). To measure mastery experiences, Klassen (2004) and Matsui and colleagues (1990) used only students' self-reports of their mathematics course grades from the previous year. Bandura (1997) postulated that it is not one's objective outcomes that count as mastery experiences, but his or her *interpretation* of those experiences that matter. For instance, a grade of a B may encourage some, leave some unaffected, or devastate others (Usher & Pajares, 2008). For vicarious experiences, Klassen's (2004) survey items in this category focused only on how the seventh-grade students compared themselves to their peers while others (Lent et al., 1991; Matsui et al., 1990) had their questions include teachers and parents along with peers. All three studies failed to explore whether other models such as non-immediate family members, experts in the field, or others had any impact on the participants, or which experiences were most meaningful to the students.

Verbal persuasion had greater correlation with students' mathematics self-efficacy levels in the three mentioned studies, but all three seemed to simplify encouragement or discouragement as positive and negative persuasion. This simplification eliminated the meaning (or consequences) those statements may have had for the recipients of them. For the last source, emotional states, Matsui and colleagues (1990) had students mark a level of agreement with the phrase "I really hated math." What is not clear, however, is whether this feeling about mathematics was an *effect* of the students' self-efficacy or a cause.

The quantitative instruments outlined above failed to investigate the experiences that *develop* an individual's self-efficacy beliefs. Qualitative instruments, however, such as a semi-structured interview, could allow participants to provide details about those experiences that they feel have been most influential on their beliefs over time (Usher & Pajares, 2008). Hence, in this paper, I report the results of a study where I explored the experiences of young adults who reported disliking mathematics to uncover the more covert aspects of the sources of their mathematics self-efficacy. For the purposes of this report, I analyzed data from semi-structured interviews intended to address the following research question: *How is the genesis of the young adults' dislike for mathematics explained by the four sources of self-efficacy proposed by Bandura?*

Methods

Participants (Elissa, Julia, Kayla, and Kyle – pseudonyms) were four young adults (ages 22-33), all of whom had attended public schools, completed at least three years of post-secondary education, and reported disliking mathematics. Three participants were female and in their post-college careers whereas the male was in his fourth year at a large Midwestern University.

Semi-structured interviews were conducted in Spring 2017 to explore the reasons people give for disliking mathematics and how they might or might not fit within Bandura's (1997) sources. Questions about the sources of self-efficacy were modified from those used by Zeldin and colleagues (2008) in their study of the self-efficacy beliefs of men and women who were successful in STEM fields. Each participant was also asked to self-report their level of self-efficacy with the following question: *On a scale of 1-10, how would you rate your confidence in doing mathematical tasks?* Interviews were audiorecorded and then transcribed. I began with open coding to look for factors mentioned as affecting their feelings of capability or confidence in doing mathematics. Surprisingly, all factors fit well within Bandura's four sources, which I had used as my initial categories (*Mastery Experiences, Vicarious Experiences, Verbal Persuasion, Emotional States*). I then created subcategories within each source, distinguishing between instances that seemed to be positive influences on their mathematics self-efficacy or feelings as well as instances that were negative influences on it. I then created tables to keep track of both the number of instances within each source for each participant and brief summaries of the instances described by the participants.

Results

Here I will focus on the results of my qualitative analysis that differ in interesting ways from prior mathematics self-efficacy research.

Mastery Experiences

Of the 20 responses coded *mastery experiences*, 13 were negatively influential, with Julia and Kyle having the most instances (n=4). Julia reported receiving A's throughout elementary and high school, yet reported a low self-efficacy level. She even said, "I think that even if I got a lower grade but still had better understanding of it, a better grasp, I think that I would feel more successful.” Julia, Elissa, and Kyle all noted grades as indicators of their own failure in mathematics. A grade of a C in Geometry for Julia, a B+ in Geometry and B- in PreCalculus for Elissa, and a C in high-school Calculus and D in college Calculus II for Kyle evoked the words “lack of understanding”, “not good”, and “failure” by the three participants, respectively.

Vicarious Experiences

Of the 18 responses coded *vicarious experiences*, 12 were negative instances, with Julia and Elissa each having the most (n=4). All four participants mentioned falling behind peers they saw as “equals” as detrimental to their confidence. Kayla, Julia, and Elissa each mentioned negative mathematics teacher models. Kayla had a mathematics teacher who taught from a piece of paper and struggled to answer her questions if they were not already on his notes. Julia’s Geometry teacher showed a willingness to help certain students, but not her. Elissa’s PreCalculus teacher was described as “convoluted” in how she taught and presented mathematics.

Verbal Persuasion

Of the 16 replies coded *verbal persuasion*, 11 were negatively influential, with Kayla and Elissa having the most negative instances (n=4). Julia once overheard her parents saying that she was not good at math and that she was “not a math person”, while her high-school guidance counselor discouraged her from taking any more mathematics than the minimum requirement. Kayla’s mom affirmed her lack of confidence by saying things like, “we don’t understand this” or “we just don’t do this” while Kayla struggled with her mathematics homework. Kayla indicated her friends’ attempts to boost her confidence verbally would in turn lower her confidence by “putting her on the spot.” Kayla had a teacher tell her that due to her success in the statistics portion of a class he taught, she should take AP Statistics the following year with him. She did not perform well in AP Statistics, however, and indicated that her failure to live up to his expectations was devastating to her remaining confidence in her mathematical capability. Kyle overheard his suite mate telling someone that Kyle was “just getting through” Calculus and told Kyle directly, “You might not do too hot but you might be able to pass.” Elissa’s PreCalculus teacher would tell the class “how bad they were.”

Emotional States

Of the 16 replies coded *emotional states*, 12 were negatively impactful, with Kayla (n=6) having the most. In elementary school, Kayla would panic, not knowing why she was being a diligent note-taker and hard worker yet not understanding the concepts. She surmised that some elementary concepts “would probably have made sense if I wasn’t in a tailspin of anxiety already, if I wasn’t sitting there crying and hyperventilating.” She mentioned shutting down when put on the spot (as mentioned previously). She even mentioned feeling anxiety when I asked her in the interview about her confidence to perform a mathematical task.

Julia used the word “stressful” to describe her experiences – including elementary math race games where she was slower than her peers and finishing her tests last. She also mentioned feeling shame in her high-school Geometry course when she realized she was cheating to pass the class because she did not understand the mathematics. Elissa mentioned her experience in PreCalculus was boring and gave her a lot of anxiety. She wanted to give up because it would be so “frustrating”. Kyle’s lone occurrence focused on his frustration when he would get stuck on mathematical tasks while his peers (especially those about whom he felt smarter) solved them.

Discussion and Concluding Remarks

Taken together, the analyses of participants’ responses regarding the sources of mathematics self-efficacy suggest that a qualitative approach (whether semi-structured interviews or other methods) afforded participants an opportunity to author *how* their experiences informed their efficacy beliefs, and how those sources were interpreted by the participants was not so predictable. For instance, positive verbal encouragement from Kayla's teacher only temporarily increased her confidence, and later debilitated it when she could not succeed in her AP Statistics course. Grades were not indicators of mastery for Julia. She indicated understanding as more important than the grades she received. Elissa saw B's as indicators of failure. Teacher models, whether by making it seem like there's only "one correct way" to "do mathematics" or by not being able to relate to or understand where some students might be in their understandings, negatively impacted participants' efficacy.

This study also differed from prior research (e.g. Usher, 2009) in that students with high achievement did not necessarily exhibit high efficacy. Julia, Elissa, and Kyle all reported successful mathematics experiences (high grades, advanced placement), but Julia had low self-efficacy whereas Elissa and Kyle had moderately high self-efficacy to complete a mathematical task. Also, there seemed to be different kinds of mastery evident in the results of this study. Participants discussed mastery as decided by others (e.g. grades, placement into advanced mathematics classes or not) and mastery perceived by one's self (e.g. Julia's self-perceived lack of understanding). Prior research only using one's grade as an indication of mastery (or not) fail to give evidence to the different types of mastery that can be influential.

Another key difference from prior studies is the prevalence of and types of emotional states in my participants' experiences. Many of the quantitative studies discussed a "lack of significance" when looking at the correlation between participants' emotional states and level of self-efficacy. Perhaps that is due to an over-simplification of emotional and physiological states to only feelings of illness, anxiety, or stress while taking exams. This study revealed there are other activities (e.g. math races, doing math homework, calculating tips) and other emotions (e.g. boredom, shame) that can bring about emotional and physiological reactions. Although this was a small-scale study with only four participants, their recollections of their past mathematical experiences point to the shortcomings of using forced-choice and Likert-scale indices to measure the sources of mathematics self-efficacy.

References

Bandura, A. (1977). Self-efficacy: toward a unifying theory of behavioral change. *Psychological review*, *84*, 191.

Bandura, A. (1997). Self-efficacy: The exercise of control. New York: Freeman

Hackett, G., & Betz, N. E. (1989). An exploration of the mathematics self-efficacy/mathematics performance correspondence. *Journal for research in Mathematics Education*, 261-273.

Klassen, R. (2004). A cross-cultural investigation of the efficacy beliefs of South Asian immigrant and Anglo non-immigrant early adolescents. *Journal of Educational Psychology*, *96*, 731-742.

Lent, R. W., Lopez, F. G., & Bieschke, K. J. (1991). Mathematics self-efficacy: Sources and relation to science-based career choice. *Journal of Counseling Psychology*, *38*, 424.

Matsui, T., Matsui, K., & Ohnishi, R. (1990). Mechanisms underlying math self-efficacy learning of college students. *Journal of Vocational Behavior, 37*, 223-238.

Pajares, F., & Valiante, G. (1997). Influence of self-efficacy on elementary students' writing. *The Journal of Educational Research, 90*, 353-360.

Usher, E. L. (2009). Sources of middle school students' self-efficacy in mathematics: A qualitative investigation. *American Educational Research Journal*, *46*, 275-314.

Usher, E. L., & Pajares, F. (2008). Sources of self-efficacy in school: Critical review of the literature and future directions. *Review of educational research*, *78*, 751-796.

Zeldin, A. L., Britner, S. L., & Pajares, F. (2008). A comparative study of the self-efficacy beliefs of successful men and women in mathematics, science, and technology careers. *Journal of Research in Science Teaching*, *45*, 1036-1058.

STUDENTS' FRAMING OF MATHEMATICAL TASKS IN CLINICAL INTERVIEWS

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Keywords: Cognition, Affect, Emotion, Beliefs, and Attitudes, Research Methods

Clinical interviews are used in mathematics education research to discover cognitive processes, identify cognitive activities, and evaluate levels of competence. Yet, interviewees' understanding of the interview impacts both how they participate and subsequently the claims we can make about their cognitive processes (Russ, Lee, & Sherin, 2012). *Framing* has been used in science education research to attend to how students understand tasks, and how that understanding impacts their performance on the task and their interactions with others. Russ, Lee and Sherin (2012) posit that students will interpret clinical interviews in "ways that will affect and be evident in how they respond and behave in the interaction" (p. 12). Thus, researchers can infer students' framing of the situation from their verbal, nonverbal, and paraverbal behaviors (Russ, Lee, & Sherin, 2012).

This study describes the framing employed by two college-Algebra students during clinical interviews. The clinical interviews analyzed were originally conducted as part of an investigation of college-level Algebra students' concept images of parabolas. The interview protocol was intended to stimulate different representations and understandings of parabolas and cue participants to move between representations. Interviews were analyzed using the three phase analysis described in Russ, Lee and Sherin (2012). Students' behaviors were compared to the three framings identified in Russ, Lee and Sherin (2012): the inquiry frame, oral examination frame, and expert interview frame.

Preliminary results suggest both participants framed the interviews as oral examinations. Unlike the students in Russ, Lee and Sherin (2012), who moved between lenses depending on the task and interviewer cues, participants in this study maintained their lenses across tasks. One possibility for the pervasion of the oral examination lens is the nature of the mathematical tasks, which may be too similar to what students experienced in their coursework (not novel) and thus they frame the tasks as an assessment of their understanding. Interestingly, when the interview shifts from mathematical tasks to questions about their experiences as mathematics learners, the college students shift to the expert interview frame. Changes in framing are indicated by the reduction in hedging language and laughter, and increased volume and confidence of tone when responding to questions. The shift in frame from oral examination to expert interview could indicate that students do not see themselves as owners of mathematical knowledge. Future analysis seeks to consider the potential influences on participants' framing, beyond interviewer cues and the specific tasks. For example, does the discipline of school mathematics constrain or facilitate how students' frame tasks, both in research settings and in classrooms?

References

Russ, R. S., Lee, V. R., & Sherin, B. L. (2012). Framing in cognitive clinical interviews about intuitive science knowledge: Dynamic student understandings of the discourse interaction. Science Education, 96(4), 537-599.

DEVELOPMENTAL DIFFERENCES IN LEARNING MATH BY EXAMPLE IN ELEMENTARY SCHOOL

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Empirical evidence has amassed supporting the effectiveness of having students explain why correct problem solutions are correct (e.g., Hilbert, Renkl, Kessler, & Reiss, 2008) as well as why incorrect problem solutions are incorrect (e.g., Durkin & Rittle-Johnson, 2012), particularly for students in middle school and above (e.g., Adams et al., 2014). However, research is only beginning to investigate how this approach impacts learning in elementary school mathematics classroom and whether effects are age-dependent. The purpose of the present study was to compare the effectiveness of worked-example worksheets in fourth vs. fifth grades.

We worked closely with elementary school teachers and mathematics coaches to construct worked-example assignments for fourth and fifth grade Common Core Content Standards. We collected data in elementary classrooms (14 5th grade, N = 300 students; 12 4th grade, N = 257 students) where students took a paper- and-pencil pretest and posttest surrounding their unit. The test contained two types of questions: isomorphic procedural items (e.g., procedural items that had the same structure and difficulty level as those trained) and items that measured conceptual understanding of the content. During the unit, teachers instructed as usual, but replaced student practice activities with study assignments where appropriate; experimental group classrooms used the worked-example assignments, while control classrooms used assignments that contained identical problems but did not include worked-examples or self-explanation prompts.

Quantitative analyses were conducted to determine whether posttest scores (controlling for pretest performance) differed for students in the experimental vs. control condition as well as to compare whether the impact of condition varied with grade level. Results indicate that while the worked-example assignments did have benefit for both fourth and fifth grade students, the pattern of results differed by grade. Fifth graders' benefit manifested mostly in gains on conceptual items, which is consistent with extant findings for middle and high-school students. In contrast, fourth graders' benefit was greater on the procedural items, which is more consistent with findings for older students when they study, but don't explain, worked examples. Grade level differences in how students approached and completed their assignments—and how those process data help illuminate findings on differential benefits—will also be discussed.

References

Adams, D.M, McLaren, B.M., Durkin, K., Mayer, R.E., Rittle-Johnson, B., Isotani, S., & van Velsen, M. (2014). Using erroneous examples to improve mathematics learning with a web-based tutoring system. *Computers in Human Behavior, 36*, 401-411.

Durkin, K., & Rittle-Johnson, B. (2012). The Effectiveness of Using Incorrect Examples to Support Learning about Decimal Magnitude. *Learning and Instruction, 22*(3), 206-214.

Hilbert, T. S., Renkl, A., Kessler, S., & Reiss, K. (2008). Learning to prove in geometry: Learning from heuristic examples and how it can be supported. *Learning and Instruction*, *18*(1), 54-65.

Siegler, R. S. (2002). Microgenetic studies of self-explanations. In N. Granott & J. Parziale (Eds.), Microdevelopment: Transition processes in development and learning (pp. 31-58). New York: Cambridge University.

NEGOTIATING FRAMES IN AN INFORMAL MATH SPACE

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According to dominant cultural narratives in the United States—including those instantiated in testing mandates—mathematics is a highly valued discipline, and mathematical competence is reserved for a select few. Despite this rhetoric about mathematical aptitude, research in the sociocultural tradition (Vygotsky, 1978) has shown that these notions of competence are inherently bound to cultural practices, and systematically exclude students from marginalized communities. Furthermore, work to correct this systemic imbalance has taken many forms. Funds of Knowledge research (e.g. Civil, 2007) identifies and draws on specific competencies of students' home communities in order to promote in-school academic achievement, while work on Third Space (Gutiérrez, 2008) promotes a more even-handed synthesis of home- and school-based practices. Relatedly, work using constructs such as Figured Worlds (Holland, 2001) has begun to specify some of the mechanisms by which students negotiate roles and responsibilities in the classroom, as well as the social power that goes along with them (e.g. Esmonde & Langer-Osuna, 2013).

Despite their different approaches and arguably different goals, each of these strands of research makes contact with a fundamental question: What happens when two or more cultural modes come into contact and must be negotiated in situ by participants? In order to begin to address this question more explicitly, this analysis draws on Positioning Theory (van Langenhove & Harré, 1999) and Framing (Goffman, 1974) to address a more circumscribed corrolary: How do students in an informal learning space negotiate answers to the question "What is going on here?" In their work to refine Positioning Theory as it applies to mathematics education research, Herbel-Eisenmann et al (2015) specifically call for attention to timescales in discussions of the storylines that students recruit, as well as more explicit empirical grounding for claims. Drawing on interactional analysis of a single pair of students within a single week of data collection at a math-focused knitting camp, initial findings suggest that 1) specific tools and constraints tend to re-Frame an interaction as being "school math", and 2) student-initiated questions are more likely to subsume mathematics instrumentally into the current Frame.

References

Civil, M. (2007). Building on community knowledge: An avenue to equity in mathematics educa- tion. In N. S. Nasir & P. Cobb (Eds.), Improving access to mathematics: Diversity and equity in the classroom (pp. 105–117). New York, NY: Teachers College Press.

Esmonde, I., & Langer-Osuna, J. M. (2013). Power in numbers: Student participation in mathematical discussions in heterogeneous spaces. Journal for Research in Mathematics Education, 44(1), 288-315.

Goffman, E. (1974). *Frame analysis: An essay on the organization of experience*. Cambridge, MA, US: Harvard University Press.

Gutiérrez, K. D. (2008). Developing a sociocritical literacy in the third space. Reading research quarterly, 43(2), 148-164.

Harré, R., & van Langenhove, L. (Eds.). (1999a). Positioning theory: Moral contexts of intentional action. Oxford, UK: Blackwell Publishers.

Herbel-Eisenmann, B. A., Wagner, D., Johnson, K. R., Suh, H., & Figueras, H. (2015). Positioning in mathematics education: Revelations on an imported theory. Educational Studies in Mathematics, 89(2), 185-204.

Holland, D. (2001). Identity and agency in cultural worlds. Harvard University Press.

Vygotsky, L. S. (1978). Mind in society. Cambridge, MA: Harvard University Press.

MATH MATTERS: FIGURING MATHEMATICS LEARNING SPACES AROUND IMPORTANCE, UTILITY, AND SMARTNESS

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Perceived mathematics ability is a valuable commodity. Those perceived to possess it are more positively positioned in mathematics learning spaces than those who do not which has implications for equity in access to mathematical ideas and participation. We know from research that perceived smartness has consequences for teacher expectations and academic identity development, among other things (Hatt, 2012). Using figured worlds (Holland, Lachicotte, Skinner, & Cain, 1998), we examine the as-if world of two groups of elementary grade students to understand their notions of smartness, mathematics utility, and mathematics importance. To further explicate the collective mathematics identity at play in this figured world, we utilize the first two components of Martin's (2000) framework of mathematics identity: beliefs about their ability and beliefs about its instrumental importance. Not much is known about elementary age student beliefs about ability or smartness in mathematics or the utility of mathematics. In this study, we employ stanzas to analyze the presence and deployment in these mathematics learning spaces of smartness and utility through cultural artifacts, actors in the space, and valued outcomes (Holland et al., 1998). We ask: How do elementary grade students figure their mathematics learning spaces in ways that demonstrate their beliefs about smartness, utility, importance, and demonstrations of mathematical competence?

Several focus groups of approximately five to six students were conducted across two urban elementary school classrooms, one second grade and one fourth grade, as part of a larger, ongoing research project. The focus group interviews contained three segments: 1) students drew diagrams of their happy places, 2) students used improvisation to act out what happens in their math classroom, and 3) students answered questions about their mathematics classroom learning. We used theoretical framing related to situated meanings, social languages, figured worlds, and Discourses (Gee, 2014; Rogers, 2011), coupled with a critical approach to discourse analysis of the focus group transcripts for our investigation. Preliminary findings reveal that teacher actions, various student behaviors, and summative assessment grades play a major role in students meaning-making about mathematical competence. These young mathematics learners overwhelmingly expressed the importance of mathematics, yet this emphasis on importance was restricted to school mathematics learning and practical uses such as counting money or use in future occupations.

References

Gee, J. P. (2014). *How to do discourse analysis: A toolkit*. London: Routledge.

Hatt, B. (2012). Smartness as a cultural practice in schools. *American Educational Research Journal*, 49(3), 438-460.

Holland, D. C., Lachicotte, W., Skinner, D., & Cain, C. (1998). *Identity and agency in cultural worlds*. Cambridge, MA: Harvard University Press.

Martin, D. B. (2000). *Mathematics success and failure among African-American youth: The roles of sociohistorical context, community forces, school influence, and individual agency*. Mahwah, NJ: Routledge.

Rogers, R. (2011). Critical approaches to discourse analysis in educational research. In R. Rogers (Ed.) *An introduction to critical discourse analysis in education* (1-20). New York: Routledge.

STEM RETENTION FOR DEVELOPMENTAL MATHEMATICS STUDENTS: AFFECTIVE TRAITS AND A PEER-MENTORING INTERVENTION

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Students beginning their mathematical studies prior to calculus must master several areas of mathematics in order to reach the level expected for a STEM degree. Yet, nationally, 42% of students enter college at a mathematical level below what is needed for science, technology, engineering and mathematics (STEM) degrees (Radford, Pearson, Ho, Chambers & Ferlazzo, 2012). From 2015-2017, we studied the success and persistence of college students not yet ready for college-level mathematics. We specifically examine mathematics anxiety of students in a pre-college algebra course and the effects of a peer-mentoring program on their success.

The potential for experiencing mathematics anxiety that will impact performance is high (Ashcraft & Krause, 2007; Hembree, 1990) and can create a barrier to entry into STEM disciplines. To examine how affective states such as anxiety impact student success and persistence, we collected survey data from 404 students and compared responses to outcomes and to persistence in their majors. We found that anxiety impacts student success in measureable ways. A one-unit increase in anxiety on our scale (Alexander & Martray, 1989) showed a 1.30% decrease in success likelihood (p=.0003), and a greater decrease in female students (2.08%, p<.0001). Additionally, we found the impact to be more significant for non-STEM intending students (1.52% decrease per unit, p=.0004) than for STEM intending (0.70% with p=.324).

To further support these students, we implemented a peer-mentoring program in the second year of this study. We found the program supported success in STEM coursework but that its impact on STEM intending students was mixed. In a one-semester implementation with N=54 students completing the mentoring process, we observed success rates in the course of 83.3% for the mentored group compared to 75% for the non-mentored population. On the other hand, we found that STEM majors in the mentored group persisted to the subsequent term at a rate of 68.8% while in the non-mentored group 80.4% continued.

Our initial conclusions indicated that as previously observed at other levels of mathematics, anxiety plays an inhibitory role in student performance, and that its impact is more pronounced in our female population and in non-STEM intending students. We find that peer-mentoring supports student success, but that its impact on STEM persistence was not as clear, suggesting that we should more pointedly target these students in the mentoring work.

References

Alexander, L., & Martray, C. (1989). The Development of an Abbreviated Version of the Mathematics Anxiety Rating Scale. *Measurement And Evaluation In Counseling And Development*, 22(3), 143-50.

Ashcraft, M. H., & Krause, J. A. (2007). Working memory, math performance, and math anxiety. Psychonomic Bulletin & Review, 14(2), 243-248.

Hembree, R. (1990). The nature, effects and relief of mathematics anxiety. Journal for Research in Mathematics Education, 21(1), 33-46.

Radford, A.W., Pearson, J., Ho, P., Chambers, E. & Ferlazzo, D. (2012). Remedial Coursework in Postsecondary Education: The Students, Their Outcomes, and Strategies for Improvement. Jefferson City, MO: Missouri Department of Higher Education

STUDENT DISENGAGEMENT IN CALCULUS: IMPLICATIONS FOR MEASUREMENT AND QUANTITATIVE RESEARCH

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Substantial research has shown that student engagement and disengagement with academic work predict student learning outcomes (Fredricks, Blumenfeld, & Paris, 2004; Skinner, Furrer, Marchand, & Kindermann, 2008). However, less attention has been paid to the ways in which students disengage and to the reasons for their disengagement. Nevertheless, understanding that may help advance the knowledge of quantitative relationships. Thus, the following question guided this exploratory study: How and why do students disengage in a Calculus classroom?

Data for this study were collected via semi-structured interviews. Participants were five demographically diverse students enrolled in the same section of the first calculus course in a three-course sequence. This section was chosen because it was taught via multiple instructional methods. Specifically, instruction included both lecturing and group work, thus allowing us to interview students about their engagement during instructionally different parts of the class.

The analysis revealed three types of student disengagement. The first type is disengagement that may be potentially detrimental for student success in the course. It can be selective when students decide to disengage in response to a particular activity (e.g., a student may stop listening to the instructor if explanations are perceived to be unnecessary long or complex). Potentially detrimental disengagement can also be non-selective when a decision to disengage does not have a specific trigger (e.g., a student may stop listening because s/he got tired or distracted). The second type is alternative engagement, which may or may not have negative implications for student success in the course. This type occurs when a student disengages from one activity but engages in a different (yet relevant) activity instead. For example, a student may decide not to answer instructor's questions out loud but, alternatively, answer them in his/her head. The third type is disengagement that may be potentially non-detrimental for student success in the course. This type can occur when engagement is perceived to be unnecessary (e.g., when a student already knows the material), unproductive (e.g., when listening leads to confusion), or not cost-effective (e.g., when asking a question outside of class is easier). Potentially non-detrimental disengagement can also occur when disengagement is perceived to be unavoidable (e.g., not working on a task because of being stuck).

The presence of these disengagement types suggests that the relationship between students' engagement and their success in the course may differ depending on how and why students disengage. In other words, the types of student disengagement may moderate this relationship. Thus, exploring it without considering the types of disengagement may not provide a full picture.

Acknowledgements

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References

Fredricks, J. A., Blumenfeld, P. C., & Paris, A. H. (2004). School engagement: Potential of the concept, state of the evidence. *Review of Educational Research*, *74*(1), 59–109.

Skinner, E., Furrer, C., Marchand, G., & Kindermann, T. (2008). Engagement and disaffection in the classroom: Part of a larger motivational dynamic? *Journal of Educational Psychology*, *100*(4), 765–781.

INFLUENCING FACTORS THAT APPEAR IN THE TRANSITION FROM MONTESSORI MATHEMATICS TO TRADITIONAL METHODS

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The purpose of this study is to investigate three particular aspects that could affect the transition between a third grade Montessori style and a fourth grade non-Montessori style classroom, specifically within the context of teaching and learning mathematics. These aspects are (a) the change in pacing and structure of the classroom, (b) the removal of manipulatives from the learning experience in favor of handwriting methods, and (c) the reversal of roles that teachers and students occupy. While post-transition effects are well documented, research is limited in analyzing learning during the transitional phase *out* of Montessori programs. The Montessori method relies on self-regulated young learners taking advantage of themselves, their environment, and their experiences—therefore metacognition is used as the theoretical framework for the research. Commonly found in Montessori classrooms, manipulatives (tangible objects used to "manipulate" and physically represent mathematical situations) are often used in three main stages: concrete, representational, and abstract—a gradual shift that maintains student understanding. This qualitative study is guided by the following research questions:

1. To what teaching practices and learning opportunities are third and fourth grade students exposed? To what extent are these practices and learning opportunities related to the Montessori approach?
2. How are three particular aspects of current teaching practices and learning opportunities in fourth grade mathematics perceived by students and teachers compared to previous exposure in the Montessori style?
3. How, and to what extent, does changing these teaching practices and learning opportunities affect the problem-solving strategies of students?

The effect of this transition on problem-solving skills is analyzed through a series of Cognitively Guided Instruction problem-solving exercises to determine mathematical understanding about key concepts within the curriculum; other tools are modified for observations and interviews related to curriculum exposure lined up to theory. Results show that students identify alternative strategies when uncertain how to proceed in a problem. Students revert to previous object-centered methods when a problem is perceived as too difficult. Documents and dialogue show that students use the concept of manipulatives in unfamiliar situations, even if they do not recognize the process. Students also need more exposure with materials for difficult and alternative topics during the Montessori ages. The use of manipulatives is one of the most influential aspects of the transition, followed by the shift in defining student and teacher roles. The pacing and structure of the classroom has minimal effect on the transition.

POWERFUL KNOWLEDGE: LANGUAGES AND IDENTITY IN CHILDREN'S MATHEMATICS LEARNING THROUGH FAMILY WORKSHOPS

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Educators and researchers concerned with equity in mathematics education have argued the need to view mathematics and schooling through parents' eyes, and in particular, to consider Latino and other immigrant parents as intellectual resources, recognizing their knowledge and experience as important resources in mathematics teaching and learning (Civil, 2007). The goal of this study was to consider what we learned from a series of family workshops in light of the following research question: How can we create diverse spaces for family engagement that support mathematics learning and draw on the wide range of knowledge and experiences that immigrant families bring to their young children's education?

The family workshops took place over a semester and were designed for Latino families with children in elementary school; six families participated regularly (6 adults, 11 children). Instruction was provided in Spanish and focused on mathematics and language acquisition. Researchers and volunteers recorded data during and after the workshops through note taking; photographs of activities and completed work were also collected. We drew on hermeneutics to understand and analyze the data. We worked together to identify significant events and then to interpret and understand them. A hermeneutic perspective allowed us to look closely at the data and to disrupt it through purposeful interpretation, questioning assumptions about the knowledge and languages Latino families bring to their children's mathematics learning.

In the family sessions, we created a space where parents and children worked together on enjoyable mathematical activities that were new and unfamiliar and that required them to draw on previous experiences. The activities were constructed so that parents and children were positioned as experts, teaching one another. For instance, during a game board construction activity parents relied on the knowledge their children had gained at school while children relied on their parents to implement complex rules and mathematical concepts in their games in both Spanish and English. As parents and children were positioned as experts, they created a space in which they were reflected in each other but also saw each other as a new reality to be learned. This process began with the understanding that constructing diverse mathematics learning spaces that support and draw on a wide range of knowledge starts with seeing families and children as people who have rich knowledge and experiences. When conceptualizing the workshops, we also considered education as an issue of equity and identity. We positioned Spanish as the dominant language in the learning space, allowing parents and children to draw on different knowledge and experiences. This challenged everyone in the workshops to explore tensions inherent to speaking Spanish in the U.S., as well as to make experiences with languages powerful knowledge in the context of mathematics learning.

References

Civil, M. (2007). Building on community knowledge: An avenue to equity in mathematics education. In N. Nassir & P. Cobb (Eds.), *Diversity, equity and access to mathematical ideas* (pp. 105-117). New York, NY: Teachers College Press.

EXPLORING THE RELATIONSHIP BETWEEN PEDAGOGICAL APPROACHES AND STUDENTS' MATHEMATICS IDENTITY DEVELOPMENT

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Introductory college calculus has been shown to be gatekeeper course for STEM majors (Suresh 2006), and research continues to show that Calculus I "lowers students' confidence, enjoyment of mathematics, and desire to continue in a field that requires further mathematics" (Bressoud 2015). One possible reason for this continued problem is a lack of research pertaining to the relationship between instructional practices and students' mathematics identity development. Cobb (2004) calls for more research on how students are developing identities in their mathematics classroom since, "mathematics as it is realized in the classroom appears to function as a powerful filter in terms of identity" (pg. 333). This is important since students' mathematics identity strongly predicts their career choice in a STEM field (Cribbs 2012).

This study will report the qualitative piece of a larger mixed methods study aimed at comparing students' mathematics identity in a traditional lecture and an active learning calculus classroom. Twelve semi-structured interviews were conducted to explore students' perceptions of the pedagogy used in their introductory calculus class. We will use the interpretive scheme developed by Cobb and Hodge (2011) to explore the normative identity established by the pedagogical methods as well as students' personal identities which are "concerned with who students are becoming in particular mathematics classrooms" (pg. 190).

Another theoretical framework that will be used to explain students' personal identities is self-determination theory (SDT). Deci and Ryan (2000) state, "human beings can be proactive and engaged or, alternatively, passive and alienated, largely as a function of the social conditions in which they develop and function" (pg. 68). They stress the importance of doing research on the design of social environments (such as a classroom) in order to determine the best conditions for optimizing "people's development, performance, and well-being" (pg. 68). Deci and Ryan identified three needs that are essential in fostering people's development: competence, autonomy, and relatedness. Preliminary analysis revealed that the active learning environment provided more opportunities that promoted students' feelings of competence, autonomy, and relatedness than students in the traditional lecture classroom described.

References

Bressoud, D. M., Mesa, V., & Rasmussen, C. L. (2015). Insights and recommendations from the MAA National Study of College Calculus. MAA Press.

Cobb, P. (2004). Mathematics, literacies, and identity. *Reading Research Quarterly*, *39*(3), 333–337.

Cobb, P., & Hodge, L. L. (2011). Culture, identity, and equity in the mathematics classroom. In *A journey in mathematics education research* (pp. 179–195). Springer.

Cribbs, J. D. (2012). The development of freshman college calculus students' mathematics identity and how it predicts students' career choice. (Doctoral dissertation, Clemson University).

Ryan, R. M., & Deci, E. L. (2000). Self-determination theory and the facilitation of intrinsic motivation, social development, and well-being. *American Psychologist*, *55*(1), 68.

Suresh, R. (2006). The relationship between barrier courses and persistence in engineering. *Journal of College Student Retention: Research, Theory and Practice*, *8*(2), 215–239.

COGNITIVE AGENCY AND COMPUTER-BASED TASKS

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As computer-focused policies and trends become more popular in schools, more students access math curriculum online. While computer-based programs may be responsive to some student input, their algorithmic basis can make it more difficult for them to be prepared for divergent student thinking, especially in comparison to a teacher. Consider programs that assess student work by judging how well it matches pre-set answers. Unless designed and enacted in classrooms with care, computer-based curriculum materials might encourage students to think about mathematics in pre-determined ways. How do students approach the process of mathematics while using online materials, especially in terms of engaging in original thought?

Drawing on Pickering's (1995) dance of agency and Sinclair's (2001) conception of students as path-finders or track-takers, I define two modes of mathematical behavior: trail-taking and bushwhacking. While trail-taking, students follow an established approach, often relying on Pickering's (1995) *disciplinary agency*, wherein the mathematics "leads [them] through a series of manipulations" (p. 115). The series of manipulations can be seen as a trail that a student may choose to follow. Bushwhacking, on the other hand, refers to actions a student takes of their own invention. It is possible that, unknown to the student, these actions have been taken before by others. In bushwhacking, the student possesses agency, which Pickering (1995) describes as active (rather than passive) and as hallmarked by "choice and discretion" (p. 117).

In this study, students worked in several dynamic geometric environments (DGEs) during a geometry lesson about the midline theorem. The lesson was originally recorded as part of a larger study designing mathematically captivating lessons. Students accessed both problems and online addresses for corresponding DGEs via a printed packet. Students interacted with the DGEs on individual laptops, but were seated in groups of three or four. Passages of group conversations in which students transitioned between trail-taking and bushwhacking were selected for closer analysis, which involved identifying evidence of each mode and highlighting the curricular or social forces that may have contributed to shifts between modes.

Of particular interest were episodes in which students asked one another to share results, which led to students reconsidering previously set approaches, and episodes in which students interacted with DGEs containing a relatively high proportion of drag-able components, which corresponded to some students working in bushwhacking mode, spontaneously suggesting and revising approaches for manipulating the DGE (e.g., "unless you make this parallel to the bottom, but I don't think you… yes you can."). Both types of episodes were found in multiple groups' conversations. Further analysis of student interactions with tasks, especially with varying levels of student control and sharing, could serve to inform future computer-based task design aimed to encourage students to productively engage in bushwhacking while problem-solving.

References

Pickering, A. (1995). *The Mangle of Practice: Time, Agency, and Science*. Chicago, UNITED STATES: University of Chicago Press. Retrieved from http://ebookcentral.proquest.com/lib/bu/detail.action?docID=648147

Sinclair, N. (2001). The aesthetic IS relevant. *For the Learning of Mathematics*, *21*(1), 25–32.

FACTORS THAT INFLUENCE STUDENT MATHEMATICAL DISPOSITIONS

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Why do secondary students in the US consistently and increasingly report a lack of interest in mathematics? Lack of interest in mathematics has been well documented by the TIMSS studies; students' dissatisfaction with mathematics more than doubled by 2011, when 40% of 8th graders reported not liking math, up from 18% as 4th graders in 2007. And, sadly, the trend appears to be worsening. In 2015, 47% of 8th graders indicated not liking math, up from 22% as 4th graders. In order to positively impact student attitudes towards mathematics, it is important to understand factors that may influence secondary students' relationship with the discipline. This poster presents findings from an exploratory study of student disposition toward mathematics.

We designed an online survey to learn about students' relationship with mathematics, including experiences and settings that contribute to both positive and negative feelings about the subject. We surveyed 275 students, grades 9 to 12, in 11 classes in three schools in three New England districts. Though not randomly chosen, this sample allows us to examine student attitudes across a variety of contexts. We asked students about their feelings towards mathematics over the years, as well as which aspects of class they most enjoyed or disliked. Finally, we included items from the TRIPOD survey (Wallace et al., 2016) and the 2015 NAEP survey, which allows us to compare our sample with the national sample.

Initial results indicate that student view their teachers and the topics of study as the central factors influencing their enjoyment of mathematics class. We found a correlation between responses that math is boring and that it is not relevant. Students who like math and those who do not reported different class activity preferences. For example, students who like math reported disliking watching a video in class, while students who dislike math reported disliking learning something new. Both groups of students (those who like math and those who do not) dislike math class when they have to present work to classmates, but hold positive views of solving puzzles and working with other students. Technology seems to appeal equally to both groups. Students who reported disliking math also look forward to playing competitive games. We saw no evidence that gender or race corresponded to students' level of appreciation math. Finally, students reported liking math class less in high school than in middle school.

Identifying factors that influence secondary student mathematical dispositions can inform curriculum designers seeking to improve mathematical attitudes. Future studies can learn if new curricular designs can change student relationships with mathematics to reverse recent trends.

References

Wallace, T. L., Kelcey, B., & Ruzek, E. (2016). What Can Student Perception Surveys Tell Us About Teaching? Empirically Testing the Underlying Structure of the Tripod Student Perception Survey. American Educational Research Journal, 53(6), 1834–1868.

CHARACTERIZING ABSTRACT ALGEBRA STUDENTS' EPISTEMOLOGICAL BELIEFS ABOUT MATHEMATICS

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Keywords: Affect, Emotions, Beliefs, and Attitudes; Post-Secondary Education

Epistemological beliefs about mathematics focus on students' views of the nature of mathematics. Researchers have asked K-12 students, non-math major freshmen, math majors, and faculty members to describe their views of mathematics directly or through metaphors (e.g. Markovits & Forgasz, 2017; Schinck, Neale, Pugalee, & Cifarelli, 2008; Szydlik, 2013; Ward et al., 2010). Students' beliefs about the nature of mathematics can affect their beliefs about what math classes should look like, including the forms of classroom and sociomathematical norms (e.g. Boaler, 1999; Cobb, Gresalfi, & Hodge, 2009). Although established classroom norms can influence students' beliefs, how students' beliefs are affected is not well understood.

This poster examines the beliefs of abstract algebra students from two classrooms, one taught using Inquiry-Oriented materials and another taught via two days of lecture and one day of "lab" in which students discussed problems in groups. Students' beliefs were analyzed through thematic analysis (Braun & Clarke, 2006) based on their responses to survey questions and open-ended questions in follow-up interviews with selected students.

Students' views of the nature of mathematics varied, but included being the study of numbers, the study of logic, and a practical problem-solving tool. Asking students to provide animal metaphors for their views of mathematics highlighted more affective views of mathematics than asking what mathematics is. (For example, "Math is like a cat because everyone either loves cats, or hates them. Everyone loves math, or never wants to touch it in their life.") Although differences between students' beliefs were expected in the two classes based on differing instructional approaches, clearer differences in responses have emerged based on students' backgrounds, such as the number of previous math classes taken, grades in previous classes, and students' majors. Reasons for such differences will be explored.

References

Boaler, J. (1999). Participation, knowledge, and beliefs: A community perspective on mathematics learning. Educational Studies in Mathematics, 40(3), 259–281.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology, 3*(2), 77–101.

Cobb, P., Gresalfi, M., & Hodge, L., L. (2009). An interpretive scheme for analyzing the identities that students develop in mathematics classrooms. *Journal for Research in Mathematics Education, 40*(1), 40–68.

Markovits, Z., & Forgasz, H. (2017). "Mathematics is like a lion": Elementary students' beliefs about mathematics. *Educational Studies in Mathematics, 96*, 49–64. doi: 10.1007/s10649-017-9759-2

Schinck, A. G., Neale, Jr., H. W., Pugalee, D. K., Cifarelli, V. V. (2008). Using metaphors to unpack student beliefs about mathematics. *School Science and Mathematics, 108*(7), 326–333. doi: 10.111/j.1949-8594.2008.tb17845.x

Szydlik, S. D. (2013). Beliefs of liberal arts mathematics students regarding the nature of mathematics. *Teaching Mathematics and Its Applications, 32*, 95–111. doi: 10.1093/teamat/hrt002

Ward, B. B., Campbell, S. R., Goodloe, M. R., Miller, A. J., Kleja, K. M., Kombe, E. M., & Torres, R. E. (2010). Assessing a mathematical inquiry course: Do students gain an appreciation for mathematics? *PRIMUS, 20*(3), 183–203.

"EXPLICAME TU ESTRATEGIA": EMERGING BILINGUAL'S DEVELOPMENT OF MATHEMATICAL AGENCY IN PROBLEM SOLVING DISCUSSIONS

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Participating in mathematical practices supports children's conceptual development and increases achievement in the classroom (Bodovski & Farkas, 2007). However, emergent bilingual children often have limited exposure to these mathematical practices, and instead instruction promotes rote memorization and procedural knowledge without conceptual connections, or a focus on English vocabulary (Moschkovich, 2015). The term "emergent bilingual" (García & Kleifgen, 2010) identifies students not as lacking language, but instead indicates the *potential* to become bilingual over time, through schooling, and while still functioning with the home language. We build on Turner's (2003) construct of mathematical agency, defined as a moment in which a child has the power to make sense of mathematics and take ownership of their mathematical thinking. This study investigated emergent bilingual children's opportunities to exhibit mathematical agency when engaging in multilingual mathematical discussions.

Using a teaching experiment methodology (Cobb, 2000), we designed a 5-week after school program for seven third grade emergent bilingual students. All sessions were video-audio recorded and transcribed. These students were recommended by their teachers for additional support due to low scores on district exams. The afterschool math sessions focused on base ten story problems, sessions consisted of understanding the story context, solving the problem using prior knowledge, and sharing students' mathematical strategies in group discussion. The instructors specifically drew upon mathematical practices and positioning moves to engage the emergent bilingual students throughout the sessions.

We uncovered three mathematical agency categories, *limited, developing and enacting*. The *limited* category describes children using procedures without connections and little to none ownership of their thinking; the *developing* category describes children sharing their own thinking with others and trying to make sense of their math ideas; and the *enacting* category describes children making sense of their strategies used and taking ownership of their ideas. Children mostly showed *limited mathematical agency* in the initial sessions and *developing mathematical agency* throughout the rest of the study. *Enacting mathematical agency* was absent in the early sessions, but visible towards the end of the sessions. Among others, a key finding was the benefit of allowing flexibility in language choice for discussion of strategies. Our study highlights the need for further analysis of classroom instruction that builds mathematical agency of emergent bilingual students by positioning them as competent learners.

References

Bodovski, K., & Farkas, G. (2007). Mathematics growth in early elementary school: The roles of beginning knowledge, student engagement, and instruction. *The Elementary School Journal, 108*(2), 115-130.

Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers.

García, O. & Kleifgen, J.A. (2010). *Educating emergent bilinguals: Policies, programs, and practices for English language learners*. New York: Teachers College Press.

Moschkovich, J. (2015). Academic literacy in mathematics for English learners. *Journal of mathematical behavior.40*, 43-62.

Turner, E. E. (2003). *Critical mathematical agency: Urban middle school students engage in mathematics to investigate, critique, and act upon their world* (Doctoral dissertation).

YOUTH'S ENGAGEMENT AS MATHEMATICIANS IN AN AFTERSCHOOL MAKING PROGRAM

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Keywords: Informal Learning Environment; Maker Education; Mathematical Practices

Making is defined as a "class of activities focused on designing, building, modifying, and/or repurposing material objects, for play or useful ends, oriented toward making a 'product' of some sort that can be used, interacted with, or demonstrated," (Vossoughi & Bevan, 2014, p. 3). Making is often pitched as a means to increase students' engagement in transdisciplinary STEM learning and career pathways (e.g., Simpson et al., 2017). In terms of mathematics, one could argue that less is known about student learning of concepts and practices in making-related environments as the majority of mathematics education and scholarship is grounded in formal learning environments (Pattison et al., 2016). This research addresses this gap in the literature.

This study was conducted in partnership with a local science museum that provided an afterschool program for six Title 1 elementary schools and set within each school. The data source from this study was video data from youths who volunteered to wear a chest-mounted Go-Pro camera. We randomly selected two videos from each day of data collection that were at least 30 minutes in length. This amounted to 44 videos. Using the work of O'Connell and SanGiovanni (2013), we developed an initial coding scheme for each mathematical practice as defined by the Common Core State Standards (CCSS, 2017; e.g., *Problem Articulation* – Determine and articulate what the problem or task is questioning).

In general, youth were more often engaged in "mathematical" practices void of mathematics. Engagement in Practice 1, *make sense of problems and persevere in solving them*, was the most common; specifically, *self-monitoring.* However, monitoring one's process and making changes when necessary (i.e., self-monitoring) looked different as youth were not engaged in doing this through mathematics. Conversely, youths were frequently engaged in doing mathematics through indirect measurement and spatial reasoning, which are not captured within the practices. Our findings highlight how the practices are not conducive to settings outside the formal classroom. This raises the question of what and whose mathematics do we privilege – school mathematics or everyday mathematics (see Civil, 2017).

References

Civil, M. (2017). STEM learning research through a funds of knowledge lens. *Cultural Studies of Science Education, 11,* 41-59.

Common Core State Standards Initiative. (2017). *Standards for mathematical practice.* Retrieved from http://www.corestandards.org/Math/Practice/

O'Connell, S., & SanGiovanni, J. (2013). *Putting the practices into action.* Portsmouth, NH: Heinemann.

Pattison, S., Rubin, A., & Wright, T. (2016). *Mathematics in informal learning environments: A summary of the literature.* Retrieved from http://www.informalscience.org/mathematics-informal-learning-environments-summary-literature

Simpson, A., Burris. A., & Maltese, A. V. (2017). Youth's engagement as scientists and engineers in an after-school tinkering program. *Research in Science Education.* [Advanced Online Publication.]

Vossoughi, S., & Bevan, B. (2014). Making and tinkering: A review of the literature. *National Research Council Committee on Out of School Time STEM* (pp. 1-55). Washington, DC: National Research Council.

Chapter 11

Teaching and Classroom Practice

Research Reports

Brief Research Reports

Posters

HIGH-QUALITY INSTRUCTION ≠ HIGH-LEVEL NOTICING: EXAMINING FACTORS THAT INFLUENCE TEACHERS' NOTICING

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In this study, we investigate the relationship between teachers' noticing and the quality of their mathematics instruction. We analyzed the conversation of seven elementary teachers around five coaching cycles and scored their level of noticing and the mathematical quality of the instruction of the lesson. We compared teachers' noticing levels with their MQI scores for each coaching cycle. One case showed discrepancy between the level of noticing and the MQI scores. In this proposal, we discuss the cognitive and psychological constructs that seemed to influence teacher's noticing including mathematical knowledge for teaching (MKT), beliefs, teaching efficacy, and emotions. Results show that each of the constructs influenced the teacher's noticing to varying degrees. Implications for professional development are discussed.

Keywords: Coaching; Mathematical Knowledge for Teaching; Teacher Efficacy, Teacher Emotion; Noticing

Effective mathematics teaching requires foregrounding students' thinking (Ball & Cohen, 1999). This requires teachers to attend to and identify how students think and use their observations to make informed decisions about how to effectively respond. Recently, a significant amount of research (e.g. Sherin & van Es, 2009) has focused on investigating teacher noticing in order to understand what and how teachers observe, how they interpret the gathered information to respond to what they observe, and how this process can be influenced. These efforts have been focused on gaining a deeper understanding of how teacher noticing supports teacher learning in efforts to improve their instructional practices (Sherin, Jacobs & Philipp, 2010).

Professional noticing of children's mathematical thinking comprises a set of interrelated skills including (a) attending to students' strategies, (b) interpreting students' understandings, and (c) determining how to respond based on these understandings (Jacobs et al., 2010). Some of the work on teacher noticing has focused on the differences in what and how teachers' notice concluding that expert teachers tend to interpret and recall classroom events with greater detail and insight than novice teachers (Sherin, Jacobs & Philipp, 2010). We also acknowledge that teachers' abilities to notice may be impacted by the nature and quality of the instruction they are observing. For example, if a teacher is lecturing with minimal input from students, which is considered to be one indicator of low quality instruction, opportunities for noticing students' thinking will be limited. In this regard, we contend that evaluations of teachers' abilities to notice should be qualified based on the opportunities the teaching event affords for high-level noticing. Thus, we were interested in examining the alignment between teachers' levels of noticing and their quality of instruction, and the factors that influence this alignment. We considered mathematical knowledge for teaching (MKT), emotions, beliefs and teaching efficacy (TE) as possible factors. We targeted the following research questions:

- Do teachers' mathematical quality of instruction align with their level of noticing?
    - If there is alignment, what factors seem to support alignment?
    - If there is misalignment, what factors seem to influence this misalignment?

Theoretical Framework

Teachers' mental lives have a significant impact on their teaching experiences (Schutz, Hong, Cross & Osbon, 2006). Surprisingly, the ways in which specific cognitive and psychological constructs, such as MKT, emotions, beliefs and TE collectively influence teachers' instructional activity including noticing, has not been investigated broadly. In what follows, we provide a brief overview of each of these constructs with a specific focus on how each may inform teachers' instructional practices.

Mathematical knowledge for teaching (MKT) encompasses deep knowledge of math concepts and the knowledge and skills to attend to students' thinking during the act of teaching, and make in-the-moment decisions about the best ways to respond to what they observe (Ball, Thame & Phelps, 2008; Hill et al., 2008.). There is also a high correlation between mathematics knowledge for teaching and the mathematics quality of instruction (Hill et al., 2008). Teaching is emotional work (Schutz, Hong, Cross & Osbon, 2006). Emotions play a significant role in teachers' relationships, instructional decision-making and overall professional well-being. According to Trigwell (2012), "there are systematic relations between the ways teachers emotionally experience the context of teaching and the ways they approach their teaching." (p. 617). In this regard, we consider that the emotions teachers experience as they engage in instructional activity, including noticing, may influence what and how they notice. Beliefs are defined as "embodied conscious and unconscious ideas and thoughts about oneself, the world, and one's position in it developed through membership in various social groups, which are considered by the individual to be true".(Cross, 2009, p. 4) They are considered precursors to actions (Pajares, 1992). Research has shown that teachers' beliefs about mathematics, learning and teaching greatly influence their practices (e.g. Cross Francis**,** 2015; Ernest, 1989). We also know that beliefs can serve as filters, orienting an individual's thoughts and ideas through a particular lens. For example, a teacher who believes that students are able to build knowledge through meaningful cognitive engagement in activity would perhaps organize her classroom activities to support inquiry and problem solving. Lastly, teacher efficacy (TE) is defined as teachers' beliefs about their capacity to affect how students learn and their perception of overall performance (Tsachannen-Moran, Woolfolk-Hoy & Hoy, 1998). It influences teachers' willingness to learn, adopt and enact particular instructional practices. TE has two components---knowledge and personal efficacy. Knowledge efficacy refers to a person's confidence in her understanding of mathematics content (Roberts & Henson, 2000), while personal efficacy describes a person's confidence in her ability to support students' learning through teaching. Teachers with strong TE tend to be flexible in their approach to teaching and are more focused in planning and organizing instructional activity.

Methods

Seven elementary teachers participated in this study. The teachers were each involved in a PD program involving five coaching cycles. Coaching involved (i) preparing teachers' for their upcoming coaching session (pre-coaching), (ii) supporting teachers during instruction (coaching) and, (iii) debriefing instruction after the coaching session (post-coaching). All conversations were audio-recorded and the coaching sessions were video-recorded. Data sources included (i) audio and videorecordings from the coaching cycle, (ii) scores on mathematical quality of instruction instrument (MQI) (see Hill et. al, 2008), (iii) interviews prior to the start of the coaching that provided data about the core constructs described, and (iv) quantitative results of their teaching efficacy and emotions. Pre-coaching conversations were analyzed to determine teachers' knowledge (about the topic to be taught in the coaching session), efficacy and emotions related to the upcoming coaching session. Coaching session videos were analyzed using the MQI instrument to determine the quality of instruction along four core dimensions. A second round of analysis of the coaching video was done to determine the MKT (specifically common content knowledge (CCK), knowledge of content and students (KCS), specialized content knowledge (SCK) and knowledge of content and teaching (KCT) – (see Ball, Thames and Phelps (2008) for full descriptions) specifically focusing on their knowledge of mathematics being taught in the lesson, instructional strategies and students' thinking and the ways they enacted this knowledge during instruction.

In preparation for the coaching conversations, teachers were asked to identify three instances during the lesson that they found interesting or significant. Post coaching conversations were used to identify the clips the teachers selected which were analyzed for their noticing levels using Van Es' (2011) framework. Then we examined both the MQI scores and noticing levels per coaching cycle (Table 1) to determine the degree of alignment. Analyses of pre- and post coaching interviews determined the emotions, knowledge and beliefs related to the coaching session. Teachers also completed the adapted SETAKIST (Self-Efficacy Teaching and Knowledge Instrument for Teachers of Mathematics) (see Roberts & Henson (2000) for description of the instrument) to determine their knowledge and personal efficacy.

Findings & Discussion

The corpus of data were examined for the seven teachers and presented in Tables 1 and 2. In Table 1, we show the data from the coaching video where they have the highest MQI score and the accompanying noticing level for that instructional video.

Table 1: Alignment between teachers' instructional quality and their noticing level

Teacher	**Noticing Level**	**MQI** (for items that align with noticing framework)	**Alignment/Misalignment**
Bill	3.7	2.5	H/H - alignment
Laura	3	3	H/H - alignment
Anthony	1.5	1	L/L - alignment
Wilma	1.6	3	L/H - misalignment
Katie	1.3	3	L/H - misalignment
Jessica	1.8	3	L/H - misalignment
Sarah	1.7	2.5	L/H - misalignment

L - low; H - high

In the cases of Bill, Laura and Anthony, there was alignment between the quality of their instruction and the level of noticing. We have discussed these findings elsewhere (Cross Francis, Eker, Lloyd, Lui & Alhaayan, 2017). In this proposal, we will focus on the cases where there was misalignment between the teachers' quality of instruction and their level of noticing. For the purposes of this proposal, we present the analyses of the case of Katie. We considered this misalignment particularly noteworthy as we expected that a teacher who was able to produce relatively high quality instruction would foreground students' thinking which would be visible in how and what she noticed. To better understand the factors that may have influenced this misalignment, we examined her MKT, efficacy, beliefs and emotions.

Table 2. Results of analyses of Katie's noticing, MKT, TE and emotions

	Noticing		MQI	Emotions			Efficacy	
	What	How	Students' Productions	Before (Coaching)	During	After	KE	PE
Katie	Level 1	Level 1	3	Indifferent/Excited	Calm	Calm	High	High

MKT

The focus of the lesson was addition and subtraction of single-digit numbers. As with all of Katie's classes she began with the daily math review which consisted of questions aligned with the instructional goal of the lesson in addition to other questions related to concepts they had previously covered (e.g., number sequences). We determined her localized MKT by rating her knowledge as low, medium or high on four of the MKT dimensions – CCK, SCK, KCS and KCT. We considered both her CCK and her SCK to be strong with respect to single-digit addition and subtraction as she made no errors in solving the problems, supporting students in solving the problems or in discussing students' solutions about the problems. Katie was cognizant of the different types of subtraction problems (total unknown, initial unknown and change unknown) and their levels of complexity. Specifically, that total unknown problems are the least complex of the three and that addition can be used to verify the solution. Subtraction was a new concept for the students so Katie began with total unknown problems. She also discussed with the students the different ways of solving addition and subtraction problems – counting one by one, counting on and making items (drawing circles) to count. We considered her KCS and KCT to be low based on specific interactions with students during the instructional video and her responses to our questions referring to her thinking during those instances. In the first instance, the students were asked to solve the following problem:

Kenneth had 6 balloons. His sister popped 2. How many does he have left?
A student solved the problem and wrote:

$6 - 2 = 4$ (student put this on the board first)
$6 - 2 \neq 4$ (Student changes to this but Katie doesn't correct the student)

When asked about the student's thinking in the post-coaching conversation, Katie says he put the slash to show that he was saying that the left side is not equal to the right side. Although Katie knew the difference between both signs, she didn't think she needed to correct it. She was then shown a clip of what was written on the board for an earlier problem.

$$7 + 8 = 12 + 7$$
$$15 \neq 17$$

After seeing these two contrasting uses of the not equal sign, Katie was better able to understand how this could lead to misunderstandings for her students.

We also observed Katie's difficulties in supporting students to solve the following:

Macy had 6 cookies and she gave 2 away. How much does she have now?

Students were supplied with manipulatives to represent the problem. However, instead of being given a set of individual manipulatives (e.g., cubes) from which to select and manipulate to solve the problem, they were given a connected stack of 6 cubes to start. One student, Macy, kept adding two cubes instead of taking two away from the stack. There were several strategies that could have been employed to identify the struggle Macy was having and then to help move her thinking forward, some of which Katie employed. For example, asking a student to show how he would add some more cookies and if that is the same as taking away some cookies. However, although Macy continued to struggle, Katie essentially kept repeating the same strategy. Katie struggled to utilize other teaching approaches that would support Macy. After several minutes the coach asked Macy to show her six cookies, then suggested that she go ahead and eat two of the cookies. Macy took two of the cookies and put them in her lap. She was then able to tell the coach that she would have four cookies left. Katie commented that she didn't think of having Macy model the story. She also didn't consider having Macy separate the stack of cubes so she could represent each cookie with a cube. It's possible that Macy was not able to conceptualize the connected stack of cubes as six cubes (indication that she is not a numeric thinker/counter (Van de Walle, Karp, & Bay-Williams, 2016) but saw the stack as one unit. If Macy was not numeric, she would need individual cubes to count and solve the problem. Based on these observations in the lesson and Katie's interpretations of the events, we considered her KCS and KCT as low.

We considered this low KCS and KCT as a possible influential factor in her noticing. If Katie was not able to recognizing the range of possible reasons why Macy may have struggled to answer the question, she may not have considered this instance of students' thinking significant. Therefore, she would not have thought of it as a particularly interesting or pertinent event to discuss in the post-coaching conversation. Considering she stated that several students were struggling with subtraction, she perhaps categorized it as a part of the normal struggle students have with subtraction problems. As such, she wouldn't have *noticed* this incident as one to unpack independent of the struggles of the class more generally. Additionally, not being aware that allowing students to use the equal sign and not equal sign interchangeably may lead to a misconception would not have been significant, and therefore unlikely she would have identified it as meaningful for discussion.

Beliefs. Katie described math as problem solving and stated that math involved quantity and exploration. She believes there is always a "right" answer, but there are multiple ways to think about and solve problems. Regarding the learning of content, Katie believes that concepts need to be understood first through concrete work and manipulatives and that real contexts are important to use when teaching. She thought math can be applied to everything and that it allowed students to be creative, referencing the approaches of her first grade students. Katie stated that students are learning when they are engaged, when they ask for more, and when they can explain a concept. Katie believes students have varied abilities and learn concepts differently and as such should be allowed to talk about their thinking and make choices about tasks and activities, and the kinds of manipulatives that would support their thinking best. She considers productive struggle to be good and that students should be able to ask questions that lead to deep thinking and good conversation. Regarding her role as a teacher, she believes that when a student doesn't understand a concept, it is her responsibility and not the child's "problem." She also believes that teaching should be thought of as a learning process and teachers should be open to learning along with his/her students. Katie thinks that "teachers don't need to know every single thing" - they should start with what they know and build on their knowledge through a range of teaching experiences; similar to the way we should engage students in learning. Mistakes are okay so teachers should model how one can reason through and struggle with a problem, as reasoning is an important part of the process to developing understanding. Although Katie thinks flexibility is important, she also thinks that consistency in routines and procedures is crucial to creating a classroom that supports learning.

We observed that in providing students with manipulatives to model the problem of giving away six cookies, Katie was adhering to her beliefs that students should use manipulatives (concrete objects) to build their understanding of new concepts. As well, starting with a total unknown problem in a context that she believed reflected their own experience allowed for students to build knowledge and develop reasoning of new concepts (in this case, subtraction). These practices aligned with what she indicates are some of her beliefs about teaching and how children learn. One belief that we considered particularly salient in this instance is Katie's belief that teachers don't need to know every single thing. Similar to the way we approach students' learning, she believes that teachers can start with what they know and continue to build on that. Katie believes that mathematics embodies a way of thinking and learning for both students and teachers. She believes that learning can be spontaneous and unplanned and that lessons should be designed to reflect this. However, while this belief aligns with what the mathematics education community would regard as healthy beliefs (NCTM, 2014) they can be problematic if the teacher does not have the MKT needed to attend to this spontaneity as learning unfolds. When working on the cookies task, Katie struggled to find a teaching approach that would benefit Macy. When the coach suggested an alternate strategy to develop understanding, Katie appreciated this step and acknowledged she didn't think of that possibility. There were a range of challenges that Macy was faced with that Katie didn't appear to know how to address. Katie saw this as a teachable moment for her – one where she learned a strategy for supporting students struggling to solve single-digit subtraction problems. However, she didn't go further into investigating why this approach worked – what was it about Macy's existing mental constructions of number, addition and subtraction that caused her to struggle when Katie posed the task? Katie also didn't see it as a shortcoming that she wasn't able to support Macy in the moment; rather, for Katie this was an opportunity to learn.

Teaching Efficacy. Katie's confidence in her mathematical knowledge, accessed shortly before the coaching experience, is relatively high with a 3.875 (5 is the highest). In the follow-up interview she stated that although she was fairly confident in her mathematical knowledge she still felt that she had "room for growth and improvement." The use of the words "growth" and "improvement" can be interpreted as indicators that Katie may be more open for learning new mathematics and expanding her knowledge of mathematics, however she was not specific about what aspects of mathematics she still needed to improve but spoke more generally. This may suggest that Katie did not have a clear idea of what she did not know or should know as a math teacher and where exactly she needed to improve her knowledge.

With respect to her personal teaching efficacy, Katie was less confident in her ability to support her students' math learning through her teaching. Her score on the survey was 2.25, which is lower than mid-range based on the scale. During the interview, she explained that: "I guess I feel like since I didn't have very strong math instruction [as a student], that I'm still trying to learn structure as a teacher. And I just question it [my teaching ability] because I know that my [own] math instruction wasn't very strong." Katie was not as confident about her teaching ability as she felt she didn't have very good teacher role models when she was a student. As such, she is still trying to develop structure as a teacher, which for her refers to classroom management, grouping strategies and lesson implementation format. It's notable that she did not include analyzing and effectively utilizing students' thinking as a core concern of teaching. In this regard, if classroom features are of primary concern, this may explain her noticing score as she would be more focused on classroom-related issues and how the students are functioning as a group.

Emotions. During the pre-coaching conversation, Kim mentioned that the score of 4 on the anger scale was because the time allotted for math was limited and she was generally displeased with the misbehavior of the students. Her concern with students' misbehavior was also raised in a subsequent pre-coaching conversation where she stated that "Oh, I'm always a bit anxious about the students' behavior." As a result, she finds that she often spends a significant amount of time dealing with disciplinary issues which leaves less time to focus on the students' mathematical ability. A third factor that contributes to this score is that fact that the expectation of the administration, according to Katie, is unreasonable. She stated that the administration expects all the schools in the school district to teach a specific number of mathematics topics within a specific time period, although the students at the respective schools are at different levels of mathematical ability.

With respect to pride (which is 2.5), Katie noted that the reason this score is mid-way is because she tends to set a high standard for students in terms of their mathematics proficiency. Therefore, if after teaching a particular lesson, there is still a gap in their understanding, then she takes full responsibility for such outcome. So, she gave that score because at this point her students tend to have more gaps than she is satisfied with. When describing her emotions with respect to the teaching of the lesson, Katie notes that she feels indifferent. She states that "teaching is what I do for a living", so there is nothing special about teaching this upcoming lesson to her students. However, she mentions that she is excited that someone is going to observe her teaching. This is because even though she has been teaching a long while, she believes that there is always room for improvement and she thinks the feedback would be useful. Moreover, she is willing to learn new ways of teaching different mathematical concepts.

During the post coaching conversation, Katie noted that she felt good about the lesson as students were more engaged and they were able to work more independently in exploring mathematical ideas. She mentioned that she was expecting this type of outcome and so she was not overly excited by the successes or challenges the students demonstrated during the lesson. It didn't appear that she had any curiosity, excitement or disappointed related to Macy's struggle. She seemed not to consider it significant so for her it seemed not to warrant any increased pleasant or unpleasant emotion. We considered her emotional response noteworthy for two reasons, (i) if she had anticipated the outcome, why was she not better prepared to support Macy, and (ii) she did appear to have some excitement and interest in the approach that worked with Macy however, the heightened emotional response did not later deem the event significant for discussion.

Implications

Our analyses showed that although a teacher is able to orchestrate instruction that is of fairly high quality, that doesn't guarantee that he/she will readily identify meaningful instances of students' thinking as significant. This may indicate that teachers may enact high-quality teacher moves drawing on knowledge that is more tacit than explicit. We also observed that there were several factors that seemed to influence Katie's ability to notice students' mathematical thinking to some degree. They seemed to serve as filters focusing attention on more general aspects of classroom activity and not on students' mathematical thinking. As such, they are important factors that need to be considered in teacher development work on noticing specifically, and on instruction broadly. This also underscores the notion that teaching is complex and multi-layered, and that there are a range of factors that influence what a teacher identifies as meaningful or worthwhile instructional events to varying degrees. Although our study focused on the factors that influenced Katie's ability to notice, we believe that the research literature supports that these factors would be influential on other core aspects of teachers' work. We suspect that across teachers the degree of influence varies which would suggest that professional development work be more individualized, first building knowledge of how these constructs influence a teacher's instructional activity, then drawing on this knowledge to inform the approach to be used to support the teacher. Coaching is one professional development model that can support this kind of work.

References

Ball, D. L. & Cohen, D. K. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In G. Sykes and L. Darling-Hammond (Eds.), Teaching as the learning profession: Handbook of policy and practice (pp. 3-32). Jossey Bass.

Ball, D. L, Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Cross, D. (2009). Alignment, cohesion and change: Examining mathematics teachers' belief structure and its influence on instructional practice. *Journal of Mathematics Teacher Education, 12*(5), 325-346

Cross Francis, D. (2015). Dispelling the notion of inconsistencies in teachers' mathematics beliefs and practices: A three-year case study. *Journal of Mathematics Teacher Education,* 18(2), 173 - 201.

Cross Francis, D., Eker, A., Lloyd, K., Lui, J. & Alhaayan, A. (2017). Exploring the relationship between teachers' noticing, mathematical knowledge for teaching, emotions and efficacy. In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pages 122-1225). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In P. Ernest (Ed.), *Mathematics teaching: The state of the art* (pp. 249-254). Philadelphia, PA: The Falmer Press.

Heather C. Hill, Merrie L. Blunk, Charalambos Y. Charalambous, Jennifer M. Lewis, Geoffrey C. Phelps, Laurie Sleep & Deborah Loewenberg Ball (2008) Mathematical Knowledge for Teaching and the Mathematical Quality of Instruction: An Exploratory Study, Cognition and Instruction, 26(4), 430-511

Jacobs, V., Lamb, L., & Philipp, R. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*(2), 169-202.

Pajares, F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research, 62*(3), 307-332.

Roberts, J. K., & Henson, R. K. (2000, November). Self-efficacy teaching and knowledge instrument for science teachers (SETAKIST): A proposal for a new efficacy instrument. In *annual meeting of the Mid-South Educational Research Association, Bowling Green, KY.*

Schutz, P. A., Hong, J. Y., **Cross, D. I**., & Osbon, J. N. (2006). Reflections on investigating emotion in educational activity settings. *Educational Psychology Review*, *18*(4), 343-360.

Sherin, M., Jacobs, V. & Philipp, R. (2010). Mathematics Teacher Noticing: Seeing Through Teachers' Eyes. New York: Routledge

Sherin, M., & van Es, E. (2009). Effects of video club participation on teachers' professional vision. *Journal of Teacher Education, 60*(1), 20-37.

Trigwell, K. (2012). Relations between teachers' emotions in teaching and their approaches to teaching in higher education. *Instructional Science,* 40, 607-621.

Tschannen-Moran, M., Woolfolk Hoy, A., & Hoy, W. K. (1998). Teacher efficacy: Its meaning and measure. *Review of Educational Research*, 68, 202-248.

van Es, E. A. (2011). A framework for learning to notice student thinking. In M.G. Sherin, V. Jacobs, & R. Philipp (Eds.) Mathematics teacher noticing: Seeing through teachers' eyes (pp. 134-151). Routledge: New York.

Van de Walle, J.A., Karp, K.S., & Bay-Williams, J.M.(2016). *Elementary and Middle School Mathematics: Teaching Developmentally,* New Jersey, NJ: Pearson Education, Inc.

EXPANDING STUDENTS' ROLE WHEN DOING PROOFS IN HIGH SCHOOL GEOMETRY

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Using mixed-effects regression, we analyzed teachers' responses to a multimedia survey of instructional practices in posing proof problems in geometry. Teachers described and rated for appropriateness three different ways of involving students in deciding what to prove, including one in which the teacher chooses the givens and the conclusion to prove, and two others that expand the students' role in different degrees. While teachers recognized the former as normative, their ratings identified an alternative as more appropriate, having more positive value, and less negative value than the normative one. This alternative has students propose the givens or the conclusion to prove, and it allows the teacher to control the complexity of instruction by endorsing one proposal before students are to write the proof.

Keywords: Instructional activities and practices, Reasoning and proof, Research methods

Proof plays many roles in mathematics; among them is that of being a method for the discovery of new knowledge (de Villiers, 1990; Lakatos, 1976). Inasmuch as mathematical knowledge is usually represented in the form of conditional propositions (i.e., a conclusion is claimed as necessarily true if certain conditions or hypotheses are taken to be true), the creation of this mathematical knowledge involves both conjecturing a conclusion that can be asserted as true and hypothesizing the conditions that need to be assumed for that conclusion to be necessitated. A conclusion may be intuited as something that is only sometimes true, and one might search for what conditions might make the conclusion is necessarily true. At times, less conditions might also be sufficient to claim the same conclusion, while at other times less conditions may only allow one to make a weaker claim. In all of that, one exercises logical deduction as the process of making a valid inference based on two or more true premises, yet one does more than that: As Lakatos (1976) explained, one engages in a heuristic process of finding out what could be reasonably true. But, while this is part and parcel of mathematical work and quite relevant to making and critiquing mathematical arguments and modeling with mathematics as expected by the Standards for Mathematical Practice (Common Core State Standards Initiative, 2010), students rarely have opportunities to engage in such work in school (Stylianides, 2009). Why would those opportunities be scarce? How could such opportunities be created?

Our work examines those questions from the perspective of the teacher's management of the complex work of having students do proofs. Our goal in this to show that, from the teacher's perspective, there is a way in which the students' share of work could be expanded without compromising the teacher's capacity to manage the work. We document below how the literature has taken care of both curricular and learning perspectives on this matter. Yet, understanding whether opportunities like that are viable also requires consideration of the demands that such work places on the teacher.

Review of Literature

The mathematical work that students do in classrooms can be understood using frameworks associated with the instructional triangle (Cohen, Raudenbush, & Ball, 2003; Herbst & Chazan, 2012). Students' interaction with content happens especially in the context of problems and tasks. The choice of such problems is made by the teacher, whose work involves having students do work that puts them in interaction with target content.

Prior research has looked at the nature of the proof tasks that are afforded to students. Herbst's (2002) historical research showed that while a wide variety of opportunities for students to do original proofs emerged in the third quarter of the 19th century, some including the search for reasoned conjectures described above, the proof exercises in use by the early 1910's already consisted of separated sets of *given* and *prove* statements provided to students. Though efforts have increased to provide more opportunities to reason and prove in a variety of ways, such as with reform-based curricula, studying one such project Stylianides (2009) found that opportunities to find patterns and pose conjectures were low compared to opportunities to engage students in providing rationales. In an analysis of six secondary geometry textbooks, Otten, Gilbertson, Males, and Clark (2014) found that only between 5% and 20% of problems involved the construction of a proof, and most of those involved proving general or particular claims rather than constructing conjectures. In contrast, in a study of grade 8 Japanese geometry textbooks, Fujita and Jones (2014) found a significant portion of exercises providing opportunities for students to conjecture and discover properties. These exploratory problems typically came at the beginning of a lesson, so that by the time students prove or justify at the end of the lesson, they had already explored and investigated the facts on their own. Cirillo and Herbst (2012) have proposed some alternative problems that could be used to engage students in an expanded scope of work, where they could either produce the conclusion of a set of givens, or the givens needed to prove a particular conclusion. We surmise that the viability of these proof problems hinges on more than having such proof problems though. In particular, one might wonder whether students are able to do such work.

The second vertex in the instructional triangle is the student whose interactions with the content happen particularly in the context of their work on tasks. The literature on students thinking and learning reveals students' potential but also their difficulties with proof. Investigations on students' thinking have revealed that students, even at the elementary level, are capable of constructing arguments and proofs (Ball & Bass, 2003; Lampert, 1992; Reid, 2002). Teaching experiments (Norton, 2008) and classroom observations (Ellis, 2007) suggest students can be engaged in making reasoned conjectures. Yet, research has also shown that students sometimes take properties to be true on account of intuition and experience, not seeing the need for proof (Chazan, 1993). This might suggest that problems in which students have to figure out what might be true, might not so easily lend themselves to engaging them in proving.

The third vertex in the instructional triangle is the teacher. One way to inquire on the viability of engaging students in better proof problems might hinge on inspecting what teachers know about proof. Knuth (2002a) found that while secondary teachers acknowledged different important purposes of proof, they failed to recognize proof as a tool for learning mathematics. They also had difficulty knowing what constitutes a robust proof, failing to recognizing non-proofs and making judgements based on the form of an argument instead of the soundness of the reasoning. In a study of the mathematical knowledge for teaching (MKT) needed to teach proof, Steele & Rogers (2012) illustrated how secondary teachers' understanding of proof affected the way the teacher positioned students - namely as creators, "but only in the sense that they provided reasons for a set of predetermined steps" (p. 175). Teachers' attitudes towards students' ability to do proofs also play a role in what proving opportunities are presented to students. We aim to investigate ways to promote the creation of these opportunities for students in a way that teachers deem appropriate.

While the literature has been progressively relying on more classroom data, it has been common to frame issues of proving in the classroom in terms of having or not having resources, be those resources curricula, abilities, attitudes, or knowledge. In our work we have been interested in addressing the problem of students' share of work from a perspective centered on the complexity of the management of classroom instruction. While a teacher may or may not have resources with which to deal with such complexity, they are likely to have means to appraise that complexity and to relate to different practices that might differ amongst themselves by the amount of such complexity. We identify one complexity associated with expanding the work of students in proving here, then we describe how we studied it.

If we start from the nature of the task, we could wonder how teachers might relate to the possibility that students might come up with the proposition to be proved. One first complexity has to do with framing the problem. A teacher is likely to need to do more than identifying the goal of the task (to prove a proposition); some specifics of the thematic territory for the proposition to be proved may need to be identified. For this reason, Cirillo & Herbst (2012) proposed problems that expected students to provide the givens or the conclusion, but providing some of those resources. A second complexity draws on Doyle's (1986) characterization of classrooms as complex environments partly on account of the simultaneity of events. This complexity points to the number of possible responses that could ensue if the teacher asked students to propose what could be the givens or the conclusion to prove: If students had to come up with givens, many students could come up with many different sets of givens. A discussion of which set of givens makes the proposition stronger could be desirable; but managing such discussion might make the work of the teacher harder, especially if students invested themselves in proving different propositions that ended up not all being equally valuable. This work might be especially difficult to manage in classrooms where the norm may be one of accepting a variety of solutions to problems.

Based on prior exploratory work (e.g., Aaron & Herbst, 2017), we conjectured that teachers can appreciate having problems in which the students' scope of work on proofs extends beyond deductive reasoning, to conjecturing a conclusion or hypothesizing the givens. We also conjectured that teachers would perceive the need to manage the complexity of the multiple responses and would appreciate the opportunity to collect the students' thoughts and endorse the proposition whose proof will be written. While such work may still maintain something of a separation between conjecturing and proving (Aaron & Herbst, 2017), it may make it manageable for a teacher to engage students in doing work that is more authentic than current work on proof.

Method

We studied this question using three sets of scenario-based instruments, all related to the hypothesized norm that in proof problems the teacher is responsible for providing the givens and the prove statement, for which we had collected empirical evidence in a pilot study (Herbst, Aaron, Dimmel, & Erickson, 2013). Each instrument consisted of four item sets and each item set included a scenario of instruction, represented using a storyboard, and questions about the actions in the scenario: Participants were asked to describe what they saw happening, then to rate the appropriateness of the teaching they saw. One set of scenarios (DP-C) included only episodes of doing proofs in which the teacher took responsibility for providing the givens and prove statement (as expected by the hypothesized norm). A second set of scenarios (DP-GP) included only episodes where the teacher allowed the students to come up with the givens or the prove statement (a breach of the hypothesized norm). And a third set of scenarios (DP-TSGP) where the teacher again allowed students to propose the givens or the prove statement but also endorsed one of those proposals as the proposition for the whole class to prove.

Open responses to all scenarios were coded in two different ways. On the one hand, they were coded for whether or not they contained evidence that the respondent recognized the teacher's enactment of the norm (in the DP-C case) or its breach (in the DP-GP and DP-TSGP cases). On the other hand, each of those descriptions were coded for the presence of positive appraisals as well as negative appraisals of what the teacher was doing in the scenario (relying on Martin & White's, 2005, appraisal theory). All those coding operations had moderate interrater reliability. Three measures were derived from such codes, which we call *norm recognition* (INR), *positive appraisal* (PA), and *negative appraisal* (NA), all of them ranging from 0 to 4 in each instrument. Additionally, participants rated each scenario for appropriateness (AT) on a scale 1-6, with 1 being very inappropriate and 6 very appropriate.

Our conjectures included that (1) scores on INR(GP) and INR(TSGP) would both be larger than INR (C), indicating that participants noticed that both sets of scenarios breached the norm, but (2) AT(TSGP) would be larger than both AT(GP) and AT(C), which would be consistent with the conjecture that teachers preferred to expand the students' scope of work if the diversity of student proposals could be made more manageable. Appraisal scores were predicted to provide additional evidence: We conjectured that (3) PA(TSGP) would be larger than PA(GP) and PA(C) while (4) NA (TSGP) would be smaller than NA(GP) and NA(C). These conjectures would align with the interpretation that teachers would see value in expanding the students' share of labor in proof problems if the complexities that ensued from such expansion could be managed. We tested these conjectures running mixed effects regression models.

Data

Data comes from a nationally distributed sample of U.S. high school mathematics teachers. Instruments were administered in 2015-2016 through the LessonSketch (www.lessonsketch.org) online platform, where they could peruse scenarios and answer questions. There were 525 participants who completed at least one of the three instruments, and 347 participants who completed all three instruments. Most of the participants who completed all three instruments were white (86%) and female (61%), which is similar to the demographics of secondary high school teachers in the US. On average, participants had been teaching secondary mathematics for 14.7 years (SD = 8.69, min = 1, max = 40).

Results

Descriptives are shown in Table 1. Of the 525 participants who completed the DP-C instrument, 83.4% ($n = 438$) recognized the compliance of the norm at least in one scenario. Of those recognizers 40.4% ($n = 177$) provided at least one positive appraisal for this compliance, while 33.3% ($n = 146$) provided at least one negative appraisal of the compliance (both groups are not necessarily disjoint as any one recognition statement could be accompanied both by positive and negative appraisals). The mean positive appraisal scores was 1.24 (SD = 0.53) and the mean negative appraisal score was 1.17 (0.43), where both measures have a possible range 0-4. In comparison, of the 395 participants who completed the DP-GP instrument, 363 (91.9%) recognized a breach of the norm. Of those recognizers 53.2% ($n = 193$) positively appraised this breach, yielding a mean positive appraisal of the breach score of 1.51 (0.75); also 53.2 % (n = 193) of recognizers appraised the breach of the norm negatively, yielding a mean negative appraisal of the breach score of 1.83 (0.98). Finally, of the 495 participants who completed the DP-TSGP, 444 (89.7%) recognized a breach of the norm. Of those recognizers, 53.2% (n = 236) positively appraised the breach, yielding a mean positive appraisal of the breach score of 1.61 (0.82); of the recognizers, also 30.6% negatively appraised the breach, yielding a mean negative appraisal score of 1.53 (0.75). These descriptives suggest that people noticed the breach of the norm more saliently than its compliance (DP-C:83.4% < DP-GP:91.9%, DP-TSGP: 89.7%), they saw more positive as well as more negative issues with merely expanding the scope of work of the students. But when considering the possibility that the teacher might control that expansion by sanctioning the proposition to be proved, positive appraisals increased and negative appraisals decreased to a level comparable to the negative appraisals of complying with the norm.

A similar tendency could be observed with the appropriateness rating scores. We examined differences in how participants responded to a breach or compliance scenario with ratings toward the low or high end of the appropriateness scale with participants who completed all three instruments and recognized the DP-GP norm (n = 343). When asked to rate the appropriateness of the teaching showed in the DP-C scenarios, average scores were 4.42(SD = 0.65, n = 343), while those average appropriateness scores were 4.56 (SD = 0.94, n = 343) for the scenarios that breached the norm by asking students to provide the givens or the prove statement. The average appropriateness score went up to 4.83 (SD = 0.70, n = 343) in the case of the DP-TSGP scenarios where in addition to expanding the students' share of work, the teacher at some point sanctioned the proposition that students would prove.

Table 1. Descriptive statistics of recognizers and appraisers for each instrument

Instrument	Obs	Recognizers	Appraisers	Mean	SD	Min	Max	% Appraisers
DP-C	525	438	POS: 177	1.24	0.53	1	4	40.41%
			NEG: 146	1.17	0.43	1	3	33.33%
DP-GP	395	363	POS: 193	1.51	0.75	1	4	53.17%
			NEG: 193	1.83	0.98	1	4	53.17%
DP-TSGP	495	444	POS: 236	1.61	0.82	1	4	53.15%
			NEG: 136	1.53	0.75	1	4	30.63%

To ascertain the significance of the differences in AT, PA, and NA scores across different instruments, we calculated a set of mixed effect regression models. Given that each participant responded to multiple instruments, average appropriateness (AT), positive appraisal (PA), and negative appraisal (NA) that come from the same participant are not independent. Therefore, we conducted mixed effect linear regression models that could account for this non-independence among the scores. In the models, the categorical variable indicating a type of instrument was entered as a fixed effect and the variable indicating a participant's ID was entered as a random effect. The analyses were conducted using the STATA statistical software with the sample of participants who responded all three instruments and recognized the DP-GP norm (n = 343).

Table 2. Mixed effects linear regressions of scores on a type of instrument

	Appropriateness (AT) B(SE)	Positive Appraisal (PA) B(SE)	Negative Appraisal (NA) B(SE)
Fixed effects			
DP-C	-0.41 (0.05)***	-0.24 (0.07)**	0.17 (0.07)*
DP-GP	-0.27 (0.05)***	-0.38 (0.071)***	-0.03 (0.07)
Constant	4.83 (0.042)***	2.84 (0.06)***	2.04 (0.06)***
Random effects			
Constant	0.19 (0.03)	0.52 (0.065)	0.37 (0.06)
Residual	0.40 (0.02)	0.86 (0.05)	0.90 (0.05)
N	343	343	343

Standard errors in parentheses; $^{*}p < 0.05$, $^{**}p < 0.01$, $^{***}p < 0.001$
*reference instrument group: DP-TSGP

As shown in Table 2 (the reference group is DP-TSGP), results generally support our conjectures. AT (appropriateness) for TSGP is higher than both AT(DP-GP) and AT(DP-C), AT is significantly lower for DP-C by about 0.41 and DP-GP by about 0.27 (in a scale 0 ~ 6). In addition, the variation associated with the participants explains about 32% of the total deviations from the predicted AT that are not due to a type of instrument. Similarly, PA is significantly lower for DP-C and DP-GP by about 0.24 and 0.38 (in a scale 0 ~ 4) than for TSGP, respectively. The participants random effect comprise about 38% of the total residual variance. NA for DP-TSGP yields significantly higher score than NA for DP-C, but it is not significantly different from NA for DP-GP. For the NA score, the participants random effect explains approximately 29% of the total residual variance.

Conclusion

The data suggests that teachers do recognize the norm that the teacher will provide the givens and the prove for proof problems, yet they do not appraise it as better than some of the alternatives presented. Instead, alternatives in which the teacher expands students' role by inviting them to propose the givens or the statement to prove are appraised as more highly positive. Apprehension for these kinds of proof problems is apparent in the fact that negative appraisals of these alternative kinds of problems are still higher than for the habitual proof problems and not significantly higher than the negative appraisals for instances of doing proofs in which students are free to prove whatever they decide. An important implication for practice of these results is that it suggests that inservice teacher education could focus on helping teachers anticipate what students could propose in response to problems such as those proposed by Cirillo and Herbst (2012) and in practicing how to bring the class to a consensus on what statement they all should be working on.

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References

Aaron, W. R. & Herbst, P. G. (2017, online first). The teacher's perspective on the separation between conjecturing and proving in high school geometry classrooms. Journal of Mathematics Teacher Education. doi 10.1007/s10857-017-9392-0

Ball, D. L. & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), A research companion to principles and standards for school mathematics (pp. 27-44). Reston, VA: National Council of Teachers of Mathematics.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content 513 knowledge for teaching what makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Bieda, K. (2010). Enacting proof-related tasks in middle school mathematics: Challenges and opportunities. *Journal for Research in Mathematics Education, 41*(4), 351-382.

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics, 24*(4), 359-387.

Cheng, Y. -H., & Lin, F.-L. (2007). The effectiveness and limitation of reading and coloring strategy in learning geometry proof. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (vol. 2, pp. 113-120). Seoul, Korea.

Cirillo, M. & Herbst, P. (2012). Moving toward more authentic proof practices in geometry. *The Mathematics Educator, 21*(2), 11-33.

Cohen, D. K., Raudenbush, S. W., & Ball, D. L. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis, 25*(2), 119-142.

Common Core State Standards Initiative. (2010). Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

de Villiers, M. D. (1990). The role and function of proof in mathematics. *Pythagoras, 24,* 17-24.

Doyle, W. (1986). Classroom organization and management. In M. Wittrock (Ed.), *Handbook of research on teaching* (*3rd edition*; pp. 392-431). New York: Macmillan.

Ellis, A. B. (2007). Connections between generalizing and justifying: Students' reasoning with linear relationships. *Journal for Research in Mathematics Education, 38*(3), 194–229.

Fujita, T. & Jones, K. (2014). Reasoning-and-proving in geometry in school mathematics textbooks in Japan. *International Journal of Education Research, 64,* 81-91.

Herbst, P. (2002). Establishing a custom of proving in American school geometry: Evolution of the two-column proof in the early twentieth century. *Educational Studies in Mathematics, 49*(3), 283-312.

Herbst, P., Aaron, W., Dimmel, J., and Erickson, A. (2013, April). Expanding students' involvement in proof problems: Are geometry teachers willing to depart from the norm? Paper presented at the 2013 meeting of the American Educational Research Association. http://hdl.handle.net/2027.42/97425

Herbst, P. & Chazan, D. (2012). On the instructional triangle and sources of justification for actions in mathematics teaching. *ZDM The International Journal of Mathematics Education, 44*(5), 601-612.

Hsu, H. Y. (2010). *The study of Taiwanese students' experiences with geometric calculation with number (GCN) and their performance on GCN and geometric proof (GP)* (Doctoral dissertation). Available from ProQuest Dissertations and Theses database. (UMI No. 3441210).

Knuth, E. (2002a). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education, 33*(5), 379-405.

Knuth, E. (2002b). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education, 5,* 61-88.

Lakatos, I. (1976). *Proofs and refutations*. New York: NY: Cambridge University Press.

Lampert, M. (1992). Practices and problems in teaching authentic mathematics. In F. K. Oser, A. Dick, & J. Patry (Eds)., *Effective and responsible teaching* (pp. 295-314). San Francisco, CA: Jossey-Bass Publishers.

Martin, J. & White, P. (2005). *The language of evaluation: Appraisal in English*. New York, NY: Palgrave Macmillan.

Norton, A (2008). Josh's operational conjectures: Abductions of a splitting operation and the construction of new fractional schemes. *Journal for Research in Mathematics Education*, *39*(4), 401-430.

Otten, S., Gilbertson, N., Males, L. & Clark, D. (2014). The mathematical nature of reasoning-and-proving opportunities in geometry textbooks. *Mathematical Thinking and Learning*, *16*(1), 51-79.
Reid, D. A. (2002). Conjectures and refutations in grade 5 mathematics. *Journal for Research in Mathematics Education, 33,* 5-29.
Shaughnessy, J. M., & Burger, W. F. (1985). Spadework prior to deduction in geometry. *The Mathematics Teacher*, *78*(6), 419-428.
Steele, M. D. & Rogers, K. C. (2012). Relationships between mathematical knowledge for teaching and teaching practice: the case of proof. *Journal of Mathematics Teacher Education, 15*, 159-180.
Stylianides, G. J. (2009). Reasoning-and-proving in school mathematics textbooks. *Mathematical Thinking and Learning, 11*(4), 258-288.
Stylianides, G. & Stylianides, A. (2008). Proof in school mathematics: Insights from psychological research into students' ability for deductive reasoning. *Mathematical Thinking and Learning*, *10*(2), 103-133.

THE IN-THE-MOMENT NOTICING OF THE NOVICE MATHEMATICS TEACHER

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Recent reforms urge teachers to support students in pursuing their own mathematical strategies. Yet, to support a student in pursuing a strategy that they themselves devise, teachers must first notice details in the student's mathematical thinking, often in-the-midst of instruction. In this study, I examined four novice teachers' noticing of students' thinking in-the-moment, as well as the types of knowledge teachers called upon to support them with this noticing. Teachers took part in a post-lesson interview, in which they discussed moments from a preceding lesson of theirs caught on video. Findings indicate that novice teachers relied most on knowledge they developed in their own classrooms (e.g., knowledge of their individual students) to support their in-the-moment noticing. These findings align with calls to prepare teachers to view their classrooms as rich sources of knowledge supportive of their instructional practice.

Keywords: Teacher Knowledge, Elementary School Education, Middle School Education, Teacher Education-Preservice

Introduction

Today's mathematics classroom looks quite unlike the mathematics classroom of old. While in the past, teachers typically modeled a single strategy for students to apply in solving a series of practice problems, today, teachers are to encourage students to author their own strategies (NCTM, 2014). Existing research has demonstrated that, when teachers teach in this way, student learning benefits (e.g., Fennema et al., 1996).

To support students in pursuing their own mathematical strategies, however, teachers must be keen observers, capable of noticing important details in students' strategies (Jacobs, Lamb & Philipp, 2010). Only after noticing what a student did and understands in using some strategy, can a teacher then respond in an informed manner (e.g., by suggesting an appropriate next step). In the absence of prior noticing, a teacher might lack an adequate understanding of the student's strategy and could thus respond by asking the student to abandon their chosen approach in favor of one recommended by the teacher.

A number of studies have examined what teachers notice. These studies, however, have typically focused on what teachers notice when watching classroom video (e.g., Sherin & van Es, 2005), and not the sort of live, in-the-moment noticing that helps teachers support students in pursuing their chosen strategies. Of the few studies that have examined teachers' in-the-moment noticing, most have involved teachers in possession of several, if not many, years of experience (Jacobs & Empson, 2016; Sherin, Russ, & Colestock, 2011). While some recent work (e.g., Dyer, 2013) has examined the in-the-moment noticing of more novice teachers, this remains an under-examined area in need of further study.

In the absence of a more refined understanding of the novice teacher's in-the-moment noticing, current programs of professional development focused on this important aspect of practice may fail to address the unique needs of the novice. Accordingly, this study examined the in-the-moment noticing of the novice mathematics teacher.

Theoretical Framework

Teacher Noticing

Framing "noticing." Teachers have likened the process of noticing to having something prick the senses, alerting one to the fact that something important and worthy of attention is occurring (Sherin, Russ, & Colestock, 2011). Some scholars have adopted a similar framing, focusing on what captures teachers' attention in their studies of noticing. For example, in their work on the noticing of pre-service teachers, Star, Lynch, and Perova (2011) examined what teachers identified as important or noteworthy when watching classroom video. Others have included not just what a teacher identifies as noteworthy in their description of noticing, but how teachers make sense of what they deem noteworthy, as well (Sherin & van Es, 2009).

A focus for noticing. While some have studied teachers' noticing of any and all "salient features of classroom instruction" (Star, Lynch, & Perova, 2011, p. 117), others have examined teachers' noticing of students' mathematical thinking, in particular. For example, in their research program, Jacobs. Lamb, Philipp, and Schappelle (2011) acknowledged being "less interested in identifying the variety of what teachers notice and more interested in how and the extent to which teachers notice children's mathematical thinking" (p. 99). To Jacobs, Lamb, and Philipp (2010), noticing children's mathematical thinking consists of three component parts: a) attending, b) interpreting, and c) deciding how to respond. "Attending" refers to the extent to which a teacher pays attention to what a student did in solving some mathematics problem. "Interpreting," on the other hand, refers to the extent to which a teacher pays attention to what a student understood in solving a problem. Finally, "deciding how to respond" refers to a teacher's intended response to a given student. Attending, interpreting, and deciding how to respond occur almost simultaneously, "as if constituting a single, integrated [act]" (p. 173).

In-the-moment noticing. In recent years, scholars have begun to focus less on what teachers notice when watching classroom video and more on what teachers notice in-the-midst of instruction. This shift appears motivated by an acknowledgement that noticing "in the classroom is quite complex – more so than [when watching video] … when teachers [are] given only a small slice of instruction to consider and extended time to do so" (Sherin & van Es, 2009, p. 33).

Jacobs and Empson (2016) developed a framework for teachers' in-the-moment noticing of children's mathematical thinking (Figure 1). According to this framework, what a teacher notices in children's mathematical thinking in-the-moment is dependent upon that teacher's knowledge of children's developmental understandings of key mathematics concepts.

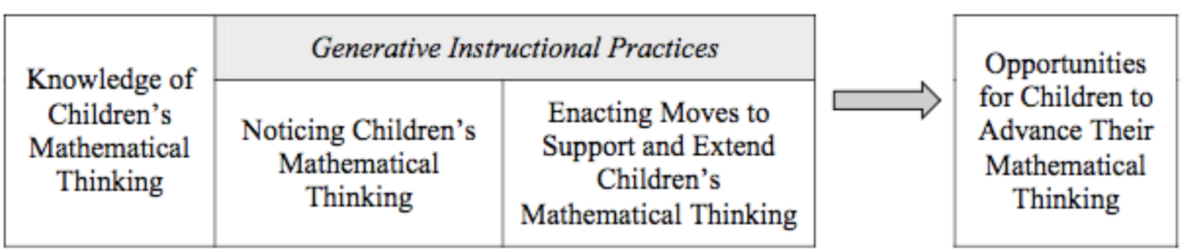

Figure 1: Framework for In-the-moment Noticing of Children's Mathematical Thinking

A significant volume of research has described this knowledge, identifying what students of a given age tend to know, think, and do when working on particular mathematics concepts. For instance, Fennema et al. (1996) identified a developmental trajectory that students in the primary grades tend to follow as they solve arithmetic problems.

Even if they have not studied such developmental trajectories explicitly, veteran teachers are likely to be familiar with what children of a given age tend to know, think, and do. For novice teachers, however, this knowledge it is likely to be less well-developed. As novices have had "the least experience with children's thinking" (Jacobs et al., 2011, p. 100), they are likely less aware of what is typical for children of a given age to know, think, and do mathematically. This raises some important questions. If novice teachers are lacking knowledge of children's developmental understandings of key mathematics concepts, are they capable of noticing students' mathematical thinking in-the-moment? And if so, what knowledge do they draw upon to support this noticing? The present study sought to answer the following:

1. To what extent do novice teachers notice students' mathematical thinking in-the-moment?
2. What knowledge do novice teachers draw upon to support them in noticing students' mathematical thinking in-the-moment?

Method

Setting and Participants

For this study, I recruited four early-career teachers from a public school serving students in grades 4-9. One teacher, Kerry, taught 4th-grade, while Taylor taught 5th-grade, Hannah taught 6th-grade, and Caroline taught 9th-grade (all names are pseudonyms). Kerry and Hannah were first-year teachers at the time of the study, while Taylor and Caroline were in their third year in the classroom. I purposefully selected these teachers because of their school's commitment to student-centered instruction. As opportunities to notice students' mathematical thinking in-the-moment arise as teachers interact with students, I sought to work with teachers in a student-centered school, where significant instructional time would be spent interacting with students.

Data Collection

Capturing video for discussion. Similar to Sherin, Russ, and Colestock (2011), I examined teachers' in-the-moment noticing by first capturing video of teachers' lessons then having them discuss moments from this video shortly afterwards. I used two cameras, a Drift Stealth™ and a GoPro Hero 4™. Teachers mounted the Stealth to the side of their head using a headband and triggered the camera to capture clips of their interactions with students in two of their lessons. I collected a second source of video using a GoPro mounted at the back of the room, as I was worried that teachers might forget to capture clips using the Stealth. Using an app, I viewed a livestream of teachers' lessons recorded using the GoPro and listened to audio of their conversations with students captured using a wireless mic that the teachers wore. As I watched the GoPro video, I tagged moments when the teacher interacted with students.

Post-lesson interview. Consistent with the procedure used by Sherin and colleagues, I asked teachers to discuss their videos in an interview shortly after their second lesson; the first lesson was solely for piloting camera gear. Between each teacher's second lesson and their post-lesson interview, I watched as many of the teacher's Stealth clips as possible, selecting those in which there was some evidence of students' mathematical thinking. I also referenced notes I had taken for the tagged GoPro moments, choosing those in which some mathematical thinking was on display. I chose Stealth clips and GoPro moments in this manner, as I worried that a more random selection would have resulted in a lack of noticing of student thinking, not because teachers were not capable of such noticing, but because there simply would have been less mathematical thinking to notice in a random selection of clips/moments.

In the interviews, I showed the teacher a still image from the start of each clip/moment then asked them the following: "can you walk me through what was happening at this point in the lesson?" When a teacher was not able to respond to this prompt, I played the video until they could. As soon as teachers were able to respond to the prompt, I stopped the video. This procedure was followed as it limited the amount of video teachers were shown, thereby reducing the likelihood that what they went on to describe was based on what they noticed in watching the video and not what they had noticed when the moment actually happened in the preceding lesson. I collected audio- and video-recordings of these interviews.

Data Analysis

Creating instances. I parsed the video of each teacher's post-lesson interview into instances, which consisted of all discussion by the teacher of a given Stealth clip or tagged GoPro moment.

Teachers' noticing. I next created two coding schemes: one for teachers' "attending" and a second for teachers' "interpreting" (Jacobs, Lamb, & Philipp, 2010). Every instance was assigned one of three codes for "attending": (1) evidence of attending provided, (2) lack of evidence of attending provided, or (3) not applicable. If a coder could describe the mathematical strategy of a student or group of students in some clip/moment based on what the teacher said, a level 1 code was assigned. When the teacher did not provide enough detail to make this possible, a level 2 code was assigned. When the teacher did not mention any mathematical strategy in a given instance, a level 3 code was assigned. A similar process was followed when coding "interpreting." Specifically, each instance was assigned one of three codes for interpreting: (1) evidence of interpreting provided, (2) lack of evidence of interpreting provided, or (3) not applicable. If in discussing a given clip/moment, a teacher mentioned what in particular a student or group of students understood (e.g., the student knows how to count by 5's), coders assigned a level 1 code. When the teacher instead made a broad claim (e.g., the student understands addition), a level 2 code was assigned. Finally, if the teacher made no mention of what a student or group of students understood mathematically, a level 3 code was assigned.

Teachers' knowledge. As teachers discussed clips/moments from their lessons, they not only attended to and interpreted students' thinking, but also referenced their knowledge of a range of things. I created a third coding-scheme describing the nine different types of knowledge that teachers referenced in their interviews.

Looking for associations between attending, interpreting, and knowledge. After all instances had been assigned a code for attending, a code for interpreting, and up to nine codes for knowledge, I looked for associations between the attending, interpreting, and knowledge codes. Specifically, I looked to see which types of knowledge were referenced most when "attending" was coded level 1, when "interpreting" was coded level 1, and when both were coded level 1.

Results

Novice Teachers' In-the-moment Noticing

Overall, the four teachers in this study were asked to discuss 49 instances of instruction. In 19 of these 49 instances, teachers provided evidence of attending (i.e., level 1 attending). Similarly, in 18 of these 49 instances, teachers provided evidence of interpreting (i.e., level 1 interpreting). Though evidence of attending was not always accompanied by evidence of interpreting, in 10 of 49 instances, teachers provided evidence of both attending and interpreting.

As an example of level 1 attending, consider the following comment made by one teacher, Kerry, in discussing an interaction she had with a student trying to solve 6×9: "So, I think what [she's] doing here is talking about ten groups of six, does that sound right? Or seven. One of the two. Like, [she's] doing ten groups then subtracting a group to get nine." Based on this comment, coders were able to describe the strategy of the student being discussed, a compensation strategy for multiplication (i.e., $6 \times 9 = 6 \times 10$, subtract 6). As such, coders determined that Kerry had provided evidence here of attending to the substantive details in the student's strategy.

Turning to interpreting, consider Taylor's comment below about an interaction she had with several students trying to place the fractions 4/5 and 3/4 on the number line:

> And, so, I think he was just … using that visual in his head, like, the four, four-fifths versus three-quarters. And then I asked them about if they were missing, they were both missing one piece and what that means, and they were able to say that the, obviously the one that's missing a fifth is a much smaller piece than the one missing a quarter … because it's split into smaller pieces.

Here, Taylor made a specific claim about the understanding of a group of students. Specifically, Taylor claimed that the students in this particular moment knew that 1/5 was a smaller piece than 1/4. As such, coders assigned a level 1 code for interpreting here.

Although teachers provided evidence of either attending or interpreting in a number of instances, this was not always the case. Indeed, in 8 of 49 instances, teachers made no mention of what a student or group of students did in solving a given mathematics problem, while in 18 of 49 instances, they made no mention of what students understood mathematically. In these instances, teachers instead discussed things like students being off-task, even though students' mathematical thinking was on display.

Knowledge Referenced When Discussing Moments of Instruction

Teachers referenced nine types of knowledge in discussing moments from their lessons. *Knowledge of an individual student* was displayed anytime a teacher referenced something in general that they knew about a student. As this knowledge was either mathematical or not, there were actually two codes for knowledge of an individual student. For example, in discussing a student in one moment from her lesson, Kerry commented, "she usually struggles in math, so she'll find the easiest way possible." As another example, when discussing a moment from her lesson, Taylor stated, "well, this one, … she's kind of a perfectionist." While Kerry's comment was focused on mathematics, Taylor's was not.

Knowledge of a group of students was displayed whenever a teacher referenced something they knew about a group of students. As an example, in discussing a moment from her lesson, Taylor stated, "this is the group that I went to first because, I, they're, they're the lowest…they're still really struggling with fractions." While Taylor here referenced knowledge of a group of students that was mathematical in nature, Hannah provided an example of this type of knowledge that was not mathematical, stating, "and the usual ones raised their hands."

Knowledge of the class was displayed anytime a teacher referenced something they knew about their class. As an example, consider the following comment made by Kerry when discussing a moment from her lesson on patterns in multiplication: "So, lots of them I noticed in their patterns, and in everything they do, assume that I can read their minds because they only put, like, half of what they're thinking and it's not enough information for me." Conversely, consider the following comment made by Hannah: "I have, like, a really gifted group and so, they tend to move very fast and they jump right ahead of many others, there's a large gap." The comment here by Kerry focused on the lack of detail her students provide in their mathematical work, so was deemed mathematical in nature, while the comment made by Hannah was not.

Knowledge of students was observed whenever a teacher mentioned something they knew about students in general. For example, consider the following comment made by Kerry in discussing her lesson: "'Cause some kids with subtraction just get, like, really stuck up, even though it is a smaller subtraction, they're like, 'muhhh,' and they get overwhelmed, so the addition really would be faster for them." By contrast, Kerry also discussed how she likes to let students work in the hallway because, "some kids really do need that quiet to focus." Hence, the first comment here was related to mathematics, but the second one was not.

Finally, *knowledge of a prior lesson or moment* was observed anytime the teacher referenced either a prior lesson or some moment that had occurred earlier in the lesson being discussed in their interview. For example, when discussing a strategy one of her students was using, Kerry, as if speaking to the actual student, said: "'I'm pretty sure you're using this strategy we talked about in class and that's really cool.'" Conversely, when discussing an interaction with a student about a lack of detail in his work, Kerry referenced knowledge of a similar moment from earlier in the lesson, stating: "Zack, when I was talking to him, … we were looking at his patterns, same thing, where I was just like, 'I don't, this isn't enough information, can you expand on that for me?'" In all cases when a teacher referenced knowledge of a prior lesson or moment, this knowledge was mathematical in nature.

Knowledge and In-the-moment Noticing

When teachers provided evidence of in-the-moment noticing, they referenced certain types of knowledge more so than others. Specifically, in 10 of the 19 instances when teachers' attending was coded level 1, teachers referenced *knowledge of individual students,* while in 9 of these 19 instances, they referenced *knowledge of a prior lesson or moment*. Similarly, in 6 of the 18 instances when teachers' interpreting was coded level 1, teachers referenced *knowledge of individual students,* while in 9 of these 18 instances, teachers referenced *knowledge of a prior lesson or moment.* In 3 of the 10 instances where both attending and interpreting were coded level 1, teachers referenced *knowledge of individual students*, while in 4 of these 10 instances, teachers referenced *knowledge of a prior lesson or moment*.

To illustrate how the knowledge teachers referenced seemed to support their in-the-moment noticing, consider the following comment made by Taylor in discussing an interaction she had with a group of students:

> This one, she's very, um, particular, like, she's, she's kind of a perfectionist, she wants to make sure everything's done well. And I think she was, she was, so, they had figured out which of the fractions, which were big–, larger than one, they had figured out which order they should go in, but they were trying to figure out where to place it on the number line. Um, and I think she was wanting to be, like, pretty exact of where it went.

In this comment, Taylor referenced non-mathematical knowledge she had of an individual student, specifically, that the student was "kind of a perfectionist." This knowledge seemed to set an expectation in Taylor's mind for what to then notice in the group's mathematical work, specifically, that this group would be quite precise in placing fractions on the number line. Indeed, shortly after providing the quote above, Taylor attended to the mathematical strategy used by the group of students being discussed, explaining that, rather than place the fractions 8/7 and 1 & 3/14 just anywhere to the right of 1 on the number line, the students converted 8/7 into a mixed number, thereby allowing for a more precise placement of these fractions:

> The, it was, like, one and two-four, they had figured out, they had, they had already found an equivalent fraction and converted it to a mixed number fraction, so I think it was 1 & 2/14 and 1 & 3/14 … I think there was 8/7, I think that they changed 8/7 to, so they figured out that it was 1 & 1/7, and then they changed it to 1 & 2/14. Yeah.

Discussion

Better Than Expected

As this study demonstrates, novice teachers are certainly capable of noticing salient details in students' mathematical thinking in-the-moment. Existing literature, however, might suggest otherwise. Studies of novice teachers' noticing have found that, at least initially, early-career teachers tend to notice things like classroom management and behavioral issues more so than the substance of students' mathematical thinking (Star, Lynch, & Perova, 2011). Similarly, research examining teachers' noticing of students' mathematical thinking in particular has found that novice teachers tend to focus on the general features of students' strategies (e.g., that a student's work lacks organization) more so than the "mathematical essence of the [strategies]" (Jacobs, Lamb, & Philipp, 2010, p. 183). And yet, in 19 of 49 instances in the present study, teachers provided evidence of attending to students' mathematical thinking, while in 18 of 49 instances, teachers provided evidence of interpreting students' thinking.

These results are particularly compelling for two reasons. First, unlike previous studies of teacher noticing (Jacobs, Lamb, & Philipp, 2010), in the present study, teachers were not asked to describe what students did or understood mathematically, yet still spoke of such things in discussing their lessons. Second, this was a study of teachers' *in-the-moment* noticing, which scholars have both theorized (Sherin & van Es, 2009) and found (Dyer, 2013) to be particularly challenging for teachers. Given this, the task presented to the novice teachers in this study was quite demanding, making the extent to which they noticed student thinking all the more impressive.

So, what could explain why the novice teachers in this study did better than might have been expected in noticing students' mathematical thinking? In previous work examining novice teachers' noticing, teachers were asked to observe students who were not their own and who they had not actually taught themselves (Jacobs, Lamb, & Philipp, 2010; Star, Lynch, & Perova, 2011). As such, teachers in these studies had no knowledge of the students they were observing, the class they were in, or the prior lessons in which they had taken part. As a result, these teachers simply could not leverage such knowledge, knowledge that the present study suggests may be critical in supporting novice teachers' noticing of student thinking.

Mine Your Experience, There Is Knowledge There

This study aligns with existing literature that suggests that teachers' own classrooms are a rich source of knowledge. At present, significant efforts are being made in teacher education programs to prepare novice teachers for the many demands of the classroom. However, scholars have cautioned that no amount of such training can adequately prepare prospective teachers for these many demands (e.g., Ball, Sleep, Boerst, & Bass, 2009). Almost two decades ago, Ball and Cohen (1999) spoke of how "much of what [teachers] … have to learn must be learned in and from practice rather than in preparing to practice" (p. 10). The findings of the present study suggest that some of the knowledge that supports novice teachers' in-the-moment noticing is acquired in teachers' very own classrooms, with their very own students.

This is not meant to suggest that all teacher learning should be postponed until the novice teacher is in their own classroom. Rather, these results suggest that, with regards to in-the-moment noticing, it may be fruitful to prepare teachers to inquire into and acquire knowledge from the spaces in which they teach (Hiebert, Morris, Berk, & Jansen, 2007).

Concluding Thoughts

There were several exceptions to the "knowledge informs noticing" story told above. Indeed, in 2 of the 10 instances in which teachers provided evidence of both attending and interpreting, none of the knowledge types in my coding scheme were referenced. Also, there were instances when teachers called upon what seemed to be the most helpful sources of knowledge (i.e., knowledge of individual students, knowledge of a prior lesson or moment), yet did not provide evidence of in-the-moment attending or interpreting. While I present these cases here as counter-examples, they could also be regarded merely as evidence that the relationship between noticing and knowledge is simply not that straightforward.

Finally, it is important to note that the teachers in this study could well have relied on other, more tacit knowledge (Eraut, 2000) to support their noticing that they simply did not, or could not, share explicitly in their interviews.

References

Ball, D. L., & Cohen, D. K. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In L. Darling-Hammond and G. Sykes (Eds.), *Teaching as the learning profession* (pp. 3–31). San Francisco, CA: Jossey-Bass.

Ball, D. L., Sleep, L., Boerst, T. A., & Bass, H. (2009). Combining the development of practice and the practice of development in teacher education. *The Elementary School Journal*, *109*(5), 458-474.

Dyer, E. B. (2013). Investigating the relationship between teacher professional vision and classroom practices: A case of misalignment. In M. Martinez & A. Castro Superfine (Eds.), *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 988-995). Chicago, IL: University of Illinois at Chicago.

Eraut, M. (2000). Non-formal learning and tacit knowledge in professional work. *British Journal of Educational Psychology*, *70*(1), 113-136.

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education, 27*(4), 403-434.

Hiebert, J., Morris, A. K., Berk, D., & Jansen, A. (2007). Preparing teachers to learn from teaching. *Journal of Teacher Education*, *58*(1), 47-61.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*(2), 169-202.

Jacobs, V. R., Lamb, L. L., Philipp, R. A., & Schappelle, B. P. (2011). Deciding how to respond on the basis of children's understandings. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 97-116). New York, NY: Routledge.

Jacobs, V. R., & Empson, S. B. (2016). Responding to children's mathematical thinking in the moment: An emerging framework of teaching moves. *ZDM Mathematics Education, 48*(1–2), 185-197.

National Council of Teachers of Mathematics (2014). *Principles to actions: Ensuring mathematical success for all.* Reston, VA: NCTM.

Sherin, M. G., & van Es, E. A. (2005). Using video to support teachers' ability to notice classroom interactions. *Journal of Technology and Teacher Education*, *13*(3), 475-491.

Sherin, M. G., & van Es, E. A. (2009). Effects of video club participation on teachers' professional vision. *Journal of Teacher Education*, *60*(1), 20-37.

Sherin, M. G., Russ, R. S., & Colestock, A. A. (2011). Accessing mathematics teachers' in-the-moment noticing. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 79-94). New York, NY: Routledge.

Star, J. R., Lynch, K. H., & Perova, N. (2011). Using video to improve preservice mathematics teachers' abilities to attend to classroom features: A replication study. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 117-133). New York, NY: Routledge.

INVESTIGATING MTE QUESTIONING AS A RELATIONAL TEACHING PRACTICE

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As mathematics teacher educators (MTEs), we enact teaching practices in our mathematics methods courses as instantiations of practice for PTs. Questioning is a core practice in teaching (and learning to teach) mathematics that interacts with relationships between MTEs and prospective teachers (PTs). We engaged in a collaborative self-study to interrogate our questioning practice as part of relational teacher education with a goal of improving our questioning to reach all PTs. Observation and recordings of our questions during mathematics methods classes, collaborative conversations, and narratives were analyzed to describe characteristics of our questioning practice (Dillon, 1990) and determine how it is relational (Kitchen 2005b). We found commonalities in our questioning practice and identified ways to improve it. This serves as an exemplar for MTEs to inquire about their own questioning practice.

Keywords: Instructional activities and practices; Teacher Education-Preservice

As mathematics teacher educators (MTEs), our teaching practice centers on the development of ambitious teaching practices (Kazemi, Franke, & Lampert, 2009) of the prospective teachers (PTs) with whom we work. We view our practice as relational in that it is complex and dependent on the strength of relationships developed between the participants (PTs and MTEs) involved (Bieda, 2016; Grossman et al., 2009a; Noddings, 2003). We extend our definition of relational practice beyond attending only to the mathematical or pedagogical thinking of the PT to include attending to the whole person (Kitchen, 2005a, 2005b; Noddings, 2003). Therefore, we strive to incorporate the experiences of our PTs as learners in our classes and interrogate our own practice as related to those experiences, thus valuing all learners.

Practices we enact in our mathematics methods courses become instantiations for PTs to experience and take up in formulating their own practices (LaBoskey, 2007). We promote the development of ambitious teaching practices by focusing on core practices (Grossman, Hammerness, & McDonald, 2009b). Asking questions of learners is one such core practice, as it occurs frequently in teaching mathematics, provides opportunities to find out about students' thinking, and is accessible to PTs learning to implement mathematics instruction. Although current research on questioning in the teaching of mathematics is available (e.g., Davis, 1997; Nicol, 1998; Parks, 2010), little research has focused on the instantiations of a questioning practice PTs experience in methods courses that prepare them to enact this practice. Yet, the form and function of questions asked in methods courses has potential for influencing PTs' views of questioning practices. We argue that the process of interrogating one's questioning practice as part of relational mathematics teacher education can inform ways to improve that practice as an instantiation which can be taken up by PTs.

Objectives of the Study

We interrogated our questioning practices as MTEs teaching mathematics methods courses to more fully understand the experiences of our PTs in these courses. The following research questions guided our self-study inquiry (LaBoskey, 2007): What are characteristics of our questioning practice? and, How is our questioning practice relational? Initial examination of our questions through the lens of Dillon (1981, 1990) illuminated tractable characteristics of our practice. To explore the relational nature of our questions, we viewed them through Kitchen's lens of relational teacher education (2005a, 2005b, 2016). This lens afforded a view of our questions as drawing from our personal histories, experiences as MTEs, and our understandings of the PTs' histories and experiences and provided us a perspective which informed improvement of this practice.

Theoretical Framework

We define practice following Pinnegar and Hamilton (2009) as "engaging with others in ways that lead to the accomplishment of goals through the use of the knowledge, theories, and understandings" (p. 16). For us, questioning as a relational practice was always a goal directed activity informed by knowledge, theories, and understanding. We sought to build "knowledge from practice" (p. 17) as a way of "knowing to" (p. 18) question as a practice. In our inquiry we were driven to become aware of what informs our questions and ways we use questioning to build our relationships with PTs.

Relationships have consistently been identified as critical in teaching and learning (Russell & Loughran, 2007). Exploration of relationships in self-study has revealed the complexity of coming to know a PT and be known by a PT (Kitchen, 2009). The significance of relationships in teaching and learning gave rise to descriptions of relational practice. Grossman et al. (2009a), drawing from the work of Fletcher (1998), described teaching as relational practice in which relationship is used as a lever in teaching. This description highlighted the utility of relationships for PTs, but not ways in which teacher educators' practices could be characterized as relational. Kitchen's (2005a, 2005b) description of *relational teacher education* as teacher educators "knowing in relationship" (2005a, p. 18) illuminated how a relational practice is constructed through knowing: knowing oneself and knowing PTs. Like Fletcher (1998), Kitchen drew from notions of empathy (Rodgers, 1961) and vulnerability to describe relational practice. Kitchen identified seven defining characteristics of relational teacher education: understanding one's own personal practical knowledge, improving one's practice in teacher education, understanding the landscape of teacher education, respecting and empathizing with PTs, conveying respect and empathy, helping PTs face problems, and receptivity to growing in relationship. We hypothesize that core practices can be viewed as contributing to an MTE's relational practice. In particular, looking at questions an MTE poses through the lens of relational teacher education has the potential to unearth factors that influence the construction and posing of questions.

Questioning can be viewed as a strand of relational practice, since questions communicate the MTE's interests and aims to PTs and provide an opportunity to elicit PT's ideas. Each strand of one's practice contributes to the development of a relational practice, but one could question whether a single strand can be seen as relational. We suggest that one strand of practice can be viewed through a relational teacher education (Kitchen, 2005a) lens to gain insight into ways understandings of self and others are drawn upon in questioning. It is from this perspective that we can label questioning practice as relational. Research points to the complexity of questioning in mathematics classrooms (Davis, 1997; Parks, 2010), particularly when moving beyond evaluative questioning, and suggests that understanding the underlying influences on questioning can inform improvements in the practice. Questioning in a methods course introduces an additional level of complexity as MTEs use questioning to understand the mathematical thinking of PTs and their views of teaching and learning. Examination of MTEs' use of the core practice of questioning in teaching is then warranted as we strive to provide instantiations of effective questioning practice for PTs to draw upon, tacitly or explicitly, as they develop their own questioning practices (Mewborn & Tyminski, 2006).

Exploring questioning with self-study methodology (LaBoskey, 2008) created space for understanding questioning as a practice informed by philosophy (Berci & Griffith, 2005; Gadamer, 1975/2004) and empirical research (Dillon, 1982) with the goal of improving questioning. Our study of questioning, like that of Olsher and Kantor (2012), focused on question-asking and characterizing our questioning practice. We draw from Dillon's (1981, 1990) definition of questions as "interrogative utterances" which are "followed by answers" (Dillon, 1981, p. 51). Dillon (1990) described two sources of assumptions: assumptions in the logic of the question (question-sentences) called *presuppositions*, and assumptions about the context in which the question is asked (question-situations) called *presumptions*. Identifying our assumptions allowed us to identify characteristics of our questioning practices.

Methodology

LaBoskey (2007) described self-study methodology as interactive and improvement aimed. Drawing from LaBoskey's view, our interactions are best described as open, collaborative, and oriented toward reframing (Samaras & Freese, 2009). We opened our questioning practices to each other for scrutiny and engaged in collaborative conversations that created opportunities for constructing new perspectives on or reframing our questioning practice. These conversations further unearthed tensions underlying our questioning practice including ways that relationships informed our practice.

Data collection and analysis occurred in multiple ways as we studied our questioning practices. First, we observed each other teaching at least one class. In-person observations allowed us to situate discussions of questioning in particular contexts and teaching practice. In addition, each of us audio-recorded at least one class session. We then each catalogued questions we posed to PTs during at least one class session. What counted as a question was informed by our view of PTs' engagement. When PTs were exploring, whether prompted by a question or a directive, we considered the activity a question. Examples of questions from the transcripts illustrate differences in the content and approach (See Table 1).

Analysis of our questioning took place in three phases. First, we each identified questions, presuppositions, presumptions, purpose, and roles for PTs associated with questions from one class session and created a table (see Table 1). *Presuppositions* are assumptions conveyed in the logic of the sentence. *Presumptions* are beliefs conveyed in the question-situations (Dillon, 1990). *Purposes* refer to the reasons the MTE posed the question. *Roles* describe the ways in which PTs are positioned in responding to the question. We then summarized our findings, describing insights and remaining dilemmas about our questions, presuppositions, presumptions, purpose, and roles. We each read the summaries in preparation for our discussions of findings in four recorded online conversations (August 10 & 29 and September, 5 & 19, 2017).

Table 1: Examples of Questions

Alyson: The hat you are going to put on is that of the students that wrote the work. And you are going to look at the feedback that the other group wrote. And you are going to decide as the student who wrote that, that is the feedback that you got, what would you do next? (9/13/2016)

<table>
<tr><td>Presuppositions:
There is a next step to which mathematics-learners can reply.</td><td>Presumptions:
PTs can pretend to be someone else and get into another person’s thinking.</td><td>Purpose:
To mimic the act of receiving feedback with a chance to critically examine it after.</td><td>Role:
Mathematics-learner receiving feedback</td></tr>
<tr><td colspan="4">Alyson: What I want you looking at now is what was the feedback and how did the math learner respond to it? Use your observations of those pairs to think about – do we have a complete list of what makes effective feedback? Are there some things you would suggest that we should not do? Are there some things you would suggest that we absolutely should do in responding? Can we use the feedback and student responses to that feedback to make some observations about, adding to or taking off of our list about feedback? (1 minute pause.) Are there some finer points that we need to add to this list about things we should consider when we are writing feedback to the students in the letter writing exchange? (9/13/2016)</td></tr>
<tr><td>Presuppositions:
Our list of characteristics of feedback is incorrect and too generic.</td><td>Presumptions:
PTs can draw conclusions from a mock example of feedback.</td><td>Purpose:
To examine instincts on providing feedback and position PTs to provide appropriate feedback in the actual activity with students.</td><td>Role:
Teacher-researchers</td></tr>
<tr><td colspan="4">Susan: So, the text on this page [reads text on a page of a counting book]. So, what I want you to do is, if this were the question and you were reading this with a group of kids, I want to know what level of cognitive demand, based on the criteria for these levels of cognitive demand, where do you think this would fall? Is it a lower cognitive demand like memorization, or is it a procedure without a connection or is it more of a higher cognitive demand where it might be something related to a procedure with a connection or something that’s more like doing mathematics, … [elaborates by continuing to rephrase the questions] … so think about that, and then of course I want you to think about how to justify your claim, there may be more than one answer here, ok, so, um I’ll give you two or three minutes to work with your group or your partner, and see what you think. (9/12/16)</td></tr>
<tr><td>Presuppositions:
Features of questions or tasks allow for classification according to cognitive demand. Depending on the question, the type of thinking is different.</td><td>Presumptions:
PTs will use the chart describing the levels of cognitive demand to classify a task posed in a children’s book.</td><td>Purpose:
Practice applying the levels of cognitive demand to questions or tasks and consider the type of thinking required in the task. Knowing the type of thinking required in a task is important in selecting tasks or questions for lesson planning.</td><td>Role:
PTs analyze tasks for thinking</td></tr>
<tr><td colspan="4">Signe: What is a confusion you have seen out in the field? (3/6/17)</td></tr>
<tr><td>Presuppositions:
Confusions can be seen in the field and are revealed through analysis of student thinking.</td><td>Presumptions:
PTs are developmentally ready to identify confusions</td><td>Purpose:
Link lesson planning and insights about children’s mathematics.</td><td>Role:
PTs conduct informal research to inform lessons</td></tr>
</table>

Our collaborative conversations served as a second phase of analysis. During this phase we focused on understanding the relational nature of our questions. We identified commonalities and differences in our presuppositions, presumptions, purposes, and roles for PTs. Our discussions focused on ways our questioning was influenced by our view of our own histories as learners and MTEs. For example, we recognized that our love of mathematics and interest in discussing mathematics learning and thinking with PTs influenced our questions. We further focused on ways our questions were influenced by the PTs, what we knew about them as people and learners. In particular, we pressed each other to explore how PTs' problems of practice were taken up or how empathy was conveyed (Kitchen, 2005b) in our questions. For example, Signe consistently wondered if the questions she was posing situated the PTs to respond from their own experiences rather than hypothesizing about the experiences of other teachers or mathematics learners. Discussions of these commonalities resulted in changes (or what we later called *fixes*) to the characteristics identified by the coding according to Dillon (1990). However, discussions of our fixes and the changes themselves provoked additional dilemmas about our questioning as a relational practice.

The third phase of analysis included writing narratives describing fixes we implemented in our questioning practice and dilemmas that remained. We read and discussed these narratives focusing on questioning as a relational practice. These discussions (December 12 & 19, 2017) were recorded and reviewed. From these discussions, we identified dilemmas, made sense of our questioning as relational, and developed assertions for action (Pinnegar & Hamilton, 2009).

Findings

We first present the common characteristics in our question practice as viewed through the lens of Dillon (1990). We then discuss the fixes applied individually in our practice as assertions for actions and highlight the dilemmas that remained relevant to our vision of relational practice.

Presuppositions and purposes

Presuppositions and purposes revealed reference to knowledge we consciously held and wanted to bring to PTs' attention. For example, Susan believed a teacher's awareness of the cognitive demand of mathematics tasks was important. The logic of her questions included the assumption that analyzing the demand of tasks is an essential part of planning lessons. For example: "I want to know what level of cognitive demand, based on the criteria for these levels of cognitive demand, where do you think this would fall?" (September 12, 2016). This example illustrates that, our questions contained expectations that PTs should accept our views of teaching and learning rather than developing views from their own experiences, including from their experiences in our courses.

Presumptions and roles

Presumptions and roles involved intent, contexts, and PTs. We consistently asked questions with the intent of learning what PTs knew. Signe asked about confusions PTs noticed in their work with children. "What is a confusion you have seen out in the field?" (March 6, 2017). Her intent was to gain insight into PTs' experiences. We wondered if we used PTs' responses in our teaching. We collected PTs' experiences with a desire to understand their insecurities and problems of practice (Kitchen, 2005b), yet our planned lessons played out without drawing from the issues PTs raised. We had empathy and respect (Kitchen, 2005b) for PTs, but struggled to convey it or help PTs face problems they described. Instead our questions focused on moving toward goals and implementing instructional activities we had planned before we knew the PTs. In this way we viewed our questions as "information-seeking" (Davis, 1997, p. 363) in that we wanted to understand PTs' thinking, but that thinking did not influence our teaching.

Additionally, our questions positioned PTs to take on a variety of roles during a single lesson. Alyson noticed that her questions positioned PTs in many different roles during one class meeting, roles such as reflective practitioner, researcher, teacher, and learner. She thought of *Whose Line is It Anyway?,* a television show where actors change roles or themes after a minute or two, generating chaos and comic situations. She felt she expected too much of PTs and wondered: What messages about teaching as reflective practice, might PTs gain through shifting roles so quickly and frequently?

Assertions for action and understanding (Pinnegar & Hamilton, 2009)

The analysis of our assumptions (Dillon, 1990) motivated fixes that improved our questioning practice. We eliminated long introductions, multiple roles, and identified the PTs as actors in questions. Dilemmas that remained focused on the relational nature of our questioning practice. Next, we describe the fixes each MTE made, improvements that resulted, and dilemmas that remained in terms of relational teacher education (Kitchen, 2005b).

Susan. Susan fixed her long introduction to questions by consciously posing planned questions. This fix unearthed a significant tension. Reducing the introduction to questions freed up time for PTs' interests and concerns. Their responses in whole class discussion surfaced their concerns and dilemmas or insecurities with teaching mathematics related to their experiences in their field classrooms. These interests and concerns turned the discussion in directions Susan had not anticipated, making it difficult to cover planned content. In addition, Susan wondered if she was listening to the PTs with empathy. Was she participating mindfully and thoughtfully, respecting PTs' contributions to the discussion (Kitchen, 2005a, 2005b)?

Susan found a dilemma between providing time and space for PTs to articulate their experiences, concerns, and insecurities about teaching mathematics (i.e., helping PTs face problems; Kitchen, 2005b) and being able to acknowledge and build on their ideas, while also trying to cover course content. Susan felt most comfortable when the discussions remained anchored around the mathematics content, and less so when the class discussions drifted into more general issues of pedagogical moves. Yet, teaching about mathematics teaching requires facility with the blending of mathematics content and teaching practices (Perks & Prestage, 2008). What is an appropriate balance and blend of content and methods that provides time and space for PTs to connect through sharing and raising their interests, concerns, dilemmas, and insecurities?

Alyson. Alyson began using a new planning method that provided for more concise questioning and positioned PTs consistently either as teachers or learners of teaching. Yet Alyson was left wondering if she was truly conveying empathy (Kitchen, 2005b) through those questions and positionings. Through the roles employed, Alyson was seeing PTs' thinking more clearly and better understanding their struggles but was left with questions about the relational characteristic of conveying empathy. Some PTs responded with favorable comments about Alyson's teaching during the semester, yet others struggled to find their way and began to doubt themselves as teachers. Alyson was left wondering whether the changes she had made had allowed her to convey empathy to *all* PTs?

Signe. Signe wondered if she was actually curious about learning about teaching. Raising this issue unearthed tensions for Signe about her receptivity to growing in relationship (Kitchen, 2005b). Signe felt that she wanted to learn from PTs, but perhaps her curiosity had a different focus than that of PTs. Wondering about her curiosity about teaching was difficult to admit and left Signe feeling distanced from her colleagues. Signe sincerely loved thinking about children's mathematics and teachers' insights about and use of this mathematics, while Signe viewed PTs as focused on improving pedagogical techniques. Neither seemed right or wrong, Signe simply worried. As she tried to enact a questioning practice aligned with Dillon's (1981) suggestion that questions should be asked only when the teacher is actually curious about a learner's thinking, she was faced with her limits. How could she be curious about the more general practices and holistic views of learners PTs shared?

Discussion

Despite significant differences in our context, we identified common characteristics in our questioning practice when we applied Dillon's (1990) framework and examined our presuppositions, presumptions, purposes, and roles. Even though we work in different institutional contexts and programs (e.g., Signe and Susan work with elementary PTs and Alyson with secondary PTs), we could describe common characteristics in our questions, allowing us to answer the first research question. We saw our own histories and experiences reflected in the presuppositions and purposes of our questions. Presumptions and roles provided insight into our exploration of PTs' histories and experiences and how we engage PTs in drawing from those experiences to inform and inspire their practice. With surface level commonalities identified and fixes applied, we could then see the different ways these factors influenced the relational nature of our practice as we developed and posed our questions (Kitchen, 2005a, b).

The examination of our questioning practice in methods courses afforded each of us the opportunity to consider ways in which we were relating to our PTs through questioning and the types of instantiations of practice PTs were experiencing. Both Alyson and Signe found issues with the ways in which they positioned PTs during class sessions. Alyson was positioning PTs in a great variety of roles and Signe was positioning PTs to draw on experiences other than their own. For each, this inappropriate positioning of PTs could hinder their relational practice. The core practice of questioning as relational, assumes that the questioner draws insights from the learner and makes use of those insights to move the lesson forward. Davis (1997) has described the reciprocal relationship involved in such questioning in the domain of mathematics. In mathematics teacher education, feedback PTs provided that is not used may suggest the MTE is not listening, thereby introducing an unintended barrier between the MTE and the PT. As a result of this finding, both MTEs have introduced more careful positioning of PTs into their courses.

All three MTEs have found the results to indicate a need for clearer communication and purposeful word choice with PTs. Signe identified a need to phrase questions differently so as to position PTs to respond from their own perspectives. She accomplished this by attending carefully to the choice of words used in her questions. Susan and Alyson identified a tendency to ask a series of related questions in quick succession rather than one purposefully chosen question. They each addressed this issue in ways that ensured one clear question is asked and then time is provided for PTs to contemplate and respond. Improving the clarity of communication with PTs removes another unintended barrier to relational practice.

Through inspection of the core practice (Grossman et al., 2009b) of questioning in a mathematics methods course, we found ways to improve that practice so that it more closely aligns with our goals of relational teacher education (Kitchen, 2005a, b) and includes more opportunities to draw on our PTs' experiences hermeneutically (Davis, 1997) during class activities. Building public exemplars of practice such as this for instructors of mathematics methods courses provides much needed opportunity for MTEs to engage in inquiry about those practices in order to seek improvements in our work.

References

Berci, M., & Griffith, B. (2005). What does it mean to question? *Interchange, 36*, 405-430.

Bieda, K. (2016). Taking stock: MTE's contribution to building a knowledge base for the practice of mathematics teacher education. *Mathematics Teacher Educator*, *5*(1), 3–7. https://doi.org/10.5951/mathteaceduc.5.1.0003

Davis, B. (1997). Listening for differences: An evolving conception of mathematics teaching. *Journal for Research in Mathematics Education*, *28*(3), 355–376.

Dillon, J. T. (1981). To question and not to question during discussion. *Journal of Teacher Education*, *32*(5), 51-55.

Dillon, J. T. (1982). Cognitive correspondence between question/statement and response. *American Educational Research Journal, 19*, 540-551.

Dillon, J. T. (1990). *The practice of questioning*. London: Routledge.

Fletcher, J. (1998). Relational practice a feminist reconstruction of work. *Journal of Management Inquiry, 7,* 163-186.

Gadamer, H. (1975/2004). *Truth and method.* (J. Weinsheimer & D. Marshall, Trans. Second ed.). London: Continuum.

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. (2009a). Teaching practice: A cross-professional perspective. *Teachers College Record, 111,* 2055-2100.

Grossman, P., Hammerness, K., & McDonald, M. (2009b). Redefining teaching, re-imagining teacher education. *Teachers and Teaching*, *15*(2), 273–289. https://doi.org/10.1080/13540600902875340

Kazemi, E., Franke, M., & Lampert, M. (2009). Developing pedagogies in teacher education to support novice teachers' ability to enact ambitious instruction. In B. Hunter, B. Bicknell, & T. Burgess (Eds.), *Crossing divides: Proceedings of the 32nd annual conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 11–29). Palmerston North, New Zealand: MERGA.

Kitchen, J. (2005a). Looking backward, moving forward: Understanding my narrative as a teacher educator. *Studying Teacher Education: A journal of self-study of teacher education practices, 1,* 17-30.

Kitchen, J. (2005b). Conveying respect and empathy: Becoming a relational teacher educator. *Studying Teacher Education: A journal of self-study of teacher education practices, 1,* 195-207.

Kitchen, J. (2009). Relational teacher development: Growing collaboratively in a hoping relationship. *Teacher Education Quarterly, 36*(2), 45-62

Kitchen, J. (2016). Looking back on fifteen years of relational teacher education: A narrative self-study. In J. Williams & M. Hayler (Eds.), *Professional learning through transitions and transformations: Teacher educators journeys of becoming.* Switzerland: Springer.

LaBoskey, V. K. (2007). The methodology of self-study and its theoretical underpinnings. In J. Loughran, M. L. Hamilton, V. K. LaBoskey, & T. Russell (Eds.), *International handbook of self-study of teaching and teacher education practices* (pp. 817-869). Dordrecht, The Netherlands: Springer.

Mewborn, D. S., & Tyminski, A. M. (2006). Lortie's apprenticeship of observation revisited. *For the Learning of Mathematics*, *26*(3), 23–32.

Nicol, C. (1998). Learning to teach mathematics: Questioning, listening, and responding. *Educational Studies in Mathematics*, *37*(1), 45–66. https://doi.org/10.1023/A:1003451423219

Noddings, N. (2003). Is teaching a practice? *Journal of Philosophy of Education*, *37*(2), 241–251.

Olsher, G., & Kantor, I. (2012). Asking questions as a key strategy in guiding a novice teacher: A self-study. *Studying Teacher Education: A journal of self-study of teacher education practices, 8*(2), 157-168.

Parks, A. N. (2010). Explicit versus implicit questioning: Inviting all children to think mathematically. *Teachers College Record, 112*(7), 1871–1896.

Perks, P., & Prestage, S. (2008). Tools for learning about teaching and learning. In B. Jaworksi & T. Wood (Eds.), *The international handbook of mathematics teacher education: The mathematics teacher educator as a developing professional* (Vol. 4, pp. 265-280). Rotterdam: The Netherlands: Sense.

Pinnegar, S. & Hamilton, M. (2009). *Self-study of practice as a genre of qualitative research: Theory, methodology, and practice*. Dordrecht.: Springer.

Rodgers, C. (1961). *On becoming a person*. Boston: Hougton Mifflin.

Russell, T., & Loughran, J. (2007). *Enacting a pedagogy of teacher education: Values, relationships and practices*. New York, NY: Routledge.

Samaras, A., & Freese, A. (2009). Looking back and looking forward: An historical overview of the self-study school. In C. Lassonde, S. Galman, & C. Kosnik (Eds.), *Self-Study research methodologies for teacher educators* (pp. 3-20). Rotterdam: Sense Publishers.

CAPTURING VARIABILITY IN FLIPPED MATHEMATICS INSTRUCTION

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Flipped instruction is being implemented in an increasing number of mathematics classes but the research base is not yet well developed. Many studies of flipped instruction involve a small number of flipped classes being compared to non-flipped classes, but this methodology fails to account for variations in implementations. To aid in the systematic attention to variation, this article presents a framework for flipped mathematics instruction that identifies key features of the videos assigned as homework as well as features of the in-class activities. The components of the framework are accompanied by proposed quality indicators to further distinguish between flipped implementations that are structurally similar but different in enactment

Keywords: Flipped Instruction; Lesson Observations; Classroom Research; Frameworks

In recent years there has been an increase in flipped instruction (Smith, 2014). Flipped instruction is characterized by teachers "flipping" the settings in which lecture and homework occur. Instead of being presented with new material in class and then completing homework problems outside of class, students in flipped classes watch videos or read new material outside of class and then complete the problem sets in class (Bergmann & Sams, 2012). Although the flipped instructional model has a single overarching label, there are a wide variety of learning opportunities it can encompass. On one hand, flipped instruction has the potential to be innovative with regard to differentiation (e.g., students can rewatch videos at their own pace and with technology-based supports) and collaboration (e.g., students can work together because of the increased time in class to solve problems). On the other hand, flipped instruction can be non-innovative, such as when students watch lecture videos at home and then work individually to complete procedural exercises in class (de Araujo, Otten, & Birisci, 2017a).

Because of the variation in flipped implementations, this theoretical article presents a research-based framework that moves beyond a general definition of flipped instruction and provides specific analytic tools that allow for meaningful distinctions to be drawn with respect to the learning opportunities afforded. In particular, the framework considers the components of flipped mathematics lessons and the quality indicators for each component.

Why a Framework is Needed

Although flipped instruction can vary in its implementation (de Araujo, Otten, & Birisci, 2017a), much of the early literature on flipped instruction seems to presume that it is a unified approach that can be directly contrasted with non-flipped instruction. For example, DeSantis, Van Curen, and Putsch (2015) investigated flipped and non-flipped geometry instruction in two high school classes. They found no differences in geometry learning, although students in the flipped class were less satisfied with the unit than students in the non-flipped class. But the flipped lessons in the study involved using the same activities as in the non-flipped lesson, so the implicit assumption was that the act of flipping itself—not the manner of flipping—is of central importance. Moreover, DeSantis and colleagues did not provide substantial detail about the non-flipped instruction, so with regard to the finding of no difference, the basis of comparison is under-specified. A similar example is Clark (2015), who compared his flipped Algebra 1 classes' year-end student outcomes with non-flipped Algebra 1 classes taught by others in the same school. Clark found, like DeSantis and colleagues (2015), no difference between flipped and non-flipped learning measures. Unlike DeSantis et al., however, Clark's students expressed a preference for flipped instruction because of the range of teaching practices it allowed. Again, because of the simplistic and small-scale comparison, the underlying assumption in Clark's study is that it is meaningful to compare flipped instruction with non-flipped instruction of the same content, without substantial regard for the variations that are possible within each category.

The fact that these studies found no difference with regard to content learning, and contradictory results with regard to student attitudes about flipped instruction, is unsurprising. Such results are to be expected if it is true that, as we argue, flipped mathematics instruction can vary widely along important dimensions of instruction, just as non-flipped instruction also varies. Nevertheless, these studies provide a positive first step with regard to empirically documenting the phenomena of flipped instruction in mathematics by showing that, in certain situations, flipping alone does not seem to influence student learning. Yet there are substantial limitations to the accumulated knowledge that can be generated from these studies because there is not yet a structural framework for identifying and interpreting the various features of flipped instruction in the cited studies. With such a framework in hand, patterns could be discerned across grade levels and even across articles that include empirical descriptions of flipped instruction without comparisons to non-flipped instruction (e.g., Talbert, 2014; Zack et al., 2015).

It should be noted that other frameworks for flipped instruction exist, but they serve different functions than ours. For example, Chen and colleagues (2014) introduced the FLIPPED model (Flexible environments, Learner-centered approach, Intentional content, Professional educators, Progressive networking, Engaging and effective learning experiences, Diversified and seamless learning platforms) but theirs is a prescriptive model meant to guide the design and implementation of a flipped course. It is not designed to capture the variability across preexisting flipped courses. Furthermore, it is not specific to mathematics instruction and it functions at the course level whereas our framework is applied at the lesson level. DeLozier and Rhodes (2017) provided a more descriptive framework, parsing the out-of-class lectures from the in-class activities and specifying different types of the latter (quizzes, pair-and-share activities, student presentations). In our work, however, we have seen that not all flipped instructors use strictly lecture videos (de Araujo, Otten, & Birisci, 2017b), and DeLozier and Rhodes do not account for the quality of the enacted in-class activities, nor is their work specific to mathematics. Yet we do incorporate their top-level distinction between out-of-class and in-class lesson phases.

To assess the quality of the flipped lessons in a mathematics-specific manner, we draw on existing instruments for lesson observations but make some key modifications. Observation protocols such as the M-SCAN (Walkowiak et al., 2014) and IQA (Boston & Wolff, 2006) presume a certain form of instruction, with cognitively demanding tasks and student discussions, and thus may not be widely applicable to all forms of mathematics instruction. Moreover, these instruments combine various sub-scores into an overall score for the quality of the lesson, even though there is scholarly disagreement about the ideal nature of some of the sub-scores. In other words, although some aspects are universally regarded as indicators of quality instruction (e.g., coherent development of ideas, effective use of multiple representations), other aspects are not universally accepted (e.g., the value of shared mathematical authority, student discussions) (Munter, Stein, & Smith, 2015). Thus, in order to capture the full range of flipped instruction variability in an even-handed way, it would be wise to separate the universally-accepted indicators of quality from those that simply mark different instructional paradigms. We represent this separation in the in-class component of our framework.

A Framework for Flipped Mathematics Instruction

Adapting the overall structure of the Mathematical Task Framework (Stein, Grover, & Henningsen, 1996), our framework (Figure 1) includes the assignment given to students to be completed outside of class (left) and the activity that takes place in class (center), which collectively produce various student outcomes (right). We distinguish between different types of videos (lecture and set-up) and between different interaction formats in class (whole-class and non-whole-class). These distinctions are rooted in our perspective on students' mathematical learning as a sociocultural process through which students are coming to participate actively in a mathematical community (Vygotsky, 1980). This process is an integration of social and individual dimensions and, mathematically, it involves becoming an active participant both in individual meaning-making through thinking or writing, for example, as well as collaborative meaning-making through listening to or interacting with the teacher and peers, for example. In other words, we view the learning of mathematics as not simply the development of an ability to solve certain kinds of problems but the development of the ability to participate in various forms of mathematical discourse and activity, and our framework distinguishes between some of these forms at a structural level. We also attend to the nature of discourse (Staples & Colonis, 2007) and the cognitive demand of tasks (Stein, Grover, & Henningsen, 1996) because these relate to the kinds of activities students have opportunities to participate in.

This framework is based on pilot analyses of flipped classes at several different academic levels and consultation with experts in specific domains (e.g., video design, classroom discourse, mathematical tasks). Due to space limitations, details on the development process are not shared.

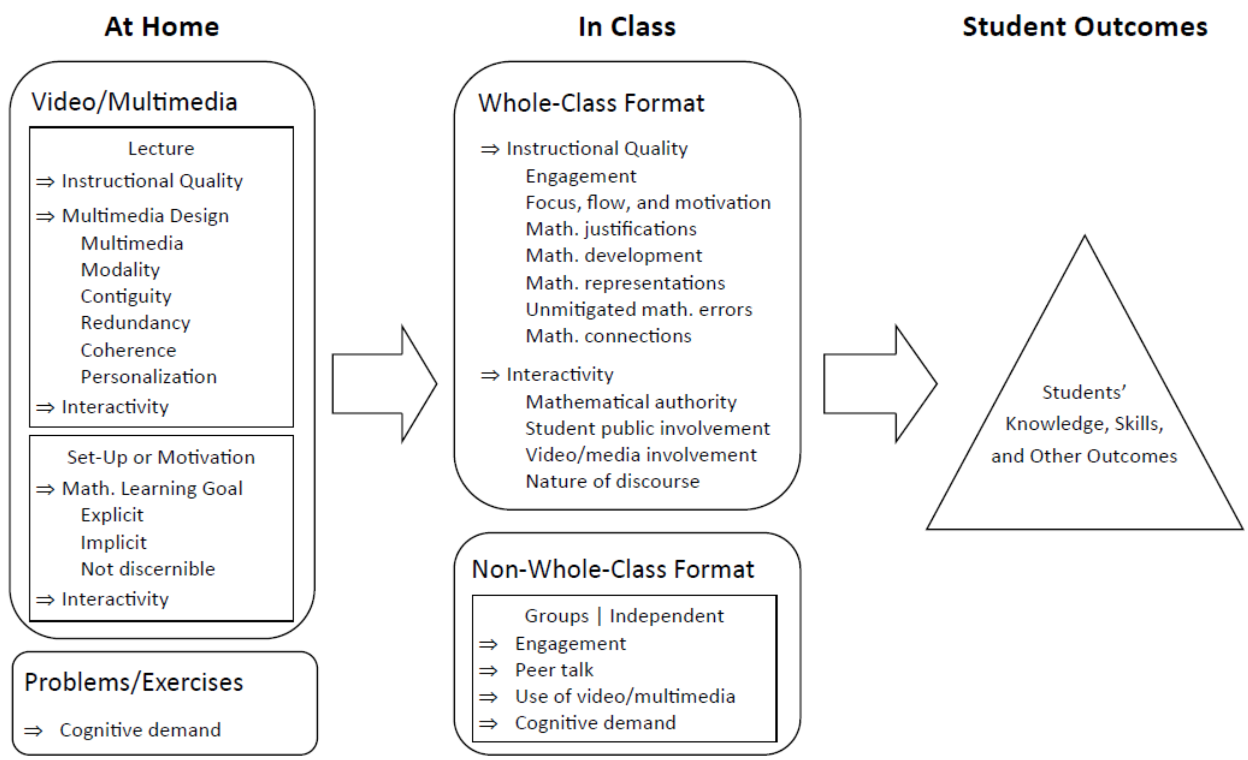

Figure 1. A framework for flipped mathematics instruction

At-Home Components

Looking more closely at the *at-home* components, the framework identifies *video or multimedia* homework in which students are assigned multimedia content to watch, read, or listen to. Although this type of homework is the definitive category for flipped lessons, teachers implementing flipped instruction are not bound to it exclusively. Teachers may, on occasion, also assign *problems or exercises* to be completed outside of class, in which case the framework could be used to show that both or either type of homework was assigned. Also, it is possible that a particular lesson has no homework assigned. Non-flipped lessons, by definition, will have only problems/exercises homework or no homework.

Videos or multimedia. Although we often use the term 'video' because flipped mathematics homework commonly takes this form, we acknowledge that some teachers might use written text, podcasts, animations, or multimedia resources such as iBooks. For video homework, the framework distinguishes between *lecture* videos, which present expository information or worked examples, and *set-up or motivation* videos, which establish a context or pose a problem that will be addressed in class. Within these two types, the framework then specifies normative quality criteria. By normative quality, we mean that instruction satisfying the criteria is hypothesized to be more predictive of positive student outcomes than instruction that does not satisfy the criteria, which could be empirically tested in future studies.

The normative quality criteria for lecture videos address three aspects: instructional quality, multimedia design, and interactivity. *Instructional quality* captures the clarity and flow of the ideas as well as features specific to mathematics, such as mathematical justifications, multiple mathematical representations, the absence of unmitigated errors, and connections to mathematical content that the students may have already learned or will learn in the future. Most of these particular aspects can be operationalized by drawing on existing instruments in the mathematics education literature. In particular, the clarity of the lecture *foci* relates to research on explicit lesson objectives (Wiggins & McTighe, 2005), and the *coherent flow* of ideas relates to all of the video content being connected to that foci. We can also attend to whether there is an intellectual or "real-world" *motivation* for the foci. With regard to mathematics-specific features, we draw on the Studying Teacher Expertise and Assignment in Mathematics (STEAM) project (Tarr et al., 2016), which provides a lesson observation coding scheme for *mathematical justifications*, the *meaningful development of mathematical ideas*, and the *integration of multiple representations*. In addition to these aspects from STEAM, we also include a dimension from the mathematical quality of instruction (MQI) instrument (Learning Mathematics for Teaching Project, 2011), namely, the *absence of unmitigated mathematical errors*. Finally, we attend to whether the video's particular mathematical foci is explicitly *connected to the mathematical ideas* of prior or subsequent lessons.

These features of instructional quality not only apply to the videos but are also used to characterize whole-class discourse in the in-class portion of lessons (see below). Aspects that are specific to videos, however, are multimedia design and interactivity. For *multimedia design* quality, we drew on Clark and Mayer's (2007) six principles of digital material design. The *multimedia principle* states that graphics (not words alone) help students learn the information in question. The *modality principle* states that the graphics should be described or explained with spoken words rather than on-screen text. The *contiguity principle* states that, if some brief on-screen text is used, it should be placed spatially near the relevant graphics. The *redundancy principle* states that the on-screen text should be minimal and should not be a full transcription of the audio narration. The *coherence principle* states that irrelevant graphics and audio should be excluded. Finally, the *personalization principle* states that audio narration should be in a conversational style and it is preferable for the speaker to be visible.

Interactivity captures the extent to which students are expected to be more than passive viewers of the video. Interactivity can be achieved in a variety of ways. For example, Webel, Sheffel, and Conner (2018) studied a teacher whose videos for flipped instruction contained embedded questions that provided the teacher with formative assessment data she could use to inform in-class activities. These embedded questions within the video are one possible form of interactivity. Other forms include requiring students to post a comment or submit a reflection after watching the video or incorporating a virtual manipulative (Moyer-Packenham & Westenskow, 2013) that the students can interact with as part of the homework.

The aspects described above apply to a lecture video, which is the most common type of video in flipped lessons (de Araujo, Otten, & Birisci, 2017b). Our framework, however, also includes *set-up or motivation videos*. Because of a lack of prior research on set-up videos, we drew on practical resources for setting up mathematical work such as the three-act tasks promoted by Meyer (2011) often involving what we call a set-up video that presents a situation and leads to a mathematical question. Our framework distinguishes between different ways in which the set-up videos connect to the mathematical goals of the remainder of the lesson. Specifically, we distinguish between three possibilities: an *explicit connection* to the mathematical goal of the in-class activity that follows (i.e., it is clear from watching the video what the specific mathematical question or problem is, even though its solution is not explicated); an *implicit connection* (i.e., it presents a context that allows for mathematical explorations but it is not clear which mathematical question or problem is intended to be pursued until it is specified later); or *no discernible connection* (i.e., it has a contextual connection or some other association with the in-class activity but does not contain the *mathematical* ideas of the lesson). We do not presume a priori that one of these is more beneficial than the others, but that could be determined empirically through applications of the framework.

Problems/exercises. If students are assigned a problem set to complete outside of class, then the framework attends to the cognitive demand (Stein, Grover, & Henningsen, 1996) of those problems. After coding each item for cognitive demand, the assignment overall can be characterized as *low* (nearly all items at a low level), *moderate* (predominantly low but a substantial minority of items at a high level), or *high* (a majority of the items at a high level). Note that if problems are begun in class and students are then expected to complete any unfinished problems at home, this would fall under the in-class component of group or independent work rather than strictly homework.

Moving on, the arrow between the at-home section of the framework and the in-class section represents the fact that what occurs outside of class potentially influences what occurs in class. For example, if students do not watch the video, the lesson will likely play out differently in class than if they had watched the video attentively. Note that, although videos are typically assigned to be watched prior to the in-class portion of the lesson, some flipped lessons may use a video as a way to summarize ideas after the in-class activity. Thus the arrow is not necessarily chronological and the left-to-right order visible in Figure 1 is not absolute.

In-Class Components

Moving to the *in-class* components, the framework first distinguishes between the interaction format the teacher employs—a *whole-class format* in which the entire class is expected to be attending to the public discourse whether that be the teacher addressing the entire class or students sharing out or asking questions, and a *non-whole-class format* which includes the expectation that students work together in *groups* or *independently*.

Whole-class format. As noted above, this framework differs from others because we separate instructional quality indicators from markers of a particular pedagogical model. For instructional quality we use the same aspects defined above (focus, flow, motivation, mathematical justifications, mathematical development of ideas, multiple mathematical representations, absence of unmitigated mathematical errors, and connections to other mathematical topics) plus an engagement (i.e., on-task) measure to gain a normative sense of the lesson flow and development of mathematical ideas. Additionally, the framework distinguishes between several descriptive aspects of the lesson. These aspects relate to the interactivity of the whole-class segments of the lesson. With regard to *mathematical authority*, we distinguish between teacher/textbook authority and shared/community authority. We also note the extent to which *students are publicly involved* in the whole-class discourse, and if they are, we attend to whether the *nature of discourse* was sharing or collaborative (Staples & Colonis, 2007). Finally, considering flipped lessons in particular, we note whether the *videos are referenced or incorporated* into the discourse (e.g., by reviewing the key ideas from the video homework).

Group or independent work. With regard to group or independent work, the framework includes *engagement* (i.e., on-task behavior), *peer talk* (i.e., the extent to which students verbally interact with one another), and the *use of video/multimedia* from the lesson homework. For example, while solving a problem in class, do students use their phones or tablets to view videos and rewatch explanations of worked examples? We also use the well-known construct of *cognitive demand* (Stein, Grover, & Henningsen, 1996) to characterize the written tasks students work on during group or independent work. Because flipped instruction can provide more in-class time for problem solving, it is plausible that teachers may feel more freedom to enact cognitively demanding tasks in their flipped lessons.

Student Outcomes

The framework does not predetermine the student outcomes of interest because those will largely depend upon the specific context in which the framework is employed, especially with regard to specific mathematical content. However, the framework does acknowledge that several different student outcomes may be of interest in studies of flipped instruction. In addition to measures of students' mathematical knowledge, one may also be interested in their skills with various mathematical practices and their attitudes or affective dispositions toward mathematics. Furthermore, researchers may wish to use multiple measures of a single outcome to increase the robustness of findings. In any event, the specification of these outcomes is somewhat independent of the enactment of lessons through the at-home and in-class phases of the lesson.

Discussion

In presenting this framework, we hope to provide a structure that can support the accumulation of empirical findings related to flipped instruction, inform the design of future studies, and facilitate connections between research and practice. This framework is needed because empirical research on the effectiveness of flipped instruction must move beyond simple flipped/non-flipped comparisons (e.g., Clark, 2015; DeSantis et al., 2015) toward an investigation of the specific features of flipped instruction and the complex interactions of those features as they relate to student outcomes.

Although the framework is built around structural components of flipped lessons, it also goes further by focusing attention on the quality with which certain structural activities occur. In particular, the framework includes quality criteria for lecture videos, set-up videos, whole-class discourse, and it incorporates the levels of cognitive demand (Stein, Grover, & Henningsen, 1996) for the independent or group tasks. This attention to quality will not only allow researchers to distinguish between different structural implementations of flipped instruction (e.g., those who use lecture videos and independent work versus those who use set-up videos and whole-class discussions) but also between different qualities of implementation within the same structure (e.g., productive whole-class discussions versus unproductive ones). To be clear, we do not mean to imply that the quality criteria specified in this article are the only sets of criteria that could be used, nor do we necessarily think they are ideal. Rather, we have proposed some characteristics that, according to our theoretical perspective, are worth considering, but those with different perspectives would likely identify different characteristics, and ultimately we hope that future research can empirically test the benefits of various implementations of flipped instruction.

With regard to empirical studies, the framework can be valuable in formulating testable hypotheses. For example, one can consider whether video features are more predictive of student learning than in-class features. (We hypothesize that the reverse is true.) Additionally, the framework can be used to aggregate across lessons rather than remain at the scope of an individual lesson. Future research could include multiple observations of a flipped class over time and compile the proportional allotments and the quality indicators for the components of flipped instruction. For instance, a teacher may 80% of the time assign lecture videos that satisfy only some of the quality criteria and may assign the other 20% of the time set-up videos that explicitly connect to the lesson's learning goals, and 50% of the in-class time may be spent solving high cognitive demand tasks in groups while the other 50% of the time is spent having high quality whole-class discussions. In this way, multiple class aggregations could be formed (for flipped and non-flipped classes) and then analyzed with respect to student outcome measures. Such work has potential to contribute to our theoretical understanding of mathematics instruction and to inform the efforts of practitioners who are implementing flipped instruction.

Conclusion

Many of the expositional and anecdotal works related to flipped instruction are written in ways that suggest flipped instruction is a unified teaching model. Even the few empirical studies that exist involved research designs that presume flipped instruction is a *singular approach* that can be meaningfully compared to non-flipped instruction. We have instead found that wide variation exists in flipped implementation (de Araujo, Otten, & Birisci, 2017b; Otten, Zhao, de Araujo, & Sherman, in press). Although it is possible that flipped instruction frees up class time for rich whole-class discussions and the enactment of cognitively demanding tasks, we know that there are secondary mathematics teachers using flipped instruction in ways that differ little from conventional non-flipped instruction aside from the change in instructional environment. As a field we do not yet know the prevalence of either of these implementations nor do we have compelling empirical evidence of the implications that either has for student learning. It is our hope that this framework will guide feature endeavors to gather this evidence.

Acknowledgments

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References

Bergmann, J., & Sams, A. (2012). *Flip your classroom: Reach every student in every class every day*. Eugene, OR: International Society for Technology in Education.

Boston, M. D., & Wolf, M. K. (2006). *Assessing academic rigor in mathematics instruction: The development of Instructional Quality Assessment Toolkit* (CSE Tech. Rep. No. 672). Los Angeles: University of California, CRESST.

Chen, Y., Wang, Y., Kinshuk, & Chen, N.-S. (2014). Is FLIP enough? Or should we use the FLIPPED model instead? *Computers and Education, 79*, 16–27.

Clark, K. R. (2015). The effects of the flipped model of instruction on student engagement and performance in the secondary mathematics classroom. *Journal of Educators Online, 12*, 91–115.

de Araujo, Z., Otten, S., & Birisci, S. (2017a). Mathematics teachers' motivations for, conceptions of, and experiences with flipped instruction. *Teaching and Teacher Education, 62*, 60–70.

de Araujo, Z., Otten, S., & Birisci, S. (2017b). Conceptualizing "homework" in flipped mathematics classrooms. *Journal of Educational Technology and Society, 20*, 248–260.

DeLozier, S. J., & Rhodes, M. G. (2017). Flipped classrooms: A review of key ideas and recommendations for practice. *Educational Psychology Review, 29*, 141–151.

DeSantis, J., van Curen, R., Putsch, J., & Metzger, J. (2015). Do students learn more from a flip? An exploration of the efficacy of flipped and traditional lessons. *Journal of Interactive Learning Research, 26*(1), 39–63.

Learning Mathematics for Teaching Project. (2011). Measuring the mathematical quality of instruction. *Journal of Mathematics Teacher Education*, *14*, 25–47.

Meyer, D. (2011, May 11th). The three acts of a mathematical story. Retrieved from http://blog.mrmeyer.com/2011/the-three-acts-of-a-mathematical-story/

Munter, C., Stein, M. K., & Smith, M. S. (2015). Is there a common pedagogical core? Examining instructional practices of competing models of mathematics teaching. *NCSM Journal of Mathematics Education Leadership, 16*(2), 3–13.

Otten, S., Zhao, W., de Araujo, Z., & Sherman, M. (in press). Evaluating videos for flipped instruction. *Mathematics Teaching in the Middle School*.

Smith, D. F. (2014, June 12). How flipped classrooms are growing and changing. *Ed Tech Magazine*. Retrieved on October 27, 2015, from http://www.edtechmagazine.com/k12/article/2014/06/how-flipped-classrooms-are-growing-and-changing

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*, 455–488.

Talbert, R. (2014). Inverting the linear algebra classroom. *PRIMUS, 24*, 361–374.

Tarr, J. E., Campbell, P. F., Webel, C., Markworth, K. A., & Gibbons, L. (2016, April). *Examining the impact of elementary mathematics specialists and coaches.* Paper presented at the 2016 National Council of Teachers of Mathematics Research Conference, San Francisco, CA.

Vygotsky, L. S. (1980). *Mind in society: The development of higher psychological processes*. Harvard university press.

Walkowiak, T. A., Berry, R. Q., Meyer, P., Rimm-Kaufman, S. E., & Ottmar, E. R. (2014). Introducing an observational measure of standards-based mathematics teaching practices: Evidence of validity and score reliability. *Educational Studies in Mathematics, 85*, 109–128.

Wiggins, G., & McTighe, J. (2005). *Understanding by design*. Alexandria, VA: Association for Supervision and Curriculum Development.

Zack, L., Fuselier, J., Graham-Squire, A., Lamb, R., & O'Hara, K. (2015). Flipping freshman mathematics. *PRIMUS*, *25*, 803–813.

ON TEACHING ACTIONS IN MATHEMATICAL PROBLEM-SOLVING CONTEXTS

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In this research report, the question of the role of the teacher in problem-solving contexts is addressed, particularly in relation to the development of mathematical concepts. Data extracts from a problem-solving session are used to draw out three sorts of teaching actions that aim to push forward the mathematics in the classroom. These teaching actions are discussed in light of verbatim extracts and of available theoretical constructs from the research literature.

Keywords: Mathematics, Problem-solving, Teaching Actions, Analytical Geometry

Introduction

The literature abounds with work conducted on mathematical problem-solving from a variety of perspectives, illustrating the beneficial outcomes for students of a sustained practice of solving problems in classrooms: on the meaning given to mathematical concepts and their relevance for everyday life; on the development of critical, logical and autonomous thinking; on the active engagement in doing mathematics; on the development of positive relationships with mathematics, and so on (Boaler, 1998; Borasi, 1992, 1996; English & Gainsburg, 2015). Through this literature, one dimension that appears in need of further research concerns the role of the teacher in problem-solving contexts. As Stein, Boaler and Silver (2004) explain, problem-solving work is mostly focused on students – who are plunged into authentic mathematical practices as are mathematicians – hence the problem-solving literature says little about teachers, their role, and classroom events or actions in which teachers interact. This leads to wonder about *What is the role of the teacher in problem-solving contexts?* In parallel, another salient issue that remains little studied, as English and Gainsburg (2015) illustrate, concerns the development of content in problem-solving contexts. There is a need to investigate the interplay of mathematical concept development and problem-solving endeavors in the classroom, and *How is problem-solving used as a powerful means to develop mathematical concepts*? It is these two issues that orient this research report, namely teachers' actions in problem-solving contexts that aim at covering content and developing concepts, that is, at pushing forward mathematics in the classroom. As Jaworski (e.g., 2011) often noted, we have numerous theories about students' learning and mathematical activity, but the same is not true for teaching and the teacher's role. There is thus an important need to conduct studies that aim to develop conceptualisations of the teacher's roles *in the action* of teaching in problem-solving contexts: this research report aims to contribute to reflections and conceptualizations on these teacher's roles.

In order to do this, an analysis is conducted on an extract taken from a session in a Grade-10 classroom. Once aspects of the research are grounded both theoretically and methodologically, the extract is presented and then analyzed in relation to three kinds of teaching actions that contributed to concept development – that is, that pushed forward the mathematics of the classroom – using theoretical concepts and verbatim extract as supporting illustrations to clarify the nature of these teaching actions.

Theoretical Grounding – Problem Solving as Engaging in a Dynamic Process

Grounded in the enactivist theory of cognition for conceptualizing problem-solving environments as non-linear endeavors (Proulx & Simmt, 2016), the research is also strongly inspired by the work of Borasi (1992) and Lampert (1990) on inquiry and problem-solving, who aim to place students in authentic problem-solving situations. In their work, mathematical problem-solving is conceived of as a process that does not follow a pre-specified thread of events – analogous to the development of mathematics itself as a discipline – where numerous questions and ideas arise amid problem-solving endeavors, these often becoming central issues that can redirect the inquiry being undertaken (see also Cobb et al., 1994).

Remillard and Kaye Geist (2003) termed these "emergent" events as openings in the curriculum, where occasions offer themselves to inquiry and (can) redirect the flow of classroom events; something akin to Van Zoest et al. (2015) notion of building on ideas unfolding in the classroom. Borasi (1992, p. 202) addressed these matters in terms of flexibility, where authentic mathematical problem-solving spaces are conceived for tackling unanticipated events:

> The open-endedness that characterizes inquiry requires extreme flexibility in terms of curriculum content and choices. A teacher will often need to deviate from the original lesson plan in order to follow a new lead, pursue valuable questions raised by the students, or let the class fully engage in a debate stimulated by difference in opinion or different solutions.

Mathematics teaching is thus here conceived as a dynamic process that emerges amid interactions between teacher and students. As Curcio and Artzt (2004) assert, teachers are themselves engaged in problem-solving when they teach through problem-solving, deploying an expertise in action, while teaching, in reaction to the events of the classroom. Teachers navigate through the "material" of the classroom, act with what happens, and attempt to push the mathematics of the classroom forward through analyzing and synthesizing on-the-spot the mathematical ideas shared and produced. It is in this sense that, in Bednarz and Proulx (2009), teachers' actions are conceived along three interconnected dimensions. *First*, following work on mathematics teachers' practices (e.g., Roditi, 2005), teaching is conceived of as an event that happens in the action of, in relation to, the task in which the teacher is engaged: *teaching is enacted in action*. *Second*, along a situated cognition perspective (Lave, 1988), teaching is conceived of as situated, deployed in relation to a specific context. As Roditi (2005) insists, teaching actions are not independent of students' learning or the classroom environment, a context that plays an immense role in the kinds of actions deployed in the act of teaching: *teaching is a situated practice*. *Third*, aligned with Mason and Spence's (1999) knowing-to act in the moment, teaching is conceived of as a practice adapted in real time to events of the classroom. Teachers constantly need to adapt their responses to the dynamics of the classroom as situations often drift from the planned script: *teaching is a practice deployed in the moment*, while it is enacted. Along these dimensions, teachers are seen as continually reflecting on possibilities, offering and inventing new avenues and representations in relation to students, thinking of additional explanations to clarify or resituate the tasks offered, choosing to emphasize some aspects and not others, knowing that an explanation or a representation may eventually benefit students' understanding, and so on. Teaching practice can thus hardly be considered as a preestablished practice designed in advance for reacting to situations, but rather an expertise enacted in context, in action, as a knowing-to act adapted to situations and deployed on-the-spot in (inter)relation to classroom events. It is along this theoretical perspective that teachers' actions for pushing forward the mathematics of the classroom, are considered.

Methodological Considerations

This research report is part of a wider research program focused on studying the teaching of mathematics through problem-solving in elementary and secondary classrooms. We collaborate with groups of teachers who regularly invite us into their classrooms to experiment various kinds of problem-solving approaches and to interact, assess and reflect with us on the teaching that goes on in these sessions. Because it inserts itself in regular classrooms, the research does not want to be disruptive and follows the teachers' teaching plans, with the tasks given in class to students being chosen by and with teachers (often coming from their teaching materials and workbooks). The problem-solving sessions usually follow the same trend, starting with a task presented to students, written on the board or handed on paper, where students are given relative amounts of time to address it. After this, in a plenary manner, students are asked to share their strategies/solutions and thoughts with the group, while making sure that these are clearly explained and justified for other students to understand and ask additional questions if necessary. Students are also invited to interact between each other in relation to the ideas shared, to question or challenge them, add to them, etc., thus aiming to create a community of inquiry (Borasi, 1992; Lampert, 1990). These various interactions in turn often provoke new inquiries, where students can be asked to explore new issues or additional questions (Cobb et al., 1994).

Data-collection focuses on classroom discussions and interactions, as well as traces left on the board, all chronologically recorded as field notes by a research assistant (RA) or videotaped. Data analysis is carried out in two phases. The *first phase* consists of on-the-spot meetings (PI, RA, and teachers) to discuss teaching events that occurred during the sessions that stood out and deserved attention (in this case, on teaching actions that pushed the mathematics forward). These meetings offer a first level of analysis, which also affords interpretations of teaching events from the teachers' perspectives and permits refinement of, and adds to, the observational notes. This first level of analysis revealed salient issues about three explicit teaching actions that succeeded in pushing forward the mathematics of the classroom, that is, actions that enabled mathematical concept development: validation practices, reformulation practices, and summarizing practices. This three-pronged orientation toward teaching actions was used to orient the subsequent data analysis. This *second phase* consisted of attending to the data in relation to Desgagné's (1998) notion of available constructs from the mathematics education literature, which could give these teaching actions deeper theoretical meaning: here, e.g., Forman and Ansell's (2001) revoicing, Shimuru's (2004) *yamada*, Lampert's (1990) establishment of a community of validation.

Data Extract from the Problem-Solving Session

The extract is taken from a session in a Grade-10 classroom of about 30 students, who were working on analytical geometry in relation to distances (points, midpoints, lines, etc.) and had been initiated to usual algebraic formulas. This extract was chosen for its capacity to illustrate patterns of teacher and student interactions that were common to almost every session conducted/experimented in classrooms. For this precise session, the teacher wished to experiment with tasks along a mental computation context (following our work on mental computation, e.g., Proulx, 2014), with the intention to see how students would engage in it. One task given to students was "*Find the distance between (0,0) and (4,3) in the plane*" (given orally, with points drawn on a Cartesian plane on the front board), who had 15 seconds to answer without recourse to paper and pencil or any other material. When time was up, students were invited to share and justify their solutions to the group. The following is a synthesis of the strategies engaged in and the discussions, questions, and explorations that ensued.

The first strategy referred to applying the usual distance formula (D=$\sqrt{(x_1-x_2)^2+\ (y_1-y_2)^2}$), leading to 5 as a distance. A second strategy suggested drawing a triangle in the plane, with sides 3 and 4, for then finding the hypotenuse by using Pythagoras (Figure 1a).

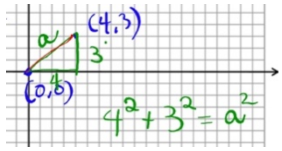


Figure 1a – Drawing the right triangle

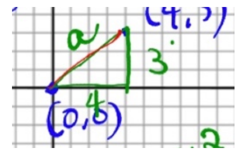


Figure 1b – Close-up on the triangle

Another student then suggested a third strategy: coming to the board to trace a red segment to count directly on it from (0,0) to (4,3) as in Figure 1b. Starting from (0,0), she counted "the number of points" to arrive at (4,3), counting the number of whole-number coordinate points from (0,0) to (4,3). While doing this at the board, she suddenly stopped and mentioned that her red segment did not go through the points she envisaged, which made the counting difficult. The teacher then traced another segment going through square diagonals linking two separate points, which could enable counting the number of (whole-number) coordinate points from one point to the next (giving 4 as a distance, Figure 2). The student agreed that for this case, it would work.

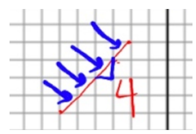
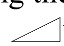


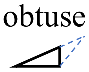
Figure 2 – Line drawn through square diagonals

The teacher then asked if the measure obtained with square diagonal lengths was identical to that obtained with the side of the square (drawing  on the board).

> One student asserted that both lengths were not identical, because the diagonal of the square was not of the same length as the square's side. Another explained that both lengths were different, because the hypotenuse is always the longer side in a triangle. Finally, a student claimed that the diagonal was longer, because it faces the wider angle.

The teacher then asked if that last assertion about facing the wider angle was always true, and if so why (drawing on the board a random right triangle ).

> One student, pointing at the triangle, stated that it was indeed the case in this drawn triangle. Another student explained that, in a triangle, the bigger the angle the longer the opposite side, mentioning that if the side-hypotenuse had been longer, the opposite angle would have been wider. And, because the sum of the (measures of the) angles in a triangle is 180º, then the 90º angle is always the wider one, the other 90º being shared between the remaining two angles.

Using the drawing of the triangle, the teacher simulated the variation of the right angle toward an obtuse one and traced the resulting side obtained, showing how it would become longer (drawing  on the board). He then moved it toward producing an acute angle, asking students if their "theory" about opposite side of the angle worked for any angle, like acute ones.

> One student asserted that it works for isosceles triangles, with equal sides facing equal angles, and another mentioned that it is the same for the equilateral triangle, since it is "everywhere the same" with same angles and same side lengths.

The teacher explained that these ideas about the diagonals being longer than the side underlined the fact that this initial strategy amounted to counting diagonals, that is, the number of diagonals of a unit square. And, that this offered a different sort of measure for the (same) distance between the two points: one in terms of units and one in terms of diagonals. A student added that if one knows the value of the diagonal, then one could find the number of unit squares for the diagonal-segment by multiplying by that factor.

One student offered a fourth strategy to find the distance, suggesting to use the sine law with angles of 45°. The teacher asked the student how he knew that both angles were 45° in the triangle. As skepticism grew in the classroom, the teacher suggested that students inquire, in small groups or individually, if the triangle's angle were 45° or not, and to be able to convince others. After 5-6 minutes of exploration, students were invited to share their findings.

> One student explained that on her exam checklist there is an isosceles right triangle with 45° angles. Thus with this triangle of side length of 4 and 3, one cannot directly assert that it is 45° because it is not an isosceles triangle as its sides are not of equal measure. Another student illustrated at the board that if one "completes" the initial triangle into a rectangle ( , see Figure 3a), then the hypotenuses of both triangles are the rectangle's diagonal which cuts it in two equal parts and thus cuts its angle in two equal 45° parts.

As the teacher highlighted that the two arguments were opposed, one student replied not in agreement with the last argument, drawing on the board a random rectangle with its diagonal (Figure 3b), and asserting that in this rectangle *it was not certain* that the angle was divided into two equal parts. Another student added that because the sides of the triangle were not identical (of 3 and 4), then the diagonal would not necessarily cut the 90° angle in two equal parts of 45°.

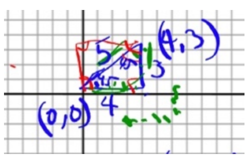


Figure 3a – The "completed" rectangle

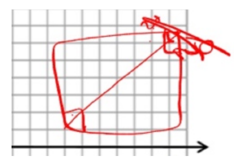


Figure 3b – The "counter" rectangle

The teacher highlighted that this last argument reused aspects of the precedent "theory" that the longer side faces the wider angle in the triangle. Hence, following this, a longer side needed to face a wider angle. Then a counter-example was offered to the group.

> The student who made reference to the checklist asserted that it happens in their exams that right triangles don't have 45° angles, for example, one with 32° and 58°; coming to the board to draw it (Figure 4). She completed her drawing to create a rectangle, explaining that the diagonal cuts as well this rectangle in two parts, but that the angles obtained are not of 45°.

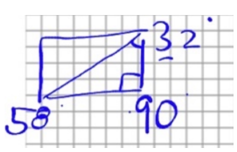


Figure 4 – The triangle counter-example with angles of 32° and 58°, and the rectangle

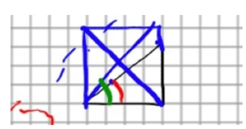


Figure 5 – Comparing triangles within a square

The teacher asserted that this offered a counter-example, with a type of right triangle frequently met that did not have angles of 45°. One student added that because all sides were different, then their associated angles would be different, the longer side needed to face a wider angle, leading to different angles. The teacher then highlighted the work of one student who drew a square in his notebook to assess the 45º situation. Drawing a triangle of sides 3-4-5, he extended the cathetus of 3 toward one of 4 to create a 4x4 square. Then, because in the previous unit-square the angles were of 45º, in this 4x4 they were 45º as well (Figure 5). Comparing hypotenuses of both triangles, it illustrated that in the initial 3-4-5 right triangle the angle is smaller than the right triangle of side 4 and 4. All this led students to appear to agree with the fact that the angle was not 45º, ending the explorations (and leading to offer another task to be solved by the students).

Pushing Forward the Mathematics: Validating Practices

One kind of teaching actions enacted in the session are validation practices. These, in problem-solving contexts, are related to the consideration of the classroom as a mathematical community of validation as Boaler (1998), Borasi (1992) and Lampert (1990) call it. In this community, members are encouraged to generate ideas, questions, and problems, to solve them, to share their understanding, to negotiate meaning, to develop explanations and justifications to support their solutions, to question others' solutions, and so on. As Lampert (1990) explains it is the teacher's role to makes sure that students' justifications are adequate, that arguments are clearly stated, that solutions are expressed in an intelligible manner and are accessible; essential conditions for the mathematical community to take shape and flourish. In sum, it is the teacher's role to create and sustain this mathematical community. As a way of example, when Max mentioned that the triangle's angle was 45º for using the sine law, the following interaction led to a request for validation from the classroom community:

Teacher: And, how do you know that the angle is 45º? [some students appeared opposed]
Max: Because it is 45º for both angles [some students agree, others express disagreement]
Teacher: Ok, so maybe 45º maybe not. We are not sure. But we need to arrive at something, we need to agree, we need to know if it is or not 45º. So, I will ask you to take a couple of minutes, alone or in small groups, to see if it is or not 45º, and to be able to explain it and even be able to convince others of it. You can use all you have, workbooks, textbooks, notes, whatever. After that, we will share your findings. Ok, go! [students start to work]

For Cobb et al. (1994) and Lampert (1990), a community of validation develops, analyzes, questions, and argues for what is or not mathematically acceptable. Hence the mathematics produced within the classroom community is validated by the community itself (in which the teacher takes part). The teacher's requirement for and establishment of these validation practices aim to give status to the mathematical productions of the classroom, to make official the ideas shared by making them accessible and reusable in the future as they are justified and argued for. These validation practices contribute to the advancement of mathematical ideas in the classroom.

Pushing Forward the Mathematics: Reformulating Practices

Another sort of teaching actions relates to the reformulation of students' ideas. While students share their strategies, as Cobb and Yackel (1998) underline, the teacher can re-describe students' reasoning and reformulate it in different terms: one in which they would not necessarily have used but that are aligned with the meaning they are making. As an example, when Sandra explained her strategy of counting the points on the red segment, the teacher not only asked her to clarify what she did, but also re-explained and reformulated the underlying idea.

Sandra: Well, I just counted the points.
Teacher: What do you mean exactly by this?
Sandra: (coming to the board) Well, if you draw a line from here to here, it gives 1, 2, 3…

Teacher: Ah, OK, you trace a segment from one point to the other, so from (0,0) to (4,3).
Sandra: Yes. But, finally, this thing does not work as I thought.
Teacher: Ok, it is difficult to count directly on the plane. So, Sandra, you wanted to count the number of coordinate points that your segment crosses from (0,0) to (4,3). And it's difficult because it does not cross them directly. But it could work! If you take this segment and link these two other points (Figure 2). Then, here, it goes though the square diagonals and we can directly count the points crossed on the plane.

The teacher here resumed and reformulated Sandra's explanations, not to correct them (even if some adjustments could be made), but mostly to clarify them, deepening them, drawing out its key elements for all to see. This is aligned with what Forman and Ansell (2001) call *revoicing*:

> there is a greater tendency for students to provide the explanations […] and for the teacher to repeat, expand, recast, or translate students' explanations for the speaker and the rest of the class. The teacher revoices students' contributions to the conversation so as to articulate presupposed information, emphasize particular aspects of the explanation. (p. 119)

These reformulating practices are a way for the teacher to insert him or herself into students' explanations and work with the mathematical ideas produced in the moment. It is also a way to make ideas accessible to the classroom community for engaging in subsequent validation practices, since some ideas are not always mathematically adequate (like numerous arguments made around the 45^o angle). These are then also clarified and reformulated for students to understand and validate as a community. By making the mathematical ideas accessible to all and highlighting them, the teacher is making the mathematics of the classroom advance.

Pushing Forward the Mathematics: Summarizing Practices

A third sort of teaching action is about summarizing practices. At different moments in the classroom, when ideas are shared in the problem-solving process, the teacher underlines explicitly some ideas produced that have important mathematical potential and to which students need to pay attention: those that will be useful or reused in the future. Hence in problem-solving contexts, the teacher has an important role to play in underlying the important mathematical ideas produced and making sure that these are clear and accessible, as Stein et al. (2004) mention. These summarizing practices can happen at varying moments during a problem-solving session, and the teacher at any moment can opt to underline, validate, put forth, thus summarize, the mathematical ideas shared for leaving traces about the mathematical productions of the classroom. For example, in relation to the query about the square diagonal, after arguments and ideas were shared, the teacher summed up the ideas and came back on Sandra's strategy, extending it to the calculation of distances:

Teacher: Let's go back to the square diagonal. It is longer than the side. In Sandra's strategy, we count the number of diagonals, the ones from one point to the other, from (0,0) to (4,3). We talked about points crossing. We would here have 4 diagonals, which is also a possible measure of the distance between (0,0) and (4,3). We would thus have two measures: one of 5 in terms of square-sides and one of 4 in terms of square-diagonals. Same distance, measured in two different ways, hence offering two different measures.

When it is done at the end of the inquiry, as Shimizu (2004) mentions, this summing up is an occasion to conclude by highlighting the mathematical ideas worked on during that inquiry, what is called in Japanese the *yamada* of the activity. For example, the strategy offered for concluding the 45^o exploration was a way of summing up most ideas shared in class and of settling the issue.

Summarizing practices in problem-solving contexts are essential acts that contributes to leaving mathematical traces of what was covered in the classroom, enabling students to pay attention to important mathematical concepts, hence pushing the mathematics forward.

Final Remarks

These three teaching actions succeeded in pushing the mathematics forward in the classroom, highlighting the important role of the teacher in the evolution of the classroom and how mathematics is tackled and develops in it. The analysis of these teaching actions offers an initial way, albeit preliminary, to address the questions that triggered this research report, namely, about the possible teacher actions that aim to contribute to the development of mathematical concepts in problem-solving contexts. Much more needs to be studied in terms of the outcomes of these practices, for example, on student's personal concept-development and how they operationalize these in various situations. But, at this point, this initial analysis offers promise for better understanding and conceptualization of teachers' roles in problem-solving contexts.

References

Bednarz, N., & Proulx, J. (2009). Knowing and using mathematics in teaching: Conceptual and epistemological clarifications. *For the Learning of Mathematics*, *29*(3), 11-17.

Boaler, J. (1998). Open and closed mathematics: student experiences and understandings. *Journal for Research in Mathematics Education*, *29*(1), 41–62.

Borasi, R. (1992). *Teaching mathematics through inquiry*. Heineman: Portsmouth, US.

Cobb, P., Perlwitz, M., & Underwood, D. (1994). Construction individuelle, acculturation mathématique et communauté scolaire. *Revue des sciences de l'éducation*, *20*(1), 41-61.

Cobb, P., & Yackel, E. (1998). A constructivist perspective on the culture of the mathematics classroom. *The Culture of the Mathematics Classroom*, (pp. 158-190). Cambridge University Press: Cambridge.

Curcio, F.R., & Artzt, A.F. (2004). Reflecting on teaching mathematics through problem solving. *Teaching mathematics through problem solving – K-6* (pp. 127-141). NCTM: Reston, VA.

Desgagné, S. (1998). La position du chercheur en recherche collaborative: illustration d'une démarche de médiation entre culture universitaire et culture scolaire. *Recherches qualitatives*, *18*, 77-105.

English, L., & Gainsburg, J. (2015). Problem solving in a 21st century mathematics curriculum. *Handbook of international research in mathematics education* (pp. 313-35). Routledge: UK.

Forman, E., & Ansell, E. (2001). The Multiple Voices of a Mathematics Classroom Community. *Educational Studies in Mathematics*, *46*, 115-142.

Jaworski, B. (2011). Situating Mathematics Teacher Education in a global context. *International Approaches to Professional Development for Mathematics Teachers,* (pp. 2- 50). Ottawa: University of Ottawa Press.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer. *American Educational Research Journal*, *27*(1), 29-63.

Lave, J. (1988). *Cognition in practice.* Cambridge: Cambridge University Press.

Mason, J., & Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing-to act in the moment. *Educational Studies in Mathematics*, *38*(1-3), 135-161.

Proulx, J. (2014). Mental mathematics and operations on functions. *Proceedings of the Joint Meeting of PME-38 and PME-NA 36* (Vol. 1, pp. 263-73). Vancouver, Canada: PME.

Proulx, J., & Simmt, E. (2016). Distinguishing enactivism from constructivism: engaging with new possibilities. *Proceedings of PME-40* (Vol. 4, pp. 99-106). Szeged, Hungary.

Remillard, J.T., & Kaye Geist, P. (2002). Supporting teachers' professional learning by navigating openings in the curriculum. *Journal of Mathematics Teacher Education*, *5*, 7-34.

Roditi, E. (2005). Les pratiques enseignantes en mathématiques. Paris : L'Harmattan.

Shimizu, Y. (2004). Problem solving as a vehicle for teaching mathematics: a Japanese perspective. *Teaching mathematics through problem solving – K-6* (pp. 205-214). NCTM: Reston, VA.

Stein, M.K., Boaler, J., Silver, E. (2004). Teaching mathematics through problem solving: Research perspectives. *Teaching mathematics through problem solving* (pp. 245-56). NCTM: VA.

Van Zoest, L.R. et al. (2015). Theorizing the mathematical point of building on student mathematical thinking. *Proceedings of PME-40* (Vol. 4, pp. 323-30). Szeged, Hungary.

EMERGENT BILINGUALS AS TEACHERS: CONSTRUCTING STORYLINES IN AN ELEMENTARY MATHEMATICS CLASSROOM

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Stereotypes and storylines for emergent bilinguals (EBs) permeate U.S. culture, have been historically deficit-oriented, and determine ways teachers and students interact with each other in classroom contexts. As a way to disrupt and challenge such narratives, one elementary teacher, Courtney, leveraged her authority to construct and foster the storyline of mathematical competence for EBs across multiple school years. One of the most frequent ways Courtney did this was to interactively position EBs as teachers in whole class mathematical interactions.

Keywords: Equity and Diversity, Classroom Discourse

Stereotypes and storylines of mathematical competence permeate U.S. culture (Nasir, 2016). For emergent bilinguals (EBs), their stereotype and storyline has historically been deficit-oriented, focusing solely on their linguistic deficiencies, need of remediation, and ill-preparedness for school, and the added challenges they pose to over-worked teachers. Such stereotypes and storylines can circulate in classrooms and determine ways teachers and students interact with EBs (de Araujo, Smith, & Sakow, 2016; Turner, Dominguez, Maldonado, & Empson, 2013; Wood, 2013; Yoon, 2008). For instance, if teachers position EBs in deficit storylines, EBs opportunities to learn, have access to, and achieve success in mathematics are restricted. In contrast, when teachers position EBs in ways that call attention to and value their mathematical and linguistic competencies and diverse cultural assets and experiences, a storyline of mathematical competence can be fostered. Too frequently, however, EBs are shut out of such storylines. Thus, it is critical for productive storylines be established and fostered for EBs in classrooms. Yet, to do this, teachers must be strategic in their use of classroom discourse to interactively position EBs in identified and desired storylines.

One theoretical lens to examine the creation of storylines for EBs via classroom discourse is positioning theory (van Langenhove & Harré, 1999). Positioning theory foregrounds discourse and proffers a way to critically analyze how language-in-use can construct and foster storylines for groups of students through moment-to-moment interactions. More specifically, positioning theory enables the researcher to examine how teachers' interactive positions of EBs individually acts to establish and foster different storylines for EBs collectively over time.

Positioning Theory and the Role of the Teacher

Positioning theory (van Langenhove & Harré, 1999) is composed of three central components: acts, storylines, and positions. *Acts* refer to the social meaning(s) of people's intended actions, (Harré, Moghaddam, Cairnie, Rothbart, & Sabat, 2009). *Storylines* are "strips of life [that] unfold according to local narrative conventions" (Harré, 2012, p. 198) that are constituted and reconstituted through social interactions. Oftentimes, these refer to the multiple categories, stereotypes, or cultural values (e.g., teacher/student, reform/traditional instruction, correct/incorrect) people draw on in social situations that are used to define the expectations and conventions of interactions in that setting (Herbel-Eisenmann, Wagner, Johnson, Suh, & Figueras, 2015). For example, a teacher in a classroom may draw on the storylines of teacher/student, reform/traditional instruction, and correct/incorrect mathematics answers simultaneously to motivate his/her interactions with students. Hence, within each social interaction there are multiple storylines at play, all drawn from and on participants' cultural, historical, and political backgrounds and experiences. Additionally, the ways individuals enact storylines are or become socially recognizable. For instance, if a teacher employs a storyline that contradicts historical or culturally shared narratives (e.g., incorrect answers are just as valuable as correct answers), the acts may not initially be conceived as socially recognizable, however, over time they become socially recognizable as the storyline is taken up.

The ways people enact storylines are referred to as *positions*. Within storylines, people assume a position, called reflexive positioning, and are positioned by others, called interactive positioning. These positions are metaphorical and determine the social expectations and range of available acts of speakers and listeners. In this way, each position is relational and the range of acts that are available is contingent upon each given situation. Some situations can further restrict the range of available positions. For example, in institutional settings, such as schools, individuals' rights and duties are socially prescribed.

Role of the Teacher

Given their position in the classroom, teachers' discursive practices represent a powerful tool that can be leveraged to create, foster, or shift storylines for groups of students in mathematics. The storylines established and fostered by the teacher can expand or restrict students' ability to exercise agency, participate, and learn. For example, if a teacher constructs a storyline of mathematical competence for an EB that is supported through her/his positioning moves (e.g., calls on student to share her/his mathematical thinking, invites student to take on the role of teacher, allows student to control the mathematical tools), the social expectations, opportunities to co-construct mathematics, and range of available positions would be different compared to a student whose storyline was of mathematical incompetence. The consequences of teacher positioning are further magnified by the appropriation of positions and storylines by peers (Turner et al., 2013; Yoon, 2008). Thus, teachers' interactive positions and storylines of EBs can affect EBs' ability to participate, contribute, and learn.

When teachers' discursive practices are employed in whole class settings, the interactive positions and subsequent storylines for EBs are made visible to everyone. Consequently, research (Turner et al., 2013; Wood, 2013) has found that these public displays affect how peers interactively position EBs in the future. Moreover, the storylines constructed by teachers can act to counter or reinforce storylines for individuals through group associations. For instance, storylines constructed for individual EBs can act to counter deficit-oriented stereotypes of EBs as a whole. In order to better understand the ways teachers' interactive positions can create and foster storylines for EBs that can counter deficit-oriented storylines, this study was guided by the question, *What storylines for EBs does an elementary teacher construct and foster through her acts in whole-class mathematical interactions?*

Methodology

Data for this study were drawn from a three-year longitudinal professional development intervention focused on EBs development of mathematic and linguistic competencies, the enhancement of mathematics curriculum for EBs, and the facilitation of productive classroom interactions for EBs (see Chval, Pinnow, & Thomas, 2014 for more information). Data collected included classroom video and audio recordings.

To answer the research question, a case study design (Stake, 1995) was used to examine the discursive practices of Courtney, a white female, monolingual third-grade teacher. Courtney taught at a school that was predominately white (>70%) with less than 10% of the student population Latinx. In addition, over half of students received free and reduced lunch. Prior to Courtney's participation in the intervention, she had taught for two years and had received no formal education in pedagogy for EBs or had opportunities to teach EBs. Consequently, Courtney's participation in the intervention coincided with her first opportunity to teach EBs. Thereafter, in each year of the study Courtney taught 1-3 Latinx EBs.

Data Selection and analysis

To identify the storylines Courtney constructed and fostered for EBs through her interactive positions, a subset of the data was analyzed. Data was restricted to the first two years of the study due to the number and retention of EBs. In each of the first two years, Courtney had three Latinx EBs (see Table 1 for a summary of student demographics). Each student was classified as an English language learner by the school district based on their scores on the Assessing Comprehension and Communication in English State-to-State for English Language Learners (ACCESS) assessment. Based on their ACCESS composite scores, the EBs in this study can be considered to be at an "intermediate" level of English language proficiency.

Table 1: Summary of student demographics

Year	Student	Birthplace	ACCESS Composite Score
1	Alonzo	Mexico	4.6*
1	Yasmin	USA	3.8*
1	Ignacio	Mexico	3.9*
2	Samuel	USA	4
2	Bryce	USA	3.8
2	Lea	USA	Not available

*ACCESS scores were only available for the following school year.

Across the two years, there were a total of 45 lessons, each approximately one hour long, that were video and audio recorded. To identify the storylines created and fostered for EBs related to mathematics, only those interactions where an EB participated, was asked to participate, or was interactively positioned by the teacher or peers in whole class settings were transcribed and analyzed. In my analysis, I first identified the Courtney's acts at the utterance and turn taking levels in MAXQDA (see Figure 1). Then, I open coded (Strauss & Corbin, 1990) the acts to identify Courtney's interactive positions of EBs and the storylines that were told via the acts.

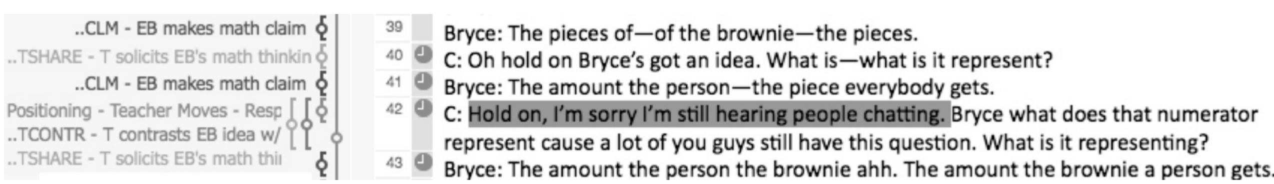

Figure 1. Example of a coded classroom transcript in MAXQDA.

Findings

Across the two years, Courtney constructed and fostered multiple storylines for EBs via her interactive positions. One of the most frequent storylines she promoted was EBs as teachers. Courtney cultivated this storyline in numerous ways, such as asking or allowing EBs to: share problem-solving strategies, control the classroom conversation, assign work, revoice a peer's thinking, and correct peers' mathematical errors. In this way, Courtney positioned EBs physically or metaphorically as teachers, which was a privileged role in the institutional space (of the classroom). To illustrate the several ways this storyline was advanced in classroom interactions, representative classroom excerpts are presented below.

Share a Problem-Solving Strategy

Courtney leveraged problem-solving sharing opportunities as a chance to position EBs as teachers. Frequently, these sharing opportunities occurred at the conclusion of the lesson and were used as a way to share different student approaches to a single problem or highlight specific aspects of a problem-solving strategy (e.g., how to draw efficient pictures). As students worked individually or in small groups on mathematics tasks, Courtney circulated and traditionally selected three students to share who represented a range of strategies. Often, each student's work was scanned, which served as a visual referent as the student shared at the board. This was a particularly useful tool for EBs since they could gesture and point while explaining in English. When students presented, they were physically and metaphorically positioned as a teacher. In this way, the student was physically located in the space routinely reserved for the teacher (i.e., standing at the front of the classroom while peers sat on the carpet) that acted to garner the attention of the class. As a result, the student was positioned as an expert who had ideas worthy of consideration and could explain those ideas to peers.

The following transcript from Oct. 22 illustrates this practice. At the close of the lesson, Courtney selected three students to share their strategy to the problem, "Clayton rolled 8 dice. Each landed on 4. What was Clayton's total? How do you know that is Clayton's total?" Lea (EB) was the second student to share and her work was shown on board (see Figure 2).

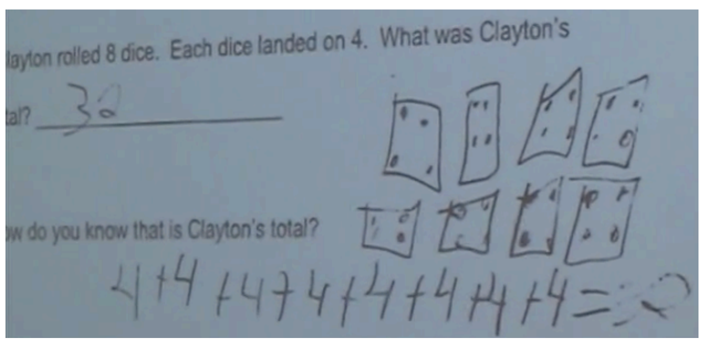

Figure 2. Lea's written work to solve the problem of calculating the sum for eight die each showing a value of four.

1 Courtney: Alright the next person to share is Lea. Lea will you get up. [Administrative talk]
2 Lea: (gets up and comes to board) First um I added four and then um I added four plus four plus four plus four. First I drew a picture of eight dice and then added four plus four plus four plus four plus four plus four plus four plus four equals 32.
3 Courtney: So why did you—how many four—how many times did you need to count up by four?
4 Lea: Well I needed to count um I needed to count eight times so I could get (inaudible)
5 Courtney: Ok so when she—when she—what'd um—any comments about Lea's strategy?
6 Lea: Janie.
7 Janie: Nice work and I like your strategy.
8 Lea: Carl
9 Carl: I like the way how you like, drew a picture of this stuff (gestures across work)—numbers.
10 Lea: Ok. Laura

11 Laura: Um I think the way that you drew your picture of the four plus four plus four kind of (inaudible) but I still think you did a great job.
12 Courtney: Alright so she wrote a number sentence and she wrote up a picture to go along with that. I like—I like your strategy a lot. Nice job Lea. (clapping and cheers)

In this excerpt, we see how Courtney invited Lea to the front of the room to share her strategy with the class with her written work as a visual referent. This act physically positioned Lea in the role of the teacher (at the board), metaphorically positioned her as an expert who had a problem-solving she could explain and peers could learn from, and provided an opportunity to be an active classroom participant. At the board, Lea described her strategy for calculating the sum of 32 (turn 2), which included drawing a representation of the dice rolled (Figure 2). Next, Courtney asked Lea for clarification of the number of times she added four (turn 3), which enabled Lea extended talk time and to maintain control of the conversational floor.
In sum, interactions such as this illustrate the ways Courtney interactively positioned EBs as teachers by allowing EBs to physically assume the space of the teacher, explain problem-solving strategies to peers, and control the classroom conversation.

Control Classroom Conversation

Another way Courtney interactively positioned EBs as teachers was to allow EBs to control the classroom conversation around their mathematical ideas, which occurred after EBs had shared their problem-solving strategies. Frequently, Courtney would ask classmates if they had any comments, questions, or compliments for the presenter, which the speaker would then field. Not only did this practice provide opportunities for EBs to retain the conversational floor but also reinforced the expectation that peers will consider and respond to EBs' mathematical ideas.

The transcript above illustrates a common way Courtney employed this practice. After Lea shared her problem-solving strategy, Courtney invited peers to comment (turn 5). This act reinforced the classroom norm that students would attend to and think about each other's mathematical thinking, signaled that Lea's ideas were worthy of further consideration, and kept Lea's mathematical thinking at the center of the conversation. Courtney then allowed Lea to control the conversation of peer comments on her problem-solving strategy, a practice that is often reserved for the teacher (Mehan, 1979). Lea fielded comments on her mathematical thinking from Janie, Carl, and Laura (turns 7, 9, and 11). Lea's peers had many comments they could make, however, in each comment there is evidence of their attention to Lea's mathematical thinking, particularly her problem-solving strategy and representation. Moreover, each peer comment included praise (e.g., "I like your strategy"), although Laura's was back-handed (turn 11). Courtney did not allow the conversation around Lea's ideas to end on this note, but concluded with a positive evaluation, "I like your strategy a lot."

Overall, this excerpt illustrates another way Courtney interactively positioned EBs in the role of the teacher, both physically and metaphorically, which advanced the storyline of EBs as teachers. These interactive positions also allowed opportunities for EBs to use mathematical discourse, have extended talk time, and control the conversational floor.

Assign Work

Assigning work to the class is a practice often reserved for the teacher, however, there were instances in the data where Courtney would allow EBs to do just this. For example, on Nov. 11, Courtney walked through a task with the class at the carpet. In the task, each student was to create their own book of stamps. To do this, they would determine the face value of a stamp, the total number of stamps in their book (shaped in an array), and the total value of the book. In the demonstration, Courtney asked Alonzo to select the face value of the stamp and then calculate the total value of the book. After Alonzo did this, Courtney asked the class what she must do next according to the checklist of instructions.

1 Courtney: So all of these altogether is 75, (writes 75 on board) 75 what, dollars?
2 Choral: 75 cents
3 Courtney: Ok, so should I just write that [the total value of the book] or
4 Student: No
5 Courtney: What's the next step that I have on my checklist, Clark what's the next step I have on my checklist?
6 Clark: Um explain, explain your strategy.
7 Courtney: Oh, I have to explain my strategy. Ok so how can I explain what Alonzo had asked us to do? Yasmin, how can I explain it?

What is important in this interaction is the way Courtney framed the next step of the instructions to "explain your strategy." Although the checklist specified students should do this, Courtney reframed this task as something Alonzo had asked them to do (turn 7). With this act, Courtney interactively positioned Alonzo as someone who asked the class to do mathematical work—acts traditionally reserved for teachers in instructional spaces. Thus, Courtney's interactive position promoted the storyline of EBs as teachers.

Revoice Peers' Mathematical Contributions

Courtney frequently revoiced students' mathematical explanations in whole class discussions. Less commonly, were explicit requests from Courtney for students to take on this practice. For instance, on Oct. 27, Courtney asked the class if someone could revoice Emily's strategy to calculate the total value of her book of stamps (see Figure 3).

Figure 3. Emily's book of stamps and her written work.

1 Emily: I counted by 5s and 10 and (inaudible) instead of going like this (gestures across a row), I got, and then I did 5 times 18.
2 Courtney: Ok um, can someone else explain her strategy?
3 Lea: (hand raised)
4 Courtney: Lea can you go up there and explain what Emily did?
5 Lea: She did like (gets up and moves to board) (5.0 second pause)
6 Courtney: Shhh.
7 Lea: She did like, she counted by fives and go like 5, like 10, 15, all the way to 90 (points to 90).

8 Courtney: Alright. Laura. Thank you Lea for explaining her strategy you must've been paying close attention.

In this excerpt, Courtney interactively positioned Lea as a student who could revoice a peer's problem-solving strategy, attended to and thought about a peer's mathematical ideas, and could articulate those thoughts publicly (turn 4). Lea paused in her explanation (turn 5) and Courtney moved to silence the class (turn 6), which positioned Lea as a student who should be respected, had an idea everyone should hear, and had the conversational floor. Lea continued (turn 7) and supported her explanation with gestures to the written work. Lea's explanation evidenced she understood Emily's strategy of repeated addition. Next, Courtney thanked Lea for her explanation and then positioned her as a student who followed the classroom norms of paying attention, stating "you must've been paying close attention" (turn 8).

Overall, this interaction represents another way Courtney interactively positioned EBs as teachers. In this case specifically, Lea was positioned physically and metaphorically as a teacher when she described Emily's strategy. As with the previous excerpt, Courtney allowed Lea to control the space in the front of the room and did not approach her.

Correct Peer Mathematical Errors

A final way Courtney interactively positioned EBs as teachers was through invitations to correct a peer's work at the board. For example, on Apr. 26, the class discussed two representations of 97 cents, "$0.97" and "¢97," written on the board and agreed "¢" was to be written on the right-hand side of the value (e.g., 97¢). Courtney then stated,

> Alright so the 97 cents, oh instead of having the 97 cents at the back of the cents sign it's in the front of this one isn't it. Alright, so how can we fix that? So, who can fix that up for me? Who can fix it? Who wants to? You want to fix it Samuel? Yeah go fix it, fix somebody's work for me. You're their editor. You're going to fix up what somebody did.

In her turn, Courtney invited an EB to come to the board to correct the error, a practice often reserved for the teacher. Courtney picked Samuel, an EB, who was asked to be an "editor" for the peer. Courtney's act invited Samuel to take up the physical space of the teacher (by coming to the board) and situated him as a person who can correct peers' mathematical work. Courtney went further by calling Samuel an "editor"—a powerful title for him in a class full of native English-speaking peers. In short, this interaction represented another way Courtney interactively positioned EBs in the role of a teacher—a person who can correct others' work.

Discussion and Conclusion

Courtney, a white female, monolingual elementary teacher presents a case of a teacher who leveraged her authority in the classroom to interactively position EBs in ways that fostered the storyline of EBs as teachers–a privileged role in the institutional space of the classroom. An examination of the data revealed that Courtney did this through interactive positions that asked or allowed EBs to: share problem-solving strategies, control the classroom conversation, assign work, revoice a peer's thinking, and correct peers' mathematical errors. As a result, Courtney employed interactive positions at the individual level, which advanced the storyline of EBs as teachers for the collective. In this way, Courtney promoted a storyline for EBs in mathematics that challenged dominant narratives that EBs require remediation and extensive mathematical support (Chval & Pinnow, 2010; de Araujo et al., 2016; Polat & Mahalingappa, 2013).

Courtney characteristically represents many elementary teachers in the U.S. As such, her case is common, yet unique. She not only represents the majority of elementary teachers in the

U.S., but her success in advancing productive storylines for EBs in mathematics situates her in the unique position to offer insight into the ways teachers can use their discursive practices to put forth storylines for EBs in mathematics that challenge dominant narratives. Thus, there is much to learn from Courtney that other monolingual teachers can integrate into their practice. Most notably, the interactive positions identified can be implemented by other teachers to advance the storyline of EBs as teachers of mathematics. For example, a teacher may begin to reflect on the storylines they construct for EBs through their acts. Alternatively, a teacher may ask EBs to take on roles and responsibilities that are typically reserved for teachers (e.g., assigning work, control the classroom conversation, teaching peers). Given the growth of EBs in the U.S., it is essential that teachers understand how their interactive positions can be used to promote storylines individually and collectively to create change in public perceptions of mathematical ability.

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References

Chval, K. B., & Pinnow, R. (2010). Preservice teachers' assumptions about Latino/a English language learners. *Journal of Teaching for Excellence and Equity in Mathematics*, *2*(1), 6–12.

Chval, K. B., Pinnow, R. J., & Thomas, A. (2014). Learning how to focus on language while teaching mathematics to English language learners: A case study of Courtney. *Mathematics Education Research Journal*, *27*(1), 103–127.

de Araujo, Z., Smith, E., & Sakow, M. (2016). Reflecting on the dialogue regarding the mathematics education of English learners. *Journal of Urban Mathematics Education*, *9*(2), 33–48.

Harré, R. (2012). Positioning theory: Moral dimensions of socio-cultural psychology. In J. Valsiner (Ed.), *The Oxford Handbook of Culture and Psychology* (pp. 191–206). New York, NY: Oxford University Press.

Harré, R., Moghaddam, F. M., Cairnie, T. P., Rothbart, D., & Sabat, S. R. (2009). Recent advances in positioning theory. *Theory & Psychology*, *19*(1), 5–31.

Herbel-Eisenmann, B., Wagner, D., Johnson, K., Suh, H., & Figueras, H. (2015). Positioning in mathematics education: Revelations on an imported theory. *Educational Studies in Mathematics*, *89*, 185–204.

Mehan, H. (1979). *Learning lessons: Social organization in the classroom*. Cambridge, MA: Harvard University Press.

Nasir, N. S. (2016). Why should mathematics educators care about race and culture. *Journal of Urban Mathematics Education*, *9*(1), 7–18.

Polat, N., & Mahalingappa, L. (2013). Pre- and in-service teachers' beliefs about ELLs in content area classes: A case for inclusion, responsibility, and instructional support. *Teaching Education*, *24*, 58–83.

Stake, R. E. (1995). *The art of case study research*. Thousand Oaks, CA: Sage.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Thousand Oaks, CA: Sage.

Turner, E., Dominguez, H., Maldonado, L., & Empson, S. (2013). English learners' participation in mathematical discussion: Shifting positionings and dynamic identities. *Journal for Research in Mathematics Education*, *44*, 199–234.

van Langenhove, L., & Harré, R. (1999). Introducing positioning theory. In R. Harré & L. van Langenhove (Eds.), *Positioning theory: Moral contexts of intentional action* (pp. 14–31). Malden, MA: Blackwell Publishers Inc.

Wood, M. B. (2013). Mathematical micro-identities: Moment-to-moment positioning and learning in a fourth-grade classroom. *Journal for Research in Mathematics Education*, *44*, 775–808.

Yoon, B. (2008). Uninvited guests: The influence of teachers' roles and pedagogies on the positioning of English language learners in the regular classroom. *American Educational Research Journal*, *45*, 495–522.

TEACHERS' RESPONSES TO INSTANCES OF STUDENT MATHEMATICAL THINKING WITH VARIED POTENTIAL TO SUPPORT STUDENT LEARNING

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We investigated teachers' responses to a common set of varied-potential instances of student mathematical thinking to better understand how a teacher can shape meaningful mathematical discourse. Teacher responses were coded using a scheme that both disentangles and coordinates the teacher move, who it is directed to, and the degree to which student thinking is honored. Teachers tended to direct responses to the same student, use a limited number of moves, and explicitly incorporate students' thinking. We consider the productivity of teacher responses in relation to frameworks related to the productive use of student mathematical thinking.

Keywords: Classroom Discourse, Instructional Activities and Practices

Recommendations for ambitious mathematics teaching have identified the importance of instruction that honors and incorporates student thinking (e.g., National Council of Teachers of Mathematics [NCTM], 2014). Such instruction involves the teacher facilitating meaningful mathematical discourse by eliciting and making public student thinking, as well as appropriately *responding* to that thinking. Research has shown that the way in which teachers respond to student mathematical thinking (SMT) affects student learning in the classroom. For example, research has found that teacher responses that press students to further engage in thinking about the mathematics in their contributions provide students with increased learning opportunities (Kazemi & Stipek, 2001). More recently, Ing et al. (2015) found that responses that encourage students to engage with each other around mathematics correlate with increased student participation and higher student achievement.

Since SMT varies in the degree to which it provides leverage for accomplishing mathematical goals (Leatham, Peterson, Stockero, & Van Zoest, 2015), it follows that not all thinking should be responded to in the same way. Research examining teachers' responses to different types of SMT has produced mixed findings. Franke et al. (2009) found that the types of questions teachers asked did not vary depending on the clarity, correctness or completeness of a student's initial explanation, but other studies have found that teachers' responses do vary based on different types of SMT. For example, Bishop, Hardison, and Przybyla-Kuchek (2016) found that short or routine student contributions were related to teacher actions that were not responsive to SMT, whereas those that included strategies or reasoning were related to responses that engaged students in conversations about the SMT. Similarly, Drageset (2015) reported that brief answers to non-complex questions were typically responded to with a recall or procedural question, whereas unexplained answers were typically followed by responses that focused on an elaboration or rationale. Although prior research provides a foundation for understanding teachers' responses to instances of SMT, more work is required to fully understand variations in such responses, including whether particular responses might be more or less productive in supporting student learning in particular situations. Fortunately, scholars have developed a number of constructs for characterizing teacher responses that support this line of research.

Researchers have characterized teacher responses in various ways based on the focus of their studies. Brodie (2011) developed a coding scheme for teacher responses that captured two key aspects of responses—responsiveness to student ideas and student engagement. Schleppenbach, Flavares, Sims, and Perry (2007) analyzed teachers' responses to student errors using a scheme that captured two additional aspects of responses—the form of the response (statements or questions) and who questions were directed towards (the same student or other student(s)). Peterson et al. (2017) aimed to develop a comprehensive coding scheme that foregrounded responsiveness while also capturing other important ideas included in existing teacher response constructs. These researchers developed the Teacher Response Coding Scheme (TRC) to both disentangle and coordinate a number of components important in a teacher response, including the actors invited to respond, the type of action, and responsiveness to the SMT. Such a scheme provides a way to study teacher responses to student contributions that simultaneously addresses components of responses valued by other scholars working in this area.

Our study extends earlier research on teacher responses by using the TRC (Peterson et al., 2017) to examine teachers' responses to a common set of SMT with varied potential to support student mathematical learning. In particular, this study focuses on answering the question: *How do teachers' responses vary depending on the potential of an instance of SMT to support student mathematical learning?* We use these findings to discuss the extent to which various teacher responses are productive given the mathematical potential of an instance of SMT.

Theoretical Framework

Our work draws on two distinct, but related theoretical constructs. To make sense of the potential of an instance of SMT to support student mathematical learning, we use the MOST Analytical Framework (Leatham et al., 2015). To interpret the productivity of teachers' responses to these instances of varying potential, we use a set of principles drawn from the literature that underlie productive use of SMT. Descriptions of these constructs follow.

Leatham et al. (2015) characterized particular high potential instances of SMT as MOSTs—**M**athematically Significant Pedagogical **O**pportunities to Build on **S**tudent **T**hinking. The MOST Analytic Framework defines three characteristics of these instances—student mathematical thinking, significant mathematics, and pedagogical opportunity—each having two criteria that are used to determine whether an instance of SMT embodies that characteristic. For student mathematical thinking the criteria are that the *student mathematics* is inferable and that one can articulate a closely related *mathematical point* that the contribution could be used to better understand. For the significant mathematics characteristic, the criteria are that the mathematical point is *appropriate* for the students in the class—not too easy or too hard—and *central* to mathematical goals for their learning. To satisfy the pedagogical opportunity characteristic, the instance must create an *opening* to build on the SMT and the pedagogical *timing* must be right to take advantage of the opening when it occurs. The six criteria are considered linearly and an instance of SMT is classified according to the last criteria it satisfies (Student Mathematics, Mathematical Point, Appropriate, Central, Opening). Instances that meet all six criteria are classified as MOSTs. Those instances that appear mathematical, but for which the student mathematics cannot be inferred, are designated *cannot infer* (CNI).

To determine whether a teacher response to an instance of SMT is likely to be productive, we focus on the extent to which it coordinates core ideas about effective teaching and learning of mathematics drawn from the literature (e.g., NCTM, 2014)—what we refer to as principles underlying productive use of SMT (see Figure 1). We see these four principles as simultaneously coordinated in the teaching practice of *building* on SMT—making student thinking "the object of consideration by the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea" (Van Zoest et al., 2017, p. 36). Further, we see MOSTs as instances of SMT that are prime opportunities for a teacher to engage in building.

1. The mathematics of the instance is at the forefront. (Mathematics Principle)
2. Students are positioned as legitimate mathematical thinkers. (Legitimacy Principle)
3. Students are engaged in sense making. (Sense-Making Principle)
4. Students are working collaboratively. (Collaboration Principle)

Figure 1. Principles underlying productive use of student mathematical thinking.

Methodology

The Scenario Interview

The Scenario Interview (Stockero et al., 2015) is a tool to investigate how teachers respond to SMT during instruction and their reasoning underlying those responses. Teachers are presented with eight instances of SMT—four each from geometry and algebra contexts. The instances represent a range of SMT that satisfy different sets of MOST criteria, including those for which the SMT cannot be inferred and those that are mathematically significant but have poor timing. Four instances—two from each context—are MOSTs. Figure 2 provides four sample instances, their contexts and the last criteria they met on the MOST Analytic Framework.

Scenario	Context	Instance	Criteria
G1	Students were sharing their solutions to the following task (a corresponding picture was on the board). *Given two concentric circles, radii 5cm and 3cm, what is the area of the band between the circles?*	Chris shared his solution: "The radius of the big circle is 5 and the radius of the little circle is 3, so the gap is 2, so the area of the band is 4π cm^2."	MOST
G2		Before the teacher had a chance to respond to Chris, Pat says, "I also got 4π cm^2, but I did it a different way."	SM
A1	Students had been discussing the following task and had come up with the equation y = 10x + 25. Task: *Jenny received $25 for her birthday that she deposited into a savings account. She has a babysitting job that pays $10 per week, which she deposits into her account each week. Write an equation that she can use to predict how much she will have saved after any number of weeks.*	Terry says, "If you deposit $20 per week instead of $10 per week, the number in front of the x in the equation would change, but the number that is added would stay the same."	Central
A2		Casey said, "You could also change the story so the number in front of the x is negative."	MOST

Figure 2. Sample Scenario Interview instances, contexts and the last MOST criteria met.

The Scenario Interview situates the interviewee as the teacher in the context presented. They are asked to describe what they might do next were the instance to occur in their classroom and to explain why they would respond in that way. The interviewee may ask for contextual information they feel is needed before giving their initial response and is later provided common contextual information, after which they can revise their response if desired. If they did so revise, the revised response was used for this analysis. Using a common set of instances of SMT and providing common contextual information allowed for direct comparisons among teacher responses to a collection of instances that satisfy different subsets of the MOST criteria. This comparison allowed us to determine whether teachers seem to differentiate their responses based on the type of SMT to which they are responding.

Data Collection and Analysis

Data consisted of video-recorded Scenario Interviews conducted with 25 grade 6-12 mathematics teachers from across the United States. We segmented each interview into the 8 instances of SMT and the 25 teachers' responses to each individual instance. A *teacher response* was defined as *the collection of actions that a teacher describes they would take immediately following an instance of SMT*. There were a total of 198 teacher responses because one teacher was not able to envision one of the instances occurring in their classroom, and another teacher's interview was cut short before the last scenario was completed.

To begin to understand teachers' responses, we focused on teachers' initial responses to the instances in the Scenario Interview. Thus, preparing data for coding required making inferences about how the teacher would respond *in the moment* to each instance by considering both the teacher's description of their initial response and their rationale. Three coders individually analyzed each instance for a participating teacher, distilled the teacher's response to its essence, and met to discuss their inferred responses until a final teacher response was agreed upon. Any disagreements were brought to the larger research team for further discussion.

The teacher responses were coded using the TRC (Peterson et al., 2017), a scheme that disentangles the teacher move (Move) from other aspects of the teacher response, including who is publicly given the opportunity to consider the SMT (Actor) and the degree to which the SMT is honored (Recognition-Student Action and Recognition-Student Ideas). Figure 3 provides the TRC coding categories and codes discussed in this paper. Note that the TRC allows for multiple moves to be present in a teacher response. To facilitate our analysis, in cases of multiple actions in a given teacher response we identified the *predominate code* for each category—*the code that the students are most likely to experience as the instructional intent of the response*. The analysis used the predominate codes for each of the 198 teacher responses.

Category	Coding Category Description	Codes
Actor	Those who are publicly given the opportunity to consider the instance of SMT	teacher, same student(s), whole class
Recognition-Action	The degree to which the teacher response (either verbal or non-verbal) uses the student action	explicit, implicit, not
Recognition-Ideas	The extent to which the student who contributed the instance of SMT is likely to recognize their idea(s) in the teacher response	core, peripheral, other, not applicable
Move	What the actor is doing or being asked to do with respect to the instance of SMT	adjourn, clarify, collect, connect, develop, dismiss, justify

Figure 3. Subset of the *Teacher Response Coding Scheme* (*TRC*) discussed in this paper.

We use two teachers' responses to Scenario G1 (see Figure 1) to illustrate our application of the TRC. T1's response, "I would just ask [Chris] to explain by using pictures and words how he came up with the 4 pi," asks the *same student* a question that is a *develop* move in that the student is asked to explain how he arrived at the answer. T1 uses the student's words so Action is coded *explicit*, and the question stays *core* to the student's Ideas because it focuses on how Chris arrived at his answer. In T2's response, "Who else has another answer? Did everybody get that? Give me some more answers," Actor is coded as *whole class* because all students are invited to participate. Move is *collect* as T2 requests that other students share their answers. The student's words are not used, but referred to (by "that"), so Action is *implicit.* Asking other students to share their answers to the same task is *peripheral* to the contributing student's Ideas.

Results and Discussion

To understand how teachers' responses vary depending on the potential of an instance of SMT to support student mathematical learning, we begin by comparing results related to the Actor, Move, and Recognition categories for MOSTs and non-MOSTs. Then, we discuss how particular responses might be more or less productive in particular situations by considering how they adhere to the principles underlying productive use of SMT. Note that there were 99 MOSTs and 99 non-MOSTs in the data set; since the frequencies and percentages are essentially equivalent, we report only the frequencies.

Comparison of Teacher Responses

Actor. Responses coded *same student* were the most prevalent in the data and occurred at about the same frequency for MOSTs (65 of 99) and non-MOSTs (63 of 99). This even distribution among MOSTs and non-MOSTs did not occur for instances with a *whole class* or *teacher* actor. More MOSTs (26) than non-MOSTs (6) were coded whole class, while the opposite was true for instances coded teacher (4 MOSTs; 30 non-MOSTs). This suggests that teachers may distinguish, at least to some extent, instances that have potential to be discussed by the class from those that the teacher might just quickly deal with and move on.

Moves. Two dominant moves in the data, *develop* and *justify*, occurred more frequently in response to MOSTs than non-MOSTs (Table 1). Develop moves accounted for 37 MOST responses and only 25 non-MOST responses, while justify moves accounted for 18 MOST and 11 non-MOST responses. Together these two moves accounted for over half of responses to MOSTs. Two other dominant moves, *adjourn* and *clarify*, occurred more frequently in response to non-MOSTs. These moves accounted for 21 and 23 non-MOST responses, respectively, and for only 3 and 5 MOST responses, respectively. Thus, adjourn and clarify moves accounted for nearly half of non-MOST responses. As with Actor, the differences in Move for MOSTs versus non-MOSTs suggest the teacher actions differ depending on the type of SMT.

Table 1: Actor and Move Totals for MOSTs and non-MOSTs

	Actor			Move			
	Same Student	Whole Class	Teacher	Develop	Justify	Adjourn	Clarify
MOST	65	26	4	37	18	3	5
non-MOST	63	6	30	25	11	21	23

Actor/Move interactions. We also considered the distribution of Moves by Actor. Three moves—*develop, clarify* and *justify*—accounted for 111 of the 128 responses with a *same student* actor. We found that the distribution of these moves between MOSTs and non-MOSTs paralleled that for the data set overall; develop and justify moves occurred more frequently in response to MOSTs and clarify moves occurred more frequently in response to non-MOSTs. The predominant moves directed to the *whole class*—*collect* and *connect*—each occurred twice as often for MOSTs as non-MOSTs (although the numbers are small). Still different moves were the most common when the *teacher* was the actor, with *adjourn* being the most common move, followed by *dismiss*. Adjourn and dismiss moves necessarily had a teacher actor since when a teacher uses these moves, they do not provide an opportunity for students to publicly consider the instance. The majority of both of these moves were in response to non-MOSTs.

Recognition of Student Actions and Ideas. The Recognition codes operationalize the extent to which the student who provided the instance would recognize their thinking in the teacher's response. More responses to MOSTs than to non-MOSTs were classified as *explicit* or *implicit* for Student Action (86 MOSTs; 65 non-MOSTs), while more responses to non-MOSTs were classified as *not* aligning with student actions (13 MOSTs; 34 non-MOSTs). In terms of Student Ideas, 139 of the 198 total responses remained *core* to the SMT, and 19 of 198 were *peripheral*. As with the student actions, more responses to MOSTs than non-MOSTs were classified as core or peripheral (89 MOST; 69 non-MOST), while more responses to non-MOSTs were classified as *other* and *not applicable* (10 MOSTs; 30 non-MOSTs). These findings suggest that teachers' responses generally valued students' contributions by incorporating their actions and ideas.

Discussion of Productivity of Responses

We consider the productivity of teacher responses by examining the extent to which a response adheres to the four principles for productive use of student mathematical thinking (see Figure 1). We discuss several instances, including both MOSTs and non-MOSTs, to illustrate how responses with different coding can be more or less productive given the type of SMT to which the response is given.

The majority of *adjourn* moves (21 of the 24) occurred in response to two particular non-MOSTs instances. The productivity of this move is not the same for each instance, however. The first instance, classified as Opening, involved a student, Sam, giving an answer before other students had time to think about the task. Here, the common response, "Let's give everyone a chance to work it out and see what everyone else gets" (T10) is productive because of the poor timing of Sam blurting out his answer. Adjourning Sam's response provides all of the students in the class sufficient time to engage in sense making. In scenario G2 (where Pat claims to have arrived at the same answer in a different way; classified as Student Mathematics), a similar adjourning response to "address Chris's [the previous student's] comment first" (T3) might be less productive. Because we do not know Pat's "different way," making a move to *develop* his idea (as 12 teachers did) could lead to an opportunity to compare and contrast two different solution methods. Thus, develop responses, such as, "Talk to us Pat. What did you do?" (T9) seem more productive than adjourning this instance. T9's develop move would position Pat as a legitimate mathematical thinker and provide an opportunity for *all* students to make sense of the relationship between Pat's and Chris's contributions.

Develop moves with a *same student* actor were common for MOSTs. However, because MOSTs are opportunities for *building* on SMT—making student thinking the object of consideration by the class—asking the same student to develop or justify their idea may not always be necessary and may actually limit other students' opportunities to jointly participate in making sense of mathematical ideas. For example, consider scenario A2 (Figure 2), for which nearly half of the develop move responses with same student actor for MOSTs occurred. The most common teacher response in this instance was to ask Casey, the student who made the suggestion, to explain how they would change the story (e.g., "Well what do you mean? What sort of an equation, or what sort of a real-life situation can you think of where that would be a negative?" (T6)). Contrast this response with a similar response directed instead to the whole class: "Interesting comment; who can come up with a story, a situation that would match what Casey is saying?" (T7). In this case, directing the response to the whole class might be more productive, as it would engage all of the students in trying to come up with a situation where the coefficient is negative, increasing the likelihood of advancing the entire class's understanding of the mathematics of linear equations. This type of response aligns with all four core principles underlying productive use of a MOST, as it positions the students as capable of collaboratively making sense of the SMT.

Sometimes it is productive, however, to direct a response back to the same student. Consider an instance in which the student, Jesse, said, "It would have to be divided by x," an imprecise statement that needs clarification because "it" is unclear. The most common move in response to this instance was *clarify*, typically by asking Jesse, "what do you want to divide by x?" (T8). This response might be quite productive in helping members of the class figure out what Jesse was saying, and in doing so, would position Jesse's statement as legitimate mathematical thinking. Although the collaboration principle for productive use of SMT (Figure 1) privileges turning SMT over to the whole class whenever appropriate, cases like this one illustrate instances where directing the teacher response to the same student may be a productive first step. Productivity also depends on the recognition of student actions and ideas. A large percentage of the responses in our data were both *explicit* and *core*, meaning that the teachers in this study honored the SMT by incorporating the student's verbal or non-verbal actions and staying focused on the student's core ideas. For example, T8's response to Jesse discussed above is *explicit* and *core* as it incorporates both the student's words (divide by x) and his ideas (what he wanted to divide). A response such as this positions the student as a legitimate mathematical thinker by keeping the students' mathematics at the forefront, important aspects of productively using SMT.

Conclusion and Implications

By studying teachers' responses to a common set of instances of SMT with varying potential for incorporation into instruction, this study contributes to the teacher response literature by illuminating (a) how teachers' responses vary depending on the potential the SMT that is shared has to support student mathematical learning and (b) why some teacher responses are more productive than others in particular situations.

The results revealed that the teachers were generally able to distinguish when different moves might be more productive, as different moves were often employed in response to MOSTs and non-MOSTs. For example, most *whole class* actor responses occurred with MOSTs and most *teacher* actor responses occurred with non-MOSTs. However, our finding that the majority of teacher responses to both MOSTs and non-MOSTs were directed to the *same student* raises some concerns, as MOSTs are prime opportunities for teachers to engage in the teaching practice of *building* on SMT. Thus, directing responses to such instances to a single student results in a missed opportunity. In terms of responsiveness, we found that teacher responses to both MOSTs and non-MOSTs often stayed *core* to the ideas in the SMT and *explicitly* incorporated the students' actions, signaling that these teachers valued students' contributions and often positioned students as legitimate mathematical thinkers who can make valid contributions to the development of the mathematics in the classroom.

Our findings have the potential to help teacher educators develop a more nuanced understanding of what teachers are doing well and where they may need support, thus providing more focus to their teacher development efforts. For example, suppose the majority of a teacher's responses honor SMT, but engage only the contributing student. Professional development work could focus specifically on expanding the ways that they honor SMT by studying the potential in directing a response to the whole class, and when it would and would not be appropriate to do so. Such focused efforts would allow teacher educators to leverage teachers' strengths and thus develop teachers' practice more effectively.

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References

Bishop, J. P., Hardison, H., & Przybyla-Kuchek, J. (2016). Profiles of responsiveness in middle grades mathematics classrooms. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (PME-NA; pp. 1173-1180). Tucson, AZ: University of Arizona.

Brodie, K. (2011). Working with learners' mathematical thinking: Towards a language of description for changing pedagogy. *Teaching and Teacher Education, 27*(1), 174-186.

Drageset, O. G. (2015). Student and teacher interventions: A framework for analyzing mathematical discourse in the classroom. *Journal of Mathematics Teacher Education,18,* 253-272.

Franke, M., Webb, N., Chan, A., Ing, M., Freund, D., & Battey, D. (2009). Teacher questioning to elicit students' mathematical thinking in elementary school classrooms. *Journal of Teacher Education, 60*(4), 380-392.

Ing, M., Webb, N. M., Franke, M. L., Turrou, A. C., Wong, J., Shin, N., & Fernandez, C. H. (2015). Student participation in elementary mathematics classrooms: the missing link between teacher practices and student achievement? *Educational Studies in Mathematics, 90*, 341-356.

Kazemi, E., & Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *The Elementary School Journal, 102*, 59-80.

Leatham, K. R., Peterson, B. E., Stockero, S. L., & Van Zoest, L. R. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. *Journal for Research in Mathematics Education, 46*, 88-124.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all.* Reston, VA: Author.

Peterson, B. E., Van Zoest, L. R., Rougée, A. O. T., Freeburn, B., Stockero, S. L., & Leatham, K. R. (2017). Beyond the "move": A scheme for coding teachers' responses to student mathematical thinking. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.), *Proceedings of the 41st annual meeting of the International Group for the Psychology of Mathematics Education*, *Vol. 4* (pp. 17-24). Singapore: PME.

Schleppenbach, M., Flevares, L. M., Sims, L. M., & Perry, M. (2007). *Teachers' responses to student mistakes in Chinese and U.S. mathematics classrooms.* The Elementary School Journal, *108*(2), 131-147.

Stockero, S. L., Van Zoest, L. R., Rougée, A., Fraser, E. H., Leatham, K. R., & Peterson, B. E. (2015). Uncovering teachers' goals, orientations, and resources related to the practice of using student thinking. In T. G. Bartell, K. N. Bieda, R. T. Putnam, K. Bradfield, & H. Dominguez (Eds.), *Proceedings of the 37th annual meeting of PME-NA* (pp. 1146-1149). East Lansing, MI: Michigan State University.

Van Zoest, L. R., Stockero, S. L., Leatham, K. R., Peterson, B. E., Atanga, N. A., & Ochieng, M. A. (2017). Attributes of instances of student thinking that are worth building on in whole-class discussion. *Mathematical Thinking and Learning*, *19*(1), 33-54.

IMPACT OF A STUDENT-ADAPTIVE PEDAGOGY PD PROGRAM ON STUDENTS' MULTIPLICATIVE REASONING

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This study examines how a PD program to promote teachers' shift toward a student-adaptive pedagogy impacts students' multiplicative reasoning. We describe the underpinnings of this pedagogy and main components of the PD program. Then, we present key features of participants, data collection/analysis methods, and the written assessment used to measure students' multiplicative reasoning (MR). We found a significant increase in students' MR, between (a) year-ends for different classes and (b) year-start to year-end for the same groups of students. We also found students of participating ("treatment") teachers outperformed those of non-participating ("control") teachers. We discuss the importance of these findings for theory, for teacher education, and for students' mathematical future.

Introduction

In the context of NCTM (2000) reform efforts, we promote students' mathematics by fostering teachers' focus on students' reasoning. Here, we examine the impact of a professional development (PD) program to foster grade-3 teachers' shift to a constructivist-based, *student-adaptive pedagogy* (Tzur, 2013) on their students' multiplicative reasoning. We stress our focus is on students' reasoning, not on their observable solutions to problems (Tzur et al., 2013). In our study, we use students' work to infer reasoning in terms of mental operations on units that could explain a *child's underlying sense* of the problem and its solution. Our study contributes to the body of research that links teacher learning to identify and build on students' mathematics with students' learning and outcomes (Visnovska & Cobb, 2009).

Conceptual framework

We explicate two components of our conceptual framework: student-adaptive pedagogy and numerical reasoning—particularly the difference between additive and multiplicative reasoning. Extending Steffe's (1990) notion of adaptive teaching, Tzur (2008, 2013) proposed student-adaptive pedagogy as a comprehensive approach rooted in a conception-based perspective on knowing and learning. Simon et al. (2000) distinguished this perspective from traditional ("show-and-tell") and perception-based perspectives identified in teachers' transition to reform-oriented practices. Perception-based perspectives differ from traditional in emphasizing the active nature of learning mathematics. However, common to both perspectives is a stance that depicts mathematical knowing as existing outside the learner.

In contrast, a conception-based perspective builds on two implications of the core constructivist notion of assimilation (Piaget, 1985). First, one's available ways of operating afford and constrain what and how one may "see" and do mathematically. Second, conceptual learning entails bringing forth available ways of operating mathematically and transforming those into more advanced ones. These implications compel pedagogical practices that adapt goals and activities for students' learning based on students' available mathematics. Tzur (2008) depicted student-adaptive pedagogy as a reflective cycle (triad) of teaching rooted in hypothetical learning trajectories (HLT; see Simon, 1995; Simon & Tzur, 2004). It begins with inferring students' available mathematics, proceeds to setting the goals for their next learning, and then to selecting tasks that can be assimilated into available schemes and help transform those into the intended mathematics. For example, a teacher may infer two different strengths of her third graders' conception of whole numbers based on the strategy they use to add two single-digit numbers: weak (count-on) or strong (break-apart-make-ten, or BAMT) (Tzur et al., 2017). Thus, she would not set the same goals and engage all of them in the same tasks. Rather, she can engage those who used count-on in tasks to strengthen their conception of number and those who used BAMT in tasks to foster transition to multiplicative reasoning.

As for numerical reasoning, we explain it in terms of mental units and operations inferred to underlie students' problem solving (Steffe, 1992; Ulrich, 2016). Two types of units inform our inferences, *singletons (1s) and composite units* (i.e., units composed of smaller units). For example, the number "8" is a unit composed of eight 1s, or of five 1s and three 1s, etc. We infer *additive reasoning* when one's operations involve no unit change—she operates on one kind of unit (e.g., 2 *keys* + 2 *keys* + 2 *keys* = 6 *keys*). Conversely, we infer the first of six *multiplicative reasoning* schemes, which Tzur et al. (2013) termed *multiplicative double counting* (mDC), when one's operations involve a change of unit (Simon et al., in press). Such a change takes place when items of one kind of unit are distributed over (and coordinated with) items of another kind of unit to yield a different kind of unit (e.g., 2 *keys-per-box*, placed in 3 *boxes*, yield 6 *keys*). To find the total, a simultaneous count of accrual of composite units and 1s takes place (e.g., first-box-is-2-keys, second-is-4, third-is-6).

Methods

Settings and Participants

For two years, eleven grade-3 teachers and their students (age ~9) in two schools in a large USA city participated in the study. School A (7 teachers) is located in a small district and School B (4 teachers) in another, large district. In School A, 3 teachers participated in the PD program (treatment) and 4 teachers did not (control). Of the participating students, ~85% identified as students of colour and ~70% were learning English as an additional language (we detail student numbers in Data Collection).

Concept-Sensitive Assessment of the mDC Scheme

To assess students' mDC scheme, we used a 5-item written measure that our team developed and validated. Validation included correlating students' written responses with an interviewer's inferences of their mDC scheme (Ktb=0.883, $p<.0005$). The first item assesses additive reasoning; the following four assess the mDC scheme (Table 1). Cronbach's α (0.91) and Rasch (0.98) indicated inter-item reliability.

Table 1: Five Word Problems Comprising the mDC Written Assessment

#	Word Problem
1	For breakfast Ana ate 8 grapes. For lunch, Ana ate 7 grapes. How many grapes did Ana eat in all? Ana ate _______ grapes in all.
2	The picture shows towers made of **3** cubes, **5** cubes, **12** cubes, and **24** cubes. In this problem, Pat only has towers with **3 cubes**. Pat cannot break apart any tower. Can Pat build a tower of **24** cubes using only towers of **3 cubes**? (If Yes, fill in the blank): Pat CAN use _____ **towers** of **3** cubes to build a tower of **24** cubes.
3	The picture to the right shows a box. Alex put **6** towers in the box. Alex made each tower with **3** cubes. The numbers on the picture show this. How many **cubes in all** did Alex use to make **6** towers? (fill in the blank): ______
4	There are **4 teams** in a school. Each team has **5 players**. The picture shows the name of each team. Joy said there are **35 players in all**, because she skip-counted by 5 (Joy started 5, 10, and kept going to 35). Is Joy Correct? (circle one): Yes No. How many teams did Joy count? (circle one): 4 5 7 20 35 Another number ______
5	Sam baked **28** smiley **cookies**. He put all of them in boxes. Sam put **4 cookies in each box**. The picture shows only **one of the boxes**. How many boxes did Sam use for all 28 cookies? (fill in the blank): ________

Job-embedded PD Program

Focusing on multiplicative and fractional reasoning enacted within the teaching triad, we engaged participating teachers in a PD program to promote their: (a) own mathematical reasoning, (b) understanding of progressions in students' reasoning, (c) use of tasks to foster such progressions, and (d) attention to language and actions used by them and by students. To these ends, we used three, job-embedded PD experiences: Two, week-long *Summer Institutes* (Sis; total ~70 hours), *Buddy-Pairs* (total ~24 hours), and School/Grade-based workshops (total ~16 hours).

In both Sis, we engaged teachers in whole groups, small group, or individual work while using tasks they could later enact in their classrooms. We frequently involved them in observing conceptually-selected videos and analysing students' reasoning, In SI-2, the segments were selected from teachers' own classrooms. Using that analysis, they would discuss goals and tailor/justify tasks to promote the next learning.

During the school year (2016-17), we worked with teachers in their own classrooms, mingling *buddy-pair* experiences and grade-based workshops. In the former, teachers teamed up to visit a buddy's classroom, while a member of our team co-taught with the hosting teacher. Then, our team member and buddy teachers reflected on: (a) what students seemed to understand, (b) what serves as evidence for such claims, (c) how tasks could foster that learning, and (d) what/why/how to teach next. In the workshops, we focused on concepts from the buddy-pair experiences to promote teachers' own mathematics, to situate the concept(s) within progressions, and to select tasks that can promote differentiated learning based on where students seemed to be conceptually. A particular emphasis was on the strength of a student's conception of number and/or on the mDC scheme, which teachers learned to glean from students' strategies when solving problems in the classroom.

Data Collection and Analysis

Collection

Five graduate research assistants (GRAs) administered the written assessment in a whole-class setting (~40 minutes). The GRA read out loud each item to enhance comprehension, monitored students' work on all sub-questions, and, before starting Problem 2, guided them to build a tower of 7 cubes to ascertain they recognized this object. We administered the mDC assessment three times: Spring '16 (pre-PD, year-end, N=81), Fall '16 (pre-PD, year-start, N=177), and Spring '17 (post-PD, N=113). The GRAs entered student responses to the mDC assessment in pairs; one read the responses out loud and another entered them into a spreadsheet.

Analysis

Scoring correct responses as 1 and incorrect as 0, we tested two main hypotheses about participating students: (a) post-PD, year-end outcomes will be better than pre-PD year-end and (b) post-PD, year-end outcomes will be better than pre-PD, year-start. We also tested a hypothesis that treatment students in School A will outperform their control counterparts. For each hypothesis, we analysed the mean of responses to all four mDC questions (ranging 0-4) and to Problem 3 alone (0 or 1). We chose Problem 3 because it is a typical multiplicative situation taught in schools that proved the hardest, and thus distinguishes teaching "multiplication as repeated addition" vs. as coordination of composite units and 1s. For the total mean on Problems 2-5 we used an independent sample t-test, a one-way ANOVA, a repeated-measure ANOVA, and Cohen's-d effect size (ES); for the non-parametric variable of responses to Problem 3 we used the Mann-Whitney test (MWz values).

Results

Three analyses show the PD impact on 3rd graders' reasoning: pre/post between two year-ends (Sp-16, Sp-17), pre/post growth between year-start and year-end (Fa-16, Sp-17), and pre/post between treatment and control students in School A.

Pre/Post PD: Year-Ends (Sp-16 vs. Sp-17)

Table 3 shows that, overall, 3rd graders at year-end of post-PD (39%) outperformed their year-end, pre-PD counterparts (28%), though this did *not* reach statistical significance (t=1.38, df=89, p=.17). For Problem #3, students in post-PD (43%) outperformed pre-PD (32%; MWz=1.02, N=91, p=.31). These results differ for each school. In School A (treatment only), a minimal change is indicated for all mDC problems, from pre-PD (25%) to post-PD (28%), with a larger (non-significant) difference on Problem #3 (22% to 36%, respectively). These non-significant results are highlighted differently when compared with changes in School A's control group. In School B, the pre-post PD increase on all four mDC problems (29% to 56%) was statistically significant (t=2.15, df=47, p=.037), but the increase on Problem #3 (36% to 52%) was not (MWz=1.07, N=49, p=.28).

Table 2: Percentages of students' correct solution (all mDC problems; Problem 3).

	All mDC Problems		mDC Problem **3**	
	Pre (Sp-16)	Post (Sp-17)	Pre (Sp-16)	Post (Sp-17)
All AdPed	28%	39%	32%	43%
School A	25%	28%	22%	36%
School B	29%	52%	36%	52%

Pre/Post PD: Year-Start (Fa-16) vs. Year-End (Sp-17)

Table 4 shows growth, from year-start (Fa-16) to year-end (Sp-17), for all students, then separately for each school. Growth in mDC reasoning, from year-start (14%) to year-end (39%), was statistically significant (t=4.91, df=147, p<.0005), with *large effect-size* (Cohen-d's ES=0.83). Similarly, results for Problem #3 show the growth from year-start (18%) to year-end (43%) was statistically significant (MWz= 3.36, N=149, p=.001), with *moderate effect-size* (Cohen-r's ES=0.58).

In School A, the growth on all four mDC problems, from pre-PD (mere 8%) to post-PD (28%), was statistically significant (t=2.96, df=62, p=.004), with *near-large effect-size* (Cohen-d's ES=0.75). Similarly, for Problem #3 the growth from year-start (6%) to year-end (36%) was statistically significant (MWz=2.87, N=64, p=.004), with *intermediate effect-size* (Cohen-r's ES=0.3). In School B, the growth on all four problems was remarkable, from pre-PD (18%) to post-PD (52%) (t=4.88, df=83, p<.0005), with a *very large effect-size* (Cohen-d's ES=1.15). Growth on Problem 3, from pre-PD (24%) to post-PD (52%), was statistically significant (MWz=2.5, N=85, p=.012), with a *moderate effect-size* (Cohen-r's ES=0.62).

Table 3: Percentages of students' correct solution (all mDC problems; Problem 3).

	All mDC Problems		mDC Problem **3**	
	Pre (Fa-16)	Post (Sp-17)	Pre (Fa-16)	Post (Sp-17)
All AdPed	14%	39%	18%	43%
School A	8%	28%	6%	36%
School B	18%	52%	24%	52%

Treatment vs. Comparison

Table 5 (Sp-16 vs. S-17) and Table 6 (Fa-16 vs. Sp-17) compare outcomes of 3rd graders' mDC reasoning, between treatment and comparison groups (School A only). In both tables, a two-way ANOVA (shown in the post-PD cells of the treatment group) is statistically significant in favour of the treatment group.

Table 4: Sp-16 vs. Sp-17 – percentages of treatment/control student correct solutions.

	All mDC Problems		mDC Problem **3**	
School A	Pre (Sp-16)	Post (Sp-17)	Pre (Sp-16)	Post (Sp-17)
Treatment	25%	28% ($F_{1,142}$=4.42, p<.037)	22%	36% (Friedman's $Q_{1,145}$=26.6, p<.0005)
Control	36%	20%	39%	15%

Table 5: Fa-16 vs. Sp-17 – percentages of treatment/control student correct solutions.

	All mDC Problems		mDC Problem **3**	
School A	Pre (Fa-16)	Post (Sp-17)	Pre (Fa-16)	Post (Sp-17)
Treatment	8%	28% ($F_{1,203}$=6.26, p=.013)	10%	36% (Friedman's $Q_{1,54}$=38, p<.0005)
Control	19%	20%	23%	22%

Summary of Analysis

We analysed third graders' responses to four, concept-sensitive items that, combined, indicate students' reasoning with the mDC scheme. We showed PD impact on that reasoning using three major comparisons: (a) pre-post increase between two consecutive *year-ends* within the same student population, (b) pre-post growth from *year-start to year-end* in the same schools within the PD year, and (c) interaction between outcomes of students of teachers in treatment and control groups within the same, single school. Those comparisons support our claim: PD to foster teachers' shift toward student-adaptive pedagogy (focus on multiplicative reasoning) can bring about desired growth in their students' multiplicative reasoning and problem solving.

A question arises of causes for between-school differences. While more data are needed, we note two plausible factors. First, teacher practices in each school reflected a different starting-point: mostly traditional at School A and mostly reform-oriented, perception-based perspective at school B (Simon et al., 2004). Second, teachers differed in their learning, and thus enactment, of assessing and using their students' conception of number, including how this conception predicts mDC (Tzur et al., 2017). Specifically, School B teachers reorganized instruction to (a) foster conception of number in students who seemed to lack it and (b) strengthen it in students with a weak conception of number. That is, they focused more on fostering students' construction of mDC by capitalizing on the strength of their conception of number.

Discussion

We found impact of a PD program to foster teachers' shift toward student-adaptive pedagogy on growth in their students' multiplicative reasoning. The scope of this paper precludes detailing the job-embedded PD. Yet, it shows that fostering teachers' initial adoption of this constructivist-based pedagogy supports the crucial conceptual advance in students' reasoning—from additive to multiplicative. Moreover, it shows the benefits of helping teachers to first conceptualize multiplication themselves, and then teach it, not as repeated addition (as did Control teachers), but rather as coordination of three different units, which we fostered in treatment teachers.

Implications for Practice

We note two main implications of this study for practice. First, our findings stress the benefits of changing teachers' understanding of multiplication, and then of teaching it, away from "repeated addition" and toward units coordination. Arguably, the most telling evidence is found in students' solutions to the *typical multiplication Problem 3*—markedly students of treatment teachers outperforming their control counterparts. Second, for mathematics teacher education, our study highlights the benefits of a dual focus on the necessary growth in teachers' own mathematical reasoning and their ability to tailor learning goals and activities to the students' available conceptions. Specifically, in this study we demonstrate the possibility of increasing students' multiplicative reasoning and problem solving by promoting teacher development as professionals who can understand, and apply, theory and research findings to alter their practice. We note that, in both schools, the principal and other instructional leaders provided constant, enthusiastic support for the intended teacher change.

Implications for Research and Theory

We note five main foci implied by this study. First, for theory building, this study implies the possibility, and need, to corroborate, statistically, the theoretically sound progression in students' schemes for whole number multiplicative reasoning (Tzur et al., 2013). In this study, we found the impact on students' mDC, an introductory level of multiplicative reasoning. Collecting and analysing data from a large sample of students would allow corroboration to more advanced schemes of multiplicative reasoning. Second, it seems important to also link the impact of our constructivist-based, job-embedded PD on students' mDC reasoning with their outcomes on district and/or state tests, as well as studying this impact for different student populations. Third, in our data analysis we grouped teachers uniformly, although they started their individual journeys toward student-adaptive pedagogy at different points and progressed in rather different paces. Linking differentiations in teacher practices to student learning and outcomes could further understandings of how the impact found in this study is related to teacher learning. Fourth, further attribution of PD impact to parts of the intervention is needed, that is, to changes in teachers': own mathematics, understanding of conceptual progressions in students' mathematics, selection and enactment of instructional activities tailored to fostering particular students' learning, and use of language as an additional focus of the intervention. Fifth, the differentiated support by school principals and coaches we witnessed indicates the importance of considering an extended-level unit of analysis (beyond teachers), namely, schools as systems. All five foci can build on the findings reported here about the promising impact that a constructivist-based PD can have on teachers and thus on students' mathematics.

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References

NCTM. (2000). *Principles and Standards for School Mathematics*. Reston, VA: Author.

Piaget, J. (1985). The equilibration of cognitive structures: The central problem of intellectual development. Chicago: The University of Chicago.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.

Simon, M. A., Kara, M., Norton, A., & Placa, N. (in press). Fostering construction of a meaning for multiplication that subsumes whole-number and fraction multiplication: A study of the Learning Through Activity research program. *Journal of Mathematical Behavior*.

Simon, M. A., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning, 6*(2), 91-104.

Simon, M. A., Tzur, R., Heinz, K., Kinzel, M., & Smith, M. S. (2000). Characterizing a perspective underlying the practice of mathematics teachers in transition. *Journal for Research in Mathematics Education, 31*(5), 579-601.

Steffe, L. P. (1990). Adaptive mathematics teaching. In T. J. Cooney & C. R. Hirsch (Eds.), *Teaching and learning mathematics in the 1990s* (pp. 41-51). Reston, VA: NCTM.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences, 4*(3), 259-309.

Tzur, R. (2008). Profound awareness of the learning paradox (PALP): A journey towards epistemologically regulated pedagogy in mathematics teaching and teacher education. In B. Jaworski & T. Wood (Eds.), *The international handbook of mathematics teacher education: The mathematics teacher educator as a developing professional* (Vol. 4, pp. 137-156). Rotterdam, the Netherlands: Sense.

Tzur, R. (2013). Too often, these children are teaching-disabled, not learning-disabled. In *Proceedings of the 11th Annual Hawaii International Conference on Education*. Honolulu, HI: Author (DVD).

Tzur, R., Johnson, H. L., McClintock, E., Kenney, R. H., Xin, Y. P., Si, L., . . . Jin, X. (2013). Distinguishing schemes and tasks in children's development of multiplicative reasoning. *PNA, 7*(3), 85-101.

Tzur, R., Johnson, H. L., Norton, A., Davis, A., Wang, X., Ferrara, M., . . . Wei, B. (2017). Conception of number as a composite unit predicts students' multiplicative reasoning: Quantitative corroboration of Steffe's model. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *PME Proceedings* (Vol. 4, pp. 289-296). Singapore: PME.

Ulrich, C. (2016). Stages in constructing and coordinating units additively and multiplicatively (part 2). *For the Learning of Mathematics, 36*(1), 34-39.

Visnovska, J., & Cobb, P. (2009). Learning about building mathematics instruction from students' reasoning: A professional development study. In R. Hunter, B. Bicknell & T. Burgess (Eds.), Paper presented at the 32nd annual meeting of the Mathematics Education Research Group of Australasia., Wellington, NZ.

HOW COGNITIVE DEMAND OF TASKS AND TEACHING PRACTICES DIFFER IN LESSONS WITH OR WITHOUT INTERACTIVE SIMULATIONS?

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This study investigates whether the use of Physics Education Technology (PhET) interactive simulations (sims) supported teachers' instructional quality depending on in enacting and maintaining cognitive demand of mathematical tasks. We report findings from two middle-school mathematics teachers who each used parallel module lesson plans across two of their sections, with one section following a sim-based lesson plan and the other following a non-sim-based lesson plan. Data collection was comprised of the teacher's tasks as planned and our classroom observations. Each lesson was analyzed in terms of instructional quality assessment framework (IQA). Preliminary findings suggest that the students using PhET sims had more opportunities for higher academic rigor and accountable talk through the engagement with cognitively demanding work, rather than focusing on procedures and reproducing knowledge.

Keywords: Instructional Quality, Interactive Simulations, Instructional Practice

In recent years, researchers have accentuated mathematical instruction that engages students as explorers in a dynamic disciplinary space (Langer-Osuna, 2017), rather than a static and structured system of facts and procedures. The Common Core State Standards in Mathematics places a high emphasis on cognitively demanding thinking skills and a more in-depth understanding of mathematical concepts (National Council of Teachers of Mathematics [NCTM], 2010; Stein, Correnti, Moore, Russell, & Kelly, 2017). However, minimal research exists on how teachers can transform their practice to help students engage with mathematics while also being faithful to the content standards they are tasked to teach (Stein et al., 2017).

The development of new technology has advanced the instructional capabilities of mathematics teachers (Goldenberg, 2000), resulting in increased use of technology in K-12 classrooms (Rehn, Moore, Podolefsky, & Finkelstein, 2013). Technology has the potential to empower teachers to transform their practice in new ways that position students to develop deeper cognitive and conceptual understanding (NCTM, 2000). The National Research Council (NRC) (2011) has highlighted simulations as among several tools that may help teachers transform their practice to meet challenges associated with recent reform efforts. Interactive simulations use dynamic visualizations to model representations and functions that lead students to sense-make about a phenomenon of interest (Hensberry, Moore, & Perkins, 2015).

For this study, we concentrate on PhET sims, which show promise as an effective tool for improving student engagement and learning in science and mathematics (Hensberry, et al., 2015). While the use of simulations provides great opportunity for students' conceptual understanding of science and mathematics, the use of computer technology cannot guarantee increased learning gains on its own (Lei, 2010). Learning to use classroom tools in tandem with reform-oriented teaching practices can assist teachers to consider the "importance of learning with technology rather than learning about technology" (Drier, 2001, p. 170). While there is some evidence indicating the use of simulations supports higher levels of student mathematical thinking (Brown, 2017), we instead consider the influence of simulations on teaching practices and its potential to push teachers to deliver a higher quality of instruction. By using instructional quality assessment (IQA) framework (Boston, 2012), we aimed to capture observable indicators of high-quality mathematics instruction. This framework included two main categories based on academic rigor and accountable talk happened during the instruction. Academic rigor included rubrics for the potential cognitive demand of the task, implementation of the task, student discussion following the task, and rigor of teachers' questions. Accountable talk included scoring of participation, teacher's linking contribution, student's linking contribution, teachers' press for knowledge or thinking, and students providing knowledge or thinking based on rubrics. Within this study, we examine the following research question: In what ways, if any, does the use of PhET sims influence each teachers' instructional quality in middle school mathematics classrooms?

Methods

Setting

This study involves two middle-school mathematics teachers in U.S. public schools, Elise and Kathy (both names are pseudonyms). Elise taught three lesson modules for sim and corresponding modules without the use of sims for grade 7 involving the topics of unit rates, scale, and probability. With the same number of modules for sim and non-sim versions, Kathy taught 6th grades on data analysis, ratio and proportion, and algebraic expressions. The non-sim lessons had the same learning goals. Each teacher had access to the lesson plans and tasks at the PhET website (https://phet.colorado.edu) and were free in deciding which sim and task they preferred to use in their lessons. For the non-sim lesson plans, the teachers each created their own sequence of tasks.

Data Collection and Analysis

Data include video recordings of both the sim and non-sim class periods, which were transcribed for ease of coding. Instructional quality was analyzed using the frameworks described previously. In IQA, academic rigor score is calculated by four rubrics AR1 (potential of the task), AR2 (implementation of the task), AR3 (student discussion following the task), and AR-Q (rigor of teachers' questions) with a scoring scale from 0 to 4. For accountable talk score, we used four rubrics named as AT1 (participation), AT2 (teacher's linking contribution), AT3 (student's linking contribution), AT4 (teacher's press for knowledge or thinking), AT5 (student's providing knowledge or thinking) with the same scoring scale from 0 to 4.

Findings

The dimensions of the IQA framework revealed some qualitative differences in the cognitive demand of tasks as planned and enacted, the quality of discussion during the whole class discussions, rigor of teacher questioning, the opportunities in making connections, teacher's press for knowledge and thinking, and students' responses to that press. Findings suggest that in sim lessons, both teachers created more opportunities for students to develop mathematical meaning, share their ideas, and justify their thinking, rather than focusing on procedures and reproducing knowledge. These findings provide a motivation to explore the role of interactive simulations in creating opportunities for some teachers to leave their comfort zone, which comprised of relatively traditional and teacher-centered practices and attempt to improve the quality of their instruction (Lampert, Boerst, & Graziani, 2008).

Conclusion

This report aimed to start an exploration of how teachers' instructional practices might be transformed by the use of tasks with cognitive demand and use of interactive simulations. Although each teacher taught the same grade level and the same content in their two classes, their moves seem to show some increase in attending student thinking and creating opportunities for students to share their ideas, justify their thinking. Despite its exploratory nature, this study offers some insight into how the use of interactive simulations might create opportunities for teachers to adopt new teaching practices that support students' mathematical understanding and reasoning.

References

Boston, M. (2012). Assessing instructional quality in mathematics. *The Elementary School Journal, 113*(1), 76-104.

Brown, J. P. (2017). Teachers' perspectives of changes in their practice during a technology in mathematics education research project. *Teaching and Teacher Education*, *64*, 52-65.

Common Core State Standards Initiative. (2010). Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Drier, H. S. (2001). Teaching and learning mathematics with interactive spreadsheets. *School Science and Mathematics*, *101*(4), 170-179.

Goldenberg, E. P. (2000). Thinking (and talking) about technology in math classrooms. *Issues in Mathematics Education*, 1-8.

Hensberry, K. K. R., Moore, E. B., & Perkins, K. K. (2015). Effective student learning of fractions with an interactive simulation. *Journal of Computers in Mathematics and Science Teaching,* 34(3), 273-298.

Lampert, M., Boerst, T., & Graziani, F. (2008). Using organizational assets in the service of ambitious teaching practice. *Teachers College Record.*

Langer-Osuna, J. M. (2017). Authority, Identity, and Collaborative Mathematics. *Journal for Research in Mathematics Education*, *48*(3), 237-247.

Lei, J. (2010). Quantity versus quality: A new approach to examine the relationship between technology use and student outcomes. *British Journal of Educational Technology, 41*(3), 455-472.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Research Council. (2011). *Successful K-12 STEM education: Identifying effective approaches in science, technology, engineering, and mathematics*. National Academies Press.

Rehn, D. A., Moore, E. B., Podolefsky, N. S., & Finkelstein, N. D. (2013). Tools for high-tech tool use: A framework and heuristics for using interactive simulations. *Journal of Teaching and Learning with Technology, 2*(1), 31-55.

Stein, M. K., Correnti, R., Moore, D., Russell, J. L., & Kelly, K. (2017). Using theory and measurement to sharpen conceptualizations of mathematics teaching in the Common Core era. *AERA Open*, *3*(1).

STUDENTS' REACTIONS TO TEAM ACTIVITIES IN A LARGE-SCALE PRECALCULUS CLASS: A MIXED METHODS STUDY

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Using active learning strategies in a large undergraduate mathematics classroom can be challenging. Minimal results from recent research on active-learning indicate that more investigation of how instructors are being effective using these strategies in these classrooms is still needed. Results from this study indicate that students who participated in Team Activities and other learner-centered activities in a large scale precalculus undergraduate classroom report a good experience and are more positive in their attitudes towards mathematics. Particularly, students valued the collaborative efforts in this mathematics classroom, something they do not often have the opportunity to do in a traditional mathematics course. This report has implications for large universities where many of the classes are taught at a large scale while instructors are focused on bringing in more active learning to improve instruction.

Keywords: Post-Secondary Mathematics; Affect, Emotions, Beliefs, and Attitudes; Instructional Activities and Practices

The number of students in the United States entering and completing undergraduate and graduate science, technology, engineering, and mathematics (STEM) programs is declining, and this decline has been connected to the nature and quality of instruction in undergraduate STEM courses (Carnevale, Smith, & Melton, 2011; PCAST, 2012). The President's Council of Advisors on Science and Technology estimates that an additional 1,000,000 STEM graduates are needed in the next decade to meet workplace demands (PCAST, 2012), with these demands being both in STEM careers and broader STEM competencies (Carnevale, Smith, & Melton, 2011). Freeman and colleagues (2014) published the results of a meta-analysis that addressed articles about using active learning in STEM and the student outcomes. The results were significant. Performance outcomes showed on average a .5 standard deviation increase when active learning strategies are used. Additionally, persistence and attitudes towards math and confidence in doing mathematics is higher when such strategies are.

This research aimed to introduce a few active learning practices in a large size Precalculus classroom and conduct a mixed method study to understand how they are perceived and affect student outcomes. Pre-calculus is a gateway course for upper level mathematics courses for students and other STEM courses, thus it should be researched to help in its implementation. Conducting this research will add to the literature as possible corroboration for the continued use for active learning strategies in mathematics courses specifically in large-scaled classrooms where direct instruction has been overused. For this study, outcomes were defined as students' attitudes towards mathematics and themselves as mathematics learners, interest in mathematics, and self-efficacy. Research questions are: (1) What are students' perceptions of active learning strategies in a large-scale pre-calculus course? And (2) How are students' self-efficacy and attitudes towards mathematics different after participation in an active learning course?

Literature Review

Student attitudes and self-efficacy are two important facets to study in mathematics education today. Núñez-Peña, Suárez-Pellicioni, and Bono (2013) found that poor success rates in mathematics course are related to negative attitudes towards mathematics in general. In addition, both student attitudes and self-efficacy play an integral part in perceived usefulness of mathematics, in choosing careers involving mathematics, and in opting to take more mathematics courses beyond those required for their degree (Sheldrake, Mujtaba, & Reiss, 2015).

There have been studies focused on ways to improve student attitudes and self-efficacy in these courses to help combat the negative repercussions as described above (Wilkins & Ma, 2003). Wilkins and Ma (2003) report on the importance of social influence, challenging curricula, and engaging activities as ways to help students see the importance of mathematics in their own lives and to see that mathematics can be engaging as well. The choices instructors make on the activities presented in class can make a big difference in how students view mathematics in general (Wilkins & Ma, 2003). As retention rates in STEM fields remain a difficult feat still today, it is important to help improve the attitudes and self-efficacy of students in undergraduate mathematics courses for the future.

Theoretical support for our work was taken from higher education research and adds a new perspective to traditional mathematics education frameworks. Fraser (1989) stated that "the strongest tradition in past classroom environment research has involved investigation of associations between students' cognitive and affective learning outcomes and their perceptions of psychosocial characteristics of their classrooms" (p. 315). So, we use this tradition of studying the associations as our research base. We overlay that plan to investigate students' outcomes and perceptions onto learner-centered classroom environments as we see this as a particularly crucial factor in the students learning as well as their self-efficacy (Peters,2013). Therefore, our investigation is of learner-centered instruction (and we define environment as part of this) which has been shown to be effective in increasing both cognitive and affective outcomes. Students' perceptions will influence the affective outcomes and we will study this relationship.

Methods

This research was carried out at a large, public university located in the southeast area of the United States. During Summer 2016, 94 students were enrolled in a four-credit hour Precalculus/Trigonometry course. Throughout the course, a total of five Team Activities were assigned to the students to complete in small groups. An entire class period was dedicated for the completion of each Team Activity. These frequently consisted of real-world scenarios, and were intended to help students develop a deeper understanding of the mathematics. Students worked in teams of 3-4 in order to solve application problems and write out complete solutions.

Participation in both pre- and post-surveys were online, voluntary, and open to all students in the course. In total, 18 students completed both pre- and post-surveys (results are in the following section). The surveys contained were Likert-scale statements where participants were instructed to select the number that best expressed their feelings as to what extent did they agreed or disagreed with each statement. Survey questions addressed mathematical self-efficacy, students perceived mathematical ability/work ethic, and students' interest in the field of mathematics in the future.

The purpose of the interview was to collect qualitative data on student perceptions of the class and the mathematics they had learned throughout the duration of the course. Fourteen students volunteered to be interviewed to share their experiences at the end of the semester. Interview participants included seven women and seven men.

The interview was split into two parts: student views on the course and student mathematical thinking.

Results

A close look at that individual responses showed that: (1) of all the changes that occurred from pre- to post-survey, 57.69% were increases, while 42.31% were decreases; (2) 11 of the 18 students increased on average from pre- to post-survey; and (3) On average, each student increased by +0.119.

Looking at the collective responses on the individual questions, 15 of the questions (65%) showed positive agreement increases, while 6 of the questions (26%) showed decreases in agreement, and 2 of the questions (9%) remained constant with regards to agreement. The questions that showed the biggest mean positive increases were: *Math is easy for me to understand* and *I am not nervous when I am in math class*, *I would like an occupation in math,* and *I am able to do the math in my class*. The questions that had the biggest decreases in mean responses were: *I have a lot of math skills*, *I work hard with math in school,* and *I don't give up even if I have a difficult math problem*.

The questions specifically centered around three distinct categories of questions. They included questions related to students' mathematics self-efficacy (Group 1), questions gauging how students' think about their own work ethic in mathematics (Group 2), and how mathematics relates to students' futures (Group 3). Participants' responses were averaged within each group to create a students' overall Group Score for that group. Paired t-tests were run using statistical software within these group averages.

Of the three groups of questions, the first was the found to be statistically significant. Just as with some findings described above, the mean values from pre- to post-survey were not practically different, only increasing from 3.07 to 3.28. The mean values in Group 2: Work Ethic in Math decreased from a 4.30 to 4.25. Group 3: Math in the Future showed an overall increase in the means from pre- to post-survey having showed a 0.24 increase.

In the interviews, students were asked if they believed that Team Activities were helpful for their learning and then were encouraged to explain their response. Several codes that emerged from this section of the interview, and will be discussed in the presentation. Overall feelings for Team Activities were very mixed. There was a similar number of students who felt positively toward Team Activities and who felt negatively.

Discussion and Conclusion

Various course components provided the students the opportunity to work with other students to aid their learning (including other aspects not discussed in this paper). The Team Activities provided this unique experience that is often not found in a typical large populated mathematics course taught at universities. By breaking away from entire class periods filled solely with lecture, this course afforded students the opportunity to benefit from active learning through these Team Activities. With the Team Activities, students had to work together in groups of 3-5 people to complete the mathematical tasks that were assigned to them. All the Team Activities got the student to get up out of their seats and find their groups. Some of the Team Activities allowed students to get out of the lecture hall, altogether. During this time, students had the opportunity to work with one another in order to strengthen and reaffirm what they should have learned. These had the students go outside and interact with their surroundings in order to solve a mathematical task. An overwhelming majority of students enjoyed this active learning aspect of the course (along with others not discussed in this paper).

An interesting aspect that came up after analysis, is that seeing the mathematics working in a "real world" context did not seem to be as an important factor to most students. We thought that this concerted effort to bring an applicable view into how mathematics in this classroom is actually used in the field is what these students wanted, but it seemed as if this was not as important to these students. The Team Activities were designed to elicit a connection between the mathematics and its applicability in the real world, yet students failed to mention how the two were connected. Potentially, the design of the Team Activities needs to be revisited to make more explicit (when appropriate) how these are real scenarios seen by professionals.

The teaching assistants (TAs) in this course were very essential to the success for many of the students. Students were able to interact with the five TAs in multiple facets of the course. TAs had a huge role in the implementation of the Team Activities. Students also lamented the fact that during the Team Activities, there were never enough of them.

In this paper, we chose to discuss just a small portion of the entire work due to lack of space. The results are significant as we are finding that it makes a difference even to include just a few instructional strategies that are considered learner-centered. As related to Fraser's (1989) work, we found evidence to support his contention that perceptions affect outcomes. This result opens up a significant question: How important is it to include learner centered instruction fully implemented or can a partial implementation work? More research has been and will be conducted to see what (if any) effect that these active learning components that were used in this course have affected students' perceptions, self-efficacy, and attitudes on learning mathematics.

References

Carnevale, A., Smith, N., & Melton, M. (2011). "STEM: Science, Engineering, Technology, and Mathematics." Georgetown University Center on Education and the Workforce.

Creswell, J. W. (2013). Five Qualitative Approaches to Inquiry. In *Qualitative inquiry and research design: Choosing among five approaches* (3rd ed., pp. 69–110). Los Angeles, CA: SAGE Publications.

Ehlich, K. (1993). HIAT: A transcription system for discourse data. In J. A. Edwards & M. D. Lampert (Eds.), *Talking data: Transcription and coding in discourse research* (pp. 123–148). Hillsdale, NJ: Lawrence Erlbaum.

Foshee, C. M., Elliott, S. N., & Atkinson, R. K. (2016). Technology-enhanced learning in college mathematics remediation. *British Journal of Educational Technology*, *47*(5), 893–905.

Fraser, B. (1989). Twenty years of classroom climate work: Progress and prospect. Journal of Curriculum Studies, 21, 307–327.

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the*

Jesse L. M. Wilkins & Xin Ma (2003) Modeling Change in Student Attitude Toward and Beliefs About Mathematics, The Journal of Educational Research, 97:1, 52-63.

Núñez-Peña, M.I., Suárez-Pellicioni, M., & Bono, R. (2013). Effects of math anxiety on student success in higher education. *International Journal of Educational Research, 58*, 36-43.

Peters, M.L. (2013). Examining the relationships among classroom climate, self-efficacy, and achievement in undergraduate mathematics: A Multi-level analysis. International Journal of Science and Math Education, Netherlands: Springer.

President's Council of Advisors on Science and Technology (PCAST). (2012). *Engage to excel: Producing one million additional college graduates with degrees in science, technology, engineering, and mathematics*.

Rittle-Johnson, B., & Koedinger, K. (2009). Iterating between lessons on concepts and procedures can improve mathematics knowledge. *British Journal of Educational Psychology,* 79(3), 483-500.

Sheldrake, R., Mujtaba, T., Reiss, M.J. (2015). Students' intentions to study non-compulsory mathematics: the importance of how good you think you are. *British Educational Research Journal, 41* (3), 462-488.

Strauss, A., & Corbin, J. (1990). Open Coding. In A. Strauss & J. Corbin (Eds.), *Basics of qualitative research: Grounded theory procedures and techniques* (2nd ed., pp. 101–121). Newbury Park, CA: SAGE Publications.

Zogheib, B., Rabaa'i, A., Zogheib, S., & Elsaheli, A. (2015). University student perceptions of technology use in mathematics learning. *Journal of Information Technology Education: Research*, *14*, 417–438.

STUDENT PARTICIPATION, AUTHORITY AND TOGETHERNESS IN A COLLABORATIVE MATH TASK: WITH AND WITHOUT THE TEACHER

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To support equitable learning opportunities during group work, it is important that teachers notice participation and authority imbalances among students. We analyze two seemingly productive student-pairs working on a math task to understand what is made available to notice about participation and positioning on two levels: (1) what a teacher can observe during teacher-group interactions, and (2) what is noticeable during group work without the teacher present. We found that while mathematical participation was imbalanced during group work for both student-pairs, for one pair, "togetherness" and balanced authority during teacher-group interaction did not reflect, and thus potentially masked, participation imbalance.

Keywords: Classroom Discourse, Equity and Diversity, Instructional Activities and Practices

Collaboration is seen as a promising approach to support equity. However, it is also recognized that group work does not always play out equitably in classrooms (Esmonde, 2009). It has long been argued that successful implementation of collaborative learning depends on strategic facilitation of learning environments and that, in particular, teacher-group interactions during small-group work are means to shift inequitable group dynamics (Cohen & Lotan, 2014). What teachers notice about students' equity-related experiences during complex collaborative learning interactions is important for such support to be possible (van Es, Hand, & Mercado, 2017; Wager, 2014). There is a need to better understand what equity-related aspects of students' collaboration are not easily available for teachers to notice during small-group check-ins.

Some aspects of group work are easier to notice than others. When student-groups are off task or silent during group work, it may be easy for a teacher to notice that collaboration is not ideal. However, even if students stay on task and talk to each other, inequitable processes such as imbalances in participation and authority can still occur (Esmonde, 2009; Langer-Osuna, 2017). Since imbalances are often subtle and evolve over time, they may not be easily observable during short teacher-group interactions. To shed more light on this issue we examine two seemingly productive student-pairs working on a collaborative math task, and compare processes consequential for equity across two settings: teacher-group interactions and group work.

Theoretical Perspectives

Esmonde (2009) draws on sociocultural theories of learning to define equity as the fair distribution of: (a) access to mathematical content and practices and (b) access to productive positional identities. In this study, we use the metaphor of participation to identify students' access to mathematical ideas and practices, and positioning theory to identify processes by which students make sense of themselves and their peers as mathematics learners.

Learning happens through participation in disciplinary practices (Lave, 1996). In collaborative settings, this means active participation in the task and mathematical discourse practices of the group, such as asking questions and providing explanations (Reinholz & Shah, 2018).

Positioning theory (van Langenhove & Harré, 1999) illuminates ways in which narratives at various scales influence and get constituted by small-scale interactions in classrooms (Herbel-

Eisenmann, Wagner, Johnson, Suh, & Figueras, 2015). During interaction, communicational acts evoke *positions*, expectations about interlocutors' rights and obligations in a given situation. For example, if one student asks another "Is this answer correct?", he creates a "right" for evaluation, positioning his partner as an *expert* and himself as a *novice*. Such interpretation of utterances in terms of rights and obligations relies on interlocutors' past experiences and familiarity with the same *storylines* (e.g. "math experts know the answers").

We draw on these analytical lenses to address the following research question: *How do student participation in mathematical practices and positioning manifest: (a) during teacher-group interaction, and (b) during group work interaction?*

Methods

Data analyzed in this study was collected at a mid-size city in central England. It is part of a large corpus of video data of students working on "Formative Assessment Lessons" (FALs; see Burkhardt & Schoenfeld, 2018) developed by the Mathematics Assessment Project (http://map.mathshell.org/). The video corpus of FAL classroom enactments was collected to support development of tablet-based versions of the FALs by the FACT project (Formative Assessment with Computational Technologies: http://fact.engineering.asu.edu/). The analyses presented in this paper emerged from a larger study within the FACT project, in which the same video data supported the development of a collaboration coding scheme for learning analytics (VanLehn, Burkhardt, Cheema, Kang, Pead, Schoenfeld, & Wetzel, submitted).

Since the purpose of our study was to analyze groups that do not exhibit easy-to-notice, undesirable small-group behaviors, we searched the data corpus for groups that seemed to be working together productively. We selected two such groups and present analyses of four episodes: two TGI episodes and two corresponding group work episodes. These episodes feature a teacher, Mr.B, and two student-pairs, Abby & Ben and Chris & Diana.

The task is part of a 7th grade FAL. Students are given four cards with printed money amounts: $100, $150, $200, $160 (to be positioned as four vertices of a square) and 12 arrow cards with a specified percent increase or decrease, e.g., "up by 50%". The task entails matching a percent increase or decrease arrow card to an appropriate space between money card vertices, e.g., matching the arrow card "down by 25%" to the space between $200 and $150.

Data Analysis

Episodes were transcribed and coded for speech and action indicators informed by our theoretical framework and prior work on participation and authority. We started our analysis with these two categories in mind, yet found that another storyline, which we call "togetherness", was helpful in capturing consequential differences between the two student pairs. Our operationalization of participation, authority and togetherness is elaborated below.

Who suggested the card placement is used as an indicator of mathematical participation in the task. To analyze participation in discourse practices, we draw on Reinholz and Shah (2018) and focus on two discourse forms: explanations and questioning. We coded *who explains* and *explanation length* (brief or extended), in addition to *who asks a question* and *question type* (a "why/how" question or a "what" question).

Our analysis focuses on two storylines and related positions: (a) *authority* storyline (experts vs. novices) and (b) *togetherness* storyline (partners vs. individual contributors). In the *authority* storyline, *experts* answer other students' questions, evaluate contributions, and explain why answers are correct. *Novices* ask questions, seek approval of experts, and defer to experts for decisions. To identify *expert* or *novice* positions we look at: Who asks and who answers "why/how", "what", and "approval" questions. The *togetherness* storyline is about how students perceive their rights and obligations as groupmates. *Partners* are expected to seek and maintain joint attention, share control over decisions, and solicit contributions from each other. For *individual contributors* joint attention and joint decisions are not expected. Though they can ask each other questions, each student is responsible for his or her contribution to the task. To identify *partner* or *individual contributor* positions we look at: the percent of card placement decisions that were made jointly, the acceptance rate of students' focus initiation moves, and the total number of questions students asked each other.

Findings

First, we present our analyses of Teacher-Group Interactions (TGIs) and Group Work focusing on mathematical participation, authority, and togetherness. We then discuss similarities and differences between what was noticeable about the two student-pairs in the two settings.

Teacher-Group Interactions

TGIs with both pairs involved one of the students explaining a card placement to the teacher. The two exchanges were different, however, in the ways authority and togetherness storylines were evoked. In the first group, Ben was positioned as an expert and as an individual contributor. In the second group, Chris and Diana positioned themselves as novices and as partners. Chris and Diana's greater similarity in authority and higher degree of togetherness give an overall impression of more balance in learning opportunities than between Abby and Ben.

Group Work

Summary of Participation, Authority, and Togetherness

	Participation						Authority			Togetherness			
	Suggestions (rejected)	Explanations		Questions Asked by			Questions Answered by			Total Number of	Focus Initiations		Joint
		Extended	Brief	Why/How	What	Approval	Why/How	What	Approval	Questions Asked	Number	Acceptance	Decision Rate
Abby	4(3)	0	2	3	1	3	0	1	0	7	16	59%	17%
Ben	9	5	2	0	2	0	4	1	3	2	11		
Chris	3(1)	1	1	2	2	3	0	1	4	8	8	76%	55%
Diana	10	2	4	0	4	4	5	6	3	10	13		

Figure 1. Summary of participation, authority, and togetherness during groupwork.

Groupwork participation was imbalanced in favor of one student in both pairs; Ben and Diana made considerably more placement suggestions and explanations than their partners. The imbalance in explaining mirrors the imbalance in asking "why/how" questions. Abby and Chris's questioning created opportunities for Ben and Diana to explain, but, since Abby and Chris were not asked "why/how" questions, they did not have similar opportunities to learn by explaining.

Authority was distributed differently in the two pairs. With Abby and Ben, imbalanced authority was stable and enduring; Ben was positioned as *expert* and Abby as *novice*. In the other pair, Chris was consistently positioned as *novice*, but Diana's position was negotiated in contrasting directions. Diana positioned herself as *novice* by repeatedly asking for approval from Chris, yet, Chris also positioned her as *expert* by asking her several "why/how" questions.

Togetherness was also different in the two pairs. Ben and Abby did not share a clearly agreed upon togetherness storyline. While Ben consistently positioned himself as an *individual contributor*, Abby made repeated moves to position them as *partners*. These ongoing negotiations were reflected in the way Ben and Abby went back and forth between having shared focus and not. In contrast, Chris and Diana accepted each other as *partners*. They maintained joint focus and joint ownership over different parts of the task by attending to each other's activity and consistently checking-in with each other to make decisions together.

Comparison between TGIs and group work

Abby and Ben's participation and authority imbalance and their overall lack of togetherness during Group Work manifested similarly during their TGI. In contrast, Chris and Diana's participation imbalance was not clearly reflected in their TGI. When interacting with Mr. B, both students positioned themselves as similarly uncertain about the math and demonstrated a high degree of togetherness. Though the pair consistently worked together and authority was relatively balanced, close examination of their work revealed that opportunities to learn were inequitably distributed as Diana suggested and explained significantly more than Chris did.

Discussion

Our study reveals that togetherness and balance in authority, salient during teacher-group interactions, does not reflect imbalances in mathematical participation that are detrimental to equitable learning opportunities. This finding has implications for teacher noticing of student collaboration for equity and the design of technology to support that noticing. Recognition that togetherness and balanced authority have the potential to mask enduring participation imbalances can help teachers critically examine their noticing of seemingly productive groups. Our findings also provide further support for the potential of using learning analytics to track participation imbalances in classrooms (Reinholz & Shah, 2018). Our analysis suggests that capturing who suggests ideas, asks questions, and explains can support noticing for equitable participation.

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References

Burkhardt, H., & Schoenfeld, A. (2018). Assessment in the service of learning: challenges and opportunities or Plus ça Change, Plus c'est la même Chose. *ZDM, 1-15.*

Cohen, E. G., & Lotan, R. A. (2014). Designing Groupwork: Strategies for the Heterogeneous Classroom Third Edition. Teachers College Press.

Herbel-Eisenmann, B. A., Wagner, D., Johnson, K. R., Suh, H., & Figueras, H. (2015). Positioning in mathematics education: Revelations on an imported theory. *Educational Studies in Mathematics*, *89*(2), 185-204.

Esmonde, I. (2009). Mathematics learning in groups: Analyzing equity in two cooperative activity structures. *The Journal of the Learning Sciences*, *18*(2), 247-284.

Langer-Osuna, J. M. (2017). Authority, Identity, and Collaborative Mathematics. *Journal for Research in Mathematics Education*, *48*(3), 237-247.

Lave, J. (1996). Teaching, as learning, in practice. *Mind, culture, and activity*, *3*(3), 149-164.

Reinholz, D. L., & Shah, N. (2018). Equity Analytics: A Methodological Approach for Quantifying Participation Patterns in Mathematics Classroom Discourse. *Journal for Research in Mathematics Education*, *49*(2), 140-177.

van Es, E. A., Hand, V., & Mercado, J. (2017). Making Visible the Relationship Between Teachers' Noticing for Equity and Equitable Teaching Practice. In Teacher Noticing: Bridging and Broadening Perspectives, Contexts, and Frameworks (pp. 251-270). Springer, Cham.

van Langenhove, L., & Harré, R. (1999). Introducing positioning theory. In R. Harré & L. van Langenhove (Eds.), *Positioning theory: moral contexts of intentional action* (pp. 14–31). Oxford, UK: Blackwell Publishers.

VanLehn, K., Burkhardt, H., Cheema, S., Kang, S., Pead, D., Schoenfeld, A., & Wetzel, J. (submitted). The effect of digital versus traditional orchestration on collaboration in small groups. Paper submitted to *AIEd 2018, the 19th international conference on artificial intelligence and education, London, England,* June 27-30, 2018.

INVESTIGATING COLLEGE CALCULUS INSTRUCTORS' KNOWLEDGE, DISPOSITION, AND RESPONSIVENESS TO STUDENT THINKING

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Effective instruction relies on an instructor's knowledge of student thinking and ability to attend and respond to student mathematical strategies. Additionally, their disposition towards student thinking further influences how they respond to students, impacting student learning. A survey and task-based interviews were used to investigate instructors' responsiveness to student thinking. Results from these analyses indicate that responsiveness to student thinking can be present during multiple phases of instruction (planning, assessing, and in-the-moment teaching) and provides specific instructional practices that relate to responsiveness to student thinking.

Keywords: Mathematical Knowledge for Teaching, Post-Secondary Education, Instructional activities and practices

Effective instruction, both in a lecture environment and in an active learning environment, relies on the instructor's ability to attend and respond to student mathematical strategies, which involves drawing on their knowledge of student thinking (Johnson & Larsen, 2012). This ability to attend, or notice, impacts student learning and is influenced by the instructor's disposition towards student thinking (Sherin & Russ, 2014). The purpose of this study is to gain insight into the instructional practices that relate to instructors' responsiveness, and to understand how college calculus instructors incorporate their knowledge of student thinking into instruction.

Research on Instructors' Knowledge with Dispositions and Responsiveness

There has been much research surrounding teachers' knowledge of student thinking, particularly at the K-12 level, by considering this as one component of instructors' mathematical knowledge for teaching (MKT). MKT includes knowledge of content and the pedagogical knowledge needed to facilitate learning and effective instruction (e.g. Ball, Thames, & Phelps, 2008). It influences instructors' decisions, and impacts what they notice and attend to during in-the-moment instruction and when they prepare for instruction (Jacobs, Lamb & Philip, 2010). Jacobs, Lamb, and Philipp (2010) describe this noticing as attending to, interpreting, and deciding how to respond to students' mathematical strategies, and can be leveraged to connect instructors' knowledge and practice with their disposition to student thinking (Hand, 2012).

An instructor's disposition influences what they notice as "significant interactions" during instruction (Goodwin, 1994). Gresalfi and Cobb (2006) describe *disposition* as that which "encompasses ideas about, values of, and ways of participating with a discipline" (p. 50). This stems from a situated cognition learning perspective which points to the importance of context in the development of understanding and knowledge. An instructor's disposition towards student thinking is continually shaped and interacts with their MKT to influence instructor responsiveness to student thinking, further impacting how they elicit and respond to student thinking (van Es & Sherin, 2002). An instructor's responsiveness to student thinking generates opportunities for effective instructional practices (Carpenter et al., 1989). For example, an instructor can support student learning by addressing difficulties, or by eliciting student thinking through purposeful activities or questioning when they attend to student thinking (Jacobs, Lamb, Philipp, & Schappelle, 2011). In this paper I investigate college calculus instructors' responsiveness to student thinking. Specifically, I investigate the following research questions: (1) What instructional practices reflect responsiveness toward student thinking? (2) How does responsiveness show up when instructors discuss their instructional decisions?

Methodology and Results

This study is part of a larger study investigating college mathematics instructor knowledge, dispositions, and responsiveness to student thinking. Quantitative and qualitative data were collected and analyzed separately, and then the results were brought together to provide a more complete picture of how responsiveness relates to instructional practices using a convergent parallel mixed methods study design (Creswell & Plano Clark, 2007).

Quantitative Data Collection and Analysis

To collect data regarding instructional practices and responsiveness to student thinking, the Postsecondary Instructional Practices Survey (PIPS) (Walter et al., 2016) was expanded to include 5 *responsiveness* items (PIPS+) (e.g. R1, listed in Table 1). The survey was distributed to precalculus and calculus I and II instructors at 12 institutions in the United States. These schools were selected as part of a larger study that is studying multiple dimensions of calculus instruction across the US. For this study, I consider results from the calculus I instructors of record, which includes 35 graduate teaching assistants and 57 faculty/lecturers from the 12 institutions.

To answer the first research question, an exploratory factor analysis with varimax rotation was conducted on the PIPS+ items. Three factors were requested, suppressing factors with loadings less than .4. Constructs were summarized considering themes from the items that loaded on each factor. In the results I discuss the two factors where the 5 *responsiveness* items loaded.

Quantitative Results

After rotation, the first factor in the factor analysis accounted for 16.4% of the variance, the second factor accounted for 12.3%, and the third factor (not related to responsiveness) accounted for 8.2%. The first factor indexed instructional practices that include student involvement in class, and had strong loadings on 20 items (including 4 *responsiveness* items). The second factor indexed instructor planning and instructional activities, and loaded highly on 8 items (including 1 *responsiveness* item), as displayed in Table 1.

Table 1. Responsiveness Items Related to Two Factors

Factor	Survey Items
1	**R1:** I use a variety of approaches (e.g., questioning, discussion, formal/informal assessments) to gauge where my students are in their understanding of concepts **R2:** I use student questions and comments to determine the focus and direction of classroom lessons **R3:** I adjust my teaching based upon what students currently do or do not understand **R4:** I use student assessment results to guide the direction of my instruction during the semester **Sample survey item:** I structure class so that students explore or discuss their understanding of concepts before direct instruction
2	**R5:** I consider students' thinking/understanding when planning lessons **Sample survey item:** I explain concepts in this class in a variety of ways

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.

The exploratory factor analysis illuminates instructional practices that are linked with responsiveness to student thinking since factors are generated by grouping items instructors rated as similarly descriptive of their instruction. For example, instructors who consider student thinking/understandings when planning lessons (R5) also explain concepts in a variety of ways.

Qualitative Data Collection and Analysis

To understand the degrees to which college calculus instructors incorporate their knowledge of student thinking into their instructional practices (answering RQ2), eight instructors (6 graduate teaching assistants and 2 teaching faculty) from one of the 12 institutions surveyed were interviewed three times over one academic year. The interview protocol was adapted from a task-based interview used to examine college instructor MKT (Speer & Frank, 2013).

The interview data were analyzed using thematic analysis (Braun & Clarke, 2006), by first identifying utterances related to instructors' attention to student thinking. These segments were then coded as either related to *responding* to specific student thinking through some action (e.g. "[I would] help him review [how to find] the minimum"), or as *responsiveness* to student thinking, which demonstrated a more general consideration of student thinking (e.g. "I just try to put myself inside [the student's] head as best as I can"). *Responsiveness* segments were coded again, beginning with a priori codes developed from the 5 *responsiveness* survey items. Eight additional codes were created for segments that could not be described by these, including two subcodes related to R5. See Table 2 in the results section for sample coded excerpts.

Qualitative Results

Instructors incorporated their knowledge of student thinking during planning, grading, and in-the-moment teaching. The coding of the interviews highlighted the attention to student thinking captured by the *responsiveness* survey items was present as instructors discussed how they adjusted their instruction in response to their students' understanding (R2, R3) or considered student thinking when planning (R5).

Table 2. Examples of Segments Coded Using A Priori Responsiveness Codes

Code	Interview Excerpt
R2	"If I have a lesson planned, … then in the middle … we'll get derailed and somebody will say 'I have a lot of confusion about this piece'... and we end up covering things that I didn't plan to cover at all."
R3	"I have been tweaking my lectures because I keep thinking, like I should do an example of this because I realize that they really are confused about [it]."
R5	"When I plan a lesson I think about what my students will struggle with and what they will feel very natural, … when I write a lesson I will try to build in examples that I hope address [students'] confusion."
R5/ Use own experience	"Every single time we try to think before the [recitation] … [thinking about] the first time I learned this- what was tough for me, and we write that on the board."

Discussion and Implications

The results from these analyses demonstrate that instructors respond to student thinking and strategies during a variety of instructional practices. The exploratory factor analysis highlights specific practices that instructors use (regardless of instructional format) to attend to students' understandings of mathematical ideas, and is supported by the interviews. These provide specific practices that can be incorporated by instructors interested in being more responsive to student thinking. This is especially relevant due to the findings that encourage the use of active learning strategies in undergraduate instruction (e.g. Freeman, et al., 2014), since this requires a rich knowledge of student thinking and the ability to respond during in-the-moment instruction.

In future work, I will further consider how an instructor's disposition is related to their responsiveness. Gresalfi and Cobb (2006) noted that developing "positive and productive dispositions … will allow us to begin to address the current well-documented gap between who actually has opportunities to be successful in the classroom" (p. 55). Although Gresalfi and Cobb were referring to cultivating students' dispositions, this also can apply to instructors' dispositions towards student thinking; when we better understand how we can foster positive dispositions and responsiveness towards student thinking, then we can better support students by providing opportunities for more effective instruction building off of student understandings (Hand, 2012).

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References

Ball, D.L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special?. Journal of teacher education, 59(5), 389-407.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. Qualitative research in psychology, 3(2), 77-101.

Carpenter, T., Fennema, E., Peterson, P., Chiang, C., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. American Educational Research Journal, 26(4), 499–531.

Creswell, J. W., & Plano Clark, V. L. (2007). Designing and conducting mixed methods research.

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. Proceedings of the National Academy of Sciences, 111(23), 8410–8415.

Goodwin, C. (1994). Professional vision. American Anthropologist, 96, 606–633.

Gresalfi, M. S., & Cobb, P. (2006). Cultivating students' discipline-specific dispositions as a critical goal for pedagogy and equity. Pedagogies, 1(1), 49-57.

Hand, V. (2012). Seeing culture and power in mathematical learning: Toward a model of equitable instruction. Educational Studies in Mathematics, 80(1-2), 233-247.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. Journal for research in mathematics education, 169-202.

Jacobs, V. R., Lamb, L. L. C., Philipp, R. A. & Schappelle, B. P. (2011). Deciding how to respond on the basis children's understanding. In M. G. Sherin, V. R. Jacobs & R. A. Philipp (Eds.), Mathematics teacher noticing: Seeing through teachers' eyes (pp. 97–116). New York, NY: Routledge.

Johnson, E., & Larsen, S. (2012). Teacher listening: The role of knowledge of content and students. The Journal of Mathematical Behavior, 31(1), 117–129.

Sherin, M. G., & Russ, R. S. (2014). Teacher noticing via video. *Digital video for teacher education: Research and practice*, 3-20.

Speer, N., & Frank, B. (2013). Building knowledge for teaching rates of change: Three cases of physics graduate students. In S. Brown, G. Karakok, K. H. Roh, & M. Oehrtman (Eds.), Proceedings of the 16th Annual Conference on Research in Undergraduate Mathematics Education, Vol. 2 (pp. 247–253). Denver, CO.

Van Es, E. A., & Sherin, M. G. (2002). Learning to notice: Scaffolding new teachers' interpretations of classroom interactions. Journal of Technology and Teacher Education, 10(4), 571-596.

Walter, E. M., Henderson, C. R., Beach, A. L., & Williams, C. T. (2016). Introducing the Postsecondary Instructional Practices Survey (PIPS): A concise, interdisciplinary, and easy-to-score survey. CBE-Life Sciences Education, 15(4), ar53.

ARGUMENTATION IN THE MATHEMATICS CLASSROOM: SOCIAL, SOCIOMATHEMATICAL, AND MATHEMATICAL ARGUMENTS

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We argue for the expansion of Krummheuer's conceptualization of argumentation in the mathematics classroom. We use Toulmin's deconstruction of arguments along with studies on social interactions (e.g. Goffman, 1959) to position argumentation as a larger social phenomenon. We provide examples from two lessons conducted by a first year secondary teacher to highlight three different types of arguments: (1) social arguments; (2) sociomathematical arguments; and (3) mathematical arguments. Compartmentalizing classroom arguments in this way provides researchers a way to examine the perpetuation of deficit narratives, construction of sociomathematical norms, and establishment of productive learning environments.

Keywords: Classroom discourse, Instructional activities and practice

In general, argumentation is defined as the attempt to convince someone of the validity of a claim (Toulmin, 2003). Researchers have focused on various aspects of argumentation in the mathematics classroom including understanding classroom events (Whitenack & Knipping, 2002), teacher's support of argumentation (Conner, Singletary, Smith, Wagner, & Francisco, 2014), and students' justification and proof (Staples, Bratlo, & Thanheiser, 2012). All of these studies build on the conceptualization of arguments by Toulmin (2003), but more importantly, on Krummheuer's (1995) ethnography of argumentation in the mathematics classroom.
In this report, we return to Toulmin's (2003) conceptualization of argumentation to describe a framework deconstructing the arguments occurring in the mathematics classroom and, as such, expand the notion of argumentation as a social action. We argue previous research on argumentation has solely focused on mathematical arguments (e.g. Ingles, Meija-Ramos, & Simpson, 2007) and less on the social arguments taking place on a regular basis (Gee, 2001; Goffman, 1959). We focus on episodes from a 1st year teacher's class to emphasize the variety of social and mathematical arguments occurring in the mathematics classroom.

Argumentation in the Mathematics Classroom

Krummheuer (1995) provided a theoretical background of argumentation and conceptualized how argumentation can be considered in the learning of mathematics. There has been, however, few previous attempts to conceptualize argumentation as a broader lens and what new understandings such a lens can provide (see Gomez, 2018). We build on previous investigations into the ways individuals attempt to convince others in the community of his or her membership to particular social groups (e.g. Gee, 2001). In other words, individuals make arguments about how they want to be seen and positioned by others. As Goffman (1959) wrote:

> When an individual plays a part he implicitly requests his observers to take seriously the impression that is fostered before them. They are asked to believe that the character they see actually possesses the attributes he appears to possess, that the task he performs will have the consequences that are implicitly claimed for it, and that, in general, matters are what they appear to be. (p. 17)

The framework we describe in this presentation is meant to bring to the forefront, not just the mathematical arguments made in the classroom, but also the social and sociomathematical arguments. The use of an argumentation lens can provide new understandings of the ways dominant narratives are perpetuated in mathematics education. With the different types of arguments described, in combination with Toulmin's (2003) deconstruction of arguments, researchers can dismantle the ways teachers and students perpetuate deficit narratives and aid in the reconstruction of more productive narratives.

Social, Sociomathematical, and Mathematical Argumentation

In this section, we provide examples of each type of argument with classroom episodes of a first year teacher. We use episodes from two lessons to highlight the three types of arguments: (1) social arguments (i.e. attempts to convince an audience about one's social positioning in the world); (2) sociomathematical arguments (i.e. attempts to convince an audience of the ways individuals do mathematics within the social order); and (3) mathematical arguments (i.e. attempts to convince an audience about the validity of a mathematical claim through the use of implicit and explicit warrants of a mathematical nature (e.g. constructs, ideas, and facts) or other social capital (e.g. authority, status)).

The classroom episodes come from a longitudinal study following a cohort of 16 prospective secondary teachers through their teacher education program. For this report, we focus on Susan (pseudonym), a white cisgender female in her early 20s during her first year of teaching 9th grade Algebra. The objective of the lessons was to have students determine and write the equations of lines. She focused the majority of her time on two tasks. The first task asked students to find the equations of lines representing roads near the school. The second task was a penny pattern task (see Figure 1). The students were asked to use the data from the penny pattern task to construct a graph on a coordinate plane and find the equation of the line.

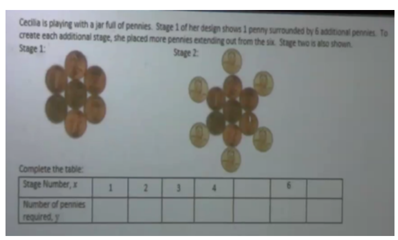

Figure 1. Penny task Susan gave to students

Social Arguments

Social arguments are attempts to convince an audience about one's social positioning in the world. In other words, social arguments are about the kind of person one wants to be seen as by others. One must argue he is the person he believes he is by using specific discourses and presenting himself in particular ways (e.g. what one chooses to wear) (Gee, 2001). We refer to arguments made to demonstrate ownership of characteristic traits or to be legitimized as a member of a community as *identity work* (Gomez, 2018). One needs to make social arguments to be recognized as a certain kind of person.

In the following episode, Susan provides a narrative to her students about her thinking process for making mathematics more interesting for them.

> Okay story time goes like this. Um, yesterday I was thinking to myself - self, how can I make math more interesting for my students because inevitably what happens when I give you guys

> something you say to me, "Ms. S when am I ever going to use this in real life?" And that is a great question. So I'm answering the question. (Susan, Lesson 1)

For this argument, Susan claimed to be attempting to make mathematics more interesting for her students warranted by her answering their questions about the application of mathematics. In the narrative, she positions herself as doing the work of a teacher and taking time outside of class for the students' benefit. She may also have been trying to provide a counter-narrative about mathematics teachers to help students reconstruct their mathematics experiences as more positive.

Sociomathematical Arguments

We define sociomathematical arguments as attempts to convince an audience of the ways individuals do mathematics within the social order. Consequently, the sociomathematical arguments made are directly related to the construction of sociomathematical norms (Yackel & Cobb, 1996), or "normative aspects of mathematics discussions specific to students' mathematical activity" (p. 361). The teacher needs to convince the students to accept, follow, and then maintain the constructed sociomathematical norms. Furthermore, these events help in the development of the students' identities as doers-of-mathematics (Cobb, Greselfi, & Hodge, 2009).

Susan made a number of sociomathematical arguments during the two lessons by attempting to convince the students learning mathematics is collaborative. She did not provide explicit warrants to the whole group during these two lessons, but did find opportunities to continue the argument while monitoring the students' small group work. While monitoring the students' work, for example, she warrants argues the benefits of collaboration:

Susan: Excellent! Yes! Great job!
Student 1: Thanks. Now I just don't know how to do this.
Susan: Yes you do.
Student 2: Yes, it's like -
Student 1: So this would be like 7 and then -
Student 2: See like 1 and 7 like that [points to paper].
Student 1: Oh.
Susan: *This is why I put you guys with each other because you can find the answer with each other.* You don't even need—well you might need me for some stuff.

Mathematical Arguments

Mathematical arguments are the attempt to convince an audience of the validity of a mathematical claim through the use of implicit and explicit warrants of a mathematical nature (e.g. constructs, ideas, and facts) or other social capital (e.g. authority, status). Mathematical arguments have been the main focus of research on argumentation (e.g. Inglis et al., 2007). This is possibly because of the strong focus Krummheuer's (1995) conceptualization of argumentation placed on the mathematical arguments happening in the classroom. These arguments are usually explicit because it is through the construction of mathematical arguments students learn to do mathematics. "Learning how to argue whether an idea or claim is true or false in a mathematically valid way is an essential part of learning to do mathematics" (Koestler, Felton, Bieda, & Otten, 2013, p. 30).

A multitude of mathematical arguments occurred during Susan's two lessons. In this episode, Susan asked a student to contribute a claim to the whole group by asking her how many pennies there are in the first stage of the penny pattern task.

Susan: Okay so this girl is playing with pennies and she's decided, hey how about I start arranging them in a particular order and keeping track of how I arranged them. So the first stage is right there [points to board]. How many pennies total does she have in the first stage? [Susan reaches for cup with popsicle sticks and selects one] Student6?

Student 6: Seven.

Susan: Seven. Yes. [writes 7 in table]. Here is her pennies [points to board; uses marker to write "1" on each penny as she counts them]. One, two, three, four, five, six, seven. She has seven total pennies from stage one.

Susan was the main contributor to this mathematical argument. While the student was providing the claim, the warrant was contributed by Susan. So she was demonstrating her mathematical reasoning and not the student's thinking. Students may infer this as Susan arguing for her authority as well especially if Susan contributes majority of warrants.

Discussion and Conclusion

In this report, we extend and build on Krummheuer's (1995) conceptualization of argumentation in the mathematics classroom. In general, more than just mathematical arguments occur in the classroom. We argue social and sociomathematical arguments need to be explored further because they recognize the complexities of teaching and learning mathematics.

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References

Cobb, P., Gresalfi, M., & Hodge, L. L. (2009). An interpretive scheme for analyzing the identities that students develop in mathematics classrooms. *Journal for Research in Mathematics Education*, *40*(1), 40–68.

Conner, A., Singletary, L. M., Smith, R. C., Wagner, P. A., & Francisco, R. T. (2014). Teacher support for collective argumentation: A framework for examining how teachers support students' engagement in mathematical activities. *Educational Studies in Mathematics*, *86*(3), 401–429.

Gee, J. P. (2001). Identity as an analytic lens for research in education. *Review of Research in Education*, *25*, 99–125.

Goffman, E. (1959). *The presentation of self in everyday life*. New York, NY: Anchor Books.

Gomez, C. N. (2018). Identity work of a prospective teacher: An argumentation perspective on identity. *Mathematics Teacher Education and Development*.

Inglis, M., Mejia-Ramos, J. P., & Simpson, A. (2007). Modeling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics*, *66*, 3–21.

Koestler, C., Felton, M. D., Bieda, K. N., & Otten, S. (2013). *Connecting the NCTM Process Standards & the CCSSM Practices*. Reston, VA: National Council of Teachers of Mathematics.

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 229–269). Hillsdale, NJ: Erlbaum.

Staples, M. E., Bartlo, J., & Thanheiser, E. (2012). Justification as a teaching and learning practice: Its (potential) multifaceted role in middle grades mathematics classrooms. *Journal of Mathematical Behavior*, *31*, 447–462.

Toulmin, S. E. (2003). *The uses of argument* (updated ed.). New York: Cambridge University Press. Originally published in 1958.

Yackel, E. & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27*(4), 458–477.

Whitenack, J. W. & Knipping, N. (2002). Argumentation, instructional design theory and students' mathematical learning: A case for coordinating interpretive lenses. *Journal of Mathematical Behavior*, *21*, 441–457.

CARING, MALE AFRICAN AMERICANS, AND MATHEMATICS TEACHING AND LEARNING

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In this paper, the authors report on a qualitative study that explored the influence a "successful" African American male mathematics teacher had on three African American male high school students' perceptions of teacher care. This critical ethnography study was guided by an intersection of an eclectic array of theoretical traditions, including care theory, critical race theory, and culturally relevant pedagogy. The study employed ethnographic methods during data collection; data analysis identified six overarching themes that the participants used to describe teacher care. Findings suggest that teachers should reconsider the ways they care for African American male students and that a caring teacher–student relationship has a positive influence on African American male students' descriptions and perceptions of teacher care.

Keywords: Equity and Diversity, High School Education

Introduction

Narratives on the achievement outcomes of Black boys in mathematics are too often negative (see Stinson, 2006). Contrary to these negative narratives, however, research affirms positive outcomes for students, including Black boys, when engaged in caring teacher–student relationships (see, e.g., Bartell, 2011; Roberts, 2009; Steele, 1992). But literature specific to caring teacher−student relationships, African American male students, and mathematics teaching and learning is all but nonexistent. Therefore, in this study, we sought to uncover the definitions that African American male high school students had of teacher care, and how, if at all, an African American male teacher might influence their perceptions of teacher care. We categorized the study as a critical ethnography where the researcher studies an intact cultural group in a natural setting over a prolonged period of time (Madison, 2005). It was guided by an intersection of an eclectic array of theoretical traditions (Stinson, 2009) that included care theory (e.g., Gilligan, 1982; Noddings, 1992), critical race theory (e.g., Tate, 1997), and culturally relevant pedagogy (e.g., Ladson-Billings, 1992).

Theoretical Frameworks

Care theory as an analytic framework has a vast history and has been described from many perspectives from a variety of disciplines. This body of knowledge is a result of contributions from several scholars who have defined care in multiple ways (see, e.g. Agne, 1999; Gilligan, 1982; Kohlberg, 1984; Noddings, 1984, 1992, 2002, 2006; Oakes & Lipton, 1999; Siddle-Walker, 1993). This multiplicity has often created somewhat of a muddled definition of care. Nonetheless, Gilligan's (1982) scholarship provided the foundation in which current research on an ethic of care is established where caring transitioned from being initially rooted in the field of nursing, to being recognized as a framework to base other research.

Noddings's (1984, see also 1992) scholarship is recognized as the first to expand Gilligan's (1982) work on care to education. Noddings believes human interaction was the central theme of care theory, which created a natural fit in education research. Noddings defines a caring relationship as inclusive of components of understanding, inter-subjectivity, and constant activity. She claims that a caring relationship is not complete unless there is some sort of confirmation given by the cared for. Therefore, Noddings argues that the teacher–student relationship should be reciprocal and requires a certain amount of trade. She also reveals concerns of cultural relevance when it comes to care in the classroom. Noddings (2006) states, “Two students in the same class are roughly in the same situation, but they may need very different forms of care from their teacher” (p. 20). The teacher must understand each individual student to better provide the care defined by Noddings’s theory.

Critical race theory (CRT) as a theoretical framework was first derived from the legal field in the 1980s, when scholars such as Derrick Bell, Richard Delgado, and Alan Freeman searched for a way to more directly and adequately address race and racism in the United States (Roberts, 2010). It was initially rooted in critical legal studies (CLS), a movement that critically examined formalism and objectivism (Tate, 1997). CLS offered critiques of the law, but failed to address issues related to race. As a result of the shortcomings of CLS, came the development of CRT. Gloria Ladson-Billings and William Tate were the first scholars to introduced CRT to the field of education (DeCuir & Dixson, 2004). CRT stresses the need to interrogate how the law as well as socio-cultural and -political structures and discourse reproduce, reify, and normalize racism in society.

Throughout the research project, male African American high school students were provided a non-colorblind opportunity to describe teacher care and explain the practices of a caring teacher (see, e.g., Thompson, 1998). This non-color-and-cultural-blind approach made it possible for the descriptions that the students had of caring teachers to include tenets of culturally relevant pedagogy (CRP). CRP is frequently identified with the work of Ladson-Billings (e.g., 1992); she defines CRP as pedagogy that “prepares students to effect change in society, not merely fit into it” (p. 382). She stresses how vital the teacher−student relationship is to CRP, and says it allows teachers to “empower students intellectually, socially, emotionally, and politically” (p. 382). This type of pedagogy views culture as a powerful variable in the success of students, and it acknowledges the importance of high standards and expectations for teachers (Irvine, 2001). Roberts (2009) argues, “CRP is a philosophical construct that discusses a set of pedagogical behaviors that identifies, values, respects, and utilizes the cultural knowledge and performance styles of ethnically diverse students” (p. 16).

Methodological Considerations

The participants in this study were three high school-aged male African Americans—Kareem, Joshua, and Michael (pseudonyms, as are all proper names)—in a tenth-grade Geometry class at Divine High School, and the African American male teacher of that class. The student body at Divine is composed of all male African Americans; it is a public charter school, located in the “inner city,” in a low socio-economic neighborhood. The three male participants are students of the same “successful” African American male teacher—Mr. Ira—who was selected by the process of “community nomination” (Foster, 1997), whereby members of a community of interest suggest individuals who they believe will be the best subjects for research. Mr. Ira had been teaching for over twenty-two years, working at Divine since it opened. He was a tenth-grade geometry teacher, 54-year-old father of three, originally from the northeastern United States, and identified as an African American man.

Data collection included semester-long, daily participant observations (and accompanying field notes) and three semi-structured interviews with each participant (all observations and interviews were conducted by the first author). The method of participant observation suggests researchers should take part in activities, rituals, interactions, and events of the participants as a means of learning about their lives and culture (DeWalt & DeWalt, 2011), while field notes assists in "constructing the case." The theoretical underpinnings of care theory, critical race theory, and culturally relevant pedagogy (described earlier) provided the lens to ground the logic of the observations. The first author spent almost every day for a school semester (16 weeks) at the research location for participant observations. Participant observation "puts you where the action is and lets you collect data…any kind of data that you want" (Bernard, 2006, p. 343). Field notes were used to record data from the observations. Field notes were crucial to recording the observations, laying a foundation for the contexts for the participant interviews; given that participant observation "is rarely, if ever, the only technique used by a researcher conducting ethnographic research" (DeWalt & DeWalt, 2011, p. 3).

Findings

During data analysis, six overarching themes were identified that the participants used to describe teacher care: (a) motivation, (b) culture, (c) confidence, (d) discipline, (e) concern for futures, and (f) environment (see Table 1). These themes were interpreted from the participants' significant statements, and of these six themes motivation and classroom environment were noted as the most important to the student participants. These African American boys wanted to feel (and be) capable and comfortable in a mathematics classroom. Fortunately for them, they had an African American male teacher, Mr. Ira, who also found these aspects of teacher care important and who would stop at nothing to encourage and create an environment designed for their success in mathematics.

Table 1: Overarching Codes/Themes and Sub Themes – Descriptions of Teacher Care

Motivation	Culture	Confidence	Discipline	Concern for Futures	Environment
Inspirational	Understands struggle	Believes in students	Rules	Will not allow students to cheat themselves	No judging
Leads by example	Fights stereotypes	Gives students hope	Expectations	Future job concern	Safe
Encouragement	Defends students	Teaches self-efficacy	Second chances	Treats students like his children	Comfortable
Makes students feel important	Diversifies lessons		Empathy	Cares about students' lives outside of school	Mistakes okay
Praises students			Patience		Family environment
			Firm but fair		Offset institutional racism

When it comes to the influence an African American male teacher had on the participants' perceptions of teacher care, there was some overlap with how they described teacher care. An African American male teacher simply softened the environment, so to speak, and made the student participants feel more comfortable. Clearly, the students perceived connections to the teacher participant, based on his appearance, and assumptions they made about him. Pursuing this further, and based on the participants' descriptions, we are all but certain that these assumptions were also rooted in their historical interactions with teachers of other races. These interactions helped garner the assumptions and shape the eventual expectations the students had of the teacher participant in the beginning. Mr. Ira called for “empathy and patience as teachers weather the storm, because they will be tested.” But when the storm is over, hindsight reveals that this is a nonissue. Because as intelligent as they are, the student participants, and Black boys in general, eventually crossed reference their pre-expectations with how they define a caring teacher.

References

Agne, K. (1999). Caring: The expert teacher’s edge. In R. Lipka & T. Brinthaupt (Eds.), *The role of self in teacher development* (pp. 165–188). Albany, NY: State University of New York.

Bartell, T. G. (2011). Caring, race, culture, and power: A research synthesis toward supporting mathematics teachers in caring with awareness. *The Journal of Urban Mathematics Education, 4*(1), 50–74.

Bernard, H. R. (2006). *Research methods in anthropology*. Lanham, MD: Altamira Press.

DeCuir, J. T., & Dixson, A. D. (2004). So when it comes out they aren’t that surprised that it is there: Using critical race theory as a tool of analysis of race and racism in education. *Educational Researcher, 33*(5), 26–30.

DeWalt, K., & DeWalt, B. (2011). *Participant observation: A guide for fieldworkers* (2nd ed.). Lanham, MD: Rowman & Littlefield.

Foster, M. (1997). *Black teachers on teaching*. New York, NY: The New Press.

Gilligan, C. (1982). *In a different voice: Psychological theory and women's development*. Cambridge, MA: Harvard University Press.

Irvine, J. (2001). Caring, competent teachers in complex classrooms. Charles W. Hunt Memorial Lecture, AACTE 53rd annual meeting in Dallas Texas. Washington D.C: AACTE Publications.

Kohlberg, L. (1984). *The psychology of moral development: Essays on moral development* (Vol. 2). San Francisco, CA: Harper & Row.

Ladson-Billings, G. (1992). Culturally relevant teaching: The key to making multicultural education work. In C. A. Grant (Ed), *Research and multicultural education* (pp. 106–121). Washington, DC: Falmer Press.

Madison, D. S. (2005). *Critical ethnography: method, ethics, and performance*. Thousand Oaks, CA: Sage.

Noddings, N. (1984). *Caring, a feminine approach to ethics & moral education*. Berkeley, CA: University of California Press.

Noddings, N. (1992). *The challenge to care in schools: An alternative approach to education*. New York, NY: Teachers College Press.

Noddings, N. (2002). *Educating moral people: A caring alternative to character education*. New York, NY: Teachers College Press.

Noddings, N. (2006). *Critical lessons: What our schools should teach*. Cambridge, United Kingdom: Cambridge University Press.

Oakes, J., & Lipton, M. (1999). *Teaching to change the world.* Boston, MA: McGraw-Hill.

Roberts, M. (2009). African American, European American, or does it make a difference? Teacher perceptions of care for African American students (Unpublished doctoral dissertation). Emory University, Atlanta, GA.

Roberts, M. (2010). Toward a theory of culturally relevant critical teacher care: African American teachers’ definitions and perceptions of care for African American students. *Journal of Moral Education, 39*(4), 449–467.

Siddle-Walker, V. (1993). Interpersonal caring in the “good” segregated schooling of African American children: Evidence from the case of Caswell County Training School. *The Urban Review, 21*(1), 63–77.

Steele, C. (1992). Race and the schooling of Black Americans. *Atlantic Monthly, 269*(4), 68–78.

Stinson, D. W. (2006). African American male adolescents, schooling (and mathematics): Deficiency, rejection, and achievement. *Review of Educational Research, 76*(4), 477–507.

Stinson, D. W. (2009). The proliferation of theoretical paradigms quandary: How one novice researcher used eclecticism as a solution. *The Qualitative Report, 14*(3), 498–523.

Tate, W. F. (1997). Critical race theory and education: History, theory, and implications. In M. Apple (Ed.), Review of research in education (Vol. 22, pp. 195–247). Washington, DC: American Educational Research Association.

Thompson, A. (1998). Not the color purple: Black feminist lessons for educational caring. *Harvard Educational Review, 68*(4), 522–555.

MULTILINGUAL PERSPECTIVES AS NEW OPPORTUNITIES FOR SEEING AND LEARNING MATHEMATICS

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This research intends to offer multilingual perspectives of the languages in mathematics classrooms and how they can be conceptualized. Through our analysis of the multilingual perspectives of translanguaging and the teachers, children, and researchers in the study we present ways to identify and understand different language practices that emerge as young multilingual children experience and make sense of mathematics.

Keywords: Equity and Diversity, Early Childhood Education, Culturally Relevant Pedagogy

The use of more than one language in mathematics classrooms has been often conceptualized as code-switching or language-switching (e.g., Barwell et al., 2016). However, neither language-switching nor code-switching represents a multilingual perspective of the world, but instead a perspective in which being monolingual is still considered the norm (Garcia & Wei, 2014; Celic & Seltzer, 2011). Although hard to know with precision, it has long been estimated that more than half the world knows more than one language (Stavans & Hoffman, 2015). These perspectives point to the increase of multiple languages in mathematics classrooms across the globe and make it necessary to understand the language practices of multilingual people[1] from a standpoint in which being monolingual is no longer considered the norm.

This paper aims to offer possibilities for the way we identify the languages used by children and educators during mathematics learning from the multilingual perspectives of the authors, children, and translanguaging, drawing on data collected for two school years in two kindergarten classrooms. For this purpose, we understand languages as any medium through which we communicate or make sense of the world. From this perspective, we consider the following questions: (1) What languages are currently being used in mathematics classrooms? (2) How can we conceptualize the languages in mathematics classrooms from multilingual perspectives?

Perspectives and Theoretical Framework

This paper takes an interpretivist approach, using hermeneutics as a way to make sense of the multilingual perspectives of the authors, the children, and translanguaging. A hermeneutic perspective allows us to look closely at the narratives collected as data for this study and disrupts them through purposeful interpretation; it allows us to question taken for granted notions about knowledge and language in the context of mathematics learning. Moules et al. (2015) explain that “hermeneutics is organized around the disruption of the clear narrative, always questioning those things that are taken for granted.” (p. 4). Hermeneutics also allows us to understand our role as researcher in the process of interpretation and to think about the contextual factors that shape understanding as well as what can be understood.

We draw on translanguaging to re-conceptualize the learning and language practices of children taking place in mathematics classrooms because it offers a view of language that is inherently multilingual. The term is used to describe the way multilingual people use language; it refers to “the ability of multilingual speakers to shuttle between languages, treating the diverse languages that form their repertoire as an integrated system” (Canagarajah, 2011, p. 401).

Translanguaging offers a way to conceptualize the language practices of multilingual people as something different than code-switching or language-switching. As explained by Celic & Seltzer (2011) "Notice that translanguaging is not simply going from one language code to another. The notion of code-switching assumes that the two languages of bilinguals are two separate monolingual codes that could be used without reference to each other. Instead, translanguaging posits that bilinguals have one linguistic repertoire from which they select features strategically to communicate effectively. That is, translanguaging takes as a starting point the language practices of bilingual people as the norm, and not the language of monolinguals, as described by traditional usage books and grammars" (p. 1).

We find it particularly helpful to think about the way translanguaging incorporates the idea of languaging, and how this allows us to conceptualize language in a dynamic way. "Language is not a simple system of structures that is independent of human actions with others, of our being with others. The term *languaging* is needed to refer to the simultaneous process of continuous becoming of ourselves and of our language practices, as we interact and make meaning of the world. […] Languaging both shapes and is shaped by context." (Garcia & Wei, 2014, p. 8.) Adopting translanguaging as a framework for understanding the language practices of multilingual people allows us to unveil the ways in which language practices are complex and ingrained in contexts that relate to identity and nationality and carry power dynamics.

Modes of Inquiry

In this paper, we draw on multilingual perspectives as the sources of our analysis and conceptualization of language through the authors, the children, and translanguaging. We also draw on the data collected during mathematics instruction in two kindergarten classrooms in two public schools in Georgia as a part of a larger study to understand the connections between mathematics and language. The focus of the study during the first school year was the bilingual students in a public school in a kindergarten classroom with a large Latino population. The focus of the study in the second school year was the children in an immersion bilingual program (Spanish/English) in a different public school. The children in the bilingual program were taught mathematics in Spanish and were native speakers of Spanish and/or English.

Cristina collected data through the role of a participant observer and volunteer teacher using journal entries to record significant events while also collecting student work. During the second year of the study we introduced audio recordings for data collection as well as periodic field notes taken by Martha through the role of reactive observer. Data was collected and analyzed through an interpretative hermeneutic approach because it provided us with ways to look deeply and understand differently, disrupting current narratives about language use in mathematics classrooms. We conducted in-depth analysis of events rendered into text, interpreting them from multilingual perspectives as a way to bring to light the dynamic, contextualized, and interactive nature of language practices in the lived experiences of multilingual children in the mathematics classroom.

Results and Discussion

In this section, we consider a lesson on word problems implemented by Cristina in the second year of the study as it allows us to see the way languages are being used. During this particular lesson, the class is being introduced to word problems as well as how to represent their mathematical processes and answers. For this purpose, in her role as a volunteer teacher, Cristina worked with the children to create a word problem and how to represent their thinking using a picture book as a common and meaningful context. The complexity of language practices occurring during this lesson is particularly visible in audio-recordings, for instance in the following transcription of the lesson:

Cristina:	Ramón hizo cinco dibujos. Su hermana Marisol encontró tres dibujos ¿Cuantos dibujos hay? [Several students raise their hand] - Kira ¿Cuál es el problema? No me digas la respuesta ¿Cuál es el problema? [*Ramon made five drawings. His sister Marisol found three drawings. How many drawings are there? [Several students raise their hand] -Kira, what is the problem? Don't tell me the answer. What is the problem?*]
Kira:	Um- uh –
Cristina:	¿Qué queremos saber? [*What do we want to know?*]
Kira:	The brother did five drawings. The little sister has three more. How many does both of them have together?

Perhaps the first idea that comes forward about language in this brief interaction is the fact the fact that mathematics is occurring multilingually, that Cristina and Kira are using Spanish and English simultaneously as a way to communicate and learn mathematics. Translanguaging allows us to see this multilingual interaction as a fully complex communicative practice. In other words, instead of seeing the interaction from a perspective in which English and Spanish are working separately to support mathematics learning, translanguaging allows to see how both languages work together to support Kira's and other students' mathematical learning. Kira's restatement of the word problem presented in Spanish not only introduces English as a strategy to support understanding but also incorporates several mathematical clues about what we are trying to know. In her answer, Kira incorporates word clues (i.e., more, both of them, and together) calling attention to the mathematical question and the way the numbers relate to each other. Kira's answer builds understanding from what is being said beyond translation, and it reflects the syntaxes of Cristina's sentences in Spanish (Ramón hizo 5 dibujos, the brother did 5 drawings); this indicates that Kyra is using both languages to understand the mathematics question and the communicative demands within the lesson.

Through the notion of languaging, as explained in the theoretical section, translanguaging also allows us to see how the contextual nature of language plays a role in our ability to communicate and make meaning of an interaction. This is noticeable in the transcription when Cristina restates her question in a different way to elicit Kira's response. In the interaction it is possible to see that the first question Cristina asks does not makes sense to Kira, as the idea of a problem in the way they are often talked about in mathematics classrooms is unfamiliar for the student. This is consistent with the findings of other researchers who have called attention to the issue of naming word problems, problems, when the situation does not appear problematic to students. This also brings attention to the way Cristina was able to identify what Kira needed to understand and to be successful at answering the question. Although Kira is a native English speaker, Cristina understood that the breakdown in communication was because of the meanings that are attributed to the word "problema/problem" in mathematics classrooms, and not because she did not understand Spanish.

The interactions in the transcription also allow us to open questions about the way language can be used symbolically and how this plays a role in the mathematics classroom. The quantities in the word problem (3/5) are originally introduced through the way they are represented by sounds in Spanish (tres/cinco). Kira then connects them to the way they are represented by sounds in English (three/five). Later in the lesson the students worked independently to connect those sounds to a numeral and a model of the quantity, as can be seen in the following examples:

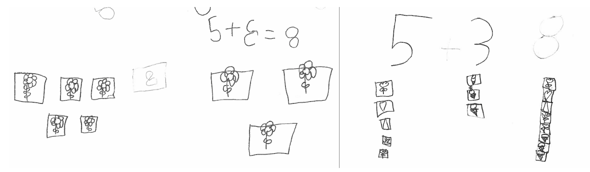

Figure 1. Connections between sounds to a numeral and a model of the quantity.

This way of using languages, to provide multiple complex representations of the same concept through different modalities, allow us to conceptualize the languages being used but also allow us to see how languages catalyze learning and make it visible. In this particular instance it is a way for children to build their number sense, making meaning of the relations that exist between numbers to successfully complete and represent ideas about addition.

This report was designed to highlight the many languages that are used in mathematics classrooms and consider how we might conceptualize them from multilingual perspectives. By offering a brief analysis of a small set of data we show some of the possibilities and richness this kind of analysis can have, particularly for multilingual people. As a final essential remark, we would like to point out that adopting translanguaging as a framework for understanding the language practices of multilingual people in mathematics classrooms is an issue of equity, particularly in a world that is increasingly multilingual. Translanguaging provides us with a way to understand the inherent power dynamics of languages and a way to invert them; this is visible even in the way it takes multilingualism as the norm instead of monolingualism. Setati (2008) makes a powerful argument for the need to focus on the political role of language within multilingual mathematics education research. She calls us to go beyond arguing for the incorporation of students' home language or thinking of language as tool for thinking and communication. In this instance, taking a stance to see language in mathematics instruction through multilingual perspectives allowed us to purposefully interpret data, disrupting taken for granted narratives and opening new and more equitable possibilities to see and understand the mathematics learning of multilingual children.

References

Barwell, B., Clarkson, P., Halai, A., Kazima, M., Moschkovich, J., Planas, N., Setati-Phakeng, M., Valero, P., & Ubiliús, M. (2016). Introduction: An ICMI study on language diversity in education, pp. 1-22. In Barwell, et al. (Eds.), *Mathematics Education and Language Diversity*. The 21st ICMI Study [electronic resource]. Cham, Switzerland: Springer International.

Canagarajah, S. (2011). Codemeshing in academic writing: Identifying teachable strategies of translanguaging. *Modern Language Journal*, 95(3), 401-417.

Celic, C. & Seltzer, K. (2011). *Translanguaging: A CUNY-NYSIEB guide for educators*. New York,NY: CUNY-NYSIEB, CUNY Graduate Center.

García, O., & Wei, L. (2014). T*ranslanguaging: Language, bilingualism and education.* Basingstoke, UK; New York, NY: Palgrave Macmillan.

Moules, N. J., McCaffrey, G., Field, J. C., & Laing, C. M. (2015). *Conducting hermeneutic research: From philosophy to practice*. New York, NY: Peter Lang.

Setati, M. (2008). Access to mathematics versus access to thelLanguage of power: The struggle in multilingual mathematics classrooms. *South African Journal of Education*, 28(1), 103-116.

INVESTIGATING THE RELATIONSHIP BETWEEN THE ENACTED CURRICULUM AND TEACHER CONCERNS IN HIGH SCHOOL MATHEMATICS

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The purpose of this study is to investigate the relationships among teacher concerns regarding the CCSSM, teachers' intended curriculum and teachers' enacted curriculum. Three high school teachers were intentionally selected as a focus group, based on their different levels of concern. Each teacher was observed during three different lessons on linear/non-linear functions. Pre- and post-observation interviews were conducted both before and after each lesson was taught. Findings suggest a complicated relationship exists between teacher concerns and their intended and enacted curriculum. Regarding teacher concerns, different factors seem to take account for the complicated relationship between teacher concerns and their enacted curricula.

Keywords: Enacted Curriculum, Teacher Concern, High School Mathematics

Introduction

The Common Core State Standards for Mathematics (CCSSM) highlight the importance of students' conceptual understanding, mathematical reasoning, and problem solving in order to prepare students for college and careers. However, the success of this reform effort largely depends on how teachers actually design and implement instruction based on the goals of the new standards (Son & Kim, 2015, 2016). Depending on the concerns teachers have, based on the new standards and curriculum, teachers may or may not consider themselves qualified to implement the desired innovations (Christou, Eliophotou-Menon & Philippou, 2004).While there exists an increasing amount of research on reform efforts, many questions still remain regarding the implementation of the CCSSM and teachers' concerns. The purpose of this study is to investigate the concerns teachers have regarding the implementation of the CCSSM and how those concerns relate to the enactment of their curricula. This study also explores how teachers address the mathematical shift of rigor defined in the CCSSM in connection with the tasks they use and types of questions they pose to their students. The research questions that guide this study include: (1) In what ways do Algebra 1 teachers implement the CCSSM to address the mathematical shift of rigor, as defined by the CCSSM, in their Algebra 1 classes? And (2) In what ways do Algebra 1 teacher concerns connect to their implementations of the CCSSM? Three high school teachers were intentionally selected as a focus group, based on their different levels of concern. In answering the two research questions, this study explored whether the three teachers' intended curricula aligned with their enacted curricula in their teaching of linear and non-linear functions. It was also determined if the tasks provided to students and the teacher questioning during instruction addressed the major shift of rigor emphasized in the CCSM.

Theoretical Perspectives

Remillard (2005) describes multiple meanings of curriculum use: curriculum use as following or subverting the text, curriculum use as drawing on the text, curriculum use as interpretation of text and curriculum use as participation with the text. Building on Remillard's second perspective, this study regards teachers' curriculum use as drawing on the text. With this view, the emphasis is placed on the teacher, and views the text as "one of the many resources in constructing the enacted curriculum" (Remillard, 2005, p. 219). We also build on three different frameworks to explore teacher concerns, mathematical tasks and teacher questions.

In exploring teacher concerns on the CCSSM, six stages of concerns identified based on the Concerns-Based Adoption Model (CBAM): A Model for Change in Individuals (Hall & Hord, 2001) are used. In exploring learning opportunities presented to students, cognitive demand of mathematical tasks was explored based on the framework by Stein, Grover, and Henningsen (1996). The four levels are used to determine the rigor of tasks teachers used in their classrooms, which refer to the thinking processes involved in solving the tasks. The framework involves two levels of low cognitive demand tasks—High cognitive demand (i.e., Procedures with Connections; Doing Mathematics) and low cognitive demand (i.e., Memorization; Procedures without Connections). In order to explore teacher questions, the framework adapted from Boaler and Brodie (2004) was used.

Methods

Table 1shows an overview of the study design. Qualitative research methods (e.g., classroom observations and teacher interviews) were utilized for the study. Three teachers were intentionally selected, based on their concerns, for the case studies in order to analyze teachers' implementation strategies of the CCSSM. Ms. Brimm, Mrs. Doherty, and Mrs. Schilling (all psuedonyms) work in the same school district. Classroom observations and pre and post-lesson interviews with these teachers were used for this study. Prior to analyzing lesson tasks, a problem analysis was conducted on the *Pearson Algebra I Common Core* textbook. Since the observed lessons for this study solely focused on linear and non-linear functions, only the corresponding chapter (chapter 4) of the textbook was analyzed.

Table 1: Overview of Study Design

	Qualitative Research Study
Participants	• 3 teachers from a rural school district in Western New York • Each teacher exhibits a different stage of concern
Data Collection	• 3 video-recorded classroom observations for each teacher • One general interview prior to any lesson enactment • Pre- and post-lesson interviews before and after each of the lessons
Data Analysis	• Relative frequencies completed for task type (application, procedural, conceptual) for (a) each lesson and (b) total across the 3 lessons • Significant patterns in teacher questioning • Differences in intended vs. enacted curriculum

Three sets of data were collected from the three teachers: teachers' planned lesson, three video-recorded observations and field notes and structured interviews. Teachers' guided notes and their taught lessons were analyzed in light of two aspects – (1) mathematical tasks and types used and (2) types and frequencies of teacher questioning. The interview data were analyzed to determine: (a) if the teachers intended curriculum differed from their enacted curriculum, (b) if they were aware of any deviations that occurred between the intended and enacted and (c) why teachers made deviations.

Summary of the Findings

All three teachers in this study used the same resources, had the same professional development opportunities and designed the curriculum collaboratively. However, the enacted curriculum for these teachers was differed based on the level of concern each teacher had.

Alignment Between Rigor of Teachers' Enacted Curriculum and the CCSSM

First, we found that the *Pearson Algebra 1 Common Core* (Charles et al., 2012) the three teachers used provide cognitively low-demand tasks and fail to address the mathematical shift of rigor highlighted in the CCSSM. All sections except section 4.1 had a heavy emphasis on procedural tasks including the sections observed in this study. This analysis suggests that if teachers closely follow the *Pearson Algebra 1 Common Core* textbook as a main resource, their students might not be exposed to a high level of cognitive demand.

Second, we found that the vast majority of tasks and teacher questions for all three teachers were low cognitive demand. Teachers at all stages of concern are not providing cognitively demanding lessons nor are they addressing the mathematical shift of rigor. Ms. Brimm stated in her initial interview that she went on the recommendation of the high school teachers to use the *Pearson Algebra 1 Common Core* textbook. Mrs. Doherty also noted that this textbook was most closely aligned with the CCSSM Algebra standards. Mrs. Schilling's comments were similar. She stated that the textbook was the best one written to address the Common Core Algebra standards. We observed that these three teachers rarely altered textbook problems. In particular, the vast majority of questions for all three teachers were low cognitive demand. Teachers only significantly increased the cognitive demand of the questions when they had students working collaboratively in groups in order to communicate their ideas to others and justify their reasoning.

The Relationship of Teacher Concerns and the CCSSM Enacted Curriculum

First, Ms. Brimm was determined to be at the self-concern level. She felt she was not knowledgeable about the mathematics content and methods she had to teach. She thus followed the textbook very closely in selecting tasks. She stated that at times, she had to learn the content before she could teach it. In addition, she rarely altered the tasks during the enactment of the lessons. Ms. Brimm's lack of content knowledge seems to lead her to have self concerns. This resulted in her following the textbook closely. Her self concerns also led her to focus mainly on content coverage and did not allow her to modify the intended curriculum.

Second, the teacher with management concerns tended to express her students' low abilities in doing mathematics. Ms. Schilling was determined to be at the management concern level. Similar to Ms. Brimm, in the selection and design of her lesson tasks, Mrs. Schilling stated that all her tasks came from the Pearson Algebra 1 Common Core textbook. However, in the enactment of the tasks, Mrs. Schilling did alter her initial plans. Mrs. Schilling's concern over the material she had to cover led her to have management concerns. She was comfortable with the content and methodology and, therefore, was able to slightly modify her intended curriculum. However, she was unable to provide any cognitively demanding lessons since her focus was on content coverage and the ability level of her students.

Third, Mrs. Doherty was found to have impact concerns. Mrs. Doherty's high interest in student understanding, as well as her desire to collaborate frequently with other teachers, led her to have impact concerns. With this higher level of concern, Mrs. Doherty was able to focus her lessons on student understanding, and therefore, was able to modify her intended curriculum. She altered the curriculum based on the needs of her students. However, her tasks and questions remained at a low cognitive level for two of the three lessons.

Discussion and Implications

This study contributes to the current literature because it has shown the relationships between teacher concerns on the CCSSM and their intended and enacted curriculum. Teacher concerns seem to be hindering their ability to provide cognitively demanding lessons, thereby showing limited success for the CCSSM. Therefore, implications of this research exist for textbook publishers, curriculum developers, professional development providers, teacher educators, teachers, administrators and future researchers. For example, textbook publishers, curriculum developers and teachers need to be aware of cognitive demand of tasks used by teachers. Professional development providers and teacher educators can help teachers focus their questioning, and address the concerns teachers have.

Reference

Boaler, J., & Brodie, K. (2004). The importance, nature, and impact of teacher questions. In *Proceedings of the twenty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 774-790) Toronto, Ontario, Canada.

Christou, C., Eliophotou-Menon, M., & Philippou, G. (2004). Teachers' concerns regarding the adoption of a new mathematics curriculum: An application of CBAM. *Educational Studies in Mathematics*, *57*(2), 157-176.

Common Core State Standards Initiative (2016). Retrieved from http://www.corestandards.org/

Hall, G. E., & Hord, S. M. (2001). Implementing Change. Patterns, Principles, and Potholes. Boston: Allyn and Bacon.

Herbel-Eisenmann, B., Wagner, D. (2007). A framework for uncovering the way a textbook may position the mathematics learner. *For the Learning of Mathematics, 27*(2), 8 – 14.

Hord, S. M., Rutherford, W. L., Huling-Austin, L., & Hall, G. E. (1987). *Taking Charge of Change, Association for Supervision and Curriculum Development.* Austin, TX: Southwest Educational Development Laboratory.

Lloyd, G. M. (1999). Two teachers' conceptions of a reform-oriented curriculum: Implications for mathematics teacher development. *Journal of Mathematics Teacher Education*, *2*(3), 227-252.

Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research*, *75*(2), 211-246.

Son, J. W., & Kim, O. K. (2015). Teachers' selection and enactment of mathematical problems from textbooks. *Mathematics Education Research Journal*, *27*(4), 491-518.

Son, J., & Kim, O. (2016). Curriculum enactment patterns and associated factors from teachers' perspectives, *Mathematics Education Research Journal*, *28*(4), 585-614.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, *33*(2), 455-488.

MATHEMATICAL MODELING COMPETENCIES ESSENTIAL FOR ELEMENTARY TEACHERS: THE CASE OF TEACHERS IN TRANSITION

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This report documents three teachers journeys as they engaged in a Teacher Research Study Group while implementing mathematical modeling in the elementary grades. In detailing three teachers' perspectives, we provide a context and describe each of the teachers' mathematical modeling journeys and their process of adopting the innovation of MM in their classrooms. Four main mathematical modeling competencies emerged as being central to the success of enacting mathematical modeling in the elementary classroom: a) finding relevance, b) identifying MM potential; c)posing MM questions; and d) capitalizing on a useable model.

Keywords: Teacher Education-Inservice/Professional Development, Modeling, Instructional Activities and Practices

Research Literature on Mathematical Modeling

As a reform-oriented initiative, we consider MM to be an **ambitious teaching** practice. Ambitious teaching is teaching that aspires to teach all students academic content and have them apply their knowledge to solve authentic academic problems. MM does exactly this as it moves students through a process of mathematizing a real-world problem and using the mathematics to solve it. MM, as an ambitious teaching practice, builds on student ideas and gives students a role of sense-makers. Our increased dependence on technology together with the complexity and interconnectedness of our society has led to significant changes in the types of mathematical thinking required in today's world (English, 2013). Educational leaders are emphasizing the importance of developing students' abilities to deal with these new demands of our society (e.g. Lesh & Doerr, 2003). These abilities include interdisciplinary problem solving, techno-mathematical literacy, flexibility in applying numerical and algebraic reasoning, thinking critically, and constructing, describing, explaining, manipulating and predicting complex systems (English, 2013). Mathematical modeling (MM) is seen as a powerful tool for advancing students understanding of mathematics and for developing an appreciation of mathematics as a tool for analyzing critical issues in the real-world, that is, the world outside of the mathematics classroom (Greer & Mukhopadhyay, 2012). Traditionally, MM has been implemented primarily in secondary schools, but recent research examines using this approach with elementary students to promote their problem solving and problem-posing abilities (e.g. English, 2010). MM provides the opportunity for students to solve genuine problems and to construct significant mathematical ideas and processes instead of simply executing previously taught procedures and is important in helping students understand the real world (English, 2010).

Method for Our Study

A qualitative case study approach was selected for this study as our intent was to gain insight and understanding of elementary school teachers' adoption of MM . The boundaries of the case were in-service, K-6, mathematics teachers who, following a PD on mathematical modeling, chose to participate in a teacher research study group. This case consisted of three mathematics teachers from our larger research project who continued in our spring teacher research study group. We were interested in gaining insights into how teachers' pedagogical practices and knowledge emerged as they enacted MM in the elementary school and how inquiry into their own practice helped them negotiate the adoption of MM. We were not looking to do a detailed study of the individual teachers as that would entail separate case studies. Instead, our unit of analysis was the similarities and differences across these teachers' experiences in implementing MM. The boundaries of the case were determined by place, time and event.

Participants. Participants were inservice, K-6, mathematics teachers, in a large mid-Atlantic school district, involved in a professional development (PD) initiative focused on mathematical modeling (MM), between August, 2015 and June, 2016. We selected three mathematics teachers from our PD who elected to join our spring teacher research study group (TRSG), demonstrated an understanding of mathematical modeling that was consistent with the project goals, and whom we identified as early adopters of MM (Rogers 2003). Our three participants taught Advanced Academic Program (AAP) students. Anne was an experienced teacher with 28 years of experience and taught sixth grade. 7% of the students in Anne's school participate in a free or reduced lunch program. Lisa, a third grade teacher, had 5 years of teaching experience at the time of our study. Matthew had four years of teaching experience and taught fourth grade and had 26 students in his math class. 4% of the students in Matthew's school participate in a free or reduced lunch program.

Research Questions. In order to understand how teachers enacted mathematical modeling skillfully in the elementary grades, we addressed these research questions: What pedagogical practices and knowledge emerged as teachers enacted MM in the elementary school classroom? How did teachers develop their modeling teaching practices as they continued on their journey as implementers of mathematical modeling in the elementary grades?

Data Sources. Debriefs of individual Lesson Study session and Lesson Study group presentation at a final symposium were captured on video. Sources also included individual teacher reflections, semi-structured researcher memo's for each Lesson Study and symposium presentation, exit passes from the summer institute and the final symposium, and participant Powerpoint presentations from the final symposium. The artifacts collected from the Lesson Study cycle included the planning agendas, actual lesson plans, student work samples, the analysis of student work, and teacher reflections. Each of these factors contributed to compiling a comprehensive picture of teachers' experiences with MM.

Data Analysis. During the first phases of the analysis, we used the observational memos from lessons with analytic researcher memos that allowed us to not only describe our teacher-designers planning and enactment of the mathematical modeling lessons but also summarize our thoughts about potential ways in which certain skills, knowledge and beliefs contributed to the teachers' ability to facilitate learning through mathematical modeling. Interviews, videoclips of classroom episodes and artifacts from lessons including lesson plans, and student artifacts also helped identity the development of the mathematical modeling processes in the elementary classroom and the various models that emerged from the mathematical modeling tasks. We created a chronological timeline for each of the teacher designers with linked data sources allowing us to focus on documenting the generative change that teachers experienced as they continued to implement mathematical modeling during the fall and spring semester of the school year. The story line provided a comprehensive profile for each of our participants for our case study analysis.

For the second phase of our data analysis, we used thematic networks as a tool for our data analysis. We started by identifying specific teacher moves in the transcripts of our interviews, in our researcher memos, and teacher reflections. We used Dedoose Version 7.5.9 to code our data and dissect the text into text segments. These text segment were analyzed qualitatively for emergent themes and patterns. In addition, the text segment were counted and compared within each theme to identify and describe any differences between the three case study participants.

Results

In detailing the teachers' perspectives of the MM lessons and reflections on the MM process using several of the episodes in these cases, we found four main categories of mathematical modeling competencies that emerged as being central to the success of enacting mathematical modeling in the elementary classroom: a) finding relevance, b) identifying MM potential; c)posing MM questions; and d) capitalizing on a useable model.

Finding Relevance. One of the strengths of MM is the authenticity and real-worldness of the problems being solved by students. We noted teachers in the TRSG able to identify engaging contexts, build on student ideas, use authentic problems and make real-life connections for their students. This was one of Anne's strengths. At the beginning of the year, Anne asked her students, "How can we impact our school and community?" With her students, she identified which ideas could be solved with mathematics and used her students' ideas and her knowledge of the curriculum to develop authentic and relevant MM projects for her class."Students are empowered by knowing they can really do something. It is unlike contrived math that comes from a book. I think it is important that they know it is something they can really do."

Identifying Mathematical Modeling Potential. We found that the MM tasks designed and developed by our participants centered around four types of models: descriptive, optimization, predictive and ranking models. This includes the notion of using mathematics to make decisions. Teachers working within their curriculums, Matthew began with his curriculum and worked backwards, for less experienced teachers, this approach supported them in adopting MM, Anne started with a problem, context and then looked at the math. For the school store task, as students compiled their surveys she asked them to predict the first week's sales in their store. This led students to compute unit rates, calculate estimated profit, differentiate between wholesale and retail costs, draw graphs to represent their predictions (predictive); with the dead-stock of pencils, she saw potential to apply linear functions by asking students to find the best sale option to get rid of stock and promote bulk sale. (predictive); with the wait-time dilemma - students computed rate and flow issues and collected data to reduce wait-time in line (optimization).In Lisa's Planning a day trip task, students had to optimize the time they could spend at sites, maximize fun but minimize cost. (optimization)

Posing Mathematical Modeling Questions. An important element of successful MM tasks is asking the right types of questions. Some were planned and spontaneous questions, driving questions while others emerged as MM questions as teachers became more comfortable asking questions. Questions varied from the main driving question of the MM task that kept students focused on the mathematics of the problem to day-to-day questions that preseed students to explain and justify their assumptions, choices and reasoning. Anne focused on questioning pattern for working with large amounts of data. Students overwhelmed by large amounts of data, Anne's focused questions (focus versus funneling, NCTM, 2014) helped them arrive at use of representative samples to process 600+ surveys. Her questioning brought out how this approach could apply to other situations.

Capitalizing on the Usefulness Principle. We found that in elementary school classrooms, not all MM tasks result in a generalizable model. Some of the tasks resulted in one model to solve a particular real-world problem, sometimes students were able to apply the principles to an additional problem and we found that less frequently, students were able to develop a generalizable model. We posit that this is a natural part of mathematical modeling in the elementary grades and that these early experiences with MM serve to develop student understanding of and comfort with MM that will eventually lead them to develop generalizable models. Thus, for elementary grades capitalizing on the useable model looks different than it would in upper grades.

Conclusion

Supporting teachers adopt an innovation as ambitious teaching practices

As shared in the teacher's profile, these teachers were knowledge-seekers, they came with the predisposition to gain knowledge "learn more about creating research-based lessons for modeling real-world mathematics" and "share this approach with other teachers". Each of our teachers sought out this innovative ambitious practice of teaching mathematical modeling as a process to use with their curriculum standards. They gained knowledge and practice by collaborating with a team of teachers during the professional development summer institute and lesson study. While they immersed as math modelers during the institute and a teacher designer and implementer, they weighed the benefits and challenges and made personal decision whether to continue with the innovative practice. This decision led these three teachers to voluntarily join the Spring Teacher Research Study Group. With additional support from the project team, these teachers gained more knowledge and continued to put this innovation into use, sometimes flailing and sometimes, finding the "sweetspot". In their continued pursuit of finding what worked for their students, they continued to evaluate not only their own teaching decisions but the learning outcome of their students.

References

Dedoose Version 6.2.7. (n.d.). Retrieved from http://www.dedoose.com/

English, L. D. (2013). Complex modelling in the primary and middle school years: An interdisciplinary approach. In G. A. Stillman, G. Kaiser, W. Blum, & J. P. Brown (Eds.), *Teaching Mathematical Modelling: Connecting to Research and Practice* (pp. 491–505). Dordrecht: Springer Netherlands. Retrieved from http://link.springer.com/10.1007/978-94-007-6540-5_42

Greer, B., & Mukhopadhyay, S. (2012). The Hegemony of Mathematics. In O. Skovsmose, O. Skovsmose, & B. Greer (Eds.), *Opening the Cage* (pp. 229–248). SensePublishers. Retrieved from http://link.springer.com/chapter/10.1007/978-94-6091-808-7_12

Lesh, R. A., & Doerr, H. M. (2003). Foundations of a Models and Modeling Perspective on Mathematics Teaching, Learning, and Problem Solving. In R. A. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 3–34). Mahwah, N.J: Lawrence Erlbaum Associates.

A CHARACTERIZATION OF STUDENT MATHEMATICAL THINKING THAT EMERGES DURING WHOLE-CLASS INSTRUCTION: AN EXPLORATORY STUDY

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This exploratory study investigated 164 instances of student mathematical thinking that emerged during whole-class instruction in a high-school geometry course. The MOST Analytic Framework provided a way to categorize these instances according to their building potential—that is, the potential for learning to occur if the student thinking of the instance were made the object of consideration by the class. The variations in the building potential of student thinking revealed in the study highlight the complexity of teaching and the need to support teachers in identifying and appropriately responding to instances with different levels of building potential.

Keywords: Classroom Discourse, Cognition, Instructional Activities and Practices

Reform documents (e.g., NCTM, 2014) advocate for the use of student thinking in instruction, and the benefits of doing so have been highlighted by various studies (e.g., Franke & Kazemi, 2001). Not all student thinking, however, has the same potential to support learning by being incorporated into instruction (Leatham, Peterson, Stockero, & Van Zoest, 2015), particularly if the teacher's goal is to enact the teaching practice of *building—making student thinking the "object of consideration by the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea"* (Van Zoest et al., 2017, p. 36). Better understanding the range of student thinking that is available during instruction could help support the development of practices that provide the most productive response to that thinking. We report here on our characterization of the student mathematical thinking that emerged during whole-class instruction in one teacher's classroom.

Literature Review

We identified research related to three types of characterizations of student thinking: (1) the context in which student thinking emerges during classroom instruction (e.g., Edwards & Marland, 1984); (2) the type of student thinking that occurs in a particular content area (e.g., Carpenter, 1985, for addition); and (3) the potential of the student thinking to further the mathematical understanding of the class if incorporated into instruction (e.g., Stockero & Van Zoest, 2013). The first two groups of studies provided insight into student thinking, but did not provide information on the potential of the student thinking to be used in instruction by a teacher. Drawing on the third group of studies, Leatham et al. (2015) developed a framework for identifying high-potential instances of student thinking that, if made the object of consideration by the class, have the potential to foster learners' understanding of important mathematical ideas.

Such instances of student mathematical thinking are called Mathematically Significant Pedagogical Opportunities to Build on Student Thinking (MOSTs). Van Zoest et al. (2017) investigated the attributes of MOSTs and how those attributes might be used to support teachers' identification and productive use of MOSTs. Their study, however, did not investigate the broader range of student thinking that is available in the classroom. The exploratory study reported in this paper extends the work of Van Zoest et al. (2017) by examining the full range of student mathematical thinking that emerges during whole-class instruction and considering productive teacher responses to instances with different potential for building.

Theoretical Framework

Leatham et al. (2015) proposed six criteria for characterizing and identifying MOSTs (see Figure 1). These criteria create a natural subcategorization of student thinking that might occur in the classroom, a subcategorization that we theorized would differentiate student thinking according to the *building potential* of that thinking—that is, *the potential for learning to occur if the student thinking of the instance were made the object of consideration by the class*. We conceptualize building as a teaching practice that involves making clear the instance of student thinking that the class is to consider, turning that instance over to the class in a way that requires them to make sense of it, orchestrating the conversation that ensues, and ensuring that the mathematical point of the instance is made explicit (Van Zoest, et. al, 2016).

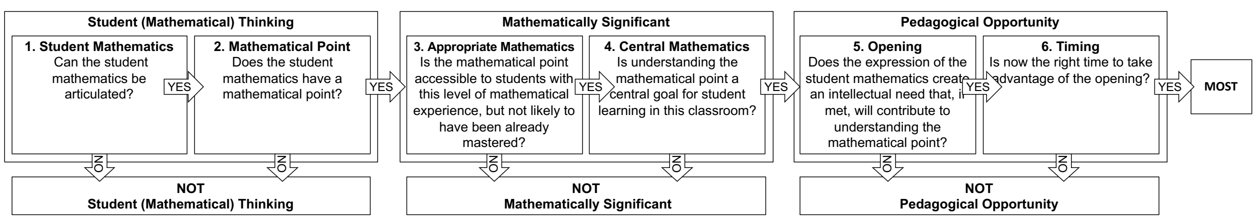

Figure 1. The Six Criteria of the MOST Analytic Framework

Methodology

This study is part of a larger project that included a data set of 11 videotaped mathematics lessons from 6-12th grade classrooms that reflected the diversity of teachers, students, mathematics, and curricula present in US schools (Van Zoest et al., 2017). Based on the analysis reported in Van Zoest et al. (2017), we identified a lesson from this data set whose rates of instances of student thinking per minute and of MOSTs per minute during whole-class instruction were at the medians of the larger data set. We analyzed the 45 minutes of whole-class instruction in this 90-minute, 11th grade geometry lesson about finding the surface area and volume of a sphere. Our unit of analysis was an instance of student thinking—an "observable student action or small collection of connected actions" (Leatham et al., 2015, p. 92)—that is potentially mathematical. Employing analytic processes described elsewhere (Van Zoest et al., 2017) resulted in 164 instances of student thinking that were coded at the last criterion they satisfied according to the MOST Analytic Framework (Figure 1). Instances that appeared mathematical, but for which the student mathematics (criterion 1) could not be inferred, were coded as Cannot Infer (CNI). We then theorized about the relationship between the location of these instances within the MOST Criteria and the potential of these instances for productively enacting the teaching practice of building. We did so by considering the potential advantages and disadvantages of enacting the building practice for such instances.

Results & Discussion

About 17% of the 164 instances were MOSTs (met all six criteria in Figure 1), and thus are, by definition, instances that are worth building on. They have *high building potential* because they are particularly opportune instances that have the potential to result in the class developing a better understanding of mathematics that is both appropriate and central to their learning. Thus, not building on a MOST would be a missed opportunity. The 34.7% of instances that met at least Central Mathematics (criterion 4) but fell short of being MOSTs, were considered to have *some building potential*. These instances are mathematically relevant, but lack pedagogical expediency—they either lack an intellectual need for the students to engage with the thinking (criterion 5) or the pedagogical timing is inappropriate to take full advantage of the thinking (criterion 6). Since these instances do involve mathematics that the students could benefit from considering, a teacher *could* use them to move the students' learning forward, but it would not be a missed opportunity were they not made an object of consideration for the class. To illustrate the difference between instances with high and some building potential, we provide an example of students engaged in sense-making for each of these categories (see Figure 2).

The teacher begins talking about great circles and a student asks a question. **Instance of Student Thinking**: Would it be kind of be like the circumference of a circle? **Mathematical Point**: A great circle is a circle of maximum circumference that can be drawn on a sphere; the great circle is the object (circle) and the circumference is the distance around the circle. **Building Potential:** High **(MOST Coding**: MOST)
The teacher asks if the surface area of a sphere is more like the area or circumference of a circle. **Instance of Student Thinking**: Area, because the circumference is just the outside, like a line [outlines a circle in the air]. And then the area's the whole thing. **Mathematical Point**: Circumference is the distance around a circle and area is the space that circle takes up. **Building Potential:** Some (**MOST Coding**: Central Mathematics)

Figure 2. Sense-Making Examples for High Building Potential and Some Building Potential

We considered the 29.9% of instances that satisfied criterion 1, 2, or 3 but fell short of meeting the criterion for Central Mathematics as having *no building potential*. These instances lack mathematical relevance to learning goals for the students in the class, so pursuing them would seem to be a pedagogical misstep. A productive teacher response to such instances might be to allow the conversation to move on. Being aware of instances that have no building potential might be as important as being aware of instances that have high building potential, since this knowledge would enable teachers to spend time where it is warranted.

The 18.9% of instances that were not inferable (CNI) were considered to have *unknown building potential*. Pursuing these instances as the object of whole-class discussion in their current form would also seem to be a pedagogical misstep. With these instances, if students attempt to infer their peers' mathematical ideas, they may arrive at very different interpretations. Participating in a discussion with different perceptions of what it is that they are discussing could lead to miscommunication and inhibit learning (Peterson et al., 2018). Although often these instances do not seem to warrant further consideration because of their limited nature, in some cases the students' idea might very well be relevant and timely to the discussion (possibly even a MOST). In these cases, a productive teacher move would be to seek clarification.

Conclusion

The results of this exploratory study showed that student thinking available in a high school geometry classroom during whole-class instruction varied substantially, both with regard to the MOST Criteria and with regard to its building potential. We identified and discussed four categories of building potential (high, some, none, and unknown), each of which suggests different types of productive responses from a teacher. The variation in building potential of instances of student thinking revealed in this study highlights the complexity of teaching, and the need to support teachers in identifying and appropriately responding to instances with different levels of building potential. Further work is needed to better understand these various response-related practices and how they might be coordinated into productive use of student mathematical thinking. Knowing more about variations in building potential and associated productive responses informs the design of professional development that supports teachers in implementing instruction that productively incorporates student mathematical thinking.

Acknowledgments

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References

Carpenter, T. P. (1985). Learning to add and subtract: An exercise in problem solving. In E. A. Silver (Ed.), Teaching and learning mathematical problem solving: Multiple research perspectives (pp. 17–40). Hillsdale, NJ: Erlbaum.

Edwards, J., & Marland, P. (1984). A comparison of student thinking in a mathematics and a science classroom. *Research in Science Education, 14,* 29-38.

Franke, M. L., & Kazemi, E. (2001). Learning to teach mathematics: Focus on student thinking. *Theory into practice*, *40*, 102-109.

Leatham, K. R., Peterson, B. E., Stockero, S. L, & Van Zoest, L. R. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. *Journal for Research in Mathematics Education*, *46*, 88-124.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: Author.

Peterson, B. E., Leatham, K. R., Merrill, L. M., Van Zoest, L. R., & Stockero, S. L. (2018). *Clarifiable ambiguity in classroom mathematics discourse*. Submitted manuscript.

Stockero, S. L., & Van Zoest, L. R. (2013). Characterizing pivotal teaching moments in beginning mathematics teachers' practice. *Journal of Mathematics Teacher Education, 16*, 125-147.

Van Zoest, L. R., Peterson, B. E., Leatham, K. R., & Stockero, S. L. (2016). Conceptualizing the teaching practice of building on student mathematical thinking. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1281–1288). Tucson, AZ: University of Arizona.

Van Zoest, L. R., Stockero, S. L., Leatham, K. R., Peterson, B. E., Atanga, N. A., & Ochieng, M. A. (2017). Attributes of instances of student mathematical thinking that are worth building on in whole-class discussion. *Mathematical Thinking and Learning*, *19*, 33-54.

TEACHER DECISION MAKING IN INSTRUCTIONAL SITUATIONS IN ALGEBRA AND GEOMETRY: DO TEACHERS FOLLOW INSTRUCTIONAL NORMS?

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Teaching, as any other social activity, is firmly rooted in the cultural context in which it takes place. Stigler and Hiebert (1999) used the notion of cognitive scripts to describe what happens in a lesson, revealing cross-national differences. Following up on this initial study, Givvin and colleagues (2005) showed that lessons across different countries converge towards common "lesson signatures." These signatures capture the (often tacit) cultural expectations of the components of a typical mathematics lesson. Building upon this idea, Jacobs and Morita (2002) have shown that American and Japanese teachers have different views about their ideal mathematics lesson and suggested that these instructional scripts are cultural in nature. Moreover, they also argued that American teachers do not hold one ideal lesson script but rather have access to multiple "culturally sanctioned options for teaching". A question that arises here is whether choice of those different scripts can be explained by distinctions between the subject matter being taught.

To operationalize the question, we conceive of scripts as being assemblages of norms, and we study whether mathematics teachers' tendency to follow a norm plays a role in teachers' instructional decision making, understood as choosing to act according to the norm. In particular, we focus on two instructional situations that are common in secondary mathematics, doing proofs in geometry and solving linear equations in algebra. The main hypothesis undergirding this poster is that these instructional situations are governed by specific norms. By engaging mathematics teachers in scenarios of classroom instruction, we measure their disposition to attend to these specific instructional norms across different scenarios. We ask: (1) Do instructional norms play a role in teacher's decisions? (2) Is the disposition to comply with norms different from individual resources?

Using multimedia scenarios depicting eight different teaching situations, we surveyed a national sample of secondary mathematics teachers. We find that our hypothesized latent constructs explain part of the variation in the kind of decisions that teachers make. In addition, we provide initial evidence in support to our claim that these dispositions to comply with instructional norms are social resources that are separate from the individual resources that have been studied up to this point in teacher decision making.

This work has raised new questions about the relationship between individual and social resources that mathematics teachers seem to rely upon when making instructional decisions. Our models provide initial evidence suggesting that decision making is subject-specific and acculturation into the norms of the teaching community could explain part of the variation we observe in teacher's decisions.

References

Givvin, K. B., Hiebert, J., Jacobs, J. K., Hollingsworth, H., & Gallimore, R. (2005). Are there national patterns of teaching? Evidence from the TIMSS 1999 video study. *Comparative Education Review*, *49*(3), 311–343.

Jacobs, J. K., & Morita, E. (2002). Japanese and american teachers' evaluations of videotaped mathematics lessons. *Journal for Research in Mathematics Education*, *33*(3), 154–175.

Stigler, J. W., & Hiebert, J. (1999). The teaching gap: Best ideas from the world's teachers for improving education in the classroom. New York: The Free Press.

EXAMINING THE SOCIAL ASPECTS OF GREENHOUSE EFFECT THROUGH MATHEMATICAL MODELING

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Keywords: Social justice, Modeling, Greenhouse Effect, Covariational Reasoning

This study focuses on the environmental topic of greenhouse effect for engaging students with issues of social injustice through mathematics. Although mathematics has been previously used by researchers to guide students towards developing an awareness about social justice issues, such as unemployment rate and racial profiling (Gutstein, 2006; Stinson, Bidwell, & Powell, 2012), it has not been used for exploring environmental justice. Environmental change is a serious threat to humanity and unfortunately it bears down disproportionately upon the poor (Agyeman, Bullard, & Evans, 2002). Consequently, any environmental issue can rightfully be considered as a social issue that requires immediate attention.

Our goal is to utilize mathematical thinking to think critically about environmental science, and express and challenge views on this topic. We exploited the power of mathematical modeling (e.g. Lesh & Lehrer, 2003) for helping students interpret and address social issues related to the greenhouse effect and the changing climate. To identify the impact of their daily actions on the greenhouse effect and understand its influence on their lives in future, we engage students in tasks in which they explore quantities that co-vary, such as the amount of carbon-dioxide (CO_2), air temperature, and the sea level rise.

In this poster, we present the three modeling tasks we developed in NetLogo (Wilensky, 1999) and our initial findings from whole class design experiments (Cobb et al., 2003) in middle school classrooms. By engaging with our modeling tasks, students were able to reason about the covariational relationships of the quantities involved, such as recognizing that "as the albedo goes down, temperature goes up" or "when the temperature rises by 0.5 degrees, the sea-level rises by 4 feet." The tasks also helped students distinguish between linear and non-linear relationships of the various quantities. By reasoning through mathematics, students were able to develop an awareness about the significance of the greenhouse effect on their lives. Examples include arguing it is important to take action about the greenhouse effect otherwise "we are gonna live with the damaged planet." In discussing the damages from the rise of sea level, they argued that they are in a worse situation than rich people because even if the rich houses get destroyed rich people "will be able to build it," something that the poor families cannot afford.

References

Agyeman, J., Bullard, R. D., & Evans, B. (2002). Exploring the nexus: Bringing together sustainability, environmental justice and equity. Space and polity, 6(1), 77-90.

Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational researcher, 32*(1), 9-13.

Gutstein, E. (2006). *Reading and writing the world with mathematics: Toward a pedagogy for social justice*: New York: Routledge-Taylor & Francis.

Lesh, R., & Lehrer, R. (2003). Models and modeling perspectives on the development of students and teachers. *Mathematical Thinking and Learning, 5*(2-3), 109-129.

Stinson, D. W., Bidwell, C. R., & Powell, G. C. (2012). Critical pedagogy and teaching mathematics for social justice. *The International Journal of Critical Pedagogy, 4*(1), 76-94.

Wilensky, U. (1999). NetLogo. http://ccl.northwestern.edu/netlogo/, Northwestern University, Evanston, IL.

DIFFERENTIATING MATH GAMES: MEETING THE NEEDS OF ALL STUDENTS

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This poster focuses on differentiating math games as an instructional strategy to meet the varying academic needs of students. It is common to see both researchers (e.g., Siegler & Ramani, 2008) and curricula, such as Everyday Mathematics, advocating for the use of math games with young children in the school setting. Often, curricula teachers' manuals suggest that all students play the same game, without differentiation. Moreover, they do not provide teachers guidance on how to meet the diverse mathematical needs of students while playing math games. Differentiation in the mathematics classroom is not new (e.g., Murata, 2013), but thus far, very few have explored the application of differentiation within math games. To address this need, my study focused on one question: "How can math games be differentiated to meet the diverse mathematical needs of individual students?"

My study is framed broadly by situated learning theories (e.g., Lave & Wenger, 1991), which allows researchers to examine students' mathematical reasoning and thinking through the lens of participation in the social classroom environment (Cobb & Bowers, 1999). In my study, I use the situated learning theory as a lens to examine the participation that occurs while students play differentiated math games in small groups of two to four. The study was situated in a first-grade classroom of a small rural school, where the majority of students were African American. For eleven sessions, I played a variety of math games that were differentiated for each students' academic level; each session lasted thirty minutes. All sessions were video recorded and later coded and analyzed following traditional methods of ethnographic analysis (Erickson, 1986).

In this study, I found that differentiating math games provided students the opportunity to engage in fun activities while solving math problems that were appropriately leveled. For example, in "The Race Car Game," Clark was able to solve double-digit problems, such as 32+16 and 36+45, while Joseph solved single-digit problems such as 5+7 and 10+9. Though their numbers differed, both students were equally engaged in the game. This was evident through their body language, such as squeals of delight and hand clapping, and the comments they made, such as "I can't wait to get to that shortcut," "I like playing," and "this game is fun."

I acknowledge that this type of differentiation would be difficult in a whole class setting. Rather, this type of differentiation would work best in math centers, when teachers have the time to focus on a small group of students and provide individualized support. By sharing this study, I hope to support teacher educators' work with pre-service and practicing teachers with regards to using games productively in the early childhood mathematics classroom.

References

Cobb, P., & Bowers, J. (1999). Cognitive and situated learning perspectives in theory and practice. *Educational Researcher*, *28*(2), 4-15.

Erickson, F. (1986). *Qualitative methods in research on teaching.* (Occasional paper no. 81).

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation.* New York, NY: Cambridge University Press.

Murata, A. (2013). Diversity and high academic expectations without tracking: Inclusively responsive instruction. *Journal of The Learning Sciences*, *22*(2), 312-335.

Siegler, R. S., & Ramani, G. B. (2008). Playing linear numerical board games promotes low-income children's numerical development. *Developmental Science*, *11*(5), 655-661.

WRITING IN MATHEMATICS CLASS: ELEMENTARY MATHEMATICAL WRITING TASK FORCE RECOMMENDATIONS

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Keywords: Elementary School Education, Classroom Discourse

The National Council of Teachers of Mathematics noted that "writing is another important component of the discourse" (1991, p. 34). Writing can enhance students' learning and foster a deeper understanding (The National Commission on Writing, 2003). In mathematics, writing can offer students an opportunity to personally make sense of ideas and use symbols to convey their mathematical thinking. Yet, descriptions are limited in how elementary teachers should utilize writing to "lay the foundation for future reasoning and proof writing" (Casa et al., 2016, p. 2).

To clarify the purposes for mathematical writing, the Elementary Mathematical Writing Task Force was convened in October 2015. The task force had three primary goals: (1) consider various purposes for writing in mathematics class, (2) reach consensus about the types and purposes for mathematical writing that leverage students' mathematics learning, and (3) account for perspectives from multiple stakeholders, evidence of students' potential for writing productively in mathematics, and multiple sets of curriculum standards. To address these goals the task force was comprised of those with expertise in mathematics, writing, scientific writing, curriculum design, assessment development. Members were also knowledgeable about English language learners, students with learning difficulties, and students identified as gifted.

To begin the meeting, members introduced themselves by sharing an artifact related to mathematical writing. These artifacts helped the task force begin to generate initial ideas. Small groups then analyzed writing-related items including standards from mathematics and writing education and student samples. As the groups worked, they developed descriptions for the types of and purposes for mathematical writing. The descriptions were then presented to the entire task force to discuss and refine; ultimately leading towards consensus about the types of writing.

The task force identified two overarching goals of mathematical writing: to reason and to communicate mathematically. The task force recommended four types with associated purposes for mathematical writing: exploratory writing - to personally make sense of a problem, situation, or one's own ideas; informative/explanatory writing - to describe and to explain; argumentative writing - to construct and to critique an argument; and mathematically creative writing - to document original ideas, problems, and/or solutions, to convey fluency and flexibility in thinking and to elaborate on ideas. The task force also identified considerations for elementary mathematical writing including: writing develops across multiple continua, the audience influences students' mathematical writing, and mathematical writing may take multiple forms.

References

Casa, T. M., Firmender, J. M., Cahill, J., Cardetti, F., Choppin, J. M., Cohen, J., …Zawodniak, R. (2016). *Types of and purposes for elementary mathematical writing: Task force recommendations*. Retrieved from http://mathwriting.education.uconn.edu.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

The National Commission on Writing for America's Families, Schools, and Colleges. (2003). *The neglected "R": The need for a writing revolution*. Retrieved from http://www.collegeboard.com/prod_downloads/writingcom/writing-school-reform-natl-comm-writing.pdf

BEGINNING MATHEMATICS TEACHERS' USES OF INSTRUCTIONAL STRATEGIES AND ORGANIZATION OF STUDENTS FOR LEARNING

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Keywords: High School Education, Instructional Activities and Practices

Reform-based instructional practices (e.g., NCTM, 2014) have been slow to be pervasive. Teachers' uses of whole class organizations, individual work, and small groups in secondary classrooms have shifted from roughly 50%, 30%, and 20% in the 1970s (Fey, 1979) to, more recently, roughly 48%, 22%, and 22% (Banilower et al., 2013). Instructional practices are important to study because of their proximity to students' opportunities to learn (Carroll, 1963).

Methods

We observed 7 beginning teachers a total of 47 times over 3 semesters to explore how they organized students for learning and the instructional strategies utilized. We investigated whether these decisions differ by course level and, when possible, examined changes over time. For every 5-minute interval observed, our dataset notes how teachers organized students and indicates whether instructional activity was teacher directed, student centered, or assessing understanding.

Results and Discussion

Collectively, teachers in this study organized students as a whole class, in small groups, and individually roughly 48%, 29%, and 23% of time. Regarding instructional strategies, teacher directed methods, assessing understanding, and student center strategies occurred during 48%, 26%, and 25% of class time. Student centered strategies occurred most often in small groups. These teachers organized their classes similarly to the collective, regardless of course level; however, teaching strategies differed. Students in intro courses were less likely to experience teacher directed methods and more likely to experience assessing understanding than peers in core or advanced courses. Students in core courses were more likely to experience teacher directed methods and less likely to experience student centered strategies than their peers. Changes in one teacher's practices stand out. She decreased the proportion of time with students organized as a whole class and individually in favor of small group time while also decreasing time using teacher directed strategies for more student centered methods.

Conclusion

Beginning teachers in this study show an alignment with more recent studies (Banilower et al., 2013) by implementing more student centered methods while lessening time students spend learning individually. Our participants are demonstrating reform practices in their classrooms, but perhaps more work needs to be done addressing how they tend to different levels of classes.

References

Banilower, E. R., Smith, P. S., Weiss, I. R., Malzahn, K. A., Campbell, K. M., & Weis, A. M. (2013). Report of the 2012 National Survey of Science and Mathematics Education. *Horizon Research, Inc.(NJ1)*.

Fey, J. T. (1979). Mathematics teaching today: Perspectives from three national surveys. Mathematics Teacher, 72(7), 490–504.

NCTM. (2014). Principles to actions: Ensuring mathematical success for all. Reston, VA: Author.

Carroll, J. B. (1963). A model of school learning. *Teachers College Record, 64*(8), 723-733.

COMPARING INTERDISCIPLINARY TEACHING APPROACHES

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Keywords: Problem Solving; Affect, Emotions, Beliefs and Attitudes; Curriculum

The purpose of this study is to test the synthetic interdisciplinary approach with the informed disciplinary approach in math and psychology (Lattuca, 2001) on its impact on math efficacy, perception of psychology as a science, and mathematics problem solving ability. In accordance with previous literature (e.g., Lattuca, Voight, & Fath, 2004) we argue that the use of a synthetic interdisciplinary approach will increase math efficacy, perceptions of psychology as a science, and academic performance than in classes where an informed disciplinary approach is used.

To test this, we constructed a survey instrument, which included math tasks. We included 10 questions from the PISA survey (OECD, 2013) so that we can make national and international comparisons. The study used a nested model (class nested in instructor) and was taken twice (within subjects approach) at the beginning of the Fall 2017 semester and again at the end of the semester. The classes that were assessed are a team-taught, synthetic interdisciplinary Math and Psychology course, as well as individually taught disciplinary Psychology in Statistics and Proof Writing classes (all classes taught by the authors).

We have preliminary results from analysis of 10 items from the PISA survey. These assessed students' feelings about their personal math performance and contained questions like: "I learn mathematics quickly." and "I am just not good at math." (reverse coded) and answered on a 1 (strongly disagree) to 5 (strongly agree) Likert scale. Students (n = 18) taking the interdisciplinary course felt that their math abilities improved by the end of the semester (M=3.34) compared to when they began the interdisciplinary course (M=3.08) with p = .003. Students (n = 9) taking the math course (without the interdisciplinary element) did not feel like their math abilities improved by the end of the semester. The surveys will be analyzed in greater depth during summer 2018 and the results of this analysis will also be included in the poster.

There is the possibility of a follow up study to this one when the authors teach the class again. Therefore, we would also like feedback on study design, and theoretical aspects. This study has implications for those teaching interdisciplinary classes and what the effects of this kind of teaching are, compared with disciplinary teaching. Because of this, we argue that this study is in line with the theme for this year's PME-NA Conference, in particular, "looking ahead to emerging opportunities in mathematics education research" while noting that "mathematics itself is increasingly interdisciplinary". This study could inform institutional decisions about whether to support instructors in team-teaching or in offering interdisciplinary courses. In this poster we will also describe the teaching methodology used in the classes studied and its potential impacts on the survey outcomes.

References

Lattuca, L. R. (2001). *Creating Interdisciplinarity: Interdisciplinary Research and Teaching Among College and University Faculty*. Vanderbilt University Press.

Lattuca, L.R., Voight, L.J., Fath, K.Q. (2004). Does Interdisciplinarity Promote Learning? Theoretical Support and Researchable Questions. *The Review of Higher Education*, 28(1), 23-48.

OECD (2013), PISA 2012 Results: Ready to Learn (Volume III): Students' Engagement, Drive and Self-Beliefs*, OECD Publishing*, Paris.

EXPLORING FIRST GRADERS' ORAL AND WRITTEN MATHEMATICAL EXPLANATIONS

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Keywords: Elementary Education, Classroom Discourse

While communicating mathematical reasoning is difficult for young children (Moyer, 2000), it is a relevant skill to develop. The following study considers the impact of adding written discourse to problem-based lessons on fifty children's ability to talk and write about their mathematical reasoning within the numbers and operations domain in four first-grade classrooms in a suburban Title I school in the southeastern United States. The four classes were randomly assigned to an intervention group or a comparison group. Both groups participated in the same problem-based lesson format that encouraged oral discourse during each of the six lessons. While both groups engaged in oral discourse, the intervention group also wrote about how they solved the problems. As the researcher and students engaged in the lessons, it became clear that it was necessary to establish sociomathematical norms so that students understood what it meant to explain how they solved problems (Cobb, Wood & Yackel, 1993).

Pre and post oral and written mathematical explanations were collected and scored by two independent scorers using a rubric adapted from Project M^3 (Gavin et al., 2007). Intervention effects on oral and written mathematical explanations were evaluated separately using a one-way Analysis of Covariance where groups (comparison and intervention) was the independent variable, post explanation scores were the dependent variable, and the pre-explanation scores were the covariate. Both analyses revealed significant differences between the post *oral* explanation scores and post *written* explanation scores of the intervention and comparison groups, $F(1,43) = 8.992$, $p = .004$ (*oral*) and $F(1,44) = 6.173$, $p = .017$ (*written*). These results suggest that adding written discourse to problem-based mathematics lessons that encourage oral discourse increased the children in intervention group's ability to talk and write about their mathematical thinking. While both groups had increased opportunity for oral discourse, the intervention group's scores were significantly different, indicating that adding written discourse also improves children's ability to talk about their mathematical reasoning. The practical benefits of these results are important. Once children are comfortable talking and writing about their solution strategies, both can be used for formative assessment by classroom teachers. Written explanations can also serve as a permanent record for mathematical thought (Lee, 2006), increasing the teacher's ability to understand each child's thinking, therefore increasing equity within the early childhood classroom.

References

Cobb, P., Wood, T., & Yackel, E. (1993). Discourse, mathematical thinking, and classroom practice. *Contexts for learning: Sociocultural dynamics in children's development*, 91-119.

Gavin, M. K., Chapin, S.H., Daily. J., & Sheffield, L. J. (2006-2008). Student Mathematician's Journal: Guides for Teaching and Assessing. *Project M3 series*. Dubuque, IA: Kendall Hunt.

Lee, C. (2006). Language for Learning Mathematics: Assessment for Learning in Practice: Assessment for Learning in Practice. McGraw-Hill Education (UK).

Moyer, P. S. (2000). Communicating mathematically: Children's literature as a natural connection. *The Reading Teacher*, *54*(3), 246-255.

AN EXPLORATION IN DOING COLLABORATIVE UNSOLVED MATHEMATICS

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Research in the field of mathematics is often done collaboratively and is open-ended, as the purpose of mathematics research is to solve previously unsolved problems. Moreover, mathematics as a field has become increasingly collaborative in recent decades (Grossman, 2002; Gowers & Nielsen, 2009). In contrast, mathematics in k-16 is still overwhelmingly viewed as a closed subject, with answers that are only right or wrong and is viewed as an individual activity. This model leaves only an elite few doing mathematics and mathematics research. Attempts to explore alternative models of learning and doing mathematics in school settings work against strong barriers of perception about what mathematics is and what the role of the learner should be. One component in the effort to overcome these barriers is a series of "existence proofs" showing (a) that collaborative mathematics is possible at a variety of levels, and (b) that working collectively can produce different kinds of mathematical experiences and results. This poster describes one such existence proof, describing collective inquiry into a celebrated problem.

In 1937, Lothar Collatz conceptualized a problem, now known by many names but most often, The Collatz Conjecture or the 3*n*+1 problem. The problem depends on a function *f(n)*, defined for any positive whole integer, *n:*

$$f(n) = \begin{cases} \frac{n}{2}, n \text{ is even} \\ 3n+1, n \text{ is odd} \end{cases}$$

The value of f(*n*) is another integer, and thus a member of the domain of f, enabling one to form f(f(*n*)), or f2(n); f(f(f(n))), or f3(n); and so on. Repeatedly applying f generates a sequence of integers {fi(n)}. The Collatz Conjecture is that for any initial value of n, this sequence reaches 1 in a finite number of iterations. This is a simple problem to state but an extremely difficult problem to solve. Since 1937, mathematicians have taken up the charge to prove the conjecture, most notably, Jeffery Lagarias. In his investigations, he has looked at modular arithmetic and creating visual representations (Lagarias, 2010).

Over a two-week period, we explored what happened when unsolved mathematics (Collatz Conjecture) was given to a group of PhD and Masters students enrolled in a course that specifically focused on inquiry-based, collaborative, open-ended mathematics. The group studied patterns that arose when exploring the 3n+1 problem. The group crafted illuminating representations, examined the problem using modular arithmetic, and explored what happened when the initial number, *n*, was negative. These explorations both recapitulated the work of Lagarias and went off in its own directions. This work describes the processes and products.

References

Grossman, J. W. (2002). The evolution of the mathematical research collaboration graph. *Congressus Numerantium*, 201-212.
Gowers, T., & Nielsen, M. (2009). Massively collaborative mathematics. *Nature, 461*(7266), 879-881.
Lagarias, J. C. (Ed.). (2010). *The ultimate challenge: The 3x+ 1 problem.* American Mathematical Society.

GROUPING FOR "DIVERSITY": AN EQUITABLE PRACTICE?

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To help meet the well-established need for a more equitable mathematics education in the U.S., it is imperative that researchers attend to teachers' conceptions and enactments of equitable instruction. In this analysis, for example, we were concerned with secondary (grades 6-12) mathematics teachers' instructional visions—or their perceptions of ideal instruction (Munter, 2014)—to understand the development of those visions as teachers engaged in equity-focused professional development.

In analyses of interviews with participating teachers, we noticed that a considerable number of them identified strategies used for assigning students to small groups as indicators of equitable instruction. Among the grouping strategies that teachers described as equitable was a subset that emphasized grouping for "diversity"—a strategy for arranging students so that each small group mirrors the demographics of the whole class. Such cases were the focus of this analysis.

Findings and Discussion

Of the 37 teachers interviewed, 15 referred to students' racial or ethnic identities in discussing equitable grouping strategies. Of those, 9 recommended grouping for "diversity" (GfD) while 6 alluded to the strategy but did not necessarily recommend it. Most teachers who promoted GfD did not provide a rationale for its use (n=6), although a few suggested that it could support students in getting to know peers outside of their own social groups (n=3). The 6 who mentioned but did not promote GfD either explained how district leaders seemed to prefer it (n=3) or defended hypothetical classroom scenarios in which GfD was not used (n=3).

We wonder whether these results are related to the price that Ladson-Billings (2004) argued was paid for the 1954 *Brown v. Board of Education* decision. As she noted, desegregation efforts were never about breaking up racial groups of students, and certainly not about getting children of color into the presence of white children. However, the U.S. Supreme Court's decision to rule in favor of Brown was grounded in the argument that the public school segregation was detrimental to Black children, as it exacerbated "Black inferiority" (p. 5). Given that teachers' GfD descriptions seem related to this narrative (e.g., "If I do see that one group is all African American males then yes, that is an issue…that needs to be broken up"), we wonder whether it might be shaping teachers' conceptions of "diversity" at the level of small-group interactions.

Regarding implications for research, we encourage those interested in teachers' learning as it relates to K-12 (in)equity to be on the lookout for practitioners' conceptions of equity as "diversity" or "desegregation." Future research might also investigate potential ways to disrupt such ideas. Regarding implications for practice, we hope that our results will encourage practitioners to take care as they conceptualize and implement equity-related efforts.

References

Ladson-Billings, G. (2004). Landing on the wrong note: The price we paid for Brown. *Educational Researcher*, *33*(7), 3–13.

Munter, C. (2014). Developing visions of high-quality mathematics instruction. *Journal for Research in Mathematics Education, 45*(5), 585–636.

DEVELOPING METHODS TO IMPLEMENT EMBODIED GAME DESIGN FOR MOBILE LEARNING TECHNOLOGIES IN STEM CLASSROOMS

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In this poster, we present a new way to engage with mathematics and design thinking that invites students to not only *play* games, but to also *create* their own math games supported by mobile technologies. Our priority is to establish a framework that can be used in all STEM classrooms, regardless of computer programming experience or tools. To achieve this goal, our team has designed and implemented an instructional activity and created an extensive coding guide to test the effectiveness of this activity in high school STEM classrooms.

The field of educational games has shown the benefits of playing well-designed games on student motivation, engagement, and performance (Gee, 2003), but only recently has research focused on the benefits of students creating their own games. By positioning students in the role of creators, rather than consumers, they can explore, discover, and construct their own learning, which could lead to deeper conceptual understanding (Papert, 1990). Some have also argued that STEM learning environments should be more embodied and social (Abrahamson, 2009). Embodied learning combines movement with higher-order cognitive activities, such as analyzing and evaluating. By grounding students through physical movement, object manipulation, and gesture, abstract concepts may become more concrete (Koedinger, Alibali, & Nathan, 2008).

Extending this evidence, we propose the Embodied Game Design framework as an instructional activity and a preliminary method for analyzing students' higher-level thinking in STEM classrooms. Our current study explores the *feasibility* of game design for high school students as well as the *pedagogical effectiveness* of the activity designs. Over the course of three days, participants played an established math game, created their own games in small teams, and then modified their games. Results showed that the project is feasible and that particular instructional design elements produce varying outcomes. Ultimately, we aim to develop a complete, instructional design that will be appropriate for K-12 students and can be used as an interdisciplinary, project-based activity that engages students in collaboration, mathematics, active and embodied learning, and technology-based design to promote computational thinking.

References

Abrahamson, D. (2009). Embodied design: Constructing means for constructing meaning. *Educational Studies in Mathematics, 70*(1), 27-47.

Gee, J. (2003). What videogames have to teach us about learning and literacy. *Computers in Entertainment (CIE), 1*(1), 20.

Koedinger, K. R., Alibali, M. W., & Nathan, M. J. (2008). Trade-offs between grounded and abstract representations: Evidence from algebra problem solving. *Cognitive Science, 32*, 366-397.

Papert, S. (1990). Introduction. In I. Harel (Ed.), *Constructionist learning*. Boston, Massachusetts: MIT.

THE RELATIONSHIP BETWEEN NOVICE TEACHERS' SELF-EFFICACY AND STUDENTS' OPPORTUNITIES FOR DISCOURSE

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Documents outlining standards and recommendations for mathematics instruction consistently emphasize the importance of high-quality mathematical discourse. Although quality mathematical discourse is valued, it can be challenging to implement, particularly for novice teachers (Bennett, 2010). Teaching self-efficacy, or one's beliefs about her/his ability to teach, has been linked to effective mathematics instructional practices for more experienced teachers (Enon, 1995); however, little is known about this relationship for *novice* teachers, the focus of the current study. Specifically, this study examines the relationship between mathematics teaching efficacy and mathematical discourse implementation for novice elementary teachers, all in their second year of teaching.

The participants in this study were 118 second-year teachers. The teachers completed a mathematics teaching log (Walkowiak & Lee, 2013) on which they reported the proportion of time that students in their classroom had opportunities to engage in mathematical discourse during a given lesson (with a mean of 38.7 lessons logged per teacher). Teachers responded on a four-point scale about students' opportunities to engage in nine "math talk" behaviors. Teachers also completed the Mathematics Teaching Efficacy Beliefs Instrument (MTEBI; Enochs, Smith, & Huinker, 2000) to measure their self-efficacy to teach mathematics. Rasch scores were computed for each scale: Personal Mathematics Teaching Efficacy (PMTE; belief about ability to teach mathematics) and Mathematics Teaching Outcome Expectancy (MTOE; belief about ability to impact student outcomes). Multiple regression analyses were utilized to assess if mathematics teaching efficacy predicted mathematical discourse, controlling for teachers' high school GPA and SAT scores. Grade band (K-2 vs. 3-5) was examined as a potential moderator. MTOE, or beliefs about ability to impact student outcomes, was a statistically significant predictor of opportunities for discourse within the mathematics lessons, $b = 0.069$, $t(117) = 2.38$, $p = .02$. The results of the regression including all variables explained a statistically significant proportion of the variance in opportunities for classroom discourse ($R^2 = 0.067$, $F(4,113) = 2.02$, $p < .05$). The moderation of grade band was not statistically significant.

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References

Bennett, C. A. (2010). "It's hard getting kids to talk about math": Helping new teachers improve mathematical discourse. *Action in Teacher Education, 32*(3), 79-89.

Enochs, L. G., Smith, P. L., & Huinker, D. (2000). Establishing factorial validity of the mathematics teaching efficacy beliefs instrument. School Science and Mathematics, 100(4), 194-202.

Enon, J. C. (1995). *Teacher efficacy: Its effects on teaching practices and student outcomes in mathematics* (Doctoral dissertation). Retrieved from ProQuest Digital Dissertations. (NN06207)

Walkowiak, T. A. & Lee, C. W. (2013). *The Instructional Practices Log in Mathematics (IPL-M).* Unpublished measure, North Carolina State University, Raleigh, NC.

DRAWING A PICTURE OF PRACTICE-BASED TEACHER COMPETENCY FOR DISCURSIVE MATHEMATICS PEDAGOGY

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Building on Shulman's (1986) work, researchers have explored different ways of categorizing and investigating teacher knowledge and competency. Our study examines a relatively unknown aspect of teacher competency in mathematics education, which we call discursive capacity. Discursive capacity is defined as the toolkit with which a teacher facilitates mathematical discourse as evident in meaningful interactions and participations with communicative forms in the classroom community as regulated by social and socio-mathematical norms. Using the construct of discursive capacity, we elaborate further the role of maxims and normative knowledge (Shulman, 1986) in teachers' effective teaching practice as an extension of Shuman's conceptualization of different forms of teacher knowledge.

We propose understanding the nature and kinds of *mathematical discourse for teaching* (MDT) as the basis for studying discursive capacity. To that end, we observed, video-recorded, and transcribed lessons of Korean master teachers in which discourse was the primary mode of teaching school mathematics. We analyzed various forms and paths of classroom discourse and discursive patterns in the videos in order to characterize the teacher's discursive practice for teaching mathematics.

Our analysis revealed that student participation is normed and nurtured for learning when the teacher established positive relationships in the classroom community and forged the kinds of connections with students that increase student motivation to learn. Further, we identified the sequential and deeply connected phases of practicing MDT: first, building a classroom community rooted in positive relationships and regulated by socio-mathematical norms; second, launching a lesson with discursive strategies to elicit student thinking and expanding it for further learning; and third, facilitating problem-solving with patterns of discursive practice depending on the nature of tasks and engaging students in collaborative problem-solving and overcoming misconceptions. These three contexts combine to afford the opportunity for us to understand how a teacher and their students co-create a community and engage in mathematics discourse. In our presentation, we will illustrate a teacher's discursive competency found in the process of these sequential phases and discuss the opportunity and challenge of theorizing the practice-based MDT.

Acknowledgments

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References

Shulman, L. S. (1986). Those who understand: knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

TEACHING NORMS FOR THE ORDER OF MULTIPLICATION IN THIRD GRADE

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Numerous different recommendations for the order that multiplication should be taught exist. Woodward (2006) suggested introducing multiplication using 0s, 1s, squares, 5s, 9s and 10s, followed by other operands. Researchers in neuroscience suggest that multiplication with operands 2-5 is simpler and uses different neural processes than operands 6-9 (Jost et al., 2004). Recommendations also vary between different third grade textbooks. Although there are numerous recommendations for the order operands are introduced, there is little to no research on the order that they are introduced. The purpose of this study is to investigate the order that third grade teachers rank introducing specific operands in multiplication. Third grade teachers (n=49) were asked to rank the order they "assign tasks/activities with these multiples [0 to 10]" in a school year. Descriptive data are reported in Table 1. The distribution of ranks was found to be independent from chance ($\chi^2(df=100)=1044.046$, $p<.001$), indicating that, there are certain reported tendencies for the order operands are introduced. Median and mode statistics suggest the most common sequence reported by teachers is in the following order: 0s, 1s, 2s, 5s, 10s, 3s, 4s, 6s, 7s, 8s, and 9s. Approximately a third of teachers reported a similar sequence. There is a large amount of variance in specific sequences reported. Preliminary analysis suggests teachers' rankings are not meaningfully associated with adopted textbook, years of experience, level of education, or other reported background characteristics. Findings suggest a need for further study to explain teachers' rankings of multiplication operands, and the reasoning behind it.

Table 1: Third Grade Teachers' Reported Rankings of Order of Multiplication Operands.

	0s	1s	2s	3s	4s	5s	6s	7s	8s	9s	10s
1st	**26**	5	14	0	0	1	0	0	0	0	3
2nd	2	**27**	4	3	0	11	0	0	0	0	2
3rd	3	1	**27**	3	3	1	0	0	0	2	9
4th	7	3	3	8	2	**18**	1	0	0	1	6
5th	4	7	1	4	9	9	0	0	0	2	**13**
6th	4	1	0	**15**	9	7	2	0	0	6	5
7th	1	4	0	11	**12**	1	13	1	2	4	0
8th	1	0	0	4	**12**	0	**16**	13	1	2	0
9th	0	1	0	0	1	0	12	**19**	14	2	0
10th	0	0	0	0	0	0	4	10	**21**	14	0
11th	1	0	0	1	1	1	1	6	11	**16**	11
Mean	2.80	3.10	2.45	5.65	6.45	4.14	8.04	9.14	9.78	8.86	5.59
Median	1	2	3	6	7	4	8	9	10	10	5
Mode	1	2	3	6	7	4	8	9	10	11	5

References

Jost, K., Beinhoff, U., Henninghausen, E., & Rosler, F. (2004). Facts, rules, and startegies in single-digit multiplication: Evidence from event-related brain potentials. *Cognitive Brain Research, 20*, 183-193.

Woodward, J. (2006). Developing automaticity in multiplication facts: Integrating strategy instruction with timed practice drills. *Learning Disability Quarterly, 29*(4), 269-289.

RECOGNIZED MATHEMATICS INSTRUCTORS' BELIEFS ABOUT LEARNING

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Keywords: Teacher Beliefs, Learning Theories, Post-Secondary Education

Prior research on teachers' beliefs and conceptions suggests a relationship between teachers' instructional practices and their beliefs about learning (Thompson, 1992). The focus of this work, however, has largely been limited to the beliefs of K–12 teachers. Only a few researchers have extended this work to the undergraduate level (e.g., Speer, 2008). In an attempt to further this line of research, this qualitative methods study tackles the question of what beliefs about learning mathematics are being held by university mathematics instructors who—through institutional teaching awards—have been recognized for their teaching excellence. The purposes of the study are to document these beliefs, as they may provide insightful, workable ways of thinking about the learning of mathematics, as well as to compare and contrast these with views espoused by mathematics educators.

The participants were five mathematics instructors at a large Midwestern university. They were randomly chosen from a pool of mathematics instructors who had won at least one of the university's teaching awards. One hour-long, semi-structured interviews were conducted with all participants, with questions seeking to unearth the instructors' conceptions of learning. For instance, questions were asked about the goals of learning, environments for learning, and the roles of teacher and student in the student's learning process. The interviews were audio-recorded and transcribed.

Analysis of the data is ongoing, but preliminary analysis suggests the emergence of a complex picture. The participants were largely able to articulate meaningful replies to the questions but seemed to fuse different viewpoints on learning. To make sense of the data, I use coding methods borrowed from grounded theory, looking for emergent leads. Furthermore, to help with untangling and delineating the mathematics instructors' views, I draw on established views of learning in mathematics education literature, such as Sfard's two metaphors for learning (Sfard, 1998) and Harel's three principles of learning and teaching mathematics (Harel, 2002).

I would like to emphasize that the mathematics instructors' views are not meant to provide a goal or a benchmark for others to attain, as neither the instructors themselves, nor I, claim that they are excellent teachers. Rather, this study is meant to scrutinize the belief systems that have allowed some mathematics instructors to celebrate teaching success and gain recognition by their students and peers. In addition, understanding the mathematics community's appointed role models may provide a greater understanding of—and mirror for—the community as a whole.

References

Harel, G. (2000). Three principles of learning and teaching mathematics. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 177–190). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher*, *27*(2), 4–13. https://doi.org/10.3102/0013189X027002004

Speer, N. M. (2008). Connecting beliefs and practices: A fine-grained analysis of a college mathematics teacher's collections of beliefs and their relationship to his instructional practices. *Cognition and Instruction*, *26*(2), 218–267. https://doi.org/10.1080/07370000801980944

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A sysnthesis of the research. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127–146). New York, NY:Macmillan.

CONNECTING A TEACHER'S USE OF SCAFFOLDING PRACTICES TO IN-THE-MOMENT NOTICING

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A majority of the studies scrutinized teacher noticing by having the teachers watch videos of lessons and reflect upon what they have noticed (Sherin & van Es, 2005); however, this is a significantly different task from the more authentic work of engaging in in-the-moment noticing to attend to and respond to student thinking during instruction. In this study, we investigated a teacher's in-the-moment noticing skills and how these supported efforts to provide effective scaffolding during interactions with a group of students.

The teacher participant in this study was a mathematics teacher with experience of 6-10 years of teaching grade 7 and 8. A total of 7 lessons were audio and video recorded and the transcripts were generated for each lesson. Direct and verbal interactions between the teacher and a group of four students in the classroom were identified from the transcripts. Three researchers coded the levels of scaffolding strategies from Anghileri (2006) to identify the frequency and context of instances of "Level 1: Environmental provisions"; "Level 2: Explaining, reviewing and restructuring"; and "Level 3: Developing conceptual thinking" (p.39). The scaffolding strategies were analyzed by examining them in relation to noticing classifications perceived by Jacobs et al. (2010), which were composed of the following components: "Attending to children's strategies", "Interpreting children's mathematical understandings", and "Deciding how to respond on the basis of children's understandings" (p.172).

Preliminary data from the 7-day lesson reveal a total 113 verbal interactions between the group of students and the teacher, many of which incorporated multiple occasions of scaffolding. Out of the 252 scaffolding instances, 58 were classified as Level 1, 180 were Level 2, and 14 were Level 3. It was noted that higher levels of scaffolding strategies, restructuring practices from the Level 2 as well as the Level 3, were mostly coincided with the other two lower strategies from the Level 2, which are explaining and reviewing.

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References

Anghileri, J. (2006). Scaffolding practices that enhance mathematics learning. *Journal of Mathematics Teacher Education, 9*(1), 33–52.

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169–202.

Sherin, M. G., & van Es, E. (2005). Using video to support teachers' ability to notice classroom interactions. *Journal of Technology and Teacher Education, 13*(3), 475-491.

INVESTIGATING GROUP PROBLEM POSING FOR SECONDARY STUDENTS IN A LINEAR FUNCTIONS-BASED LESSON INTERVENTION

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Keywords: group problem posing, representations, mathematical connections, positioning theory

Rationale and Theory

Looking back, the NCTM (1991) has advocated for instruction that promotes *representations* and *mathematical connections*. Yet, this has been an enduring challenge as U.S. mathematics lessons often lack such opportunities (Hiebert et al., 2005). A more contemporary 21st-century expectation from industry is *collaboration*. However, students may *position* one another in ways that hinder shared intellectual work (Wood, 2013). As problem posing is deemed a valuable mathematical practice (NCTM, 1991) and a tool for representation use (Cai & Hwang, 2002), I used the activity of *group problem posing* to examine how high-school student groups incorporated representations and mathematical connections into their problem scenarios. I also explored how intellectual work was shared as roles were negotiated and renegotiated.

Methods and Analysis

Five groups of Integrated Mathematics 2 students participated in a five-lesson algebra intervention. Groups were heterogeneously formed by race, gender, mathematical disposition, and performance. Each lesson began with a 30-minute review session on linear functions, followed by a 30-minute group problem-posing session. Groups were asked to create problems including lesson mathematics, representations, and cultural artifacts (e.g., fast-food menu prices). Groups also evaluated and re-negotiated their roles each day. Problems were coded using a priori and grounded categories based on representations and mathematical connections. One group's problem-posing sessions were video-recorded. Transcribed videos, field notes, and written reflections about the use of roles were analyzed to understand the negotiation of student roles.

Results and Discussion

Analysis revealed that groups often incorporated one or more representations but struggled to purposefully use them and/or make mathematical connections. Additionally, many problems only focused on finding "total cost." In spite of the focus on roles, the analyzed group typically worked *cooperatively* rather than *collaboratively*. Looking ahead, these findings provide insights into challenges with using representations, making mathematical connections, and sharing in the intellectual work of group problem posing. Understanding such challenges can help identify principles for refining the practice and supporting the effective engagement of student groups.

References

Cai, J., & Hwang, S. (2002). Generalized and generative thinking in US and Chinese students' mathematical problem solving and problem posing. *The Journal of Mathematical Behavior*, *21*(4), 401-421.

Hiebert, J., Stigler, J. W., Jacobs, J. K., Givvin, K. B., Garnier, H., Smith, M., ... & Gallimore, R. (2005). Mathematics teaching in the United States today (and tomorrow): Results from the TIMSS 1999 video study. *Educational Evaluation and Policy Analysis*, *27*(2), 111-132.

National Council of Teachers of Mathematics. (1991). Professional standards for teaching mathematics. Reston, VA: Author.

Wood, M. B. (2013). Mathematical micro-identities: Moment-to-moment positioning and learning in a fourth-grade classroom. *Journal for Research in Mathematics Education, 44*(5), 775-808.

TRADITIONAL VERSUS STANDARDS-BASED GRADING: A COMPARISON OF THE TEACHING PRACTICES OF SECONDARY MATHEMATICS TEACHERS

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Keywords: Assessment and Evaluation, Standards

There has been a movement in recent years towards developing alternative systems for measurement and evaluation of academic achievement and performance. By using a standards-based grading system, Iamarino (2015) argued that a teacher "is better able to determine a student's grade based on the single most important aspect of education – how well the student comprehends the content of the course" (p. 2). According to Iamarino (2015), implementation of standards-based grading can be difficult and uncomfortable for teachers as well as other stakeholders such as district supervisors, students, and parents. Furthermore, Vatterott (2015) cautioned that standards-based grading is more than a modification to the course grading scheme and many teachers fail to modify all aspects of the learning process.

Methodology

This quantitative survey study sought to determine and describe the impact of standards-based grading practices on instruction in traditional, secondary (Grades 7-12) mathematics classrooms across a state in the Rocky Mountain region of the United States. To answer their research questions, the research team developed an online questionnaire based on sample questions from the TIMSS survey (NCES, 2011; NCES, 2015) and the Math Task Analysis Guide (Smith & Stein, 1998). It included basic demographic questions as well as specific questions focused on how teachers implement and assess instruction as part of their practice as a secondary mathematics teacher. Using information available on publically accessible websites, the research team compiled a list of teachers who were clearly identifiable as teaching secondary mathematics. This process resulted in a sampling frame of 2565 secondary mathematics teachers all of whom were invited via email to participate in the study.

Results and Discussion

The survey received a response rate of 24.9% ($N = 638$). Preliminary data analysis suggests that there was no difference between the practices of secondary mathematics teachers who self-reported utilizing standards-based grading and those who self-reported traditional grading practices. The results of this study suggest a need for follow-up research studies as well as a need for pre-service and in-service teacher preparation and support.

References

Iamarino, D. L. (2014). The benefits of standards-based grading: A critical evaluation of modern grading practices. Current Issues in Education, 17(2), 1-10.

National Center for Education Statistics (NCES). (2011). *TIMSS 2011 teacher questionnaire mathematics grade 8*. Washington, DC: U.S. Department of Education.

National Center for Education Statistics (NCES). (2015). *TIMSS 2015 teacher questionnaire mathematics grade 8*. Washington, DC: U.S. Department of Education.

Smith, M. S., & Stein, M. K. (1998). Reflections on practice: Selecting and creating mathematical tasks: From research to practice. *Mathematics Teaching in the Middle School*, *3*(5), 344-350.

Vatterott, C. (2015). Rethinking grading: Meaningful assessment for standards-based learning. Alexandria, VA: ASCD.

THE EFFECTS OF INQUIRY-BASED PEDAGOGY ON STUDENTS' ATTITUDES IN MATH AND INTENT TO PURSUE STEM

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The education research community has long recognized that teachers' pedagogical choices are key to the development of learning environments that can either promote or hinder equity. This is especially evident in mathematics classrooms where inequality between groups of students (i.e., boys and girls and majority and minority racial/ethnic groups) is pronounced. Specifically, we know from prior research that pedagogy in mathematics that is based on traditional modes of direct and lecture-based instruction can actually reproduce and exacerbate current patterns of inequality (Ellis, 2008). As a result, mathematical reform efforts call for more inquiry-based modes of instruction (NCTM, 2000) as they are beneficial for student outcomes and for fostering equitable experiences and outcomes (Boaler & Staples, 2008).

Both quantitative and qualitative studies have provided valuable insight into the positive ways that inquiry-based mathematics instruction can affect not only student achievement but also their attitudes towards mathematics and STEM in general (Boaler, 2006; Goos, 2004). Importantly, qualitative studies often consider the viewpoints of students in their assessment of the processes of classroom instruction as well as the outcomes of that instruction. Yet there is a lack of large-scale quantitative research that captures students' experiences of inquiry-based activities in their mathematics classrooms and examines its potential impact on their math-related attitudes. Furthermore, as female students continue to confront obstacles to their pursuit of math in the form of stereotypes and messages from others about a lack of belonging, it is critical to examine whether and how inquiry-based math classrooms may be particularly important in shaping favorable math attitudes for girls. And while there is some research that examines whether teachers' pedagogical choices may impact girls' attitudes more than boys, it is generally limited in scope, and further does not simultaneously consider differences by race/ethnicity.

Thus, our study seeks to fill current gaps in the literature by investigating whether students' experiences of inquiry-based learning in their mathematics classrooms are related to their attitudes towards math, and whether and how such patterns might vary by gender as well as by race/ethnicity. We utilize data from the U.S. portion of the Trends in International Mathematics and Science Study (TIMSS) of 2007 to conduct our investigation. This dataset is ideal as it includes a nationally representative sample of students at the critical stage of adolescence, and queries them about their math attitudes as well as the range of activities that characterize their math classroom.

References

Boaler, J. (2006). Urban success: A multidimensional mathematics approach with equitable outcomes. *Phi Delta Kappan*, *87*(5), 364-369.

Boaler, J., & Staples, M. (2008). Creating mathematical futures through an equitable teaching approach: The case of Railside School. *Teachers College Record*, *110*(3), 608-645.

Goos, M. (2004). Learning mathematics in a classroom community of inquiry. *Journal for Research in Mathematics Education*, 35(4), 258-291.

National Council of Teachers of Mathematics (Ed.). (2000). *Principles and standards for school mathematics* (Vol.1). National Council of Teachers of Mathematics.

CONTEXTUAL PROBLEMS AS CONCEPTUAL ANCHORS?: A STUDY OF TWO CLASSROOMS

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Mathematics teachers have been advised that contextual problem-solving experiences can be used to develop new mathematical ideas, and can serve as anchors for understanding new mathematical ideas (Meyer, Dekker, & Querelle, 2001). But research suggests students are often conditioned to strip away contextual elements of problems; thus, the contexts are unlikely to serve as anchors for these students. In this study, we ask: *What are the different ways students and teachers make reference to the context or prior contexts to support understanding? When does this occur?*

We conducted a multiple case study of two seventh grade mathematics teachers over the course of six weeks. We analyzed video recordings of the enacted curriculum through the lens of Wernet's (2017) framework for exploring classroom interactions around contexts.

First, the depth of referencing previous contexts varied across teachers. One teacher referenced previous contexts through brief transitional statements; the other teacher prompted students to compare contexts to previous storylines throughout lessons. Second, we identified two subcategories of Wernet's (2017) "references to the context while solving or making sense of the problem" category. Students made *conceptually-oriented contextual references* when reasoning informally or giving contextualized rationales for their reasoning (e.g. grouping infants in fives because in the day care each adult could only watch five infants). When more formal representations were discussed, the teachers and students made p*rocedurally- oriented contextual references* by explaining the contextual meaning of quantities in the tables, but not the conceptual meaning underlying their procedures (e.g. "what you do to the bottom you have to do to the top"). Third, we note implicit references to prior contextualized experiences, in the form of student-generated inscriptions mimicking those developed in response to previous problems.

Our additions to Wernet's (2017) framework add to the theoretical knowledge base of classroom interactions around contextual problems by providing specificity about different ways teachers and students reference contexts, but further research is necessary to understand whether the patterns found within these two classrooms are representative of larger trends in contextual-problem based instruction. Our findings also have practical implications. Teachers should be supported in understanding which types of contextual references support conceptual reasoning and should consider providing opportunities for students to compare various contexts. Reflecting across contexts gives students the opportunity to identify relevant mathematical connections and generalize previously developed strategies.

References

Meyer, M., Dekker, T., & Querelle, N. (2001) Context in mathematics curricula. *Mathematics Teaching in the Middle School,* 6(9), 522-527.

Wernet, J. L. (2017). Classroom interactions around problem contexts and task authenticity in middle school mathematics. *Mathematical Thinking and Learning*, *19*(2), 69-94.

PROBLEM POSING AND STUDENT ENGAGEMENT IN A UNIVERSITY DEVELOPMENTAL MATHEMATICS COURSE

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Keywords: Engagement, Attitudes, Problem Solving, Problem Posing

With the continued increase of enrollment at colleges and universities, students entering college mathematics underprepared is increasing. The National Assessment of Educational Progress, found only 25% of 12 grade students had scored at or above the proficient level on the NAEP math assessment (NAEP, 2015). A portion of these students entering post-secondary institutions, are being required to enroll in developmental mathematics courses to better prepare them for college mathematics coursework (Larnell, 2016).

Developmental mathematics students learn to focus on the outcomes of their problem-solving efforts and often have little or no time to focus on the problem formation where Ellerton (2013) noted that the formulation of the problem is more essential than its solution. Silver (1997) found that instruction involving problem solving and problem posing tasks will assist students in developing more creative approaches to mathematics. With this, problem posing is seen to not only engage students with problem solving but use the individual student interest to help increase the level of engagement, reduce common fears and anxieties, and help students develop improved attitudes towards mathematics (Lavy & Bershadsky, 2003).

The research is guided by the question: *What problems do developmental mathematics students design based on their personal experiences and interests as they engage in problem posing?* The researcher will use Ellerton's (2013) *Active Learning Framework* and categorize problems posed based on Mji & Glencross' (1999) notion of surface and deep approaches to mathematical understanding. Students will be observed and recorded discussing and creating word problems based on their interests and approach. These problems will give instructors insight of types of problems that developmental mathematics students prefer to engage in and create more potential in understanding course content at a higher level and improve attitudes towards mathematics.

The findings of the study will be reported, and student work will be presented at the poster session. Implications of the study will be to observe how to better serve students entering underprepared for post-secondary math. It will also give a training ground for pre service teachers to better prepare themselves for working with students at the K-12 levels.

References

Ellerton, N. F. (2013). Engaging Pre-Service Middle-School Teacher-Education Students in Mathematical Problem Posing: Development of an Active Learning Framework. *Educational Studies in Mathematics*, *83*(1), 87,101.

Larnell, G. (2016). More than just skill: examining mathematics identities, racialized narratives, and remediation among black undergraduates. *Journal for Research in Mathematics Education*, *47*(3), 233,269.

Lavy, I., & Bershadsky, I. (2003). Problem posing via. *Journal of Mathematical Behavior*, (4), 369,387.

Mji, A., & Glencross, M. J. (1999). An Examination of First-Year University Students' Attitudes toward and Approaches to Learning Mathematics. *Psychological Reports*, *85*(3), 809–816.

NAEP - 2015 Mathematics & Reading at Grade 12 - Mathematics - National Results Overview. (n.d.). Retrieved March 22, 2018, from https://www.nationsreportcard.gov/reading_math_g12_2015/#mathematics?grade=12

Silver, E. A. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *ZDM*, *29*(3), 75–80.

SECONDARY STRINGS: MATH TALKS IN HIGH SCHOOL

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Keywords: Instructional Activities and Practices, Classroom Discourse, High School Education

Background: The Problem

The authors work closely with a group of K-12 teacher leaders who have become excited about engaging students in reasoning and problem-solving in their classes. Despite this enthusiasm, secondary teachers find it particularly difficult to move beyond occasional problem-solving lessons while teaching their mandated curriculum. Their default has been using guided notes and practice, focusing on procedures.

One support for change is instructional routines that provide structure for norms and interactions (Lampert & Graziani, 2009) We engaged our teacher leaders in number/pattern talks, a widely used structure used by k-8 teachers to engage students in "making sense of mathematics" (Parrish 2011, p.204). For our secondary colleagues, number talks have been pivotal in shifting them towards a student-centered practice. This has led to a new challenge - designing talks that focus on secondary content.

Secondary Strings: A Work in Progress

Together, we are exploring/studying the question: ***What might math talks look like in high school classrooms?*** For us, a secondary math talk (or string) is a series of problems that support the construction of conceptual understanding and procedural fluency. We begin with a problem that all students can do, connected to a fundamental concept. Ensuing problems gradually move students to new ideas, while positioning the students as problem solvers and sense-makers, and prompt them to make connections and generalizations. We will share several prototype secondary strings. We will also identify questions that we still have connected to the structure, design and use of these strings, including:

- What are basic design features of effective secondary strings?
- What skills and knowledge are needed to plan and enact effective secondary strings?
- What factors contribute to successful uptake of secondary strings?

Connection to Conference Theme

Our efforts to extend the work of elementary mathematics teacher educators into secondary classrooms is connected to the conference theme in two ways. First, we are hopeful that secondary strings can provide supports for teachers hoping to respond to "changing mathematical and pedagogical demands" by helping them provide opportunities for students to reason and make sense of secondary curriculum. In so doing, we also hope support equitable teaching by broadening access to opportunities for reasoning and sense-making for all students.

References

Lampert, M., & Graziani, F. (2009). Instructional activities as a tool for teachers' and teacher educators' learning. *The Elementary School Journal, 109*(5), 491-509.

Parrish, S. D. (2011). Number talks build numerical reasoning. *Teaching Children Mathematics, 18*(3), 198-206.

Chapter 12

Technology

Research Reports

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.

Brief Research Reports

Posters

EXPANDING STUDENTS' CONTEXTUAL NEIGHBOURHOODS OF MEASUREMENT THROUGH DYNAMIC MEASUREMENT

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This study is part of a larger project exploring students' thinking of Dynamic Measurement (DYME), an approach to area measurement that engages students in dynamic digital experiences of measuring rectangular surfaces through sweeping lengths. The goal of this study was to evaluate the extent to which students could bridge the mathematical knowledge they gained from these dynamic experiences to other activities that are more static in nature. A classroom of 19 third grade students participated in an 8-period design experiment (DE) centered on DYME. Data to evaluate the bridging were obtained from pre- and post-assessments administered before and after students' participation in the DE. The results suggest that by working with the DYME tasks, students were able develop a conceptual connection between multiplicative reasoning and area measurement that were able to apply to solve static tasks.

Keywords: measurement, technology, situated learning, learning trajectory

> *Researcher*: So, what did you learn from interacting with these tasks?
> *Student*: We learned how to paint inside the lines.
> (*Excerpt from a design experiment after a series of sessions using DYME tasks*)

As researchers, we carefully design tools and tasks to engage students in experiences that would support their learning of mathematics. However, similar to the excerpt above we notice that "unfortunately mere experience is not sufficient for learning, for integrating into one's functioning, or for making, more and more effective actions available to be enacted in the future" (Mason 2015, p. 334). Indeed, students learn how to play with the tool, engage with the task and make some generalizations, which a researcher may describe as powerful for developing advanced mathematical ideas, yet there is "little evidence that students can abstract beyond the modeling context" (Doerr & Pratt 2008, p. 272) or whether students are able to use this knowledge to make sense of other tools or apply it in other contexts.

In fact, many researchers have tried to describe the situated nature of the generalizations that students develop as they engage with contextual problems and digital tools. An example is the work of Hoyles and Noss (1992) on the notion of "situated abstractions" in which they describe the gap between the generalizations that students form in one context but not in others. Many researchers tried to describe this as a failing "transfer" of knowledge. For instance, Broudy (1977) discussed transfer as the ability of students to apply their prior knowledge in order to solve new problems, while diSessa and Wagner (2005) discussed it as re-using the knowledge gained in one situation (or class of situations) to a new situation (or class of situations).

In exploring students' transition from a constructionist learning environment to formal algebra, Geraniou and Mavrikis (2015) raised the issue of what exactly is that "knowledge" being transferred, and chose to focus on "bridging" instead, a metaphor first used by Perkins and Salomon (1988) to describe "a process of abstraction and connection making" (p. 28). Following this notion, Geraniou and Mavrikis (2015) designed a series of "bridging activities" which assisted students in making the connections between the digital tool and the mathematics. These bridging activities included *consolidation tasks* that asked students to reflect on their interactions with the software, *collaborative tasks* that focused on asking students to justify whether their rules were equivalent or not to each other, *tool-like paper* tasks that looked like the software tasks but were on paper, and finally textbook or exam-like tasks. This variety in the design of the bridging activities can be seen as aiming to expand what Pratt and Noss (2010) refer to as students' *contextual neighborhood*, or the range of contexts and variety of circumstances in which the students' knowledge is made relevant and accessible.

In this paper, we describe our efforts in assisting students to expand their contextual neighborhood of measurement through dynamic measurement (which we describe in the next section), and discuss how our task design may have helped them bridge their dynamic digital experiences with typical static measurement tasks.

Dynamic Measurement (DYME)

DYME draws on research on visualizing area as 'sweeping' (e.g. Lehrer, Slovin, Dougherty, & Zbiek, 2014; Thompson, 2000) to engage students in digital experiences of visualizing the multiplicative relationship of length *times* width that underlies the area formula of a rectangle. To understand this approach, imagine a paint roller of length *a* being swept over a distance of *b* (width) to generate a rectangle of area *ab* (Figure 1). DYME presents area as a continuous dynamic quantity that depends on both the length of the roller and the length of the swipe. Aiming to explore students' DYME reasoning for area, we conducted two cycles of design experiments (DE) (Cobb, Confrey, Lehrer, & Schauble, 2003) and developed an interactive book of tasks on Geogebra [www.montclair.edu/csam/DYME] and a learning trajectory (LT) (Simon, 1995), illustrating how students' thinking of DYME may progress over time.

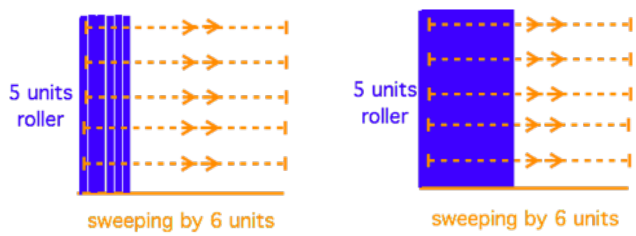
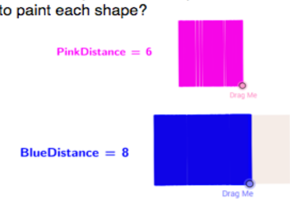
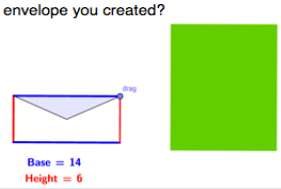
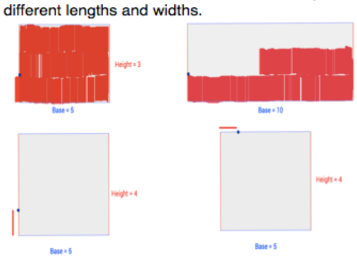
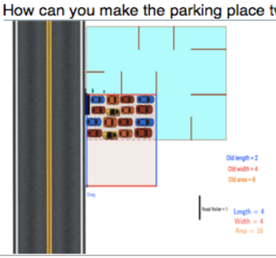
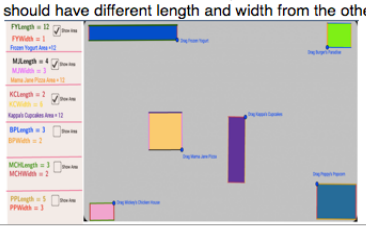
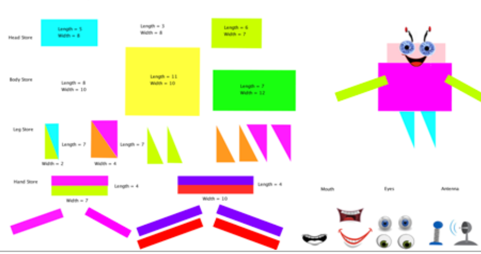


Figure 1. Area as a continuous structure using the 'sweeping' approach (Panorkou, 2017)

Table 1 presents the LT, which consists of a set of DYME constructs from less sophisticated to more sophisticated (levels), a sample Geogebra task for each construct, and a set of observable student generalizations that are more likely to occur at each construct. The goals of Levels 1-3 are for the students to build the idea of 2D space by visualizing area as a continuous structure that can change dynamically through 'sweeping,' recognize that the measurement of a surface requires the coordination of two dimensions (e.g. Reynolds & Wheatley, 1996), and recognize the multiplicative relationship between the two dimensions of a rectangle and its area (Izsak, 2005). To help students identify this relationship, the tasks involve the use of a 1-inch roller to paint shapes of different lengths and widths and constructing a repeating pattern for covering the shape (e.g. Outhred & Mitchelmore, 2000), by considering the distance covered in one swipe with the number of swipes. Subsequently, Levels 4-6 present an exploration of what else is possible to learn after students develop their DYME thinking, therefore they do not follow a specific order (although in this paper we refer to them "levels".) For instance, in Level 4 students use their DYME knowledge of area to identify the effects on the dimensions when the area of a shape is scaled, or in Level 5 students explore dimensions as factors that may give the same area.

Table 1. The hypothetical learning trajectory for DYME

Stage	Example of a task from the Geogebra book	Example of Students' behavior
1. Exploring dimensions and area as continuous attributes: Students visualize area as a continuous structure that can change dynamically.	Drag the paint roller to paint each shape. How far did you drag the paint roller to paint each shape? 	**a.** Students recognize that the dimensions, length of the roller and the distance dragged, define how big or small a shape is, e.g. "*The more I drag the roller, the more space I cover*" "*The longer the roller, the more space it covers.*" **b.** They also form relationships between the dynamic action of painting and the dimensions of the shape being painted, e.g. "*The length of the roller needs to be the same as the height of the rectangle*" "*The distance of paint is same as the base of the rectangle.*"
2. Coordinating two dimensions to compare area: Students recognize that the measurement of a surface requires the coordination of two dimensions.	Modify the envelope to fit the size of the card! How big is the envelope you created?  What do we need to change? What stays the same?	Students recognize that to compare two shapes, they need to compare both dimensions, e.g. "*The height needs to change, the base needs to stay the same.*"
3. Multiplicative Relationship of length, width and area: Students recognize the multiplicative relationship between the two dimensions of a rectangle and its area.	Students are asked to use a 1-inch roller to paint surfaces of different lengths and widths.  How far did you drag the roller? How many swipes did you need to cover the wall? How much space did you cover?	**a.** Students use the multiplicative 'times' language to find the space covered, e.g. "*This is 30 because the base is 10 and we are going to swipe three times.*" **b.** Students recognize that a roller of size *l* covers the same area as *n* rollers of size *l/n* dragged for the same distance, e.g. "*A 3-inch roller covers the same as three 1-inch rollers. If we cut it into 3 parts and you go across one time it is 10 and then if you go across another time it will be 20 and if you go one more time it will equal 30.*" **c.** Students recognize that the space covered can be found by the length of the roller *times* the base of the rectangle or the distance of swipe, e.g. "*The length of the roller is 3 and the distance is 10, so area is 3 times 10.*" **d.** Students using "length of roller" and "height" interchangeably, and this intuitively leads to *height times base*, e.g. "*The length is 3 and the base is 10. To find area, I did 3 times the base, which is 30.*"
4. Multiplicative Coordination of length and area: Students recognize that length, width and area are coordinated through multiplicative comparisons.	How can you make the parking place twice as big as it is now? 	**a.** Students identify that they can double or triple areas by multiplying only one of the dimensions by the same factor, e.g. "*To double the area, we can double the length or the width.*" **b.** Students recognize that in order to split area (fractional thinking), they need to split the length or the width, e.g. students create a cafeteria which is 1/4 of an 8 by 5 inches garden and argue: "*If we split this [the height of the cafeteria] into four parts, then one of the parts will be the cafeteria. It would be 2 inches because the if we use only 1-inch roller it would go 8 times across but if you use 2-inch roller then it would go 1,2,3, and that would go 4 parts.*"
5. Identifying area as a multiple of its dimensions: Students recognize area as a multiple of its dimensions and identify factors that give same area.	Please design the stores in such a way so that each of the food stores will have an area of 12 sq. meters. Also, each food store should have different length and width from the other stores! 	**a.** Students use the commutative property to compare areas, e.g. "*A rectangle with length 4 and width 3 has the same area with a rectangle of length 3 and width 4.*" **b.** Students identify factors that give same area, e.g. "*Length 4 and width 3 is doing 4 swipes of 3. This is same as two swipes of 6, so length 2 and width 6*"
6. Multiplicative Coordination of relative areas: Students recognize that if a shape is 1/n of another shape, then its area is 1/n of the area of that shape.	Make the robot as fancy as you like but its area should be no more than 190 sq. inches. 	Students recognize that if a right triangle is half a rectangle, then the area of that triangle is half the area of the rectangle, e.g. for calculating the area of each blue leg, the students argued: "*You have to do a half of 7 and 2. We do 7 times 2 and a half of that.*"

The first two cycles of DE showed DYME's potential for making the multiplicative relationship of the area formula more intuitive and accessible to students (e.g. Panorkou, 2017). However, the nature of DYME tasks is very different than any of the measurement tasks that students encounter in their classrooms and many of the generalizations they make are situated in the specific context of DYME. Therefore, in Cycle 3 we wanted to test out whether students could connect these situated experiences to their other measurement experiences. Our task design and questioning included many tasks that could help them do that. First, we used consolidation tasks that asked students to reflect on their interactions with the software, reflect on their learning in every session and generalize their strategies. An example of the latter was when we asked the students to advise a painter of how to measure the space of any rectangular surface. Second, we used whole class discussions at the end of each session that included collaborative tasks asking students to share different strategies of solving a task and discuss why different strategies generate the same answer. For instance, finding the space of a rectangular surface of 4 inches by 5 inches, by using 4 one-inch swipes of 5 inches or 2 two-inch swipes of 5 inches or using 4 inches *times* 5 inches. Third, in contrast to Geraniou & Mavrikis (2015) that used software-like paper tasks, we used paper-like dynamic tasks, that are similar to what students would encounter on paper but with technological affordances, such as presenting them with a rectangular surface which they can make bigger or smaller by modifying the length and width through dragging (e.g. Level 2 task in Table 1). We conjectured that this kind of design would help students build connections between DYME and other types of measurement.

Aims

This article describes our efforts to investigate how a whole class design experiment (DE) (Cobb et al., 2003) on DYME could help students develop their thinking of area as a multiplicative relationship and whether students are able to transfer knowledge gained from interacting with the DYME tasks to the static traditional area tasks they would encounter in the classroom. More specifically, our goals were to explore:

1. To what extent did the students develop their thinking of area as a multiplicative relationship as a result of the design experiment and the use of the DYME tasks?
2. To do extent did the students bridge the experiences gained from the dynamic environment of DYME tasks to traditional area tasks they encounter in the classroom?

Methods

Nineteen third-grade students from an elementary school in the Northeast participated in the whole class DE (Cobb et al., 2003). The students already had formal instruction on multiplication and area measurement using the common approach of using square units. The DE consisted of six 50-minute periods of instruction using the DYME Geogebra tasks.

Assessment Design

To answer our research questions, we designed and administered an assessment to students at the very beginning of the DE and at the end of the learning experience. The items in both pre- and post-assessments were identical aiming to create an initial and final "profile of strengths and weaknesses" (Huhta, 2008, p. 470) for each student. The assessments were pilot tested with a separate group of students and revised before they were used for the present study. Aiming to examine both the development of students' thinking of area and the extent of bridging, we designed the items based on typical measurement assessments used in the literature (e.g. Battista, 2004) and on questions found in standardized test assessments, such as PARCC. All the items were aligned to a level of the LT and were designed to provoke responses that could indicate how students' reasoning has changed following their interaction with the tasks during the DE.

For example, one of the DYME goals is for students to recognize that *a* swipes of *b* cover the same amount of space as *b* swipes of *a*, what we refer to as commutativity in DYME. To do that, we engage students in dynamic tasks, such as the one presented in Figure 2 (left), where they use paint rollers of different orientations (horizontal/vertical) in order to find area and generalize that 4 swipes of 5 cover the same space as 5 swipes of 4. Instead, typical measurement assessments evaluate the commutative property in static tasks similar to the Question A in Figure 2 (right), which assesses if students recognize that a 3ft by 4ft tabletop has the same area as a 4ft by 3ft tabletop. Question A was used as an item in our pre- and post-assessment. Each assessment included three to four items corresponding to each construct of the LT and these items were evenly distributed throughout the paper so that no two consecutive items were associated with the same LT construct.

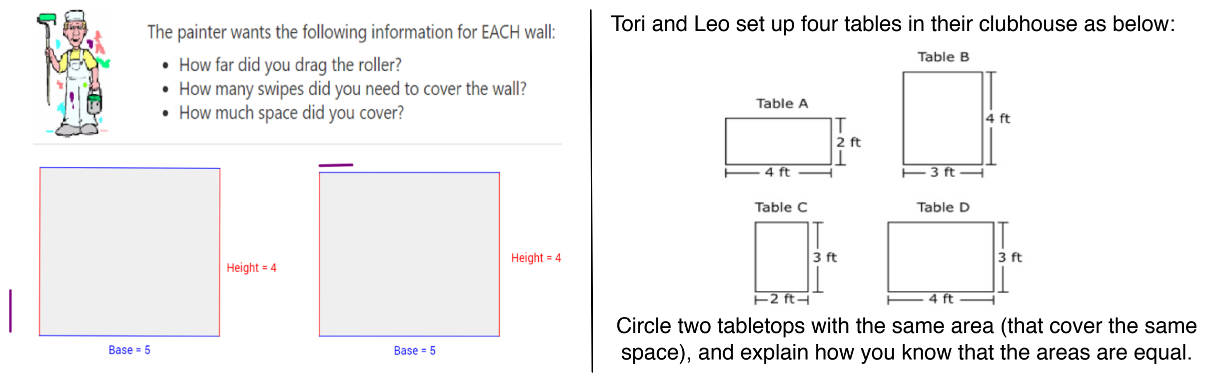


Figure 2. DYME Geogebra task (left); Question A, adopted from the PARCC assessment (right).

Assessment Analysis

For analyzing the students' work, we read every student's response and generated categories to capture the themes in their responses. We then developed a scoring protocol to measure the range of sophistication in their responses. To score the responses, we adopted Norton and Wilkins' (2012) suggestion: 1 for indication of a particular reasoning level and 0 for counter-indication of a particular reasoning level. The levels of reasoning adopted were consistent with the LT levels in Table 1. For instance, a response showing a Level 5 understanding was receiving a point for Level 5, while a response showing a Level 3 understanding was receiving a point for Level 3. The responses were scored by four researchers independently and then negotiated aiming to maximize reliability.

For an illustrative example of the whole process, consider Question A in Figure 2 (right). Table 2 presents the scoring rubric we developed for Question A. In this question, we noticed three different levels of reasoning corresponding to constructs 2, 3 and 5 of the LT. Depending on how the student responded, they would get a point for that construct. For example, if a student used the commutative property to find equal rectangles, they received a point only under level 5.

Results

After each student response was scored, we summed the total scores received by all students under each level (L1 - L6) and color-coded the responses based on these scores in a contingency table (Norton & Wilkins, 2012) (Table 3). As aforementioned, each assessment included three to four items corresponding to each level of the LT. For instance, we included four different items in which students could use level 5 reasoning to respond. If a student received 3 points of level 5 that meant that the student used level 5 reasoning to respond to 75% of the level 5 questions.

Table 2. Rubric for Question A created based on students' responses

Rubric	Example of students' responses	Scoring justification
The student expresses commutative reasoning to find the two rectangles with same area. (1pt for Level 5)	"Table B and table D, because 3x4 = 12, 4x3=12."	Response is aligned to LT level 5. The student exhibits an understanding of the commutative property.
The student multiplies the length and width of the rectangles and compares their area to find rectangles of equal area. (1pt for Level 3)	"I know they are equal because if you multiply them they have the same area."	Response focuses on students' calculation and comparison of area (LT level 3), but does not explicitly state that the commutative property is used.
The student coordinates the length and width of rectangles simultaneously to make judgement about their area. (1pt for Level 2)	"They are same because there is a 4 and a 3."	Response focuses on the numerical values of the sides of the rectangles to make judgements (LT level 2), but does not discuss the order of the numbers.
The student answers correctly but provided vague or no reasoning. (0 points)	"Because when I circle the table. So, I think I should circle the same table."	Response does not provide any description of students' thinking.
The student answers incorrectly. (0 points)	6x5, 3x5, 6x3, 5x6.	Response is illegible.
The student does not respond. (-)		No response.

If a student's overall raw score for a given level was above 60%, then it was inferred that the student reached proficiency in that level and the cell under that level was colored dark grey. If a student's overall score for a particular level was between 30%- 60% the cell was colored grey. For students whose total scores in any level of understanding were less than or equal to 30%, the cells were colored light grey to indicate that students showed some instances of the particular level of understanding. Some students did not answer all of the questions, and thus white cells indicate missing responses. Dotted lines (--) indicate students who did not take the assessment. Table 3 compares students' responses to the pre- and post-assessments showing the extent to which the students' thinking of area as a multiplicative relationship was developed due to their interaction with the DYME tasks. As Table 3 illustrates, out of the 19 third-grade students in the whole class design experiment, only 3 were able to think multiplicatively about area in the pre-assessment (represented by level 3). However, in the post-assessment, 15 students showed an understanding of area as a multiplicative relationship of length *times* width. These results showed that not only students' thinking of area as a multiplicative relationship was developed by engaging with the DYME tasks, but also, that they were able to use their DYME experiences to solve traditional area tasks.

Table 3 also shows that in each level, students understanding was developed from the pre-assessment to the post-assessment (with the exception of level 1 showing that students moved beyond that level.) This is illustrated both by the darker shading, which is more prominent for the post-assessment, and also by the increase in the average score in every level in post-assessment compared to pre-assessment. For instance, in level 3, students scored an average of .2 during pre-assessment, which increased to 2.0 during post-assessment. Similar development in students' thinking of area is observed in other levels, especially in level 4, 5, and 6 where the average score of students' responses increased from 0 to 0.6, 0.14 to 1 and 0.07 to 0.6 respectively.

Table 3. Contingency table showing the total scoring of the assessment questions

Student	Pre-assessment						Post-assessment					
	L1	L2	L3	L4	L5	L6	L1	L2	L3	L4	L5	L6
A							--	--	--	--	--	--
B												
C												
D												
E												
F												
G												
H	--	--	--	--	--	--						
I	--	--	--	--	--	--						
J												
K												
L												
M	--	--	--	--	--	--						
N	--	--	--	--	--	--						
O												
P												
Q												
R	--	--	--	--	--	--						
S												
Average	1.35	1.93	0.2	0	0.14	0.07	1.11	1.94	2	0.6	1	0.6

Although students exhibited an overall improvement in their level of understanding in the post-assessment compared to the pre-assessment, we found some exceptions in some students' responses. For instance, though students E and J showed proficiency in coordinating the two sides of a rectangle (Level 2) during pre-assessment, they showed a decrease in their responses in the particular level during the post-assessment. This was most likely because during the post-assessment, the level of understanding of the two students in the certain questions moved beyond level 2 and they showed a higher degree of understanding in level 3 and 5. Another case is student L, who reached level 5 during the pre-assessment but did not proceed beyond level 4 during the post-assessment. On further analysis, we found that student L did not answer the last 9 questions in the post-assessment, which explains the decrease in the level of understanding.

Significance

Researcher: What have you learned all these days?
Student 1: I learned that there are other ways to measure length and width. You can use objects like paint rollers.
Student 2: Yes, and that to double area we need to double one of the measurements.
(*Excerpt from a design experiment after a series of sessions using DYME tasks*)

The findings from the pre- and post-assessment analysis show that students' engagement with the DYME tasks helped them improve their understanding of area. The data analysis also showed that students were able to bridge their DYME experiences and generalizations with typical area tasks. Students extended their contextual neighborhood of measurement to include sweeping-based reasoning as well as reasoning about area as length *times* width. Students not only developed their understanding of area as a multiplicative relationship but they also used this knowledge to respond to more advanced questions in levels 4-6. Among our future goals are to examine the order in the upper LT levels (4-6) and also to further examine the type of activities that assist students in bridging these connections between DYME and other area generalizations.

Acknowledgements

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References

Battista, M. T. (2004). Applying cognition-based assessment to elementary school students' development of understanding of area and volume measurement. *Mathematical Thinking and Learning*, 6(2), pp.185-204.

Broudy H. S. (1977) Types of knowledge and purposes of education. In Anderson R. C., Spiro R. J. & Montague W. E. (eds.) *Schooling and the acquisition of knowledge*. Erlbaum, Hillsdale NJ, pp. 1–17.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational researcher*, 32(1), pp. 9-13.

diSessa A. A. & Wagner J. F. (2005) What coordination has to say about transfer. In Mestre J. (ed.) *Transfer of learning from a modern multi-disciplinary perspective*. Information Age Publishing, Greenwich, CT, 121–154.

Doerr H. M. & Pratt D. (2008) The learning of mathematics and mathematical modeling. In Heid M. K. & Blume G. W. (eds.) *Research on technology and the teaching and learning of mathematics*. Volume 1: Research syntheses. Information Age Publishing, Charlotte NC, pp. 259–285.

Geraniou, E., & Mavrikis, M. (2015). Building bridges to algebra through a constructionist learning environment. *Constructivist Foundations*, 10(3), pp. 321-330.

Huhta, A. (2008). Diagnostic and formative assessment. In B. Spolsky & F. Hult (Eds.), Handbook of educational linguistics (pp. 469–482). Malden, MA: Blackwell.

Izsak, A. (2005). "You Have to Count the Squares": Applying Knowledge in Pieces to Learning Rectangular Area. *The Journal of the Learning Sciences*, 14(3), pp. 361-403.

Lehrer, R., Slovin, H., Dougherty, B., & Zbiek, R. (2014). *Developing essential understanding of geometry and measurement for teaching mathematics in grades 3–5*. Reston, VA: NCTM.

Mason J. (2015) Bringing Reflection to the Fore Using Narrative Construction. *Constructivist Foundations,* 10(3), pp. 334–335.

Norton, A., & Wilkins, J. L. (2012). The splitting group. *Journal for Research in Mathematics Education*, 43(5), pp. 557-583.

Noss R. & Hoyles C. (1996). *Windows on mathematical meanings: Learning cultures and computers*. Kluwer, Dordrecht.

Outhred, L. N. and Mitchelmore, M. C. (2000). Young children's intuitive understanding of rectangular area measurement, *Journal for Research in Mathematics Education*, 31(2), pp. 144–167.

Panorkou, N. (2017). Dynamic Measurement: The crossroad of area and multiplication. In Galindo, E. & Newton, J. (Eds.) *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Indianapolis, INL Hoosier Association of Mathematics Teacher Educators (pp. 339-346).

Perkins D. N. & Salomon G. (1988) Teaching for transfer. *Educational leadership*, 46(1), pp. 22–32.

Pratt D. & Noss R. (2010) Designing for mathematical abstraction. *International Journal of Computers for Mathematical Learning*, 15(2), pp. 81–97.

Reynolds, A., & Wheatley, G. (1996). Elementary students' construction and coordination of units in an area setting. *Journal for Research in Mathematics Education*, 27(5), pp. 564–581.

Salomon G. & Perkins D. N. (1987) Transfer of cognitive skills from programming: When and how? *Journal of Educational Computing Research*, 3, pp.149–169.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 26(2), pp.114-145.

ENTERING EQUATIONS: COMPARISON OF HANDWRITING RECOGNITION AND EQUATION EDITORS

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Once a novelty, Digitally-Based Assessments (DBA) have become commonplace in the USA. With mathematics, it is often a necessity to include items that require the student to input a mathematical formula, equation, or expression. Many of these responses, especially in the upper grades, cannot be input with a standard keyboard, but must use some type of equation entry. In this study, we compare ninth-graders' entry of mathematical expressions using an equation editor versus using handwriting recognition on a tablet. While neither method is currently without flaws, we discuss the benefits and drawbacks of each as well as potential methods for improvement and the implications for mathematics assessment.

Keywords: Assessment and Evaluation, Technology

Introduction

Mathematics has long been a subject of paper and pencils. Scratch work, diagrams, and mark-ups are all parts of solving mathematical problems that need to be addressed as part of developing Digitally-Based Assessments (DBA). One artifact of the paper-and-pencil world is that of the mathematical response itself. Most mathematical formulae, equations, and expressions cannot be input properly using a standard keyboard. Even at the elementary level, a simple fraction requires an equation editor, as most students are used to seeing the numerator directly above the denominator ($\frac{1}{2}$), not the orientation that would result from a sideways slash (1/2). At the middle-school level, exponents, square roots, and π further complicate the mix. Finally, at the high-school level, all of these are combined together in various embedded formats that can confuse even those who are comfortable with the individual symbols.

The most common solution to this open-ended response problem is to use an equation editor (see Figure 1).

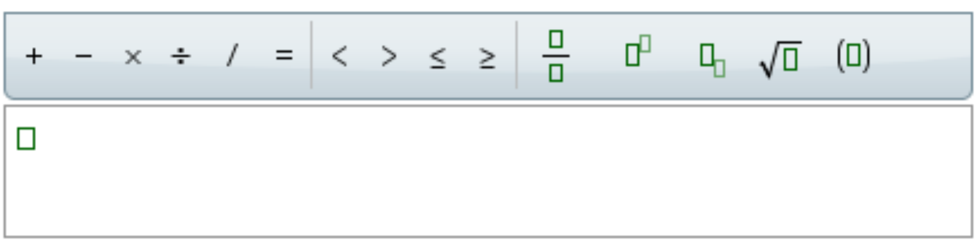

Figure 1. Sample Equation Editor Entry Box

While equation editors allow for precise entry of mathematical expressions, they do add an extra burden on the student. These additional difficulties can be both construct-related (i.e., students who struggle with mathematics may struggle more to use the equation editor due to not understanding the various mathematical symbols, orders of entry, etc.; see Noyes, Garland, & Robbins, 2004) and construct-irrelevant (i.e., students who have less exposure to equation editors, regardless of mathematical ability, may require extra time to identify and select the proper symbols and where to click or type; see Leeson, 2006). Hargreaves et al. (2004) also showed that students may solve problems differently when presented assessments through different media.

Tablets and other devices that allow for handwritten digital entry could resolve some of this burden, but unless those handwritten responses can be automatically scored with the same ease as the typed equation editor responses, the cost (in both time and money) of scoring the assessment becomes too great for this to be a reasonable solution. Thus, we cannot administer assessments with digital handwritten entry without automated handwriting recognition. In this paper, we discuss a study intended to be the first in a series of experiments aimed at first identifying differences between equation editor entry and automatically translated handwritten entry and then addressing the challenges of each towards the development of a solution that provides the most authentic experience as possible for students that minimizes construct-irrelevant difficulties.

Study and Research Questions

As stated above, this study describes the first year of a multi-year study aiming to understand and improve equation entry for mathematics assessments. Our goal for the first year of the project was 1) to demonstrate that we can score equations and expressions that have been captured on a tablet (Apple iPad) via handwriting with a stylus, then automatically translated into MathML (Ausbrooks et. al., 2010) via the translator MyScript (Vision Objects, 2017) and 2) to determine if there is a difference in performance when students enter responses on a computer, using an equation editor (WIRIS, see Maths for More, 2017), versus entering responses on a tablet. If we suppose that prior evidence with paper and pencil is an appropriate stand-in for tablets, research would suggest that it is easier for students to handwrite responses on a tablet using a stylus than to enter them on a computer using an equation editor and that students are less likely to make construct-irrelevant errors (see Hargreaves et al., 2004). The ability to administer assessments on a tablet, and to score the responses automatically, will increase flexibility and the attractiveness of digitally based assessments to both stakeholders and test-takers. For this study, we had students copy equations directly from a screen to study only the entry of the equation itself and minimize any effect that the method of entry may have on the solution strategy prior to equation entry.

We designed a study to answer the following research questions:

1. What are the potential causes of errors and variability in score differences with automatically scored equation responses that have been captured on a tablet via handwriting with a stylus?
2. Is there a difference in student performance between responses handwritten on a tablet and automatically translated into MathML and responses typed on a computer using the equation editor WIRIS?
3. Are the error rates of the handwriting recognition comparable to (or better/worse than) the errors students make when entering equations into an equation editor?

Methods

We developed two parallel forms (termed Form A and Form B) of a mathematics assessment that would each be administered on computer (using an equation editor) and on an iPad (using handwriting recognition). In both forms, students are asked to copy equations directly from the screen. The forms had different equations, yet we developed them to be as parallel as possible in terms of mathematical complexity, the necessary use of equation editor templates, and the number of characters. For the first half of the assessment, students are looking at the equation while they copy it, and for the second half, students must hide the stimulus while copying, but they are allowed to go back to view the stimulus as many times as necessary. All students took both forms. One-half of the participants handwrote their responses to the item on the first assessment on a tablet with a stylus and entered their responses to the items on the second assessment on a computer using an equation editor. The other half of participants completed the two assessments in the opposite order. The two groups of students were further subdivided into Form A or Form B and counterbalanced for order. Thus there were four groups, into which participants were randomly assigned (see Table 1).

Table 1: Counterbalanced design

	First Task: Computer Second: iPad	**First Task: iPad Second: Computer**
Form A: iPad Form B: computer	Group 2	Group 1
Form A: computer Form B: iPad	Group 3	Group 4

In each form, students were asked to copy 20 mathematical equations that ranged from very simple to enter (e.g., 6+2=8) to complex (e.g., one form of the quadratic formula). No equation exceeded in difficulty or complexity that which a student would see as part of a typical Algebra I class. For the computer (equation editor) form, all responses were automatically scored using a proprietary mathematical scoring engine (m-rater) that works through MathML. For the tablet (handwritten responses), all responses were automatically translated into MathML and then scored using the same scoring engine. For this study, we used the WIRIS equation editor, which is widely used in K-12 mathematics assessments and the MyScript Handwriting recognition tool, which has a popular iOS handwritten mathematics calculator and is also used in various mathematical contexts by some leading tech designers.

See Figure 2 for the process of translating the handwritten responses into MathML.

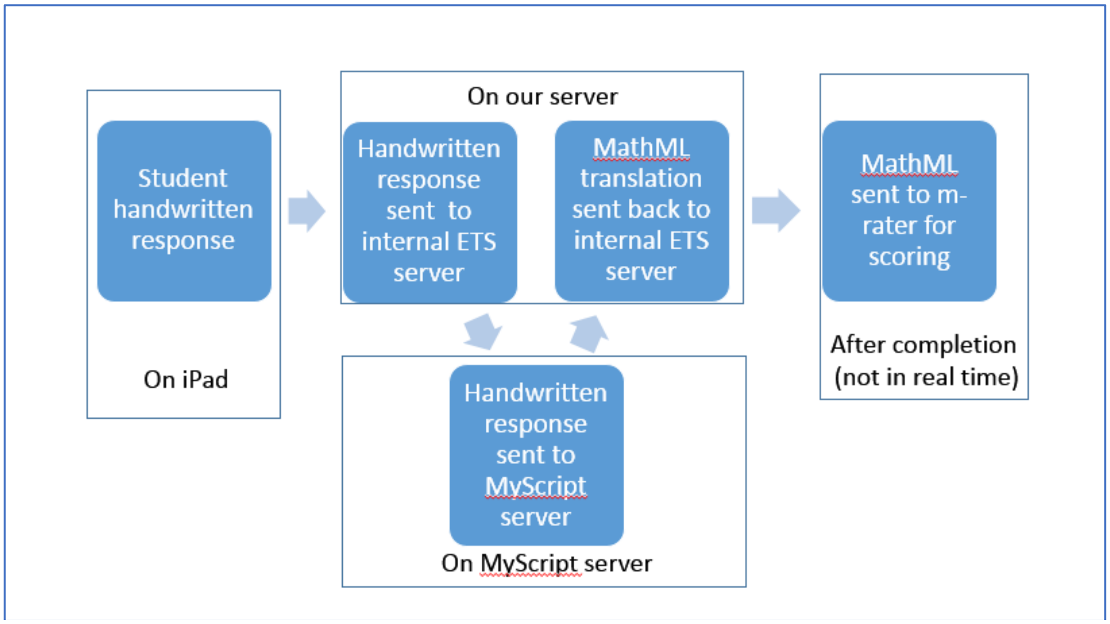

Figure 2. Handwriting to MathML process

Sample and Data Collection

We recruited 9th-grade students from 4 high schools in the USA from racially, culturally, and socioeconomically diverse areas (35% of students qualified for Free/Reduced Lunch of those that reported). Overall, 474 9th-grade Algebra students completed the study (204 Male, 265 Female, 5 Nonreported). The final sample was 45.8% White, 25.9% African American, 10.1% Asian American, 9.7% Hispanic/Latino, and 8.4% other). All students participated during their regular mathematics class and we randomized the experimental groups to which students were assigned within classes (i.e., each class had students placed into each of the four groups). We encountered some connectivity issues discovered after administration whereby 143 students in varying classes and schools did not receive MathML translations from the MyScript server as well as some various missing data with 22 students. Therefore, analyses were conducted with the remaining 309 students so as to maintain the counterbalanced design.

Results

The students received one point for each response that was an exact match to the target equation (in the case of the handwritten responses, one point was given per response in which the computer translation was an exact match to the target). No credit was given for partial responses. Overall, the automatically-produced scores were higher with the equation editor than with the translated handwriting (Form 1: iPad average 10.96 (out of 20), Computer Average, 15.46 (out of 20); $t(156) = 13.4$; $p < .001$; Form 2: iPad average 10.59 (out of 20), Computer Average, 15.5 (out of 20); $t(145) = 11.5$; $p < .001$).

It is worth reiterating that these scores are a reflection of the computer translated handwriting not a score obtained from the writing itself. In other words, we are not comparing what a student wrote with what they typed, but rather how well their handwriting translated into the correct scoreable form versus what they typed. Thus, we can state that the student responses entered via equation editor far outperformed the computer translated handwritten responses, but we make no statement as to the writing on the iPad itself and whether it would have been scored correct or incorrect by a human rater. We do this because it is that final translated response that is the subject of the viability of this as an option for large-scale assessment, which is the focus of this multi-year project.

Despite the higher overall scores for the equation editor, not all items are created equal. For some items, scores between the conditions were about equal (and not significantly different) while for others there were large differences. Figure 2 shows the differences for all scores in both Form A and Form B. As can be seen, both forms produced very similar scores per item, no differences between Forms are significant.

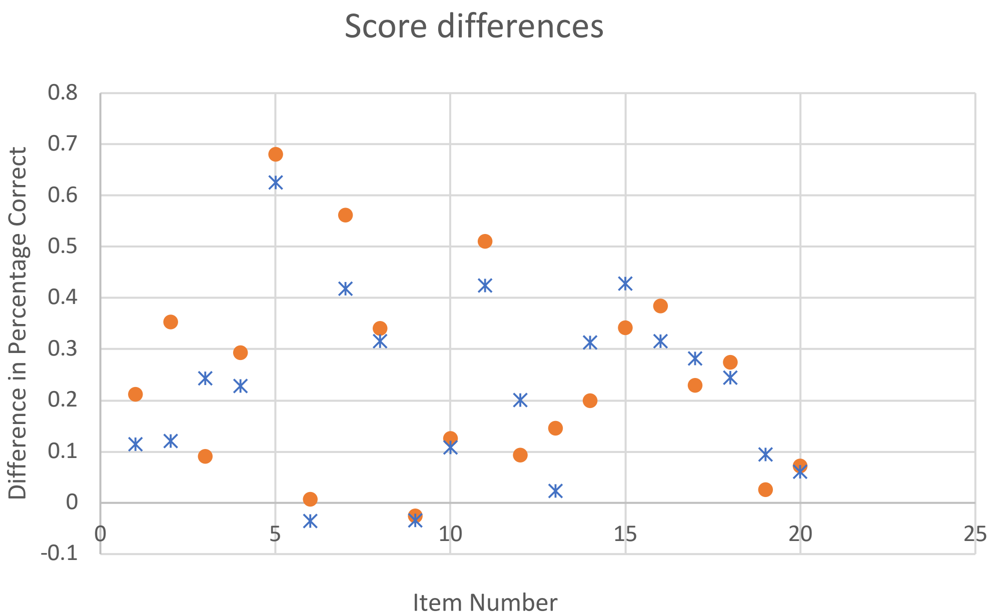

Figure 3. Score differences, equation editor minus tablet (Form A in blue square and Form B in orange circle)

As stated in our first research question, we wished to not only look at overall performance, but to better understand the potential causes of the variability in score differences. To do so, we looked at individual examples of high and low difference items. As an illustration, Table 2 lists the three items with the smallest and largest differences in percentage correct over both Form A and Form B.

Table 2: Items with Greatest and Smallest Differences in Percentage Correct

Item	Equation editor mean	Handwritten mean	Difference (Equation Editor minus Handwriting)
Smallest Differences			
$x+y=y+x$	87.4%	86.7%	0.7
$4 \times 2 = 8$	88.7%	86.3%	2.4
$y = ax^2 + bx + c$	75.4%	78.0%	-2.6
Greatest Differences			
$28 \div 4 = 7$	98.7%	30.7%	68.0
$36 \div 4 = 9$	92.0%	29.4%	62.6
$x(y+z)=xy+xz$	84.9%	28.7%	56.2

In taking a deeper dive into the characteristics of the equations with higher and lower handwriting recognition rates, we found that while variables (x and y) were recognized reasonably well, the multiplication symbol (×) was occasionally recognized as an x. Additionally, the equation $y = ax^2 + bx + c$ was one of the only items in which the handwritten mean was higher than the equation editor mean, largely because many students had difficulty using the template for the superscript, thus resulting in errors such as $y = ax^{2+}bx + c$ and similar errors.

As can be seen in two of the examples with the greatest difference in response type, the division symbol (÷) had a fairly low recognition success rate, and even though variables were recognized well, once parentheses were used along with the variables, recognition dropped substantially.

However, this is clearly only one part of the story. As part of this exploration of the differences, we felt it was also important to look at timing differences. Our rationale was that if equation editors do allow for higher performance but also take much longer to use, there may be some tradeoff between time and accuracy.

Figure 3 shows the difference in average time taken on the item for both Form A and Form B (computer minus iPad). As can be seen in Figure 3, timing scores for both forms were nearly identical, with both forms having high outliers for items 1 and 11. Both of these items occurred at the beginning of a new section in each form, and thus it appears that the time difference is more a product of the students on computers taking longer to read the directions and begin typing than it is of the item itself. For the rest of the items, differences tended to range between zero and ten seconds, though we did see a great amount of variability between students such that some students had virtually no difference while others took much longer for the computer entry than the iPad. The exploration of individual student characteristics will be the focus of a future analysis.

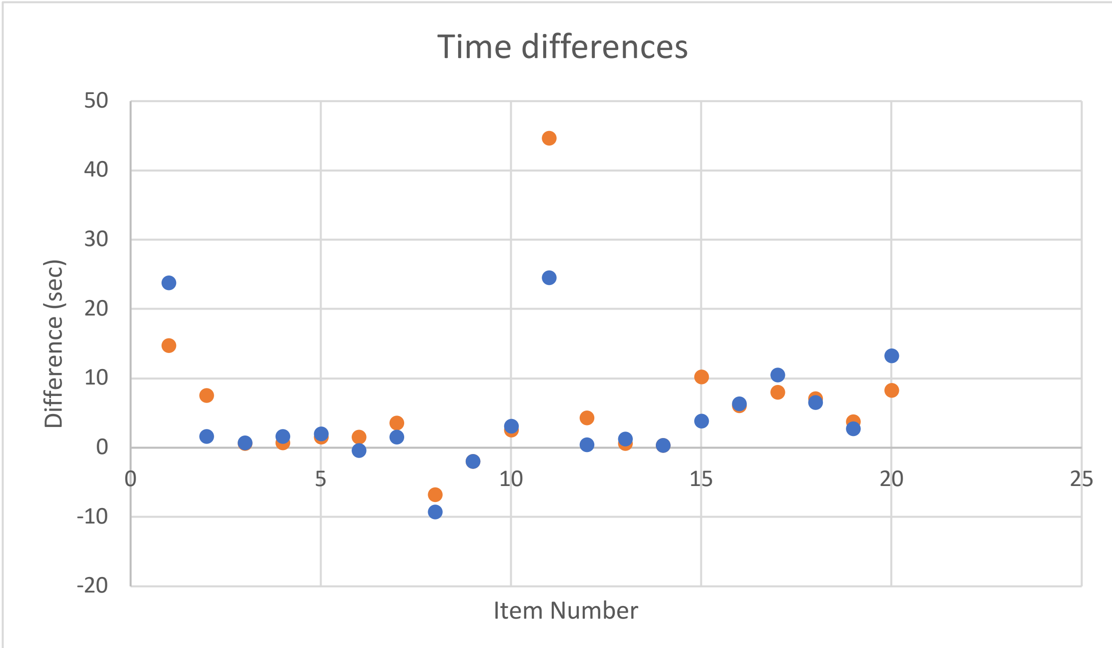

Figure 4. Timing Differences, equation editor minus tablet (Form A in blue and Form B in Orange)

Overall, using current technology, automatically produced scores were much higher when equations were entered using a standard equation editor than when using handwriting recognition software. While those equations did take slightly longer to enter, the timing differences were not large enough for this mode of entry to result in more than a few minutes of extra testing time (depending on the number of equations being entered). One caveat is that this was for copying equations. It would be interesting to see if timing differences were similar when students were asked items in which they had to generate equations, as we could see some compounded differences with uses of scratch paper, etc.

Discussion

What should we conclude about the future of DBA mathematics response entry on the basis of this study? While on the surface, we seem to have found that equation entry is currently superior to handwritten responses, we also uncovered important difficulties with equation editor responses that are particularly troubling considering this is the current preferred response method. For example, equation editor responses ranged between 70% and 95% correct. Considering that students were copying equations, a 70% correct response rate means that 30% of students did not accurately copy an equation using a typical editor that has been the standard of DBA. While these rates did outperform translated handwriting, we see this as more of an indicator that handwriting translation technology needs to be improved so that it can eventually replace the much troubled equation editors, as opposed to an indicator that equation editors are superior for response.

We are currently planning a follow-up study in which students are able to see their handwritten translations in real time and make corrections to their writing to see if this capability enables students to raise their own rates of recognition. We are also looking into ways to limit the lexicon of the recognition software to only include those characters which are part of the Algebra I curriculum (i.e., exclude most Greek letters, integrals, and derivatives). We also plan to see if a tutorial on how to use the equation editor may improve equation editor responses. These improvements to both of these entry methods would need to be considered along with how much time it may cost the test-taker. For instance, watching a tutorial would be a one-time cost of a few minutes, while the ability to see the handwriting translation could potentially cost test-takers a lot of time if they need to correct every individual item (some more than once).

Conclusion

Overall, it is true that we should have some concerns about responses to DBA mathematics items that require students to enter mathematical equations or expressions, regardless of entry mode. Equation editor items should potentially allow for some leeway in scoring on responses that indicate the student may have had entry difficulties (e.g., where it appears students have had trouble knowing how and where to use a template). Additionally, we should also not discount the future of handwritten entry, though the technology is not currently up to the state it should be for assessment use. Since DBA mathematics assessment is here to stay, and growing in use, we need to continue this line of work to improve the entry capabilities to the point that they seamlessly allow students to show what they truly know in the mathematics and not be limited by the technology.

References

Ausbrooks, R., Buswell, S., Carlisle, D., Chavchanidze, G., Dalmas, S., Devitt, S., & Kohlhase, M. (2010). *Mathematical markup language (MathML) version 3.0*. Tech. rep. World Wide Web Consortium (W3C), 2010. url: http://www. w3. org/TR/MathML3.

Hargreaves, M., Shorrocks-Taylor, D., Swinnerton, B., Tait, K., & Threlfall, J. (2004). Computer or paper? That is the question: does the medium in which assessment questions are presented affect children's performance in mathematics?. *Educational Research*, 46(1), 29-42.

Leeson, H. V. (2006). The mode effect: A literature review of human and technological issues in computerized testing. *International Journal of Testing*, 6(1), 1-24.

Maths for More (2017). WIRIS equation editor. url: http://www.mathsformore.com

Noyes, J., Garland, K., & Robbins, L. (2004). Paper-based versus computer-based assessment: is workload another test mode effect?. *British Journal of Educational Technology*, 35(1), 111-113.

Vision Objects (2017). Myscript handwriting recognition engine.

RAZONAMIENTO DIGITAL: REPRESENTAR, EXPLORAR Y RESOLVER PROBLEMAS VERBALES CON EL USO DE GEOGEBRA

DIGITAL REASONING: REPRESENTING, EXPLORING AND SOLVING WORD PROBLEMS THROUGH THE USE OF GEOGEBRA

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En este estudio se analizan y contrastan acercamientos que futuros profesores de matemáticas de bachillerato muestran al resolver problemas de palabras con el uso de papel y lápiz y, posteriormente, con el uso de un Sistema de Geometría Dinámica (SGD). Se analizan los recursos, representaciones, estrategias y formas de razonamiento matemático que exhiben los participantes cuando utilizan GeoGebra en el proceso de resolución de los problemas. Los resultados muestran que el uso de la herramienta favorece la exploración dinámica de los conceptos involucrados, la formulación de conjeturas y la búsqueda de distintos argumentos para validar la solución.

Palabras clave: Resolución de Problemas, Álgebra y Pensamiento Algebraico, Tecnología

Introducción

Los problemas verbales o de palabras aparecen en los planes y programas de estudio desde el nivel básico y representan un reto para los estudiantes en términos de desarrollar conceptos y estrategias eficientes para resolverlos. Este tipo de problemas, en general, se destacan como una forma de aplicar los contenidos que se estudian a ese nivel con la resolución de problemas situados en contextos de la vida real (Verschaffel, Depaepe, & Van Dooren, 2014). En este sentido, en algunos libros de texto se propone a los estudiantes un camino que se considera es el que debe seguir para resolver los problemas verbales de manera competente: 1) comprender el problema; 2) identificar los datos conocidos y desconocidos; 3) asignar a un dato desconocido la incógnita (generalmente representada con la letra x); 4) expresar los datos desconocidos que restan en términos de la incógnita; 5) formular la ecuación; y, 6) resolver la ecuación y comprobar el resultado (Rees & Sparks, 2005). Sin embargo, cuando un estudiante se enfrenta a la resolución de problemas verbales, la primera fase (comprender el problema) suele carecer de sentido, ya que los enunciados tienden a usar palabras clave como “más” o “la diferencia de” que guían inmediatamente la selección de alguna operación aritmética, fórmula geométrica o expresión algebraica (Verschaffel, Greer, & De Corte, 2000), la cual se resuelve sin contemplar la situación planteada. En este sentido, se identifica una necesidad por promover tareas que involucren problemas verbales donde los estudiantes puedan dar significado a los conceptos u objetos matemáticos involucrados en los problemas. De acuerdo con Santos-Trigo y Reyes-Martínez (2014), el uso coordinado de las tecnologías digitales puede ser un factor fundamental para alcanzar los objetivos deseados en la enseñanza de la matemática, ya que ofrece la posibilidad de representar, explorar, identificar, formular y resolver problemas. Además, Santos-Trigo, Reyes-Martínez y Aguilar-Magallón (2015) resaltan que, “los sistemas escolares en todo el mundo se enfrentan a un desafío de incorporar sistemáticamente la utilización coordinada de las tecnologías digitales en las propuestas curriculares y entornos de aprendizaje" (p. 298). ¿En qué medida el uso de un Sistema de Geometría Dinámico (SGD) influye en el desarrollo de recursos, estrategias y formas de razonamiento matemático en los estudiantes cuando resuelven problemas verbales? ¿Qué tipo de razonamiento matemático (representaciones, exploraciones, conjeturas, explicaciones parciales, etc.) caracterizan los acercamientos basado en el uso de tecnologías digitales? Con el objetivo de responder estas preguntas, en este estudio se presenta una ruta donde se incorpora el uso de un SGD en la resolución de problemas verbales y se analizan las formas de razonamiento que los estudiantes desarrollan sobre los conceptos y la manera en que influye en las actividades que caracterizan a la resolución de problemas.

Marco Conceptual

Resolución de problemas y el uso de tecnología digital

Schoenfeld (1985) propuso un marco que explica el comportamiento de los estudiantes en la resolución de problemas. Éste se centra en el domino de conocimiento conceptual y procedimental (recursos), estrategias de búsqueda para el análisis y transformación de problemas que aumenten la probabilidad de encontrar una solución (heurísticas), monitoreo y control de los procesos cognitivos (metacognición) y actitudes y emociones positivas hacia la tarea que se está desarrollando (creencias). No obstante, el marco se diseñó en un ambiente donde las herramientas principales eran el papel y lápiz. Cuando se incorpora el uso de tecnologías digitales en la resolución de problemas, no solo facilita aspectos como representar, buscar patrones o invariantes entre los objetos de un problema, que es esencial para que los estudiantes identifiquen relaciones y planteen conjeturas, sino también simplifica la implementación de estrategias y ayuda a extender el repertorio de heurísticas (Santos-Trigo, 2008).

Con base en estas ideas, Santos-Trigo y Camacho-Machín (2013) propusieron un marco que, además de permitir analizar los procesos que siguen los estudiantes cuando resuelven problemas con el uso de tecnología digital, da la posibilidad de transformar problemas rutinarios en no rutinarios. El marco consta de cuatro episodios: 1) comprensión: identificar los conceptos que se relacionan con el problema y plantear algunas preguntas que orienten la actividad; 2) explorar el problema: valorar diferentes caminos para alcanzar la solución; 3) búsqueda de distintas soluciones: analizar y discriminar los distintos resultados encontrados para proponer una solución argumentada, así como plantear nuevos problemas a partir de dichos resultados; y, 4) integración y reflexiones: discutir en torno a las actividades y resultados obtenidos.

Bajo este enfoque, los modelos dinámicos de los problemas se vuelven importantes para explorar comportamientos matemáticos de una familia de objetos. En este camino, la construcción de un modelo dinámico ofrece una oportunidad para que los estudiantes busquen y exploren relaciones entre los elementos del problema vía el uso de estrategias que involucran el movimiento controlado de objetos, la generación de lugares geométricos, la cuantificación de atributos y la búsqueda de propiedades que validan relaciones y soluciones de los problemas (Santos-Trigo, Reyes-Martínez, & Aguilar-Magallón, 2015).

Metodología

En este estudio participaron ocho estudiantes de un curso de resolución de problemas que forma parte de un programa de Maestría en Educación Matemática. Se trabajó durante diez semanas, una sesión de tres horas por semana. El desarrollo de las sesiones se llevó a cabo en un aula de computo donde cada estudiante tuvo acceso a una computadora con el SGD, GeoGebra, instalado. Durante las primeras dos sesiones, se implementaron tareas que tenían la finalidad de que los estudiantes comenzaran a familiarizarse con GeoGebra, posteriormente, se abordaron diversos problemas (rutinarios y no rutinarios) con el objetivo de que los estudiantes desarrollaran diversas estrategias durante el seguimiento de los episodios planteados por Santos-Trigo y Camacho-Machín (2013). En este reporte se analiza el trabajo de los estudiantes en un problema que se abordó en dos sesiones al final del curso.

El problema

Un grupo de deportistas efectúan un recorrido de 380 km en siete horas durante una expedición de caza. Durante cuatro horas viajan a lo largo de una carretera pavimentada y el resto del tiempo por un camino de herradura. Si la velocidad media en el camino de herradura es de 25 km/h menor que la velocidad media en la carretera, encuéntrese la velocidad media y la distancia recorrida en cada uno de aquellos tramos de camino (Rees & Sparks, 2005, p. 60).

La implementación de la tarea se dividió en dos momentos: (1) primero se les solicitó a los estudiantes resolver el problema solamente con papel y lápiz y, (2) después, se les pidió que lo resolvieran con el uso de GeoGebra. Los datos se recolectaron a través de los reportes escritos, las videograbaciones de las sesiones, archivos de GeoGebra con las construcciones dinámicas que elaboraron los estudiantes y las notas de campo de los investigadores.

Resultados

En esta sección se discuten los recursos, estrategias y formas de razonamiento que exhibieron los estudiantes al resolver el problema: (1) con el uso de papel y lápiz; y, (2) con el uso de GeoGebra. En la Tabla 1 se muestran las principales características de los acercamientos que llevaron a cabo los estudiantes.

Tabla 1: Resultados con el uso de papel y lápiz y con el uso de GeoGebra.

Estudiante	Resultados con el uso de papel y lápiz	Resultados con el uso de GeoGebra (primer modelo)
Felipe	- Plantea un sistema de ecuaciones con ambas velocidades. - Encuentra la solución.	- Representa dinámicamente las condiciones del problema. - Encuentra solución de manera empírica. - Analiza el lugar geométrico que describe el punto (distancia del camino pavimentado, diferencia entre las velocidades) - Justifica algebraicamente la solución encontrada en el modelo dinámico.
Homero	- Diagrama (segmento de recta). - Plantea ecuación en términos de la velocidad del camino pavimentado. - Encuentra la solución.	- Representa dinámicamente la condición de las distancias. - No construye el modelo dinámico de manera individual.
Alfonso	- Plantea ecuación en términos de la velocidad del camino pavimentado. - Muestra dificultades procedimentales. - No encuentra la solución.	- Representa dinámicamente las condiciones del problema. - No presenta un análisis del comportamiento de los objetos matemáticos involucrados.
David	- Diagrama (áreas relacionando tiempo y distancia). - Plantea ecuación a partir del diagrama (interpretación errónea). - No encuentra la solución.	- Representa dinámicamente las condiciones del problema. - Encuentra solución de manera empírica. - Justifica algebraicamente la solución encontrada en el modelo dinámico.
Rocío	- Diagrama (segmento de recta). - Expresa las velocidades en términos de las distancias sin llegar a plantear la ecuación correspondiente.	- Representa dinámicamente las condiciones del problema. - No presenta un análisis del comportamiento de los objetos matemáticos involucrados.

	- No encuentra la solución.	
Raúl	- Diagrama (segmento de recta). - Plantea ecuación (relaciona la suma de las velocidades con la velocidad promedio). - No encuentra la solución (Figura 1).	- Representa dinámicamente las condiciones del problema. - Encuentra solución de manera empírica. - Analiza el lugar geométrico que describe el punto (distancia del camino pavimentado, diferencia entre las velocidades)
Uriel	- Diagrama (segmento de recta). - Plantea ecuación en términos de la distancia recorrida en el camino de herradura. - Muestra dificultades procedimentales. - No encuentra la solución.	- Representa dinámicamente las condiciones del problema. - Encuentra solución de manera empírica. - Justifica algebraicamente la solución encontrada en el modelo dinámico.
César	- No muestra la solución al problema.	- Representa dinámicamente las condiciones del problema. - Encuentra solución de manera empírica.

Se observa que, en los distintos acercamientos con el uso de papel y lápiz, el diagrama es insuficiente en la búsqueda de la solución. En general, cinco de los estudiantes usan esta estrategia (los diagramas son muy similares) y, aunque solo uno de ellos obtuvo la solución correcta, en ningún caso el diagrama aportó un dato relevante para resolver correctamente el problema, por ejemplo, el acercamiento propuesto por Raúl (Figura 1). De los estudiantes restantes, que no hicieron un diagrama, solo uno resolvió correctamente el problema. Al respecto, Schoenfeld (1985) menciona que el uso de heurísticas no garantiza la solución de un problema si no se cuenta con los recursos necesarios, sin embargo, en los problemas verbales parece tener mayor peso el dominio de recursos (procedimientos rutinarios) que los otros aspectos que se consideran en la resolución de problemas. También, se observan dificultades para representar la situación problemática mediante ecuaciones. Por ejemplo, Raúl plantea la igualdad entre la suma de las velocidades y la velocidad promedio, y David plantea la igualdad entre la suma de los productos de tiempos por distancias y distancia total recorrida. Es decir, ambos casos carecieron de sentido.

Respecto a los acercamientos con el uso de GeoGebra, se observa que todos los participantes lograron representar geométricamente las condiciones iniciales que se deben cumplir en el problema: distancias y tiempo (Figura 2). Cinco de los estudiantes encontraron la solución de manera empírica mediante la exploración del modelo dinámico del problema. Únicamente dos participantes realizaron un análisis dinámico del lugar geométrico que describe el punto que relaciona, de manera funcional, la distancia del camino pavimentado y la diferencia entre las velocidades. Por último, la justificación algebraica de la solución encontrada en el modelo dinámico solo fue presentada por tres estudiantes.

El primer modelo dinámico del problema sirvió de plataforma para la construcción de un segundo modelo dinámico basado en recursos, estrategias y formas de razonamiento diferentes a los planteados en el primero. Estos modelos dinámicos se describen en las siguientes secciones.

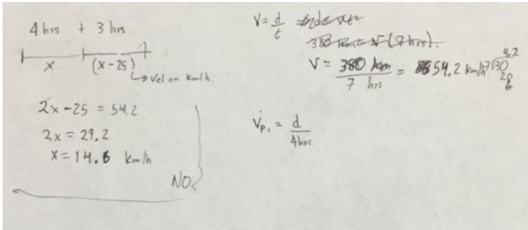


Figura 1. Diagrama y ecuación propuestos por Raúl.

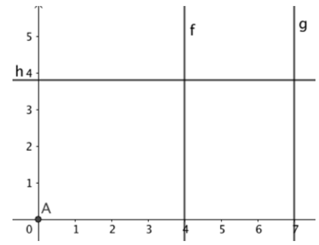


Figura 2. Representación de datos conocidos en el sistema cartesiano.

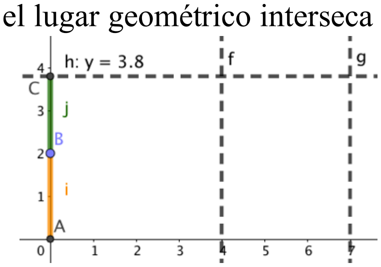
Acercamientos con el uso de un SGD

Cuando los estudiantes comenzaron a resolver el problema con GeoGebra, tuvieron que pensar el problema en términos de propiedades y los comandos de la herramienta. Por ejemplo, se plantearon preguntas como: ¿de qué manera se pueden representar el tiempo, distancia y la velocidad en este ambiente?, ¿cómo se construye un modelo dinámico asociado al problema?, ¿qué estrategias permiten llegar a la solución? Para resolver el problema los estudiantes desarrollaron dos modelos dinámicos, el primero, fue común entre los estudiantes (Figura 3), mientras que el segundo (Figura 4) fue desarrollado con la guía del investigador e implica una solución distinta a la que puede obtenerse en un ambiente donde solo se utiliza papel y lápiz. Ambos modelos se basaron en el uso del sistema cartesiano, donde se representaron las unidades de tiempo en el eje x y en el eje y la distancia recorrida (dividida entre 100 para facilitar la construcción y análisis del modelo dinámico). Las condiciones iniciales del problema: recorrido de 380 km en siete horas, de las cuales cuatro fueron en carretera pavimentada y el resto del tiempo por un camino de herradura, fueron representadas como rectas perpendiculares a los ejes que pasaban por el valor numérico correspondiente al dato conocido (Figura 2).

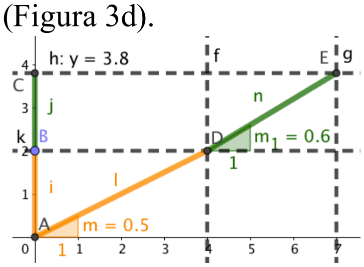
Primer modelo

Este modelo parte de la idea de representar las distancias como segmentos sobre el eje y (Figura 3a) con la finalidad de satisfacer la condición de que sumen 3.8, la cual está relacionada con la distancia total que se recorre. Para lograrlo, un estudiante coloca un punto sobre el eje y y define los segmentos AB y BC como las distancias recorridas en la carretera pavimentada y el camino de herradura, respectivamente. Enseguida, traza una perpendicular sobre el eje y que pasa por B, esta recta está asociada a la distancia del primer tramo del camino, y construye los segmentos AD y DE, en los cuales obtiene las pendientes que interpreta como las velocidades de cada tramo (Figura 3b).

Una vez que cuenta con el modelo dinámico, se enfocó en encontrar la posición del punto B en la que se satisface la condición: la velocidad media en el camino de herradura es de 25 km/h menor que la velocidad media en la carretera, es decir, la diferencia de las pendientes (m-m_1) es 0.25 (Figura 3c). El estudiante encuentre y analiza el lugar geométrico del punto F que relaciona la ordenada del punto B y la resta de las pendientes, e identifica que la solución al problema es cuando el lugar geométrico interseca a la recta y=0.25 (Figura 3d).

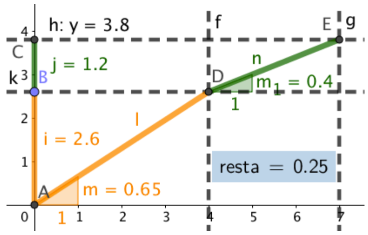


a)

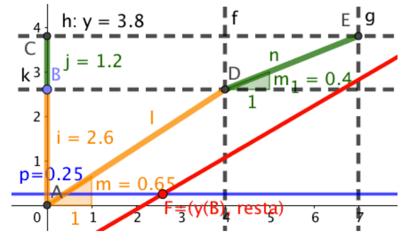


b)

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.



c)



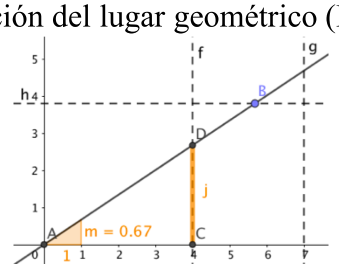
d)

Figura 3. Construcción del primer modelo dinámico.

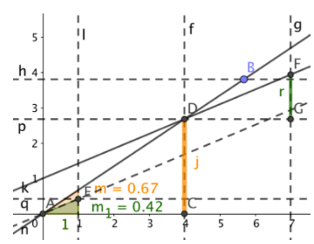
Para justificar el modelo dinámico, los estudiantes representaron algebraicamente la idea desarrollada en GeoGebra. Para encontrar la ecuación del lugar geométrico escribieron la *resta* en términos de la variable (en este caso $y(B)$ que denotaron como x). Luego, expresaron las pendientes como $m=\frac{y(B)}{4}=\frac{x}{4}$ y $m_1=\frac{3.8-y(B)}{3}=\frac{3.8-x}{3}$. Así, la función que define a la recta es $f(x)=m(x)-m_1(x)=\frac{x}{4}-\frac{3.8-x}{3}=\frac{3x-15.2+4x}{12}=\frac{7x-15.2}{12}$. Para buscar la solución, encontraron la coordenada en x del punto de intersección de $f(x)$ y la recta p (definida como $y=0.25$), resolvieron la ecuación $\frac{7x-15.2}{12}=0.25$ y obtuvieron que la solución se obtiene en $x=2.6$, es decir, 260 km recorridos en el primer tramo. Con este resultado obtienen las demás respuestas.

Segundo modelo

La idea esencial detrás de este modelo fue interpretar las velocidades como pendientes asociadas a rectas. El camino inicial que siguió un estudiante fue definir el punto B sobre la recta h, trazar la recta AB y medir su pendiente (Figura 4a), que está relacionada con la velocidad del primer tramo recorrido. Para representar la velocidad en el segundo tramo del recorrido, el estudiante construyó un triángulo rectángulo con base igual a una unidad y altura igual a $m-0.25$, lo que le permitió trazar la recta AE cuya pendiente es $m-0.25$ (Figura 4b). Luego, trazó una recta paralela a la recta AE que pasara por el punto D y una perpendicular al eje y que también pasara D. Así, trazó el segmento FG, el cual está relacionado con la distancia recorrida en el camino de herradura. Posteriormente, analizó el comportamiento del punto H, que relaciona la abscisa del punto B y la suma de las distancias de los recorridos (segmentos r y j) mediante la visualización del lugar geométrico (Figura 4c).



a)



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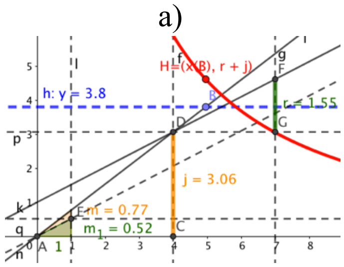

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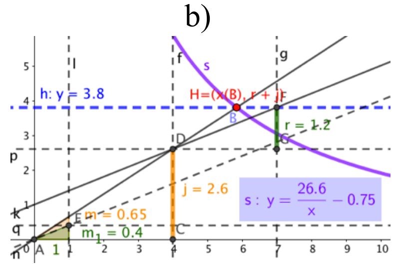

d)

Figura 4. Construcción del segundo modelo dinámico.

De esta manera, encuentra que la solución es cuando el lugar geométrico se interseca con la recta $y = 3.8$. Con base en estas ideas y exploraciones, el estudiante fue capaz de representar algebraicamente este acercamiento. Representó con x la coordenada de la abscisa del punto B, el cual está asociado a la pendiente m. Para obtener la función debe expresarse a $j + r$ en términos de x, es decir, dado que $m = \frac{3.8}{x}$ y $m = \frac{j}{4}$ entonces al igualar y despejar j se obtiene que $j = \frac{15.2}{x}$. Similarmente, dado que $m_1 = m - .25$ y $m_1 = \frac{r}{3}$ entonces $r = \frac{11.4}{x} - .75$. Así, la función asociada al lugar geométrico descrito por H es $f(x) = \frac{26.6}{x} - .75$. Al igualar la función a 3.8, se tiene que $x = \frac{26.6}{4.55} \approx 5.84$. Cuando se grafica en GeoGebra la función encontrada, es posible corroborar que se los resultados coinciden (Figura 4d).

Discusión de los resultados

Cuando los estudiantes intentan resolver el problema en un ambiente donde solamente cuentan con papel y lápiz, se observa que para llegar a la solución es necesario que posean los recursos que les permitan plantear las ecuaciones adecuadamente y, mediante procedimientos rutinarios, obtener el resultado correcto. Sin embargo, si no logran comprender la situación problemática será difícil que logren plantear la ecuación o modelo matemático que les permita llegar a la solución e incluso implementar la heurística de representar el problema por medio de un diagrama para observar cómo se relacionan los datos del problema.

Cuando se incorporó GeoGebra para resolver el problema, se observó que el nivel de apropiación del SGD y los recursos con los que contaba cada estudiante determinaba el tipo de estrategia que desarrollaban. De manera general, los dos modelos dinámicos que se construyeron involucraron recursos y conceptos matemáticos diferentes, lo que implicó el desarrollo de diferentes formas de razonamiento. Cada uno de los modelos permitió que los estudiantes interpretaran geométricamente las condiciones que se deben cumplir en el problema planteado. Así, el SGD permitió que los estudiantes dieran significado a los conceptos y objetos matemáticos involucrados en el problema y, a través de la exploración de los modelos dinámicos, encontraran la solución sin necesidad de plantear un acercamiento algebraico de manera inicial. En la Tabla 2 se presenta una comparación de las actividades matemáticas que se llevaron a cabo en cada uno de los modelos exhibidos por los estudiantes, con base en los episodios propuestos por Santos-Trigo y Camacho-Machín (2013).

Tabla 2: Comparación de los modelos dinámicos del problema

Episodio	Primer modelo	Segundo modelo
Comprensión	Interpretar las distancias geométricamente como segmentos sobre el eje *y* que sus longitudes sumen 3.8 y representar las velocidades en términos de las distancias.	Interpretar las velocidades geométricamente como pendientes de rectas donde una sea 0.25 unidades menor a la otra y representar las distancias en términos de las velocidades.
Exploración del problema	Buscar para qué longitudes de los segmentos se cumple la condición de las velocidades (pendientes).	Buscar para qué pendientes de las rectas se cumple la condición de las distancias (segmentos).
Búsqueda de distintos acercamientos a la solución	Soluciones empíricas obtenidas por: - Analizar los valores numéricos de los atributos del modelo - Visualizar lugar geométrico (recta) que relaciona la longitud del segmento AB con la diferencia de las pendientes m y m_1 (ver Figura 2d)	Soluciones empíricas obtenidas por: - Analizar los valores numéricos de los atributos del modelo - Visualizar lugar geométrico (hipérbola) que relaciona la abscisa del punto B con la suma de los segmentos j y r (ver Figura 3d) Solución algebraica obtenida por:

	Solución algebraica obtenida por: - Representar el modelo dinámico mediante ecuaciones.	- Representar el modelo dinámico mediante ecuaciones.
Integración y reflexión	Dar sentido a la resta de velocidades respecto a la situación problemática.	Relacionar la pendiente de rectas verticales con la indeterminación de la función.

En la Tabla 2 se puede observar que hay recursos y estrategias que coinciden en cada episodio de ambos modelos con el uso del SGD. En el primer episodio, fue importante el uso del sistema cartesiano para representar los datos del problema geométricamente y para, posteriormente, analizar las relaciones y atributos de la construcción. Lo anterior, implicó aplicar una escala en el eje de las ordenadas para tener la posibilidad de visualizar el modelo con mayor facilidad y explorarlo. También, cobró sentido el concepto de pendiente cuando se interpretó geométricamente la razón de dos cantidades (como es el caso de la velocidad). El episodio de exploración del problema es resultado de implementar la heurística de punto con movimiento controlado que permite el SGD. Es decir, a partir de un punto que se definió sobre una recta o un eje, se construyó un modelo dando la oportunidad de explorar y analizar las relaciones entre los atributos de la construcción. En el tercer episodio, utilizar la herramienta de lugar geométrico es una estrategia esencial para encontrar soluciones geométricas en términos de la covariación de atributos de la configuración dinámica, que conlleva a plantear conjeturas por medio de la observación y, por lo tanto, a buscar argumentos. Por último, el cuarto episodio se centra en el dominio del problema donde se reflexiona sobre el comportamiento y la interpretación del modelo para todos los valores del punto móvil. Por otro lado, se evidencian las diferencias de ambos acercamientos que muestran los conceptos que se enfatizan y los distintos caminos que involucran reflexiones diferentes para interpretar la solución del problema.

La construcción y exploración de los modelos dinámicos permitió que los estudiantes lograran representar algebraicamente el problema al dar seguimiento a los pasos de la construcción. Específicamente en el segundo modelo dinámico, se desarrolló un modelo algebraico que no sería fácil de plantear sin la exploración de la construcción. De esta manera, los estudiantes pudieron interpretar y contextualizar las expresiones algebraicas, y dar sentido a diferentes objetos y conceptos matemáticos como la pendiente.

Conclusiones

El uso del SGD para la resolución de problemas verbales permitió que los estudiantes prestaran más atención a la comprensión del problema para poder construir el modelo dinámico del problema y a la exploración del problema, para analizar el comportamiento de los objetos matemáticos involucrados. La búsqueda de distintos acercamientos a la solución favoreció el desarrollo de diferentes formas de razonamiento, especialmente, se observó un tránsito de lo empírico a lo formal cuando los estudiantes llevaron su solución dinámica a la representación algebraica. Al integrar y reflexionar sobre las estrategias de solución, fue posible analizar, dentro del contexto del problema, estrategias que se identificaron en las exploraciones como la resta de velocidades, además de ideas matemáticas como la indeterminación de una función.

In this study we analyze and contrast approaches that future high school math teachers show when solving word problems with the use of pencil and paper and later, with the use of a Dynamic Geometry System (DGS). The resources, representations, strategies and forms of mathematical reasoning that the participants exhibit when they use GeoGebra in the process of

problem solving are analyzed. The results show that the use of the tool favors the dynamic exploration of the concepts involved, the formulation of conjectures and the search for different arguments to validate the solution.

Keywords: Problem Solving, Algebra and Algebraic Thinking, Technology.

Introduction

Word problems are an important theme in school plans and programs of study from the basic level and represent a challenge for the students in terms of developing concepts and strategies to solve them. This type of problems, in general, stand out as a way of applying the contents that are studied at that level by solving problems situated in real-life contexts (Verschaffel, Depaepe, & Van Dooren, 2014). Likewise, in some textbooks students are encouraged to follow a path that involves the following stages: 1) understand the problem statement; 2) identify known and unknown data; 3) assign the unknown to an unknown data (usually represented by the letter x); 4) expressing the unknown data remaining in terms of the unknown; 5) formulating the equation; and, 6) solving the equation and check the result (Rees & Sparks, 2005). However, when a student is faced with solving word problems, the first phase (understanding the problem) is often meaningless, since statements tend to include keywords such as "more" or "difference" that push them immediately the selection of some arithmetic operation, geometric formula or algebraic expression (Verschaffel, Greer, & De Corte, 2000), that then is solved without making sense of the situation posed. In this way, a need is identified to promote tasks that involve word problems where students can give meaning to the concepts or mathematical objects involved in the problems. According to Santos-Trigo and Reyes-Martínez (2014), the coordinated use of digital technologies can be a fundamental factor in achieving the desired objectives in the teaching of mathematics, since it offers the possibility of representing, exploring, identifying, formulating and solving problems. In addition, Santos-Trigo, Reyes-Martínez and Aguilar-Magallón (2015) highlight that, "school systems throughout the world face a challenge of systematically incorporating the coordinated use of digital technologies in curricular proposals and learning environments " (p. 298). To what extent does the use of a Dynamic Geometry System (DGS) affect the development of resources, strategies and ways of mathematical reasoning in the students, when they solve word problems? What kind of mathematical reasoning (representations, explorations, conjectures, partial explanations, etc.) characterize the approaches based on the use of digital technologies? In order to answer these questions, this study presents a route that incorporates the use of DGS in the word problem solving and discusses the forms of reasoning that students develop about concepts and the way in which it influences the activities that characterize the problem solving.

Conceptual Framework

Problem Solving and the use of digital technology

Schoenfeld (1985) proposed a framework that explains the students´ behavior in problem solving. It focuses on the domain of conceptual and procedural knowledge (resources), search strategies for the analysis and transformation of problems that increase the probability to find a solution (heuristics), monitoring and control of cognitive process (metacognition) and attitudes and positive emotions towards the task that is been developed (beliefs). However, the framework was designed in an environment where the main tools were paper and pencil. When the use of digital technologies is incorporated into problem solving, it not just facilitates aspects such as representing, looking for patterns or invariants among objects of a problem, which is essential for the students to identify the relationships and raise conjectures, but also it simplifies the strategies implementation and helps to extend the heuristics repertoire. (Santos-Trigo, 2008).

Based on these ideas, Santos-Trigo and Camacho-Machín (2013) proposed a framework that besides allowing the analysis of procedures that the students follow when they solve problems with the use the digital technology, it gives the possibility to transform routine problems into non-routine problems. The framework consists of four episodes: 1) understanding: to identify the concepts that relate to the problem and ask some questions that guide the activity 2) explore the problem: to evaluate different ways to find the solution 3) search for different solutions: to analyze and discriminate the different results found to propose an argued solution, as well as to pose new problems from these results 4) integration and reflections: to discuss about the activities and the results obtained.

Under this approach, the dynamic models of the problems become important to explore mathematical behaviors of a family of objects. In this way, the construction of a dynamic model offers an opportunity for the students to search and explore relationships between the elements of the problem through the use of strategies that involve the controlled movement of objects, locus generation, attributes qualifications and the search of properties that allow relationships and solutions to problems. (Santos-Trigo, Reyes-Martínez, & Aguilar-Magallón, 2015).

Methodology

In this study, eight students participated from a problem solving course that is part of a Master´s program in Mathematics Education. It worked for ten weeks, with a session of three hours per week.

The development of the sessions was done in a computer lab where each student had access to a computer with the DGS, GeoGebra, installed. During the first two sessions, tasks aimed at getting the students familiarized with GeoGebra were implemented, after that, various problems (routine and non-routine) were addressed with the aim of having the students develop various strategies during the follow-up of the episodes proposed by Santos-Trigo and Camacho-Machín (2013). In this report it is analyzed the students' work in a problem that was addressed in two sessions at the end of the course.

The Problem

A group of athletes make a 380 km tour in seven hours during a hunting expedition. For four hours they travel along a paved road and the rest of the time by a bridle path. If the average speed in the bridle path is of 25 km/h lower than the average speed on the road, find the average speed and the distance traveled in each of those road sections. (Rees & Sparks, 2005, p. 60).

The implementation of the task was divided in two stages: (1) the students were first asked to solve the problem just with the use of pencil and paper and (2) later, they were asked to solve it with the use of GeoGebra. The data was collected through the written reports, the video recordings of the sessions, GeoGebra archives with the dynamic constructions created by the students and the field notes of the researchers.

Results

In this section are discussed the resources, strategies and ways of thinking that the students showed when they solve the problem: (1) with the use of pencil and paper; and, (2) with the use of GeoGebra. Table 1 shows the main characteristic of the approaches carried out by the students.

Table 1: Results with the use of pencil and paper and with the use of GeoGebra.

Student´s Name:	Results with the use of pencil and paper	Results with the use of GeoGebra (first model)
Felipe	- He proposes a system of equations with both speeds. - He found the solution.	- He represents in a dynamic way the conditions of the problem. - He found the solution in an empiric way. - He analyses the locus that describes the point (paved road distance, difference between speeds) - He justifies algebraically the solution found in the dynamic model.
Homero	- Diagram (line segment). - He proposes an equation in terms of the speed of the paved road. - He found the solution.	- He represents in a dynamic way the conditions of the distances. - He does not create the dynamic model in an individual way.
Alfonso	- He proposes an equation in terms of the speed of the paved road. - He has procedural difficulties. - He does not find the solution.	- He represents in a dynamic way the conditions of the problem. - He does not present an analysis of the behavior of the mathematical objects involved.
David	- Diagram (areas relating time and distance). - He proposes an equation from the diagram (wrong interpretation). - He does not find the solution.	- He represents in a dynamic way the conditions of the problem. - He found the solution in an empiric way. - He justifies algebraically the solution found in the dynamic model.
Rocío	- Diagram (line segment). - She expresses the speeds in terms of the distances without proposing the corresponding equation. - She does not find the solution.	- She represents in a dynamic way the conditions of the problem. - She does not present an analysis of the behavior of the mathematical objects involved.
Raúl	- Diagram (line segment). - He proposes an equation (he relates the sum of the speeds with the average speed) - He does not find the solution (Figure 1)	- He represents in a dynamic way the conditions of the problem. - He found the solution in an empiric way. - He analyses the locus that describes the point (paved road distance, difference between speeds)
Uriel	- Diagram (line segment). - He proposes and equation in terms of the distance traveled in the bridled path. - He has procedural difficulties. - He does not find the solution.	- He represents in a dynamic way the conditions of the problem. - He found the solution in an empiric way. - He justifies algebraically the solution found in the dynamic model.
César	- He does not find the solution.	- He represents in a dynamic way the conditions of the problem. - He found the solution in an empiric way.

It is observed that, in the different approaches with the use of pencil and paper, the diagram is insufficient in the search of the solution. In general, five of the students used this strategy (the diagrams are very similar) and, although just one of them obtained the correct solution, in no case the diagram gave relevant data to solve correctly the problem, see for example, the approach proposed by Raúl (Figure 1). Of the remaining students, who did not make a diagram, just one of them solved correctly the problem. According to that, Schoenfeld (1985) says that the use of heuristics does not guarantee the solution of a problem if the necessary resources are not available, however, in the word problems, the domain of resources (routine procedures) seems to have greater weight than the other aspects that are considered in the problem solving. Also, there are some difficulties as to represent the problematic situation through equations. For example, Raúl proposes the equality among the sum of the speeds and the average speed, and David proposes the equality among the sum of the products of times by distances and the total of distance traveled. In other words, both cases suffered from lack of meaning.

Regarding the approaches with the use of GeoGebra, it is observed that all the participants managed to represent geometrically the initial conditions that must be satisfied in the problem: distances and time (Figure 2). Five of the students found the solution in an empiric way through the exploration of the dynamic model of the problem. Only two participants did a dynamic analysis of the locus that describes the point that relates, in a functional way, the distance of the paved road and the difference between the speeds. Finally, the algebraic justification of the solution found in the dynamic model was only presented by three students.

The first dynamic model of the problem was used as a platform to the construction of a second dynamic model based on resources, strategies and different forms of reasoning to the ones proposed in the first one. These dynamic models are described in the following sections.

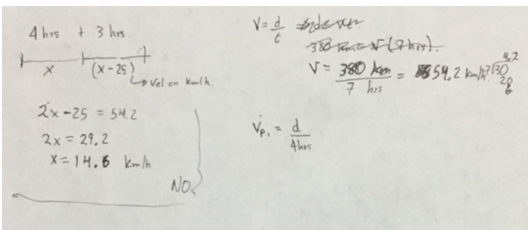

Figure 1. Diagram and equation proposed by Raúl.

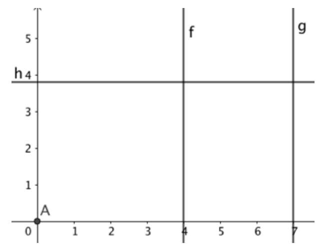

Figure 2. Representation of known data in the cartesian system

Approaches with the use of a Dynamic Geometry System

At the moment that the students started to solve the problem with GeoGebra, they had to think the problem in terms of properties and the commands of the tools. For instance, they asked questions like: In which way the time, the distance and the speed in this environment can be represented? How does a dynamic model related to the problem is constructed? What strategies allow to reach the solution? In order to solve the problem the students developed two dynamic models, the first one, was common among the students (Figure 3), while the second one (Figure 4) was developed with the guide of the researcher and it involves a different solution to the one that can be obtained in an environment where is only used pencil and paper.

Both models were based on the use of a cartesian system, there were represented the time units in the *x*-axis and in the *y*-axis the distance traveled (divided between 100 to facilitate the construction and analysis of the dynamic model). The initial conditions of the problem: 380 km route in seven hours, in which four of them were traveled in a paved road, and the remaining time by a bridle path, were represented as perpendicular lines that pass through the numerical value corresponding to the known data (Figure 2).

First model

This model is based on the idea of representing the distances as segments on the y-axis (Figure 3a) with the objective of satisfying the condition that they sum 3.8, which will be related to the total distance traveled. To achieve this, a student places a point on the y-axis and defines the segments AB and BC as the distances traveled in the paved road and in the bridle path respectively. Then, trace a perpendicular to the y axis that passes by B, this line segment is associated to the distance of the first section of the road, and they build the line segments AD and DE, from which they obtain the slopes that are interpreted as the speed of each section. (Figure 3b).

Once a dynamic model is obtained, they were focused to find the position of the point B in which the condition is satisfied: the average speed in the bridle path is of 25 km/h less than the average speed in the paved road, in other words, the difference of the slopes (*m-m$_1$*) is 0.25 (Figure 3c). The student finds and analyses the locus of the point F that relates the ordinate of the point B and the subtraction of slopes, and identifies that the solution to the problem is where the locus intersects the line y=0.25 (Figure 3d).

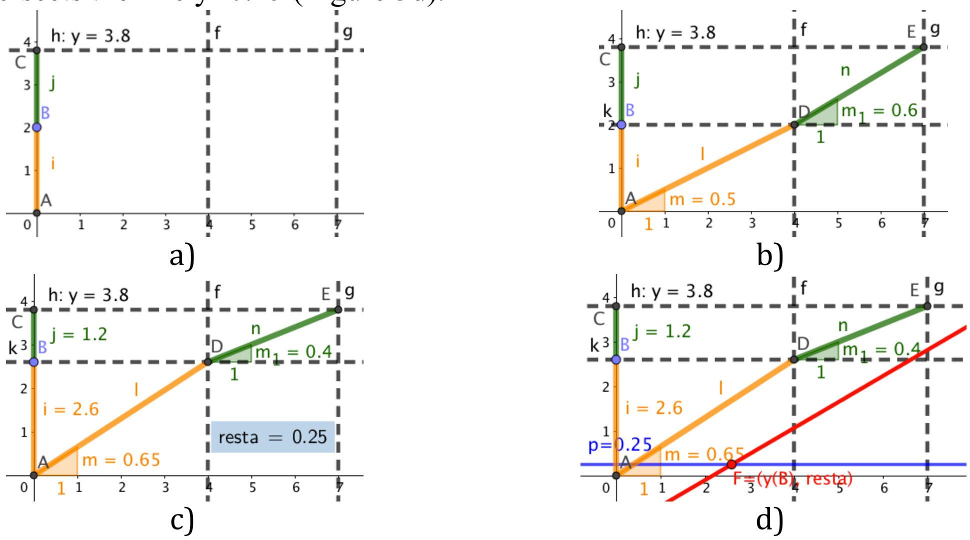

Figure 3. Construction of the first dynamic model

In order to justify the dynamic model, the students represented algebraically the idea developed in GeoGebra. To find the equation of the locus they wrote the subtraction in terms of the variable (in this case *y*(*B*) that is denoted as *x*). Later, they express the slopes as $m = \frac{y(B)}{4} = \frac{x}{4}$ and $m_1 = \frac{3.8-y(B)}{3} = \frac{3.8-x}{3}$. Hence, the function that defines the line is $f(x) = m(x) - m_1(x) = \frac{x}{4} - \frac{3.8-x}{3} = \frac{3x-15.2+4x}{12} = \frac{7x-15.2}{12}$. To look for the solution, they found the coordinate in x of the intersection point of $f(x)$ and the line p (defined as $y = 0.25$), they solved the equation $\frac{7x-15.2}{12} = 0.25$ and got that the solution is obtained in $x = 2.6$, in other words, 260 km were traveled in the first section. With this result they got the others answers.

Second Model

The essential idea of this model was to interpret the speeds as slopes related to straight lines. The initial path that a student followed was to define the point B on the line *h*, trace the line AB and measure its slope (Figure 4a), which is related to the speed of the first section traveled. To represent the speed in the second section of the travel, the students built a right triangle with a base equal to one unit and height equal to $m - 0.25$, which gave him the opportunity to trace the line AE, its slope is $m - 0.25$ (Figure 4b). After that, he traced a line parallel to the line AE that goes through the point D and a perpendicular to the y-axis that also goes through D. In this way, he traced the segment FG, which is related to the distance traveled in the bridle path. Later on, he analyzed the behaviour of the point H that relates the abscissa of the point B and the sum of the distances of the travels (segments r and j) through the visualization of the locus (Figure 4c)

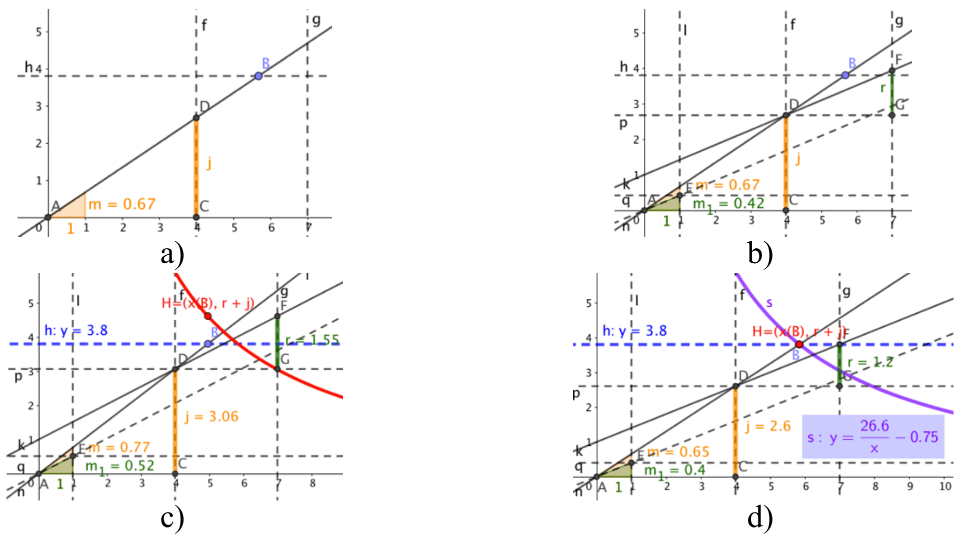

Figure 4. Construction of the second dynamic model.

In this way, he finds that the solution is when the locus intersects the line $y = 3.8$. Based on these ideas and explorations, the students were able to represent this approach algebraically. He represented with x the coordinate of the abscissa of the point B, which is associated with the slope m. In order to get the function, $j + r$ must be expressed in terms of x, in other words, since $m = \frac{3.8}{x}$ and $m = \frac{j}{4}$ then when equalizing and clearing j we obtain that $j = \frac{15.2}{x}$. Similarly, since $m_1 = m - .25$ and $m_1 = \frac{r}{3}$ we get $\frac{11.4}{x} - .75$. In this way, the function associated to the locus is described by $\frac{11.4}{x} - .75$. At the moment to equalize the function to 3.8, we obtain that $x = \frac{26.6}{4.55} \approx 5.84$. When the function found is plotted in GeoGebra, it is possible to corroborate that the results match (Figure 4d).

Discussion of the Results

When the students try to solve the problem in an environment where they can only use pencil and paper, it is observed that in order to get the solution is necessary that they have the resources that allow them to propose the equations in a correct way and, through routine procedures, to get the correct result. However, if they do not understand the problematic situation it will be difficult for them to manage to propose the equation or mathematical model that allows them to find the solution and even, to implement the heuristic of representing the problem by a diagram to observe how the data of the problem is related.

When GeoGebra was incorporated to solve the problem, it was observed that the level of appropriation of the DGS and the resources that each student could use determined the type of strategy that was developed. In general, the two dynamic models that were built involved different resources and mathematical concepts, which in turn involved the development of different ways of reasoning. Each model allowed the students to interpret geometrically the conditions that must be fulfilled in the problem given. So, the DGS allowed students to give meaning to the concepts and mathematical objects involved in the problems and through the exploration of the dynamic problems, they found the solution without the need to propose an algebraic approach from the beginning.

Table 2 presents a comparison of the mathematical activities that were done in each model showed by the students, based in the episodes proposed by Santos-Trigo and Camacho-Machín (2013).

Table 2: Comparison of the dynamic models of the problem

Episode	First model	Second model
Understanding	To interpret the distances geometrically as segments on the y axis such that all its lengths sum 3.8 and represent the speeds in terms of the distances.	To interpret the speeds geometrically as slopes of straight lines where one is 0.25 less than the other and represent the distances in terms of the speeds.
Exploration of the problem	To search for which lengths of segments does the condition of the speeds is met (slopes).	To search for which slopes of the straight lines does the condition of the distances (segments) is met.
Search of different approaches to the solution	Empiric solutions obtained by: - Analyzing the numeric values to the model attributes. - Visualizing the locus (line) that relates the length of the segment AB with the difference of the slopes m y m_1 (see Figure 2d) Algebraic solution obtained by: - Representing the dynamic model through equations.	Empiric solutions obtained by: - Analyzing the numeric values to the model attributes. - Visualizing the locus (hiperbola) that relates the abscissa of the segment B with the sum of the segments j and r (see Figure 3d) Algebraic solution obtained by: - Representing the dynamic model through equations.
Integration and reflection	To give sense to the subtraction of the speeds related to the problematic situation.	To relate the slope of the vertical line with the indetermination of the function.

In Table 2 it can be observed that there are resources and strategies that match in each episode in both models with the use of the DGS. In the first episode, it was important the use of the Cartesian system to represent the data of the problem geometrically, and after that to analyze the relations and attributes of the construction. The last process, involved the application of a scale on the axis of the ordinates to get the possibility of visualizing easier the model and explore it. Also, the concept of the slope got sense when it was interpreted geometrically as the ratio of two quantities (as in the case of the speed). The episode of the exploration of the problem is the result of implementing the heuristic of the point with controlled movement that allows the DGS. In other words, from a point that was defined on a line or an axis, a model was built giving the opportunity to explore and to analyze the relations between the attributes of the construction. In the third episode, to use the locus tool is an essential strategy to find the geometric solutions of the covariation of the attributes of the dynamic setting, that allows to propose conjectures through the observation and hence, to search arguments. Finally, the fourth episode is focused on the handling of the problem where is analized the behavior and the interpretation of the model for all the values of the free point. On the other hand, the differences between both approaches are pointed out, these shows the emphasized concepts and the different ways that included different reflections in order to interpret the solution of the problem.

The construction and exploration of the dynamic models allowed the students to achieve to represent algebraically the problem by following the steps of the construction. Specifically, in the second model it was developed an algebraic model that will not be easy to propose without the exploration of the construction. In this way, the students could interpret and contextualize the algebraic expressions and give sense to different mathematical concepts and objects as the slope.

Conclusions

The use of DGS for the solution of verbal problems allowed the students to pay more attention to the understanding of the problem in order to build the dynamic model of the problem and the exploration of it, in order to analyze the behavior of the mathematical objects involved. The search for different approaches to the solution promotes the development of different forms of reasoning, especially, it was observed a change from the empiric to the formal when the students carried their dynamic solution to the algebraic representation. By integrating and reflecting about the strategies of the solution, it was possible to analyze in the context of the problem, strategies that were identified in the explorations as the subtraction of the speeds, and also the mathematical ideas as the indetermination of a function.

References

Rees, P. & Sparks, F. (2005). Álgebra. McGraw-Hill/Interamericana.

Santos-Trigo, M. (2008). An inquiry approach to construct instructional trajectories based on the use of digital technology. *EURASIA Journal of Mathematics, Science & Technology Education,* 4(4), 347-357.

Santos-Trigo, M. & Camacho-Machín, M. (2013). Framing the use of computational technology in problem solving approaches. *The Mathematics Enthusiast*, 10(1), 279-302.

Santos-Trigo, M. & Reyes-Martínez, I. (2014). The coordinate use of digital technologies in learning environments. In L. Uden, J. Sinclair, Y. Tao & D. Liberona (Eds.), *Learning Technology for Education in Cloud* (pp. 61-71). Switzerland: Springer.

Santos-Trigo, M., Reyes-Martínez, I., & Aguilar-Magallón, D. (2015). The use of digital technology in extending mathematical problem solving reasoning. In L. Uden, D. Liberona & T. Welzer (Eds.), *Learning Technology for Education in Cloud* (pp. 298-309). Switzerland: Springer.

Schoenfeld, A. (1985). *Mathematical Problem Solving*. New York: Academic Press.

Swetz, F. (2009). Culture and the development of mathematics: a historical perspective. In B. Greer, S. Mukhupadhyay, A. B. Powell y S. Nelson-Barber (Eds.) *Culturally responsive mathematics education* (pp. 11–42). Taylor and Francis, Routledge.

Verschaffel, L., Depaepe, F., & Van Dooren, W. (2014). Word problems in mathematics education. In *Encyclopedia of mathematics education* (pp. 641-645). Springer Netherlands.

Verschaffel, L., Greer, B., & De Corte, E. (2000). *Making sense of word problems*. Swets & Zeitlinger, Lisse.

TEACHERS' ANALYSIS OF STUDENT THINKING IN A TEACHING MATHEMATICS WITH TECHNOLOGY MASSIVE OPEN ONLINE COURSE

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The research study examined the ways participants analyzed students' mathematical thinking in a massive online open course focused on teaching mathematics with technology. Across units, participants' posts showed increased attention to student thinking and less focus on issues related to technology.

Keywords: Technology, Teacher Professional Development

Introduction

Effective professional development is intensive, ongoing and connected to practice. It connects to teachers' professional practice and develops relationships among teachers (Darling-Hammond, Wei, Andree, Richardson, & Orphanos, 2009; Loucks-Horsley, Hewson, Love, & Stiles, 1998). Research provides support for seven key features that are essential for providing effective professional development for teachers (e.g., American Federation of Teachers, 2002; Borasi & Fonzi, 2002; Darling-Hammond, et al., 2009; Garet, Porter, Desimone, Birman & Yoon, 2001; Sparks & Hirsh, 1997). These features include assurance that professional development: broaden and deepen teachers' understandings of the content they are teaching; strengthen teachers' pedagogical skills specific to teaching a particular discipline; provide opportunities for teachers to understand how students learn and appreciate differences in student thinking; be connected to the practice of teaching (lesson planning, assessing student thinking, curriculum development, implementation of activities); be ongoing, job-embedded and site specific and continuously interwoven in the daily work of teachers; engage teachers in ongoing discussions within professional communities focused on content, teaching, and learning; provide supports for change that include personnel, materials, equipment, and time.

With technology rapidly changing, teachers often seek out professional development opportunities to learn about new tools. However, research shows that just adding technology to the classroom is not sufficient (e.g., Ertmer, 1999). Teachers need to understand how technology can be used to support students' mathematical thinking. This requires teachers to attend carefully to students' work and thinking.

Building on the work of van Es and Sherin (Sherin & van Es, 2005; van Es & Sherin, 2002; van Es & Sherin, 2010) who examined teachers' noticing of student thinking from video recordings of classrooms, Wilson, Lee, and Hollebrands (2011) developed a model for characterizing prospective teachers' attention to students' work and actions and interpretations of students' mathematical thinking while analyzing a video of students' work with technology. The model facilitated the identification of four distinct processes teachers use to make sense of student work and thinking: describing, comparing, inferring, and restructuring. **Describing** is characterized by an explicit focus on the actions and words of students. Teachers may repeat the same words that were spoken by the students or provide a detailed account of the menus that were selected while using the technology. **Comparing** is characterized by teachers considering their own work on the task with students' actions, either implicitly or explicitly. **Inferring** involves teachers in making assumptions about students' mathematical knowledge to make inferences about what students are thinking. The final category, **restructuring**, occurs when teachers' own knowledge about mathematics, teaching, or technology is modified based on their analysis of students' work and thinking.

The purpose of this study was to examine what teachers focused upon when attending to student work and thinking while participating in online discussion forums within a teaching mathematics with technology professional development course. When teachers focused on student thinking, we were interested in examining the processes teachers used to make sense of students' work with technology.

Context

With a focus on strengthening teachers' pedagogical skills, knowledge of technology, and appreciating differences in student thinking, the first two authors and members of the design team created a Massive Open Online Course, "Teaching Mathematics with Technology." The goals of the course are for teachers to become more familiar with different types of technology and pedagogical strategies that can be used to support mathematics teaching and learning. In addition, we desire for teachers to become aware of the ways students reason and think by providing opportunities for them to analyze videos of students' working on mathematical tasks with technology. A summary of the course and opportunities included for teachers to analyze students' thinking is included in Table 1.

Table 1: Description of Units and Opportunities for Teachers to Analyze Student Thinking

Unit Title	Opportunities to Analyze Student Thinking
Unit 1: Affordances of Technology for the Teaching and Learning of Mathematics	Two video recordings of pairs of students completing the Penny Circle task are provided. In one video students use Demos, in the other video students use paper and pencil to complete the task.
Unit 2: Capitalizing on the Power of Technology	A video recording of a pair of students completing the "Three Animals" task using Geogebra is provided.
Unit 3: Interacting with Engaging Mathematical Tasks	A video of two students completing a "Mystery Transformations" task using WebSketchpad is provided. Three videos of students solving a geometry, statistics, and algebra task are also provided. Teachers can select one to view and analyze.
Unit 4: Using Multiple-Linked Representations	An animation of a classroom scenario involving a discussion of a statistics task using CODAP is provided.
Unit 5: Assessing Students' Mathematics Thinking	A video of a class using Plickers to conduct formative assessment is provided.

The course was first offered in Fall 2016. The first unit opened October 3rd. Subsequent units were opened weekly and the entire course remained open until mid-December. Each unit contained two discussion forums where participants were encouraged to respond to prompts and to each other. One discussion forum was included in the "Essential Exploration" section where a technology-based mathematical task was presented for teachers to solve. It was followed by a video or animation of students' work on that same task. The second discussion forum was included in the "Connect to Practice" section where additional tasks and/or videos of students or classrooms were provided. Although each unit was opened weekly, participants could work at their own pace and choose to complete activities that were of interest to them. Participants interested in obtaining a certificate of completion were required to access the Essentials (readings and videos), complete the Essential Exploration, and post at least one discussion or comment in each of the two discussion forums within each unit.

Participants and Methods

A total of 534 individuals registered for the course, while 337 accessed the course and 236 posted to the discussion forums. At the conclusion of the course, all discussion forum posts were downloaded into a .csv file. Data were cleaned to remove html codes and personal identifiers. Because we were interested in how participants analyzed student thinking, we focused on discussion forum posts from units 1, 2, and 3 that included opportunities for participants to analyze student thinking. This included five discussion forums (two from Unit 1, one from Unit 2, and two from Unit 3) resulting in 271 discussion threads. From these 271 discussion threads, a decision was made to focus on 35 participants who posted in two or more of the five forums.

These 35 participants resided in four countries, with the majority in the US (89%). Within the US, most of the participants resided in the South (48%), while 23% lived in the Midwest, 19% resided in the Northeast, and 10% resided in the West. The majority were female (89%). The highest degree completed by 23% of the participants was a 4-year college degree, while most held advanced degree (69% and 9% had obtained a master's degree and PhD, respectively). The majority of participants were classroom teachers (82%); other participants included administrators (6%), undergraduate and graduate students (3%), university faculty (3%), and other (6%). Participants background with respect to number of years of experience was diverse, with a mean of 14.59 years and a median of 11.5 years. Most of the participants specialized in high school (46%) and middle grades (37%), while others specialized in post-secondary or a combination of the specialization already mentioned.

The data were organized by each of the 35 participants to examine their posts across units. A total of 174 posts made by the 35 participants were analyzed across three units. Of those 174 posts, 79 came from Unit 1, 27 from Unit 2, and 68 from Unit 3. With a total of 174 posts and 35 participants, the average number of posts per participant across all units was 4.97.

Each post was then coded to indicate whether the post was about student thinking or not (yes or no), what the post was focused on (technology, task, students, mathematics, teaching) and if the post was about student thinking the way they were analyzing student thinking (describing, comparing, inferring, restructuring, other).

The **describing** category was used if the post provided details about what the students did or said. For example, one participant posted, "The technology using pair of students used interactive coordinate plane, click and drag (mouse) digital data tables." **Inferring** was defined as a post that includes an assumption about student thinking or motivation. For example, one participant posted "The technology option allowed them to see the patterns much easier and with more possibilities." **Comparing** was defined as a post that included a direct relationship between the work of the participant and the work of the student. For example, one participant posted, "When I was doing the task, I was trying to ask myself the same questions the boys asked in the video." **Restructuring** was defined as a post that included evidence that the participant learned something about mathematics, teaching, or student thinking from their analysis of student thinking. For example, one participant wrote, "What I noticed with the video of the students who used paper and pencil, they actually collected data for various circles. They used this data to create a graph. By looking at the data, they determined that a quadratic equation was a best fit for the data. I think that the students should do this activity first and then go to Desmos to determine the equation that would be a best fit of the data." We also had a code, **Other**, that was used to code posts that did not fit the four existing categories. Within the Other category there were two themes: **contrasting paper/pencil to technology** and **anticipating or comparing to own students or students in general**. This post was coded as contrasting paper/pencil to technology, "The paper-pencil method definitely took a lot longer and there was not enough data to be seen." The following post was coded comparing to students, "I have a similar issue with my students and it is tough to not give them too much, but enough to get them exploring to a point they have enough to make good conjectures on."

The researchers coded an initial subset of posts collaboratively. Once there was consistency in coding, each post was coded by three researchers, these codes were discussed until a consensus was reached. If there was disagreement about a particular code, definitions were refined to provide greater clarity.

The Focus of Teachers' Attention

For those posts that were focused on student thinking 66% were about students or task, which is not surprising. For those posts that did not address student thinking, the focus was somewhat equally distributed among technology, teaching, task, and students (See Figure 1).

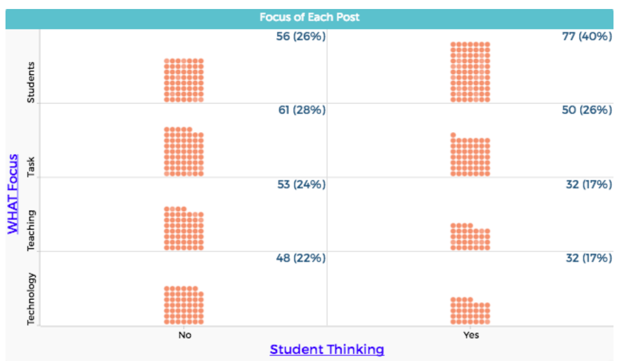


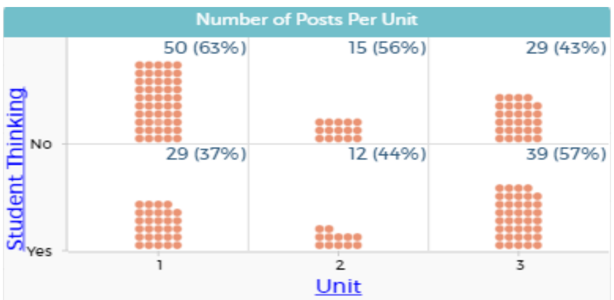
Figure 1. The focus of each discussion forum post and its consideration of student thinking.

It appears that when participants did not address student thinking, they still attend to task but focus more attention on issues related to teaching and technology. Themes within the posts that did not address student thinking were: participants own solution of the task with technology, contrasting paper/pencil and technological approaches, general discussion about students' mathematical work, teaching concerns.

Several posts referred to teachers' own work with technology. For example, one participant noted, "I also had difficulties with software, but it means I need to practice more." Another participant stated, "I started with the linear model too! The exponential model got crazy large. I like that when I picked that one, the simulation questioned that one and prompted me to go back." Participants also discussed how different tools (paper/pencil, technology) can be used to support students' learning. For example, one participant exclaimed, "Yes! I agree that students need a combination of both methods as technology should enhance the experience and not do it completely for them!" and another participant commented, "The technology allows learners to use more functions and visualise the circle with unlimited pennies and may notice the large difference when the functions and graph are done by hand writing." Several posts addressed general teaching concerns such as, "I teach developmental math at college level. I think technology is good for teaching and learning. But I have classes in a room without computers. So

I mostly think about using technology and demonstration on the screen for the whole class to see." While others referred to students in a general way, "Students sometimes have a hard time figuring out if their predictions are good or bad. With this program, they are able to immediately determine if the estimates were good or bad."

Because opportunities to analyze students' thinking were included in each unit, teachers' attention to student thinking across the units was examined. As shown in Figure 2a, there was increased attention to student thinking across units (from 37% to 44% to 57%). While the majority of the posts were identified as "not student thinking" in Units 1 and 2 (63% and 56%, respectively), the majority of the posts were identified as "student thinking" in Unit 3 (57%). In addition to analyzing whether a post was about student thinking or not, researchers examined the focus of each post across units. In Figure 2b, it is evident that of all the posts that were coded as focusing on technology, the majority occurred in Unit 1 (61%) and decreased in Units 2 and 3.



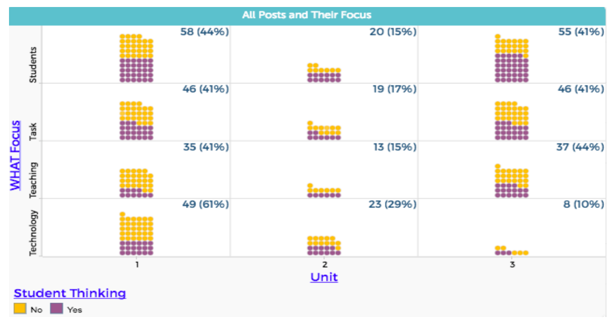


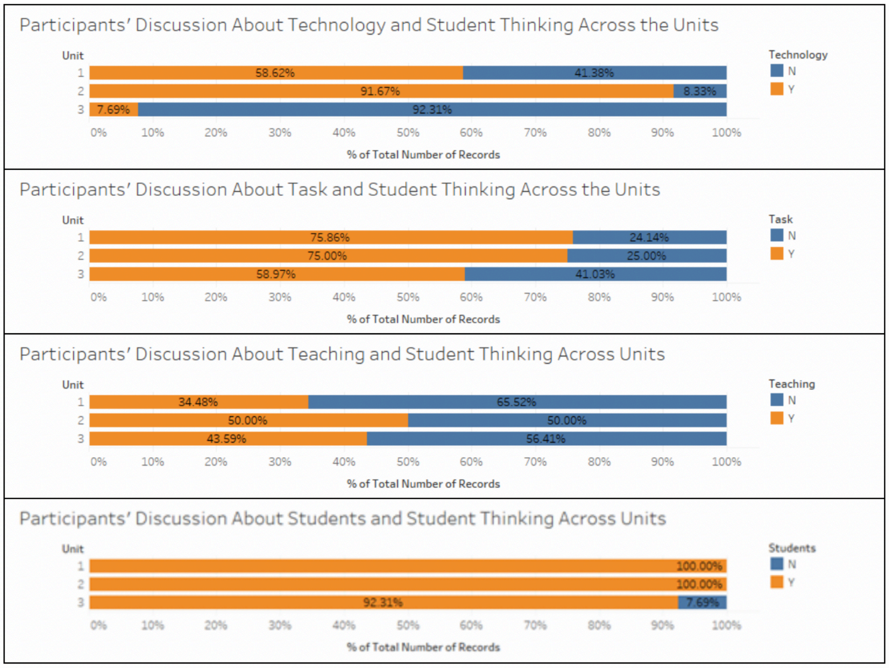
Figure 2a. Number of discussion forum posts focused on student thinking across units.

Figure 2b. Proportion of coded discussion forum posts in each of the units.

Focusing only on those posts that were coded as addressing student thinking, we see that the number of posts that also address technology increased from Unit 1 to Unit 2 and then decreased in Unit 3 (Figure 3a). Although not as drastic, there was a similar result for the discussion of tasks, where 59% of the discussion in Unit 3 of the posts identified as student thinking were also discussing the task, whereas those percentages were slightly higher in Units 1 and 2 (76% and 75%, respectively- Figure 3b). In addition to discussing the technology and the task related to student thinking, participants also discussed students and teaching. Posts about teaching and student thinking increased from Unit 1 to Unit 2 (See Figure 3c) and posts about students and student thinking stayed fairly consistent across units (Figure 3d).

One of the purposes of this research study was to examine what teachers focused upon when attending to student work and thinking while participating in online discussion forums within a teaching mathematics with technology professional development course. We found an increase in the number of posts that analyzed student thinking across units. When posts were not focused on student thinking, they tended to address teaching or technology. Within posts that addressed student thinking we saw different trends in the ways teachers discussed technology, teaching, task, and students.

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.



Figures 3a-3d. Focus of posts coded student thinking.

The other purpose of this study was to examine processes teachers used when examining student thinking. If a post was coded as addressing student thinking, it was coded for the process that was used (describing, comparing, inferring, restructuring, and other) to make sense of student thinking. Findings to this question are presented in the following section.

How Teachers Analyze Students' Work and Thinking

Posts that included an analysis of student thinking were coded in terms of describing, comparing, inferring, restructuring or other. Many of the teachers analyzed student thinking and responded by: 1) describing and inferring, 2) inferring and restructuring, or 3) describing, inferring, and restructuring. However, others indicated that they had solved the task themselves using technology, anticipated student strategies, made comparisons to their own thinking, and/or described/made inferences about student thinking (See Figure 4). We found that some teachers made two other types of comparisons: comparisons to their own students or students in general.

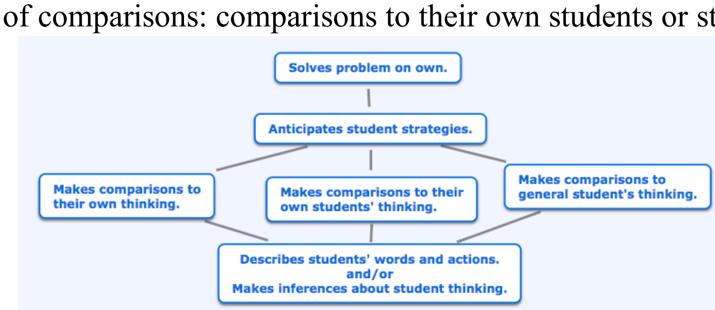


Figure 4. Teachers' patterns of analysis of student thinking.

Across all units, posts typically included inferences about students. They also included descriptions of students' actions or words. Less often, we saw instances of restructuring (Figure 5).

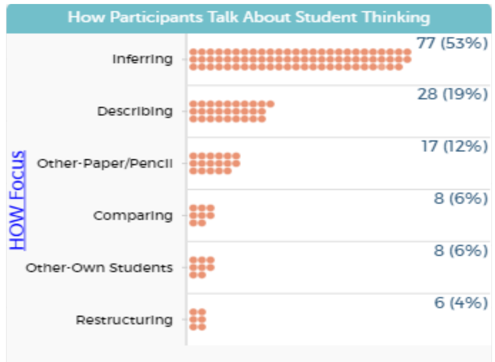


Figure 5. How teachers made sense of student thinking in discussion forum posts.

If we look at the processes used in the posts across units, posts are identified as "Inferring" 61% of the time in Unit 3, whereas the percentages are lower in Units 1 and 2 (45% and 57%, respectively). Discussion forum posts are also more likely to compare the paper/pencil students to the technology students in Units 1 and 2, whereas only 2% of the posts were assigned this code in Unit 3. This is not surprising since we noted earlier that the posts focused more on technology first two units, and this decreased in Unit 3. Additionally, posts were more likely to compare the students in the video to their own students in Unit 3 (about 11% of the time), than in Units 1 and 2 (about 2% and 0%, respectively).

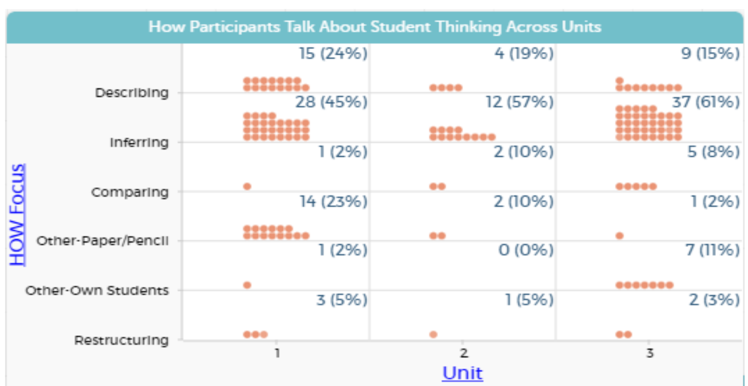


Figure 6. How participants discuss student thinking across units.

Discussion

With more content and courses for teachers available online, it is likely that this will be a venue where teachers seek out professional learning experiences. Massive open-online courses transcend geographic distances and time zones to bring people together around a common interest across the globe. With asynchronous MOOCs, teachers can make decisions about when and how to participate. While research provides evidence about how teachers can improve in their ability to notice and analyze students' thinking in face-to-face professional development, little research is available that examines how this transpires online.

Other studies that examined how participants noticed classroom events while participating online found a shift in attention from factors related to classroom implementation to features of the task (McGraw, Lynch, Koc, Budak, & Brown, 2007) or a shift in the sophistication of analysis of student thinking about specific mathematics (Fernández, Llinares, & Valls, 2012).

Our study provides evidence that in our online course teachers' attention to student thinking increased over time. In addition, teachers' posts provided evidence of going beyond what students were doing with the technology to making inferences about students' thinking. We also found that while many of the initial posted focused on issues related to the technology, this shifted over time to focus on other aspects related to mathematics teaching and learning.

Acknowledgments

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References

American Federation of Teachers (2002). Principles for professional development. Washington, DC: Author. Available: http://www.aft.org/pubs-reports/downloads/teachers/PRINCIPLES.pdf

Borasi, & Fonzi (2002). Foundations (vol. 3): Professional Development That Supports School Mathematics Reform. National Science Foundation.

Darling-Hammond, L., Wei, R. C., Andree, A., Richardson, N., & Orphanos, S. (2009). Professional learning in the learning profession. *Washington, DC: National Staff Development Council*.

Ertmer, P. A. (1999). Addressing first-and second-order barriers to change: Strategies for technology integration. *Educational technology research and development*, *47*(4), 47-61.

Fernández, C., Llinares, S., & Valls, J. (2012). Learning to notice students' mathematical thinking through on-line discussions. *ZDM*, *44*(6), 747-759.

Garet, M., Porter, A., Desimone, L., Birman, B., & Yoon, K. (2001). What Makes Professional Development Effective? Results from a National Sample of Teachers. *American Educational Research Journal, Vol. 38(4)*, 915-945

Loucks-Horsley, S., Hewson, P., Love, N., & Stiles, K. (1998). *Designing professional development for teachers of mathematics and science*. Thousand Oaks, CA: Corwin Press.

McGraw R, Lynch K, Koc Y, Budak A, & Brown, C. (2007). The multimedia case as a tool for professional development: an analysis of online and face-to-face interaction among mathematics pre-service teachers, in-service teachers, mathematicians, and mathematics teacher educators. *Journal of Mathematics Teacher Education 10*(2):95–121

Sherin, M. G., & van Es, E. A. (2005). Using video to support teachers' ability to notice classroom interactions. *Journal of Technology and Teacher Education*, *13*(3), 475-491.

Sparks, D. & Hirsh, S. (1997) A vision for staff development. Association for Supervision and Curriculum Development, Alexandria, VA.

Van Es, E. A., & Sherin, M. G. (2002). Learning to notice: Scaffolding new teachers' interpretations of classroom interactions. *Journal of Technology and Teacher Education*, *10*(4), 571-596.

van Es, E. A., & Sherin, M. G. (2010). The influence of video clubs on teachers' thinking and practice. *Journal of Mathematics Teacher Education*, *13*(2), 155-176.

Wilson, P. H., Lee, H. S., & Hollebrands, K. F. (2011). Understanding prospective mathematics teachers' processes for making sense of students' work with technology. *Journal for Research in Mathematics Education*, *42*(1), 39-64.

DESIGN PRINCIPLES FOR THE DEVELOPMENT OF PROFESSIONAL NOTICING OF STUDENTS' TECHNOLOGICAL MATHEMATICAL PRACTICES

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In support of standards for the learning and teaching of mathematics and statistics that advocate for the use of technology to promote reasoning and sense making, and to elicit student thinking, we draw on the use of authentic student work in the form of video case instruction to develop prospective secondary mathematics teachers' [PSMTs] knowledge of students' understanding, thinking, and learning with technology in mathematics. Specifically, we draw on the extant literature related to TPACK, video cases as learning objects, and noticing to propose a set of design principles intended to guide the development of materials to support PSMTs' acquisition of TPACK. Here we explicate six design principles situated in the literature, provide an example of a module designed based on these principles, and share findings from pilot studies utilizing the module.

Keywords: TPACK; Professional noticing; Preservice teacher education

The National Council of Teachers of Mathematics (NCTM, 2000) has long advocated that "technology is essential in teaching and learning mathematics; it influences what is taught and enhances students' learning" (p. 24). Given the impact that meaningful incorporation of technology tools can have on students' understanding of mathematics, it is important for teachers to develop a model of teaching and learning that goes beyond the specifics of a technology tool so they are able to make informed decisions about appropriate use of technology to develop mathematically proficient students. This was most recently articulated in the Association of Mathematics Teacher Educator's (AMTE) *Standards for Preparing Teachers of Mathematics* which states, "well-prepared beginning teachers of mathematics are proficient with tools and technology designed to support mathematical reasoning and sense making, both in doing mathematics themselves and in supporting student learning of mathematics" (2017, p. 11). This requires teachers to not only be proficient users of technologies, but also to understand how to use technology in meaningful ways to support students' thinking about mathematics. Whether or not the use of technology will enhance students' learning depends on teachers' decisions when using technology tools to design and implement meaningful tasks. These decisions are informed by teachers' knowledge of mathematics, technology, and pedagogy.

Consider the context of teaching trigonometric functions in high school. Teachers need to know how triangle and unit circle models of trigonometric functions are related and connected to each other (knowledge of content). They also need to know how to use technology to create connected representations of right triangle models, unit circle models, and representations of trigonometric functions (knowledge of technology specific to the content). Finally, teachers need to be able to design activities that align with the approaches that students might take when asked to make sense of the connections between right triangle and unit circle models of trigonometry (knowledge of pedagogy specific to the content). The intersection of these forms of knowledge has been identified as technological pedagogical content knowledge (TPACK).

Building off of the work of Grossman (1989), Niess (2005) articulated four components of TPACK: 1) an overarching conception of what it means to teach a particular subject while integrating technology in the learning; 2) knowledge of instructional strategies and representations for teaching particular topics with technology; 3) knowledge of students' understandings, thinking, and learning with technology in a specific subject; and 4) knowledge of curriculum and curriculum materials that integrate technology with learning in the subject area. It is the third component, knowledge of students' understandings, thinking, and learning with technology in a specific subject that is the focus of this paper. Specifically, drawing on the extant literature related to TPACK, video case instruction, and professional noticing we propose a set of design principles for the development of video-enhanced modules for PSMTs with an eye toward the development of their *knowledge of students' understandings, thinking, and learning with technology in mathematics,* an important aspect of TPACK.

Theoretical Foundations

Technological Pedagogical Mathematical and Statistical Knowledge

Within teacher education, many have built upon Shulman's (1986) idea of teachers' pedagogical content knowledge (PCK). Research and teacher education in mathematics education have been greatly influenced by Simon's hypothetical learning trajectory (1995), Even's (1990) work on the essential features of subject matter knowledge in mathematics (particularly for functions), and Ball and colleagues (e.g., Ball, Thames, & Phelps, 2008) work on the components of mathematical knowledge for teaching. However, none of this work considered the knowledge that comes with teaching with technology. The particular knowledge needed when technology is added has been identified as TPACK (e.g., Niess, 2005). This construct has been used by several researchers as a frame for their work to describe the development of PSMTs' abilities in using technology in mathematics teaching (e.g., Hollebrands, McCulloch, & Lee, 2016; Lee, Kersaint, Harper, Driskell, & Leatham, 2012).

When considering TPACK within the context of preparing prospective secondary mathematics teachers [PSMTs], we believe it is *most* essential to focus on the intersections of content (mathematical and statistical) knowledge with technological and pedagogical knowledge. So although we acknowledge the importance of general knowledge of technology and pedagogy, we focus on thinking about mathematics and statistics content, and the use of technology tools specific to teaching mathematics and statistics, as well as pedagogical and technological knowledge that is central to teaching and learning mathematics and statistics (Figure 1). This means developing specific types of knowledge for teaching secondary mathematics/statistics with technology: 1) Mathematics/Statistics Knowledge (Content Knowledge of mathematics and statistics); 2) Technological Mathematical/Statistical Knowledge (Technological Content Knowledge with appropriate tools used in mathematics and statistics); 3) Pedagogical Mathematical and Statistical Knowledge (Pedagogical Content Knowledge for teaching mathematics and statistics); and 4) Technological Pedagogical Mathematical and Statistical Knowledge (as a specific type of Technological Pedagogical Content Knowledge). With a focus on content, notice that the largest circle represents our approach in that mathematics/statistics knowledge is foundational to developing the other three knowledge types. In designing modules to focus on examining students' mathematical practices with technology, we also necessarily concentrate on increasing undergraduate PSMTs' Mathematical and Statistical Knowledge (MSK) and Technological Pedagogical Mathematics and Statistics Knowledge (TPMSK).

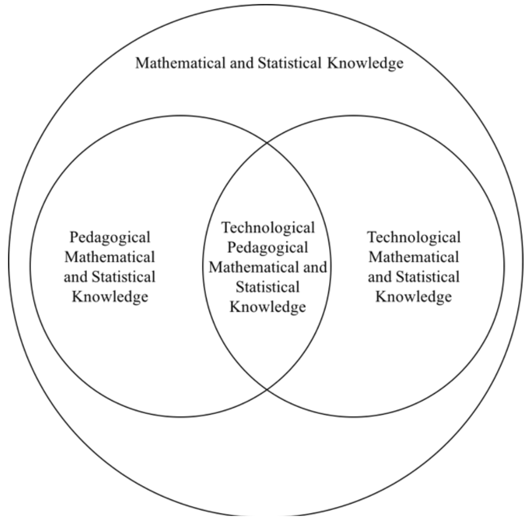

Figure 1. Framing Technological Pedagogical Mathematical and Statistical Knowledge

Analyzing Student Work

Research has pointed to the important role that students' mathematical thinking plays in high-quality instruction (e.g., Jacobs & Spangler, 2017). This points to the need for PSMTs to have opportunities to grapple with and make sense of how students think about mathematics. NCTM's publication, *Principles to Action*, identified "elicit and use evidence of student thinking" as one of the eight mathematics teaching practices (2014, p. 10). For PSMTs this skill must be purposefully developed via teaching practice. One method that has been shown to help PSMTs develop an understanding of student thinking is analysis of authentic student work (e.g., Jansen & Spitzer, 2009; Philipp, 2008).

Authentic student work can come in the form of written artifacts or video cases. Here we focus on video cases and their corresponding written artifacts as together they provide insight to student thinking as they are engaged in mathematical work. Video cases have been shown to improve PSMTs' ability to critically observe classroom practice, attending to teacher choices and student thinking rather than merely content delivery (e.g., Star & Strickland, 2008; Sherin & van Es, 2005). Additionally, a focus on student thinking through video case analysis has been shown to improve PSMTs' abilities to draw attention to and describe teachers' instructional moves to make student thinking visible, to reason about impact of teacher's decisions on student learning, and to propose alternatives to what was observed in the video (Santagata & Guarino, 2011).

While video case instruction has been shown to be very beneficial for PSMTs, researchers caution that the *selection* of video clips (e.g., Kurz, Llama, & Savenye, 2005; Sherin, Linsenmeier, & van Es, 2009) and *how* video cases are used is critical to promoting teacher learning (e.g., Brophy, 2004). To this end, Sherin et al. (2009) articulated a framework for selecting video clips to develop cases that attend to the extent to which a clip provides a *window* into student thinking, the *depth* of student thinking, and *clarity* of student thinking. To ensure the best video clip possible they suggest that all three criteria are considered. It is also suggested that cases be designed so that they focus on aspects of student work in which there are elements of confusion or surprise (Shulman, 1996; Sherin et al. 2009). Once video clips are selected, the activities that surround their use must be carefully designed, articulating clear goals to focus the analysis of the video (Borko, Jacobs, Eiteljorg, & Pittmann, 2008). A method often used to guide PSMTs' analysis of student work in video cases is the *professional noticing* construct developed by Jacobs, Lamb and Philipp (2010). The three components of the professional noticing construct are attending to students' strategies, interpreting students' mathematical thinking, and deciding how to respond on the basis of students' understandings.

Much of the research on PSMTs analyzing student work has been completed through the lens of professional noticing. Within professional noticing work more attention has been paid to professional noticing of whole class video (e.g., Krupa, Huey, Lesseig, Casey & Monson, 2017; McDuffie et. al, 2013), with less on prospective teachers' noticing of student written work (e.g., Dick, 2017; Goldsmith & Seago, 2011). In terms of PSMTs' professional noticing of students' understanding, thinking, and learning mathematical tasks in technological environments, very few studies have been conducted (e.g., Wilson, Lee & Hollebrands, 2011).

The seminal work on PSMTs' professional noticing of students' understanding, thinking, and learning of mathematics with technology comes from Wilson et al. (2011). They not only indicated that engaging preservice teachers in analyzing video cases of students' technological mathematical work resulted in identifying different ways of constructing models of student thinking, but also made a call for the need of research to more fully understand the role of these models in the development of PSMTs' TPACK. Even so, a broad search of the literature (including unpublished dissertations) indicates there has been very little continued work in this direction. This might be due to the complex nature of designing such materials for PSMTs. To address this, and support others who are aiming to support PSMTs in their development of TPMSK, we draw on the literature described here to propose a set of design principles for engaging PSMTs in professional noticing of students' mathematical technological practices.

Design Principles for Supporting Professional Noticing of Students' Technological Mathematical Practices

The design principles we propose draw on the integration of the literature on developing TPACK, video case pedagogies, and the construct of professional noticing. Specifically, we propose that by beginning with the philosophy of an integrated approach to develop skills in a specific content area, pedagogy and technology, the development of MSK and TPMSK can be done through the use of video cases of student practices on technology-based mathematics and statistics tasks. Guiding PSMTs' analysis of video cases is the use of the professional noticing construct. Specifically, we propose the following design principles.

1. PSMTs need to observe secondary students engaged in technology-based tasks of high-cognitive demand. As such, the selected tasks must be of high cognitive demand (Smith & Stein, 1998) and position the use of technology to develop mathematical or statistical understanding (Dick & Hollebrands, 2011).
2. Video clips (and associated written artifacts) should focus on aspects of student work in which there are elements of confusion or surprise, as is suggested by Sherin et al. (2009) and Shulman (1996).
3. Final clips should be selected based on Sherin et. al's (2009) recommendations for dimensions of video clips that support teacher discussion of students' mathematical thinking (i.e., *window* into student thinking, *depth* of student thinking, and *clarity* of student thinking). This includes use of picture-in-picture so that students' technological work is visible as well as any gestures they are making in relation to their technological work (e.g., pointing).
4. To support PSMTs' development of TMSK they must engage with the technology-based task first as learners (Lee, Hollebrands, & McCulloch, 2015).

5. To support PSMTs' development of TPMSK through the analysis of video clips, guiding questions should be designed based on Jacobs et al. (2010) framework for professional noticing. This means specifically asking PSMTs to attend, interpret, predict, and make pedagogical decisions based on their analysis of the video cases.

We conjecture that careful selection of tasks, technology, video case clips, and the questions included in a video case will work together to provide PSMTs an opportunity to develop their knowledge of students' understandings, thinking, and learning with technology in mathematics.

An Example: The Function Concept - Functions and Non-functions

To illustrate our vision for video cases that can promote PSMTs' professional noticing of students' technological mathematical practices we provide an example. We begin by having PSMTs engage with a preconstructed GeoGebra applet designed to provoke a dilemma in relation to their understanding of function (Design Principle 4) PSMTs are asked to engage with this applet and answer questions related to their own understanding of function, representations of function, and consider how they might use the applet with students. (Figure 2). This applet has been designed using a vending machine metaphor so that PSMTs grapple with making sense of function, domain, and range in a context that does not use traditional algebraic representations (Design Principle 1) (see McCulloch, Lovett, & Edgington (2017) for a full discussion of the design of this applet and one study of its use with undergraduate students).

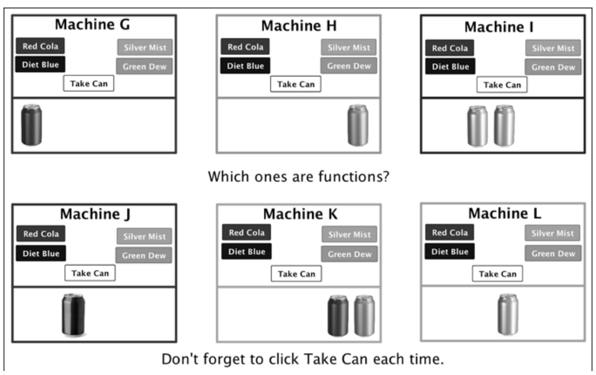

Figure 2. Vending machine applet

Figure 3. Students working on the applet

A second version of the Vending Machine task was designed specifically for secondary students (Design Principle 1). It was designed to be used as an introduction to function with students who had no previous experience with the term *function*. The goal of this version of the task is that students develop a definition of function based on their exploration of the machines in the applet. The applet was then implemented in an 8th grade math class with pairs of students working together and their work on the task was screen captured. Next, the students' videos were analyzed for the purpose of selecting examples of work. This included identifying video clips for episodes of confusion or surprise, followed by the analysis of the window, depth of student thinking, and clarity of student thinking (Design Principle 2 & 3). Once video episodes were selected, they were packaged as a case with questions designed based on Jacobs et al. (2010) professional noticing construct (Design Principle 5). For example, PSMTs were asked to attend to and interpret the coordination of student thinking and technological actions when analyzing the video cases. Further, they were asked to make predictions about student thinking and technological action based on their attending and interpreting. Using these materials, the video case was implemented with PSMTs as follows:

- PSMTs were asked to discuss revisions they would make to the original applet if they intended to use it with secondary students to develop the definition of function.
- After exploring the version of the applet that was used with secondary students, PSMTs anticipate how secondary students would engage with the task - considering actions with the technology and students' verbal and written responses to task prompts.
- PSMTs then analyze carefully selected screen capture clips of multiple pairs of students engaging with the applet (and their associated written work). These video clips show the pairs of students' engagement with some of the machines in the applet, but not all. PSMTs then predict how each student pair might identify each machine as a function or non-function and how they might engage with the remaining machines given their analysis of the first few.
- Given written definitions of function from five pairs of secondary students, PSMTs select which they would use to start a class discussion of the definition of function, why they selected that sample, and which machines in the applet they would draw upon for the discussion given their selection.

As is evident, the video case provided ample opportunities for the PSMTs to consider their own and students' technological mathematical work through engaging with a carefully selected task and analyze carefully selected video cases of students' technological mathematical work by engaging in professional noticing.

Discussion

Our proposed design principles for the development of professional noticing of students' technological mathematical practices are grounded in the literature and have been successfully used to frame the design of a module for PSMTs in the context of the use of a vending machine applet to build an understanding of the function concept. The vending machine task module has been piloted with 98 PSMTs in secondary mathematics education methods courses at six different universities. Data included PSMTs' pre and post function definitions, screen casts of their own mathematical work with the vending machine task, and written artifacts from the analyzing student work assignments. These studies indicate that this module was successful in eliciting PSMTs' MK, TMK, and TPMK (Lovett et al., Under Review). Specifically, within the realm of TPMK, not only were PSMTs able to show an understanding of *students' understandings, thinking, and learning with technology in mathematics*, but also as they stated their predictions of students' technological mathematical work they showed evidence of being able to conceive of how technology can support mathematical thinking (Lovett et al., Under Review), both important aspects of TPACK (Niess, 2005).

Pilot study results also indicate the use of professional noticing to frame the analysis of the video cases was important in that it elicited the specific ways in which prospective teachers drew upon different aspects of student work evident in the video cases. We found that some drew only upon students' spoken and written words, others drew upon only students' actions with the technology. However, those that coordinated both aspects of the students' work were better prepared to predict students' responses on related tasks and to make decisions to support student learning (Lovett, Dick, McCulloch, Sherman, & Martin, 2018). This coordination was especially important as they were making sense of a particular video clip in which students were confused.

Finally, it was found that PSMTs' knowledge of the mathematics deepened through their own engagement with the vending machine task as a learner, and for many this knowledge was expanded even more through their analysis of the video cases (Lovett et al., 2017). Thus, we have substantial evidence that these proposed design principles for developing video cases for examining students' practices with technology are promising.

Conclusion

Strong preparation of mathematics teachers must include opportunities to engage with technology-based mathematics tasks as learners as well as opportunities to develop an understanding of how to support students' learning in mathematical technological environments. As we consider the conference theme, *looking back* and understanding theories and methods that have been successful in supporting prospective teachers as they learn to make sense of student thinking can help us *look ahead* and move forward by drawing on this work to propose new theory about continuing this development in technological contexts. In this paper we have articulated a set of 6 principles to frame the design of video case materials to support PSMT development of TPACK. Results from pilot studies provide empirical support for the promise of these design principles. We now challenge ourselves and others in the field to keep these principles in mind as we work to ensure PSMTs are well-prepared to teach with technology in ways that support students' mathematical reasoning.

References

Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics*. Available online at amte.net/standards.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*, 389-407.

Borko, H., Jacobs, J., Eiteljorg, E., & Pittman, M. E. (2008). Video as a tool for fostering productive discussions in mathematics professional development. *Teaching and Teacher Education, 24*, 417-436.

Brophy, J. (2004). Discussion. In J. Brophy (Ed.), *Using video in teacher education* (pp. 287-304). New York: Elsevier Science.

Dick, L. K. (2017). Investigating the relationship between professional noticing and specialized content knowledge. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 339–358). Cham, Switzerland: Springer International Publishing.

Dick, T. & Hollebrands, K. (2011). Focus on High School Mathematics: Reasoning and Sense Making with Technology. Reston, VA: National Council of Teachers of Mathematics.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics, 21*, 521-544.

Goldsmith, L. T., & Seago, N. (2011). Using classroom artifacts to focus teachers' noticing. In M. G. Sherin, R. Jacobs, Victoria, & R. A. Phillipp (Eds.), *Mathematics Teacher Noticing: Seeing through teachers' eyes* (pp. 169–187). New York, NY: Routledge.

Grossman, P. L. (1989). Learning to teach without teacher education. *Teachers College Record, 91*(2), 191-208.

Hollebrands, K. F., McCulloch, A., & Lee, H. S. (2016). Prospective teachers' incorporation of technology in mathematics lesson plans. In M. Niess, S. Driskell, & K. F. Hollebrands (Eds.), *Handbook of research on transforming mathematics teacher education in the digital age* (pp. 272-292). Hershey, PA: IGI Global.

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*(2), 169-202.

Jacobs, V. R., & Spangler, D. A. (2017). Research on core practices in K-12 mathematics teaching. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 766–792). Reston, VA: National Council of Teachers of Mathematics.

Jansen, A., & Spitzer, S. M. (2009). Prospective middle school mathematics teachers' reflective thinking skills: descriptions of their students' thinking and interpretations of their teaching. *Journal of Mathematics Teacher Education*, *12*(2), 133–151. http://doi.org/10.1007/s10857-009-9099-y

Krupa, E. E., Huey, M., Lesseig, K., Casey, S. & Monson, D. (2017). Investigating secondary preservice teacher noticing of students' mathematical thinking. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks (pp. 49–72). Cham, Switzerland: Springer.

Kurz, T. L., Llama, G., & Savenye, W. (2005). Issues and challenges of creating video cases to be used with preservice teachers. *TechTrends, 49*(4), 67-73.

Lee, H. S., Hollebrands, K. F., & McCulloch, A. W. (2015). *A framework to guide a research-based approach to teacher education materials*. Raleigh, NC: NC State University. Retrieved from https://d3cxmgqg4e7xkm.cloudfront.net/ptmt_guiding_framework.pdf

Lee, H. S., Kersaint, G., Harper, S., Driskell, S. O., & Leatham, K. R. (2012). Teachers' statistical problem solving with dynamic technology: Research results across multiple institutions. *Contemporary Issues in Technology and Teacher Education, 12*, 286-307.

Lovett, J. N., Dick, L. K., McCulloch, A. W., Sherman, M. F., & Martin, K. (2018). Preservice mathematics teachers' professional noticing of students' mathematical thinking with technology. In E. Langran & J. Borup (Eds.), *Proceedings of Society for Information Technology & Teacher Education International Conference* (pp. 1802-1811). Washington, D.C., United States: Association for the Advancement of Computing in Education (AACE). Retrieved from https://www.learntechlib.org/primary/p/182772/.

Lovett, J.N., McCulloch, A.W., Dick, L.K., Sherman, M.F., Reed, S.D., Wanner, C.A., & Edgington, C. (under review). Developing preservice mathematics Teachers TPACK of the function concept. Book chapter proposal submitted to Cases on Developing Teachers Technological Pedagogical Content Knowledge (TPACK) in the Digital Age.

McCulloch, A. W., Lovett, J. N., & Edgington, C. (2017). Developing preservice teachers' understanding of function using a vending machine metaphor applet. In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1281-1288). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

McDuffie, A. R., Foote, M. Q., Bolson, C., Turner, E. E., Aguirre, J. M., Bartell, T., … Land, T. (2014). Using video analysis to support prospective K-8 teachers' noticing of students' multiple mathematical knowledge bases. *Journal of Mathematics Teacher Education*, *17*, 245–270. http://doi.org/10.1007/s10857-013-9257-0

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2014). *Principles to action*. Reston, VA: Author.

Niess, M. L. (2005). Preparing teachers to teach science and mathematics with technology: Developing a pedagogical content knowledge. *Teaching and Teacher Education, 21*, 509-523.

Philipp, R. A. (2008). Motivating prospective elementary school teachers to learn mathematics by focusing upon children's mathematical thinking. *Issues in Teacher Education*, *17*(2), 7–26.

Santagata, R., & Guarino, J. (2011). Using video to teach future teachers to learn from teaching. *ZDM, 43*(1), 133-145.

Sherin, M. G., Linsenmeier, K. A., & van Es, E. A. (2009). Selecting video clips to promote mathematics teachers' discussion of student thinking. *Journal of Teacher Education, 60*(3), 213-230.

Sherin, M. G. & van Es, E. A. (2005). Using video to support teachers' ability to notice classroom interactions. *Journal of Technology and Teacher Education, 13*, 475-491.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Shulman, L. S. (1996). Just in case: Reflections on learning from experience. In J.A. Colbert, P. Desberg, & K. Trimble (Eds.), *The case for education: Contemporary approaches for using case methods* (pp. 197-217). Boston: Allyn & Bacon.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.

Smith, M. S., & Stein, M. K. (1998). Selecting and creating mathematical tasks: From research to practice. *Mathematics Teaching in the Middle School*, *3*, 344-350.

Star, J. R., & Strickland, S. K. (2008). Learning to observe: Using video to improve preservice mathematics teachers' ability to notice. *Journal of Mathematics Teacher Education, 11*(2), 107-125.

Wilson, P. H., Lee, H. S., & Hollebrands, K. F. (2011). Understanding prospective mathematics teachers' processes for making sense of students' work with technology. *Journal for Research in Mathematics Education, 42*(1), 39-64.

RESOLUCIÓN DE PROBLEMAS Y USO DE TECNOLOGÍAS DIGITALES EN UN MOOC: DISEÑO E IMPLEMENTACIÓN

PROBLEM SOLVING AND THE USE OF DIGITAL TECHNOLOGIES IN A MOOC: DESIGN AND IMPLEMENTATION

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En este estudio se analizan y discuten los elementos del diseño y resultados de la implementación de un curso masivo abierto en línea (MOOC) que promueve la resolución de problemas basado en el uso de tecnología digital. Los resultados muestran que el diseño de las actividades, la intervención del equipo de diseño del MOOC en los foros de discusión y la interacción entre los participantes permitió y favoreció la creación de un ambiente de colaboración en la resolución de problemas. Los participantes compartieron y discutieron sus ideas matemáticas como parte de una comunidad virtual de aprendizaje, lo que les permitió formular conjeturas, explorar y buscar propiedades para sustentar relaciones y comunicar resultados.

Palabras clave: Resolución de Problemas, Tecnología.

Introducción

Durante o después de sus clases, es común que los estudiantes consulten diversas plataformas digitales en Internet para acceder a información sobre contenidos disciplinarios, compartir y discutir ideas, tomar cursos especializados en algún tema e interactuar con expertos. Así, los espacios y tiempos en los que se produce aprendizaje han cambiado y han sido alterados por la conectividad en la red (Gros, 2016).

Diversas universidades han puesto a disposición del público general, Cursos Masivos Abiertos en Línea (*Massive Open Online Course*, MOOC por sus siglas en inglés) a través de diversas plataformas digitales. La comunidad virtual que participa en estos cursos comprende un conjunto grande (generalmente miles) de personas con diferentes niveles de estudios, edad, dominio o conocimiento previo de la materia, entre otros. En el desarrollo de las actividades de un MOOC no existe un profesor encargado de responder o dar seguimiento puntual a cada participante. Cada integrante define su grado de compromiso y participación en las actividades. Para este estudio, se construyó el MOOC: Resolución de Problemas Matemáticos y uso de Tecnologías Digitales, basado en los marcos: resolución de problemas (Santos-Trigo, 2014) y RASE (Churchill, King, & Fox Churchill, 2016). Churchill, et al., (2016) argumentan que un ambiente de aprendizaje en línea debe incluir elementos que permitan a sus participantes trabajar y colaborar en equipo con otros. Las actividades deben fomentar la participación activa de los estudiantes en un ambiente de reflexión, de discusión y centrarse en un contexto donde las tareas involucren a los estudiantes en los episodios de resolución de problemas: formulación de preguntas, búsqueda de diversos métodos de solución, exploración de diferentes representaciones, búsqueda de patrones, variantes y relaciones entre objetos matemáticos, presentación de argumentos, comunicación de resultados, planteamiento de preguntas y formulación de nuevos problemas (Santos-Trigo, 2014). El uso sistemático de tecnologías digitales resulta importante en la representación, exploración, comunicación y comprensión de conceptos matemáticos en la resolución de problemas (Santos-Trigo, Moreno-Armella, 2016).

En este estudio interesó analizar el diseño de las actividades matemáticas y la formas de razonamiento de los participantes relacionado con: (1) el planteamiento de preguntas y la búsqueda de diversas maneras de responderlas; (2) la formulación de conjeturas basadas en el movimiento de objetos matemáticos y la cuantificación de atributos como medida de segmentos, ángulos, áreas, etc.; y (3) la búsqueda de argumentos que validen esas conjeturas transitando desde los argumentos empíricos y visuales hasta la construcción de argumentos geométricos y algebraicos. Así, la pregunta de investigación que sirvió de guía para el desarrollo de este estudio fue ¿De qué manera el diseño e implementación de las actividades en un MOOC basado en resolución de problemas y uso coordinado de tecnologías digitales influye en la construcción y el desarrollo del pensamiento matemático de los participantes?

Marco Conceptual

La construcción del pensamiento matemático está relacionada con la resolución de problemas ya que es un medio que permite identificar, explorar, probar y comunicar las estrategias de solución (Santos-Trigo, 2014; Schoenfeld, 1992). Diversas tecnologías digitales ofrecen a los estudiantes diversos caminos para representar y explorar conceptos y problemas matemáticos que extienden aquellos acercamientos basados en el uso de lápiz y papel (Santos-Trigo, 2014; Aguilar-Magallón & Poveda, 2017).

Un Sistema de Geometría Dinámica (SGD) puede utilizarse para integrar los procesos que intervienen en la resolución de problemas ya que permite generar representaciones o modelos dinámicos de los problemas matemáticos donde el movimiento de objetos particulares (puntos, rectas, segmentos, polígonos, etc.) puede ser explorado y explicado en términos de relaciones matemáticas. Así, el uso sistemático de tecnologías digitales resulta importante en la representación, exploración, comunicación y comprensión de conceptos matemáticos en la resolución de problemas.

Churchill et al. (2016) argumentan que se necesita un modelo para el diseño de ambientes de aprendizaje que proporcione, a profesores e investigadores, pautas para utilizar tecnologías digitales en el contexto de la enseñanza y aprendizaje. Proponen el modelo de diseño RASE que integra Recursos, Actividades, Soporte y Evaluación. Los Recursos, se refieren a los materiales disponibles para los estudiantes: videos, imágenes, documentos digitales, calculadoras, herramientas para la representación de situaciones matemáticas, etc. Las Actividades deben permitir a los estudiantes involucrarse en procesos de discusión, análisis y reflexión en colaboración con otros.

El Soporte establece que se deben incluir diversos medios de consulta y comunicación entre los estudiantes con el objetivo de que puedan obtener ayuda o retroalimentación en el momento en que lo necesiten y, así, fomentar su independencia del profesor o tutor. La evaluación debe favorecer que los estudiantes mejoren constantemente su aprendizaje, por ello, es necesario que cuando expresen sus ideas, analicen la retroalimentación recibida, a través de los medios de soporte, para refinar o ampliar los conceptos o ideas iniciales.

En un MOOC, los foros de discusión se convierten en un medio de conversación entre sus participantes y les ofrece la oportunidad de plantear, aclarar, conocer, contrastar ideas propias y de otros (Poveda & Aguilar-Magallón, 2017). Ernest (2016) argumenta que en la conversación intervienen: un hablante/proponente, un oyente/crítico y un Texto Matemático. El hablante/proponente plantea una idea (Texto Matemático) y el oyente/crítico responde proporcionando su punto de vista, aceptando o modificando la idea original. Posteriormente, el hablante/proponente puede asumir el rol de oyente/crítico, de esta manera, se alternan sus roles.

Metodología

El objetivo del diseño de las actividades del curso fue enfatizar que el aprendizaje de las matemáticas requiere problematizar o cuestionar las tareas o situaciones, pensar distintas maneras de resolver un problema, comprender y utilizar diversas representaciones, interpretar la solución y comunicar los resultados. Mediante este proceso, la formulación de preguntas se convierte en un medio que permiten a los participantes construir, desarrollar, refinar, o transformar sus formas de comprender y resolver problemas.

Los Recursos incluyeron representaciones o modelos dinámicos de problemas construidos en GeoGebra, vínculos a la plataforma Wikipedia y videos de KhanAcademy para el estudio y consulta de conceptos o relaciones matemáticas. Las Actividades comprendieron tres etapas: 1) *Movimiento*: los participantes tuvieron la oportunidad de explorar un modelo dinámico que representa un problema y explorar el comportamiento de algunos de sus objetos que resulta al mover otros elementos, 2) *Conjetura:* el objetivo es que, a partir del movimiento, los participantes formularan conjeturas que puedan ser sustentadas o refutadas a partir de argumentos visuales o empíricos y 3) *Justificación*: se buscó que toda conjetura fuera justificada mediante argumentos que involucraran propiedades y resultados matemáticos.

Se utilizó el foro de discusión para que los participantes tuvieran la posibilidad de plantear sus dudas e ideas las veces que consideraran necesarias. Así, el trabajo de los integrantes puede ser un punto de referencia para que otros retomen o extiendan las ideas, las contrasten y las discutan. Además, en los foros de discusión los participantes pueden recibir retroalimentación acerca de las ideas que plantean, esto puede favorecer el análisis y reflexión como parte del proceso de Evaluación.

Implementación del MOOC, sus participantes y procedimientos

El MOOC se construyó e implementó en la plataforma *Open edx* (https://open.edx.org/about-open-edx), tuvo una duración de siete semanas y estuvo compuesto por cinco Actividades. El requisito fue estar cursando o haber terminado estudios del nivel K-12. Participaron 2669 personas. Al inicio del curso, únicamente estuvo visible la Actividad 1, una semana después se mostraba la Actividad 2 y así sucesivamente, sin restringir el acceso a las anteriores.
Durante las siete semanas en que el curso estuvo disponible a los participantes, el equipo de diseño (ED) monitoreó el desarrollo de las Actividades y la participación en los foros. El trabajo consistió en lo siguiente:

1. Al inicio de la Actividad (primer y segundo día), clasificó los comentarios en cuatro categorías: respuestas a las preguntas que planteaba cada Actividad, acercamientos hacia la solución del problema (correctos e incorrectos), nuevas preguntas planteadas por los integrantes y propuestas para extender el problema. Posteriormente, se eliminaron los comentarios con ideas similares; se tomaron dos o tres de cada categoría y fueron colocados de tal forma que se mostraran al inicio de las conversaciones, así los participantes les podrían dar prioridad en su análisis.
2. Durante la Actividad intervenía en los foros solo cuando se requería orientar y extender la discusión. Nunca se respondía de manera directa a las preguntas de los participantes, sino que, se planteaban preguntas con el objetivo de generar discusión y que ellos mismos buscaran diferentes formas de solucionar la situación.

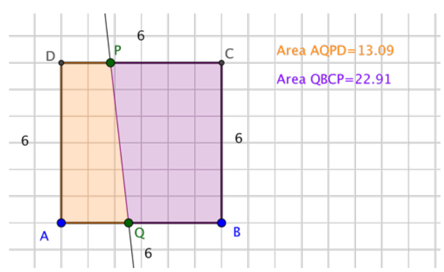
3. Al final de la Actividad se plantearon preguntas para promover la ampliación del tema y que los participantes buscaran extender los problemas iniciales.

Los datos de este estudio se recolectaron por medio de los foros de discusión. La unidad de análisis fueron las conversaciones de los participantes en cada Actividad. Al finalizar el curso, el equipo de diseño analizó y documentó cómo el diseño de las Actividades, la discusión e interacción entre los participantes y las acciones que tomó el equipo de diseño en el foro influyeron en el desarrollo de los episodios de resolución de problemas, según el marco de resolución de problemas y uso de tecnologías digitales de Santos-Trigo (2014).

Presentación de Resultados

Se discute una de las Actividades propuestas en el MOOC: Dos granjeros desean sembrar un terreno que tiene forma de un cuadrado. ¿Cómo dividir el terreno para que cada granjero siembre exactamente la misma área? La Actividad consistió desarrollar dos soluciones del problema, en cada una se proporcionó un modelo dinámico y un conjunto de preguntas para guiar el trabajo de los participantes. La Tabla 1 muestra la etapa en donde los integrantes del MOOC tuvieron la oportunidad de mover objetos en búsqueda de relaciones o invariantes y un resumen de las conversaciones en el foro.

Tabla 1: El movimiento y los primeros resultados de la exploración dinámica.

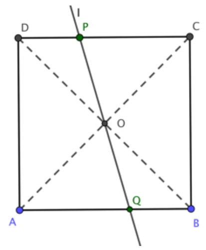
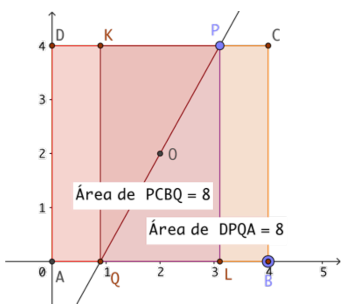
El movimiento y la exploración	Resultados de las interacciones en los foros
 **Movimiento:** Al mover los puntos A, B, P y Q ¿Qué ocurre con las áreas de las regiones? ¿Dónde situar los puntos P y Q para que las áreas de las regiones AQPD y QBCP sean las mismas?	Todos los participantes: 1. Coincidieron que, al mover los puntos P y Q, es posible obtener regiones de áreas iguales, sin importar la medida del lado del cuadrado. 2. Se basaron en el movimiento de objetos y la medición de áreas para determinar algunas soluciones: $P = C$ y $Q = A$; $P = D$ y $Q = B$; y PQ mediatriz de DC. En esta parte de la Actividad, los participantes no lograron observar que una solución general es cuando PQ pasa por el centro del cuadrado.

La siguiente pregunta que se planteó a los participantes fue: ¿Resulta importante el centro del cuadrado y la posición de la recta que divide al terreno en regiones de la misma área? Todos formularon su primera conjetura basada en argumentos visuales y empíricos (mediciones de las áreas de $AQPD$ y $QBCP$): "*Si la recta PQ pasa por el centro del cuadrado ABCD entonces lo divide en dos áreas iguales*". El ED cuestionó: *"¿Cuántas soluciones existen?*" Las respuestas coincidieron en que existe un número infinito de soluciones.

En la búsqueda de una justificación de la conjetura, se cuestionó a los participantes: ¿Qué conceptos, propiedades y recursos matemáticos se pueden usar para sustentar la conjetura? Todos coincidieron, en las conversaciones, en que era necesario trazar las diagonales del cuadrado y utilizar sus propiedades (Justificación 1 de la Tabla 2).

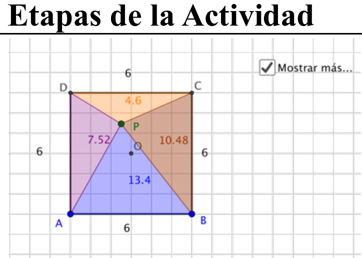
El participante P_1 presentó otra forma de justificar la conjetura: construyó un modelo dinámico, planteó sus ideas, contestó las dudas de otros y proporcionó retroalimentación a los que decían no comprender la justificación (Justificación 2, Tabla 2).

Tabla 2: Recursos y estrategias de exploración y solución.

Justificaciones	Recursos y estrategias de exploración y solución
Justificación 1 	**Recursos:** Propiedades del cuadrado y sus diagonales, ángulos entre paralelas, ángulos opuestos por el vértice y congruencia de triángulos. **Estrategia**: Colocar un punto móvil P sobre el lado DC, construir la recta PO y trazar las diagonales del cuadrado. **Justificación:** $\Delta DOP \cong \Delta BOQ$ (ALA), así $DP = BQ$, por lo tanto, los trapecios $AQPD$ y $QBCP$ son congruentes y tienen área igual.
Justificación 2 de P_1 	**Recursos:** Propiedades del cuadrado, congruencia de triángulos, ángulos entre paralelas, propiedades de rectas paralelas y perpendiculares. **Estrategia:** Colocar el punto móvil P sobre DC, trazar la recta PO y construir rectas perpendiculares a DC y AB que pasan por P y Q, respectivamente. **Justificación:** $\Delta QKP \cong \Delta PLQ$ (LAL) y los rectángulos $AQKD$ y $LBCP$ son congruentes (no proporciona argumentos). Preguntas de los participantes en el foro: (1) *¿Qué sucede si P=C y Q=A?* P_1: "*Se tienen dos triángulos rectángulos congruentes*", (2) "*¿Cómo sabes que AQ=PC?*". P_1 hizo referencia a la Justificación 1.

La siguiente parte de la Actividad guio a los participantes en la exploración y búsqueda de otra solución, los detalles se muestran en la Tabla 3.

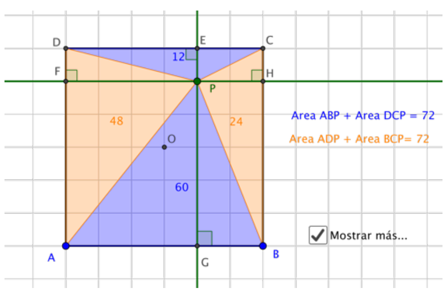
Tabla 3: La importancia del movimiento y la exploración.

Etapas de la Actividad	Interacciones en los foros
P es un punto móvil dentro del cuadrado. **Movimiento.** ¿Qué regiones se les puede asignar a los grajeros para que cada uno siembre la misma área?	Los participantes coincidieron en que: • Al mover P se pueden obtener las soluciones cuando: P coincide con alguno de los vértices, P es el punto medio de un lado o P es el centro del cuadrado. • Otra solución es cuando P pertenece a una de las diagonales del cuadrado o a la recta que une los puntos medios de AB y DC. Luego de las conclusiones anteriores, un participante mencionó que la suma de las áreas de los triángulos ABP y CDP se mantiene contante e igual a la mitad del área del cuadrado. Los demás estuvieron de acuerdo y afirmaron que no habían visto dicha invariante.

Conjetura: ¿Al variar la posición del punto P, qué ocurre con la suma de las áreas de los triángulos ABP y CDP?	Los participantes formularon la conjetura: "*Si P es un punto que está dentro del cuadrado, la suma de las áreas de los triángulos opuestos es la mitad del área del cuadrado $ABCD$*". Retroalimentación: "*Al mover P se forman 4 triángulos*", respuestas: "*Mueve el punto P de tal forma que coincida con un vértice, o bien, coloca P sobre sobre uno de los lados*".

Luego de que los integrantes formularon la conjetura, la siguiente parte consisitió en buscar relaciones matemáticas para justificarla, por ello, se incluyó, en el modelo dinámico del cuadrado, las rectas FH y EG. Los participantes discurtieron y presentaron la Justificación 1 que muestra la Tabla 4. La Justificación 2 de la Tabla 4 fue construida y compartida por el participante P_2. Los demás aprobaron las ideas y construyeron modelos algebraicos como otra ruta para sustentalo.

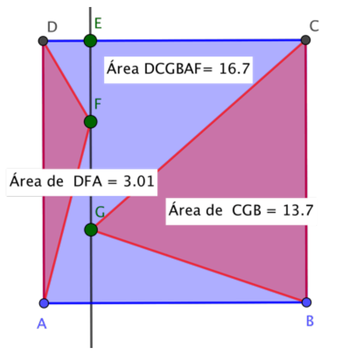
Tabla 4: Recursos y estrategias de exploración y solución.

<table>
<tr><th>Justificación de la Solución 2</th><th>Recursos, estrategias y justificación</th></tr>
<tr><td>

¿Qué propiedades son importantes para presentar un argumento que sustente la conjetura? ¿Qué propiedades tienen los triángulos que se generan al trazar las rectas perpendiculares a los lados que pasan por el punto P?</td><td>Recursos: Propiedades de rectas paralelas y perpendiculares, triángulos rectángulos congruentes
Estrategia 1: Trazar las rectas EG y FH perpendiculares a los lados del cuadrado que pasan por P.
Justificación 1. $\Delta AGP \cong \Delta AFP$ (LLL) y $\Delta GBP \cong \Delta HBP$ (LLL), del mismo modo $\Delta CEP \cong \Delta CHP$ y $\Delta EDP \cong \Delta FDP$. Por lo tanto, $Area\, \Delta DCP + Area\, \Delta ABP = Area\, \Delta BCP + Area\, \Delta DPA$.
Estrategia de P_2. La suma de las alturas de ΔAGP y ΔDPE, sin importar la posición de P, es constante e igual a BC.
Justificación de P_2: La suma de las alturas de ΔPCD y ΔABP es igual a la suma de las alturas de ΔDAP y ΔBCP, además, los 4 triángulos tienen la misma base, por lo tanto, $Area\, \Delta PCD + Area\, \Delta ABP = Area\, \Delta DAP + Area\, \Delta BCP$.</td></tr>
</table>

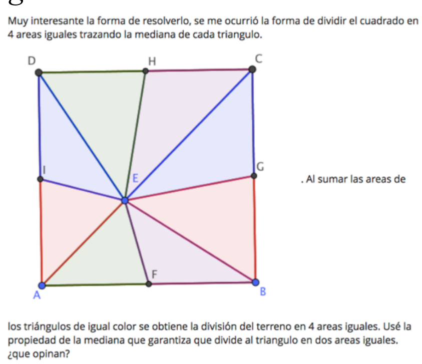
El ED cuestionó en el foro: *"¿Existe otra forma de solucionar el problema? ¿Qué otras preguntas se pueden plantear?"*. El participante P_3 retomó las ideas P_2 (Jjustificación 2 de la Tabla 4) y construyó un modelo dinámico del cuadrado con otra solución (Justificación 1 de la Tabla 5). Los demás participantes aprobaron las ideas de P_3 y discutieron otra forma de sustentar la conjetura (Justificación 2, Tabla 5). Todos los participantes validaron las ideas expuestas y proporcionaron ayuda a otros.

Tabla 5: Otra solución del problema.

<table>
<tr><th>Otra solución</th><th>Recursos, estrategias y justificación</th></tr>
<tr><td>
</td><td>Recursos: Propiedades de rectas paralelas y perpendiculares.
Estrategia: Colocar un punto móvil E sobre DC, trazar la recta m perpendicular a DC que pasa por E. Colocar dos puntos móviles F y G sobre m.
Justificación 1 de P_3: La suma de las alturas de ΔAFD y ΔBCG es constante e igual a AB sin importar la posición de E, F y G, por lo tanto, $Area\ \Delta AFD + Area\ \Delta BCG$ es la mitad del área del cuadrado.
Justificación de otros participantes: Trazar rectas paralelas a los lados que pasan por los puntos móviles para obtener triángulos congruentes.</td></tr>
</table>

La Tabla 6 muestra dos de las preguntas que formularon los participantes y las respuestas que encontraron.

Tabla 6: Nuevos problemas planteados por los participantes.

<table>
<tr><th>Preguntas formuladas</th><th>Justificación</th></tr>
<tr><td>Extensión 1
“¿Cómo dividir el terreno en 4 regiones con la misma área?”.

</td><td>En la búsqueda de respuestas, un participante utilizó la propiedad: Al trazar una mediana de un triángulo, éste queda dividido en dos áreas iguales. Así, construyó y presentó un modelo dinámico similar al de la Solución 2 y en donde trazó la mediana de cada triángulo (EI, EH, EG y EF). Al sumar las áreas de los triángulos DEI y CGE, se obtiene que su valor es un cuarto del área del cuadrado.
Los demás participantes en la conversación estuvieron de acuerdo con las ideas y recalcaron el uso de un SGD en la exploración y búsqueda de soluciones.</td></tr>
<tr><td>Extensión 2
“¿Será posible determinar un cuadrilátero inscrito dentro de cualquier otro tal que sea la mitad de su área?”.</td><td>Otro participante compartió información de Wikipedia relacionada con el teorema de Varignon: En cualquier cuadrilátero, los puntos medios de los lados forman un paralelogramo cuya área es la mitad de la del cuadrilátero original.</td></tr>
</table>

Discusión de los resultados

El diseño de las Actividades guio el trabajo de los participantes hacia la búsqueda de diversas formas de explorar los modelos dinámicos. Cada modelo representó para los participantes un punto de partida que les permitió identificar conceptos, plantear conjeturas basadas en el movimiento de los objetos matemáticos presentes en la configuración dinámica y sus relaciones o invariantes. El uso de los foros de discusión favoreció la comunicación, análisis y contraste de ideas matemáticas, además permitió a los participantes proporcionar u obtener retroalimentación de otros. Lo anterior fue importante para que transitaran de soluciones visuales y empíricas (asociadas con el uso de las herramientas como movimiento de objetos dentro de la configuración, la cuantificación de atributos como medida de segmentos, ángulos, áreas, etc.) hacia la presentación de argumentos geométricos y algebraicos en la validación de las conjeturas formuladas.

Durante el desarrollo del curso, el monitoreo que realizó el ED en la clasificación y la jerarquización de los comentarios favoreció la discusión, refinamiento de conceptos e ideas matemáticas y los episodios de la resolución de problemas. Por ejemplo, los participantes formularon conjeturas, presentaron argumentos para justificarlas y formularon nuevos problemas. El planteamiento de preguntas que presentó el ED fomentó la discusión de las ideas matemáticas, la formulación de nuevas soluciones y extensiones del problema (Ver Tabla 2, 5 y 6).

Algunas personas participaron en el foro asumiendo el rol específico de proporcionar retroalimentación a las preguntas de otros, esto, junto con el diseño de las actividades, promovió el trabajo colaborativo y fomentó la independencia de los participantes en el proceso de la construcción y su desarrollo del pensamiento matemático.

Conclusiones

Los resultados muestran que las diversas tecnologías digitales utilizadas en este estudio y la integración de los componentes Recursos, Actividades, Soporte y Evaluación basados en la resolución de problemas permitieron crear un ambiente de trabajo colaborativo. La plataforma digital permitió incluir representaciones dinámicas de los problemas elaboradas en GeoGebra, en las cuales los participantes tuvieron la oportunidad de explorar, identificar conceptos, buscar conjeturas y diversos argumentos para sustentarlas. En este proceso, utilizaron estrategias asociadas con el uso de la herramienta como movimiento de objetos dentro de la configuración dinámica y la cuantificación de sus atributos (longitudes y áreas).

Durante el desarrollo de las Actividades, un grupo de participantes proporcionó retroalimentación a otros, lo que favoreció aclarar o refinar dudas o ideas relacionadas con conceptos matemáticos, con la exploración del modelo dinámico, con la formulación de conjeturas y su justificación. Lo anterior permitió a los participantes avanzar en el desarrollo y comprensión de las Actividades en forma colaborativa y sin depender de un tutor.

Las acciones que tomó el ED en sus intervenciones en los foros expandieron las discusiones en tres direcciones: (1) favorecieron la creación de grupos de trabajo donde se discutían los diferentes episodios de la resolución de problemas, (2) dirigieron las discusiones y fomentaron la comprensión de conceptos e ideas matemáticas y (3) permitieron a los participantes formular preguntas y buscar respuestas, esto los llevó a plantear nuevas soluciones y extensiones al problema. Es importe reconocer que durante la etapa del diseño y la implementación de un MOOC se debe buscar que los participantes creen la conciencia de que ellos mismos monitoreen sus avances en la comprensión y uso de las ideas matemáticas en la resolución de problemas.

Un factor por considerar en el trabajo a futuro es la posibilidad de que los participantes construyan y presenten sus propios modelos dinámicos de los problemas, ya que, en este estudio, pese a que no se solicitó explícitamente, algunos construyeron y compartieron sus construcciones dinámicas fomentando la discusión de ideas, las formas de resolver y extender el problema.

In this study, the elements of the design and results of the implementation of a Massive Open Online Course (MOOC) based on the problem solving and the use of digital technology are analyzed and discussed. The results show that the design of the activities, the intervention of the MOOC design team in the discussion forums and the interaction among the participants allowed and favored the creation of a collaborative environment in problem solving. Participants shared and discussed their mathematical ideas, as part of a virtual learning community, which allowed them to formulate conjectures, explore and search properties to sustain relationships, and communicate results.

Key Words: Problem Solving, Technology.

Introduction

During and after their lessons, it is common for the students to use different digital platforms on Internet to have access to information about disciplinary contents, share and discuss ideas, take specialized courses in a topic and interact with experts. The spaces and times in which the learning is developed have been changed and altered by the connectivity of the network (Gros, 2016). Some universities have made available to the general public Massive Open Online Courses through different digital platforms. The virtual community that participates in these courses comprises a large group (usually thousands) of people with different levels of studies, age, domain or prior knowledge of the subject, among others. In the development of the activities of a MOOC there is not a professor in charge to answer or provide timely follow-up to each participant. Each member defines their degree of commitment and participation in the activities. For this study, the MOOC: Mathematical Problem Solving and use of Digital Technologies was built, based on the frameworks: Problem Solving (Santos-Trigo, 2014) and RASE (Resources, Activities, Support and Evaluation) (Churchill, King, & Fox Churchill, 2016). Churchill, et al., (2016) argue that an online learning environment must include elements that allow its participants to work and collaborate in team with others. The activities must motivate the active participation of the students in a reflecting environment of discussion and focus in a context in which the tasks involve the students in the episodes of problem solving: formulation of questions, search for different solution methods, exploration of different representations, search for patterns, variables, and relations between mathematical objects, arguments representation, communication of results and formulation of new problems (Santos-Trigo, 2014). The systematic use of digital technologies is important in the representation, exploration, communication and understanding of mathematical concepts in problem solving (Santos-Trigo, Moreno-Armella, 2016).

In this study, it was interesting to analyze the design of the mathematical activities and the forms of mathematical reasoning of the participants related to: (1) the question formulation and the search for different ways to answer them (2) the formulation of conjectures based on the movement of mathematical objects and qualification of attributes, e.g. measure of segments, angles, areas etc. and (3) the search for arguments that validate those conjectures moving from the empiric and visual arguments to the geometric and algebraic arguments. In this way, the research question that guided the development of this study was: How does the design and implementation of activities in a MOOC based on problem solving and coordinated use of digital technologies influence the construction and development of the participants' mathematical thinking?

Framework

The construction of the mathematical thinking is related to problem solving because it is a way that allows to identify, explore, test and communicate the solution strategies (Santos-Trigo, 2014; Schoenfeld, 1992). Diverse digital technologies offer the students different ways to represent and explore concepts and mathematical problems that extend those approaches based on the use of pen and paper (Santos-Trigo, 2014; Aguilar-Magallón & Poveda, 2017).

A Dynamic Geometry System (DGS) can be used to integrate the processes that are present in problem solving that allow to generate representations or dynamic models of the mathematical problems where the movement of particular objects (points, lines, segments, polygons, etc.) could be explored and explained in terms of mathematical relations. Hence, the systematic use of digital technologies is important in the representation, communication, and understanding of mathematical concepts in problem solving.

Churchill et al. (2016) argue that a model is needed for designing the activities that provides professors and researchers, guidelines to use digital technologies in the teaching and learning context. They propose the design model RASE, the Resources refer to the available materials for the students: videos, images, digital files, calculators, tools for the representation of mathematical situations, etc. The Activities must allow the students to be involved in the processes of discussion, analysis and reflection in collaboration with others.

The Support establishes that some means of consultation and communication must be included between the students, with the objective that they can get help or feedback at the moment that they need it and, in this way, to promote the independence from the professor or tutor. The Evaluation should encourage the improvement of the students' learning, therefore, it is necessary that when they express their ideas, they analyze the feedback received through support media, to refine or expand the initial concepts or ideas. In a MOOC the discussion forums become a mean of conversation between their participants and give them the opportunity of proposing, getting to know, contrast own ideas and those of others (Poveda & Aguilar-Magallón, 2017). Ernest (2016) argues that in the conversation are involved: a speaker/proponent, a listener/critic and a Mathematical Text. The speaker/proponent proposes an idea (Mathematical Text) and the listener /critic answers by giving his point of view, accepting or changing the original idea. After that, the speaker/proponent can assume the listener /critic role, in this way the roles are reversed.

Methodology

The objective of the activities design of the course was to emphasize that the learning of mathematics requires problematizing or questioning the tasks or situations, thinking about different ways to solve a problem, understanding and using different representations, interpreting the solution and communicating the results. Through this process, the formulation of questions becomes a medium that allows the participants to build, develop, re-think or transform their ways to understand and solve problems. The Resources included representations or dynamic models of problems built in GeoGebra, links to the Wikipedia platform and the videos of KhanAcademy for the study of mathematical concepts and relationships.

The activities included three stages: 1) *Movement*: the participants had the opportunity to explore a dynamic model of a problem and the behaviour of some of its parts that results from the movement of other elements, 2) *Conjecture*: the objective is that from the movement, the participants will formulate conjectures that can be supported or refuted from visual or empirical arguments and 3) *Justification*: it was tried that all conjectures were justified through arguments that involved mathematical properties and results. The forum was used to give the participants the possibility to propose their doubts and ideas to others as many times as needed. In this way, the students' work can be used as a reference point for others to take up or extend the ideas, contrast them and discuss them. In addition, in the forums the participants can receive feedback about their proposed ideas, it can be useful for the analysis and reflection as part of the Evaluation process.

Development of activities, participants and procedures

The MOOC was built and implemented in the platform *Open edx* (https://open.edx.org/about-open-edx), it has a duration of seven weeks and it was composed by six activities. The only requirement for registration was that interested individuals had a minimum schooling level equivalent to grade 12. 2669 people signed up for the MOOC. At the beginning of the course, only Activity 1 was visible, Activity 2 was shown one week later and so on, without restricting access to the previous ones. During the seven weeks in which the course was available to the participants, the Design Group (DG) monitored the development of the activities and the participation in the forums. The work consisted in:

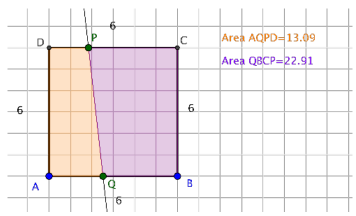
1. At the beginning of the activity, the comments were classified in four categories: answers to the questions that were proposed in each activity, approaches to problem solving (correct and incorrect), new questions asked by the members and proposals to extend the problem. Later, the comments with similar ideas were eliminated, two or three were chosen from each category and were placed in such way that they were shown at the beginning of the conversations, in this way the participants could give them priority in their analysis.
2. During the Activity, it intervened in the forums only when it was necessary to orient and extend the discussion. Questions from the participants were never answered directly, but some different questions were asked in order to generate more discussion and to allow the participants to find different ways to create the solution by their own.
3. At the end of the activity, questions were raised to promote the extension the topic and for the participants to look to extend the initial problems.

The data of this study were collected through the discussion forums. The unit of analysis were the conversations of the participants in each activity. At the end of the course, the design team analyzed the conversations. Interested to analyze and document how the activities design, the discussion and interaction between the participants and the actions that the design team took in the forum influenced in the development of the episodes of problem solving, according to the framework of the problem solving and the use of dynamic technologies of Santos-Trigo (2014).

Presentation of results

One of the Activities proposed in the MOOC is discussed: Two farmers want to plant land that is shaped like a square. How to divide the land so each farmer can plant exactly the same area? The activity consisted in developing two solutions to the problem, in each one a dynamic model was given and a set of questions to guide the work of the participants. Chart 1 shows the stage in which the members of the MOOC had the opportunity to move the objects to look for relations or invariants and a summary of the conversations in the forum.

Chart 1: The movement and the results of the dynamic exploration.

The movement and the exploration	Results of the interaction in the forums
 **Movement:** When we move the points A, B, P & Q, what happens with the areas of the regions? Where to place the points P and Q so that the areas of the $AQPD$ & $QBCP$ regions are the same?	All the participants pointed: 1. It is possible to move the point P and Q and obtain regions of same areas regardless of the size of the square side. 2. The object movement and the measurement of the areas can give some solutions: $P = C$ y $Q = A$; $P = D$ y $Q = B$ & PQ perpendicular bisector of DC. In this part of the activity, the participants could not observe that a general solution is when $O \in PQ$ ($O$is the square center).

The next questions asked to the participants was: Is the center of the square and the position of the line that divides the land in regions of the same area important? All of them made their first conjecture based in visual and empiric arguments (measurement of the areas $AQPD$ y $QBCP$): "*If $O \in PQ$ then it divides the square into two figures of the same area*".

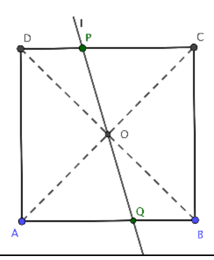
The DG asked: *"How many solutions are there?"* The answers agreed that there is an infinite number of solutions. In the search for a justification of the conjecture, the participants were asked: What concepts, properties and mathematical resources can be used to support the conjecture? In the conversations all of them pointed out the need to trace the diagonals of the square and use its properties (Justification 1, Chart 2).

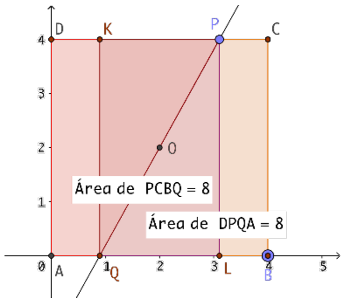
The participant P_1 presented a way to justify the conjecture: he built a dynamic model, answered the doubts of others and gave feedback to the people that could not understand the justification. (Justification 2, Chart 2).

The next section of the activity guided the participants in the exploration and searching of another solution, the details are showed in Chart 3.

Later, the members formulated the conjecture, the next section consisted in searching for mathematical relationships to justify it, therefore, in the dynamic model of the square the lines FH and EG were included. All the participants discussed and presented the Justification 1 that is shown in Chart 4.

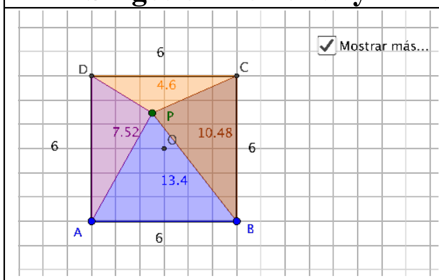
Chart 2: Resources and strategies of exploration and solution.

Justifications	Resources and strategies of exploration and solution
Justification 1 	**Resources:** Properties of the square and its diagonals, angles between parallel lines, angles opposed by the vertex and congruence of triangles. **Strategy:** Put P on the side DC, build the line PO and trace the diagonals of the square. **Justification:** ΔDOP≅ΔBOQ (SAS), so $DP = BQ$, therefore, the polygons $AQPD$ y $QBCP$ have equal area.

<table>
<tr>
<td>Justification 2 of P_1


</td>
<td>Resources: Properties of the square, congruence of triangles, angles between parallel lines, properties of parallel lines and perpendicular lines.
Strategy: Placed P on DC, trace the PO line and build perpendicular lines to DC and AB that pass through P and Q, respectively.
Justification: $\Delta QKP \cong \Delta PLQ$ (SAS) and the rectangles $AQKD$ and $LBCP$ are congruent (arguments are not given). Participants' questions in the forum: (1) What happens if P=C & Q=A? P_1: "We have two congruent right triangles", (2) "¿How do you know that AQ=PC?" P_1 reference Justification 1.</td>
</tr>
</table>

Chart 3: Movement and exploration importance

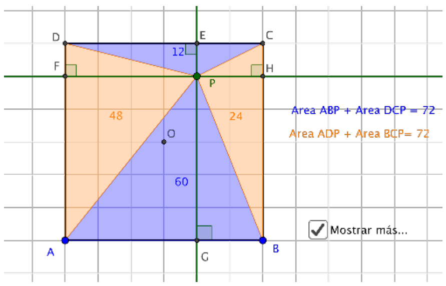
<table>
<tr>
<th>Stages of the activity</th>
<th>Interaction in the forums</th>
</tr>
<tr>
<td>

P is inside the square.
Movement. What regions can be assigned to the farmers so that each one can plant the same area?</td>
<td>Ten participants pointed out some solutions:
1. P matches with any vertex, P is the central point of a side or P is the center of the square.
2. P belongs to a diagonal of the square or to the line that joins the middle points of AB and DC.

In this part of the activity, none of the participants observed solutions based on the movement of objects, for instance, the solution is obtained for any position of point P inside the square.</td>
</tr>
<tr>
<td>Conjecture: Is it possible to identify any relationship between the values of the areas? When varying the position of the point P, what happens to the sum of the areas of the triangles ABP and CDP?</td>
<td>All participants formulated and shared the conjecture: "If P is a point that is inside of the square, the sum of the areas of the opposite triangles is the half of the square ABCD". Some participants gave feedback to others, for example, someone mentioned: "P always generates four triangles", it was suggested: "Move point P in such a way that it matches with a vertex, or P over one of the sides".</td>
</tr>
</table>

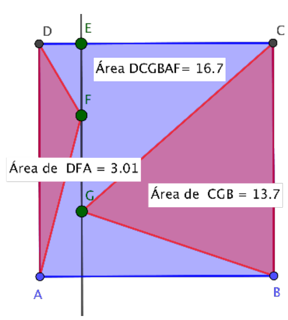
The Justification 2 of Chart 4 was built and shared by participant P_2. The other participants approved the ideas and built algebraic models as another way to support it.

Chart 4: Resources and strategies of exploration and solution.

Justification of the Solution 2	Resources, strategies and justifiaction
 What properties are important to present an argument that supports the conjecture? What properties have the triangles that are generated when we trace the perpendicular lines to the sides passing through the point P?	**Resources:** Properties of parallel and perpendicular lines, congruence of triangles. **Strategy 1**: Trace the perpendicular lines EG and FH to the sides of the square that pass by P. **Justification 1.** $\Delta AGP \cong \Delta AFP$ (SSS) and $\Delta GBP \cong \Delta HBP$ (SSS), at the same time $\Delta CEP \cong \Delta CHP$ & $\Delta EDP \cong \Delta FDP$. Therfore, Area ΔDCP + Area ΔABP = Area $\Delta BCP + Area\Delta DPA$. **Strategy of P_2.** the sum of ΔAGP and ΔDPE, without taking into account the position of the point P, is constant and equal to BC. **Justification of P_2:** The sum of the heights of ΔPCD and ΔABP is equal to the sum of the heights of ΔDAP y ΔBCP, so, the four triangles have the same base and Area ΔPCD + Area ΔABP = Area $\Delta DAP + Area\Delta BCP$.

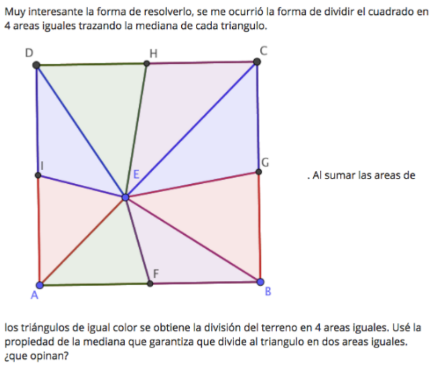
The DG asked in the forum: Is there another way to solve the problem? What other questions can you ask? The participant P_3 revisited the ideas from Justification 2 of Chart 4 and built a dynamic model of the square with other solution (Justification 1 of the Chart 5). Other participants approved the ideas and discussed another way to support the conjecture (Justification 5 Chart 5).

Chart 5: Another Solution

Another solution	Resources, strategies and justification
	**Resources:** Properties of the parallels and perpendicular lines. **Strategy:** Placed the mobile point E on DC, trace the perpendicular line m to DC that passes by E. Placed two mobile points $F$and G on m. **Justification 1:** The sum of the heights of ΔAFD and ΔBCG is constant and equal to AB for all positions of E, F and G, therefore, Area ΔAFD + Area ΔBCG is half of the square's area. **Justification 2.** Trace parallel lines to the sides that pass by the mobile points F & G to generate congruent triangles.

The Chart 6 shows two of the questions the participants formulated and the answers they found.

Chart 6: Extensions of the problem.

Formulated questions	Justification
Extension 1 *"How to divide the land into 4 regions with the same area?".*  	During the search for the answers, one participant used the property: Each median divides the area of the triangle in half. In this way, he built and presented a dynamic model similar to the one of the Solution 2 where he traced the medians of triangles ($EI, EH, EG \& EF$). The sum of the areas of triangles DEI and CGE is a quarter of the square's area.
Extension 2 *"Is it possible to determine a quadrilateral inscribed within any other such that it has half of its area?".*	The participants shared information of Wikipedia related to the theorem of Varignon: *The midpoints of the sides of an arbitrary quadrilateral form a parallelogram. If the quadrilateral is convex or concave, then the area of the parallelogram is half the area of the quadrilateral.*

Discussion of the Results

The activity design guided the participants' work towards searching different ways to explore the dynamic models. Each model represented to the participants a starting point that allowed them to identify concepts, propose conjectures based on the movement of the mathematical objects present in the dynamic setting and their relationships or invariants.

The use of discussion forums helps the communication, analysis and the contrast of mathematical ideas, in addition, they allowed the participants to give and receive feedback from others. This was important for transiting from visual and empirical solutions (associated with the use of tools such as movement of objects within the setting, quantification of attributes as measures of segments, angles, areas, etc.) to the presentation of geometric and algebraic arguments in the validation of the formulated conjectures.

During the development of the course, the monitoring that the DG did in the classification of the comments helped the discussion, re-think of mathematical concepts and ideas and the episodes of problem solving. For example, the participants formulated conjectures, they presented arguments to justify them and also proposed new problems. The proposal of questions made by the DG helped the discussion of mathematical ideas, the proposal of new solutions and extensions of the problem. (See Chart 2, 5 and 6).

In addition, a group of participants assumed the role of giving feedback to the questions of other people, this promoted the collaborative work and fomented the independence of the participants in the construction process and the development of mathematical thinking.

Conclusions

The results show that the different digital technologies used in this study and the integration of the components: Resources, Activities, Support and Evaluation based in problem solving allowed to create a collaborative work environment. The digital platform allowed the inclusion of dynamic representations of the problems elaborated in GeoGebra, in which the participants had the opportunity to explore, identify concepts, look for conjectures and different arguments to justify them. In this process, they used strategies associated with the use of the tool like object movement within the dynamic setting and the quantification of its attributes (lengths and areas). During the development of the Activities, a group of participants provided feedback to others, which helped to clarify ideas related to mathematical concepts, with the exploration of the dynamic model, with the formulation of conjectures and their justification. This allowed the participants to advance in the development and understanding of the activities in a collaborative way and without depending on a professor or tutor.

The actions taken by the DG in its interventions in the forums increased the discussions in three directions: (1) they helped the creation of working groups where the different episodes of problem solving were discussed, (2) led discussions and fomented the understanding of mathematical concepts and ideas and (3) allowed the participants to propose questions and look for answers, this led them to propose new solutions and extensions to the problem. It is important to recognize that during the design and implementation stage of a MOOC, participants should be expected to create an awareness that they themselves monitor their progress in understanding and using mathematical ideas in solving problems.

One factor to be considered in future work is the possibility for participants to build and present their own dynamic models of the problems, since, in this study, although it was not explicitly requested, some built and shared their dynamic constructions helping the discussion of ideas, the solutions and extensions for a problem.

References

Aguilar-Magallón, D. & Poveda, W. (2017). Problem Posing Opportunities with Digital Technology in Problem Solving Environments. In *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*.

Churchill, D., Fox, B., & King, M. (2016). Framework for Designing Mobile Learning Environments. In D. Churchill, B. Fox, & M. King (Eds.), *Mobile Learning Design, lecture Notes in Educational Technology* (pp. 3-25). Singapore: Springer.

Gros, B. (2016). The Dialogue Between Emerging Pedagogies and Emerging Technologies. In B. Gros, Kinshuk, & M. Maina (Eds.), *The Future of Ubiquitous Learning Designs for Emerging Pedagogies* (pp. 3-24). Berlin Heidelberg: Springer.

Santos-Trigo, M. (2014). *La resolución de problemas matemáticos: fundamentos cognitivos*. Segunda edición. México: Trillas, Asociación Nacional de profesores de matemáticas.

Santos-Trigo, M., & Moreno-Armella, L. (2016). The Use of Digital Technology to Frame and Foster Learners' Problem-Solving Experiences. In Posing and Solving Mathematical Problems (pp. 189-207). Springer International Publishing.

Poveda, W. & Aguilar-Magallón, D. (2017). Mathematical Problem Solving and Digital Technologies in a Massive Online Course. In *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*.

Schoenfeld A. (1992). Learning to think mathematically: Problem Solving, metacognition, and sense making in mathematics. En D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-371.) New York: Macmillan.

LA FORMULACIÓN Y RESOLUCIÓN DE PROBLEMAS EN LA RECONSTRUCCIÓN DE FIGURAS MEDIANTE MODELOS DINÁMICOS

POSING AND SOLVING PROBLEMS IN FIGURE RECONSTRUCTION THROUGH DYNAMIC MODELS

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Se reportan y analizan episodios de formulación y resolución de problemas relacionados con la construcción de configuraciones dinámicas durante un curso de Maestría en Educación Matemática ¿Cuáles son las heurísticas y estrategias que exhiben profesores y futuros profesores en el proceso de construir representaciones dinámicas de la figura presente en un problema geométrico de demostración? Los resultados muestran que el uso de GeoGebra puede ser útil para motivar e involucrar a los profesores en diversos episodios de formulación y resolución de problemas. En este camino, algunas estrategias importantes fueron relajar las condiciones del problema, la exploración de casos particulares, la búsqueda de patrones e invariantes y la visualización de lugares geométricos de puntos de intersección.

Keywords: Resolución de Problemas, Tecnología, Geometría y Pensamiento Geométrico y Espacial, Preparación de Maestros en Formación.

Introducción

La resolución de problemas es un tema importante en la agenda de investigación en Educación Matemática. Actualmente se reconoce que el uso de tecnologías como un Sistema de Geometría Dinámica (SGD) genera oportunidades para desarrollar conocimiento matemático, pero también permite transformar escenarios de enseñanza alrededor de la resolución de problemas (Santos-Trigo, Reyes-Martínez & Aguilar-Magallón, 2016). Un principio fundamental de este enfoque de resolución de problemas y uso de tecnologías digitales es que el individuo reconozca la resolución de problemas como una oportunidad para plantear preguntas relevantes de forma constante y sistemática que le ayude en su actividad matemática (Santos-Trigo, Reyes-Martínez & Aguilar-Magallón, 2015). Para este estudio, resultó importante investigar en qué medida el uso de un SGD (GeoGebra), permite a los participantes involucrarse en actividades de planteamiento de problemas. En este camino, la pregunta de investigación que guio el desarrollo de este estudio es: ¿Cuáles son las heurísticas y estrategias que exhiben profesores y futuros profesores en el proceso de construir representaciones dinámicas de la figura presente en un problema geométrico de demostración?

Marco Conceptual

Por un lado, la formulación de preguntas relevantes o actitud inquisitiva al enfrentarse a situaciones problemáticas es un aspecto central de la actividad matemática de un individuo. Comúnmente el planteamiento de preguntas durante todo el proceso de resolución de problemas conduce a la formulación de nuevos problemas (Osana & Pelczer, 2015). Por otro lado, el planteamiento de preguntas y problemas es importante no solo para el desarrollo de habilidades matemáticas en profesores y futuros profesores, sino también para el desarrollo de habilidades didácticas. En la literatura se reconoce que, en general, profesores y futuros profesores tienen serias dificultades al enfrentarse a tareas de planteamiento de problemas (Rosli et al., 2015).

¿Cuál es el papel de las herramientas digitales en el planteamiento de preguntas y problemas y en la formación de profesores para desarrollar habilidades de formulación y reformulación problemas? Leikin (2015) argumenta que el uso de un SGD es útil para formular preguntas y problemas relacionados con el análisis de relaciones matemáticas presentes en figuras que se usan para resolver problemas geométricos. Aguilar-Magallón y Poveda-Fernández (2017) afirman que las permisibilidades de un SGD (como el arrastre de objetos, la medición de atributos, el uso de deslizadores y la visualización de lugares geométricos) en conjunto con heurísticas de resolución de problemas (como relajar las condiciones del problema, analizar casos particulares y visualizar patrones e invariantes) pueden motivar diversos episodios de formulación y resolución de problemas.

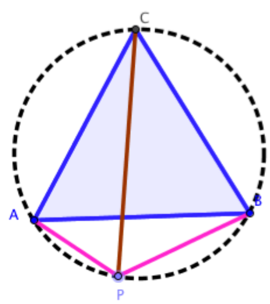
Metodología

Participantes

En este estudio participaron nueve estudiantes de un curso de Maestría en Educación Matemática durante cuatro sesiones semanales con duración de tres horas cada una. El grupo estuvo conformado por seis profesores en servicio y tres futuros profesores. Todos los participantes tenían formación académica relacionada con matemáticas.

El problema

Para este reporte se analizan los episodios de formulación y resolución de problemas que surgieron a partir del análisis de la figura presente en un problema geométrico de demostración (Figura 1).



Sea ABC un triángulo equilátero y P cualquier punto de la circunferencia que lo circunscribe, demostrar que $AP + BP = CP$.

Figura 1. El problema

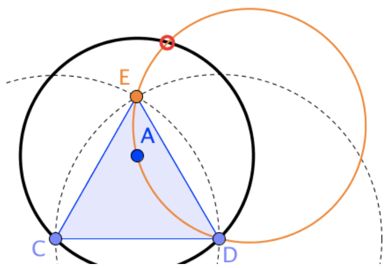
Resultados

En esta sección se expone un episodio de formulación y resolución de problemas que fue motivado por el uso de GeoGebra. En particular, el análisis se centró en las formas de razonamiento, recursos y heurísticas que exhibieron los participantes en este proceso.

Construcción de una representación dinámica del problema

¿Cómo obtener una representación dinámica del problema utilizando el SGD? Los participantes formularon dos problemas relacionados con dos caminos para obtener una representación dinámica del problema: 1) construir un triángulo equilátero y luego trazar su circunferencia circunscrita y 2) inscribir un triángulo equilátero en una circunferencia dada. En la Tabla 1 se muestra un acercamiento dinámico a la solución del segundo problema mostrado por los participantes. Este acercamiento estuvo basado en la heurística de relajar las condiciones del problema y la visualización de lugares geométricos por medio del arrastre de objetos de la configuración dinámica. Los participantes trazaron un triángulo que cumplía de forma parcial con las condiciones del problema; trazaron un triángulo equilátero con dos vértices móviles sobre la circunferencia. Posteriormente, mediante el movimiento de la configuración plantearon las condiciones necesarias para resolver el problema.

Tabla 1: Acercamiento dinámicos y solución empírica mostrada por los participantes.

Configuración	***Análisis***
 1. Puntos C y D móviles. Se traza el triángulo CDE equilátero por medio de circunferencias. Lugar geométrico descrito por el punto E cuando se mueve el punto C.	***Recursos.*** Triángulo equilátero y lugar geométrico. ***Estrategia dinámica.*** Relajar las condiciones del problema. EL triángulo CDE es equilátero, pero tiene sólo dos vértices inscritos en la circunferencia. Así la estrategia de exploración consiste en mover el punto C hasta que el punto E pertenezca a la circunferencia inicial. ***Solución empírica.*** La intersección del lugar geométrico descrito por el punto E con la circunferencia inicial definen uno de los vértices del triángulo equilátero inscrito (otro es el punto D y el último se encuentra por construcción).

En este acercamiento se observa el uso de lugar geométrico como herramienta para resolver problemas, sin embargo, en una primera instancia la solución se quedó en un nivel visual y empírico debido a la imposibilidad de GeoGebra para intersecar lugares geométricos. Para poder intersecar lugares geométricos (en GeoGebra) se requiere transformarlos en objetos geométricos robustos; en otras palabras, que su construcción dependa de ciertos elementos bien definidos. Por ejemplo, para robustecer un lugar geométrico que se supone circunferencia se necesita su centro y radio. En la búsqueda del centro para trazar de forma robusta la supuesta circunferencia usada en el acercamiento dinámico de la Tabla 1, los participantes se dieron cuenta que la intersección F de la mediatriz del segmento variable DE con la circunferencia original era invariante (Figura 2). Así supusieron que el punto F era el centro de la circunferencia y midieron la distancia al punto E para verificar que era constante. La detección de invariantes en la exploración del modelo dinámico de la Tabla 1 permitió a los participantes encontrar una solución sintética (que puede obtenerse con regla y compás) al problema (Tabla 2).

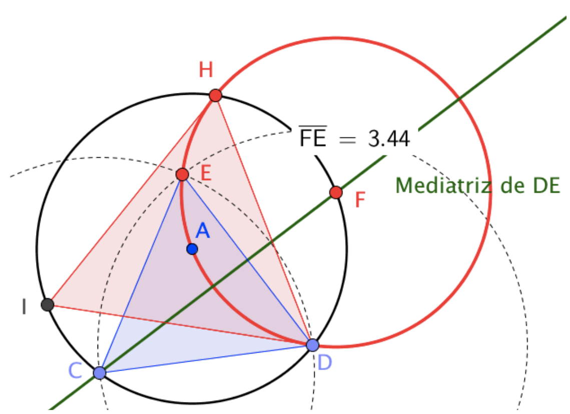
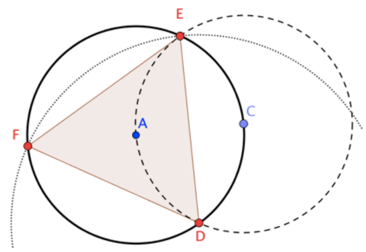


Figura 2. El punto $\boldsymbol{F}$ de intersección de la mediatriz de $\boldsymbol{DE}$ con la circunferencia inicial es invariante y es el centro de la circunferencia descrita por el punto $\boldsymbol{E}$.

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.

Tabla 2: Solución sintética encontradas por los participantes al analizar modelo dinámico.

Solución sintética	***Descripción***
 1	Punto C cualquiera sobre la circunferencia inicial de centro en A. Circunferencia con centro en C y radio CA. Los puntos de intersección E y D de las dos circunferencias formarán uno de los lados del triángulo equilátero inscrito. El tercer vértice F se encuentra con la circunferencia de centro en D y radio DE.

Discusión de resultados

Con ayuda del SGD, los participantes pudieron encontrar tres tipos de soluciones al problema de inscribir un triángulo equilátero en una circunferencia: soluciones empíricas, robustas y sintéticas. En una primera instancia, debido a que en GeoGebra no se pueden intersecar lugares geométricos, las solución encontrada fue empírica. En esta dirección, para encontrar la solución robusta (en términos de intersecciones) se requirió transformar (robustecer) el lugar geométrico. Posteriormente, la exploración de las propiedades del lugar geométrico mostrado en Tabla 1 permitió a los participantes encontrar una solución sintética al problema, es decir, una solución que puede obtenerse con trazos básicos de regla y compás. Tanto para las soluciones empíricas como para las soluciones exactas (robustas y sintéticas), fueron importantes las heurísticas de visualizar patrones o lugares geométricos, explorar casos particulares y detectar invariantes.

Conclusiones

Existe una gran variedad de problemas geométricos en los cuales la información o condiciones son presentados por medio de figuras o configuraciones geométricas. Comúnmente, cuando se resuelven problemas de forma tradicional (con papel y lápiz) la tarea de reconstruir las figuras que aparecen en los enunciados no es relevante, pues únicamente interesan las relaciones matemáticas implícitas en dichas figuras. Cuando se trabaja dentro de un SGD resulta primordial pensar en las formas de obtener representaciones dinámicas para explorar los problemas. Así, los resultados del estudio mostraron que considerar distintas maneras de representar una situación problemática puede motivar episodios de planteamiento de problemas.

This document reports the analysis of problem-solving and problem-posing episodes related to constructing dynamic configurations in a Math Education Masters course. What are the heuristics and strategies that teachers and prospective teachers exhibit in the process of constructing dynamic representations of the figure in a geometric problem? The results show that the use of GeoGebra motivates teachers to pose and solve problems. In this way, some important strategies were to relax the conditions of the problem, the exploration of particular cases, the search of patterns and invariants and the visualization of loci of intersection points.

Keywords: Problem Solving, Technology, Geometry and Geometric and Spacial Thinking, Teachers` Preparation in Training

Introduction

Problem solving is an important issue in the research agenda for Mathematics Education. Currently, it is recognized that the use of technologies such as a Dynamic Geometry System (DGS) generates opportunities to develop mathematical knowledge, but also allows transforming teaching scenarios into problem-solving scenarios (Santos-Trigo, Reyes-Martínez & Aguilar-Magallón, 2016). A fundamental principle of this approach for problem solving and use of digital technology is that the individual should be able to recognize that it provides an opportunity to, constantly and systematically, formulate relevant, helpful questions in their mathematical activity (Santos-Trigo, Reyes-Martínez & Aguilar-Magallón, 2015). For this study, it was important to investigate to what extent the use of a DGS (GeoGebra) allows participants to get involved into problem-solving activities. The research question that guided the development of this study was: What are the heuristics and strategies that teachers and future teachers develop in the process of constructing dynamic representations of a figure in a geometric-demonstration problem?

Conceptual Frame

A critical characteristic in a mathematical activity is the capacity to formulate relevant questions or possessing an inquisitive attitude when facing problematic situations. Frequently, formulating questions throughout the problem-solving process leads to formulation of new problems (Osana & Pelczer, 2015). Furthermore, formulating questions and new problems is a key factor not only for developing mathematical skills in teachers and prospective teachers, but also for the development of didactic skills. Previous studies recognize that, in general, professors and prospective teachers have serious difficulties when facing problem-posing tasks (Rosli et al., 2015). What is the role of digital tools in formulating questions and problems, and in training teachers to develop problem-posing skills? Leikin (2015) argues that using DGS helps to formulate both questions and problems related to the analysis of existing mathematical relationships in figures used to solve geometric problems. Aguilar-Magallón and Poveda-Fernández (2017) affirm that the affordances of DGS (such as dragging objects, measuring attributes, use of sliders, and loci visualization) together with problem-solving heuristics (such as relaxing problem conditions, analyzing particular cases, and visualizing patterns and invariants) motivate diverse episodes of problem formulation and resolution.

Methodology

Participants

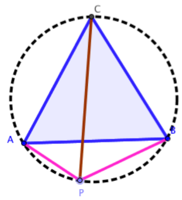
Nine students from a Math Education Masters course participated in this study, over four three-hour, weekly sessions. From this group, six were in-service teachers, and three were prospective teachers. All the participants had academic training related to mathematics.

The problem

The study consisted in analyzing the problems that emerged from the analysis of Figure 1, presented in a geometric-demonstration problem.

Results

This section presents an episode of problem posing and solving motivated by the use of GeoGebra. In particular, the analysis focused on the reasoning forms, resources and heuristics exhibited by the participants in this process.



Let ABC be an equilateral triangle and P be any point of the circumscribing circle, show that $AP + BP = CP$.

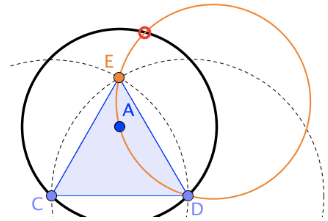
Figure 1. The Problem

Construction of a Dynamic Representation of the Problem

How to construct a dynamic representation of the problem using the DGS? The participants formulated two problems for obtaining dynamic representations of them: 1) construct an equilateral triangle and then trace its circumscribed circumference and 2) inscribe an equilateral triangle on a given circumference.

Table 1 shows a dynamic approach for the solution of the second problem. This approach was based on the heuristic of relaxing the problem conditions, and the visualization of loci through dragging objects allowed by dynamic configuration. Participants drew a triangle that partially met the problem conditions; they drew an equilateral triangle with two dynamic vertices on the circumference. Afterwards, through moving the configuration, they formulated the necessary conditions to solve the problem.

Table 1: Dynamic Approach and empirical solution shown by the participants.

Configuration	***Analysis***
 1. Dynamic Points C and D. The equilateral triangle *CDE* is traced through circumferences. Locus described by point *E* when the point *C* moves.	***Resources.*** Equilateral Triangle and locus. ***Dynamic Strategy.*** Relaxing the problem conditions. The triangle *CDE* is always equilateral, but only two of its vertices are inscribed in the circumference. Thus, the exploration strategy consists of moving point C until point E belongs to the initial circumference. ***Empirical Solution.*** The intersection of the locus described by point E with the initial circumference defines one of the vertices of the inscribed equilateral triangle (another one is point D and the last one is by construction).

In the participants' approach, the use of locus as a problem-solving tool is observed; however, in a first instance, the solution remained at a visual and empirical level due to the impossibility of GeoGebra to intersect geometric places. This intersection (in GeoGebra) is only possible if the objects are transformed into robust geometric objects; meaning that its construction depends on well-defined elements. As an example, in order to make robust a locus that is assumed as a circumference, its center and radius are required. While searching for the center, in order to draw the robust circumference used in this approach, participants realized that the intersection of the perpendicular bisector of the variable segment DE with the original circumference was invariant (point F in Figure 2). Thus, they assumed that point F was the center of the circumference, and measured the distance to point E to verify that it was constant. Detecting invariants while exploring the dynamic model shown in Table 1, allowed the participants to find a synthetic solution (which can be obtained with ruler and compass) to the problem (Table 2).

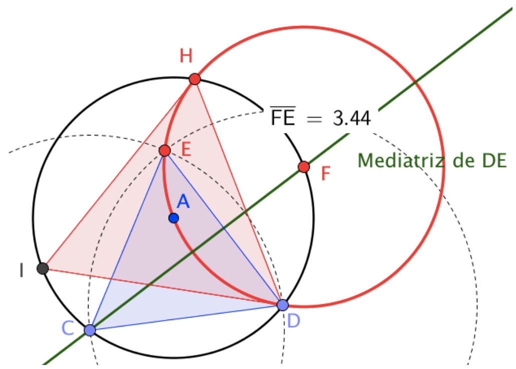
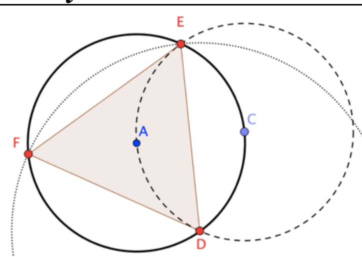


Figure 2. The intersection point, F, of the ED perpendicular bisector with the initial circumference is invariant and the center of the circle described by point E.

Table 2: Synthetic solution found by the participants when analyzing dynamic model.

Synthetic Solution	***Description***
 1	Dynamic Point C on the initial circumference with center in A. Circumference with center in C and radius CA. The points of intersection, E and D, of the two circles form one of the sides of the inscribed equilateral triangle. The third vertex F was found whit the circumference of center at D and radius DE.

Discussion of Results

Using the DGS, the participants were able to find three types of solutions to the problem of inscribing an equilateral triangle in a circumference: empirical, robust and synthetic solutions. The empirical solution was the first one to be found, since GeoGebra cannot intersect loci. In this direction, to find the robust solution (in terms of intersections), it was required to transform (make robust) the locus. Exploring the properties of the locus shown in Table 1 allowed the participants to find a synthetic solution to the problem, that is, a solution that can be obtained with basic rules and compass traces. For both empirical and exact solutions (robust and synthetic), three problem-solving heuristics were essential: pattern or loci visualization; exploring particular cases; and detecting invariants.

Conclusions

There are a variety of geometric problems in which the information or conditions are presented by means of figures or geometric configurations. Commonly, when problems are solved in a traditional way (with paper and pencil), reconstructing the figures that appear in the statements is irrelevant, since only the mathematical relationships implicit in these figures are important. When working within a DGS, it is essential to think about ways to obtain dynamic representations to explore problems. The study results showed that considering different ways of representing a problematic situation by dynamic models motivates the formulation and resolution of new problems.

References

Santos-Trigo, M., Reyes-Martínez, I., & Aguilar-Magallón, D. (2015). The Use of Digital Technology in Extending Mathematical Problem Solving Reasoning. In *Learning Technology for Education in Cloud* (pp. 298-309). Springer International Publishing.

Santos-Trigo, M., Reyes-Martínez, I., & Aguilar-Magallón, D. (2016). Digital Technologies and a Modeling Approach to Learn Mathematics and Develop Problem Solving Competencies. In International Workshop on Learning Technology for Education in Cloud (pp. 193-206). Springer International Publishing.

Aguilar-Magallón, D., & Poveda-Fernández, W. (2017). Problem Posing Opportunities whit Digital Technology in Problem Solving Environments. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1432-1446). Tucson, AZ: The University of Arizona.

Leikin, R. (2015). Problem Posing for and Through Investigations in a Dynamic Geometry Environment. En F. Singer, N. Ellerton & J. Cai (Eds.), *Mathematical Problem Posing* (pp. 373-391). New York: Springer.

Osana, H. P., & Pelczer, I. (2015). A Review on Problem Posing in Teacher Education. En F. Singer, N. Ellerton & J. Cai (Eds.), *Mathematical Problem Posing* (pp. 469-492). New York: Springer.

Rosli, R., Capraro, M. M., Goldsby, D., y Gonzalez, E. G., Onwuegbuzie, A. J., & Capraro, R. M. (2015). Middle-Grade Preservice Teachers' Mathematical Problem Solving and Problem Posing. En F. Singer, N. Ellerton & J. Cai (Eds.), *Mathematical Problem Posing* (pp. 333-354). New York: Springer.

USING TEXT MESSAGES TO CONNECT LINGUISTICALLY-DIVERSE FAMILIES WITH THEIR CHILD'S MATHEMATICS CLASSROOM LEARNING

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This study investigated how a weekly mathematically-focused text message to parents, which shared 1) what children were learning about in mathematics class, and 2) an idea to build on this mathematics topic at home, were perceived and used by parents. We found that parents highly valued this regular communication, not only because they appreciated hearing from the school but because it connected them more with their child and their child's mathematical learning. Parents also shared that the texts influenced how they thought about their child's mathematical learning and increased mathematical talk at home.

Keywords: Elementary School Education, Equity and Diversity, Technology

Objective of the Study

The overarching goal of the study was to investigate how mathematically-focused text messages can be used to connect linguistically-diverse families with what their child is learning in mathematics. We first created relevant and mathematically-focused text messages that had the potential to support parents to build on their child's school mathematical learning. After creating the texts, our goal was to investigate how families used the texts at home to see if parents perceived them as supporting their child's learning, and how the texts affected parents' feelings of connection to schools and their children.

Perspective/Rationale

Home-school partnerships and communication are critical for educational growth and development in children (Epstein, 2001) and connecting the home and school settings develops a cohesive learning environment for children (Christenson, 2004). Essential to developing home-school partnerships is active, productive, effective, regular communication.

Although parents want better communication with teachers and schools (Miretzky, 2004), Davies (1991) found that families often wait for schools to initiate contact. Text messaging is one promising way to better communicate with parents. This innovative pathway for building home-school partnerships and communication opens an efficient, nearly universal avenue for schools to communicate with and engage parents. Ninety-one percent of American adults own cell phones (Rainie, 2013). Of these users, 98% can receive text messages of which 95% are opened (Ehrlich, 2013). Additionally, Hispanics (88%), African Americans (93%) and whites (90%) have similar cell phones ownership rates mitigating the Internet access gap that has plagued the use of other electronic communications such as email, student information systems, and websites (Rainie, 2013).

Studies on parent communications using text messages have promising results. York and Loeb (2014) found using text messages with tips for parent engagement in literacy increased parent engagement with their children's early literacy development and increased in student literacy assessment scores. Kraft and Monti-Nussbaum (2017) used text messages to engage parents as partners in successfully mitigating the summer slump in early literacy. While these results show potential, there is a dearth of literature implementing similar communication strategies and support for parents and teachers in developing early mathematics skills and concepts. Additionally, little of this research captures qualitative data from parent perspectives.

A review of the literature points to a need to investigate how mathematically-focused texts can be used to connect parents and teachers, which leads to the research questions: How do we create relevant and useful mathematically-focused text messages that support parents to build on students' mathematical learning? How do families use the texts at home? How do the texts affect parents' feelings of connection to schools and their children?

Methods

Context

The study took place in a culturally and linguistically diverse school. The school's student population during the 2017-18 academic year was almost 500 (43% Black, 35% Hispanic, 8% White, 9% Asian, 1% Native American, and 4% multiracial). The overall English language learner population was 40% (District Report, 2018). Of that population, about half are Spanish speakers, and the remaining ELL students speak Vietnamese, French, Lao, Nilo-Saharan, Bosnian, Burmese, Kru, Karen, Nepalese, Arabic, Kunama, Mabaan, Nuer, Swahili, Somali, Dinka, Tigrinya and Zapoteco (teacher communication, 2015). Ninety-seven percent of the school's students receive free or reduced lunch (District Report, 2018).

Within the school, the study worked with the school's four kindergarten teachers and their parents. During the two years of the study, seven to 12 different primary languages were spoken by kindergarten parents.

Participants

The researchers designed mathematically-focused text messages for the parents and/or guardians of four kindergarten classes throughout the school year. Data was collected during the 2016-2017 and 2017-2018 school years.

The four kindergarten teachers taught all the kindergartners in the school. Three of the teachers had taught kindergarten at the school for over 15 years and the fourth teacher, while new to the school and to kindergarten, had previously taught preschool for over 10 years in the same district.

All kindergarten students' parents or guardians were recruited to sign up for the text messages during the school's Meet-the-Teacher night, which was held the day before school began, and at parent conferences in fall and spring. During the 2016-2017 school year, 22 out of 62 parents signed up (35%). All text messages were sent only in English that year. During the 2017-2018 school, 43 parents out of 84 signed up for the texts (51%). This year, recipients were able to choose to receive text messages in English only (21 parents), Spanish only (5), or both languages (17). In both years, every parent asked has signed up to receive the texts, with one exception. The challenge was not convincing parents to participate in receiving the texts; the challenge was finding the parents to ask them.

The Text Messages

The researchers developed mathematically-focused text messages for the parents and/or guardians of students in the kindergarten classes at the school. The weekly texts included 1) what the children where learning about in mathematics class and, 2) an idea for how to engage in similar mathematics at home. These suggestions for engagement were designed to take only a short time to implement (3-5minutes) but could be adapted and repeated as often as a parent or child choose. Each text was limited to 128 characters, so being concise and clear was a challenge. An example from November 8, 2016 shares what the children were learning about in mathematics and a related activity:

> Text 1: In math class this week we've been counting and writing numbers up to 10.
> Text 2: Activity-Ask your child to write his/her name. Write your own name. Ask how many letters are in each name. Which name has more?

The mathematically-focused text messages were designed to encourage mathematical discussions between the parent and child at home. In this way, the child would have an opportunity to share his/her mathematical thinking with his/her parents and the parents will have an opportunity to extend their child's mathematical understanding.

Teacher Focus Groups and Classroom Observations

To ensure the text messages are aligned with the content, readable by parents, and relevant to the children, we conducted a teacher focus group to get feedback from the teachers on text message design and content. In year 1, we also visited each of the four classrooms two times per month to observe and take field notes of the mathematics that kindergartners were learning to align the text messages with the timing and nature of the content taught in the classroom.

Parent Focus Groups

Parent focus groups were held at two different points in the study (mid-year 2017-18 (n=7) and end-of-the-school-year 2016-17 (n=5); 2017-18 (n=10)) to gather information from parents about their perceptions and uses of the text messages.

Data Analysis

All the transcripts of parent and teacher focus group interviews were analyzed using NVivo software for qualitative research to support systematic procedures for coding and categorizing data. The data interpretation and reporting used an inductive approach incorporating systematic methods of managing data through reduction, organization, and connection (Dey, 1993). This approach allowed us to identify common themes, as well as divergent cases, looking across categories to see if there are underlying patterns to the responses.

Results

First, creating the first draft of the text messages involved reviewing the kindergarten curriculum and curriculum map for mathematics topics and timing, creating texts with specific activities, and reviewing these texts with the kindergarten teachers in a focus group to gather their feedback. We adapted the texts based on their feedback, which was primarily about aligning the math language to the math language they use in their classrooms and how they felt parents may understand the language. Then on a weekly basis, after observing in the classrooms and before the weekly texts were sent, we tweaked the texts again to make sure timing, activities, and language were aligned to the teachers' instruction.

Parent focus groups were held to gather information on parents' use and perspectives of the mathematically-focused texts. We discovered that parents found the text useful in fostering communication with their child; they appreciated knowing what their child was learning about in mathematics because it gave them background so they could discuss the content of what their child was learning about in school. As one parent said, "I really liked that, knowing what's going on. 'Cause he may, [child's name], may not be able to describe what's going on to me" (Parent focus group, May 30, 2017). Another parent shared a similar sentiment,

> When they come home I always ask them what they learned. And for their age, they're just like, "Um." And they just list all the fun stuff. 'We play. We ate.' And I'm just like, 'Okay.

> What did you actually learn?' The text messaging helped because I'm like, 'Okay, so you're making ways to make 10? Make five?' They're like, 'Oh yeah. We learned that.' So, then we started doing it. (Parent focus group, December 20, 2017)

Also, most parents shared ways that they adapted and extended the activities.

> We do [the math activity] in the car. And then um, when I do laundry on the weekends, my daughter's favorite thing to do is count out five dryer sheets to put in the dryer, because I always put extra. But she counts out five and then she looks at the knob and she puts it on 70 and turns it on. So, she's helping with chores while she's counting. (Parent focus group, December 20, 2017)

Parents said they felt more connected to their child's learning because of the texts and that it had increased their mathematical talk with not only their kindergarten child, but also other children in the family. "I put some stuff out for him to try and count and stuff and, and like she said, it's a learning thing because not only is my five-year-old learning, he, my two-year-old now is counting" (Parent focus group, December 20, 2017).

Conclusion

Using mathematically-focused text messages to communicate with parents is rare in the research literature. Working to communicate with linguistically diverse parents using texts is also rare in the educational research literature. These two gaps in the mathematics education research show the need for figuring out how we can use technology such as text messaging to better meet parents where they are and connect them more closely to their child's learning. This study began to address this need and found that receiving the texts made parents feel more connected to the school and their child's learning. Parents appreciated knowing what their child was learning, were able to do the math activities in the texts with their child, and often times incorporated the math ideas into regular household activities or extended the math ideas. Next year we are ready to expand beyond English and Spanish to the next three most popular languages in the school – Arabic, Swahili, and Vietnamese.

References

Christenson, S. L. (2004). The family-school partnership: An opportunity to promote the learning competence of all students. *School Psychology Review*, *33*(1), 83.

Davies, D. (1991). Schools Reaching Out: Family, School, and Community Partnerships for Student Success. *Phi Delta Kappa*, *72*(5), 376.

Dey, I. (1993) *Qualitative data analysis: A user-friendly guide for social scientists*. London: Routledge.

Ehrlich, S. (2013, February 27). Mogreet releases best practices guide for successfully navigating text marketing rules and regulations. *The Wall Street Journal*. Retrieved from http://online.wsj.com/article/PR-CO-20130227-908303.html

Epstein, J. L. (2001). Introduction to the special section. New directions for school, family, and Community partnerships in middle and high schools. *NASSP Bulletin, 85*(627), 3.

Kraft, M. A., & Monti-Nussbaum, M. (2017). Can schools enable parents to prevent summer learning loss? A text-messaging field experiment to promote literacy skills. *The ANNALS of the American Academy of Political and Social Science*, *674*(1), 85-112.

Miretzky, D. (2004). The communication requirements of democratic schools: Parent-teacher perspectives on their relationships. *Teachers College Record*, *106*(4), 814-851.

Rainie, L. (2013). Cell phone ownership hits 91% of adults. *Pew Research Center*. Retrieved from http://www.pewresearch.org/fact-tank/2013/06/06/cell-phone-ownership-hits-91-of-adults/

York, B. N., & Loeb, S. (2014). One Step at a Time: The Effects of an Early Literacy Text Messaging *Program for Parents of Preschoolers* (No. 20659). National Bureau of Economic Research, Inc.

STUDENT ONTOLOGY AND EPISTEMOLOGY IN AXIOMATIC GEOMETRY USING TECHNOLOGY

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In this theoretical paper, I address students' ontological and epistemological shifts that have been discussed in the existing literature of student learning of geometry. First, students make a shift in the ontological view of geometric models from Euclidean to non-Euclidean geometry, in which the geometric models are considered conscious artifacts of mathematical design. Second, students make a shift in the epistemological view of mathematical proofs from absolutism to fallibilism, in which proofs can be characterized with a variety of functions and forms. Drawing on the prior literature, I argue that making successful shifts can benefit students in axiomatic geometry and that such shifts can be facilitated by engaging in mathematical activities with supports of dynamic geometry environments. Examples of those activities that involve constructing and analyzing geometric proofs are illustrated.

Keywords: Epistemology, Geometry, Technology

The purpose of this theoretical paper is to address students' ontological and epistemological shifts that have been discussed in the existing literature of mathematics education regarding student learning of axiomatic geometry. Drawing on the prior literature, I argue that making successful shifts can benefit students in axiomatic geometry and that such shifts can be facilitated by engaging in mathematical activities with supports of Dynamic Geometry Environments (DGEs). In this paper, axiomatic geometry means a study of geometry with a focus on the axiomatic approach to understand various geometries with different axiomatic systems including Euclidean and non-Euclidean geometries.

A Shift in Ontology of Geometric Models for Axiomatic Geometry

In axiomatic geometry, students need to make a shift in ways of perceiving and interacting with geometric models. Understanding axiomatic systems and models involves knowing how geometric models are constructed with sophisticated mathematical designs, and why those models satisfy the given postulates of the systems. In many cases, especially for non-Euclidean geometry, the models are not just natural outcomes resulting from physical observations of existing figures. Rather, they are conscious artifacts produced by creative and elaborative mathematical design. This nature of non-Euclidean models indicates that students need to change their ways of perceiving geometric models, which have been developed through their prior experience in Euclidean geometry.

Hegedus and Moreno-Armella (2011) introduced *Euclidean ontology* in their analysis of historical development of non-Euclidean geometry. In Euclidean geometry, a student can draw a line on paper, which is an iconic representation of the pure abstraction of all lines. A diagram is merely a mirror of an *a priori* geometric figure. In axiomatic geometry, however, a diagram results from an interpretation of the figure in a given geometric model. For instance, Poincaré constructed a model of hyperbolic geometry with a particular interpretation of lines–arcs orthogonal to the unit circle–whereas Klein used Euclidean line segments inside the unit circle in his model. Those models are not natural beings that reflect abstractions of figures, rather artifacts to be constructed, verified, and explored by students investigating characteristics of the figures emerging across all possible interpretations.

The shift in ontology of geometric models from Euclidean to non-Euclidean geometry can be supported by DGE-activities that provides students with opportunities of conceptual embodiment of geometric models. The conceptual embodiment means to develop perceptual representations for figures in non-Euclidean geometry, which are not static mental images, rather dynamic mathematical structures that realize abstract concepts of geometry (Tall, 2008). In this regard, exploring behaviors of dynamic diagrams of figures in DGEs can facilitate students to develop perceptual representations of figures in non-Euclidean geometry. In this exploration, students can drag the figures and observe the dynamic responses to their manipulations. It enables students to build up their perceptual representations that they can access as engaging in formal reasoning. For example, Guven and Karatas (2009) showed that interactive diagrams of a DGE can support students' development of a conceptual embodiment of triangles in spherical geometry. In their study, the participants who engaged in DGE-activities investigating triangles on a sphere were able to create paper-and-pencil diagrams of spherical triangles in explaining the properties used in their proofs.

An Example: DGE-Construction of Poincaré Model

Constructing an interactive model of hyperbolic geometry in DGEs can provide an opportunity to experience both exploration of dynamic figures and design of mathematical models highlighted in the literature (Guven & Karatas, 2009; Otten & Zin, 2012). In this activity, students can develop perceptual representation of hyperbolic figures with its underlying mathematical structures and connect the observed properties of figures and mathematical structures. For example, students may recognize a hyperbolic line in Poincaré model becomes straight as the two distinct points getting close to a diameter of the unit disk (Figure 1). Knowing the underlying design of the lines in this model, students can explain this observation by imagining that the radius of the orthogonal circle diverges as the two points lie on the diameter. In addition, students can discover new concepts in hyperbolic geometry, which they have never seen in Euclidean geometry. For example, students can explore infinitely many parallel lines in hyperbolic geometry and discover particular set of parallel lines distinguished from others.

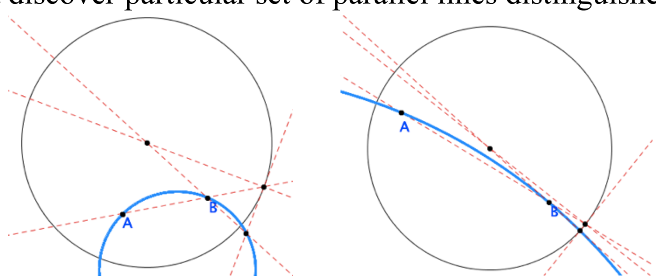

Figure 1. Hyperbolic lines in Poincaré model

A Shift to Fallibilist View of Mathematical Proof

In the axiomatic approach to geometry, college students encounter mathematical proofs less absolute and deterministic than what they have seen in their prior experience. Proving is not for generating an absolute certainty of a given mathematical statement as usually pursed in the absolutist view of mathematics. Instead, students can benefit from analyzing why a proof works in a system but not in others and investigating why a statement is not provable in a system. In this regard, students' views on proof in axiomatic geometry need a shift from the absolutist to the fallibilist view, which provide a variety of functions and forms of proofs.

Researchers have compared the two contrasting views of mathematics, absolutism and fallibilism, in terms of how proofs are perceived differently in each view (de Villiers, 1998; Ernest, 1991, Öner, 2008). In the fallibilist view, also frequently referred to as quasi-empiricist (Lakatos, 1976), mathematics is considered a product of social processes in which its concepts and proofs can be revised (Ernest, 1991). This view contrasts to the absolutist view, in which absolute certainty exists in mathematics. A proof in the absolutist view is a requirement for verification or conviction of mathematical knowledge. On the other hand, in the fallibilist view, personal conviction motivates one to seek a proof to convince oneself and/or intended audiences of the proof (Öner, 2008). As a quasi-empiricist, de Villiers (1998) argues that proving involves a variety of functions, including explanation, discovery, communication, and systematization, which constitute a broader set of mathematical activities.

These various functions of proofs in the fallibilist view can be promoted in axiomatic geometry by introducing an alternative form of proofs in DGEs. DGEs enable students to transform a written proof into a geometric construction. This visual and interactive proofs in DGEs is called a robust construction (Leung, 2008; Stylianides & Stylianides, 2005). Whereas a diagram in the written proof is a particular instantiation of generic objects from the abstract procedure of the construction, the constructed figures in DGEs serve as generic examples that can vary to represent any other particular example. This generalizability of the robust constructions in DGEs enables us to transform written proofs to geometric constructions without loss of generality. Creating and analyzing robust constructions provide students with opportunities to discover hidden assumptions, logical flaws in proofs, and axiomatic feature of the geometry.

An Example: Proof Analysis of Exterior Angle Theorem in Spherical Geometry

From the fallibilist view of proofs, I illustrate a DGE-activity of generating and analyzing robust construction of a proof for the exterior angle theorem–an exterior angle of a triangle is greater than either of the two opposite interior angles–that holds in Euclidean geometry but not in spherical geometry. Analyzing a proof of this theorem in those two geometries allows students to investigate the axiomatic natures of the systems related to the proof. By creating and interacting with robust constructions of a written proof in the two different geometric models (Figure 2), students can identify when the proof fails in spherical geometry as they stretch out the triangle. This activity can enable students to figure out what feature of the spherical geometry–two distinct lines intersect at two distinct points–causes this inconsistent result between two geometries. Furthermore, this activity can lead student to experience developing new mathematical knowledge. Students generate their own definitions of a family of triangles that do not satisfy the theorem in spherical geometry then refine the original statement by adding a specified condition describing such triangles.

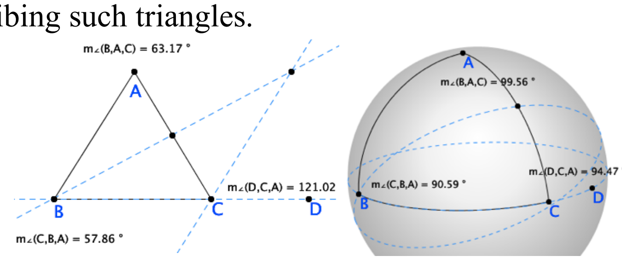

Figure 2. Exterior angle theorem in Euclidean (left) and spherical geometry (right)

Discussions

In this paper, I addressed students' ontological shift in geometric models and epistemological

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.

shift in mathematical proofs with respect to student learning of axiomatic geometry using technology. The DGE-activities illustrated in this paper show the potential of pedagogical role of technology in pursuing such goals in axiomatic geometry classrooms. To provide an implication to teaching practices, I propose an alternative way of mathematical communication promoting those activities–proving as performing. The act of proving in this view is not just to produce a tangible material (e.g., written proofs or images of geometric constructions), but can be to perform a proof that is transformed into the robust construction (Leung, 2008; Stylianides & Stylianides, 2005). Using screencast software, students can create video recording of their proofs in DGE and verbal annotation, showing how they created this generalizable geometric construction that is equivalent to the formal written proof.

Screencast presentations of DGE practices are beneficial for implementing student-centered and discourse-oriented geometry classrooms. First of all, video recordings afford a way for instructors to listen to their students verbally explaining their thinking while interacting with dynamic geometry construction. Second, practices of creating and sharing screencast presentations can empower teacher and students by establishing a mathematics register (Pimm, 1987) and enrich the communicative classroom community. The use of student-generated presentations can promote students' agency in classroom by sharing their own mathematics, understanding each other, and interpreting these ideas into formal statements. Lastly, screencast presentations can support equitable power dynamics between students (e.g., Esmonde & Langer-Osuna, 2013) in discourse-based geometry classrooms for all students by allowing them to share their voices with each other, especially for those who are marginalized in classroom discussions.

References

de Villiers, M. (1998). An alternative approach to proof in dynamic geometry. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 369–393). Mahwah, NJ: Lawrence Erlbaum Publishers.

Ernest, P. (1991). *The philosophy of mathematics education.* London: Falmer.

Esmonde, I., & Langer-Osuna, J. M. (2013). Power in numbers: Student participation in mathematical discussions in heterogeneous spaces. *Journal for Research in Mathematics Education, 44*(1), 288–315.

Guven, B., & Karatas, I. (2009). Students discovering spherical geometry using dynamic geometry software. *International Journal of Mathematical Education in Science and Technology, 40*(3), 331–340. https://doi.org/10.1080/00207390802641650

Hegedus, S. J., & Moreno-Armella, L. (2011). The emergence of mathematical structures. *Educational Studies in Mathematics, 77*(2), 369–388. https://doi.org/10.1007/s

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery.* Cambridge: Cambridge University Press.

Leung, A. (2008). Dragging in a dynamic geometry environment through the lens of variation. *International Journal of Computers for Mathematical Learning, 13*, 135–157. https://doi.org/10.1007/s10758-008-9130-x

Öner, D. (2008). Supporting students' participation in authentic proof activities in computer supported collaborative learning (CSCL) environments. *International Journal of Computer-Supported Collaborative Learning, 3*(3), 343–359. https://doi.org/10.1007/s11412-008-9043-7

Otten, S., & Zin, C. (2012). *In a class with klein: Generating a model of the hyperbolic plane. PRIMUS, 22*(2), 85–96. https://doi.org/10.1080/10511970.2010.494153

Pimm, D. (1987). *Speaking mathematically*. Londan and New York: Routledge and Kegan Paul.

Stylianides, G. J., & Stylianides, A. J. (2005). Validation of solutions of construction problems in dynamic geometry environments. *International Journal of Computers for Mathematical Learning, 10*(1), 31–47. https://doi.org/10.1007/s10758-004-6999-x

Tall, D. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal, 20*(2), 5–24. https://doi.org/10.1007/BF03217474

TECHNOLOGY INTEGRATION IN SECONDARY MATHEMATICS TEXTBOOKS

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As the presence of instructional technology in secondary mathematics classrooms increases, the need for teacher resources supporting its use becomes of greater importance. Mathematics textbooks are one resource that teachers rely on for instruction, so it is important to know how they support teachers in this regard. We analyzed a nationally representative sample of 20 high school textbooks, including integrated/non-integrated and conventional/investigative curricula, to compare the level of technology integration and the types of technology utilized. Findings indicate a relatively low percentage of tasks integrating technology, and that calculators are the predominant technology within this sample.

Keywords: Technology, Curriculum, Curriculum Analysis, High School Education

A wealth of technologies exists for mathematics instruction, and research shows that the strategic use of technology has the potential enhance students' mathematical thinking and understanding (e.g., Hollebrands & Dove, 2011). As access to instructional technologies increases in secondary classrooms, the question arises as to the extent to which instructional technologies are integrated into contemporary curricula. As mathematics teachers rely on textbooks to guide their instruction (e.g., Banilower et al., 2013), it is important to understand how often and the types of technology that are included in mathematics curricula. This understanding may be used to guide professional development efforts and provide important guidance to curriculum developers.

This study examines the integration of technology use in current secondary mathematics curricula. We conducted a textbook analysis using a nationally representative sample of secondary mathematics curricula. We noted how often technology is incorporated and the types of technology used with respect whether the textbook was 1) Algebra 2 vs. Geometry vs. Integrated, and 2) investigative versus conventional.

Background

A number of textbook analyses have been published in the last ten years (e.g. Sherman, Walkington, & Howell, 2016; Otten, Gilbertson, Males, & Clark, 2014; Tarr, Grouws, Chavez, & Soria, 2013), but very few have examined the extent to which instructional technology is incorporated into curricula (Oner, 2009). We aim to address this gap in the literature regarding the content of current curricula.

In this study, we followed Stein, Remillard, and Smith (2007) in differentiating between two types of texts: "Investigative textbooks reflect a reform philosophy that is consistent with the recommendations of the *NCTM Curriculum and Evaluation Standards* (1989). Conventional textbooks refer to commercially developed textbooks that were not influenced by the earlier reform documents mentioned above" (Sherman et al., 2016, p. 118). Recent studies of integrated versus subject-specific curricula demonstrate an advantage for integrated over the subject-specific on standardized tests of student learning in high school algebra (Grouws et al., 2013) and geometry (Tarr et al., 2013)

Methods

This study examines current secondary mathematics textbooks to answer the questions: 1) how often do secondary textbooks incorporate tasks that utilize technology, and 2) what types of technologies are utilized?

To characterize technology use in secondary textbooks, a sample of twenty textbooks varying with respect to curricular focus, integrated versus non-integrated, and conventional versus investigative was chosen. Geometry and Algebra 2, and corresponding integrated textbooks, were selected because of the differing technologies that may be used. Textbook state adoption lists were examined for all states that had current, publicly available adoptions posted online. These criteria yielded in the sample noted in Table 1.

Table 3: Textbooks Used for Analysis

	Textbook Publisher/Series	Conventional/ Investigative/ # Texts	Reference Information
1	Carnegie Learning	Investigative (4)	Barte Dengler et al.
2	McGraw Hill	Conventional (2)	Carter
3	Houghton Mifflin Harcourt	Conventional (2)	Burger et al.
4	Houghton Mifflin Harcourt	Conventional (2)	Larson et al.
5	Pearson/Prentice Hall	Conventional (2)	Bass et al.
6	Pearson/Prentice Hall	Investigative (2)	CME Project
7	McGraw Hill	Investigative (2)	Hirsch & Fey
8	College Preparatory Mathematics	Investigative (4)	Dietiker et al. Kysh et al.

The unit of analysis for coding was a mathematical task. We were interested in tasks that could expect to be implemented in classroom instruction. To this end, main narrative sections, extensions of narrative sections, stand-alone labs, stand-alone activities were included, while exercise sections, review sections, summary sections, practice assessments, standardized test prep sections were excluded. Previous textbook analysis (Sherman et al., 2016), demonstrated that this distinction could be made reliably in a large set of textbooks. Each task was coded with respect to whether or not it included the use of technology, and the type of technology that was used. Secondary coding was conducted for each of these distinctions to ensure reliability.

Results

Across the 20 textbooks a total of 10,100 tasks were coded, with an average of 14.88% (*SD* = 8.25%) of tasks within a textbook using technology, ranging from 1.84% to 30.70% across the sample. Table 2 depicts how the inclusion of technology differed across textbook types and subjects.

Table 2: Technology Integration by Textbook Type and Subject

		% of Tasks Integrating Tech	Total Tasks
Conventional	Algebra 2	16.23%	2834
	Geometry	7.43%	2558
	Overall	**12.05%**	**5392**
Investigative	Algebra 2	14.98%	1202
	Geometry	7.61%	1262
	Integrated	17.47%	2244
	Overall	**14.19%**	**4708**

Mixed effects logistic regression models were fit to predict whether a given task utilized technology or not (coded as a 0/1) with textbook title as a random effect. Whether the textbook was conventional or investigative did not significantly predict the presence of technology tasks (p = 0.726). However, when comparing Geometry versus Algebra 2 versus Integrated texts, significant differences were found. Geometry texts were significantly less likely to include technology tasks than Algebra 2 texts (Odds = 0.39, d = -0,52, p = 0.0012) or Integrated texts (Odds = 0.33, d = -0.61, p < .001).

Technology tasks in each textbook were also coded for the type of technology used, as summarized in Table 3 below.

Table 3: Technology Type by Textbook Subject and Type

	Tech Tasks	Graphing Calculator	Scientific Calculator	DGS	Custom eTool	CAS	Sprea dsheet	Other
Algebra 2	640	69.53%	20.00%	0.31%	6.41%	1.09%	1.56%	1.09%
Geometry	286	14.69%	37.06%	26.92%	19.23%	0.00%	2.10%	0.00%
Integrated	390	52.05%	6.15%	11.79%	15.38%	4.87%	1.54%	8.21%
Conventional	650	55.08%	33.08%	8.15%	0.00%	1.08%	2.46%	0.15%
Investigative	666	49.85%	6.46%	10.81%	23.42%	2.85%	0.90%	5.71%

Conventional and investigative texts used graphing calculators and DGS at a relatively similar rate, while conventional texts were more likely to use scientific calculators. Although custom eTools constitute nearly a quarter of the technology tasks found in investigative texts, they are unique to the CPM texts.

Discussion

In terms of the quantity of technology integration in current curricula, across all textbooks, an average of about 15% of tasks utilized technology in some way, a result that indicates that the inclusion of technology in secondary mathematics textbooks is relatively rare. Regarding the types of technology utilized in texts, calculators are by far the most commonly used technology, while the use of DGS is much less common. The relatively rare use of DGS, even in Geometry texts, replicates earlier findings (Oner, 2008) for a larger and more current sample. Taken together, the potential of DGS to support exploring, generalizing, and conjecturing, the fact that Geometry texts use technology significantly less than Algebra 2 texts, and the predominant use of calculators in this sample of textbooks, suggest that technology might primarily be used as a computational tool, rather than for investigating and interacting with the mathematics. Furthermore, in light of current standards (NCTM, 2000; Common Core, 2010) articulating the important role of technology in a reformed vision of mathematics education, it is surprising that the investigative texts in this sample did not demonstrate a greater inclination toward integrating instructional technology. One of four components of Technological Pedagogical Content Knowledge (TPACK) articulated by Niess (2005) is "knowledge of curriculum and curriculum materials that integrate technology with learning in the subject area" (p. 511). Thus, these results should have important implications for mathematics teachers and teacher educators.

What the current study does not address is *how* these textbooks incorporate technology in reference to supporting students' high-level thinking. Our ongoing analysis of the tasks presented here includes the use of technology as an amplifier vs. reorganizer (Pea, 1985), as well as level of cognitive demand (Stein & Smith, 1998) in order to answer questions regarding the ways in which technology is used across this sample of texts.

References

Sherman, M. F., Walkington, C., & Howell, E. (2016). A comparison of Symbol-Precedence View in investigative and conventional textbooks used in Algebra courses. *Journal for Research in Mathematics Education*, *47*(2), 134-146.

Banilower, E. R., Smith, P. S., Weiss, I. R., Malzahn, K. A., Campbell, K. M., & Weis, A. M. (2013). *Report of the 2012 National Survey of Science and Mathematics Education.* Chapel Hill, NC: Horizon Research, Inc. http://www.eric.ed.gov/contentdelivery/servlet/ERICServlet?accno=ED548238

Grouws, D. A., Tarr, J. E., Chávez, Ó., Sears, R., Soria, V. M., & Taylan, R. D. (2013). Curriculum and implementation effects on high school students' mathematics learning from curricula representing subject-specific and integrated content organizations. *Journal for Research in Mathematics Education*, *44*(2), 416-463.

Hollebrands, K. F. and Dove, A. (2011). Technology as a tool for creating and reasoning about geometry tasks. In T. P. Dick & K. F. Hollebrands (Eds.), *Focus in high school mathematics: Technology to support reasoning and sense making* (pp. 33-52). NCTM: Reston, VA.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Governors Association Center for Best Practices, Council of Chief State School Officers. (2010). *Common core state standards for Mathematics*. Washington, DC: Author.

Niess, M. L. (2005). Preparing teachers to teach science and mathematics with technology: Developing a technology pedagogical content knowledge. *Teaching and Teacher Education*, *21*, 509-523.

Oner, D. (2008) A comparative analysis of high school geometry curricula: What do technology-intensive, standards-based, and traditional curricula have to offer in terms of mathematical proof and reasoning? *Journal of Computers in Mathematics and Science Teaching* Vol. 27 Issue 4

Otten, S., Gilbertson, N.J., Males, L.M., & Clark, D.L. (2014). The Mathematical Nature of Reasoning-and-Proving Opportunities in Geometry Textbooks, Mathematical Thinking and Learning, 16:1, 51-79

Pea, R. D. (1985). Beyond amplification: Using the computer to reorganize mental functioning. *Educational Psychologist*, *20*(4), 167–182.

Stein, M.K., Remillard, J., & Smith, M.S. (2007). How curriculum influences student learning. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 319-370). Greenwich, CT: Information Age Publishing.

Stein, M. K., & Smith, M. S. (1998). Selecting and creating mathematical tasks: From research to practice. *Teaching Mathematics in the Middle School*, *3*(5), 344–350.

Tarr, J. E., Grouws, D. A., Chávez, Ó., & Soria, V. M. (2013). The effects of content organization and curriculum implementation on students' mathematics learning in second-year high school courses. *Journal for Research in Mathematics Education*, *44*(4), 683-729.

USING TABLET TECHNOLOGY TO PROMOTE PARENT/CHILD MATHEMATICAL DIALOGUE

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As an important player in their child's education, parents should discuss mathematics with their young children. Since children who are read to regularly by their parents have higher literacy rates; we posit that positive interactions between children and their caregivers would increase children's competence in mathematics and lower rates of mathematics anxiety. We present findings from a design based experiment of original technology developed to encourage parents to engage in mathematical dialogue with their children. Three rounds of research were conducted with mother and preschool child pairs. Our findings indicated that an iPad app can present rich tasks for preschool children and prompts for the parents to encourage productive discussion.

Keywords: Early Childhood Education, Design Experiments, Pre-School Education, Technology

Mobile technology is now a constant part of our lives. These devices, particularly smart phones and tablets, can cause distraction and disconnection between parents and their young children if not used in a conscious and thoughtful manner (Park, 2016; Thorton, Faires, Robbins, & Rollins, 2014). As this technology continues to emerge and develop, parents face difficult decisions about the role they will allow these devices to play in their children's lives.
While parents and educators may be worried about the amount of "screen time" that children are exposed to, many also recognize that these devices have the potential to be powerful educational tools. The American Academy of Pediatrics posited recently that educational apps can be beneficial for children if "they are appropriately engaging, but not distracting; that they are designed to be used by a dual audience (i.e., both parent and child) to facilitate family participation in media use and modeling of more effective social and learning interactions" (Reid Chassiakos, Radesky, Christakis, Moreno, & Cross, 2016, p. e5). A randomized experiment conducted by Berkowitz et al. (2015) of 587 first-graders showed gains in mathematics competence after using an app that encouraged parents and elementary school children to solve word problems together.

We believe that educational apps that facilitate family mathematics interaction could also benefit younger children, yet such apps do not currently exist. However, a review of the apps available through Apple's App Store and Google Play revealed that the majority the mathematics apps for young children are based on math drills and memorization or have very little educational benefit (Reid Chassiakos, Radesky, Christakis, Moreno, & Cross, 2016)
In this paper, we present the preliminary development and analysis of an iPad app which fosters interaction between a parent and young child. This app, Family Math Moments, leverages the same mobile technology that modern parents are addicted to as a tool to help them utilize research-based mathematics education discussion techniques to unlock and support their child's deep mathematical thinking.

Our research asked the following questions during three research cycles:

1. How can a mobile app provide a tool for parents to better uncover and support their child's complex mathematical thinking?

2. What supports help parents engage their children in mathematical dialogue?

Theoretical Framework

Cognitively Guided Instruction (CGI), play-based mathematics in early childhood, and the importance of math talk to develop children's thinking frame this work. CGI in early childhood education emphasizes the child's thinking in their own learning of mathematics and posits that "eliciting and responding to children's mathematical ideas is central to supporting student learning" (Carpenter, Franke, Johnson, Turrou, & Wager, 2017, p. 98). Teachers using CGI in their teaching of young children are empowered to listen carefully to children's thinking and use questions to help students work from what they know. The task presented in the app was chosen because it was situated in a context familiar to young children and crafted to allow children to develop and visually explain their sophisticated mathematical thinking. The app provides a task and creates an environment for parents and child to engage in these necessary discussions.

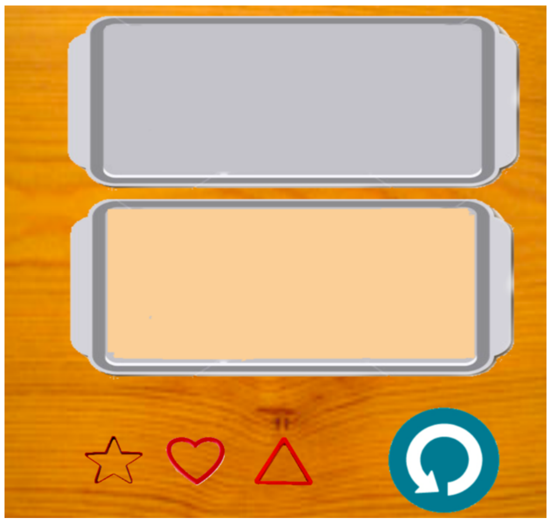

Figure 1. A Screenshot of the Cookies Activity in Family Math Moments. The child uses the cookie cutters to create cookies, with the goal of maximizing the amount of cookies made. The parent/family-member engages the child in mathematical discussion while the child explains their thinking.

Methods

This study used a design experiment methodology with lean thinking rapid iterations (Cobb, Confrey, DiSessa, Lehrer, & Schauble, 2003). Lean thinking allows for quick build-measure-learn feedback loops (Ries, 2011) that allow for research to be conducted on a minimum viable product (MVP) that is constantly updated based on the assessment from each research cycle. Before the research interviews began, an MVP prototype of this app was developed. The prototype was situated in the context of a mother and child baking cookies together. An opening page included cartoon figures and a voice-over that asked children if they would help the characters cut out as many cookies as possible out of a piece of cookie dough. Then the app transitioned to a screen that included a rolled out piece of cookie dough, cookie cutters, and an empty baking pan. Using the touch features of the iPad, the cookie cutters can be picked up and moved to the dough. After placing the cookie cutter on the dough, the user can move a cutout cookie to the baking pan. Figure 1 shows a screenshot of the app during the task.

Consistent with the CGI framework, the child is not provided with direct instructions on how to accomplish the task. In the spirit of play-based mathematics learning, the activity was intentionally not designed to be a video game experience. There is no scoring or time feature in the app. A child using the app is never designated as a "winner" or a "loser." The mathematics involved in this task includes counting, shape recognition, and optimization. The context of baking cookies was chosen based on the primary investigator's own experience as a parent.
To conduct the three research cycles, three mother/child pairs were recruited to complete interviews where the dyad tested the MVP and provided feedback. First, the lead author conducted a pilot round with her four-year-old son, Logan (all names used are pseudonyms). The other two pairs were four-year-old girls, Ada and Grace, and their mothers, Jessica and Amanda. All three children were enrolled in the same Pre-Kindergarten class. Table 1 provides demographic information about the participants.

Table 1: Participants

Research Cycle	Mother	Child
1 (Pilot)	First Author	Logan, 4 years old Considerable experience using iPad
2	Jessica Daycare administrator	Ada, 4 years old Limited experience using iPad
3	Amanda Works in Higher Education	Grace, 4 years old Limited experience using iPad

Results and Discussion

The results from this research are promising. The children were interested in using the app and engaged with the task quickly. The children and their mothers provided helpful feedback that could be quickly incorporated to improve the app's user experience. The feedback also advanced our understanding of how mobile technology can support parents to engage their children in mathematical dialogue. Table 2 provides and overview of the results from each research cycle and the impact that those results had on the development of Family Math Moments.

Table 2: Results

Participants	What Happened	Revisions
First Author and Logan	Logan liked that the task was not timed. Logan wanted to try the task multiple times.	Start over button added Parent handout created

Jessica and Ada	Jessica pushed parent handout to side Jessica focused on giving directions on how to use app. Researcher asked questions and Ada answered, Jessica never joined in the questioning.	Suggested questions were put on notecards
Amanda and Grace	Verbal introduction and tutorial was provided by researcher. Researcher handed Amanda notecards with questions to ask. Amanda initiated math talk with Grace.	Tutorial and “pop-up” questions for parents added to app

These first three cycles revealed that parents had difficultly discussing mathematics with their children without any direction, but when provided with sample questions they can use their intuition and knowledge of their child to extend the discussion and draw out their child’s thinking. The results from the first three research cycles will now be used to more fully develop iPad app.

Young children’s mathematical thinking is often diminished or dismissed. This study showed thinking that increases a parent’s understanding of their own children’s mathematical prowess. These results show that just as parent participation in reading to and with their children has led to a generational elimination of illiteracy in North America, an app such as Family Math Moments has the potential to further mathematical thinking for all children and eliminate mathematics anxiety.

References

Berkowitz, T., Schaeffer, M.W., Maloney, E.A., Peterson, L., Gregor, C., Levine, S.C.,…Collins, M.F. (2015) Math at home adds up to achievement in school. *Science* (New York, N.Y.), 350(6257), 196-8.

Carpenter, T. P., Franke, M., Johnson, N., Turrou, A., & Wager, A. (2017). *Young children’s mathematics: Cognitively guided instruction in early childhood education*. Portsmouth, NH: Hainemann.

Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design Experiments in Educational Research. *Educational Researcher*, *32*(1), 9–13. https://doi.org/10.3102/0013189X032001009

Park, A. (2016). Cell-phone distracted parenting can have long-term consequences: study. TIME *Health*.

Reid Chassiakos, Y., Radesky, J., Christakis, D., Moreno, M.A., & Cross, C. (2016). Children and Adolescents and Digital Media. *PEDIATRICS*, 138(5), e201625930e20162593. https://doi.org/10.1542/peds.2016-2593

Ries, E. (2011). *The Lean Startup*. New York: Crown Business.

Thorton, B., Faires, A., Robbins, M., & Rollins, E. (2014). The mere presence of a cell phone may be distracting implications for attention and task performance. *Social Psychology*, 45(6), 479-488. https://doi.org/10.1027/1864-9335/a000216

PROMOTING COLLABORATION AND MATHEMATICAL ENGAGEMENT IN A DIGITAL LEARNING ENVIRONMENT

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Promoting mathematics learning without developing students' engagement is a critical issue in the teaching and learning of mathematics. This study reports on the student and teacher perceptions on mathematics engagement in digital collaborative settings. Emerging themes arose through open coding of the student survey responses and the teacher interview. Findings revealed two themes: (a) the digital learning environment holds promise for promoting real-time collaboration and productive disciplinary engagement in mathematics and (b) in the digital learning environment, some students requested explicit opportunities for initial individual work before accessing a shared workspace on the digital platform.

Keywords: Technology, Problem Solving, and Middle School Education

Introduction

The 40th Annual Meeting of the North American Chapter of Psychology in Mathematics Education invites the mathematics education community to consider emerging opportunities related to technology use that can support each and every student. To this end, we are exploring ways in which digital technologies can support student learning of and engagement in mathematics. Through the use of design research methodologies, we report on initial work that offers promising directions for collaboration when teaching and learning mathematics.

Theory of Productive Disciplinary Engagement

While it often appears to educators and scholars that students are engaged in problems in the classroom, students may not be involved in the mathematical underpinnings of that engagement. Students are productive in their disciplinary engagement when they make intellectual progress or demonstrate change over time on the disciplinary learning goal (e.g., Hiebert, et al., 1996). This notion is referred to as productive disciplinary engagement. Four design principles of productive disciplinary engagement are needed to be embodied in learning environments: problematizing, authority, accountability, and resources.

Good problem solvers are not passive recipients of knowledge. If students are to engage in the content, issues, and practices of mathematics, then something must exist that is of genuine uncertainty for which there is sufficient space for students to make progress. Problematizing in mathematics (Hiebert, et al., 1996) involves addressing problematic situations that encourage learner "perplexity, confusion, or doubt" (Dewey, 1910, p.12).

If learners are involved in the content, issues, and practices of mathematics, then they must have some degree of intellectual authority when addressing problematic situations (e.g., Lampert, 1990). "As learners are authorized to share their thinking, they become recognized as authors of the ideas and contributors to the ideas of others, leading to students becoming local authorities on a subject" (Williams-Candek & Smith, 2015, p. 3).

Accountability refers to the notion that students are self-regulated in their learning. This means that they are responsible for how their ideas make sense amongst the ideas of others. The goal is that students will make ongoing revisions in their work, communicate their ideas, and consider how the ideas do or do not make sense in the discipline so that they are better positioned to improve them when more thoroughly challenged.

Learning environments need to provide access to sufficient resources so that students can engage in the other principles of productive disciplinary engagement. Resources might include sufficient time and location, technology tools, or classroom artifacts. Resources are going to vary dramatically as they depend on the topic, learning goal, classroom setting, and other factors.

Methods

This study is part of a larger design research (e.g., The Design-Based Research Collective, 2003) project focused on iteratively developing and enacting digital environments. The study is guided by thinking about how productive disciplinary engagement can be fostered in digital learning environments. The research reported in this study addresses the following research question: *In the digital learning environment, what are student and teacher perceptions of mathematics engagement in collaborative settings?* To this end, we report on data collected from four classes (approximately 25 students per class) that tested the developed digital resources over two days. The mathematics problem that was tested was from the seventh-grade unit, *Stretching and Shrinking: Understanding Similarity*. Problem 2.2 Hats Off to the Wumps: Changing a Figure's Size and Location comes from the *Connected Mathematics3* (CMP) curriculum materials (Lappan, Phillips, Fey, & Friel, 2014). While the teacher and students are familiar with CMP, the use of the prototype digital platform was the first experience for the students and the teacher. The platform supports students to make their thinking visible to others, to see and make changes in real time, and to publish their work (at any point in time).

The student survey was administered electronically after the testing of the mathematics problem. A total of 37 responses were captured from the students in the four classes. The teacher interview was also conducted after the testing of the mathematics problem. Interview questions focused on the experience using the digital platform and its features, student engagement in mathematics (problematizing, authority, accountability, and resources), student learning about similarity, and similarities and differences about the instructional model (Launch, Explore, Summary) from past teaching experiences. Student survey responses and teacher interview transcriptions were coded using an open coding approach (Strauss & Corbin, 1998). We report on the emerging themes on mathematical engagement in collaborative settings.

Findings

In this section, we report on findings of student and teacher perceptions on mathematics engagement in collaborative settings for an open mathematics problem.

Theme 1. The digital learning environment holds promise for promoting real-time collaboration and productive disciplinary engagement in mathematics.

From the analysis of the interview, the teacher spoke towards the positive aspects of the digital platform for student learning and engagement in mathematics. Below are excerpts of questions and responses between the interviewer (I) and the teacher (T).

As shown on Table 1, students found that the real-time collaborative digital resources were helpful for mathematics engagement and learning. However, some students raised issues on the technical lags (e.g., freezing and refreshing) and the cultural norms (i.e., hesitation of mistakes/errors now being public and sharing of the work as "copying" of others work). Although issues were raised, no students indicated they did not want to work collaboratively with the digital resources.

I: In your own words, tell me about the different features of the digital platform and the different ways you and your students used those features.
T: So, the main feature was the collaborative space screen, and the students slowly get used to using that to be able to share their thoughts and the answers on the same screen. When one student had an answer, they were able to input it and then share with their group and the whole class.
I: Did you notice changes in how students were accountable for their ideas, to the ideas of their group mates, or to the ideas of the class?
T: I think students were held a lot more accountable to each other and group. Because they think their work is much more visible, they cannot just write. They have to discuss, talk, and share. They were accountable to the whole class with the publish feature.
I: Have you noticed changes in how students share their thinking or contribute to the ideas of others?
T: They were just a lot more specific. They had things to reference. They felt a little more comfortable with each other. I am excited to see as we keep going how that changes. The next day, we did a lesson in class just out of the book paper pencil, I can still definitely feel the vibe of the cooperation thing as we were doing. So, I am interested in seeing this throughout the rest of this book if they continue to keep the same tightness they forced to do with the digital platform.

Figure 1. Teacher interview

Table 1: Select survey items and student responses

Did the technology help you be more confident, less confident, or as confident as usual in mathematics and in sharing your ideas with others? Explain why. (n=37)	More Confident 30%	Less Confident 22%	No Difference 40%	Other 8%
Was the technology helpful to explore similarity by producing similar and non-similar hats? Explain. (n=37)	Helpful 84%	Not Helpful 8%	Other 8%	
The technology allowed you to publish your work. Was this helpful or not? Explain. (n=37)	Helpful 62%	Not Helpful 30%	Other 8%	
The technology allowed you to look at your classmate's published work. Was this helpful or not? Explain. (n=37)	Helpful 65%	Not Helpful 24%	Other 11%	

Theme 2. In the digital learning environment, some students requested explicit opportunities for initial individual work before accessing a shared workspace on the digital platform.

Responses from the student survey surfaced an important consideration for working in collaborative groups. This consideration may be connected to an affordance of paper/pencil contexts. For example, below are sample student responses from students.

What changes, if any, would you like to see in the technology?

- *It should be individual and then you share your answers with your table group.*
- *You do a private thing then share with your group then publish*
- *I would make changes so that you can individually write things then discuss it with your group.*
- *I would have it work better; for example, everyone could have their own separate work and then share it when they were all done to compare answers.*

What did you not like about the technology?

- *It was hard to get work done because someone was trying to work on the table, and someone was working on the graph. I think each person should have their own interactive thing and then share with just their group of table.*

Figure 2. Sample of survey questions and students' responses

Discussion

In this study, we reported on student and teacher perceptions related to mathematics engagement in collaborative settings. Our analysis of the data was limited to student and teacher perceptions around one mathematical problem. The two themes highlight the pivotal role of accountability in the theory of productive disciplinary engagement. As Engle (2011) noted, developing and increasing accountability builds by asking learners to account for how their ideas make sense from the “inside out" - oneself, safer peers, more challenging peers, internal authorities providing increasing challenges, and external disciplinary authorities providing increasing challenges. In this version of the prototype, the digital platform began with the assumption that students would share a collaborative space. While students had the opportunity to create a space for themselves individually, it was an optional feature. The findings from the study suggest some need by students to work individually before moving to collaborative space. This finding may be interpreted as the students needing the opportunity to make sense of the problem themselves before sharing it with safe peers (group members). This study provides some evidence that resources and opportunities are needed to support accountability for the learner itself prior to moving outwards with others.

Despite the early stage prototype having technical glitches where students and the teacher had limited familiarity with the digital resources, this study underscores the important design consideration of collaboration when designing and enacting digital learning environments for promoting student engagement and learning of mathematics. Interestingly, the digital platform provided an option of an individual workspace for students (where group mates could not access their work), but its usage did not surface in our analysis of the survey.

While the digital learning environments reported in this study are different than paper/pencil contexts, students were still expected to be accountable to their ideas by justifying their work in their small group, to the whole class, and the teacher. Almost all students appreciated the value of working in a collaborative workspace, highlighting the benefits of real-time communication and sharing of work with others.

Acknowledgments

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References

Dewey, J. (1910). How we think. Boston, MA: Heath.

Engle, R.A. (2011). The productive disciplinary engagement framework: Origins, key concepts, and developments. In D.Y. Dai (Ed.), Design research on learning and thinking in educational settings: Enhancing intellectual growth and functioning (pp. 161-200). London: Taylor & Francis.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Human, P., Murray, H., Olivier, A., & Wearne, D. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. *Educational Researcher*, *25*(4), 12-21.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, *27*(1), 29-63.

Lappan, G., Phillips, E.D., Fey, J.T., & Friel, S. N. (2014). Stretching and Thinking: Understanding Similarity. Boston, MA: Pearson Student Edition and Teacher Edition.

Strauss, A. L., & Corbin, J. M. (1998). *Basics of qualitative research: Grounded theory procedures and techniques* (2nd ed.). Thousand Oaks, CA: Sage.

The Design-Based Research Collective. (2003). Design-based research: An emerging paradigm for educational inquiry. *Educational Researcher*, *32*(1), 5-8.

Williams-Candek, M., & Smith, M.S. (2015). Principles of productive disciplinary engagement: Framework for design. Paper presented at the Research Pre-Session of the National Council of Teachers of Mathematics conference, Boston, MA.

TEACHER KNOWLEDGE FOR TECHNOLOGY IMPLEMENTATION

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This study is centered around a year-long professional development intended to improve teachers' technological, pedagogical, and content knowledge for middle school mathematics. Seven teachers' responses and asynchronous module work were analyzed to measure teachers' beliefs about their ability to implement technology in the classroom and the alignment between teacher interaction with technology and their understanding of mathematics. Results suggest that 1) teachers were more efficacious teaching mathematics without technology than with it; 2) teachers' beliefs about technology and their ability to implement it were generally positive and increased after professional development activities; 3) gaps in teacher knowledge and teachers' ability to use technology were evident.

Keywords: Teacher knowledge, technology, beliefs

Purpose of the Study

The use of technology in the classroom has gained momentum over the last decade, putting a focus on teachers' understanding and implementation of a technology-infused classroom. While much attention has been focused on building a framework of teachers technological, pedagogical, and content knowledge (TPACK), little attention has been paid to the possible factors that guide the implementation process. The year-long professional development (PD) focused on providing technological, pedagogical and content support for middle school mathematics teachers. The purpose of this study, then, is to examine possible factors influencing teachers' ability to implement technology in the middle school, mathematics classroom with two guiding questions: 1) What are middle school mathematics teachers' beliefs about their own ability to implement technology in the classroom? and 2) How do measures aimed at quantifying teachers' TPACK align with teachers' interaction with technology and their understanding of mathematics?

Theoretical Framework

In *Adding It Up,* the National Research Council (2001) calls out five components to effectively teach mathematics, three of which describe aspects of the knowledge base required to teach mathematics stemming from to the notion of Mathematical Knowledge for Teaching (MKT, Ball & Bass, 2003). While MKT theorizes the working parts of the knowledge required to teach mathematics, it does little to describe how technology integration fits into instruction. In an attempt to create a framework for the overlapping knowledges required to implement technology, Koehler and Mishra (2009) developed technological pedagogical and content knowledge (TPACK). To more effectively teach mathematics using technology, then, one must consider both frameworks. Only then can the influence of knowledge on technology integration be more fully understood.

Unlike MKT, most all measures of TPACK are related to teachers perceived levels of knowledge and are self-reported. While this can be reflective of actual levels of knowledge, there is clearly the possibility for disparity among the two levels. Instruments used to assess TPACK may be a better measure of beliefs than actual knowledge.

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.

In addition to the possible relationship between knowledge and beliefs, a compelling relationship between mathematical knowledge for teaching and instruction exists (Hill et al, 2008). Teachers with a strong knowledge of mathematics, a strong knowledge of how to teach it, and how to teach it within the context of a mathematics classroom were found to avoid mathematical content errors and implement more rigorous lessons. It would be unjust to assume that a relationship between MKT and implementation and not question the relationship between a measure of knowledge that includes technology integration and instruction.

Subsequent research also shows that teacher understanding of curriculum, a construct associated with both MKT and pedagogical content knowledge within TPACK, is an essential construct that must be addressed in future research on the relationship between knowledge and instruction (Hill et al, 2008). Therefore, the implementation of technology should be studied to fully understand how knowledge supports the integration of technology into classroom instruction.

Synthesizing the previous needs and findings, it becomes clear that too little research focuses on what factors support the integration of technology in a mathematics classroom. The impact of knowledge, beliefs and experience emerge as crucial constructs, as displayed in Figure 1, impacting how mathematics teachers select, evaluate, and implement technology into instruction.

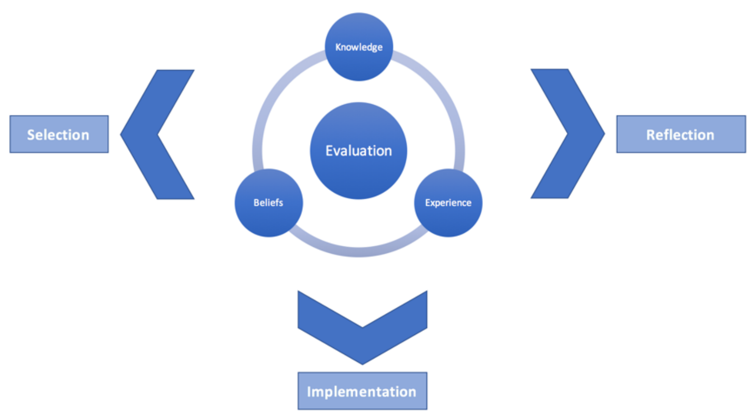

Figure 1. A theoretical framework for the implementation of technology in a mathematics classroom.

Methods

Participants

Middle school mathematics teachers from a large Midwestern city were recruited to participate in a year-long professional development. Five different school districts, spanning from urban to rural, provided 31 teachers and mathematics coaches to participate. For the purpose of this study, seven teachers were chosen to examine more closely. Emergent trends among the qualitative data were identified and analyzed.

Professional Development

Teachers began their PD experience with a six-day, summer institute that aimed to support teachers in building algebra content knowledge, building pedagogical content knowledge, and supporting the selection, evaluation, and implementation of digital resources in the mathematics classroom. Monthly, asynchronous modules were delivered online and modeled using the same focus as the summer institute. Email and site visits provided participants with additional support while they completed the online module portion of the project.

Data Collection

Initial pre-surveys were administered to measure participant TPACK, self-efficacy, and other measures of motivation. These surveys were again administered after the completion of the summer institute. Teacher work samples serve as a large component of the data set.

Results

Teacher beliefs

Teachers were given a survey with several items aimed at measuring their beliefs. With respect to self-efficacy to teach mathematics with technology versus without technology, five of the seven teachers had higher levels of self-efficacy for teaching mathematics without technology than with technology. All seven teachers had average scores over 3 (out of 5) for measures of self-efficacy to evaluate digital resources, self-efficacy to design, and perceived TPACK scores. Self-efficacy to evaluate digital resources, TPACK, self-efficacy to design curriculum, ability, expectancy, and attainment saw an increase after the summer institute.

Knowledge

Outcomes related to teacher content knowledge were mixed. Evidence suggests that while teachers understood basic slope content, other content provided more issues. Topics on exponents, multiple representations of equations, and fractions and rational numbers were littered with errors.

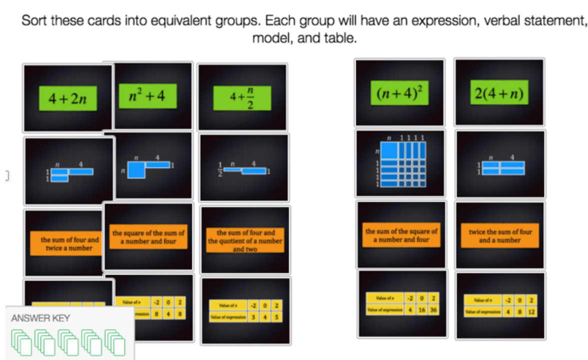

Figure 2. Work example of incorrect sorting of equivalent expressions.

The majority of teachers (4 out of 7) were unable to correctly match equivalent expressions that utilized different exponential rules. The teachers who were incorrect reported that they had difficulty remembering the rules associated with exponents. Figure 2 provides a different example, in which the teacher looks to be correct, but has switched two of the cards incorrectly. Similarly, the teacher did not fully submit their work due to difficulties utilizing the grouping mechanism of the digital resource.

Although teachers struggled with exponents, they excelled with notions of slope. Participants were able to correctly use and create slope triangles to answer questions about slope using Geogebra as demonstrated in the figure 3.

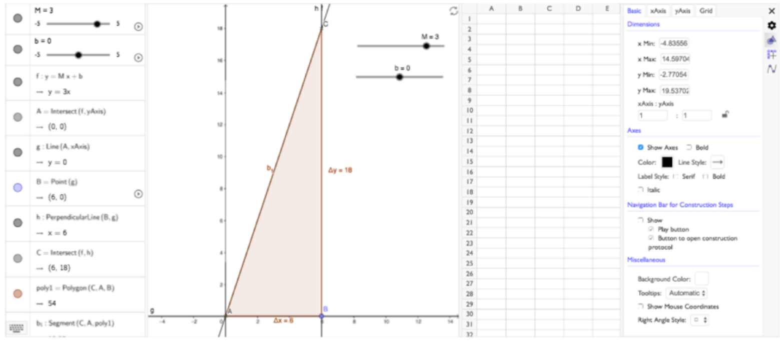

Figure 3. Example of the creation of a slope calculator using sliders.

Discussion

The study aimed to explore possible factors influencing teachers' ability to implement technology in the middle school, mathematics classroom. This study was unique in that looked at teachers' TPACK beyond popular survey methods by examining teacher work in asynchronous, online modules. While teachers reported above median values for TPACK, self-efficacy, and other measures, they reported more confidence in their ability to teach without technology. Teachers work in the asynchronous modules show teacher error in both content and technology utilization raising the question of whether erroneous responses were reflective of content knowledge or technology knowledge.

References

Ball, D. L., & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching (Adobe PDF). In B. Davis & E. Simmt (Eds.), Proceedings of the 2002 Annual Meeting of the Canadian Mathematics Education Study Group (pp. 3–14). Edmonton, AB: CMESG/GCEDM.

Hill, H.C., Blunk, M.L., Charalambous, C.Y., Lewis, J.M., Phelps, G.C., Sleep. L, & Ball, D.L. (2008) Mathematical Knowledge for Teaching and the Mathematical Quality of Instruction: An Exploratory Study, Cognition and Instruction, 26:4, 430-511, DOI: 10.1080/07370000802177235

Koehler, M., & Mishra, P. (2009). What is technological pedagogical content knowledge (TPACK)? Contemporary Issues in Technology and Teacher Education, 9(1), 60–70.

National Research Council. (2001). Adding it up: Helping children learn mathematics. J.Kilpatrick, J. Swafford, and B.Findell (Eds.). Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.

TEACHERS' IMPLEMENTATION OF VIRTUAL MANIPULATIVES AFTER PARTICIPATING IN PROFESSIONAL DEVELOPMENT

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The following study investigated why and how teachers implemented virtual manipulative (VM) tasks after participating in a professional development opportunity aimed at teaching with VMs. Findings indicate that teachers used VM tasks to support students' developing understanding and engagement and because they were supported in teaching with VM tasks. Mediating factors (e.g., time) influenced why and how teachers chose to use a particular technology tool.

Keywords: Professional Development, Technology, Middle and High School Education

Introduction

As access to various technology tools increases, so too has the expectation for teachers to use technology to enhance student engagement and understanding. Unfortunately, teachers often report that they are not prepared to use technology in an innovative manner and effectively in their instruction (Albion et al., 2015), a critical and longstanding issue in mathematics education. Effectively teaching *with* technology entails teachers using technology to promote opportunities for students to develop conceptual understanding through reflection and communication, as well as through using and connecting mathematical representations (Reiten, 2018). Which leads to the question, *when teachers are supported in teaching* with *a technology tool, why and how do they integrate the tool into their instructional practices?*

The following study highlights two teachers to report the findings of a professional development (PD) opportunity for secondary mathematics teachers aimed at supporting their efforts to teach *with* virtual manipulatives (VMs). A VM is an interactive, technology-enabled visual representation of a dynamic mathematical object, including all of the programmable features that allow it to be manipulated, that presents opportunities for constructing mathematical knowledge" (Moyer-Packenham & Bolyard, 2016). Visit http://bit.ly/VirtManips for an annotated list of VMs and http://bit.ly/VMTasks for a repository of VM tasks used during the PD. VMs were the focus of this PD due to the widespread availability of the tool and the increased attention that has been given to the development of VMs in recent years. Many VMs provide immediate feedback and the interactive environment can promote student perseverance and exploration for students who get frustrated or disengage with paper and pencil investigations (Moyer-Packenham & Westenskow 2013). Despite these benefits for using VMs, limited resources exist to support teaching *with* VM tasks (Reiten, 2018). A VM task is the VM and all accompanying instructions/prompts whether onscreen or in printed form.

Theoretical Background

Teachers' implementation of VM tasks were considered to be mediated by their mathematical goals for the task, the tools available related to their implementation, the students they teach, the teachers with whom they teach, and school/district initiatives. Therefore, to understand why and how teachers implemented VMs, the following study drew from activity theory (Engeström, 1999). An activity consists of a subject, object, and actions. The *subject* is the person or people engaged in an activity (i.e., the teachers in the PD). The *object* motivates the activity and gives the activity specific direction (i.e., teachers' practices related to implementing

VMs). *Actions* are goal-directed processes undertaken by the subject to achieve the object (i.e., teachers' instructional practices related to implementing VM tasks). Other aspects of this activity system include the tools/mediating artifacts (e.g., the VMs and tasks, the task analysis framework), rules (e.g., curriculum, instructional style), community (e.g., other teachers in the school), and division of labor (e.g., do teachers work collaboratively or primarily individually).

Methods

For the larger study, 14 mathematics teachers participated in the PD opportunity aimed at supporting their efforts to teach *with* VMs (see Reiten, 2018). The focus in this paper is on a 6th /7th grade mathematics intervention teacher (i.e., Tracy, 20 years of experience) and a high school Geometry/AP Statistics teacher (i.e., Daron, 21 years of experience). These participants were chosen due to representing different grade bands and student populations. Data collected included online background and final surveys, video and audio recordings of the PD sessions, and teachers' work during the sessions. Tracy also participated in semi-structured interviews at the beginning and end of the PD.

Transcripts, teachers' work, and their reflections were initially coded to identify themes related to how they used the tools introduced in the PD, how and why they implemented technology (codes drawing from literature), and related tensions (emergent codes). Focused coding (Saldaña, 2013) then progressed after sorting codes vis-à-vis components of the activity system (e.g., division of labor, community, tools, tensions, etc.), re-examining data within codes, and linking teachers' work during the PD with their conversations. This analysis gave insight into how and why teachers implemented VM tasks (i.e., the object of the activity system).

Findings and Discussion

The goal of the PD was to support teachers in teaching *with* VM tasks. Teachers' appropriation of tools (i.e., guiding questions, task analysis framework, annotated VM list, folder of tasks) used during the PD supported their implementation efforts (Reiten, 2018). Tracy implemented over 12 VM tasks and Daron implemented at least four VM tasks with students.

Why Teachers Implemented VM Tasks

Engagement and exploration. Teachers implemented VMs and tasks because they thought their students' learning might be enhanced due to students' engagement with the VM tasks. For example, during Tracy's final interview, she said

> It piques their interest, in all of it. All of a sudden we'll [i.e., the students] TRY because we're not writing on pencil and paper. Like the one that we did with the area and the perimeter when they're making the gardens [i.e., Fido's Flower Bed from ExploreLearning] all of a sudden, "*OH! A man walked by.*" Or "*Oh, did you see that dog?!*" You know, I mean … they get excited about it and they can compare [with each other].

On Daron's final survey, he said that he used the VM tasks because "students can discover and explore concepts in a dynamic way. [The VM tasks] allow for a deeper level of engagement and more efficient way to investigate deep mathematical ideas." Tracy's and Daron's reflections highlight how they were using VM tasks to give students opportunities to explore content in a more dynamic way and with less restrictions than if the task had been done with paper and pencil. Additionally, the interactive environment helped to promote students' engagement (e.g., Moyer-Packenham and Westenskow 2013) with the tasks as well as reflecting on the mathematics through comparing solution strategies with their peers.

Instructional guides. Many of the VM tasks that teachers chose to implement had accompanying guided questions or instructional guides to help students engage in the task and support their developing understanding. During the October PD session, Tracy said

> I just think that it [the VM with an instructional guide] is more GUIDED and that makes it a little more REAL to them where they are actually having to do what they would have to do on a math test, you know. That they are practicing those skills and writing it down and maybe making it a little more concrete than just click, click, click, click, click, click, click. And see if we get an answer eventually that says, *"Hey, you're right!"*

The accompanying questions or instructional guides supported teachers in how they may want to use a given VM task and assisted students' exploration of the given topic.

However, not all VM tasks had accompanying questions. Therefore, at times teachers developed instructional guides to accompany a VM. During the January PD, Daron said

> Some of the ones that I had been looking at don't really have a guide, so … I kind of keep these things in mind [i.e., the affordances in the task analysis framework] as I try to write a guide or put together some questions…to try to make sure as many of these were covered so it was a meaningful activity for them as well. Where otherwise you might just throw the questions out, you just, I think it allows you to think more like, "How do I move them up to those levels that we're trying to get them to?"

Therefore, teachers' efforts to develop instructional guides that further supported student learning and engagement were also supported via a tool introduced during the PD (i.e., the task analysis framework (Reiten, 2018)). Hence, when investigating why teachers choose to implement VM tasks, it is important to consider how teachers are supported in developing accompanying materials that they find helpful for supporting students' developing understanding.

How Teachers Implemented VM Tasks

Instructional role. Teachers used VM tasks for a variety of purposes or instructional roles. During the January PD session, Daron specifically looked for a VM exploring a confidence interval for a proportion to use in his AP Stats class that might enhance student understanding by connecting the confidence interval and the graph of a distribution. During the spring months, Tracy often looked for VM tasks that allowed students to investigate and develop area formulas (e.g., calculating the area of a parallelogram) rather than simply being told the area formula.

Implementation efforts. Although the PD aimed to support teachers in teaching *with* VM tasks, there was no requirement that teachers had to implement the VM tasks that they were critiquing and modifying during the PD session.

During the January PD session, Daron brought up his experience implementing a VM with his AP Statistics students. According to Daron, after doing the gizmo, students had less questions about the binomial theorem compared to years past and a better understanding of the formula. Students were not relying on their calculators to evaluate the formula and were not making some of the common errors that previous students had made (e.g., leaving off the beginning part of the binomial formula). After the PD session, Daron emailed the following thoughts related to his implementation of the Binomial Probabilities gizmo:

> I think students were much more engaged in learning the binomial pattern. They were able to see where the pattern was coming up from the most basic situation to more complex examples. I think students were able to see the pattern in the factorials and Pascal's triangles quickly, then were able to adapt them to more difficult problems. I asked for student feedback and they said they really liked using the gizmo and it helped them to see the pattern.

Daron's thoughts demonstrate how both he and his students found the VM helpful for students' developing understanding. This early positive experience implementing a VM task encouraged Daron to continue implementing VM tasks during the school year.

At times, teachers found it challenging to implement VM tasks. When asked during her final interview if there was something that prevented her from using VM tasks, Tracy stated, "Time!" Time related to having the instructional time within a class period as well as time for finding the tasks and getting them ready to implement with her students. Although the PD could not affect the instructional time teachers had, the PD provided teachers time to find, critique, and modify VM tasks specifically related to their upcoming units. As demands on teachers' time continue, integrating supported time within PD for teachers to find and develop VM tasks directly connected to their curriculum can help support teachers' integration efforts by strengthening the connection between the PD and teachers' instructional practices (Wilson, 2008).

Conclusion

Although VMs are not new, little resources exist to support teachers' efforts to teach *with* VM tasks. Therefore, this PD opportunity was developed to support teachers as they found, critiqued, and modified VM tasks to implement with their students. Teachers implemented VM tasks due to benefits they experienced in relation to their students' developing understanding and engagement, as well as the ways in which they were supported in designing instructional guides to support students' emerging understanding. The PD aimed to support teachers' efforts to teach *with* technology by providing teachers' the time and support needed to not only learn about VM tasks, but also time to critique the tasks and prepare the tasks to implement with their students. Supporting teachers to teach *with* technology goes beyond providing them with access to technology tools. Rather, it includes providing opportunities for teachers to interact with technology tools related to their current curriculum. Additionally, it means providing teachers resources that support their integration efforts and emerging understanding regarding knowledge growth in how to teach *with* technology. Supporting teachers also means acknowledging the tensions that exist related to integrating technology tools (e.g., time) student needs (i.e., community factors), tool limitations and supports (i.e., tools/mediating artifacts), etc.

References

Albion, P. R., Tondeur, J., Forkosh-Baruch, A., & Peeraer, J. (2015). Teachers' professional development for ICT integration: Towards a reciprocal relationship between research and practice. *Education and Information Technologies, 20*(4), 655-673.

Engeström, Y. (1999). Activity theory and individual and social transformation. In Y. Engeström, R. Miettinen, & R. Punamaki, (Eds.), *Perspectives on activity theory* (pp. 19–38). Cambridge, MA: Cambridge University Press.

Moyer-Packenham, P. S., & Bolyard, J. J. (2016). Revisiting the definition of a virtual manipulative. In P. S. Moyer-Packenham (Ed.), *International perspectives on teaching and learning mathematics with virtual manipulatives* (pp. 3-24). Switzerland: Springer International Publishing.

Moyer-Packenham, P. S., & Westenskow, A. (2013). Effects of virtual manipulatives on student achievement and mathematics learning. *International Journal of Virtual and Personal Learning Environments, 4*(3), 35-50.

Reiten, L. (2018). Teaching WITH (not near) virtual manipulatives. In E. Langran & J. Borup (Eds.), *Proceedings of Society for Information Technology & Teacher Education International Conference* (pp. 1826-1835). Washington, D.C., United States: Association for the Advancement of Computing in Education (AACE). Retrieved May 17, 2018 from https://www.learntechlib.org/primary/p/182776/.

Saldaña, J. (2013). *The coding manual for qualitative researchers.* Second Edition. Sage Publications Ltd.: Los Angeles, CA.

Wilson, P. S. (2008). Teacher education: A conduit to the classroom. In G. W. Blume & M. K. Heid (Eds.), *Research on technology and the teaching and learning of mathematics: Vol. 2. Cases and perspectives* (pp. 415-426). Charlotte, NC: Information Age.

MEASURING SELF-REGULATED LEARNING IN CALCULUS I

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Calculus 1 has been and continues to be a key gateway course to STEM majors, which contributes to a loss of students in the STEM pipeline. Self-regulated learning (SRL) competencies have widely been found to be related to academic achievement (e.g. Zimmerman, Moylan, Hudesman, White, & Flugman, 2011), though common tools to measure SRL have fallen under scrutiny (Winne & Jameison-Noel, 2002). Ways in which students interact in the learning process impacts performance and, in turn, the student experience. Using an SRL framework, online tools were designed to collect data that can be interpreted to create a behavioral SRL score based on in-course student activity. This brief report presents initial findings on the relationship between a behavioral SRL score and academic achievement in Calculus I.

Keywords: Metacognition, Measurement, Technology

Calculus I has often been identified as a gatekeeper course for STEM disciplines. Aside from incoming ability, evidence suggests that how students engage with their studies impacts their success (Vandamme, Meskens, Superby, 2007). This research aims to investigate the impacts of student's online interactions on academic performance by interpreting students' use of purposefully-designed online tools using a self-regulated learning (SRL) framework. For the purpose of academic performance prediction and at-risk student detection, SRL theory provides a lens to understand how students engage in online settings.

This brief research report will describe how an SRL framework can be used to more deeply understand students' behaviors in online settings and how student interactions with resources developed around SRL theory can be organized into a behavioral SRL score. We will then analyze relationships between this score and academic achievement in a Calculus I class.

Theoretical Framework: Self-Regulated Learning

SRL - "the self-directed process by which learners transform their mental abilities into academic skills" (Zimmerman, 2002, p. 65) - enables students to better understand how they engage with content and adapt their study strategies to overcome academic difficulties and become more successful students. For example, when a mathematics student receives a bad grade on a homework assignment, they often do not reflect on why or how to change their study habits to be more successful in the future. Zimmerman (2000) presents a three-phase process model of SRL describing an ideal process through which students can support their learning as they engage in a task. This model is cyclic, with each phase - forethought, performance, self-reflection - informing the next.

Forethought is the planning phase, which ties in a student's beliefs and motivation, such as self-efficacy, outcome expectations, and task value, around the task (Zimmerman, 2008). During this stage a student will analyze a task and determine goals and strategies to implement. Following forethought, the performance phase centers around the task itself. The student then implements planned strategies, observing and adapting strategies during the task (Zimmerman, 2000). Self-observations also provide information to the student about effectiveness of strategies on performance as students enter the self-reflection phase. The self-reflection phase analyzes the gathered information, drawing causal attribution to conditions surrounding the task so that adjustments and improvements can be made, informing future task cycles (Zimmerman 2000). As an example, consider a student preparing for an exam via a practice exam. Before sitting down, a self-regulating student would determine what resources are needed working on the practice exam (e.g. time, location, notes, textbooks, technology). They would identify test-taking strategies to use and then select ones that are relevant. During the practice exam, identified strategies are implemented and resources are used. Before sitting down, a self-regulating student would identify resources and strategies to use during the practice exam. After selecting and implementing these strategies, the student would reflect on their performance and process to make judgements regarding the effects of their methods on their performance and goals for the practice exam. Successful strategies could then be used in the subsequent exam, while others might be modified for future use.

Measuring SRL

The Motivated Strategies for Learning Questionnaire (MSLQ) is the most commonly used instrument for measuring SRL. The MSLQ is an 81-item self-report questionnaire that assesses "college students' motivational orientations and their use of different learning strategies for a college course" as well as their "goals and value beliefs" (Pintrich, Smith, Garcia, & McKeachie, 1991, p. 3). The MSLQ has primarily been used to study the relationship between components of SRL and academic performance (e.g. Pardo, Han, & Ellis, 2016). Unfortunately, self-reports do not always align with observed behaviors (e.g. strategy usage), which has caused the validity of the MSLQ to be called into question (Winne & Jamieson-Noel, 2002). To address this concern, we designed online tools specifically to coax evidence of the SRL phases into observable, measurable events that can be recorded and formed into a behavioral SRL score, which will be referred to as the *B-SRL score*.

Methods

As students often struggle with precalculus content (e.g. Carlson, Madison, & West, 2015), self-regulation around the task of assessing and addressing one's knowledge of precalculus topics at the start of the semester was the focus of this research. Three optional online tools were developed and made available to students through the university's learning management system (LMS) based on the three SRL phases: forethought, performance, and self-reflection.

The Prerequisite Self-Assessment (SA) was developed to capture evidence around the forethought phase. This eight-item survey asks students to rate their confidence in correctly answering questions on relevant prerequisite material on a Likert Scale from one (no confidence) to five (very confident). The SA determines if students are engaging in the task analysis component of forethought by assessing their confidence in precalculus topics.

Students' participation in the performance phase of SRL was determined by how they scored on the optional Prerequisites Content Quiz (CQ) - a 12-item multiple-choice and multiple-answer quiz about prerequisite material essential for Calculus I. Topics included in the SA and the CQ related to function notation, graphs of functions, algebra, and trigonometry. Upon completion, students received immediate feedback on the correctness of each response and, if incorrect, the relevant topic to review, as well as available resources (e.g. specially developed review videos).

The Prerequisite Reflection Tool (RT) was designed to provide evidence of students engaging in the self-reflection around their CQ performance. This five-item survey asks students to identify what precalculus topics they plan to revisit in light of their CQ performance, as well as how and when they plan to review. Student precalculus remediation was tracked through provided digital and physical resources. Subsequent follow through of remediation provides evidence of cyclic nature of SRL, capturing the transitions from reflection into forethought of remediation. As all tools and resources were optional for the students, their' mere use of these online tools and resources provide evidence of students engaging in the different phrases of SRL.

Formulating a Behavioral SRL Score

Using the data from the online tools, the B-SRL score was formulated by first categorizing the usage data for each online tool, and then coding each student's set of behaviors with the tools onto a six-point scale representing students SRL strategy usage. It is worth noting that SRL is a process that can stretch semesters, so the B-SRL score is a snapshot of the SRL process at the beginning of the semester around the task of assessing one's precalculus knowledge and potential remediation of skills within the first two weeks of class.

Since the online tools were all optional, each tool was either used by a student or not. For the both the SA and CQ, students who used the tool were further subdivided. Of students that used the SA, they were considered to have either high confidence (mean confidence score of three or greater) or low confidence. Students that completed the CQ were considered to have either high precalculus ability (score of eight or greater) or low ability. Results from the SA and CQ were combined with use of the RT (used or did not) and precalculus resource access (accessed or did not) to formulate a B-SRL score. Each set of behaviors was then evaluated using Zimmerman's three-phase model with the guiding question 'How appropriately is the student responding to the feedback about their impressions on their precalculus abilities?'. Further, the B-SRL score took into account whether or not the student was appropriately calibrated with regard to their confidence and academic ability, where more accurate calibration has found to be a product of SRL training in the literature (Zimmerman et al., 2011). Each behavior was assigned B-SRL scores of 0 (no self-regulation), 1, 2, 3, 4, or 5 (highly self-regulating).

To highlight the coding from behaviors into the B-SRL score, consider three students: Derek, Matilda, and Jay. Derek has high precalculus confidence, but low precalculus ability. Further Derek has not engaged with the RT nor the precalculus resources. His confidence and ability regarding precalculus ability are miscalibrated, and Derek is not using the low CQ score as a prompt to revisit precalculus material. For these reasons, Derek would receive a B-SRL score of two. If Derek were to instead use the RT and use one of the precalculus resources, he would receive a B-SRL score of four.

Matilda showed low confidence with precalculus content despite showing high precalculus ability. She then did not use the RT nor the precalculus resources. Matilda has high precalculus ability, so she does not need to remediate her precalculus abilities. Despite this, Matilda's ability and confidence are not appropriately calibrated since she has high ability but low confidence. Further Matilda has not reflected on her CQ performance, so she receives a B-SRL score of three. Were Matilda to reflect on the CQ (use the RT), she would receive a B-SRL score of four. Lastly, Jay is a student who uses none of the provided online tools. Though Jay may be self-regulating, there is no evidence of this. Hence, Jay would receive a B-SRL score of zero.

Data and Initial Findings

Data was collected from 376 consenting university Calculus I students who competed the course. Data sources included course performance data and B-SRL score data.

Toward considering the relationship of the B-SRL score with academic performance, DF rates were inspected for each set of students with a particular B-SRL score. For the 376 students, the DF rates for each B-SRL score generally tends to decrease as the B-SRL score increased. Particularly, the groups of students who received a B-SRL score of 0, 1, 2, 3, 4, and 5 had DF rates of 46.9%, 32%, 27.8%, 31.2%, 24%, and 17%, respectively. Note that the group with a B-SRL score of two deviates slightly from this pattern. It is worth noting that this B-SRL group is smaller compared to other B-SRL groups (n=18, compared to, on average, n=71.2). Further investigation of this group is warranted, but outside the scope of this report.

In comparing students who had high B-SRL scores (B-SRL score ≥ 3) with students with low B-SRL scores (B-SRL score < 3), Spearman's correlation revealed significant differences between the groups on all course exams and the final course grade ($0.14 \leq r \leq 0.23$ and $0.001 \leq p \leq 0.005$), suggesting that students with high SRL have higher scores than students with low SRL.

Discussion

These preliminary findings show promise for being able to use an SRL framework to develop tools that measure students' SRL through observable means. Using purposefully-designed tools, we developed a method for generating a B-SRL score for students by analyzing their online interactions, specifically those around prerequisite remediation and readiness for Calculus I. The relationship between B-SRL scores and academic performance metrics suggests that students who tend to self-regulate their precalculus abilities more tend to be more successful in Calculus I, which aligns with what is seen in the literature (e.g. Zimmerman et al., 2011; Zimmerman & Schunk, 2001). Similarly, students who self-regulate more tend to perform better on exams compared to those who self-regulate less.

Some foundational STEM courses implement 'gateway exams' at the beginning of a course to assess student readiness for the course. The methodology discussed in this paper expands on this strategy, going beyond just prerequisite content knowledge by incorporating student SRL strategies and remediation tendencies as well. The B-SRL score and discussed tools can further aid in early detection of at-risk students, informing intervention to support and improve student success in Calculus I and STEM. Future efforts in this project will be to utilize results to inform course design and pedagogical decisions to better support at-risk and underprepared students.

References

Carlson, M. P., Madison, B., & West, R. D. (2015). A study of students' readiness to learn calculus. International Journal of Research in Undergraduate Mathematics Education, 1(2), 209-233.

Pardo, A., Han, F., & Ellis, R. A. (2016). Exploring the relation between self-regulation, online activities, and academic performance: a case study. In Proceedings of the Sixth International Conference on Learning Analytics & Knowledge, 422-429.

Pintrich, P. R., Smith, D. A., García, T., & McKeachie, W. J. (1991). A manual for the use of the Motivated Strategies for Learning Questionnaire (MSLQ). DOI: 10.13140/RG.2.1.2547.6968.

Vandamme, J. P., Meskens, N., & Superby, J. F. (2007). Predicting academic performance by data mining methods. Education Economics, 15(4), 405-419.

Winne, P. H., & Jamieson-Noel, D. (2002). Exploring students' calibration of self reports about study tactics and achievement. Contemporary Educational Psychology, 27(4), 551-572.

Zimmerman, B. J. (2000). Attaining self-regulation. In Boekaerts, M., Pintrich, P. R., & Zeidner, M (Eds.), Handbook of self-regulation (pp. 13-39). San Diego, CA: Academic Press.

Zimmerman, B. J. (2002). Becoming a self-regulated learner: An overview. Theory into practice, 41(2), 64-70.

Zimmerman, B. J. (2008). Investigating self-regulation and motivation: Historical background, methodological developments, and future prospects. American Educational Research Journal, 45(1), 166-183.

Zimmerman, B. J., Moylan, A., Hudesman, J., White, N., & Flugman, B. (2011). Enhancing self-reflection and mathematics achievement of at-risk urban technical college students. Psychological Test and Assessment Modeling, 53(1), 141-160.

Zimmerman, B. J., & Schunk, D. H. (Eds.). (2001). Self-regulated learning and academic achievement: Theoretical perspectives. Routledge.

AN EXPLORATORY APPROACH TO ANALYZING STUDENTS' EYE MOVEMENTS WHEN SOLVING MATH PROBLEMS

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Eye movement is important data to consider when studying students' cognition processes while solving math problems. However, the large amount of data makes it challenging for educators and researchers to discover meaningful patterns. Thus, this paper proposes using visualization and the Shneiderman visual information-searching theory —"Overview first, zoom and filter, details on demand"—to enable educators to easily and logically discover temporal and spatial patterns in eye movement data.

Keywords: Data Analysis and Statistics, Research Methods, Problem Solving, Technology

Introduction

Eye tracking is an important technology that has been increasingly used in recent education research. An eye tracking system can capture students' eye movements, reflecting their visual attention distribution in real time. From these data, researchers link students' eye movements to their internal cognition processes, which are also the processes of decision making. Eye tracking systems have successfully provided eye movement data support for educators to elucidate many aspects of students' cognition processes, such as visual attention (Takacs & Bus, 2016), problem-solving strategies (Thibaut & French, 2011), and comprehension and learning processes (Liu & Shen, 2011).

However, since the big data size challenges people's data exploration and analysis abilities, determining how to analyze eye movement data is difficult. Hundreds of data points per second are produced by eye tracking systems, making it impossible to analyze the data line by line. We need an effective method for simplifying the data and extracting valuable information. A typical method for simplifying eye movement data is choosing metrics such as fixation numbers and fixation duration to conduct statistical analysis (Lai et al., 2013). However, this method discards rich spatial and temporal information related to eye movement data. Here, the authors propose a method that uses information visualization techniques and a visual information-searching mantra to help teachers explore students' eye movement data efficiently. Specifically, this study applies the Shneiderman visual information-searching theory to educational eye movement data analysis in order to facilitate data exploration and pattern discovery. The Shneiderman visual information-searching theory introduced three steps for exploring visual data: "overview first, zoom & filter, details-on-demand" (Shneiderman, 1996). In the following sections, the authors apply visualization techniques to eye movement data from a mathematical problem-solving project to illustrate how this theory can be used to analyze eye movement data.

Methods

Stimuli and Participants

The computer program used in this study is a Conceptual Model-based Problem-Solving

(COMPS) tutor, which was developed to promote additive mathematics problem-solving. Figure 1 is a typical task that students were given. They were asked to represent the information from the word problem task in the Part-Part-Whole (PPW) diagram equation. Then, based on the PPW equation, students were required to set up the math equation. Ten third-grade students (five boys, five girls) participated in the study. Their eye movements were recorded by Tobii Pro X3-120 eye trackers during their interactions with the computer program.

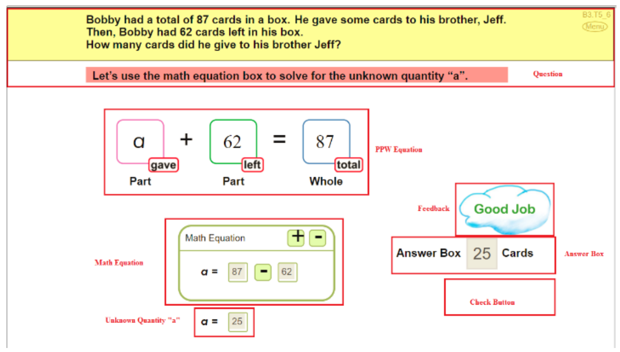

Figure 1. Layout of A COMPS Task with AOIs (Area of Interest)

Visualization Exploration

The visualization exploration process follows the Shneiderman information visualization-searching theory. First, the overview stage reveals students' eye movement patterns under each question structure. Then, zoom and filter closely observes eye movement data and discovers patterns worth future exploration. This process examines details on demand to investigate outliers, patterns, and differences. We hypothesize that by using Shneiderman, educators can quickly understand students' behaviors, identify concerns, and determine differences among them.

Overview first. Visualizing eye movements without considering the problem-solving context is neither accurate nor complete. Consolidating eye movement data within the context of the question and students' performance is necessary for obtaining an integrated and comprehensive view of students' attention distribution that reflects their internal cognitive process. In Figure 2, the first column signifies the task content. Each circle represents the length of time a student spent on the task. The outer ring of a circle is separated into arcs, according to the different stages of each student's interaction. For example, the light gray arc shows the percentage of time a student spent reading the question, while dark gray indicates the operating segment. The red/green arc is the feedback segment, and its color is determined by the correctness of the students' operation (incorrect-red /correct-green). Since students have two opportunities to try, some circles have two operation and feedback segments. The inner sectors indicate the total fixation duration of each interaction segment, while the color of each sector corresponds to the area of interest (AOI) color within which the fixations were distributed.

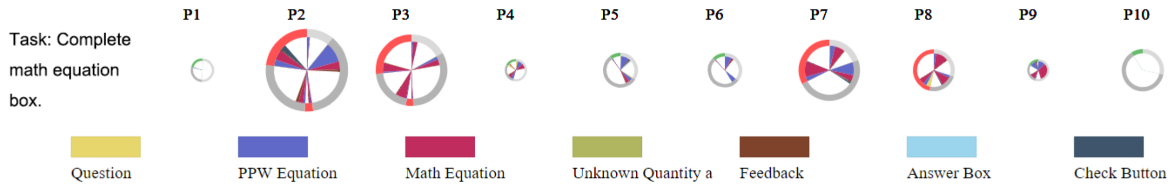

Figure 2. Students' Eye Movement Data Overview

As shown in Figure 2, six students completed the task fluently, indicating that they have no difficulty in transferring the PPW equation into the math equation. Four students failed the task. Two of them (P2 and P3) tried twice and still failed. It seems that, compared with the other students, those who struggled to solve the problems tend to spend much more time looking at the math equation area. To get more accurate information, we needed to zoom in and filter out unnecessary information.

Zoom and filter. The visual graphs of P4 and P5 are small in Figure 2, since students spent relatively less time on these tasks. Because of this, zoom is needed to observe the graphs. Even closer exploration can be supported by using filters. Figure 3 shows students' fixation distribution when reading the questions. Operation segment, feedback segment, and participants (P1 & P10) who have no eye movement data in the reading segment are filtered out. Obvious eye movement differences between high-performance and low-performance students can be observed. High-performance students focused on the filled PPW equation, while low-performance students spent little time on the PPW equation, indicating that they could not build connections between the PPW equation and equations. They looked at the math equation but were confused about how to fill it in. This pattern suggests that low-performance students have difficulty understanding abstract equations.

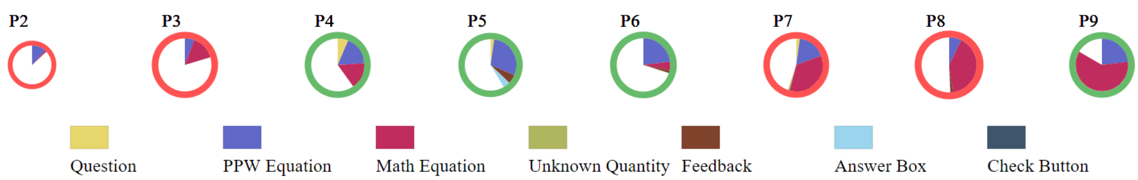

Figure 3. Fixation Distribution in Reading Segment

In the overview stage, educators gain a global understanding of students' performance and eye movement distributions. The zoom and filter techniques strengthen the exploration process by filtering noise and providing a clear view of information. Students' eye movement patterns can be observed at this stage. As the data is still aggregated, a more detailed exploration of the spatial and temporal acuities of the eye movement patterns is needed.

Details on demand. The "Zoom and Filter" stage reveals different fixation patterns between high- and low- students. Figure 4 pinpoints the moment when students' attention switches between the PPW equation and the math equation. This provides useful information: 1) all students looked at the math equation at the beginning of the task, but high-performance students switched their attention to the PPW diagram equation within two seconds, while low-performance students kept their attention on the math equation for at least three seconds; 2) Low-performance students kept looking back at the math equation during the reading segment and paid little attention to the PPW equation, while high-performance students focused on the PPW equation for a relatively long period of time. Although only four participants' visual graphs are presented in Figure 4 due to space limitations, similar patterns can be observed in all participants' eye movement visualizations. This suggests that the timing patterns extracted from this visualization can successfully capture differences between high- and low- performers.

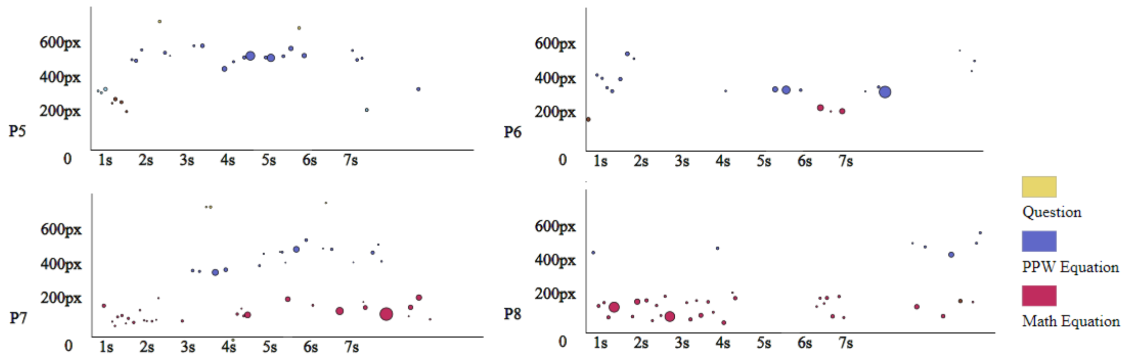

Figure 4. Visualizing Eye Movement Data from Reading Segment

Conclusion

This study presents a visual exploration approach to investigating students' eye movements when solving math problems. The visual exploration example from the COMPS project illustrates how the Shneiderman visualization searching-theory can be used to analyze complex eye movement data. In the past, research has relied exclusively on statistical models, causing valuable patterns to be ignored. The proposed approach can help educators and researchers track patterns in students' eye movements and discover valuable research directions related to students' cognition process.

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References

Lai, M. L., Tsai, M. J., Yang, F. Y., Hsu, C. Y., Liu, T. C., Lee, S. W. Y., … Tsai, C. C. (2013, December 1). A review of using eye-tracking technology in exploring learning from 2000 to 2012. Educational Research Review. Elsevier.

Liu, C. J., & Shen, M. H. (2011). The Influence of Different Representations on Solving Concentration Problems at Elementary School. *Journal of Science Education and Technology*, 20(5), 621–629.

Shneiderman, B. (1996). The eyes have it: a task by data type taxonomy for information visualizations. *Proceedings 1996 IEEE Symposium on Visual Languages*, 336–343.

Takacs, Z. K., & Bus, A. G. (2016). Benefits of motion in animated storybooks for children's visual attention and story comprehension. An eye-tracking study. *Frontiers in Psychology*, 7(OCT), 1591.

Thibaut, J., French, R., Missault, A., Gerard, Y., & Glady, Y. (2011). In the eyes of the beholder: What eye-tracking reveals about analogy-making strategies in children and adults. *Proceedings of the thirty-third Annual Meeting of the Cognitive Science Society,* 453–458.

FAILING TO REWIND: STUDENTS' LEARNING FROM INSTRUCTIONAL VIDEOS

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In this study we investigate how students watch and learn from a set of calculus instructional videos focused on reasoning about quantities needed to graph the function modeling the instantaneous speed of a car. Using pre- and post-video problems, a survey about the students' sense-making and data about the students' interactions with the video, we found that many students did not appear to make significant gains in their learning and that students appeared to not recognize their own moments of confusion or lack of understanding. These results highlight potential issues related to learning from instructional videos.

Keywords: Technology, Instructional Activities and Practices, Post-Secondary Education

In recent years, "flipped" classrooms and massive open online courses have been promoted as effective ways to deliver content to students and to support active learning in the classroom (e.g., Schroeder, McGiveny-Burelle, & Xue, 2015). Although there is increased interest in using these techniques and a growing body of research literature on student learning in flipped classrooms (e.g., Maxson & Szaniszlo, 2015), there is still relatively little empirical data to support the claims of the efficacy of these instructional innovations.

With a few exceptions (e.g., Weinberg & Thomas, 2018), there have been virtually no studies that have investigated how students utilize the out-of-class resources or how students' experience with the videos supports their construction of particular mathematical meanings. Instead, the research has been based on an implicit empiricist epistemology, assuming that not only do students actually watch and learn from the out-of-class resources, but that the students uniformly construct the meaning the instructor believes the video to convey. Thus, it is important for us to investigate how students engage with and learn from instructional videos.
Our research questions are:

- How often do students pause or re-watch sections of the videos?
- What do students learn from watching instructional calculus videos? How is students' learning connected with their video-watching activity?
- What aspects of the calculus videos do the students find confusing? How is this connected to their learning?

Theoretical Framework

Sense-Making Gaps

Sense-making research (e.g., Dervin, 1983) has been used in the fields of information systems and, more recently, in mathematics education (e.g., Weinberg, Wiesner, & Fukawa-Connelly, 2014) to understand the ways individuals perceive, act within, and make decisions in situations. From this perspective, students experience gaps—moments of confusion or questions that must be answered or overcome in order to construct meaning for the video. Gaps are not a feature of the video, but rather are a product of the interaction between the video and the student's knowledge, beliefs, and purpose for watching the video.

Covariational Reasoning

To make sense of dynamic situations modeled by calculus, students construct relationships between conceived quantities that co-vary (i.e., change together), that is, they develop and apply *covariational reasoning* (e.g., Thompson & Carlson, 2017). Carlson, Jacobs, Coe, Larsen, and Hsu (2002) defined this as "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354).

Methods

Video-watching was assigned in three of the authors' first-semester calculus classes. There were 29 volunteer student participants, with only 23 also completing the post-video survey. We describe student activity and learning from one set of three instructional videos that focused on graphing derivatives. The first video described how to use ideas about amounts of change to construct a distance-versus-time graph; the second video described how to construct and graph rates of change; the third video provided another example of constructing a graph of speed. All the videos were hosted on the Ximera online platform (https://ximera.osu.edu/), which recorded the timestamp of each student interaction with the videos—playing, pausing, and skipping backward or forward. In order to (potentially) identify places where students experienced a gap, we classified each pause and skip-back as a "revisit"—a place where the student felt that some aspect of the video was either important, unclear, or confusing.

Prior to watching the set of videos, students were presented with a graph of a cubic function $y=g(x)$ and asked to solve three problems related to approximating values of $g'(x)$. After watching the videos, the students were shown a graph of a quartic function $y=f(x)$ and asked to solve eleven problems that were similar in nature to those in the pre-video assessment. In order to further identify students' gaps, the students completed a sense-making survey in which they were asked to describe aspects of the video that were confusing or could use additional explanation. We used thematic analysis (Braun & Clarke, 2006) to generate initial descriptions and categories of the students' responses.

Results

Student Learning

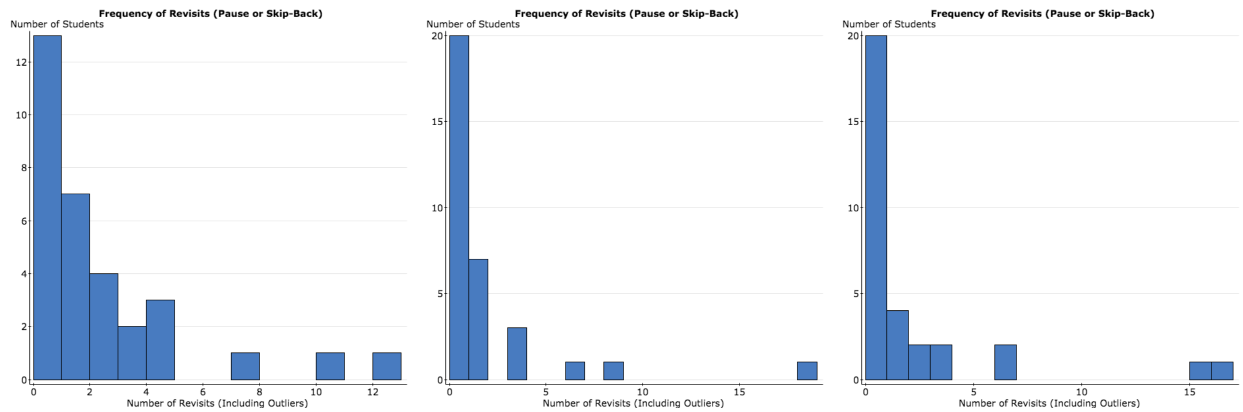
As shown in Table 4, the students correctly answered 41% of the pre-video problems and did not improve their scores significantly on their first attempt at the post-video problems ($t(28) = 0.9184$, $p=0.3662$)). Their mean score on their second attempt at the post-video problems was 82%, which was significantly higher than their pre-video scores ($t(28) = 7.7617$, $p<.0001$). The students' mean normalized gain (Bao, 2006) scores was 4.9% when comparing the pre-video problems and first attempt at the post-video problems and was 63% with the second attempt.

Revisits

The histograms in Figure 5 show the number of "revisits" (i.e., times each student paused or skipped backward) for Videos 1, 2, and 3. This shows us that, particularly in Video 2 and Video 3, most students never revisited during the video; in Video 1 roughly half of the students paused at least once. There were a handful of students for each video who revisited relatively frequently (i.e., their number of revisits was an outlier for the video).

Table 4: Student scores and normalized gains on the pre- and post-video problems

Group	Mean Pre-Video Score	Mean Post-Video Score (First Attempt)	Mean Post-Video Score (Second Attempt)	Mean Normalized Gain (Pre to First Post)	Mean Normalized Gain (Pre to Second Post)
Overall	41% (*SD*=30%)	46% (*SD*=17%)	82% (*SD*=15%)	4.9% (*SD*=37%)	63% (*SD*=34%)
Students with >50% on Pre		56% (*SD* =16%)	90% (*SD*=11%)	-22% (*SD* =32%)	47% (*SD* =49%)
Students with <50% on Pre		41% (*SD* =17%)	78% (*SD* =16%)	19% (*SD* =31%)	71% (*SD* =20%)



Video 1 Video 2 Video 3

Figure 5. Histograms of number of revisits by students per video.

Comparing Revisits and Learning

We performed a linear regression on the normalized gain scores (on pre-video problems to post-video problems) versus the number of revisits produced positive slopes; for the purpose of performing this regression, we eliminated one high-leverage data point (one student had a total of 46 revisits). When comparing the pre-video and first attempts at the post-video problems, the slope was not significantly different from zero (β =-.353, $t(26)$ = -0.353, p=0.727); this was also the case for the second attempt at the post-video problems (β = 0.00943, $t(26)$ = 0.649, p=0.522).

Survey Results

On the sense-making survey, 16 out of 23 students said that the video didn't need any additional clarification; these students only averaged getting 48% of the post-video problems correct on the first try, and only 85% after the second try. Of the six remaining students, three suggested clarifications that weren't directly related to graphing the derivative or the presentation of the video. The remaining three students indicated confusion about the triangle depicting amounts of change in distance and time together with a secant line; the relationship between the function increasing/decreasing and the derivative graph; and the relationship between the shape of the graph of the function and the shape of the derivative graph, respectively.

Discussion

The results for the students' performance on the pre- and post-video problems—in particular, their relatively poor performance on their initial attempts at the post-video problems—suggest that the students' learning was not particularly significant. Potential explanatory factors, such as the frequency of pausing or skipping backward (an indicator that the student experienced gaps in their understanding) were not associated with the students' normalized gain scores.

Most students never revisited the videos. In their responses to the sense-making surveys—which were written after the students had completed and received feedback on the post-video problems, the students indicated that they generally felt that the videos were clear. We found this surprising since the students tended to struggle with the post-video problems.

There are several potential explanations for these results. Perhaps the explanations in the video could be improved or there could be better alignment between the mathematical content of the video and the pre- and post-video problems. However, it also could be because the students didn't experience gaps or recognize their own lack of understanding thereby neglecting to revisit moments within the video that were critical for their own learning. Students' insistence that the videos were clear—even after they struggled with the post-video problems—could be attributed to either their inability to reflect on their own understanding and a propensity to attribute their struggle to a perceived inherent difficulty of mathematics.

Despite creating instructional materials guided by research-based recommendations, the students did not appear to construct an understanding of the underlying concepts sufficient for successfully solving to the post-video problems. This calls into question the effectiveness of instructional videos as stand-alone teaching tools. Moreover, that students did not experience gaps or recognize their lack of understanding exposes one of the most commonly-proposed benefits of a flipped class—the students' ability to re-watch videos. If students do not recognize their lack of understanding, they will not take advantage of this aspect of a flipped classroom. The scope of conclusions we can draw is limited by the relatively small sample size, the focus on a single calculus topic, and the use of a single set of videos. It is important to further investigate these conclusions by exploring how students watch, interact with, and learn from other instructional videos and other mathematical topics.

Acknowledgments

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References

Bao, L. (2006). Theoretical comparison of average normalized gain. *American Journal of Physics 74*, 917–922.

Braun, V. & Clarke, V. (2006) Using thematic analysis in psychology. *Qualitative Research in Psychology, 3* (2), 77-101.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education,* 352-378.

Dervin, B. (1983). An overview of sense-making research: Concepts, methods and results. Paper presented at the annual meeting of the International Communication Association. Dallas, TX.

Maxson K, Szaniszlo Z, (Eds) (2015). Special issue on the flipped classroom: Effectiveness as an instructional model. *PRIMUS 25*(9–10).

Schroeder, L. B., McGivney-Burelle, J., & Xue, F. (2015). To Flip or not to flip? An exploratory study comparing student performance in Calculus I. *PRIMUS, 25*(9-10), 876-885.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.

Weinberg, A., & Thomas, M. (2018). Student learning and sense-making from video lectures. *International Journal of Math Education in Science and Technology*. Advance online publication. Doi: 10.1080/0020739X.2018.1426794

Weinberg, A., Wiesner, E., & Fukawa-Connelly, T. (2014). Students' sense-making practices in a proof-based mathematics lecture. *Journal of Mathematical Behavior 33*,168–79.

THE EFFECT OF A COMPUTER-ASSISTED MODEL-BASED PROBLEM-SOLVING PROGRAM FOR STUDENTS WITH LEARNING DIFFICULTIES IN MATHEMATICS

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According to recent National Assessment of Educational Progress (2015), 60% of American 4th graders performed below proficiency level in mathematics. For students with disabilities, about 85% performed below the proficiency. In the meantime, there is a serious shortage of curriculum materials and teachers in providing high quality Response-to-Intervention (RtI) programs to students who are struggling in mathematics. The purpose of this single-case research design study was to evaluate the effect of a computer-assisted RtI program that emphasizes mathematical model-based problem solving on enhancing problem-solving performance of three elementary students with learning disabilities or difficulties. Findings from this study indicate promising effects of the program.

Keywords: mathematical model-based problem solving, computer-assisted instruction, students with learning difficulties, elementary mathematics problem solving.

Background and Conceptual Framework

While mathematics problem-solving skills are well recognized as critical for virtually all areas of daily life and for successful functioning in school, at home and in the community, many students with learning disabilities/difficulties in mathematics (LDM) fail to acquire key math concepts, such as additive and multiplicative reasoning, during the early school years. According to recent National Assessment of Educational Progress (NAEP, 2015), 60% of American 4th graders performed below proficiency level in mathematics. For students with disabilities, about 85% performed below the proficiency. In fact, about 5-10 % of school age children have been identified as having mathematics disabilities (Fuchs, Fuchs & Hollenbeck, 2007) and students whose math performance was ranked below the 20 to 35 percentile are often considered at risk for learning disabilities or for having learning difficulties in mathematics (LDM) (Bryant, et al., 2011). Given the increases in sophisticated and affordable technology for in and out of school, parents and teachers often turn to the many games, ‘apps,’ and web-based teaching/learning programs for help. However, there is a lack of intervention programs that focus on nurturing *conceptual understanding* of fundamental mathematical ideas--concepts that are essential to enabling students with LDM to understand and solve word problems.

Conceptual Model-based Problem Solving (COMPS) Tutor

Supported by National Science Foundation (NSF), we have developed a prototype of a computer-assisted COnceptual Model-based Problem Solving (COMPS) tutor, as a Response-to-

Intervention (RtI) program, to address the skill deficit of second- and third-grade students with LDM in meeting the challenging mathematics curriculum standards, particularly in the area of additive word problem solving. The COMPS-RtI program reflects a shift from traditional problem-solving instruction, which focuses on the *choice of operation* for solution, to a mathematical model-based problem-solving approach that emphasizes an understanding and representation of mathematical relations in algebraic equations. The explicit strategy teaching also makes the reasoning behind mathematics explicit to the students, so that they are able to make sense of what they are doing with mathematical models and abstract symbols.

The purpose of this study was to explore the effect of computer-assisted COMPS-RtI program that emphasizes mathematical model-based problem solving involving elementary students with LDM. Specific research questions were: (1) what was the functional relationship between the intervention delivered by the COMPS-RtI tutor and students' performance on solving addition and subtraction word problems measured by a researcher-developed criterion test?; and (2) were students able to transfer their learned strategy to solve problems with different story situations?

Participants and Setting

This study was conducted within the larger context of a National Science Foundation (NSF) funded project (Xin, Kastberg, and Chen, 2015). Participants were three 3rd graders with LDM, who enrolled in an urban elementary school in the United States. Three students (two girls and one boy) were included in the general education classrooms for 80-100% of the school day and they were all receiving Tier III Response to Intervention (RtI) support. All the instruction and testing were conducted in the school's computer lab during the afterschool program Monday through Thursday. Specific participant selection was based on the following criteria: (a) students identified as having difficulties in mathematics by the school and (b) students scored below the 25th percentile on the *Mathematics Problem Solving* subset of the Stanford Achievement Test (SAT-10; Harcourt Assessment, 2004).

Dependent Measures

The primary dependent measures is a researcher-developed *criterion test* (adapted from Xin, Wiles, and Lin, 2008) involving eight part-part-whole additive word problem solving items (including situations such as *combine*, *change-join*, and *change-separate* with unknown quantity placed at all possible positons), similar to problems included in the training module. The generalization test (adapted from Xin et al., 2008) not only includes part-part-whole problem structures but also additive comparison problem types (involving the "…more than..." or "…less than…" situations).

Experiment Design

An adapted multiple-probe-design (Horner & Baer, 1978) across participants was employed to evaluate potential functional relationship between the intervention and participants' word problem-solving performance. We chose a single-subject research design because the design provides a methodological approach well suited to conduct single case or group studies. In a multiple-probe-design, the intervention effects can be demonstrated by introducing the intervention to different baselines or participants at different points in time. "If each baseline changed when the intervention is introduced, the effects can be attributed to the intervention rather than to extraneous events" (Kazdin, 1982, p.126). The design included a *Baseline* (students were assessed on the criterion test and generalization [G] test), COMPS-RtI

Intervention (repeatedly measured on criterion test), followed by posttests and maintenance tests (criterion test and G test). Multiple equivalent test forms were used for the assessment during baseline, intervention, and post-intervention phases.

Procedure

All three participants completed one criterion test during the baseline condition. Then one student (Amelia) took another two equivalent criterion tests. Following the baseline, the intervention was first introduced to Amelia. Once the data for Amelia showed an accelerating trend, the intervention was introduced to the second student Sam immediately after he took two additional baseline tests. The same sequence was followed until all three participants were introduced to the intervention. Posttests and maintenance tests follow the intervention. Participating students worked with the COMPS-RtI tutor one–on-one on a laptop computer four times a week, with each session lasting about 20-30 minutes. Graduate research assistants supervised all the sessions. Their roles included administering pre-post assessment, recording computer/program's "bugs" and guiding students to appropriate part of the program after any unexpected "interrupt." Participants received about a total of 11-12 sessions during the spring semester.

Intervention Components

The COMPS-RtI program addresses Common Core State Standards (CCSS): Content.2.OA.A.1 *Represent and solve problems involving addition and subtraction*: "Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem" (CCSS Initiative, 2012). As part of the NSF-funded project, the COMPS-RtI tutor used in this specific study focused on solving part-part-whole problem types (e.g., *combine*, *change-join*, *change-separate*) but not comparison problems. Before solving real world problems, Part-I of the program engaged students in a series of activities involving the use of virtual manipulatives such as unifix cubes and coverings guided by a constructivist view of mathematics learning. The goal of this part of the program was to nurture fundamental mathematical ideas that are crucial for the development of additive reasoning and problem solving. It focuses on students' conception of "number as the composite unit" and the development of multi-digit numbers as quantities of tens and ones.

Building on these fundamental ideas, Part-II of the program focused on representing various additive word problem situations in a cohesive mathematical model equation. Part-III of the problem engage students in representing and solving contextualized real world problems using the mathematical model equation. The goal of Part-II and Part-III of the program was to engage students in constructing the big idea of "*Part* and *Part* makes up the *Whole,"* which leads to conceptual understanding of the mathematical model equation "P + P = W." Early in the program, Singapore bar models (Ramakrishnan & Soon, 2009) were used to facilitate students' transition from isolated unit of ones to the composite unit (CU); the bar model also facilitates students' conceptual transition to symbolic level of operation as they use the mathematical model equation to represent and solve various part-part-whole (PPW) word problems. The program engages students in decontextualizing mathematical relations from various real-world story problems, and then representing them in a single mathematical model, P + P = W, which *drives* the solution plan including the decision on the choice of operation.

Results and Discussion

All three participants' baseline performances were low and stable indicating the need for intervention. During the intervention, participants gradually improved their performance only after they worked with each part of the COMPS-RtI program. Although there are some variations across three participants in their performance after each part of the program, overall changes in performance during and after the intervention indicate students' improved performance following the COMPS-RtI intervention. Across all three participants' data path, there is no overlapping between baseline performance and the performance during the intervention as well as post-intervention assessment. This is an evidence of a strong treatment effect (Percentage of Non-overlapping Data [PND] = 100%, Scruggs & Mastropieri, 1998). In addition, all participants maintained their improved performance at an average of 90%, 87%, and 90% correct respectively during posttest and follow-up test phase. In addition, all three participants improved their scores on the generalization (G) test. However their performances were around 50 or 60% correct. It should be noted that the G test included additive comparison problem types that were not covered in the current version of the COMPS-RtI program. Systematic programing and training for skill transfer may be essential for students with LDM in particular.

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References

Xin, Y. P., Wiles, B. and Lin, Y. (2008). Teaching conceptual model-based word-problem story grammar to enhance mathematics problem solving. *The Journal of Special Education, 42*, 163–178.

Xin, Y. P. (PI), Kastberg, S., & Chen, V. (2015-2018). *Conceptual Model-based Problem Solving (COMPS): A Response to Intervention Program for Students with LDM*. National Science Foundation (NSF) funded project.

Bryant, D. P., Bryant, B. R., Roberts, G., Vaughn, S., Pfannenstiel, K. H., Porterfield, J., & Gersten, R. (2011). Early numeracy intervention program for first-grade students with mathematics difficulties. *Council for Exceptional Children, 78*(1), 7-23.

Common Core State Standards Initiative (2012). *Common core state standards for mathematics practice*. Retrieved on September 19, 2012, from http://www.corestandards.org

Fuchs, D. and Deshler, D.D. (2007). What we need to know about responsiveness to intervention (and shouldn't be afraid to ask). *Learning Disabilities Research and Practice*, *22*, 129-136.

Horner, R. D., & Baer, D. M. (1978). Multiple-probe technique: A variation of the multiple baseline. *Journal of Applied Behavior Analysis, 11*, 189-196.

Kazdin, A. E. (1982). *Single-case research designs: Methods for clinical and applied settings*. New York: Oxford.

National Assessment of Educational Progress result (NEAP, 2015). Extracted from http://www.nationsreportcard.gov/reading_math_2015/#mathematics/acl?grade=4

Pearson Inc. (2004). Stanford Achievement Test, Tenth Edition (SAT-10). San Antonio, TX: Author.

Ramakrishnan, C. and Soon, G. K. (2009). *Math in Focus: The Singapore Approach.* Singapore: Marshall Cavendish Education.

Scruggs, T. E., & Mastropieri, M. A. (1998). Summarizing single-subject research: Issues and applications. *Behavior Modification, 22*, 221-242.

INVESTIGATING SPATIAL AND TEMPORAL REASONING OF ELEMENTARY STUDENTS THROUGH GAMIFIED MATHEMATICS SOFTWARE

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The purpose of this study is to explore how elementary school students engage with the mathematical concepts embedded within "Spatial-Temporal" (ST) representations. I found a wide spectrum of conceptual understanding of ST Math, from strong connections with the mathematical concepts to little connections. When students had more access to the mathematical concepts and the explicit structure of the task, they tended to articulate the mathematical meaning of ST representations. However, some students appeared to make little sense of the apparent mathematical concepts. These findings suggest that ST Math tasks could improve engagement with the operational mathematical concepts, but a sense-making approach would additionally support a better relationship between ST representations and mathematical concepts.

Keywords: Technology, Elementary School Education, Number Concepts and Operations

Introduction

The integration of technology in mathematics education has increased; however, mathematical thinking is not driven by technology alone (Heid, 2005). Investigating the way students interpret technology through a mathematical lens and connect mathematical concepts is vital to use technology effectively. Regarding sense-making and reasoning, technologies can be categorized as either *conveyance* or *mathematical action tools*, based on their purposes within mathematics instruction (Dick & Hollebrands, 2011). *Conveyance technologies* are used to convey information such as presentation, communication, sharing/collaboration, and assessment/monitoring/distribution not oriented to mathematics content. *Mathematical action technologies* enable students to interact and receive feedback during the performance of mathematical tasks such as computational /representational tool kits, dynamic geometry environments, micro-worlds, and computer simulations.

Many researchers consider mathematical action tools as cognitive technologies which Pea (1985) defined as those which help users "transcend the limitations of the mind...in thinking, learning, and problem-solving activities (p. 168)." Also, these cognitive technologies are articulated either as *amplifiers* or as *reorganizers*. *Amplifiers* enhance the processes such as students' ability to solve problems in an efficient way, but do not change their thinking. *Reorganizers* engage cognitive processes by modifying the process of cognition (Barrera-Mora & Ryes-Rodriguez, 2013). This type of technology allows establishing the dialectic relationship among our actions, the forms of thinking, and the functions of technology which influence the acquisition of mathematical knowledge. According to What Works Clearinghouse (U.S. Department of Education, 2013), only one out of 40 mathematical programs pertained to high-quality elementary mathematics technology research. Much technology in the elementary school level is focused on drill-and-practice as *amplifiers* not conceptual understanding.

Recently, Spatial-Temporal (ST) Math, a game-based instructional software, is used in mathematics classrooms from kindergarten to middle school. In the same spirit of *reorganizers*, helping student's cognition enhance, the main feature of ST Math is to utilize interactive visual representations to build conceptual understanding with little language and symbols. ST Math is designed to aid students with learning spatial and temporal reasoning through visualized representations in which mathematical concepts are infused with dynamic action introducing mathematical concepts and procedures. Spatial and temporal reasoning means making a mental image and thinking ahead in space and time (Peterson et al., 2004). The game-style design and exercises are intended to get students engaged and motivated when solving each task (Rutherford et al., 2014). If students successfully complete a lower level in the task, then more challenging and difficult levels are proposed gradually.

This study aims to explore the spatial and temporal reasoning of elementary school students when they are engaged with ST Math software. In particular, this study seeks to examine the following questions: (a) What connections do students make between the spatial-temporal representations and the mathematical concepts they are meant to represent? (b) How do different activities appear to provide different access to the mathematical concepts?

Methodology

In a mathematics education methods course, elementary pre-service teachers (PSTs) participated in a series of one-to-one task-based interactions with elementary students in a Midwest elementary school. The goal of the interactions were for the PSTs to help support elementary students' mathematics learning and to make sense of children's mathematical thinking.

Twenty-one PSTs had a training session about how to use ST Math prior to an actual interview. PSTs selected specific ST Math tasks such as Pie Monster, How many petals?, and Building Expressions (Figure 1) based on the belief that which task was well-suited to their assigned student's mathematical proficiency level; the PSTs also prepared a script to use for questioning. PSTs interviewed the assigned child for 10-30 minutes and reported reflections on the interview as a course assignment. The audio and screen display were recorded via iPad's screen-casting function. This screen-casting enables the real-time recording of students' explanation about the task including their touching and dragging on the screen.

To analyze the elementary students' ST Math activity, I evaluated screen-casted videos and made a condensed file to extract information (e.g., task, level, attempted problems, and the rate of correct answers). Each video was transcribed and analyzed one at a time. A research memo was created and compared to find the common themes across the participants (Yin, 2013).

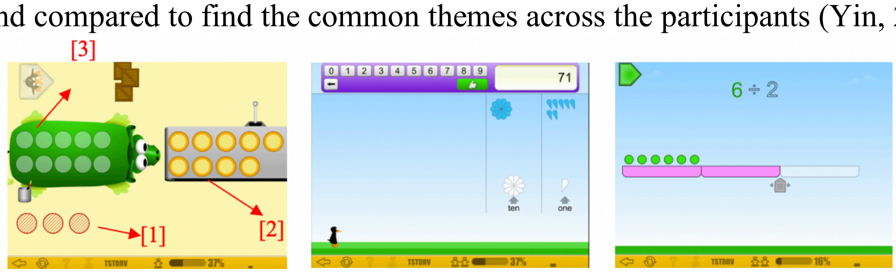

Figure 1. Pie monster (left), How many petals? (middle), and Building Expression (right)

Findings

Ways in which the students engaged with the spatial-temporal representations were analyzed by focusing on the mathematical interpretations students articulated when solving ST Math tasks. The following section presents the cases of one child (Amy) across the samples, selected on the basis of how children connected mathematical concepts with the ST representations in the given tasks.

Strong Connection with Mathematical Concepts

Clear relation to the ST representations. When some students solved the ST Math tasks, they exhibited strong connection with the mathematical concepts embedded within the ST representations. For example, the first level in Pie Monster task represents part-part-whole relationship (Carpenter et al., 2015). In the Pie Monster task, the problem type is separate and there are three types of direct modeled representations. The screen (see Figure 1- left side) represents three red-circles [1] as the *change* (subtrahend), nine orange-circles [2] as the *start* (minuend), and the white circles [3] in the Monster's belly as the result (difference).

Amy showed strong mathematical understanding in this task. To find a solution, Amy was supposed to figure out how many pies would be put in the Pie Monster's body once the amount of orange circles was reduced. She perceived that the discrete pie models represented as the *change* (red) to subtract from the *start* (orange) when she said, "So, you see the number at the bottom? Those are how much you're going to subtract." In addition, she had clear understanding of the structure of the task as subtraction. The function of the red balloons is to remove the number of orange pies as the number of red pies the student chosen. After she selected the pie models on Monster's belly, she noticed the animated effect of balloons. At that time, she explicitly described that the orange pies took away from the number of red pies: "If you do three, it's going to take away three of them… count the rest of them." This shows that she distinguished the between the change, the start and the result in the Pie Monster context. From the ST representations, she extracted her strategy to count the left over after subtracting.

Little Connection with Mathematical Concepts

Weak connection with mathematical concepts. The students' ability to explain the relationship between ST representation and mathematical concepts was not consistent across different ST Math tasks. For example, the *Building Expression* task begins with choosing the size of the group, selecting the number of groups, then deciding the result from division and multiplication respectively.

Though Amy explained well her problem-solving strategy in the Pie monster task, she was struggling with the Building Expression task. At first, Amy chose six green dots, matching the first number 6 on the screen (see Figure 1– right side). Then, she selected two groups of pink boxes. Here, she was possibly interpreting 6 ÷ 2 as two groups of six, or she might have been trying simply to represent the number 2 by manipulating the bar, without considering the meaning of the operation. In either case, Amy did not appear to be connecting the division symbol to the concept of division as she was engaged with the task. The result she selected was two groups of six, a total of twelve dots. When Amy was asked to explain her reasoning about the Building Expression task, she was relatively reluctant to express her idea from the beginning of the interview—unlike with the Pie Monster task. She was not engaging with solving the task or responding to probes from the interviewer. In order to explore Amy's understanding, the interviewer asked about the meaning of the pink box since Amy was struggling with making a decision between two big boxes and small parts boxes. She used the negative words five times such as "I don't know!" and "It doesn't matter." Even though she strived to solve the task with dragging and tapping at first, she was struggling with connection between the ST representations and mathematical concepts.

Discussion

With regard to the mathematical concepts, the students show the wide spectrum of understanding from the ST representations. Across the examples, most students showed the ability to quantify the pictorial representations with numbers and vice versa. For example, in the

Pie monster task, Amy could interpret the discrete representation as the total number. She perceived the discrete red-colored pies as the number 3 to take away from nine orange pies. Also, Amy could visualize the number with the concrete representations in the Building Expression task. She matched six green dots with the first number 6 and two pink boxes with the following number 2. This fundamental skill to solve ST Math tasks is to match the numerical value with the ST representations (Sarama & Clements, 2009). Potentially, the visual models in technology can help the children see such connections (Moyer, Niezgoda, & Stanley, 2005).

ST Math activities have potential to support engagement with the mathematical concepts operationally. For example, through the exploration between red, orange and white circles in the Pie Monster task, Amy was able to access the operational concepts of the relationship between minuend, subtrahend and difference. The impact on students' conception should be considered at the heart of the program (Webel, Krupa, & McManus, 2015). Meanwhile, to become reorganizer, the comprehensive approach is required such as sense-making of the ST representations, verbalizing what the students think, and concentrating on the animated feedback (Barrera-Mora & Ryes-Rodriguez, 2013).

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References

Barrera-Mora F., & Reyes-Rodriguez, A. (2013). Cognitive processes developed by students when solving mathematical problems within technological environments. *Mathematics Enthusiast. 10.* 109-136.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (2015). *Children's mathematics: Cognitively guided instruction.* Portsmouth, TX: Heinemann.

Dick, T. P., & Hollebrands, K. F. (2011). *Focus in high school mathematics: Technology to support reasoning and sense making*. Reston, VA: The NCTM.

Heid, M. K. (2005). Technology in Mathematics Education: Tapping into Visions of the Future. In W. J. Masalski and P. C. Elliot (Eds), *Technology-Supported Mathematics Learning Environment*. Reston, VA: The National Council of Teachers of Mathematics, Inc.

Moyer, P. S., Niezgoda, D., & Stanley, J. (2005). Young children's use of virtual manipulatives and other forms of mathematical representations. In W. Masalski & P. C. Elliott (Eds.), *Technology-supported mathematics learning environments: 67th yearbook* (pp. 17–34). Reston, VA: National Council of Teachers of Mathematics

Pea, R. D. (1985). Cognitive technologies for mathematics education. In A. H. Schoenfeld (Ed*.), Cognitive science and mathematics education* (pp. 89-122). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Peterson, M. R., Balzarini, D., Bodner, M., Jones, E. G., Phillips, T., Richardson, D., & Shaw, G. L. (2004). Innate spatial-temporal reasoning and the identification of genius. *Neurological Research, 26*, 2–8.

Rutherford, T., Farkas, G., Duncan, G., Burchinal, M., Kibrick, M., Graham, J., . . . Martinez, M. E. (2014). A randomized trial of an elementary school mathematics software intervention: Spatialtemporal math. *Journal of Research on Educational Effectiveness, 7*(4), 358–383.

Sarama, J., & Clements, D. H. (2009). "Concrete" computer manipulatives in mathematics education. *Child Development Perspectives, 3*, 145–150.

U.S. Department of Education, Institute of Education Sciences, What Works Clearinghouse. (2013). *Elementary school mathematics intervention report: DreamBox Learning*. Retrieved from http:// ies.ed.gov/ncee/wwc/pdf/intervention_reports/wwc_dreambox_ 121013.pdf

Webel, C., Krupa, E., McManus, J. (2015). Benny goes to college: Is the "Math Emporium" reinventing Individually Prescribed Instruction? *The MathAMATYC Educator, 6*(3), 4-13.

Yin, R. K. (2013). *Case study research: Design and methods.* Los Angeles, CA: SAGE Publications.

SHEARING TRIANGLES AND PYRAMIDS IN AN IMMERSIVE VIRTUAL SPACE: A STUDY OF PRE-SERVICE TEACHERS' CONCEPTIONS

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Pre-service elementary teachers (PSETs) have difficulties extending measurement concepts to from two-dimensional to three-dimensional figures (Tossavainen et al., 2017). This matters because how teachers understand mathematical topics shapes how they present those topics to their students (Murphy, 2012). If the geometry activities expected of young children are to move beyond shape recognition and naming tasks (Bruce & Hawes, 2015), PSETs need to be prepared for deeper investigations of the measurement properties of plane and solid figures. The affordances of immersive spatial displays –where participants can engage directly with virtual, three-dimensional figures – may provide new opportunities for PSETs to explore measurement concepts (Dimmel & Bock, 2018). We ask: How do the affordances of immersive spatial displays allow pre-service elementary teachers to reason about relationships between plane and solid figures?

Immersive spatial displays (i.e. HTC Vive) allow learners to interact with solid mathematical figures in their native dimension, visually explore parameters that grow without bound, and directly interact with figures. A pilot study found that pre-service teachers might draw connections between plane and solid figures when exploring shearing in immersive spatial displays (Bock & Dimmel, 2017).

In an ongoing study, we are conducting task-based, semi-structured interviews, during which participants work in small groups to explore the shearing of plane and solid virtual manipulatives. In the case of the triangle, participants can shear the triangle, watching its perimeter grow without bound, by standing in its plane. First person and mixed reality views of the environments will be recorded with the audio from participant dialog. These interviews will be analyzed with Balacheff's Conception-Knowing-Concept model of the conception (2013).

References

Balacheff, N. (2013). cK¢, a model to reason on learners' conceptions. In A. C. S. Mara V. Martinez (Ed.), PME-NA 2013 - Psychology of Mathematics Education, North American Chapter (pp. 2–15). Chicago, IL, United States.

Bock, C., & Dimmel, J. (2017). Explorations of Volume in a Gesture-Based Virtual Mathematics Labratory. Presented at the Psychology of Mathematics Education North America, Indianapolis, IN: PMENA.

Bruce, C. D., & Hawes, Z. (2015). The role of 2D and 3D mental rotation in mathematics for young children: what is it? Why does it matter? And what can we do about it?. *ZDM*, *47*(3), 331-343.

Clements, D. H., & Sarama, J. (2011). Early childhood teacher education: the case of geometry. *Journal of mathematics teacher education*, *14*(2), 133-148.

Dimmel, J.K., & Bock, C.G. (2018). Dynamic mathematical figures with immersive spatial displays: The case of *HandWaver.* To appear in G. Aldon and J. Trgalova (eds.), *Selected papers of the 13th international conference on technology in mathematics teaching.* Berlin: Springer

Murphy, C. (2012). The role of subject knowledge in primary prospective teachers' approaches to teaching the topic of area. *Journal of Mathematics Teacher Education*, *15*(3), 187-206.

Tossavainen, T., Suomalainen, H., & Mäkäläinen, T. (2017). Student teachers' concept definitions of area and their understanding about two-dimensionality of area. *International Journal of Mathematical Education in Science and Technology*, 48(4), 520–532.

PRE-SERVICE TEACHERS' USE OF TPACK THROUGH THE CREATION OF VIDEO LESSONS USING IPADS

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Curriculum policies set expectations for technology to be incorporated in teaching of mathematics so that teachers can better support their students in comprehending mathematics using the technology. This scenario reinforces the importance in higher education to prepare future teachers to employ technology in their teaching to foster mathematical understanding of their future students. Pre-service teachers' creation of video lessons using iPads emerges as a new use of technology in which they can actively explore mathematical content and develop communication skills that will be required in their future practice (Niess & Walker, 2010). In this paper, we investigate how pre-service secondary mathematics teachers incorporate their knowledge of technology, mathematics, and pedagogy in a project of creating video lessons using an app called ExplainEverything (EE) on an iPad. We characterize their experiences using the TPACK framework (Mishra & Koehler, 2006).

The context of this study was a methods course focused on teaching mathematics with technology offered at a large mid-Atlantic university. Seven pre-service teachers participated in this study, and this paper presents results on three participants. Data sources include participants' lesson plans, video lesson files, their reflection upon their own videos, and online surveys. Using qualitative research methods, students' video lessons were analyzed by the researchers in tandem with their respective lesson plans, reflections, and surveys and coded by TPACK components.

Results and Implications

The project of creating video lessons provided an opportunity for pre-service teachers to use their mathematical, pedagogical, and technological knowledge in an integrated way towards developing their TPACK. In their video lessons, participants showed effective use of technology (e.g., EE, GeoGebra, CAS), pedagogical techniques (e.g., pausing, questioning, and building on students' prior knowledge), and mathematical representations using technology (e.g., graphical representations). Results suggested that pedagogical aspects of participants' TPACK were the most salient feature of their videos when compared to the other components. Implications from this study support that using video lessons in teacher preparation provides opportunities for instructors to give specific feedback and guidance for preservice teachers to learn and reflect upon their practice as suggested by the literature (e.g., Sherin & van Es, 2009). It is recommended that teacher educators implement a cycle of learning in which preservice teachers can refine their pedagogical strategies, the rigor in their mathematics, and their use of mathematical action technology, if a similar project is to be used in teacher education programs.

References

Mishra, P., & Koehler, M. J. (2006). Technological pedagogical content knowledge: A framework for teacher knowledge. *Teachers College Record, 108*(6), 1017-1054.

Niess, M. L. & Walker, J. M. (2010). Guest editorial: Digital videos as tools for learning mathematics. *Contemporary Issues in Technology and Teacher Education, 10*(1), 100-105.

Sherin, M. G., & van Es, E. A. (2009). Effects of video club participation on teachers' professional vision. *Journal of Teacher Education, 60*, 20-37.

WIDE WALLS, COMPUTATIONAL THINKING, AND MATHEMATICS

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Gadanidis et al. (2016) expressed concerns that CT in the K-12 curriculum is often viewed as its own objective, rather than used to promote other subjects. According to Resnick et al. (2009) one of the benefits of CT is that it promotes "wide walls", meaning it allows for a variety of projects and learners to engage and benefit from CT skills. Gadanidis et al., propose that mathematical thinking is an appropriate context for these wide walls. In contrast, Wong et al. (2015) say learning must trigger a students' interest so that the interest can be extended to new areas. In our research we explored the concept of wide-walls using two different approaches, one which used CT skills to trigger interest and later used math as the framework for the walls of CT The first activities used both unplugged activities and SCRATCH and were highly engaging for the students. The class was able to understand and communicate the importance of learning the basics of CT. Through the activities the researchers noticed that, when students asked for help, they were willing to answer questions about the reasons certain pieces of code influenced the outcome and were willing to explain to the instructor and themselves how to go about finding a solution. This optimism persisted throughout the classes.

The next set of lessons required students to build on their understanding of how to calculate area and perimeter by making their own code There was a significant lack of interest from the students that became evident in the results. Only one third of students ended up completing the activity. One new component was introduced in this process but all students worked through this new process on their own. Phrases such as "I can't do it" began to be prevalent in the classroom. Our results suggest that caution should be exercised in defining wide walls of CT by preset curriculum. Seeing as wide walls are intended to allow all students to engage, when CT is incorporated into a subject where students are not fully engaged or the connection is forced, the future benefit of CT to the curriculum may be lost, as was observed in our research. This is not to say that triggering students' interest in the learning loop is not possible with mathematics as the starting point; of course, mathematical problems can be creatively contrived to do just that.

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References

Gadanidis, G., Hughes, J. M., Minniti, L., & White, B. J. G. (2016). Computational thinking, grade 1 students and the Binomial Theorem [Electronic Version]. *Digital Experiences in Mathematics Education*,

Resnick, M., Maloney, J., Monroy-Hernandez, A., Rusk, N., Eastmond, E., Brennan, K., et al. (2009). Scratch: Programming for all. *Communications of the ACM., 52*(11), 60-67.

Wong, L. H., Chan, T. W., Chen, Z. H., King, R. B., & Wong, S. L. (2015). The IDC theory: Interest and the interest loop. *Workshop proceedings of the 23rd international conference on computers in education*, 804–813.

TEACHING MATHEMATICS WITH IMMERSIVE SPATIAL DISPLAYS: A FOCUS GROUP STUDY

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Room-scale virtual reality (e.g., the HTC Vive) is an example of an immersive spatial display—a technology where *space itself,* rather than a two-dimensional surface, is a canvas for making inscriptions (Dimmel & Bock, 2018). Spatial displays make it possible to readily interact with figures at different scales and explore them from new perspectives. Despite their educational promise, spatial displays are likely to follow the slow, complex process of technology acceptance and adoption that is standard in schools (Ertmer, 1999; Inan & Lowther, 2010). The objective of our research is to look ahead to a time when spatial displays are ubiquitous and to begin, now, to prepare teachers to adapt their instructional practices to these new technologies. If such adaptations are to be accepted by teachers, teachers themselves will have to play a generative role in their design (Cobb, Zhao, & Dean, 2009). We thus cast teachers as partners in the process of designing instructional activities that are possible with spatial displays. We ask: How do secondary mathematics teachers anticipate using spatial displays as instructional tools in their classrooms?

To investigate this question, we developed an immersive, gesture-based virtual environment that places users inside a 10x10x10 grid of points, each of which has integer coordinates. The resulting lattice is a three-dimensional analog of the Geoboard (Gattegno, 1954). The environment defines a spatial canvas where learners can use gestures to construct and investigate lattice polygons or polyhedrons. We convened focus groups of secondary mathematics teachers, during which participants explored the environment, critiqued its capabilities, and designed instructional activities students could engage in through the environment. Two groups of four secondary mathematics teachers participated in the focus group study. We report how participants' explored the environment, their discussion of its potential for use in mathematics classrooms, and the instructional activities they designed.

References

Cobb, P., Zhao, Q., & Dean, C. (2009). Conducting design experiments to support teachers' learning: A reflection from the field. *The Journal of the Learning Sciences*, *18*(2), 165 - 199.

Dimmel, J.K., & Bock, C.G. (2018). Dynamic mathematical figures with immersive spatial displays: The case of *HandWaver.* To appear in G. Aldon and J. Trgalova (eds.), *Selected papers of the 13th international conference on technology in mathematics teaching.* Berlin: Springer

Ertmer, P. A. (1999). Addressing first-and second-order barriers to change: Strategies for technology integration. *Educational Technology Research and Development*, *47*(4), 47-61.

Gattegno, Caleb. (1954). The Gattegno Geoboards, in *Bulletin of the Association for Teaching Aids in Mathematics*, No. 3.

Inan, F. A., & Lowther, D. L. (2010). Factors affecting technology integration in K-12 classrooms: A path model. *Educational Technology Research and Development*, *58*(2), 137-154.

LEARNING TO NOTICE AND NAME STRENGTHS WITH LESSONSKETCH

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Teachers who adopt a strengths-based approach identify and rely upon the assumed valuable resources that students bring to their classroom communities (Aguirre et al., 2013; AMTE, 2017). While the rhetoric of a strengths approach is relatively easy to endorse, the intentional practice of interpreting and leveraging student actions as mathematical strengths is far more difficult to enact (Jilk, 2016), especially for prospective teachers (PTs) given the broader culture of power and exclusion that has persisted in U.S. classrooms for over 40 years (Louie, 2017). An enduring challenge for mathematics teacher educators is to develop course-based approximations of practice (Grossman et. al, 2009) that provide PTs with meaningful entry points into this work. Recent advances in technology-based methods of teacher preparation suggest that the Lesson*Sketch* (LS) platform is a promising medium for helping PTs to learn this complex practice (Amador et. al, 2016; Bannister et. al, 2018). Our research questions were: What do PTs notice and name about students' strengths in a LS experience before and after they are given explicit support for using strengths-based language? Are there any changes in the quality and quantity of the claims and evidence PTs used in their noticing statements?

We used a sentence frame intervention in the LS experience to explicitly help PTs learn how to notice and name students' mathematical strengths (Bannister et. al, 2018). We analyzed the quality and quantity of the claims 22 undergraduate PTs in an elementary mathematics methods course made about student strengths and the mathematical evidence they used to justify these claims. We found statistically significant increases in the quality of their claims and the quantity of evidence they used to support their claims, which provides empirical backing for a course experience that MTEs can use to approximate and support PTs to learn strengths-based practice.

References

Bannister, N., Kalinec-Craig, C., Bowen, D., Crespo, S. (2018). Learning to notice and name students' mathematical strengths: A digital experience. *Journal of Teacher Education, 26*(1), 13-31.

Aguirre, J. M., Turner, E. E., Bartell, T. G., Kalinec-Craig, C., Foote, M. Q., Roth McDuffie, A., & Drake, C. (2013). Making connections in practice: How prospective elementary teachers connect to children's mathematical thinking and community funds of knowledge in mathematics instruction. *Journal of Teacher Education, 64*(2), 178-192.

Amador, J., Weston, T., Estapa, A., Kosko, K., & De Araujo, Z. (2016). Animations as a transformational approximation of practice for preservice teachers to communicate professional noticing. *Journal of Technology and Teacher Education*, *24*(2), 127-151.

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. (2009). Teaching practice: A cross-professional perspective. *Teachers College Record*, *111*(9), 2055-2100.

Jilk, L. M. (2016). Supporting teacher noticing of students' mathematical strengths. *Mathematics Teacher Educator*, *4*(2), 188-199.

Louie, N. L. (2017). The culture of exclusion and its persistence in equity-oriented teaching. *Journal for Research in Mathematics Education, 28*(5), 488-519.

DEVELOPMENT OF THE ADDITIVE PROBLEM-SOLVING STRATEGIES IN A COMPUTER-BASED LEARNING ENVIRONMENT: A CASE STUDY OF A STUDENT WITH MATHEMATICS DIFFICULTY

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To address the needs of children with learning difficulties in mathematics (LDM), a mathematics computer tutoring program was developed to nurture conceptual understanding of fundamental mathematical ideas in students with LDM (Xin, Kastberg & Chen, 2015). As an intervention for this study, we used the Numerical Structuring Module (NSM) of the mathematics computer tutoring program, which was designed for developing the composite unit in additive problem-solving by provoking changes in children's mental operations.

Steffe and Cobb (1988) suggested a model that explains how children come to understand quantity by proposing the stages of number sequence development. In each stage, children show unique counting acts (e.g., Double-counting). The composite unit, defined as a unit composed of unit items, is developed by the result of mental operations developed through challenges to counting acts (Olive, 2001).

The purpose of our study is to understand the learning paths of one student with LDM, Sofia, as she engages in NSM. Two research questions are posed in our study: 1) What strategies does Sofia use and how does she develop her strategies when she engages in NSM?

The sources of data for our analysis included pre- and post-tests, video recorded tutoring sessions, field notes and the performance data on NSM. We used constant comparative analysis to analyze the video recorded sessions and field notes.

Through her interaction with a computer tutoring program, our participant exhibited the use of the composite unit consistent with Steffe's idea although she relied on using perceptual material (i.e., her fingers) when she kept track of her counting acts. Our finding indicates that our participant developed the effective double counting strategy using a perceptual material for compensating her working memory challenges.

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References

Xin Y. P., Kastberg, S., & Chen, Y. (2015). *Conceptual model-based problem solving (COMPS): A response to intervention program for students with LDM*. National Science Foundation funded project.

Steffe, L. P. & Cobb, P. (1988). *Construction of arithmetical meanings and strategies.* New York: Springer-Verlag.

Olive, J. (2001) Children's number sequences: an explanation of Steffe's constructs and an extrapolation to rational numbers of arithmetic. *The Mathematics Educator* 11(1), 4–9.

SEEKING MATHEMATICS HELP IN PHYSICAL AND VIRTUAL SPACES

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This study investigates students' help-seeking practices and their perceptions toward peer interactions as they solve mathematics tasks in a hybrid physical/virtual space. This pilot study paves the way for creating an equitable learning technology platform to facilitate classroom interactions and discussions in which ALL students participate with confidence.

In 2013, the report of Programme for International Student Assessment (PISA) suggested that students' poor mathematics performance relate to their mathematics anxiety. To deal with this anxiety, struggling students can seek help not only from their teachers but also their peers. Research has shown peers can provide rich and meaningful feedback for learning (Gielen et al., 2010; Colley & Windschitl, 2016). However, help-seeking practices can be intimidating to students. Therefore, we aim to understand what would occur as students interact with peers to get mathematics help in different contexts and how they perceive the practices.

Using design-based methodology and a clinical-interview protocol, we revised our tasks and questions each round based on participants' responses and interactions. This study involved six (two boys, four girls) children ranging from 3rd-7th grade over three sessions. The activities were mainly facilitated by the first author, a 10-year veteran elementary mathematics and science teacher. Each session including interviews lasted approximately 45 minutes. In the first round, two subjects were seated together. The math questions were presented to each participant on a sheet of paper. In the second round, three subjects were seated separately. Each child was provided with a laptop or tablet with math questions posted on a Google Doc. In the third round, three subjects were seated separately and asked to work on one same math problem in a simulation of collaborative virtual environment, i.e., Google Docs.

This pilot study reveals various factors that influence an individual's help-seeking practices. The virtual space, Google Doc, was tested but failed to serve as an effective means because students were impeded by the linguistic demands of conventional semiotic registers of standard virtual arenas. Moreover, some students were cautious about who they interacted with in virtual space, others were open to interact with someone they did not know. These findings inform us of how a safe and interactive online learning environment could be designed to help discover silent students' voice, facilitate quality discussions, and create a more equitable math classroom.

References

Colley, C., & Windschitl, M. (2016). Rigor in Elementary Science Students' Discourse: The Role of Responsiveness and Supportive Conditions for Talk: Role of Supportive Conditions in Student Discourse. *Science Education*, *100*(6), 1009–1038.

Gielen, S., Peeters, E., Dochy, F., Onghena, P., & Struyven, K. (2010). Improving the effectiveness of peer feedback for learning. *Learning and Instruction*, *20*(4), 304–315.

TOUCHSCREEN TECHNOLOGY WITH A DYNAMIC MATH NOTATION TOOL

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As technology becomes more prevalent in the classroom, it is important that we fully explore how students interact with learning technology such as laptops and tablets through native or peripheral inputs. Prior research comparing the two found that user's proficiency with each device depended on their prior experience using computers with experts were more efficient with a mouse while beginners were more efficient with a touchscreen (Thomas & Milan, 1987). When estimating the performance of each input device, it is important to also consider how each will alter the student's learning experience. This paper explores how students interact with a dynamic technology using a touchscreen versus a traditional mouse and how that affects learning gains.

From Here to There! (FH2T) is a dynamic math game that uses perceptual based interventions to introduce foundational algebraic concepts (Ottmar et al., 2015). The FH2T program was given to 23 first grade students, all within a single classroom with a single teacher. Of the 23 first graders, 11 students used a mouse on a computer and 12 students used an iPad tablet. Our in-class observations noted that students struggled to use the computer mouse and would instead try to physically move numbers manually with their hands out of frustration. These findings support Thomas & Milan (1987) in that touchscreen devices are preferred by beginners than the traditional computer mouse. However, we examined learning gains using a two-tailed t-test on the pre and post assessment test scores. This data suggests that learning gains are not significantly affected by either input device. This could indicate that deployment on a regular computer is just as effective as on a touchscreen device, thus not hindering learning due to inaccessibility. We believe this is an important finding as touchscreen devices such as iPads are more expensive and are not as readily available as laptops and desktop computers in schools.

References

Thomas, C., & Milan, S. (1987). Which input device should be used with interactive video?. In *Human–Computer Interaction–INTERACT'87* (pp. 587-592).

Ottmar, E., Landy, D., Goldstone, R. L., & Weitnauer, E. (2015). Getting From Here to There!: Testing the Effectiveness of an Interactive Mathematics Intervention Embedding Perceptual Learning. In *CogSci*.

MATHEMATICAL OPPORTUNITIES FOR LEARNING IN FREELY AVAILABLE PBS APPLICATIONS

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Although recent evidence suggests that well-designed applications can positively impact mathematics learning (Foster, Anthony, Clements, Sarama & Williams, 2016; Schacter & Jo, 2017), few studies have examined the impact of mathematics applications that teachers or parents can find for free on the internet. Yet, these sorts of resources often get used in schools and at home. Broadly, the goal of this study is to explore the affordances of freely available resources by answering the question: What opportunities for learning mathematics do mathematics applications designed and made available by the Public Broadcasting Company make possible for urban kindergarteners?

This poster reports Phase 1 of the study, which included a mathematical content analysis of PBS applications, an analysis of initial teacher feedback, and a preliminary analysis of usage data that revealed how frequently children engaged with mathematics games in their classrooms. The study is not yet complete, but Phase 2 will include an ANOVCA analysis to identify relationships between usage and performance on district mathematics assessments. This mixed methods study began when national PBS educators produced a suite of 26 games with mathematics content and the ability to collect user analytics (such as duration of use, answers, etc.) Mathematics educators then reviewed the games to identify mathematics content. This analysis identified 10 games likely to provide productive mathematical interactions for kindergarteners. Treatment tablets were loaded with these math games, while control tablets contained only literacy and drawing games. Approximately 900 tablets were distributed to all kindergarteners in a single district.

The results of the content analysis demonstrated that about half of the designed games were not mathematically productive. For example, one application allowed users to succeed by repeatedly clicking the mouse to fill a given container without requiring attention to size or quantity. Another application asked users to wash, dry, and cut a character's hair. The only mathematics was the labeling of the steps as first, second, and third, and a character reciting the counting sequence while having his hair washed. For this reason, only ten math games were loaded onto tablets. Initial analysis of the usage data and of conversations with teachers suggests that teachers are regularly using these games. For example, about one month after distribution, approximately 300 unique users played the games in a two-day period. This analysis suggests that teachers are willing to incorporate technology into their instruction and feel comfortable doing so on their own, but that initial assessments of the quality of resources are important. Once the assessment data is collected as part of this study, we will have been insight about whether the games identified as potentially productive had any impact on student mathematical performance.

References

Foster, M. E., Anthony, J. L., Clements, D. H., Sarama, J., & Williams, J. M. (2016). Improving mathematics learning of kindergarten students through computer-assisted instruction. *Journal for Research in Mathematics Education*, *47*(3), 206-232.

Schacter, J., & Jo, B. (2017). Improving preschoolers' mathematics achievement with tablets: a randomized controlled trial. *Mathematics Education Research Journal*, *29*(3), 313–3.

CALCULUS II STUDENTS' DEFINITIONS OF FUNCTION: ATTENTION TO CORRESPONDENCE

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Keywords: Function, Calculus, Technology

For students studying upper-level mathematics, particularly those majoring in STEM fields, having a robust concept definition of function is an essential part of their continued study of topics in advanced mathematics (e.g., Thompson & Carlson, 2017). Thompson and Carlson (2017) note that it would benefit students learning calculus to have a more developed, conceptual understanding of function; and technology can be an effective way for students to make connections between different ways of thinking about function (Dick & Hollebrands, 2011). Students' definition of function is varied, with many being misaligned with the mathematical definition (Vinner & Dreyfus, 1989). Often students rely on the use of continuous functions or think of functions only in terms of a graph or algebraic formula (e.g., Vinner & Dreyfus, 1989). Thus, it is worthwhile to give students an opportunity to reexamine their definition of function using technology, especially in a manner which does not involve graphs, or equations.

We designed an applet to challenge Calculus II students' understanding of function. Our applet (https://ggbm.at/J3mJaU6H) featured a vending machine design, as this allowed us to put the concept of function in a context that did not make use of numbers or other traditional algebraic language. We specifically built the machines to problematize students' understanding of function. Thus we analyzed the students' pre and post definitions to answer the following question: How do students change their definition of function after engaging with the applet?

Pre- and post-definitions were collected from 87 Calculus II students from three different universities. Definitions were coded for many characteristics including accounting for correspondence with elements of the range. Our results indicated that 33 of the 64 students (52%) who did not account for correspondence in their pre-definition did so in their post. Our poster will share full details of our analysis and associated results.

References

Dick, T. P., & Hollebrands, K. F. (2011). Focus in high school mathematics: Technology to support reasoning and sense making. Reston, VA: National Council of Teachers of Mathematics.

Vinner, S. & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20,* 356-366.

Thompson, P.W., & Carlson, M.P. (2017). Variation, Covariation, and Functions: Foundational Ways of Thinking Mathematically.

CONSTRUCTING MICROWORLDS, CONSTRUCTING KNOWLEDGE, CONSTRUCTING COMMUNITIES: MATHEMATICAL MICROWORLDS AND CONSTRUCTIVIST THEORY

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Mathematical microworlds are computer-based exploratory environments with a restricted set of operations but which have no fixed goals. Theories of learning take on a particular importance, as the way in which these environments are used in the classroom depends on how a particular instructor believes that students learn. This literature review surveys nine key articles on learning theory and microworlds, with a particular focus on how social constructivist theories can be used to explain the interactions between the student and the computer.

Keywords: Technology, Learning Theory

The purpose of this literature review is to examine theoretical research on mathematical microworlds and constructivism to determine the contribution that the study of mathematical microworlds (computer environments with restricted sets of possibilities but no fixed goals) has to offer to the ongoing discourse in the area of developmental and social constructivism.
For this literature review, we searched Google Scholar and ERIC for various combinations of the following phrases: math, mathematical, microworlds, constructivism, sociocultural theory, situated cognition, Piaget, and/or Vygotsky. We also used a snowball methodology to look at who cited the articles we found and who those articles were cited by. From this, we created a database of 27 articles. We excluded articles that were empirical and did not have a significant theoretical contribution to offer, narrowing the list down to n=9 articles.

Microworld theorists coming from a developmental constructivist perspective argue for microworlds that are open-ended and focus on studying individual student's explorations and processes. (e.g. Biddlecomb, 1994). Social constructivists raise concerns about the fixed nature of the tools (Cerulli & Mariotti, 2003) but are also intrigued by how the microworlds embed accumulated cultural knowledge (Chiappini, Pedemonte, and Robotti, 2003). Furthermore, they focus on the ways in which students interact with each other when working with these tools in a social context (Chiappini, 2003; Hackenberg & Sinclair, 2007).

Where we choose to focus our attention, on the individual (as the radical constructivists do) or on the social (as the social constructionists do), affects what sort of tools we will create.

References

Biddlecomb, B. D. (1994). Theory-based development of computer microworlds. Journal of Research in Childhood Education, 8(2), 87–98. https://doi.org/10.1080/02568549409594858

Cerulli, M., & Mariotti, M. A. (2003). Building theories: working in a microworld and writing the mathematical notebook. In J. Z. Neil, A. Pateman & Barbara J.Dougherty (Ed.), 2003 Joint Meeting of PME and PMENA (p. Vol. II, pp. 181–188). Honolulu, United States. Retrieved from https://telearn.archives-ouvertes.fr/hal-00190307

Chiappini, G., Pedemonte, B., & Robotti, E. (2003). Mathematical teaching and learning environment mediated by ICT. In C. Dowling & K.-W. Lai (Eds.), Information and Communication Technology and the Teacher of the Future (pp. 217–228). Springer US. https://doi.org/10.1007/978-0-387-35701-0_24

Hackenberg, A., & Sinclair, N. (2007). Talking about embodiment and caring in relation to computer use in mathematics education. For the Learning of Mathematics, 27(3), 12–16.

ORIENTATION OF PEDAGOGY WHEN EXPLOITING CAS TECHNOLOGY IN THE DEVELOPMENT OF MATHEMATICAL KNOWLEDGE

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Keywords: Technology, Instructional Activities and Practice, Algebra and Algebraic Thinking

The purpose of this study was to understand (a) what pedagogical opportunities mathematics teachers exploited with the presence of CAS and (b) how teachers aligned lessons to develop mathematical understandings. The following research question guided the study: How do secondary mathematics teachers orient their instructional practices to exploit computer algebra systems (CAS) in the development of mathematical knowledge? This case study considered two secondary teachers pedagogy as they administered CAS technology. Data included lesson observations, interviews, and reflective writings. Pierce and Stacey's (2010) pedagogical framework (P-Map) illuminated the opportunities that teachers situate their utilization of CAS as a pedagogical tool. The data was matched to the P-Map and analyzed for emergent themes.

Background

The NCTM *Principles to Action* technology standard stated that "an excellent mathematics program integrates the use of mathematical tools and technology as essential resources to help students learn and make sense of mathematical ideas, reason mathematically, and communicate their mathematical thinking" (NCTM, 2014, p. 5). CAS tools have continued to advance in their simplicity, functionality, and availability, but are still not a part of standard educational practice (Heid et al., 2013). The mathematics education community is ripe for technological infusion with the teacher at the heart of implementation (NCTM, 2014; Heid & Blume, 2008). Teachers are the agent of change for technology use in classrooms (Ertmer & Ottenbreit-Leftwich, 2010). It is logical to first consider teacher instructional practice that utilized CAS to develop MK.

Results & Discussion

Teacher pedagogy was adjusted at these levels: subject, classroom, or task. The taxonomy presented by Pierce and Stacey (2010) formed the basis for the analysis. Findings indicate that these two teachers were creative in their methodological approaches. Teachers oriented lessons through outsourcing procedures, providing guidance, verifying answers, regulating access, investigating with multiple representations, and viewing CAS as a mathematical consultant. Motivations were grounded in the CAS' functional capabilities of accuracy and efficiency.

References

Ertmer, P. A., & Ottenbreit-Leftwich, A. T. (2010). Teacher technology change: How knowledge, confidence, beliefs, and culture intersect. *Journal of Research on Technology in Education*, *42*(3), 255-284.

Heid, M. K., & Blume, G. W. (2008). *Research on technology in the teaching and learning of mathematics* (Vols. 1-2*)*. Charlotte, NC: Information Age Publishing.

Heid, M. K., Thomas, M. O. J., & Zbiek, R. M. (2013). How might computer algebra systems change the role of algebra in the school curriculum? In Clements et al. (Eds.) *Third International Handbook of Mathematics Education*. New York, NY: Springer.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all.* Reston, VA: National Council of Teachers of Mathematics.

Pierce, R., & Stacey, K. (2010). Mapping pedagogical opportunities provided by mathematics analysis software. *International Journal of Computers for Mathematical Learning*, *15*, 1-20.

KINECTING GEOMETRIC PROOF CONCEPTS USING GESTURES

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We explore whether *directed actions*—body movements that learners are instructed to formulate—enhance mathematical reasoning during proof production. Evidence is mounting that sensorimotor activity can activate neural systems, which can in turn alter and induce cognitive states (Nathan, 2014). New interventions are using motion-sensing technology to track actions that support geometric reasoning (e.g., Smith, King, & Hoyte, 2014). We directed learners to perform mathematically-relevant (vs. irrelevant) motions through motion-capture video game play and hypothesized (H1) that directed actions will facilitate production of dynamic gestures, which will, in turn, (H2) improve students' nonverbal mathematical insights and the production of multimodal transformational proofs. Moreover, we hypothesized (H3) that adding pedagogical hints explicitly connecting directed actions to the conjectures enhances proof performance.

Thirty-five middle and high school students played the *Hidden Village* game for the Kinect. Students played through 6 conjectures, with 2 to 4 conjectures (with relevant motions) revisited where the interviewer revealed to students how the motions related to the conjectures. Students' responses were scored along 4 dimensions: (1) making spontaneous *depictive* gestures, (2) making spontaneous *dynamic* depictive gestures, (3) recognizing key mathematical *insights*, (4) formulation of a *valid* transformational proofs (Harel & Sowder, 1998).

Participants were more likely to make depictive gestures when performing mathematically relevant (vs. irrelevant) directed actions (Odds=4.2, d=0.8, p=.008). Participants who performed relevant directed actions were *not* more likely to make dynamic gestures (H1), demonstrate the mathematical insight, or provide a valid proof (H2; $p > 0.1$). However, after receiving the pedagogical hint (H3), participants were more likely to make depictive gestures (Odds=5.4, d=0.9, p<.001), dynamic gestures (Odds=4.0, d=0.8, p=.001), more likely to express the correct insight (Odds=3.1, d=0.6, p<.001), and more likely to formulate a valid proof (Odds=4.7, d=0.9, p<.001). Producing depictive gestures predicted mathematical insight (Odds=3.0, d=0.6, p=.007), but not formulating a transformational proof. However, making *dynamic* depictive gestures (H2) predicted both insight (Odds=8.1, d=1.2, p<.001) and proof (Odds=11.5, d=1.3, p<.001).

Results suggest that dynamic gestures may be associated with reasoning deductively about generalizable properties of space and shape and that pedagogical hints related to the directed actions are beneficial for insight and learning geometric proof.

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References

Harel, G., & Sowder, L. (1998). Students' proof schemes. In E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), *Research on collegiate mathematics education* (Vol. III, pp. 234–283). Providence, RI: American Mathematical Society.

Nathan, M. J. (2014). Grounded mathematical reasoning. In L. Shapiro (Ed.). The Routledge Handbook of Embodied Cognition (pp. 171-183). Routledge: New York.

Smith, C. P., King, B., & Hoyte, J. (2014). Learning angles through movement: Critical actions for developing understanding in an embodied activity. *The Journal of Mathematical Behavior, 36*, 95–10

FLIPPED MATHEMATICS INSTRUCTION OBSERVATION PROTOCOL

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Although flipped instruction is often viewed as a unified instructional model, many variations of flipped implementations exist in mathematics (de Araujo, Otten, & Birisci, 2017). Some teachers assign videos for homework and follow that with individual activities in class, whereas others assign videos with embedded questions followed by group projects in class. Extant observation protocols do not adequately capture the nuances between different variations of flipped lessons. Thus, we developed a classroom observation protocol robust enough to capture variations in flipped and non-flipped lessons.

Our Flipped Mathematics Instruction Observation Protocol (FMIOP) draws upon existing frameworks (e.g., Stein, Grover, & Henningsen, 1996), observation instruments (e.g., MQI), and advice from experts in educational technology and mathematics education. We then iteratively revised the protocol based upon teacher interviews and observations of flipped lessons. FMIOP consists of two components: in-class and at-home. The in-class component captures two aspects of the lesson: instructional quality and interactivity. Each aspect has sub-characteristics (e.g., mathematics development, video involvement). The protocol also distinguishes the whole-class and non-whole-class formats. In a departure from prior protocols, aspects such as the nature of authority are not combined into the instructional quality score but instead held separately as descriptive features of the lesson since evidence from different disciplines is contradictory about which authority pattern will be predictive of student learning. The at-home components are examined along three aspects: instructional quality, multimedia design, and interactivity. In addition to lecture videos, FMIOP captures the use of set-up videos, which are those that establish a non-mathematical context to intrigue students about what will happen in class.

Looking back, we see that instruments for lesson observations have progressed as new instructional models took hold. Looking ahead, the use of technology will continue to grow, so our observational tools must advance to account for key features of instructional videos and how videos are used in lessons so that we might distinguish implementations of flipped instruction and draw meaningful conclusions about how the instruction relates to student learning.

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References

de Araujo, Z., Otten, S., & Birisci, S. (2017). Teacher-created videos in a flipped mathematics class: Digital curriculum materials or lesson enactments? *ZDM Mathematics Education, 49*, 687–699.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks. *American Educational Research Journal, 33*(2), 455–488.

Chapter 13

Theory and Research Methods

Research Reports

Brief Research Reports

Posters

DEVELOPMENT AND USE OF A CONJECTURE MAP FOR ONLINE PROFESSIONAL DEVELOPMENT MODEL

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In this paper we discuss our development and use of a conjecture map for a large research project on the design and implementation of an online professional development model. Following Sandoval's (2014) model, we built the conjecture map to reflect our high-level conjectures (overarching goals of the project), the embodiment of the learning design, the mediating processes, and the outcomes. After roughly half a dozen iterations to ensure that we accurately and fully captured the features of the learning environment and the mediating processes, we used the conjecture map as an anchor for a number of our data analysis activities, particularly of our online course modules, and for initiating theory-building processes. We also found that it was an expedient way to communicate internally and externally the core assumptions and learning principles of our multifaceted online professional development model.

The project described in this paper involves a three-part model whose primary goal is to transition multi-faceted face-to-face professional development experiences to online environments in order to provide rural teachers access to high-quality professional development experiences. For example, the Teaching Lab lessons (similar to Lesson Study) have traditionally been conducted entirely face-to-face; we subsequently redesigned these experiences so teachers from disparate rural environments can participate without having to travel great distances. Moving face-to-face experiences to online environments is not straightforward and requires an iterative process to understand and respond to the logistic, technological, and theoretical challenges that arise. Consequently, we conceptualized our project as design research in that we planned to use lessons gained from our initial attempts to inform subsequent revisions to the original designs and sequencing of the components of the model. We turned to Sandoval's (2014) description of *conjecture maps* to guide our efforts to unpack the assumptions and theories on which we operated as we designed and redesigned the components.

In this conceptual paper, we discuss the development and use of a conjecture map (Sandoval, 2014) to help us reflect on and better understand our own assumptions regarding the overall conjectures for the project, the design of the learning environment, and the processes that mediate between the learning environment and the outcomes related to our high-level conjectures. Wozniak (2015) used a conjecture map for a project involving online learning and noted that the refinements made through design research due to the conjecture map enhanced their "theoretical understanding about transitioning to online distance learning" (p. 608). In considering the relevance of this model to our own context, we asked ourselves the following questions:

1. To what extent do the conjecture maps have face validity to the actual processes engaged by our participants?

2. How have the conjecture maps informed both the revisions of the professional development model and the data analysis?

Frameworks

Design Research

We follow a design-based research model (Barab & Squire, 2004; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Edelson, 2002; The Design-Based Research Collective, 2003) to guide our research. Design research is intended to engineer learning environments, study the impact of that learning environment on desired outcomes, and revise the learning environment as needed, with the goal of testing and building theory. The goal of design research is to be able to explain learning that occurs and what supports that learning by systematically and iteratively studying the design and the impact of the design (Cobb et al.).

Design research departs from naturalistic research in that it purposefully engineers features in the learning environment, though it retains some of the messy complexity found in naturalistic settings. This type of research departs from experimental design and other research done in highly controlled, even contrived, settings in that it seeks to understand the role of context and the situated nature of activity to explain learning. Design research operates at an intermediate level of theory, to produce useful explanations that extend beyond the context in which the study is situated, but not to pose universal theories of learning (Cobb et al., 2003). Additionally, the design research process purposefully facilitates (engineers) particular interactions in order to produce useful explanations (Barab & Squire, 2004).

In this paper, we describe a model of online mathematics professional development for teachers in rural contexts, the hypothesis and conjectures we developed and are testing, and the way in which we articulated and operationalized features of the design experiment we are conducting. Our goal is to not only understand how our model works with our participants, but how it may inform broader efforts to conduct high-quality professional development online, how to engage rural mathematics teachers in such experiences, and how it leads to theory-building.

Conjecture Maps

We follow Sandoval (2014) in developing and using conjecture maps to articulate our model, guide our research, and build theory. According to Sandoval, "*Conjecture mapping* is a means of specifying theoretically salient features of a learning environment design and mapping out how they are predicted to work together to produce desired outcomes" (p. 19) and is intended to reify the conjectures regarding the learning environment and how they interact to promote learning. There are four main elements to a conjecture map. The first element involves *high-level conjectures* about the learning context and it supports learning. Those conjectures are then operationalized in the *embodiment* of the learning design, which is the second element. In the third element, this embodiment in turn is intended to generate *mediating processes* that produce desired *outcomes*. The desired outcomes constitute the fourth and final element of the conjecture map. The conjectures about how the designed learning environment (the embodiment) results in the mediating processes—or the process from the second to third element—are called *design conjectures*, which take the form "if learners engage in this activity (task + participant) structure with these tools, through this discursive practice, then this mediating process will emerge" (p. 24). The conjectures about how those mediating processes produce desired outcomes—or the process from the third to final element—are *theoretical conjectures*, which take the form "if this mediating process occurs it will lead to this outcome" (p. 24).

High-level conjectures are the abstracted ideas about the learning principles evident in the design of the learning environment and are produced by an analysis of the needs evident in a certain context with respect to the desired outcomes. The embodiment of the learning design involves four elements, according to Sandoval (2014): tools and materials, task structures, participant structures, and discursive practices, described in further detail below. Mediating processes involve the kinds of observable interactions between participants and the design of the environment, though artifacts created through learning activities can be used as well. Observable interactions can show how the learning environment facilitates or mediates participants' interactions, particularly those conjectured as leading to desired outcomes. Analysis of artifacts can show how participants interpret designed activity structures and tools to explain their interactions and engagement. Mediating processes are intended to produce desired *outcomes*. Sandoval notes that different design research projects can utilize "a wide variety of outcomes and could take a wide variety of approaches to gathering evidence of those outcomes" (p. 23).

Online Learning Model

We designed a three part online professional development model with the goal of providing rural mathematics teachers access to high quality professional development. We chose to pursue an online model due to the difficulties for rural teachers to attend face to face professional development. The three components had been originally designed and implemented in face to face formats and were moved to fully online versions for the purposes of this project. Our project utilizes a series of synchronous online experiences, which departs from the typical asynchronous nature of much of the current online professional development, educational coursework, and virtual teacher communities. The three parts of the model include online course modules, Teaching Labs (akin to a lesson study approach), and online coaching. We based our model on research on teacher learning, described in more detail below, which informed our conjectures.

The online course modules were based on discourse practices that orient teachers toward high-leverage discourse practices that facilitate mathematically productive classroom discussions (Smith & Stein, 2011). These discourse practices are facilitated by five practices emphasized in the course, entitled *Orchestrating Mathematical Discussions (OMD)*, anticipating, monitoring, selecting, sequencing, and connecting. The modules also emphasize key aspects of lesson planning, such as goal-setting, in addition to having teachers solve and discuss high-cognitive demand tasks. The modules are designed to develop awareness of specific teacher and student discourse moves that facilitate productive mathematical discussions, to understand the role of high cognitive demand tasks in eliciting a variety of approaches worthy of group discussions, and to further develop participants' mathematical knowledge, particularly the rich connections around big mathematical ideas that are helpful to teach with understanding (Ball, 1991; Ma, 1999). The modules involve a combination of synchronous and asynchronous work, in order to minimize the amount of time teachers must virtually meet together (Robinson, Kilgore, & Warren, 2017). This minimizes logistical challenges and maintains a high degree of teacher effort and attention due to the shortened synchronous time. Hrastinski (2008) found that the combination of synchronous and asynchronous complement each other by offering opportunities for cognitive participation (asynchronous) and personal participation (synchronous). Cognitive participation allows for reflection on complex instruction, while personal participation involves collaborative opportunities for immediate feedback, community building, and collaborative learning.

The Teaching Labs follow a lesson study design, modified to decrease time commitments and, in our case, with the need to be physically present in the classroom of the target lesson. Research on lesson study (e.g., Amador & Carter, 2018; Stigler & Hiebert, 1999) has led to an emphasis on demonstration lessons where teams of teachers collectively plan, enact, and reflect on lessons in ways that make public the features of the lessons and teachers' instructional practices (Saphier & West, 2009). The benefits of lesson study are to help teachers incrementally expand their knowledge base, promote collective professional discussions, and to improve instructional practices (Choskshi & Fernandez, 2005; Stigler & Hiebert, 1999). Our work takes this traditionally in-person model and translates it into a synchronous and asynchronous online professional development opportunity.

Coaching has a relatively short history in education, and online coaching is just now emerging as something that is beyond experimental. Coaching, as characterized by the interactions between coaches and practitioners, facilitates *deliberative practice* in that there are repeated opportunities to reflect on practice in principled and formative ways (Ericsson, Krampe, & Tesch-Romer, 2003). Over the last two decades, there has been an increasing focus on coaches to provide teachers with personalized and content focused professional development (Campbell & Malkus, 2011; Cobb & Jackson, 2011; Hartman, 2013), which is highly valued by the teacher (Chval et al., 2010). Common aspects of coaching include working with teachers to model instructional practices, reflect on observed instruction, study student work, and plan lessons (Batt, 2010; Matsumura, Garnier, & Spybrook, 2012). Research on the impact of coaching has found positive effects of content-focused coaching on teachers' instructional practices and student achievement in the area of literacy (Matsumura et al., 2012). Other research has shown that literacy coaches, when they collaborate closely with teachers around core instructional practices, can have positive impacts, though this impact was mediated by the roles of the coaches and the access of teachers to coaches with expertise in the instructional intervention (Coburn & Russell, 2008; Penuel, Riel, Krause, & Frank, 2009). Again, our work focuses on moving in-person coaching to an online context.

Methods

Development of the Conjecture Map

To develop the conjecture map, we first articulated the high-level conjecture and *desired* outcomes (Edelson, 2002), the bookends of the conjecture map. Our high-level conjecture was that *teachers transform their instructional practices by engaging in collegial interactions related to core instructional practices across multiple online contexts*. This conjecture captured aspects of the goal of transforming practice, the ways in which we were engineering opportunities for that to occur, the kinds of interactions we envisioned, the context in which the designed learning environment was going to be situated, and the theories we hoped to test and build. This conjecture reflected the core hypotheses of our online professional model.

We then articulated four desired outcomes related to our model. These outcomes were that teachers would become more adept at: (1) attending to student thinking in productive ways; (2) noticing key aspects of instructional practices; (3) engaging in high-leverage discursive practices with the goal of eliciting and refining student thinking; and (4) reflecting on one's own practices, using evidence from the classroom, leading to instructional change. The outcomes reflected our understanding of the kinds of instructional practices that result in vibrant and productive classroom learning environments (e.g. Jacobs, Lamb, & Philipp, 2010; Smith & Stein, 2011).

The next step was to describe the embodiment of our conjectures, the designed learning environment or what Cobb et al., (2003) term the *learning ecology*, which they describe as "a complex, interacting system involving multiple elements of different types and levels" (p. 9). Following Sandoval (2014) and Cobb et al., we focused on four features of the learning environment as: (1) tools and materials, (2) task structures, (3) participant structure, and (4) discursive practices. The tools and materials included the online platforms we used, the tools available within those platforms, and the protocols for structuring interactions, such as planning templates and the Noticings and Wonderings prompts for reflecting on mathematical tasks and episodes of instructional practice. The task structures included engaging with and reflecting on mathematical tasks, fictional and real accounts of instructional practices, and planning and reflecting on lessons. The participant structures included a range of whole group, small group, and one-on-one settings. Discursive practices included the type and tenor of interactions we intended to facilitate, such as collegial collective discussions and critical reflection on one's own practices. We identified these four features for each of the three components of the model. For example, for the Orchestrating Mathematical Discussion online model, we listed the Zoom environment (online based video conferencing software) and protocols for interaction as the Tools and Materials, the mathematical tasks and cases of classroom instruction as Task Structures, collective work in small and whole groups in an online space as the Participant Structures, and collegial discussions in which participants share their reflections on the learning activities as the Discursive Practices we were trying to engineer.

Our design conjecture was that the designed learning environment would facilitate the development of four mediating processes. These four mediating processes were: (1) reflection on one's own engagement in mathematical processes, discursive processes, and task characteristics; (2) discussion of core instructional practices in relation to attention to student thinking; (3) collective observation and non-evaluative reflection on concrete instances of practice; and (4) supported observation and reflection on one's own practices. These mediating processes constituted a progression of a sort, in that we felt that participants needed to discuss instructional practices in the abstract initially, discuss how those practices helped to lead to increased attention on student thinking, reflect on instances of actual practice collectively observed, and then reflect on their own practice. We saw these as requiring increased capacity as well as development of non-evaluative norms for noticing and discussing practice.

These mediating processes were the practices in which we hoped to productively engage the participants and which we viewed as essential if we were to realize the outcomes we had identified. While we imagined that we would need to revise the design of the learning environment to better facilitate these mediating processes, we saw the mediating processes as stable features of the overall design and conjectures, and which would be revised only after considerable discussion of the design and systematic analysis of data. We indicate the conjectured contribution of each component of the learning environment to the development of the mediating processes by the arrows shown in the conjecture map (See Figure 1). The width of the arrow indicates the strength of contribution for a particular component to the development of a mediating process. Our theoretical conjecture was that these mediating processes collectively would lead to the desired outcomes noted above.

Results

Below, we first present the conjecture map we constructed and then describe how we use the map to guide our analytic and design processes. The conjecture map is seen in Figure 1.

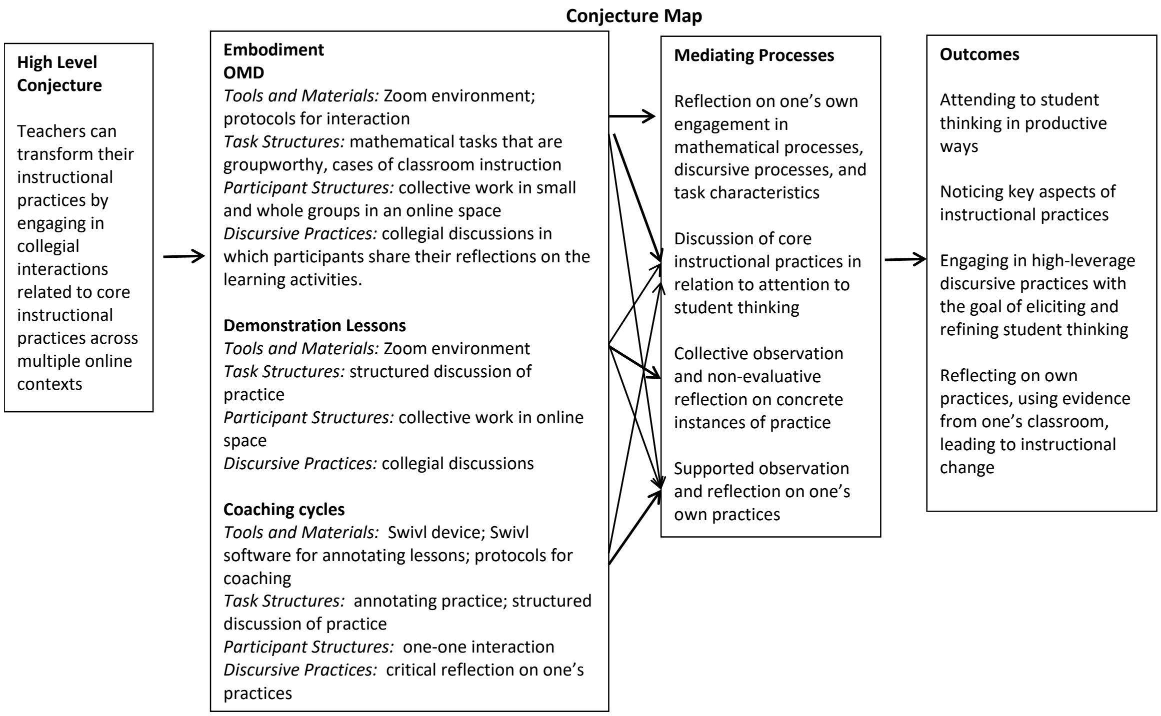

Figure 1. Conjecture Map for the Project

Conjecture Map as Reification of Our Model

Developing the conjecture map helped to reify our core conjectures about the design of the learning environment and the relationship between the design of the model, the intermediate processes we needed to develop, and our desired outcomes. Constructing the model engendered clarifying discussion about the learning environment and what each component was intended to accomplish. In our group, there is a distinction between the design and research teams, with several people dedicated entirely to the design and implementation of the learning environment, and several people dedicated to research, though there is some overlap between the groups. The researchers have focused primarily on the design conjecture at this point in the project: they have focused on how the learning environment facilitates the development of the mediating processes. Developing the map was one way for the design and research teams to come into greater contact with each other. The map provides an explicit articulation of the design and intended impact of the model as well as the conjectures and hypotheses on which we are basing our work.

Conjecture Map Guides Research Activities

The conjecture map has served as an anchor artifact in the discussions around data analysis activities. The map has focused the research team discussions on the design conjectures: we make sure we are documenting the design and we explore the affordances and constraints in the online learning environment; and we explore the impact of the learning environment on the development of the mediating processes. When we find a feature of the learning environment missing from the map, we revise the map to more accurately reflect our observations, and we check back in with the design team to verify the revised map. We develop our data analysis activities keeping in mind that they must speak directly to the nature of the learning environment, the relation between the learning environment and mediating processes, and the nature of the mediating processes in terms of our observations of the participants.

Conjecture Map Guides Documentation and Revision of Model

We use the conjecture map to anchor discussions about whether our designed learning environment facilitates the desire mediating processes and, if not, what needs to be revised within the learning environment. We recognized that it was crucial to fully document the enacted design at key intervals, to note the changes made to the learning environment, and to ensure that those changes are reflected in any revision of the conjecture map. This process has also identified unanticipated processes that have influenced the design of the project. For example, working in rural environments requires much more flexibility on our part in terms of the timing of classes and meetings, and the lack of internet infrastructure in some cases has led to complications. As a result, our design team has made changes on the fly without the benefit of using analysis derived from the research team, who have typically lagged six months behind in terms of processing and analyzing data. Thus, in documenting our design processes, we have audio-recorded and transcribed the design team meetings to understand the motivations behind the changes in the learning environment. Ultimately, however, the design and research teams are accountable to the conjecture map, and thus the teams will collectively revise the map to account for realizations and evidence gleaned through the practical contingencies of implementing a complex online model in rural environments and through systematic data analysis.

Role of Conjecture Map in Theory Building

The work in the project thus far has focused on our design conjectures (the link between the learning environment and mediating processes). However, as we wrap up the initial cohort at the end of the current project year, we will begin testing and revising the theoretical conjectures (the relationship between the mediating processes and outcomes). Ultimately, we will refine our high-level conjecture as we seek to generalize our findings beyond the context of the project, to address broader claims of learning, and, in particular, professional learning in online contexts.

Discussion and Implications

In this conceptual paper, we describe an ongoing design experiment that demonstrates the dynamic nature of the development and use of conjecture maps. Similar to Wozniak (2015), we found that our ongoing use of the conjecture map is strengthening the iterative nature of our design research project. As we regularly revisit the conjecture map and make modifications to our professional development design, we use the map as a framework for planning and evaluating the actual processes engaged by our professional development participants and the theoretical implications of the design. Through this process, the conjecture map is a key artifact for researchers engaging in design experiments (Sandoval, 2014; Wozniak, 2015). Developing and using a conjecture map holds us accountable to the assumptions of design experiments and to our own conjectures and hypotheses about improving classroom instruction of mathematics teachers in rural environments via an online professional development model. The map has anchored discussions about the design of the learning environment and how we research the design conjecture, specifically the data collected and the analysis process. Developing and using the map has helped us to identify misunderstandings between the design and research teams, to understand our lived processes with respect to revising the design of the learning environment, and to be more explicit about the specific mediating processes in which we seek to engage our participants. It has also helped us to be aware of key features of the design process, especially around the intended consequences of our work and the messiness of conducting research in naturalistic, if designed, environments. Ultimately, it will focus our theory-building discussions, especially as we explore the theoretical conjectures.

References

Amador, J., & Carter, I. (2018). Audible conversational affordances and constraints of verbalizing professional noticing during prospective teacher lesson study. *Journal of Mathematics Teacher Education, 21*, 5-34.

Ball, D. L. (1991). Teaching mathematics for understanding: What do teachers need to know about subject matter? In M. M. Kennedy (Ed.), *Teaching academic subjects to diverse learners*. New York: Teachers College Press.

Barab, S., & Squire, K. (2004). Design-based research: Putting a stake in the ground. *Journal of the Learning Sciences, 13*(1), 1-14.

Batt, E. (2010). Cognitive coaching: A critical phase in professional development to implement sheltered instruction. *Teaching and Teacher Education, 26*, 997-1005.

Campbell, P. F., & Malkus, N. N. (2011). The impact of elementary mathematics coaches on student achievement. *Elementary School Journal, 111*(3), 430-454.

Chokshi, S., & Fernandez, C. (2005). Reaping the systemic benefits of lesson study: Insights from the U.S. *Phi Delta Kappan, 86*(9), 674-680.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in education research. *Educational Researcher, 32*(1), 9-13.

Cobb, P., & Jackson, K. (2011). Towards an empirically grounded theory of action for improving the quality of mathematics teaching at scale. *Mathematics Teacher Education and Development, 13*(1), 6-33.

Coburn, C. E., & Russell, J. L. (2008). District policy and teachers' social networks. *Educational Evaluation and Policy Analysis, 30*(3), 203-235.

Edelson, D. C. (2002). Design research: What we learn when we engage in design. *Journal of the Learning Sciences, 11*(1), 105-121.

Ericsson, K. A., Krampe, R. T., & Tesch-Romer, C. (2003). The tole of deliberate practice in the acquisition of expert performance. *Psychological Review, 100*(3), 363-406.

Hartman, S. (2013). Math coaching in a rural school: Gaining entry: A vital first step. *Journal of Education, 193*(1), 57-67.

Hrastinski, S. (2008). Asynchronous and synchronous e-learning. *Educause quarterly*, *31*(4), 51-55.

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional Noticing of Children's Mathematical Thinking. *Journal for Research in Mathematics Education, 41*(2), 169–202.

Ma, L. (1999). *Knowing and teaching elementary mathematics*. Mahwah, NJ: Lawrence Erlbaum.

Matsumura, L., Garnier, H., & Spybrook, J. (2012). The effect of content-focused coaching on the quality of classroom text discussions. *Journal of Teacher Education, 63*(3), 214-228.

Penuel, W., Riel, M., Krause, A., & Frank, K. (2009). Analyzing teachers' professional interactions in a school as social capital: A social network approach. *Teachers College Record, 111*(1), 124-163.

Robinson, H.A., Kilgore, W. & Warren, S. J. (2017). Care, communication, learner support: Designing meaningful online collaborative learning. *Online Learning, 21*(4), 29-51. doi: 10.24059/olj.v21i4.1240

Sandoval, W. (2014). Conjecture mapping: An approach to systematic educational design research. *Journal of the Learning Sciences, 23*(1), 18-36. doi:10.1080/10508406.2013.778204

Saphier, J., & West, L. (2009). How coaches can maximize student learning *Phi Delta Kappan*, 46-50.

Smith, M. S., & Stein, M. K. (2011). *5 practices for orchestrating productive mathematical discussions*. Reston, VA: National Council of Teachers of Mathematics.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom* (Second ed.). New York: Free Press.

The Design-Based Research Collective. (2003). Design-based research: An emerging paradigm for educational inquiry. *Educational Researcher, 32*(1), 5-8.

Wozniak, H. (2015). Conjecture mapping to optimize the educational design research process. *Australasian Journal of Educational Technology, 31*, 597-612.

NAVIGATING DILEMMAS OF STUDYING MATHEMATICS ENGAGEMENT IN SECONDARY CLASSROOMS

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To better understand how to support high school students' engagement, advancements in research methods that provide greater understandings of malleable factors of engagement and conditions that affect students' engagement are needed. In this conceptual paper, we introduce four dilemmas that researchers need to navigate to study secondary students' engagement with mathematics: How can we concurrently capture engagement in-the-moment and at scale? What counts as a moment or experience? What sorts of experiences could be engaging? Whose perspectives on the experience should be privileged? We propose approaches for navigating these dilemmas in the context of a current research project.

Keywords: Research Methods; Affect, Emotions, Beliefs, and Attitudes; High School Education

The decline of students' motivation and engagement in mathematics as they move through levels of education is a persistent problem. Students' self-efficacy, their enjoyment, and their sense of the utility of mathematics tends to decrease as they move from elementary school into junior high and this trend continues through high school (Chouinard & Roy, 2008; Wigfield, Eccles, Mac Iver, Reuman, & Midgley, 1991). To develop insights about how to create learning environments that engage more students, more deeply, researchers must develop approaches that allow for a greater understanding of students' engagement.

Historically, research on engagement and its associated constructs has been conducted either at a small, descriptive scale, *or* at a larger, probabilistic scale. This issue of scale ranges on two dimensions: idiographic → nomothetic and momentary → longitudinal. Ideally, learning environments would be designed to promote positive experiences at all levels. From the learner's perspective, which is primarily idiographic and momentary, we want to find ways to "catch and hold" their engagement. From the perspective of the education system, which is primarily nomothetic and longitudinal, we want to increase the chance that learners will identify with mathematics-related disciplines and choose to continue learning mathematics far into the future. In particular, greater insights are needed about how to foster engagement with mathematics among high school students. Prior research suggests that instructional practices such as teachers' demonstrations of warmth and focus on understanding over performance engage students (Stipek, Salmon, Givvin, Kazemi, Saxe, & MacGyvers, 1998), but this research was conducted in classrooms with elementary school students. Alternative approaches that address nuanced differences, are likely needed to engage high school students; they are in a different developmental stage both cognitively and socially, and are learning more abstract or otherwise different mathematics. What *is* known is that high school students are more engaged with mathematics when they experience authentic work such as being asked interesting questions, solving novel problems, digging deeply into understanding a single topic, applying the subject to problems and situations in life outside of school, and discussing ideas about the subject with the teacher or students, along with pedagogical strategies that support students socially (Marks, 2000). However, what counts as authentic (i.e., *interesting, novel, applicable*) for some students may not be so for others, and means of social support have numerous facets that are worth understanding more deeply in high schools.

The purpose of this paper is to provide a critical analysis of concepts and methods for studying engagement with mathematics in the moment in high school classrooms. We propose an approach, which we are currently undertaking, that goes beyond fine-grained analyses of a few students to seek out larger trends in a manner that is still situated in the context of classroom opportunities to engage in mathematics. This approach also delineates variability within and between classrooms that might explain some of the reasons students both turn on and turn off from mathematics during the high school years. Below, we describe our approach to investigating students' engagement in relation to four dilemmas, but before we do so, we share our orientation on engagement.

What Is Engagement?

Previous work has considered engagement as both an anthropological and a psychological construct. Anthropologically, engagement can be understood as students' opportunities to participate in a particular activity. Psychologically, engagement is often conceptualized as a personal endeavor, emphasizing how psychologically present a person is during a moment in which they are actively involved (including, but not limited to, a pedagogically relevant experience in a math class) (e.g., Shernoff, Csikszentmihalyi, Schneider, & Steele, & Shernoff, 2003). Engagement can also combine the two approaches, emphasizing sustained behavioral involvement (i.e., making use of opportunities to engage) combined with some emotional tone, either positive or negative, in context. By this view, all students are engaged in some fashion and to some degree, though not all engagement is positive. For example, when engagement occurs by the demands of the educator rather than the will of the student, it may instead manifest as anti-engagement, sometimes termed as *disaffection* (e.g., Skinner & Belmont, 1993).

A useful conceptualization of engagement in mathematics classrooms is a person's investment in a pedagogically relevant object of engagement, such a mathematics task or lesson, as situated in the relationship between the self, the object of engagement, and others in the environment (Middleton, Jansen, & Goldin, 2017). Engagement is dynamic and multi-dimensional. It manifests itself in affect, cognition, behavior (Fredricks, Blumenfeld, & Paris, 2004), and under some definitions, social interactions (e.g., Rimm-Kaufman, Baroody, Larsen Curby, & Abry, 2015; Wang, Fredricks, Ye, Hofkens & Lin, 2016).

Although these four components of engagement (affective, cognitive, behavioral, and social) interact dynamically, they are often defined separately. For instance, the affective dimension of engagement involves both more immediate positive and negative affective reactions to stimuli such as math activities, teachers, or classmates, (Fredricks, Blumenfeld, & Paris, 2004), and higher-order evaluation of those reactions (e.g., Goldin, 2002; Goldin, 2014); for example, perceiving the struggle involved in solving a math problem as enjoyable. By contrast, the cognitive dimension of engagement involves effortful cognitive coordination between prior knowledge and current information (Middleton, Jansen, & Goldin, 2017). Exemplars of cognitive engagement include concentration, and memorization. The behavioral dimension of engagement, then, includes the positive *behaviors* associated with a student's productivity in the math classroom (Rimm-Kaufman, Baroody, Larsen Curby, & Abry, 2015). This entails the actions students employ working on math problems and collaborating with peers. Finally, the social dimension of engagement relates to the quality and investment in social relationships and nature of interactions, such as those with peers and teachers (Wang, Fredricks, Ye, Hofkens & Lin, 2016). We now turn to a critical analysis of four essential dilemmas researchers must face when studying engagement from this perspective.

Dilemma 1: How Can We Concurrently Capture Engagement In The Moment And At Scale?

Engagement is highly dynamic, particularly on a moment-by-moment basis. An uplifting mood, a novel mathematics problem, and an open and welcoming classroom environment might each increase engagement in the moment. However, over time, these dynamics may become less extreme as they become habituated (e.g., a mathematics problem becomes less novel the longer one works on it). It is important, therefore, to consider whether the goal is to capture students' *tendencies to engage* (which may stabilize over time) or to understand conditions that impact the more *malleable* aspects of students' engagement to foster change in engagement patterns. When researchers investigate students' tendencies to engage, they learn about variations between individual students, understanding how different students experience mathematics learning. Alternatively, when researching engagement in the moment, researchers can understand factors that can impact why and how engagement becomes more and less productive for different people at different times. Moreover, sometimes researchers are interested in both the stable and malleable aspects of engagement, trying to piece out what aspects are productive, and what features of classroom practice support productive engagement patterns.

Different pursuits require somewhat different methodological approaches. Researchers interested in students' more stable tendencies to engage have relied on the use of long-term surveys and proxies for successful outcomes, such as grades (Pinxten, Marsh, De Fraine, Van Den Noortgate, & Van Damme, 2014). In contrast, research on engagement in the moment has relied on qualitative data collection techniques such as observations (focused on individuals, small groups, or classrooms), videos, and observer field notes followed up by interviews involving video viewing sessions (e.g., Esmonde, 2009; Gresalfi, Martin, Hand, & Greeno, 2009; Webel, 2013) to capture students' rich experiences in the moment. Although work in this qualitative tradition has contributed considerably to knowledge about student engagement, the intensive nature of such qualitative data collection has typically necessitated the use of small samples, which does not enable seeking wider trends. Efforts to understand students' experiences in the moment have been scaled up using the Experience-Sampling Method (ESM) (e.g. Shernoff, Csikszentmihalyi, Schneider, & Shernoff, 2003), in which respondents are signaled at random intervals or around pre-determined experiences, and complete a series of (closed-ended and/or open-ended) questions about their experience in the moment (Shernoff, 2013). This method has the added benefit of capturing students' impressions in real time relative to later retrospective measures (such as after-the fact interviews) (Shiffman, Stone, & Hufford, 2008). Researchers interested in both the stable and malleable, in-the-moment, aspects of student engagement benefit from triangulating methods. For example, we are currently combining methods ideal for measuring changes in stable traits, such as longer-term surveys, with more in-the-moment methods, such as ESMs, interviews, and observation, as in Figure 1. In Figure 1, our data collection begins with the administration of a long-term survey assessing students' more stable perceptions and traits. This initial survey serves two purposes. First, it can be used to establish a baseline measurement of initial perceptions. These baseline measurements are then repeated at critical times, most notably at the end of data collection, with a post-test version to examine change over time. This provides a check on the validity of the processes going on between the two points of study as well as a means of assessing the effect size of the impact of the processes occurring between long-term survey measurements. Second, responses to the long-term survey can be analyzed using cluster analysis to select focal students to follow up with using qualitative methods, such as observation and interviews (e.g., Patrick & Middleton, 2002).

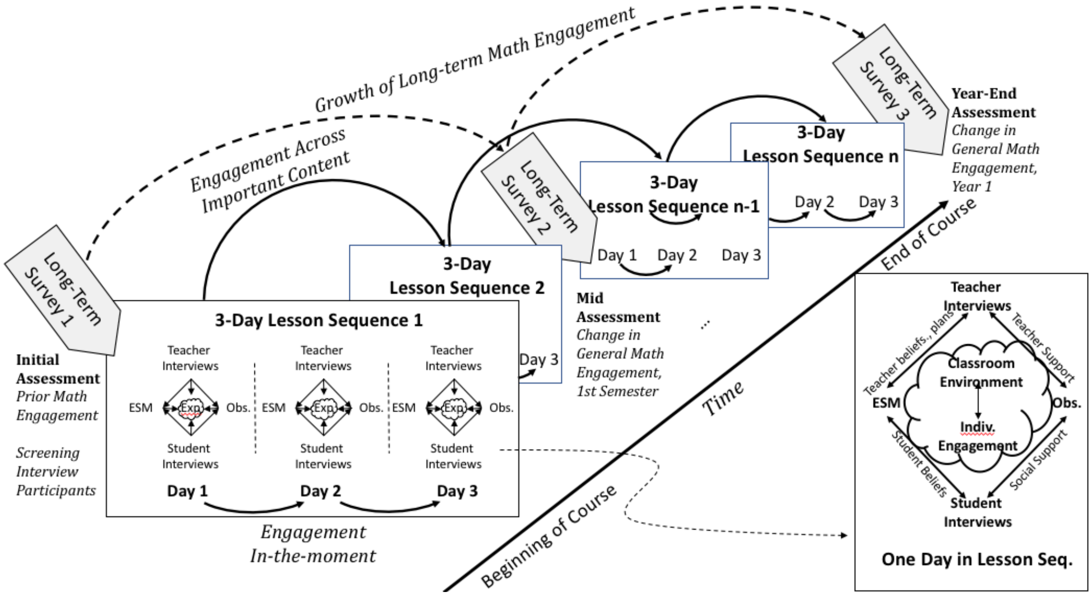

Figure 1. Mixed methods model for studying student engagement over time in one classroom.

Note: In our current project, some classes use a block schedule, completing a course in one semester. Others follow the traditional year-long course model. The number of lesson sequences and long-term survey administrations vary accordingly, with fewer administrations in the block scheduled model.

Data collection then focuses in on a series of pedagogically relevant experiences during a given class period, often identified in advance by the teacher and researcher (a dilemma described further below). During each of these experiences, a series of measurements—ESM surveys, observation, student interviews, and teacher interviews—are taken in parallel in a convergent design (e.g., Creswell & Clark, 2018), characterizing the student's engagement in all of its nuances. By this view, engagement in the moment in Ms. Smith's freshman algebra class is determined by triangulating the teacher's thoughts on the observed session with researcher observations and videos of each experience, with students' in-the-moment ESM reports of their affect and behavior, and finally, with retrospective interviews asking the students about their impressions of each experience. Each of these data points measures and helps describe a given experience and students' engagement in it. Over time, this can uncover the potential factors of classroom practice that *cause* change in engagement, or that *support* engaged behavior over a relatively long period of time, such as a course or academic year. Such mixed methods can characterize the moments of mathematics engagement, provide an explanatory narrative of their development, and estimate their effect size.

Dilemma 2: What Counts As A Moment Or Experience?

When investigating students' engagement, grain size matters. Doing mathematics in school can be viewed as a set of experiences strung together by time, topic, practices, and roles. Thus, an experience could be an entire school year or semester, a single class period, an activity during a class period, or a mathematics problem within an activity. Many studies at a larger scale focus on the longer-term, such as a course, semester, or year (e.g., Skinner & Belmont, 1993). Smaller scale studies focus on the class period, activity, or task-level (e.g., Gresalfi, 2009). The more fine-grained the timescale, the more detailed insights can be gained about engagement. But to understand how to impact students' engagement, whether long-term or short-term, researchers must record the conditions giving rise to engagement patterns, the engagement patterns themselves, and finally, the ways in which students conceptualize their engagement and use it to direct future mathematics-related activity.

By experience, then, we mean an interactive situation (with mathematics content and with others in the classroom) structured by some mathematical task, *cued* by the teacher or students. A *cue* is a social act proffered by a person that indicates an *intended shift in the group interactive behavior* in the mathematics task. Experiences could be operationalized in the context of a teacher cueing, such as launching or introducing a task with various forms of participation, such as having students work on a task (in groups or alone), or discuss a task in small groups, with partners, or as a whole class. Experiences can could also be cued by a student's query about a mathematics concept or procedure.

Yet, this raises another question: is the experience bounded by an activity or by a mathematics problem? We could consider students' engagement during an entire activity, such as the timeframe when students are working together in a small group, as an "experience." Alternatively, we could monitor students' engagement problem-by-problem when working in small groups, considering each problem to be an "experience" in itself.

Ultimately, our definition of experience fundamentally shapes our research methods. For instance, we can take a problem-centered approach by using an ESM to ask participating students to reflect upon the same experience at the same grain size. We can ask for such input either at the end of a given problem, or an entire group task—reflecting either a problem-centric or more holistic approach to the "experience." Such methods could support comparison of students' perceptions across particular experiences, as well as researchers' attempts to track the coincidence of behaviors and reported feelings across particular experiences.

We can also have students or teachers participate in a video viewing session to reflect back on students' engagement, using criteria to select a particular moment in the video record to reflect upon that participants will recognize as an experience, with some beginning, end, and flow of activity. The selection of a clip for a video viewing session can be based on a range of criteria. A focal experience could be one that the teacher conjectured would be productive for students' learning and engagement. Alternatively, researchers could choose a moment that they conjecture could be engaging (or not) for students based on students' displayed affect, and ask the teacher and students to reflect back on that moment.

Each of these potential selection criteria reveal different information regarding what the engagement patterns of the experience are, and what the causal impact of teacher behaviors, mathematics curriculum, and the social setting might be. Defining criteria for selection of experiences that matter, then, must be done carefully, with clear theoretical guidance to generate coherent epistemological claims.

Dilemma 3: What Sorts of Experiences are Engaging?

Following Shernoff et al. (2016), we conjecture that engaging experiences are situated in a learning environment that includes both academic and social support. It is worth investigating the degree to which mathematics learning environments provide students with (a) opportunities to engage in sense-making and reasoning and (b) opportunities to experience positive social interactions. Figure 2 presents hypotheses of classrooms that are likely to result in very different patterns of engagement among students.

A. High sense-making & reasoning Negative social interactions	B. High sense-making & reasoning Positive social interactions
C. Low sense-making & reasoning Negative social interactions	D. Low sense-making & reasoning Positive social interactions

Figure 2. Four patterns of academic and social support

One hypothesis is that when classrooms provide students with academic support through opportunities to engage in sense-making and reasoning (as in A. and B. in Figure 2), students will be more engaged with learning mathematics. For example, prior work has suggested that learning environments that promote making sense of challenging mathematics and opportunities to reason mathematically engage learners in developing mathematical understandings, and, in turn, raise students' self-efficacy, interest, and mastery goals (Stipek et al, 1998).

Another hypothesis is that when classrooms provide students with social support through relatedness (as in B. and D. in Figure 2), students will be more engaged with learning mathematics. Learning environments that are warm and welcoming such that they promote positive social relationships between the teacher and students, and among the students, allow students to feel safe to take intellectual risks and to develop positive emotional well-being (Stipek et al, 1998). Students will then exert more effort spent toward learning mathematics.

Realistically, most classrooms fall in the in-between space of academic and social support, such that sense making is pretty good at some times for some students, and pretty poor at others times for others. Social support may vary likewise at some times and for some students. This makes investigating the dynamics of the pedagogical and social situating of mathematics classrooms so critical for the study of engagement.

To investigate the degree to which such learning environments are engaging, a number of questions could be pursued, such as: How prevalent are learning environments that are high in sense-making and reasoning in these schools; how prevalent are learning environments that focus on positive social interactions in these schools; and how does this relate to engagement? Although learning environments that are high in sense-making and reasoning as well as positive interactions should lead to greater engagement, and learning environments that are low in sense-making & reasoning and have negative social interactions should lead to lower engagement, what does students' motivation and engagement look like in learning environments that are high in sense-making & reasoning and negative in social interaction OR in learning environments that are low in sense-making and reasoning and positive in social interaction?

Dilemma 4: Whose Perspective On The Experience Should Be Privileged?

Multiple perspectives on a school mathematics experience are likely to provide broader insights not only about how or why an experience is engaging, *but for whom*. Nevertheless, consulting different viewpoints also offers the opportunity for discord. When different viewpoints depict differing accounts of who is engaged in the classroom, whose account should be given precedence: our own observations as researchers or students' self-reports?

One answer is to focus on students' own perspectives (in contrast to researchers' perspectives). This seems defensible; even though engagement has dynamic components and is influenced by classmates and teachers in the classroom, engagement is fundamentally a personal, psychological phenomenon. However, the different levels of analysis we prioritize (whole class, a small group, individual student, interactions between these), have implications for the content of interview questions and on the conclusions we are able to draw. For example, in his high school case study of goal development, Webel (2013) found that although engagement behaviors were relatively stable at the group level, they varied considerably at the individual level. Each student's goal-seeking behaviors were expressed differently based on the extent to which they perceived a match between their own goals and that of the group. Had perceived goal match not been a measure in Webel's work, this relationship would have been obfuscated. This suggests a need to capture a broad array of student perceptions, both at the group and individual levels.

Another dilemma that comes with prioritizing students' impressions of engagement is which temporal student account to prioritize. In the mixed methods approach we advocate above, for example, researchers can choose to prioritize in-the-moment assessments, such as the ESM, or retrospective reports, such as student interviews. Each of these methods have been used in meaningful ways across different studies. For example, research using ESMs has suggested that high school students experience more concentration, but less interest and enjoyment when they are in class compared to other places (Shernoff 2013), while research using semi-structured interviews has suggested that seventh-grade students' beliefs about participation (e.g., beliefs constraining or supporting it) influenced the goals they had during participation (e.g., to help classmates and behave appropriately, or to demonstrate competence and complete tasks, respectively) (Jansen, 2006).

Moreover, within the context of a single study, it is possible that a student's in-the-moment impression of an experience may be different than her after-the-fact account. A quiz that seemed difficult in the moment may seem easy retrospectively once the student learns that she had earned an A. What then? Prior research from other domains has suggested that in some cases, willingness to re-engage in an activity (such as an invasive medical procedure) is better predicted by a person's after-the fact impressions than in-the-moment impressions (Shiffman, Stone, & Hufford, 2008). Such discoveries require the measurement of both in-the-moment and later impressions to determine how such memories are consolidated. This suggests yet another utility of using mixed methods, as well as interesting possibilities that may arise from comparing the results of different measurements on educational and persistence-related outcomes of interest, such as grades, and desire to continue in math or pursue a STEM career.

Conclusions

Through this essay, we explored a range of considerations for researchers who are interested in the study of secondary students' engagement in school mathematics classrooms. We reflect upon a current project that attempts to address these considerations. A premise guiding this work is that students' voices should be solicited through multiple methods: long-term surveys of tendencies, ESM surveys, and interviews. We urge for the study of engagement to go beyond observational methods so that students can share the degree to which they experienced the observed incident as engaging. The dilemmas we explore support research that examines engagement in the moment but in ways that also explore trends over time and allows for uncovering variations within individual students as well as between them.

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References

Chouinard, R., & Roy, N. (2008). Changes in high-school students' competence beliefs, utility value and achievement goals in mathematics. *British Journal of Educational Psychology, 78*, 31–50.

Creswell, J. W., & Clark, V. L. P. (2018). *Designing and conducting mixed methods research*. Thousand Oaks, CA: Sage Publications.

Esmonde, I. (2009). Mathematics learning in groups: Analyzing equity in two cooperative activity structures. *Journal of the Learning Sciences*, *18*, 247–284.

Fredricks, J., Blumenfeld, P., & Paris, A. (2004). School engagement: Potential of the concept, state of the evidence. *Review of Educational Research, 74*(1), 59-109.

Goldin, G. A. (2002). Affect, meta-affect, and mathematical belief structures. In G. Leder, E. Pehkonen, & G. Torner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 59–72). Dordrecht, The Netherlands: Kluwer.

Goldin, G. A. (2014). Perspectives on emotion in mathematical engagement, learning, and problem solving. In R. Pekrun & L. Linnenbrink-Garcia (Eds.), *Handbook of Emotions in Education* (pp. 391–414). New York, NY: Taylor & Francis.

Gresalfi, M., Martin, T., Hand, V., & Greeno, J. (2009). Constructing competence: An analysis of student participation in the activity systems of mathematics classrooms. *Educational Studies in Mathematics, 70*(1), 49-70

Jansen, A. (2006). Seventh graders' motivations for participating in two discussion-oriented mathematics classrooms. *The Elementary School Journal, 106*(5), 409–428.

Marks, H. M. (2000). Student engagement in instructional activity: Patterns in the elementary, middle, and high school years. *American Educational Research Journal*, *37*(1), 153-184.

Middleton, J., Jansen, A., & Goldin, G. (2017). The complexities of mathematical engagement: Motivation, affect, and social interactions. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 667-699). Reston, VA: NCTM.

Patrick, H. & Middleton, M. J. (2002) Turning the kaleidoscope: What we see when self-regulated learning is viewed with a qualitative lens, *Educational Psychologist, 37*(1), 27-39.

Pinxten, M., Marsh, H. W., De Fraine, B., Van Den Noortgate, W., & Van Damme, J. (2014). Enjoying mathematics or feeling competent in mathematics? Reciprocal effects on mathematics achievement and perceived math effort expenditure. *British Journal of Educational Psychology*, *84*(1), 152-174.

Rimm-Kaufman, S., Baroody, A., Larsen, R., Curby, T., & Abry, T. (2015). To what extent do teacher–student interaction quality and student gender contribute to fifth graders' engagement in mathematics learning? *Journal of Educational Psychology, 107*(1), 170-185.

Shernoff, D. J. (2013). Measuring student engagement in high school classrooms and what we have learned. *Optimal Learning Environments to Promote Student Engagement*. New York, NY: Springer New York.

Shernoff, D. J., Csikszentmihalyi, M., Schneider, B., & Shernoff, E. S. (2003). Student engagement in high school classrooms from the perspective of flow theory. *School Psychology Quarterly, 18*(2), 158-176.

Shernoff, D. J., Csikszentmihalyi, M., Schneider, B., & Shernoff, E. S. (2014). Student engagement in high school classrooms from the perspective of flow theory. In *Applications of Flow in Human Development and Education* (pp. 475-494). Dordrecht, The Netherlands: Springer Netherlands.

Shiffman, S., Stone, A.A., & Hufford, M.R. (2008). Ecological momentary assessment. *Annual Review of Clinical Psychology, 4*, 1-32.

Skinner, E. A., & Belmont, M. J. (1993). Motivation in the classroom: Reciprocal effects of teacher behavior and student engagement across the school year. *Journal of Educational Psychology*, *85*(4), 571-581.

Stipek, D., Salmon, J. M., Givvin, K. B., Kazemi, E., Saxe, G., & MacGyvers, V. L. (1998). The value (and convergence) of practices suggested by motivation research and promoted by mathematics education reformers. *Journal for Research in Mathematics Education*, *29*(4), 465-488.

Wang, M.-T., Fredricks, J. A., Ye, F., Hofkens, T. L., & Linn, J. S. (2016). The math and science engagement scales: Scale development, validation, and psychometric properties. *Learning and Instruction, 43*, 16–26.

Webel, C. (2013). High school students' goals for working together in mathematics class: Mediating the practical rationality of studenting. *Mathematical Thinking and Learning, 15*(1), 24–57.

Wigfield, A., Eccles, J., Mac Iver, D., Reuman, D., & Midgley, C. (1991). Transitions at early adolescence: Changes in children's domain-specific self-perceptions and general self-esteem across the transition to junior high school. *Developmental Psychology, 27*, 552 – 565.

USES OF COORDINATE SYSTEMS: A CONCEPTUAL ANALYSIS WITH PEDAGOGICAL IMPLICATIONS

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Conventional coordinate systems are often considered representational tools for reasoning about mathematical concepts. However, researchers have shown that students experience persistent difficulties as they engage in graphing activity. Using examples from research and textbooks, we present a framework based on a conceptual analysis of the use of coordinate systems. We discuss the implications of the framework for student learning, curriculum design, and teaching.

Keywords: Cognition, Curriculum Analysis, Modeling

Conventional coordinate systems, such as the Cartesian and polar coordinate systems, are often considered representational tools for reasoning about mathematical concepts such as number systems, geometric figures, ratios and proportional relationships, and equations and functions (e.g., Common Core State Standards for Mathematics). Not only are coordinate systems used in the learning, teaching, and doing of mathematics, they are also commonly used in other fields like science, technology, and engineering as a means to communicate information (Paoletti et al., 2016; Roth, Bowen, & McGinn, 1999; Rybarczyk, 2011).

Despite a widespread use of coordinate systems, researchers have shown that students experience persistent difficulties as they engage in graphing activity—constructing and interpreting graphs—in their mathematics (e.g., Leinhardt, Zaslavsky, & Stein, 1990) and science courses (Potgieter, Harding, & Engelbrecht, 2008). For instance, when constructing graphs, students struggle with establishing axes and scales and transitioning from discrete graphs to continuous graphs (Herscovics, 1989), connect points without considering what happens between points (Yavuz, 2010), and adhere to familiar forms of regularity like linearity (Leinhardt et al., 1990). When interpreting graphs, students show common difficulties such as treating graphs as literal representations of a situation (e.g., interpreting a graph of a biker's speed vs. time as the biker's traveled path) (Clement, 1989; Oehrtman et al., 2008), and focusing on one quantity while tacitly overlooking the other quantity when prompted to interpret relationships between two quantities (Leinhardt et al., 1990; Oehrtman et al., 2008).

A frequently overlooked aspect accounting for students' persistent difficulties in graphing activity is students' understanding of underlying coordinate systems. Researchers and curriculum developers often take coordinate systems for granted as a primitive structure. Little focus is given to students' construction of coordinate systems or their understanding of what is being represented within coordinate systems. For example, most prior research on students' understandings of graphs presumes the Cartesian plane as a given structure in students' reasoning without exploring what meanings students hold for the underlying coordinate system (e.g., Levenberg, 2015). In this report, we outline a framework distinguishing two uses of coordinate systems: situational coordination and quantitative coordination. This explicit distinction has received little, if any, attention in extant mathematics education research and curricula. By bringing attention to the underlying coordinate systems on which students are asked to reason, we intend to provide an explanatory construct as to why students may have difficulty constructing or interpreting graphs represented on coordinate systems. We discuss implications our framework could have and pose future research directions.

Conceptual Analysis

The framework we propose is based on our conceptual analysis (Thompson, 2008) of students' potential uses of coordinate systems as representational tools. Our conceptual analysis is informed by research on spatial cognition (e.g., Levinson, 2003; Tversky, 2003), Thompson's (2011) theory of quantitative reasoning, research examining students' construction and use of coordinate systems (e.g., Lee, 2016; Lee & Hardison, 2016; Lee, 2017), research examining students' graphing understandings (e.g., Moore, Paoletti, & Musgrave, 2014; Joshua, Hatfield, Musgrave, & Thompson, 2015), and our experiences working with students.

Two Contrasting Examples

To illustrate our distinction between two kinds of coordinate systems, we present two tasks from textbooks (Figures 1 and 2). Both tasks involve a Ferris Wheel context and, from our perspective, both tasks involve establishing a coordinate system. Task A (Figure 1) prompts students to find the coordinates of the car located at the loading platform and in other positions, with the axle of the Ferris Wheel defined to be at the origin. In this case, students are asked to coordinate the location of each car within the space in which the Ferris Wheel is *situated*. In other words, the coordinate system in this task is used for spatially organizing the location of each car in reference to the position of the axle of the wheel.

31. **Entertainment** The Ferris Wheel first appeared at the 1893 Chicago Exposition. Its axle was 45 feet long. Spokes radiated from it that supported 36 wooden cars, which could hold 60 people each. The diameter of the wheel itself was 250 feet. Suppose the axle was located at the origin. Find the coordinates of the car located at the loading platform. Then find the location of the car at the 90° counterclockwise, 180°, and 270° counterclockwise rotation positions.

Figure 1. Task A (Holliday, Cuevas, McClure, Carter, & Marks, 2006, p. 95).

1. *Ferris Wheel Problem:* When you ride a Ferris wheel, your distance, $y(t)$, in feet from the ground, varies sinusoidally with time t, in seconds since the wheel started rotating. Suppose that the Ferris wheel has a diameter of 40 ft and that its axle is 25 ft above the ground (see Figure 3-8j). Three seconds after it starts, your seat is at its high point. The wheel makes 3 rev/min.
   a. Sketch the graph of function y. Figure out the particular equation for $y(t)$.
   b. Write an equation for $y'(t)$.
   c. When $t = 15$, is $y(t)$ increasing or decreasing? How fast?
   d. What is the fastest $y(t)$ changes? Where is the seat when $y(t)$ is changing the fastest?

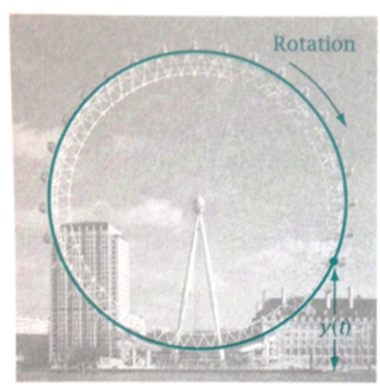


Figure 3-8j

Figure 2. Task B (Foerster, 2005, p. 112).

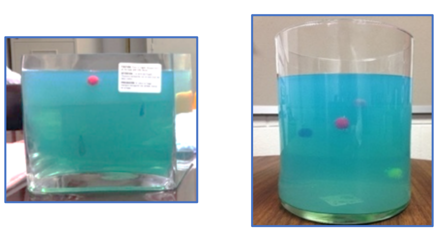
Now consider Task B (Figure 2). Rather than asking students to use a coordinate system to spatially represent the situation, Task B prompts students to graph the relationship between the time elapsed since the wheel started rotating and a rider's distance from the ground. To solve this task, students must extract two quantities (Thompson, 2011), time and distance, from the Ferris Wheel situation and coordinate them in a new space in order to produce a graph. This new space does not entail the spatial situation from which the quantities were extracted. In other words, the Ferris Wheel situation is *not* contained in the graph students are asked to produce or in the coordinate system containing this graph.

Two Uses of Coordinate Systems

As demonstrated through the contrasting examples above, coordinate systems can serve different purposes, which potentially lead to different graphing activities. We propose a framework to distinguish between two uses of coordinate systems, *situational coordination* and *quantitative coordination*. We emphasize we are distinguishing uses of *coordinate systems*, which are different from distinctions others have made about students' *graphing* activities (e.g., Moore & Thompson, 2015); coordinate systems can, but need not, contain graphs. We describe each use of coordinate systems with examples from our research and from textbooks; textbook examples are used to illustrate our interpretations of the textbook author's intended use of coordinate systems, which do not necessarily coincide with how students might perceive of the coordinate system. Through hypothetical examples, we also illustrate how a student might engage in graphing activity in each case.

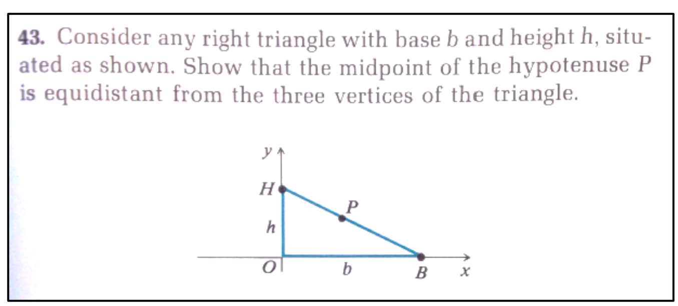
Situational Coordination

Situational coordination refers to an individual using a coordinate system to represent or mathematize a space or physical phenomena, as in Task A (Figure 1). Constructing a coordinate system for situational coordination (i.e., constructing a *situational coordinate system*), involves establishing frames of reference (e.g., Levinson, 2003) to gauge extents of various attributes of objects (e.g., relative location or orientation of an object) within the space or phenomena. When constructing a situational coordinate system, an individual can produce quantities by measuring attributes of the space/situation using their frames of reference and coordinate such measurements to represent attributes of objects *in the space or situation*. An everyday example of a situational coordinate system is used is a map. Graphs constructed on situational coordinate systems can be viewed as projections or traces of physical objects or phenomena onto the *same space* containing the original objects or phenomena (e.g., the movement of a car shown on a GPS).



Locate the four fish in the fish tank models.

Task C: Fish Tank Task (Lee, 2017)



43. Consider any right triangle with base b and height h, situated as shown. Show that the midpoint of the hypotenuse P is equidistant from the three vertices of the triangle.

Task D: (Keedy & Bittinger, 1981, p. 159)

Figure 3. Two Examples of Coordinate Systems Used for Situational Coordination.

As an example of a task designed for students to construct a situational coordinate system, consider Task C (Figure 3) from the first author's research involving students' organization of space. In this task, the researcher asked students to locate four fish figures in three-dimensional tanks. The students' responses to this task (see Lee, 2017) provide an example of how a student might construct a coordinate system to represent objects in space by establishing a frame of reference and locating points within the space using coordinated measurements. As a second example, consider Task D from a college algebra text (Keedy & Bittinger, 1981, p. 159), which asks students to reason about a geometric figure and its characteristics situated in a coordinate system. Here, the x-axis, y-axis, and origin suggest a frame of reference used to locate the points or line segments within the geometrical figure (i.e., triangle). Tasks C and D exemplify situational coordinate systems as the coordinate system is (or can be) used to locate or mathematize objects (e.g., the fish in Task C or right triangle in Task D) in a given space.

The examples above highlight how situational coordinate systems can be used to mathematize a single moment in time (i.e., a snapshot). Situational coordinate systems can be used when representing dynamic situations as well. In other words, a student may think of overlaying a frame of reference on an imagined movie. For example, in Task A, one can observe or imagine a rider's position on the Ferris Wheel changing over time within a situational coordinate system (e.g., Williams, 2018). Similarly, in Task C, one can observe or imagine a fish's position in the fish tank changing over time within a situational coordinate system.

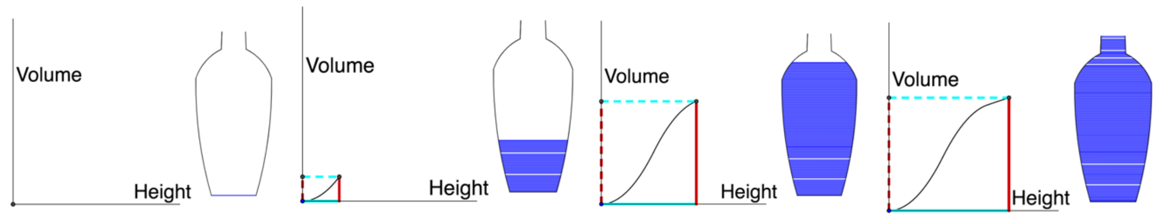
When constructing a situational coordinate system, one can establish and insert a multitude of quantities onto the situation. We have found it helpful to imagine a situational coordinate system as mentally tagging an object with an n-tuple and imagining the values within the n-tuple changing over different situational instances. For example, for a Ferris Wheel car's location in Task A, one can mentally tag a *4*-tuple such as (θ, h, a, t), where θ represents the angle through which a car on the Ferris wheel has rotated, h represents the height of the car, a represents the arc length the car has traveled, and t represents the time the car has been in motion. The quantities one inserts onto the situation need not be purely geometric or temporal in nature, as in Tasks A, C, and D. For example, an individual might also consider inserting the blood pressure of the rider in the car into the situation and therefore the tuple. The critical distinguishing feature of a situational coordinate system is that frames of reference are established and used to construct and tag quantities *onto* the situational space or physical phenomena.

Quantitative Coordination

Quantitative coordination refers to an individual using a coordinate system to coordinate sets of quantities by obtaining a geometrical representation of the product of measure spaces, such as in Task B above. To establish a quantitative coordinate system, an individual must establish quantities within the given space/situation, disembed (Steffe & Olive, 2010) these quantities (i.e., extract them from the situation while maintaining an awareness of the quantities within the situation), and project them onto some new space, which is different from the space in which the quantities were originally conceived. Graphs as emergent traces (Moore & Thompson, 2015) representing relationships between disembedded quantities can be constructed within a quantitative coordinate system. These graphs are not projections of physical objects or phenomena from the same space containing the original objects or phenomena.

Consider Task E (Paoletti & Moore, 2017), a variation of Swan's (1985) bottle problem, which has been used to examine students' graphing activity (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). In this problem, students are asked to imagine a bottle being filled with liquid and sketch a graph relating the volume of liquid in the bottle and height of liquid in the bottle. To solve this task, a student must first conceive of two quantities, liquid volume and liquid height. The student may then consider how the two quantities vary as liquid is poured into the bottle.

Next, the student may conceive of two number lines, one for liquid height and one for liquid volume, and use these number lines to represent the varying magnitudes or values of each quantity. The student can then construct a Cartesian plane with the two number lines—the horizontal axis representing liquid height and the vertical axis representing liquid volume—and imagine a point in this space as simultaneously representing the liquid height and liquid volume as water is poured into the bottle (Figure 4a-d). The two-dimensional space that is made by the product of the two axes form a new space (a {height×volume} plane), which is different from, but related to, the space containing the bottle itself. The infinitude of traced points (height, volume) in the graph (Figure 4d) do not depict the physical bottle being filled but represents the covarying quantities.



(a) (b) (c) (d)

Figure 4. Task E: Representing the height and volume of liquid in a bottle.

For an individual to construct a quantitative coordinate system she must first establish a situational coordinate system in terms of the relevant quantities in the imagined situation before disembedding these quantities to create a quantitative coordinate system (Lee & Hardison, 2016). For example, in Task B, in order for a student to coordinate the time traveled and vertical distance of car from the ground, she will first need to establish a situational coordinate system on the Ferris Wheel situation (as in Task A) through which she could gauge the time traveled and vertical distance from the ground before disembedding those quantities into a new space. Similarly, in Task E, a student will first need to establish a situational coordinate system on the bottle situation through which she could gauge the liquid height and liquid volume as the water fills the bottle.

Implications of the Framework and Future Research Directions

We encourage researchers to focus their attention on the two uses of coordinate systems when studying students' construction of coordinate systems and also to use this framework as an analytical tool in future studies examining students' graphing activity. Findings from such studies can support curriculum development and classroom instruction related to coordinate systems and students' graphing activities, which we further discuss in the sections below.

Using the Framework to Better Understand Students' Graphing Activities

The proposed framework may help explain some of the students' challenges in constructing or interpreting graphs represented on coordinate systems as reported in the literature. That is, identifying whether a student is aware of the purpose of the coordinate system he has constructed and/or is reasoning upon can provide insight for his graphing activity. For example, when a student constructs axes, examining whether he is (a) attempting to quantitatively organize a particular situation, (b) representing magnitudes of quantities abstracted from an outside situation, or (c) attempting to reproduce an image from their prior classroom instruction can be informing. Similarly, for a student who draws graphs by connecting points without considering what happens between points (Yavuz, 2010), it could be insightful to investigate whether the student understands points as representing the coupling of two quantities (Saldanha & Thompson, 1998) disembedded from the situation.

The lack of explicit attention to the two uses of coordinate systems can become problematic when a student interprets what a teacher or researcher intends to be a quantitative coordinate system as a situational coordinate system. In such a case, the student will be constrained, at best, to reasoning about points and lengths in graphs as opposed to reasoning about the contextualized quantities outside of the perceptually available graph. Such an interpretation may explain why researchers report students treating graphs as literal representations of a situation (e.g., interpreting a time-speed graph of a biker as the biker's traveled path) (Clement, 1989) or why some students have difficulty "distinguishing between visual attributes of a physical situation and similar perceptual attributes of the graph of a function that models the situation" (Oehrtman et al., 2008, pp. 153).

Looking forward, we intend for our conceptual analysis to be a tool for empirical research examining students' construction of coordinate systems and graphing activities. Furthermore, because a quantitative coordinate system presupposes some situational coordinate system, we hypothesize that many of students' difficulties interpreting and constructing graphs within quantitative coordinate systems can be explained by students' challenges in constructing situational coordinate systems. Even if an individual is capable of constructing a situational coordinate system, transitioning to a quantitative coordinate system is nontrivial. If an individual has constructed a situational coordinate system in the bottle problem, for example, establishing a quantitative coordinate system requires transforming a volume in the situational coordinate system to a directed length on a number line; this transformation cannot be taken for granted when teaching and researching students' graphing activity. As such, we argue students need experiences constructing situational coordinate systems prior to constructing quantitative coordinate systems and we echo others' calls (e.g., Thompson & Carlson, 2017) for more research on how students map quantities onto number lines.

Using the Framework to Reflect on Curriculum and Teaching

Our framework can be used to analyze mathematical tasks. For example, students' thinking of a graph as a picture of a physical situation may be unsurprising when students are, at times, presented with graphs that seem to conflate the two uses of coordinate systems. In Task F (Figure 5), we see a quantitative coordinate system used for relating time and distance. Yet, the use of arrows and labels in the text is problematic from our perspective. The stoplight, school, and school zone are *not* represented physically on this graph as the arrows seem to indicate. These objects exist in a space different from the coordinate plane established by the quantities time and distance. We hypothesize that conflations between quantitative and situational coordinate systems in mathematical tasks can lead to students' difficulties in graphing activity. In analyzing tasks, we also noticed that many graphing exercises in textbooks did not explicitly include quantitative referents for axes, which is consistent with the analysis of pre-calculus and calculus textbooks reported by Paoletti et al. (2016). They found that popular pre-calculus and calculus textbooks almost exclusively used graphs to represent decontextualized functions of x and y. Absent of any quantitative referents, students are likely left with little choice but to interpret these decontextualized graphing tasks in terms of situational coordinate systems. This is problematic as a common way STEM fields communicate information is through graphical representations of two (or more) covarying contextualized quantities.

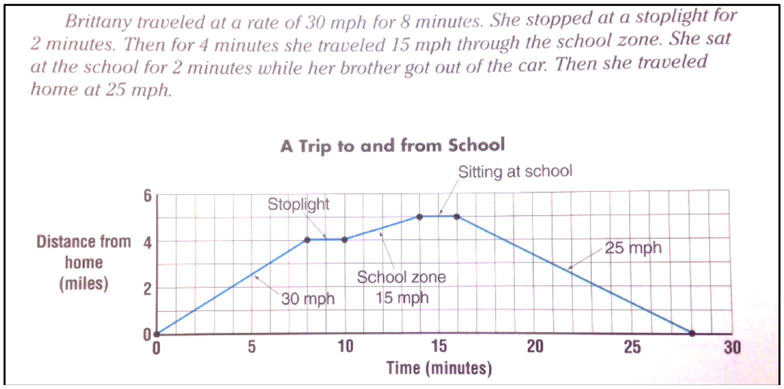


Figure 5. Task F: (Holliday, Cuevas, McClure, Carter, & Marks, 2006, p. 95)

Also relevant to our distinction, Paoletti et al. (2016) found that although many STEM fields (e.g., Biology, Chemistry, medicine) often use quantitative coordinate systems to represent two or more covarying quantities in their practitioner journals and textbooks, other fields (e.g., engineering, physics) commonly employ situational coordinate systems to mathematize a situation or phenomena. This difference between the use of coordinate systems in pre-calculus and calculus textbooks and other STEM fields may help explain why students often do not see connections between the mathematics they learn in mathematics classes and the mathematics they use in STEM courses (Britton et al, 2005). We encourage mathematics educators to provide students with opportunities to develop a balanced understanding of both uses of coordinate systems because it is important for their mathematical development as well as potential future STEM courses and careers.

References

Britton, S., New, P. B., Sharma, M. D., & Yardley, D. (2005). A case study of the transfer of mathematics skills by university students. *International Journal of Mathematics Education in Science and Technology, 36*(1), 1-13.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 352–378.

Clement, J. (1989). The concept of variation and misconceptions in Cartesian graphing. *Focus on Learning Problems in Mathematics, 11*(1-2), 77-87.

Foerster, P. A. (2005). Calculus: Concepts and applications. Key Curriculum Press. Emeryville, CA.

Herscovics, N. (1989). Cognitive obstacles encountered in the learning of algebra. In S. Wagner, C. Kieran (Eds.), *Research issues in the learning and teaching of algebra*, Vol. 4 (pp. 60–86). Hillsdale, NJ Reston, VA EnglandUS: Lawrence Erlbaum Associates, Inc.

Holliday, B, Cuevas, G..J, McClure, M. S., Carter, J. A., & Marks, D.. (2006). Advanced mathematical concepts: Precalculus with applications. Glencoe/McGraw-Hill.

Joshua, S., Musgrave, S., Hatfield, N., & Thompson, P. W. (2015). Conceptualizing and reasoning with frames of reference. In T. Fukawa-Connelly, N. E. Infante, K. Keene & M. Zandieh (Eds.), *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education* (pp. 31–44). Pittsburgh, PA: RUME.

Keedy, M. L., & Bittinger, M. L. (1981). Fundamental college algebra. Addison Wesley Publishing Company.

Lee, H. (2016). Just go straight: Reasoning within spatial frames of reference. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 278–281). Tucson, AZ: The University of Arizona.

Lee, H. (2017). Students' construction of spatial coordinate systems (Unpublished doctoral dissertation). University of Georgia, Athens, GA.

Lee, H., & Hardison, H. L. (2016). Spatial coordination as a prerequisite for representing quantitative coordination in two dimensions. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual*

meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (p. 304). Tucson, AZ: The University of Arizona.
Leinhardt, G., Zaslavsky, O., & Stein, M. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research*, 60(1), 1-64.
Levenberg, I. (2015). Information on coordinate system as a tool for developing mathematical thinking. *International Journal of Contemporary Educational Research, 2*(1), 46–53.
Levinson, S. C. (2003). Space in language and cognition: Explorations in cognitive diversity (Vol. 5). United Kingdom: Cambridge University Press.
Moore, K. C., Paoletti, T., & Musgrave, S. (2014). Emerging Complexities in Students' Construction of the Polar Coordinate System. *The Journal of Mathematical Behavior, 36*, 135–149.
Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In T. Fukawa-Connelly, N. Infante, K. Keene, & M. Zandieh (Eds.), *Proceedings of the Eighteenth Annual Conference on Research in Undergraduate Mathematics Education* (pp. 782–789). Pittsburgh, PA: West Virginia University.
Oehrtman, M. C., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' understandings of function. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics* (pp. 27–42). Washington, DC: Mathematical Association of America.
Paoletti, T., & Moore, K. C. (2017). The parametric nature of two students' covariational reasoning. *The Journal of Mathematical Behavior, 48*, 137–151.
Paoletti, T., Rahman, Z., Vishnubhotla, M., Seventko, J., & Basu, D. (2016). Comparing Graph Use in STEM Textbooks and Practitioner Journals. *Proceedings from the 20th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1386–1392).
Potgieter, M., Harding, A., & Engelbrecht, J. (2008). Transfer of Algebraic and Graphical Thinking between Mathematics and Chemistry. *Journal of Research in Science Teaching, 45*(2), 197–218.
Roth, W. M., Bowen, G. M., & McGinn, M. K. (1999). Differences in graph related practices between high school biology textbooks and scientific ecology journals. Journal of research in science teaching, 36(9), 977–1019.
Saldanha, L., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensah & W. N. Coulombe (Eds.), *Proceedings of the Annual Meeting of the Psychology of Mathematics Education - North America* (pp. 1–7). Raleigh, NC: North Carolina State University.
Steffe, L. P. & Olive, J. (2010). *Children's fractional knowledge*. New York: Springer.
Swan, M., & the Shell Centre Team (1985). *The language of functions and graphs*. Nottingham, U.K.: Shell Centre Publications.
Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sépulveda (Eds.), Plenary Paper presented at the Annual Meeting of the International Group for the Psychology of Mathematics Education, (Vol 1, pp. 31–49). Morélia, Mexico: PME.
Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), New perspectives and directions for collaborative research in mathematics education, WISDOMe Monographs (Vol. 1, pp. 33–57). Laramie, WY: University of Wyoming Press.
Thompson, P. W., & Carlson, M. P. (2017). Variation, Covariation and Functions: Foundational Ways of Thinking Mathematically. In J. Cai (Ed.), *Handbook of Research in Mathematics Education*. Reston, VA: National Council of Teachers of Mathematics.
Tversky, B. (2003). Structures of mental spaces: How people think about space. *Environment and behavior*, *35*(1), 66–80.
Rybarczyk, B. (2011). Visual literacy in biology: A comparison of visual representations in textbooks and journal articles. *Journal of College Science Teaching*, *41*(1), 106–114.
Williams, D. (2018) Relationships between Precalculus students' engagement and shape thinking. *Paper presented at the Twentiy-first Annual Conference on Research in Undergraduate Mathematics Education* . San Diego, CA.
Yavuz, I. (2010). What does a graphical representation mean for students at the beginning of function teaching? *International Journal of Mathematical Education in Science and Technology, 41*(4), 467–485.

STATIC AND EMERGENT THINKING IN SPATIAL AND QUANTITATIVE COORDINATE SYSTEMS

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The body of research examining students' graphing understandings across STEM fields indicates students are not developing productive meanings for graphs. We conjecture such failings may, in part, be explainable by features of students' use of coordinate systems and graphing activity that are under examined. In this theoretical report, we present a conceptual analysis of different ways students may reason about graphs and coordinate systems. Specifically, we describe two different uses of coordinate systems—spatial and quantitative—students might leverage and two ways of reasoning—static and emergent—students might engage in as they construct or interpret graphs. We characterize how a student may engage in each kind of reasoning in each use of coordinate system. We intend this paper to serve as a theoretical lens for future empirical studies examining students' developing graphing understandings.

Keywords: Cognition, Learning Theory, Spatial Thinking

There is a need for U.S. students to enter STEM fields, with mathematics often serving as a "gatekeeper" for student success in these fields (Crisp, Nora, & Taggart, 2009). As such, it is important for students to have experiences in their K–12 schooling that are attentive to the needs of potential STEM coursework and careers. A common way STEM fields communicate information is through graphical representations. Paoletti, Rahman, Vishnubhotla, Seventko, and Basu (2016) analyzed the graphs depicted in commonly used STEM textbooks and practitioner journals, finding the most common uses of graphs were to mathematize a spatial situation or phenomena (e.g., Figure 1, left) or to represent two covarying quantities (e.g., Figure 1, center). In contrast, the researchers noted most graphs in popular precalculus and calculus textbooks were used to represent two decontextualized quantities (e.g., Figure 1, right). Hence, there are discrepancies between the graphs students experience in their math classes and are expected to interpret in other STEM fields.

Researchers have identified persistent difficulties students experience as they engage in constructing and interpreting graphs in their mathematics (e.g., Leinhardt, Zaslavsky, & Stein, 1990) and science courses (e.g., Potgieter, Harding, & Engelbrecht, 2008). For instance, researchers have identified a range of difficulties including (a) drawing graphs by connecting points without considering what happens between points (Yavuz, 2010); (b) treating graphs as literal representations of a situation (e.g., interpreting a time-speed graph of a biker as the biker's traveled path) (Clement, 1989); and (c) attending to one quantity while ignoring other quantities (i.e. reasoning variationally) (Leinhardt et al., 1990). Collectively, this research indicates instructional approaches have not provided students sustained opportunities to develop meaningful ways of understanding and interpreting graphs.

Dewey (1910) stated, "vagueness disguises the unconscious mixing together of different meanings, and facilitates the substitution of one meaning for another, and covers up the failure to have any precise meaning" (p. 130). In this report, we attempt to clarify a vagueness, in Dewey's terms, regarding how students may think about graphs and coordinate systems. We hypothesize supporting students becoming explicitly aware of the subsequent distinctions may alleviate some difficulties students experience as they engage in graphing activity. In this theoretical report, we present a conceptual analysis of different ways students may reason about coordinate systems and graphs. Specifically, we describe two different uses of coordinate systems—spatial and quantitative (Lee & Hardison, 2016)—and two ways students might reason as they construct or interpret graphs within these coordinate systems—static and emergent thinking (Moore & Thompson, 2015). For each use of coordinate system, we characterize how a student may engage in each kind of graphical reasoning, creating a framework that will be useful for future studies.

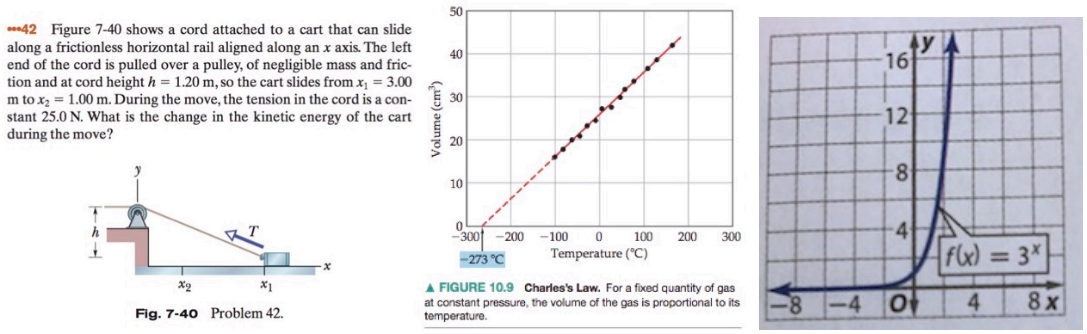

Figure 1. Examples of using coordinate systems to (left) mathematize space (Halliday, Resnick, & Walker, 2011), (center) represent two covarying quantities (Brown et al., 2012), and (right) represent two decontextualized quantities (Glencoe, 2014).

Conceptual Analysis

In this theoretical report, we present a conceptual analysis of ways students may reason about graphs and coordinate systems; we intend for this conceptual analysis to inform future empirical research. Thompson (2008) characterized several uses of conceptual analyses; we leverage two of these uses, namely "describing ways of knowing that might be propitious for students' mathematical learning," and "describing ways of knowing that might be deleterious to students' understanding of important ideas and in describing ways of knowing that might be problematic in specific situations" (p. 46). We present our first-order models (Steffe & Olive, 2010) of how students may reason about graphs and underlying coordinate systems. These models are based on research examining students' graphing understandings (e.g., Moore & Thompson, 2015), research examining students' construction and use of coordinate systems (e.g., Lee, 2017), principles of quantitative reasoning (Thompson, 2011), and our experiences working with students. We point the reader to examples from extant literature in which we infer students are engaging in reasoning compatible with our conceptual analysis.

Bridging Two Theoretical Frameworks

In this report, we examine how a distinction between two uses of coordinate systems and a distinction between two ways of thinking about graphs (i.e., traces within coordinate systems) can create four different ways students may construct or reason about graphical representations within coordinate systems. Below, we describe two frameworks: one for students' understandings of coordinate systems and one for students' thinking about graphs.

Two Uses of Coordinate Systems

Lee and Hardison (2016) found curricular materials often give students rules for "generating" a Cartesian plane and plotting points within it, and these materials rarely address students' conceptions of coordinate systems. They (Lee, 2016; Lee & Hardison, 2016; 2017) have distinguished between two uses of coordinate systems in students' thinking: spatial coordination and quantitative coordination.

Spatial coordination refers to an individual using a coordinate system to represent or mathematize a space or physical phenomena. This involves establishing spatial frames of reference (Levinson, 2003; Rock, 1992), such as a reference point or orienting vectors, to locate objects within the space or physical phenomena (e.g., Figure 1, left). In this case, an individual can produce quantities by measuring attributes of the space/situation using the established frames of reference and thus coordinate such measurements to locate objects in the space or situation (e.g., a map).

Quantitative coordination refers to an individual coordinating sets of quantities by obtaining a geometrical representation of the product of measure spaces. In this case, the quantities being coordinated are already established and abstracted from the space/situation and superimposed onto some new space. This use of coordinate system, as a result of coordinating quantitative frames of reference (Joshua, Musgrave, Hatfield & Thompson, 2015), allows the individuals to coordinate quantities and construct graphs representing relationships between these quantities (e.g., Figure 1, middle). These graphs are not projections of physical objects or phenomena onto the same space containing the original objects or phenomena.

Static and Emergent Shape Thinking

Moore and Thompson (2015, under review) differentiated between students' static and emergent shape thinking. A student's *static shape thinking* entails his thinking of a displayed graph as a shape (i.e. graph-as-wire) that can possibly be manipulated (e.g., translated, rotated). In such a case, properties of the perceptual shape and the shape itself are the focus of the student's thinking. For instance, a student emphasizing the properties of a shape may argue that a straight line moving up from left-to-right unquestionably represents a linear function with a positive slope even if positive *x*-values are represented to the left of the *y*-axis in the Cartesian coordinate system or if the line is represented in the polar coordinate system.

Moore and Thompson also noted static shape thinking can take the form of students' making iconic or thematic associations (e.g., Clement, 1989; Leinhardt et al., 1990). A student reasons iconically when he incorporates visual features of an event in a graph (e.g., drawing a graph resembling a hill because a biker is traversing a hill). A student reasons thematically when he incorporates aspects of a phenomena in his graph that are unnecessary from the researcher's perspective (e.g., an object traveling at a varying speed necessarily implying a curved graph).

In contrast to static shape thinking, Moore and Thompson characterized *emergent thinking* as conceiving a displayed graph simultaneously in terms of "what is made (a trace) and how it is made (covariation)" (2015, p. 785). Critical to such a conception is a student's construction of a point on a graph as a multiplicative object; when using the term multiplicative object, Thompson and colleagues draw on Piaget's notion of "and" as a multiplicative operator. Specifically, Thompson, Hatfield, Yoon, Joshua, and Byerley (2017) noted, "A person forms a multiplicative object from two quantities when she mentally unites their attributes to make a new attribute that is, simultaneously, one and the other" (p. 98). Hence, when reasoning emergently, a student understands a point as simultaneously representing two quantities and imagines a graph being created by the trace of the point as the quantities vary.

A student with sophisticated emergent thinking may, at times, appear to treat a graph as a static shape (e.g., 'shifting' the graph in a direction) for various reasons whilst being able to unpack the static thinking in terms of the graph's emergence. Although such 'shifting' activity could indicate static thinking, if the student is able to unpack his new graph in terms of representing an emergent trace constituted by the new distances, such reasoning would constitute emergent, rather than static, thinking in the quantitative coordinate system.

Static or Emergent Thinking in Spatial or Quantitative Coordinate Systems

In combining the frameworks elaborated above, we differentiate between static and emergent thinking, which characterizes a student's reasoning when producing or interpreting a graph (i.e., a trace within a coordinate system), and quantitative and spatial coordinate systems, which characterizes a student's understanding of the coordinate system potentially containing a graph. We provide an example of each kind of reasoning using a billiard context at the Infinity Pool Hall. Additionally, we provide a description of student reasoning for each case. We reiterate these are first-order models of how students may reason; we point the reader to empirical examples where appropriate.

Table 1: Four ways of reasoning about graphs within coordinate systems.

		Ways of Reasoning About a Graph (Moore & Thompson, 2015)	
		Emergent Reasoning	Static Reasoning
Uses of Coordinate Systems (Lee & Hardison, 2016)	Spatial Coordination	Case A: Emergent thinking within a spatial coordinate system	Case B: Static thinking within a spatial coordinate system
	Quantitative Coordination	Case C: Emergent thinking within a quantitative coordinate system.	Case D: Static thinking within a quantitative coordinate system

Case A: Emergent thinking in a spatial coordinate system (Emergent, Spatial)

To imagine a student reasoning emergently in a spatial coordinate system, consider a student seeing the red 3-ball moving from the left wall to the top middle pocket in a straight line (Figure 2, top). If asked to describe the location of the red ball throughout its journey toward the middle pocket, the student may establish a spatial frame of reference consisting of a horizontal axis and a vertical axis through which the student can gauge the horizontal and vertical locations of the ball. Using this spatial frame of reference, the student can describe the ball's movement in terms of varying horizontal and vertical distances within a spatial coordinate system. By conceiving of the ball's location as simultaneously composed of vertical and horizontal components, the student conceives of the ball's location as a multiplicative object and therefore can reason emergently about the ball's path in terms of its horizontal and vertical components. Figure 2 (bottom) depicts instances of a student's potential emergent imagery when coordinating the ball's trajectory in relation to its horizontal and vertical components.

There are several features critical to a student reasoning emergently in a spatial coordinate system. First the student must conceive of an object or phenomena happening in a spatial system and imagine the object or phenomena as producing a trace in this space. The student then overlays a coordinate system onto the spatial system in order to explicitly coordinate and/or represent how the object or phenomena is producing the imagined trace in terms of the quantities represented in the coordinate system. Hence, the student must conceive of the object or phenomena as representable by a multiplicative object, which she can decompose as simultaneously representing the orienting quantities in the spatial coordinate system. The student can then explicitly coordinate how the orienting quantities are changing as the object moves or phenomena occurs to mathematize the situation. For an empirical example of a student reasoning emergently in a spatial coordinate system, see Lee (2016).

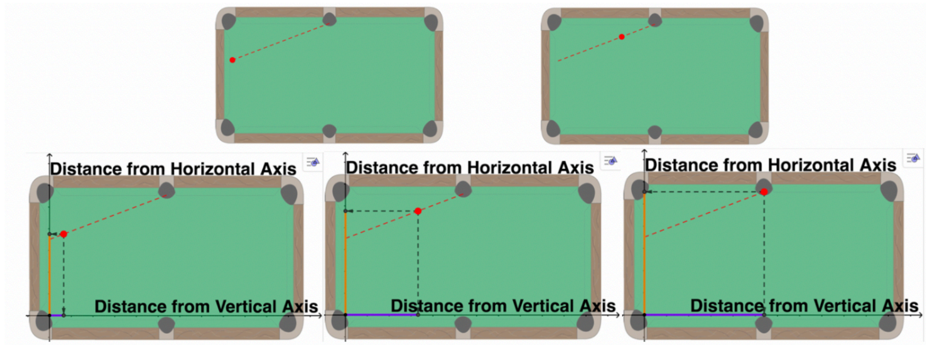

Figure 2. (top) Two instantiations of the red 3-ball traveling from the wall of the table to the pocket and (bottom) three instantiations of how a student may reason about the ball's path as an emergent trace by coordinating its vertical and horizontal components.

Case B: Static thinking in a spatial coordinate system (Static, Spatial)

To imagine a student reasoning statically in a spatial coordinate system, consider the logo on an Infinity Pool Hall table (Figure 3, left). A student sees the logo, which he interprets as composed of two circles tangent at a point (i.e., shapes), and is tasked with describing the shape of this logo mathematically. To do this, the student defines a coordinate system by establishing a spatial frame of reference through which he could describe the location and shape of the logo. This includes choosing a reference point and defining orienting quantities. After constructing the spatial coordinate system, the student may mathematically describe the shapes in the logo within the coordinate system using known equations. For example, in Figure 3 (middle) a student decides to use Cartesian coordinates with the origin at the intersection of the circles, and describes each circle using memorized rules related to the general form of a circle, $(x - h)^2 + (y - k)^2 = r^2$. Alternatively (Figure 3, right), a student might define a polar coordinate system as in Figure 3c, with the pole at the intersection of the circles and use the recalled formula $r = a\cos\theta$.

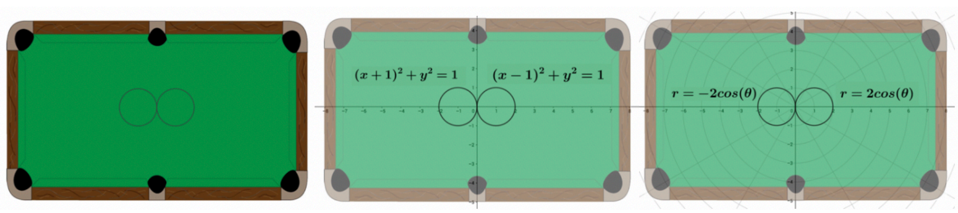

Figure 3. (left) A pool table with logo, and two possible ways to spatially orient the pool table and mathematize the circles in the logo in (middle) Cartesian and (right) polar coordinates.

As exemplified in the above examples, static thinking in a spatial coordinate system entails (a) conceiving of a static shape or object to be located or described mathematically in a situation and (b) establishing a spatial frame of reference through which one can situate, coordinatize, and mathematize the shape. For an example of a teacher using coordinate systems in ways compatible with this description in their classrooms, see Disher (1995).

In this example, we emphasize a student using memorized rules to mathematize the conceived shape, rather than reasoning about an emergent trace. We note that a student *could* demonstrate an understanding of analytic rules as representing an emergent trace of covarying quantities; in such a case, the student would not necessarily be reasoning statically when representing the 'shape' via an analytic rule if the shape's emergence is implicit in his understanding.

Case C: Emergent thinking in a quantitative coordinate system (Emergent, Quantitative)

To imagine a student thinking emergently in a quantitative coordinate system, consider a student seeing the red 3-ball's path, now with the yellow 1-ball and blue 2-ball on the table (Figure 4, we positioned the balls and trajectory to mimic the cities and path of the car in Saldanha and Thompson's (1998) Car Problem). The student may be asked to create a graph representing the red 3-ball's distance from the yellow 1-ball and blue 2-ball as it moves to the pocket. After conceiving the quantities and how they vary, the student may construct a Cartesian coordinate system with the horizontal and vertical axes representing the red 3-ball's distances from the blue 2-ball and yellow 1-ball, respectively. Having constructed a quantitative coordinate system, the student may construct an emergent trace within this coordinate system to represent the relationship between the 3-ball's distance from the other balls (e.g., Figure 4a–d).

We highlight several critical features in this example. First a student needs to conceive of two quantities covarying and intend to represent the relationship between the quantities. The student must then disembed (Steffe & Olive, 2010) the two quantities from the context and insert them onto two number lines to construct a quantitative coordinate system with the intention of simultaneously representing the two quantities in a product space. Each quantity would then be represented by one of the orienting quantities in the coordinate system (see Moore, Paoletti, and Musgrave (2013) for examples in Cartesian and polar coordinate systems). The student understands a point on the emergent trace in this quantitative coordinate system as a multiplicative object simultaneously representing the two quantities.

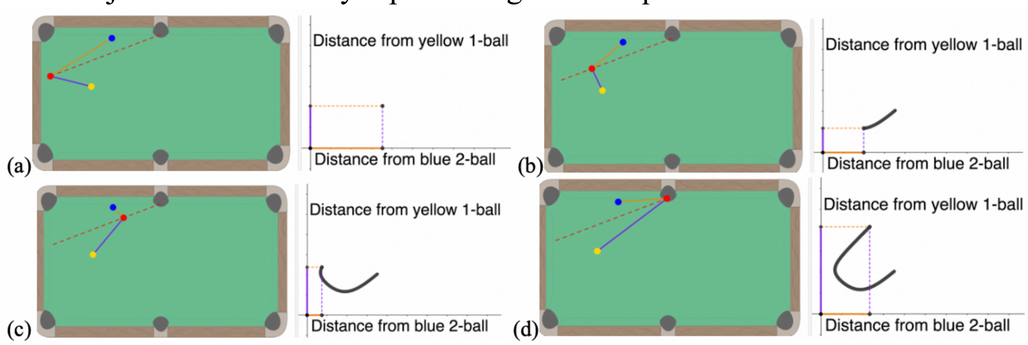

Figure 4. Four instantiations of how an individual may reason about and represent the red 3-ball's distance from the blue 1-ball and yellow 2-ball as an emergent trace.

Case D: Static thinking in a quantitative coordinate system (Static, Quantitative)

We modify the same situation described in Case C to characterize how a student may reason statically while using a quantitative coordinate system. Recall, static thinking may entail graphically representing features of the situation. For instance, Figure 5 (center) shows a student duplicating the 3-ball's path as a representation of the ball's distance from the other two balls, and Figure 5 (right) shows a student using the balls' and pocket's relative locations as points on his graph. In both cases, we highlight that although the coordinate system is meant to represent (from the researcher's perspective) the 3-ball's distance from the 1-ball and 2-ball, the student may not explicitly use the coordinate system in this way. One indication that a student is interpreting the coordinate system quantitatively is if the student interprets her constructed graph by describing the 3-ball's relative distance from the other two balls. For instance, in Figure 5 (center), the student may describe that the 3-ball's distance from the 1-ball increases at a constant rate with respect to its distance from the 2-ball (see Paoletti, 2015 for an empirical example).

Reasoning statically in a quantitative coordinate system can be unproductive in several ways: (a) treating a graph in a quantitative coordinate system as a shape which can be moved around the coordinate system without considering how the translation relates the underlying relationship between quantities, (b) engaging in an iconic translation while interpreting or creating a graph in a quantitative coordinate system, or (c) engaging in a thematic association while interpreting or creating a graph in a quantitative coordinate system. In each case, in order for the system to be a quantitative coordinate system *from the student's perspective*, the student needs to make some indication that he is representing the quantities defined on the coordinate axes.

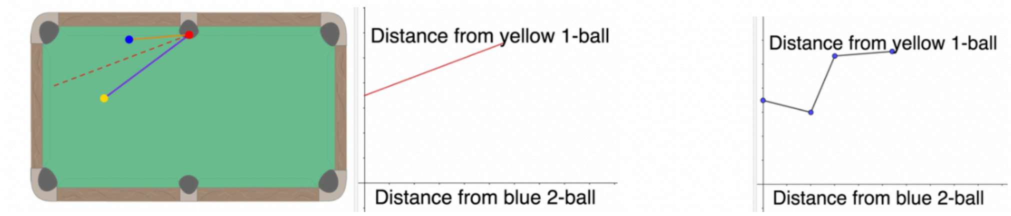

Figure 5. Two examples of static thinking in a quantitative coordinate system.

Concluding Remarks and Areas for Future Research

In this report, we have presented a conceptual analysis that distinguishes meanings students may hold as they engage in graphing activity within coordinate systems. These distinctions can provide insights into some difficulties students experience as reported in extant literature. For instance, students representing the path of a biker (Clement, 1989) may be indicative, not of a misconception, but of the students leveraging a spatial coordinate system to mathematize a situation. In this case, and others, misconceptions identified in the literature may be partially explainable by students' reasoning about coordinate systems and graphs in ways inconsistent with what the researcher (or teacher) intended.

If we intend for curriculum designers, teachers, and students to maintain and convey particular meanings for graphs within coordinate systems, mathematics education researchers must be explicit about these meanings. We intend for the hypothetical models we elaborated, which explicitly address students' meanings for both coordinate systems and graphs, to serve as a resource for future research into how students may develop understandings of spatial and quantitative coordinate systems, as well as graphs in these systems. Further, the similarity between the two uses of coordinate systems discussed here and the uses of coordinate systems observed by Paoletti et al. (2016) in STEM resources (e.g., Figure 1, left, middle) underscores the importance of students explicitly understanding and using coordinate systems for each purpose. Hence, there is a need to develop and test curricular materials that support students in explicitly understanding the differences between spatial and quantitative coordinate systems, as such differences are relevant for their potential future STEM studies and careers.

References

Brown, T. L., LeMay, H. E., Bursten, B. E., Murphy, C. J., & Woodward, P. M. (2012). *Chemistry: The Central Science* (12 ed.). Boston, MA: Prentice Hall.

Clement, J. (1989). The concept of variation and misconceptions in Cartesian graphing. . *Focus on Learning Problems in Mathematics, 11*(1-2), 77-87.

Crisp, G., Nora, A., & Taggart, A. (2009). Student characteristics, pre-college, college, and environmental factors as predictors of majoring in and earning a STEM degree: An analysis of students attending a hispanic serving institution. *American Education Research Journal, 46*(4), 924–942.

Disher, F. (1995). Graphing art. *The Mathematics Teacher, 88*(2), 124-128,134–136.

Glencoe precalculus. (2014). Bothall, WA: McGraw-Hill Education.

Halliday, D., Resnick, R., & Walker, J. (2011). *Fundamentals of physics* (9th ed.). Jefferson City, MS: John Wiley & Sons, Inc.

Joshua, S., Musgrave, S., Hatfield, N., & Thompson, P. W. (2015). Conceptualizing and reasoning with frames of reference. In T. Fukawa-Connelly, N. Infante, E., K. Keene, & M. Zandieh (Eds.), *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education* (pp. 31-44). Pittsburgh, PA: RUME.

Lee, H., & Hardison, H. L. (2017). Foregrounding the background: Two uses of coordinate systems. *Abstracts of Papers Presented to the American Mathematical Society, 38*(1), 563–564.

Lee, H. (2016). Just go straight: Reasoning within spatial frames of reference. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 278–281). Tucson, AZ: The University of Arizona.

Lee, H., & Hardison, H. L. (2016). Spatial coordination as a prerequisite for representing quantitative coordination in two dimensions. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (p. 304). Tucson, AZ: The University of Arizona.

Leinhardt, G., Zaslavsky, O., & Stein, M. (1990). Functions, Graphs, and Graphing: Tasks, Learning, and Teaching. *Review of Educational Research, 60*(1), 1–64.

Levinson, S. C. (2003). Space in language and cognition: Explorations in cognitive diversity (Vol. 5). United Kingdom: Cambridge University Press.

Moore, K. C., Paoletti, T., & Musgrave, S. (2013). Covariational reasoning and invariance among coordinate systems. *The Journal of Mathematical Behavior, 32*(3), 461–473.

Moore, K. C., & Thompson, P. W. (2015). *Shape thinking and students' graphing activity.* Paper presented at the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education, Pittsburgh, PA.

Paoletti, T. (2015) Students' reasoning when constructing quantitatively rich situations. In T. Fukawa-Connolly, N. E. Infante, K. Keene, & M. Zandieh (Eds.), *Proceedings of the Eighteenth Annual Conference on Research in Undergraduate Mathematics Education* (pp. 845-852). Pittsburgh, PA: West Virginia University.

Paoletti, T., Rahman, Z., Vishnubhotla, M., Seventko, J., & Basu, D. (2016). Comparing Graph Use in STEM Textbooks and Practitioner Journals. *Proceedings from the 20th Annual Conference on Research in Undergraduate Mathematics Education*, 1386–1392.

Potgieter, M., Harding, A., & Engelbrecht, J. (2008). Transfer of algebraic and graphical thinking between mathematics and chemistry. *Journal of Research in Science Teaching, 45*(2), 197–218.

Rock, I. (1992). Comment on Asch and Witkin's "Studies in space orientation II.", *Journal of Experimental Psychology: General*, 121 (4), pp. 404–406.

Saldanha, L. A., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensen, K. R. Dawkings, M. Blanton, W. N. Coulombe, J. Kolb, K. Norwood, & L. Stiff (Eds.), *Proceedings of the 20th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 298–303). Columbus, OH: Ohio State University.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sépulveda (Eds.), *Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 31–49). Morélia, Mexico: PME.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^e*. Laramie, WY: University of Wyoming.

Thompson, P. W., Hatfield, N. J., Yoon, Y., Joshua, S., & Byerley, C. (2017). Covariational reasoning among U.S. and South Korean secondary mathematics teachers. *Journal of Mathematical Behavior, 48*, 95–111.

Steffe, L. P. & Olive, J. (2010). *Children's fractional knowledge*. New York: Springer.

Yavuz, I. (2010). What does a graphical representation mean for students at the beginning of function teaching? *International Journal of Mathematical Education in Science and Technology, 41*(4), 467–485.

ANALYSIS OF TEACHERS' QUESTIONING IN SUPPORTING MATHEMATICAL ARGUMENTATION BY INTEGRATING HABERMAS' RATIONALITY AND TOULMIN'S MODEL

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Teachers' questioning is a pivotal contributing factor to support students' engagement in productive mathematical collective argumentation. Following Habermas'(1998) construct of rational behavior, we attempted to demonstrate how teachers' questioning can be framed based on this construct. Adapting Habermas' construct with Toulmin's (1958/2003) model for argumentation, we conducted a case study to analyze how a prospective secondary teacher used questions to support collective argumentation in one of her student teaching classrooms. We also explored how the prospective teacher's interpretations of argumentation related to her use of rational questioning. The results suggested that this theoretical integration can be a useful tool to frame teacher questioning based on teachers' intentions and organizations of argumentation. Some educational implications for teacher professional development are provided.

Keywords: Classroom Discourse, Reasoning and Proof, Research Methods, Teacher Education-Preservice

Introduction

Purpose Statement

Developing mathematical reasoning is central to mathematics education. Considerable evidence suggests the beneficial effects of students articulating their mathematical reasoning and challenging others when engaged in collective mathematical argumentation (e.g., Krummheuer, 1995). *Collective mathematical argumentation* can be defined broadly as "any instance where students and teachers make a mathematical claim and provide evidence to support it" (Conner, Singletary, Smith, Wagner, & Francisco, 2014, p.44). Collective mathematical argumentation occurs in a mathematics class when the teacher and students (or a small group of students working independently) work together to construct or reject mathematical arguments.

Facilitating productive mathematical argumentation is challenging for many new teachers. As stated in *Principles to Actions: Ensuring Mathematics Success for All* (National Council of Teacher of Mathematics, 2014), effective mathematics teachers are expected to use purposeful questions to access students' conceptual understanding and to advance students' reasoning and sense making about important mathematical ideas and relationships. Teachers' questioning has the potential to bring students into a conversation and promote participation, and the type and quality of questions can have significant impact on students' engagement in productive argumentation. By examining teachers' actions in support of student participation in mathematical argumentation, Conner et al. (2014) found that posing questions to elicit parts of arguments is a useful strategy that supports collective argumentation. Therefore, it is necessary to understand what factors influence teachers' questioning and how teachers use different types of questioning to support collective argumentation in mathematics classrooms.

Boero (2006) proposed that Habermas' (1998) construct of *rational behavior* can become a frame to deal with the complexity of discursive practices in the intersection of three kinds of rationality: epistemic (inherent in the control of validation of statements), teleological (inherent in the strategic choice of tools to achieve the goal of the activity), and communicative (inherent in the conscious choice of suitable means to communicate understandably within a given community). Researchers have highlighted the importance of Habermas' (1998) construct as a tool to analyze didactical obstacles inherent in proving and argumentative activities (e.g., Boero, 2011; Boero & Planas, 2014). Under the vygotskian didactical perspective, Douek suggested using *rational questioning* as a method to "organize the mathematical discussion according to the three components of rationality" (Douek in Boero & Planas, 2014, p. 210). The teacher plays an essential role in creating a suitable context to promote the vygotskian dialectics and develop argumentation.

An interesting aspect of this construct is that Boero, Douek, Morselli, and Pedemonte (2010) illustrated the possible adaptation of combining Habermas' (1998) construct and Toulmin's (1958/2003) model for argumentation to study discursive practices related to proving and argumentative activities; they claimed that the combination of these two theoretical frameworks might provide a comprehensive frame that would allow one to better analyze students' proving and argumentative activities in mathematics classrooms and support teachers as they plan and carry out rational classroom interventions.

Research Questions

The goal of this study is to build on Habermas'(1998) construct of rational behavior, and to integrate Toulmin's (1958/2003) model for argumentation to ascertain the potential to use this tool to analyze teacher questioning through the lens of supporting mathematical collective argumentation. In addition, we are interested in how teachers' interpretations of argumentation relates to their pedagogical choices for rational questioning. The following research questions guided this study:

1. How does a prospective secondary teacher use rational questioning when guiding collective mathematical argumentation?
    - Which component of rationality is privileged in class?
    - What does rational questioning tell us about a teacher's support for argumentation in class?
    - What combinations of components of rational questioning support students' contribution of argument components?
2. How does a teacher's interpretations of argumentation align with his or her use of rational questioning?

Theoretical Framework

In order to make sense of classroom experiences, we need to broaden our interpretative stance by developing a sociological perspective on mathematical activity. For this purpose, in this study we employed the emergent perspective which emphasizes the idea that knowledge is created through actions and interactions, and that the development of an individual's reasoning and sense-making processes cannot be separate from his or her participation in the social interaction of taken-as-shared mathematical meanings (Cobb, Yackel & Wood, 1992).

According to Habermas'(1998) construct of rational behavior, we developed a Teacher Questioning Framework (see Table 1) by inferring the teacher's intentions and consciousness in asking the question. The framework consisting of three components of rationality and we defined *rational questioning* as a question that contains at least one component of rationality. At times, for clarity, we call a question epistemic rational questioning if it contains an epistemic rationality component.

Table 1: Teacher Questioning Framework

Component of Habermas' Rationality	Description of Questions	Examples
Epistemic Rationality	The questions used not only allow students to provide and evaluate arguments or ideas but also intend to challenge students by asking them to reason and make sense of their arguments and ideas.	Can you tell me why?
Teleological Rationality	The questions used allow students to show or reflect on the strategic choices that they used to achieve their arguments or ideas.	What do we need to do to the cards in the hat to make sure that every single family member has an equally likely chance of being drawn?
Communicative Rationality	The questions used allow students to communicate or reflect on the steps involved in their reasoning and arguments to ensure that their ideas can be understandable in the given community.	So will somebody raise their hand and tell me in their own words what they think it means to be equally likely?

In education literature, Toulmin's (1958/2003) model has been widely used to identify argument components (*claims, data, warrants, rebuttals and backing*) and analyze argumentative activities in classes (see Krummheuer, 1995 for more details on Toulmin's model). Our adaptation of Toulmin's diagrams (see Figure 1) includes the use of color and line style to record the contributor(s) of a component for a given argument and uses "Teacher Support" to denote teachers' contributions and actions (for this specific study, we focused only on teachers' questioning) that prompt or respond to parts of arguments (see Conner et al., 2014 for more details on the development of the framework).

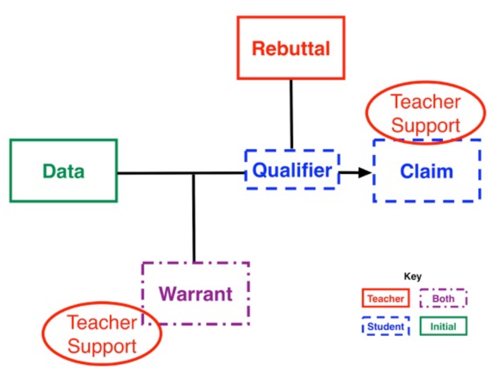

Figure 1. Adaptation of Toulmin's (1958/2003) Diagram for an Argument

In this study, we adapted Habermas'(1998) construct of rational behavior to analyze the rationality of teachers' questioning and employed Toulmin's (1958/2003) model to examine how a teacher used these rational questioning to teaching.

Context and Methods

Context

This study is part of a larger project in which we are examining how teachers learn to support collective mathematical argumentation. The participant, Cathy (a pseudonym) was purposefully selected in this study based on her good understanding of argumentation as she participated in a unit on argumentation during her course work and be willing to support student engagement in argumentation during her student teaching. We observed Cathy's classroom teaching during a statistics and probability unit which focused on chance processes and probability models. The main data sources in this study included (1) video recordings and transcriptions of one representative day of Cathy's classroom teaching, (2) Cathy's interactions during a unit of instruction on collective argumentation (the argumentation unit) at the beginning of her pedagogy course in the mathematics teacher education program, and (3) a reflection that Cathy wrote about argumentation based on her observations in secondary classrooms.

Methods

We used Teacher Questioning Framework (see Table 1) to establish our starting codes to examine teachers' questioning in Cathy's class. A simple enumerative approach was also used to quantify verbal data in order to explicate the patterns that emerged from the open-coding process. Because we want to see how Cathy used her rational questioning to organize arguments, we used Toulmin's (1958/2003) model to diagram every episode of argumentation in Cathy's class, and we noted the parts of an argument that each question prompted. Finally, we used thematic analysis to understand Cathy's priorities and thoughts about argumentation based on her classroom discussions in the argumentation unit at the pedagogy course and her written reflection about argumentation a few weeks into the course.

Results

Use of Rational Questioning

According to our Teacher Questioning Framework (see Table 1), 66% (80/122) of Cathy's questions involved rational questioning. Among these rational questions, the largest component was communicative rationality (83%), followed by teleological rationality (81%); and the portion that involved epistemic rationality was smallest (40%). Although Douek's rational questioning seems to presuppose all three components of rationality, it is apparent that communicative rationality occupied a dominant position in Cathy's class. According to our analysis, only 34% (27/80) of the rational questioning contained all three components of rationality. Moreover, we found that Cathy used a variety of combined forms of rational questioning: some questions included two or three components of rationality and others only involved one.

Our diagrams of argumentation indicated that 59% (72/122) of Cathy's questions were asked during argumentative activities. Among these questions, most of them (81%) contained at least one component of rationality, which showed that Cathy had the intention to use rational questioning to support collective argumentation. Additionally, for these rational questions that were involved in argumentation, most of them had a communicative rationality component (85%) and a teleological rationality component (75%), but only 33% (19/58) had an epistemic rationality component. An interesting aspect of this result is that compared to the percentage of each rationality component Cathy used in her whole class, as reported previously, we found that little difference existed for Cathy in her use of rational questioning during argumentative activities and non-argumentative activities.

According to the above results, Cathy showed a strong emphasis on developing her students' communicative rationality. Some of her communicative rational questioning asked students to explain their reasoning steps (e.g., Can you explain a little bit?). Some of them asked students to communicate ideas more clearly (e.g., And which numbers are they?) or to rephrase a mathematical definition to ensure that their explanations were understandable in the given community (e.g., What does it mean for a coin to be fair?). Cathy also used communicative rational questioning to encourage more students to engage in collective argumentation; this process was evidenced by Cathy often waiting a couple of minutes to see more students' raise their hands before she asked a student to answer her question. In addition, Cathy also pushed her students to evaluate and make sense of the claims and warrants that were provided by other students (e.g., Does it make sense why these two are counted separately?). According to our diagrams of argumentation, for rational questioning containing a communicative component, 24 (out of 44) prompted claims and data/claims while only 13 (out of 44) prompted warrants, which means that Cathy mainly used communicative rational questioning to promote student participation in the construction of arguments.

Within the rational questioning that contained a teleological component in Cathy's class, we found that most all of these questions were strategically goal-oriented reasoning: Cathy first let the students find the possible results of an event, and she next asked them to conjecture whether these events had equally likely chances of happening. Sometimes she asked students to justify their answers by using the classical approach to probability. On a few occasions, Cathy wanted to highlight the importance of the strategic choice of using examples, and she asked her students to provide different examples of what it meant to be "equally likely": "Is any, can anybody think of a different example? That might give you [an] equally likely [chance]?"

Cathy used most of her epistemic rational questioning to push her students to justify why their arguments hold. For example, she asked, "Would you mind sharing your reasoning with the class?" However, only 33% of the rational questioning that contained epistemic components. Cathy's goal for students was apparently less focused on epistemic rationality than on teleological and communicative rationality. This difference was also evidenced by our diagrams of argumentation, within which only 17 (out of 72) questions prompted warrants while 45 (out of 72) prompted claims and data/claims.

As mentioned above, some rational questions involved all three components of rationality (e.g., Can you explain to me why it's going to be fifty fifty?), and we found that more than half (56%) of these questions helped students contribute warrants, which means these questions provided students opportunities to reason about and make sense of their arguments — one of the most important aspects of argumentation for students. Some of Cathy's rational questions contained only teleological and communicative components (e.g., Can somebody remind me how I find the volume of any prism? I will wait until I see a few more hands.), and among these questions, 11 (out of 19) prompted students' claims and data/claims. Cathy used this kind of rational questioning to ask students to communicate their arguments and to focus on the strategic tools that they used to solve the problem.

Interpretations of Argumentation

By examining portions of classroom discussions during the argumentation unit that Cathy participated in and her written reflections about argumentation, we discovered two clear themes.

First, Cathy regarded the role of teachers as asking questions and calling on students to participate. For example, during an in-class discussion in the argumentation unit, Cathy stated, "The role of the teacher is to facilitate, asking questions, and choosing students to participate…" Another theme that emerged concerned Cathy's desire that students be the main contributors to the arguments. In her written reflection, she explained that an effective argument should involve multiple students, and she labeled as an ineffective argument as one that involved only one student.

Our analysis suggests that Cathy's interpretations of argumentation align with her use of rational questioning to support collective argumentation during her student teaching classroom practice. Most of the rational questioning that Cathy asked contained a communicative rationality component, which aimed to bring students' voices into the discussion and encouraged more students to participate in collective argumentation. Data from Cathy's student teaching showed that she used communicative rational questioning not only to encourage students to communicate their ideas more clearly but also to provide opportunities for her students to lead the classroom discussion. For example, she would ask questions such as the following: "Can I have a volunteer go to the board and help us find the unit rate?" Several times after a student gave a claim and warrant, Cathy used communicative rational questioning — "Does anybody have any questions for Ashley?" — to intentionally slow down the presentation to make sure it made sense to others and to inspire students to critique one another's arguments. The large number of communicative rational questions and the way Cathy used them throughout her student teaching seemed consistent with her belief that the teacher was responsibility for promoting student engagement. Cathy also used teleological and epistemic rationality questioning to support of her idea of the teacher as a facilitator and used teleological rationality in her questions to strategically develop goal-oriented reasoning, and then followed up with epistemic rational questioning to request further explanations from her students.

In addition, we noticed that 34% (42/122) of the questions were non-rational, which could be categorized into two types of questions. One type of non-rational questions was either/or questions that were situated in argumentative activities, which prompted claims and asked students to construct results for the final step of argumentation. An example of this type of question would be as follows: Is it equally likely or not equally likely? Another type of non-rational questions was used to present information. For example, Cathy often asked students to read the task. She also asked several factual questions in order to create a probability problem context. We labeled these types of questions as contextual questioning, which may be specific to teaching probability. The way in which Cathy used non-rational questioning showed that instead of giving all the information herself, she asked students to provide data and final claims in the arguments, which aligned with her goal of promoting participation among the students.

Discussion

Many researchers have used Habermas'(1998) construct to conduct studies that centered on *students* (e.g., Morselli & Boero, 2011), and our study showed that this theoretical framework can also be used to investigate *teachers'* intentions when they ask specific kinds of questions. Some researchers have categorized types of teachers' questioning within classroom discourse (e.g., Wood, 1998). The integration of Habermas' construct with Toulmin's (1958/2003) model as a tool to analyze teachers' questioning provides us with a more comprehensive perspective for understanding the roles of teacher questioning within collective argumentation. The two models complement each other in the following sense: Habermas' lens help us to identify fine grained rationality components of teachers' questioning and also how teachers' questioning is constrained in relation to the three components of rational behavior; the teacher then using rational questioning to control the fundamental steps of argumentation is seen through Toulmin's lens. We see the function of rational questioning as fully implementing an approach that pushes students to make sense of arguments, to strategically choose effective tools or methods in problem-solving, and to communicate understandably in a given community, which are essential goals of collective argumentation. This study provides empirical evidence to support Boero et al.'s (2010) idea about the value of integrating Habermas' elaboration of rational behavior with Toulmin's model for argumentation to deal with the complexity of learning and teaching activities, particularly with respect to proving and argumentative activities.

Douek, in Boero and Planas (2014) suggested that teacher's rational questioning can support students' "rationalization" of discourse and develop mathematical discussions along the three components of rationality. Based on our analysis, most of the questions that were posed by Cathy to support collective argumentation were connected to the three components of rationality. Moreover, in this study we found that Cathy used different combinations of rational questioning in her class, and most of them involved only one or two rationality component. According to our analysis, these kinds of rational questioning can also support collective argumentation, and the function of different kinds of rational questioning may vary across different classroom activities. More research is needed to explore whether any other forms of combinations can support argumentation and when are the best times and contexts for teachers to use them.

Although the study showed that Cathy's rational questioning included all three components of rationality, we found that the epistemic component made up a relatively small portion. This result raises an important question: Why do some components of rationality occur less frequently than others? One reasonable explanation from our study is that teachers attempt to use different kinds of rational questioning based on their interpretations of argumentation during course work. We also considered that a teacher's use of different kinds of rational questioning may depend on different mathematical domains, content, and various contexts in the classroom (e.g. whole class discussions or small group discussions). In this study, we only analyzed one of Cathy's probability classes, and more studies are needed to explore what factors influence a teacher's intentional choices of different kinds of rational questioning.

Implications

Our results indicated that Cathy was primarily concerned about her students' participation and thought that the role of teacher was a facilitator; these themes remained consistent with the way she adopted rational questioning during her student teaching. Cathy seemed to take her interpretations of argumentation into her student teaching placement, and it is reasonable to infer that the argumentation unit in her pedagogy course may have influenced her use of questioning for instruction. Our results also suggest that teacher education programs should demonstrate the power of collective argumentation and promote teachers' understanding of the roles of a teacher in questioning.

As mentioned above, Cathy used almost the same proportion of different kinds of rational questioning when she led argumentative activities and non-argumentative activities, which showed that Cathy's intention to strategically use rational questioning as a tool to support collective argumentation spilled over into her regular instruction. More research is necessary to examine how to use the three components of rationality in questions at the moment when argumentation does not develop or when argumentation is begun but breaks off. We hypothesize that understanding specific components of rational questioning could help teachers to integrate multiple questioning strategies to support collective argumentation in a specific content and in a specific context. Future studies should continue to analyze the benefits of rational questioning and how can it be incorporated into teacher education program to make it more approachable to prospective teachers.

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References

Boero, P. (2006). Habermas' theory of rationality as a comprehensive frame for conjecturing and proving in school. In *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol.2, pp. 185-192).

Boero, P., Douek, N., Morselli, F., & Pedemonte, B. (2010). *Argumentation and proof: A contribution to theoretical perspective and their classroom implementation.* Paper presented at the Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education.

Boero, P. (2011). Argumentation and proof: Discussing a "successful" classroom discussion. In M. Pytlak, T. Rowland & E. Swoboda (Eds), *Proceedings of CERME7* (pp. 120-130). Rzeszów, Poland: ERME.

Boero, P. & Planas, N. (2014). Habermas' construct of rational behavior in mathematics education: New advances and research questions. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol.1, pp. 205-235). Vancouver, Canada: PME.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics education,23*(1), 2-34.

Conner, A., Singletary, L. M., Smith, R. C., Wagner, P. A., & Francisco, R. T. (2014). Teacher support for collective argumentation: A framework for examining how teachers support students' engagement in mathematical activities. *Educational Studies in Mathematics, 86*(3), 401-429.

Habermas, J. (1998). *On the pragmatics of communication* (M. Cooke Ed.). Cambridge, MA: MIT Press.

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp.229-269). Hillsdale, NJ: Lawrence Erlbaum Associates.

Morselli, F., & Boero, P. (2011). Using Habermas' theory of rationality to gain insight into students' understanding of algebraic language. In *Early algebraization* (pp. 453-481). Springer, Berlin, Heidelberg.

National Council of Teachers of Mathematics (2014). *Principles to Actions: Ensuring Mathematical Success for All.* Reston, VA: Author.

Toulmin, S.E. (2003). *The uses of argument* (updated ed.). New York: Cambridge University Press. (First published in 1958).

Wood, T. (1998). Alternative patterns of communication in mathematics classes: Funneling or focusing? In H. Steinbring, M. G. B. Bussi & A. Sierpinska (Eds.), *Language and communication in the mathematics classroom* (pp.167-178).

THEORY, ETHICS, AND EQUITY IN INTRA-ACTION IN MATHEMATICS EDUCATION: LOOKING BACK, LOOKING FORWARD

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This paper considers the intra-actions between poststructural theories and mathematics education over the last 40 years and considers how these theories have resulted in different ways to think students, teachers, and knowledge production. I argue that thinking in intra-action with various and different theories can allow us to ask different questions and radically rethink school mathematics.

Keywords: Research Methods, Equity and Diversity

At the opening plenary of PME-NA Rochelle Gutiérrez (2017) proposed to the mathematics education research community that "interaction between different knowledges, different ways of knowing and different knowers" (p. 2) could serve to respond to and perhaps address the precarious state of our planet and our relationship with it. Gutiérrez is not the first leader in the field of mathematics education research to call on the community to consider not only the what of mathematics education but also the how. William Tate (1995) called on the community to consider policy in relation to equity in mathematics education. He commented that he found the "paradigmatic boundaries of mathematics education somewhat narrow," and he intentionally modeled his work after scholars who "crossed the epistemological boundaries of their fields to provide a more cogent analysis of important issues facing African Americans" (pp. 425-426). In this paper, I will consider research in mathematics education that has already crossed epistemological boundaries to open up spaces for mathematics educators, teachers, and students to think themselves, mathematics, and schools differently. I will then explore Barad's (2007) construct of intra-action and the potential possibilities it offers for qualitative methodology, specifically in mathematics education. Finally, I will come back to calls by Gutiérrez (2017), de Freitas and Sinclair (2014), and Martin (2015) to consider how possibilities for ethical action are structured by the ways we do and think research.

Theory and Qualitative Research in Mathematics Education

de Freitas and Walshaw (2016) describe their approach to theory as impacting their thinking and meaning-making, explaining that "the act of defining or creating new concepts is precisely what theory has the potential to do. Thus, theory is a creative tool, an inventive approach to making meaning, as well as being an intervention into current cultural practices" (p. 4). In this frame, theory becomes not just something that you think about prior to research or something that you apply to research, but something that inevitably impacts the meaning that is made through research.

Stinson and Walshaw's (2017) chapter on theoretical frontiers in mathematics education research explains that theory has not always been considered a foundational aspect of mathematics education research. As theory gains space and attention in the field of mathematics education, the question of which theory to use will continue to surface. Rather than thinking about which theory is best, perhaps mathematics educators can consider how theories have functioned to allow us to "move toward the unthought" (St. Pierre, 1997, p. 185) and ask, what other important thoughts have we yet to think or unthink that matter in mathematics education?

Poststructuralism has intersected with mathematics education for decades, and mathematics educators use of poststructural theories have made it possible to consider how meaning and knowledge get made and whose "interests are privileged, marginalized, or silenced" (Stinson & Walshaw, 2017, p. 139). Poststructural theories have also been taken up to allow a different view of teachers and students as subjects that are constituted through interactions with the powerful discourses of school mathematics, education, and gender.

Walkerdine (1994) drew on poststructural theory to consider the production of the "appropriate" mathematical subject, arguing that "theories of the development of reasoning when incorporated into education become 'truths' which actually serve to produce the desired kinds of subjects as normal and pathologizes differences" (p. 65). Walkerdine was particularly interested in the effects of gender and class on subject formation in mathematics classrooms. She pointed to the ways in which girls were positioned as compared to boys. Her work pointed to the need to recognize the meaning in the practices that mathematics educators carry out and the limitations they, sometimes inadvertently, put forward in how students can live. Similarly, Mendick (2005) questioned why and how particular girls seemed to freely choose paths that reified their oppression. She found that girls were less likely to enroll in accelerated mathematics classes despite equal or higher achievement on math assessments. She used poststructuralism to question the assumptions that math is legitimately powerful and that gender is a natural binary.
Margret Walshaw (2004) describes the "postmodern analytical edge … invites a less certain space for research, pedagogy and practice" (p. 4) that has allowed mathematics educators to recognize and disrupt taken for granted assumptions.

Though poststructuralism has taken on the humanistic stable subject and the power of discursive formations, it has been critiqued for its focus on the linguistic and lack of attention to the material. In the materialistic turn, where the question of what matter matters has been raised, new understandings and theorizations of quantum mechanics and environmental concerns have come together to produce theories that undo the nature culture divide and decenter the human as privileged caretaker or dominator of the earth. van der Tuin and Dolphijn (2010) explain that new materialism is fascinated by affect, force and movement as it travels in all directions. It searches not for the objectivity of things in themselves but for an objectivity of actualisation and realisation... It is interested in speeds and slownesses, in how the event unfolds according to the in-between. (p. 169)

Like poststructuralism, the key tenants of these new materialisms function to trouble binaries and distinctive boundaries. In addition, new materialist theories take seriously what matter matters and how it comes to matter.

Barad—a feminist, philosopher and quantum physicist—introduced many useful and important figurations in the last two decades as she imagined her agential materialism into being. Through her concept of intra-action, Barad (2007) denies the existence of individual separate beings and objects through the exploration and study of Niels Bohr's "philosophy-physics" (p. 24). Intra-action is born out of the recognition that things are not discrete but are always already entangled. Interaction implies separate entities that take individual action towards or away from each other. Instead, Barad considers intra-action that is always taking place between "two mutually entailed folds of the same realm" (de Freitas and Sinclair, 2014, p. 46).

In Barad's view, matter and meaning are co-constituted and inseparable. Just as matter and meaning cannot be separated, so too epistemology, ontology, and ethics cannot be thought apart. We are "part of that nature that we wish to understand" (Barad, 2007, p. 26). Since, things are not thought of as separate and discrete, neither can they have individual agency, "rather what is understood as 'agency' in the relational materialist approach is a quality that emerges in-between different bodies involved in mutual engagements and relations" (Hultman & Lenz Taguchi, 2010, p. 530). In a research setting then, we can no longer think the researcher as an objective separate observer, who studies from afar to know a subject. Instead, researcher, students, teachers, materials are mutually entangled and constituted and come to know "from a direct material engagement with the world" (Barad, 2007, p. 49).

New materialisms and in particular Barad's agential realism are beginning to be taken up by mathematics educators and are effecting/affecting the types of knowledge that are being produced through research. de Freitas and Sinclair (2014) bridge the fields of philosophy, mathematics and feminism, pulling the threads of various (and differing) theories to put forward a new form of materialism that they term *inclusive materialism*, which troubles traditional humanist and rationalist notions and takes up the aesthetic, affective, and material as mattering. Inclusive materialism functions in their work to allow them to conceive of school mathematics as reconfigurable into what they imagine as a minor mathematics that is "not the state-sanctioned discourse of school mathematics but that might be full of surprises, non-sense and paradox" (de Freitas & Sinclair, 2014, p. 226). This is an ethical move for them, though they recognize that this mathematics will be "at odds with current institutional demands. However, a minor mathematics is likely to engage students and teachers in more expansive ways, and our hope is that it would engage more students in mathematics" (de Freitas & Sinclair, 2014, p. 226).
As in the review of poststructural theories, in thinking with Barad and inclusive materialism a new conception of methodology was necessary. de Freitas and Sinclair (2014) show how inclusive materialism might alter the way we think about embodiment of mathematical concepts, offering alternate ways of studying how students learn concepts and of how we might choose and order concepts as part of a curriculum sequence. When concepts are animated differently, learning is similarly altered. Inventive acts in classrooms become part of a growing material assemblage, a process of embodiment in which the potentiality of the body is emphasized.

Implications for Enactment of Qualitative Methodology in Mathematics Education

Though Barad's conception of onto-epistemology and the collapsing of knowing and being are important and productive in how we think mathematics education, given Gutiérrez's (2017) and Martin's (2015) demands, it is Barad's inclusion of ethics and her view on responsibility that could really matter for students and researchers in mathematics education. Her concept of intra-action demands a relational ethics, as being and knowing are entangled, so, too, is living well and in respons-ability to all others. Barad (2007) proposes, ethico-onto-epistem-ology—an appreciation of the intertwining of ethics, knowing, and being— since each intra-action matters, since the possibilities for what the world may call out in the pause that precedes each breath before a moment comes into being and the world is remade again, because the becoming of the world is a deeply ethical matter. (p. 185)

Our ontologies, epistemologies, and ethics cannot be separated out. They are entangled in the production of our worlds and our lives. As we make choices in how we live and research we are, according to Barad's (2007) agential realism, making cuts. We are engaged in boundary-making practices that categorize and classify: "Cuts are enacted not by willful individuals but by the larger material arrangements of which 'we' are a 'part'" (p. 178). These cuts have material effects. For example, in Gutiérrez's (2008) work around the achievement gap, she points to cuts that are made around black and brown bodies that produce them as deficient and lacking. Cathy O'Neil (2016) argues convincingly that mathematics has material effects on people's lives and discriminatively negative effects on the poor. Far from being an abstract and static discipline that it is sometimes assumed to be, mathematics is intimately entangled in our lives as it continues to serve as a proxy for truth and privilege. The way that we do and use mathematics and the way that we do qualitative research matters. The models that we set up, in Barad's (2007) terms—the apparatus within which we are entangled—determine reality (O' Neil, 2016). In each intra-action, we determine reality and reconfigure our worlds. These determinations cannot be made ahead of time and cannot be rule-bound or universalized. This brings us again to calls by Gutiérrez (2017), Martin (2015), and de Freitas and Sinclair (2014), especially as educational researchers. How do we work to continually pose questions to ourselves/each other that take into consideration how we might all live differently? I propose that we as mathematics educators be open and uncertain about what possibilities thinking/being differently in ethical relation with mathematics education might open up. Research over the last forty years has shown that reconfiguration is possible, and we may have to unlearn some of what we know to achieve the needed radical reconfigurations in the next forty.

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References

Barad, K. (2007). *Meeting the universe halfway: Quantum physics and the entanglement of meaning and matter.* Durham, NC: Duke University Press.

de Freitas, E., & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom*. New York, NY: Cambridge University Press.

de Freitas, E., & Walshaw, M. (2016). *Alternative theoretical frameworks for mathematics education research: Theory meets data.* Switzerland: Springer.

Gutiérrez, R. (2008). A "gap-gazing" fetish in mathematics education? Problematizing research on the achievement gap. *Journal for Research in Mathematics Education, 39*(4), 357–364.

Gutiérrez, R. (2017). Living mathematx: Towards a vision for the future. *Philosophy of Mathematics Education, 32*(1).

Hultman & Lenz Taguchi, H. (2010). Challenging anthropocentric analysis of visual data: a relational materialist methodological approach to educational research. *International Journal of Qualitative Studies in Education, 23*(5), 525-542.

Martin, D. B. (2015). The collective Black and *Principles to Actions*. *Journal of Urban Mathematics Education, 8*(1), 17–23.

Mendick, H. (2005). Choosing maths/doing gender: A look at why there are more boys than girls in advanced mathematics classes in England. In L. Burton (Ed.), *Which Way Social Justice in Mathematics Education* (pp. 105–132). Westport, CT: Information Age Publishing, Inc.

O'Neil, C. (2016). *Weapons of math destruction: How big data increases inequalities and threatens democracy*. New York, NY: Crown.

St. Pierre, E. A. (1997). Methodology in the fold and the irruption of transgressive data. *International Journal of Qualitative Studies in Education*, *10*(2), 175–189.

Stinson, D. W., & Walshaw, M. (2017). Exploring different theoretical frontiers for different (and uncertain) possibilities in mathematics education research. In J. Cai (Ed*.). Compendium for Research in Mathematics Education* (pp. 128-155). Washington, DC: National Council of Teachers of Mathematics.

Tate, W. F. (1995). School mathematics and African American students: Thinking seriously about opportunity-to-learn standards. *Educational Administration Quarterly, 31,* 424-448.

van der Tuin, I., & Dolphijn, R. (2010). The transversality of new materialism. *Women: A Cultural Review, 21*(2), 153-171. DOI: 10.1080/09574042.2010.488377

Walshaw, M. (Ed.) (2004). *Mathematics education within the postmodern*. Greenwich, CT: Information Age Publishing.

UNDERSTANDING THE NATURE OF UNCERTAINTY IN PROBLEM SOLVING SITUATIONS

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Productive disciplinary engagement (PDE; Engle & Conant, 2002) describes classroom situations where students publicly engage in disciplinary practices. Researchers have argued that PDE is fostered when students engage in problematizing, where they grapple with genuine uncertainty about mathematical objects, among other characteristics. Building on work by Zaslavsky, this paper advances a framework to capture the nature of uncertainty in mathematics classrooms.

Keywords: Problem Solving, Research Methods, Curriculum, Middle School Education

Introduction

Problematizing is a core principle of classrooms that embody *productive disciplinary engagement* (PDE) (Engle & Conant, 2002). Engle (2011) articulates problematizing as situations involving "uncertainties...related to the discipline...to be taken up by students." Although some existing work has operationalized other principles of PDE, such as authority and accountability, measuring the extent to which students are engaging in problematizing is less well-defined in existing literature. While existing research has modeled uncertainty when students participate in group work (Wood & Kalinec, 2012) and the linguistic basis for verbal expressions of uncertainty (Martin & Rose, 2003; Rowland, 1995), uncertainties that arise during problematizing are tied to the mathematical work students are engaged in. Engle (2011) suggests that uncertainties which arise in situations of PDE include (1) competing claims; (2) unknown path; (3) questionable conclusion; (4) non-readily verifiable outcome; and (5) other. Our work seeks to extend these initial categories to generate a comprehensive framework of situations of uncertainty when students grapple with open mathematical problems.

Methods

To generate the framework, we used an iterative process to code video recordings of classroom episodes where students were engaged in solving open problems. Our video data came from one lesson taught across two class periods in an 8th grade mathematics classroom in a rural school setting, collected in the late spring. The lesson was based on an investigation in the Connected Mathematics (CMP) curriculum (Lappan, Fey, Fitzgerald, Friel & Phillips, 2006), a curriculum used in this school from 6th to 8th grades. The teacher had taught from CMP in prior years. The specific lesson features students engaged in finding and generalizing patterns to find an algebraic expression for the number of tiles around a square pool, given any side length. The video shows the teacher's launch of the problem, small group work as students explore the problem, and a summary discussion of different groups' solutions and strategies.

We used iterative processes (Shaffer, 2017) to generate categories in our framework. The process began with individual open coding by each member of the research team to describe the uncertainties observed through students' spoken words, gestures, postures, and silences. Next, we engaged in several iterative cycles of axial coding (Strauss & Corbin, 1998) to collectively compare codes against the data and existing codes in order to restructure and refine the framework to account for students' processes. In the final iterative cycles, we looked for overlap in our descriptions of types of uncertainty and worked to eliminate any redundancies. We then compiled the framework and individually re-coded the video data to determine that all uncertainties could be coded with the refined framework.

The Framework

The process described in the Methods section yielded the framework shown in Figure 1. Building from work by Engle (2011) and Zaslavsky (2005), our modified categories include uncertainty about: (1) What action to take - students have uncertainty about what to do, where they may feel unable to begin or continue working; (2) Justifying actions or outcomes - students don't know how to verify, check, or justify what they are doing or some outcome of mathematical work; (3) Meaning of conclusion - students have or are presented with a final result, but are uncertain about what it means or how to apply it more broadly; and (4) Competing alternatives - two or more mathematical ideas arise that cause hesitation or uncertainty. Competing alternatives may occur during any stage of problem solving and is not necessarily independent of other types of uncertainty. Further, uncertainties do not necessarily arise in a linear fashion, and the resolution of one uncertainty might give rise to others.

Categories of Uncertainty	
Code	**Definition**
A. What Action to Take	
1. What a mathematical object represents	Students have hesitation/questioning about the possible meaning conveyed by a mathematical object (i.e. a diagram, shape, expression/equation, graph, table)
2. What does [part of problem context] mean?	Students are unsure about what is meant by some detail of the problem statement or part of the problem context.
3. How to create or use something	Students are uncertain about how to use the tools they have to achieve their desired goal.
4. What is the goal? What is to be created?	Students are unsure about what the problem is asking them to do or what they are trying to accomplish.
B. Justifying Actions or Outcomes	
5. Connecting to prior work	Students are uncertain on which prior knowledge they can draw.
6. Whether strategy is productive	Students are pursuing one method or strategy but are unsure if it will help them achieve their desired mathematical goal or solve the problem.
7. Representing thinking	Students have a partially developed idea or conjecture but are not sure how to represent it physically or verbally.
8. Justifying strategy	Students are uncertain of how to justify that the mathematical actions they have taken are correct.
9. Has solution been reached?	Students are unsure if the solution has been reached or if there is more to do.

C. Meaning of Conclusion	
10. What to conclude	Students are uncertain of their solution's broader meaning or why it works.
11. Whether solution makes sense	Students are unsure if their solution makes sense or is reasonable, or how to determine this or explain it to someone else.
12. Strategy works for other cases	Students have an idea or conjecture, but are unsure what it means more generally or in other contexts
D. Competing Alternatives	
13. Relating one mathematical object to another	Students are uncertain if two mathematical objects are related or how they might be related.
14. Finding another way to represent idea/work	Students are uncertain if there is another way to represent their existing mathematical idea/work, or how they would go about doing that.
15. Revising work	Students are uncertain of how to revise or change their work or strategy, once they recognize it needs to be revised.
16. Finding another way to solve problem	Students are uncertain if there is another way to solve the problem, or how they would go about doing that.
17. Seeing commonality among strategies	Students are uncertain if two mathematical strategies are related or how they might be related.

Figure 1. The framework and definitions

Using the Framework: Capturing Uncertainty when Problematizing the Pool Problem

We present some of the uncertainties that arose in one pair of students' efforts on the Pool Problem (Lappan et al., 2006a, p. 6), where they are asked to find an algebraic expression for the number of tiles around a square pool given any side length. This focal pair of students initially created an algebraic expression to determine the correct answer for a 2x2 pool, but when asked if it would work for a larger pool, one student said "I'm thinking… but probably not." They struggled to calculate appropriate examples with the expression s^2+4s+s, unsure which sizes of the pool it might work for. This illustrates the uncertainty Whether a Strategy is Productive (B-6). With the teacher's assistance, the pair tried their expression with a larger pool and determined it was incorrect. One of the students attempted to explain where the structure of their expression came from, saying "I don't know, I tried to take the… what's it called… not the expanded form…", and the teacher asks, "The factored form?" The student agreed, but that they were not sure if it would apply. This illustrates the uncertainty Connecting to Prior Work (B-5). The teacher then directed the pair's attention to the initial problem statement and the use of the variable s, and one student quickly gestured to a diagram in the textbook and asked, "Is it the outside side or just the inside one?" This uncertainty about the labeling of a diagram illustrates What a Mathematical Object Represents (A-1). Moreover, this example shows how addressing one uncertainty may involve one or more additional uncertainties arising.

After resolving this uncertainty about the initial diagram, the teacher asked the pair about an equation they wrote down earlier. One of the students confirmed that they used 4s to represent the sides, and the teacher asked, "Now what do you need?" The student responded, "The… corners." The teacher asked, "So how could you write that into the equation?" and the student answered, "Um… maybe you could, like, put it as a second power?" The student had a mental representation of the problem relating to sides and corners but struggled to represent it as an equation, illustrating the uncertainty Representing Thinking (B-7). The pair knew that they needed to include 4s in their expression, but they were not sure how to modify this expression to account for the corners. One of the students began to try various other multipliers in their calculator and realized that multiplying by a different number for each pool size produced the correct number of tiles. However, the pair tentatively proceeded to produce what they considered as an appropriate equation. This illustrates the uncertainty Revising Work (D-15).

In many cases, not all uncertainties were resolved quickly. Uncertainty about revising work, for example, extended through the rest of the small group work time. To further understand the nature of problematizing, we acknowledge several key categories of meta-data important to correlate with codes for uncertainty, such as *source* (others, technology teacher, self), *indicator* (gesture, statement, question), *phase* (launch, explore, summary), *part of problem* (initial challenge, explore, check, verify), and *number of students* (whole group, small group, individual).

Discussion

An essential part of achieving the goal of increasing students' opportunities to engage in PDE is understanding the mechanisms that allow PDE to emerge in classroom settings. In this paper, we advance a framework that may help the field get closer to identifying situations of problematizing that are essential for PDE by articulating kinds of uncertainty that are empirically grounded from a video-recorded sample of classrooms engaging in rich, open problem solving. This framework provides a starting point for further research, including investigating problematizing as it arises in classrooms with enhanced technology supports for collaborative problem solving as well investigating the relationship between authority and accountability and the emergence or scarcity of particular kinds of uncertainty.

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References

Engle, R. (2011). The productive disciplinary engagement framework: Origins, key concepts, and developments. In D. Dai (Ed.), Design research on learning and thinking in educational settings. New York: Taylor & Francis Educational Psychology Press.

Engle, R. & Conant, F. (2002). Guiding principles for fostering productive disciplinary engagement: Explaining an emerging argument in a community of learners classroom. Cognition and Instruction, 20(4), 399-483.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S., & Phillips, E. D. (2006). Connected Mathematics 2. Boston: Pearson Prentice Hall.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S., & Phillips, E. D. (2006a). Say It with Symbols: Making Sense of Symbols. Boston, MA: Pearson Student Edition.

Martin, J. R. & Rose, D. (2003). Working with discourse: Meaning beyond the clause. London: Continuum.

Rowland, T. (1995). Hedges in mathematics talk: Linguistic pointers to uncertainty. Educational Studies in Mathematics, 29(4), 327-353.

Shaffer, D.W. (2017). Quantitative ethnography. Madison, WI: Cathcart Press.

Strauss, A. L., & Corbin, J. M. (1998). Basics of qualitative research: Grounded theory procedures and techniques (2nd ed.). Thousand Oaks, CA: Sage.

Wood, M. B., & Kalinec, C. A. (2012). Student talk and opportunities for mathematical learning in small group interactions. International Journal of Educational Research, 51, 109-127.

Zaslavsky, O. (2005). Seizing the opportunity to create uncertainty in learning mathematics. Educational Studies in Mathematics, 60(3), 297-321.

CHALLENGING THE STIGMA OF A SMALL N: EXPERIENCES OF STUDENTS OF COLOR IN CALCULUS I

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Because students of color are underrepresented in undergraduate mathematics classes, their experiences are often ignored in studies drawing on large data sets or are inferred based on the experiences of other populations, specifically women. This exclusion and misrepresentation of students of color is often attributed to methodological limitations. We reexamine the data studied for a previous analysis attending to student race and ethnicity rather than to gender. Due to the smaller numbers of non-white students, we utilize different analytic tools, and draw on students' open-ended responses to a survey question asking about their experiences in Calculus I. In addition to adding to the literature on students from marginalized populations in undergraduate mathematics, this paper argues for a reframing of how we value papers with a small n, and what this value indicates about our value of the students making up the small samples.

Keywords: Equity and Diversity, Research Methods, Gender, Post-Secondary Education

A number of recent studies have been published that draw on a large data set to make strong claims about students' gendered experiences in undergraduate mathematics (see e.g., Ellis, Fosdick, & Rasmussen, 2016; Laursen et al., 2011). In each of these studies, the researchers had access to the students' race and ethnicity, but were unable to conduct the same statistical analyses differentiating by race and ethnicity as they did by gender because of the small sample of students from non-white populations. For instance, in discussing learning gains between the active learning courses (specifically Inquiry Based Learning; IBL) and non-active learning courses, Laursen and her colleagues (2011) state: "We could only compare [the learning gains of] white and Asian students, because the number of other students of color in our sample was very low (see Appendix A3)" (p. 55). Similarly, in a recent publication investigating factors related to students' and instructors' experiences in calculus: Hagman, Johnson, and Fosdick (2017) state: "We do not investigate the association between race or ethnicity and [opportunities to learn] due to the small proportion of non-white students and instructors in our study" (p. 5).

Because of the underrepresentation of students of color in undergraduate mathematics courses, their experiences in these courses are made invisible in studies that draw on large data sets, or are inferred based on the experiences of other underpenetrated populations, specifically women. For instance, researchers use Laursen and her colleagues' (2011) paper as evidence that active learning benefits students from "underrepresented" groups. As a prototypical example of this, Webb (2016) states: "Research has shown that undergraduate students who are involved in active learning techniques can learn more effectively in their classes, resulting in increased achievement and dispositions… particularly so for underrepresented groups (Laursen et al., 2011)" (pp. 1-2). While Laursen et al. (2011) are able to make substantive claims about the benefits of IBL for women and typically low-achieving students, which are both underrepresented groups in STEM, these findings are being generalized to make claims about underrepresented students in general, which is often taken to specifically include students from underrepresented racial and ethnic minorities.

While there are some studies about such students' actual experiences in undergraduate mathematics (see Adiredja & Andrews Larson, 2017 for a review of this literature), such studies are limited and, due to the underrepresentation of such students in undergraduate mathematics, draw on a smaller data set and often employ qualitative methods. For instance, McGee and Martin (2011) studied the experiences of 23 Black mathematics and engineering college students, Levya (2016) studied the experiences of five Latinx engineering college students, and Adiredja and Zandieh (2017) studied the experiences of 8 Latina's in a Linear Algebra course. In comparison, Laursen et al. (2011) drew on survey data from 1,100 students and Ellis, Fosdick, and Rasmussen (2016) analyzed data from 2,266 students, with about 50% identifying as female in each study.

Because students of color are underrepresented and their numbers are small, their experiences are often ignored or are inferred based on the experiences of other underrepresented populations, specifically women. This "exclusion and misrepresentation of [students of color] in education research" is often attributed to methodological limitations (Teranishi, 2007, p. 38). In this study, we reexamine the data studied for a previous analysis (Ellis, Fosdick, & Rasmussen, 2016) attending to student race and ethnicity rather than to gender. Due to the smaller numbers of students of color, we utilize different analytic tools, and draw on students' open-ended responses to a survey question asking about their experiences in Calculus I. In addition to adding to the literature on students from marginalized populations in undergraduate mathematics, this paper seeks to argue for a reframing of how we value papers with a small n, and what this value indicates about our value of the students making up the small samples.

Students of color continue to face systemic racism and racialized negative narratives in mathematics classrooms (Spencer & Hand, 2015). Despite the large and still-growing academic research and literature unveiling these disparities and systems of oppression, dominant mathematics education narratives continue to defend the "objectivity" of mathematics as a color-blind discipline (Martin, 2013). While this color-blind narrative put forth in defense of the continued practice of marginalization of students of color has been challenged for many years, the issue remains of low concern and priority in higher education, perhaps largely due to the continued low numbers of students of color in college mathematics courses. This underrepresentation, while itself an area of high concern, has set up those students of color who do gain access to college mathematics to have their experiences and even their entire presence often completely erased in research surrounding college/university mathematics education research. One significant contributor to this phenomenon is the practice of removing would-be outliers and/or subgroups within large data sets that have small sample sizes.

Methods

Background and Data Collection

The data used for these analyses draws on a national data set made available by the Mathematical Association of America. This data was collected by surveys sent to Calculus I students at the beginning and the end of the course. We draw on the larger data set containing survey responses (n=9793) and a smaller data set containing students' free responses on the survey (n=520). Students were grouped into three race/ethnicity groups. We understand and acknowledge the problems in these groupings based on a number of issues. Based on students' responses to race/ethnicity questions, students were categorized as White (68% of large data set, 67% of smaller), Asian (14%, 12%), and Underrepresented Students of Color (17%, 19%), comprised of students who identify as Black, Pacific Islander, Native American or Alaska Native, Hispanic/Latinx, multiple race/ethnicity identities, and race/ethnicities not listed on the survey.

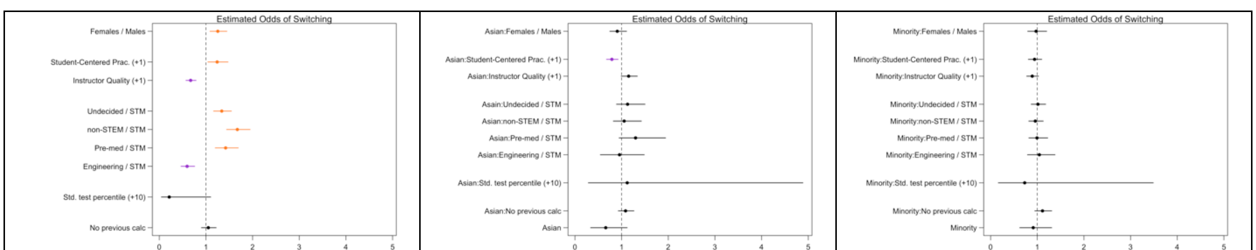
Sample Data Analysis

The first analysis is the coarsest, and attempts to mimic the analysis of the Ellis, Fosdick, & Rasmussen (2016) analysis as much as possible by grouping students into three race and ethnic categories. In this analysis, we run a regression with the outcome of persistence in calculus, and use the same predictors as those used in the initial gender comparison. i.e. previous calculus experience, math standardized test percentile (ACT/SAT), major, scores of teaching quality and student centered instruction in their course, and gender. Additionally, we added interaction terms with each of the predictor variables for Asian students and underrepresented Students of Color; we explain the significance of these interaction terms below. The second analysis investigates students' reasons for switching, and we display student responses grouped by gender and the three race/ethnicity groups identified in the first analysis

Sample Results

The output variable used for the gender-comparison study was student persistence through the calculus sequence. In that study, we looked at the relationship between gender and student persistence, controlling for a number of factors that may be related, such as career intentions and previous calculus experience. Through that analysis, we found that female students were 50% more likely to be identified as Switchers compared to male students, after controlling for a number of factors. This result was very statistically significant, which we were able to test for because of the large number of students and, more specifically, the large number of students who identified as male (n=1,236) or female (n=1,030).

For the first analysis in this paper, we attempt to mimic the gender-comparison analysis as much as possible by identifying students as either White (n=6,674), Asian (n=2,921), or underrepresented Students of Color (n=1,699). Due to the large majority of White students in our sample, we wondered if the results found in Ellis, Fosdick, & Rasmussen (2016) really represented the experiences of only White students. To test this, we rerun the analysis for Asian students and unrepresented Students of Color and compare this to the original analyses of all students.



1a. All students	1b. Asian	1c. UR SOC

Figure 1a-1c. Odds ratios of switching for all students, Asian students, and underrepresented Students of Color (URSOC).

Figure 1 displays the estimated odds ratios of switching out the calculus sequence for each of the predictor categories, for each of the three populations. An estimated odds ratio bigger than 1 (highlighted in orange) indicates a student with this quality has a higher chance of switching out of calculus and if the odds ratio is smaller than 1 (highlighted in purple), then the student has a lower chance of switching.

In our second analysis, we investigate Switchers' reasons for switching, disaggregated their gender and race/ethnicity group. On the end of term survey, all students who said that they either did not intend to take Calculus II, or weren't sure, we given a list of possible reasons for why, and were asked to select all that apply for them. While the numbers of responses are small for some items, we can compare relative frequency and see that, for example, a larger ratio of URSOC women (17/53) attribute a lack of understanding to their decision compared to Asian women (1/53) and White women (40/176, which would be equivalent to approximately 12/53). This second analysis compliments the first by showing that, while the regression analysis did not indicate significant differences related to persistence based on the factors we investigated, students from different populations attribute their decision for leaving to very different aspects of their experience. While this analysis allows us to identify trends between the different racial and ethnic groups of students, we cannot identify the strength of these trends due to the small n's. Our third analysis analyzes students' free responses to the question "Is there anything else you want to tell us about your experience in Calculus I?"

Brief Discussion

The goal of this paper is to bring attention to the normative practice in our community of ignoring the experiences of students of color in our quantitative studies. While our qualitative colleagues work to richly understand and document the experiences of students of color in our undergraduate classes, and while work to increase the representation of students of color in our classes and in our data sets, we must challenge and overcome the stigma of a small n. This paper indicates that the experiences of students of color are (a) different from the experiences of women, (b) not all the same, and (c) are more complex than statistics alone indicate.

References

Adiredja, A. P., & Andrews-Larson, C. (2017). Taking the Sociopolitical Turn in Postsecondary Mathematics Education Research. *International Journal of Research in Undergraduate Mathematics Education,* 1-22.

Adiredja, A. P., & Zandieh, M. (2017) Using Women of Color's Intuitive Examples to Reveal Nuances about Basis. *Paper presented at the 20th Annual Conference on Research in Undergraduate Mathematics Education*, San Diego, CA.

Creswell, J. W., & Clark, V. L. P. (2007). Designing and conducting mixed methods research.

Ellis, J., Fosdick, B.K., Rasmussen, C. (2016). Women 1.5 Times More Likely to Leave STEM Pipeline after Calculus Compared to Men: Lack of Mathematical Confidence a Potential Culprit. *PLoS ONE* 11(7): e0157447.

Hagman, J.E., Johnson, E., & Fosdick, B. (2017). Factors Contributing to Experiencing a Lack of Time in College Calculus. *International Journal of STEM Education, 4*(12).

Laursen, S., Hassi, M. L., Kogan, M., Hunter, A. B., & Weston, T. (2011). Evaluation of the IBL mathematics project: Student and instructor outcomes of inquiry-based learning in college mathematics. *Colorado University*.

Martin, D. B. (2013). Race, racial projects, and mathematics education. *Journal for Research in Mathematics Education,* 44(1), 316–333. https://doi.org/10.5951/jresematheduc.44.1.0316

McGee, E. O., & Martin, D. B. (2011). "You would not believe what I have to go through to prove my intellectual value!" Stereotype management among academically successful Black mathematics and engineering students. *American Educational Research Journal*, *48*(6), 1347-1389.

Spencer, J. A., & Hand, V. M. (2015). The racialization of mathematics education. In *The race controversy in American education* (Vol. 1, pp. 237–258). Santa Barbara: ABC-CLIO.

Teranishi, R. T. (2007). Race, ethnicity, and higher education policy: The use of critical quantitative research. *New Directions for Institutional Research*, *2007*(133), 37-49.

Webb, D. C. (2016). Applying Principles for Active Learning to Promote Student Engagement in Undergraduate Calculus.

NETWORKING THEORIES TO DESIGN A FULLY ONLINE ASSESSMENT OF STUDENTS' COVARIATIONAL REASONING

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Networking theories of different grain sizes, we designed a fully online assessment of students' covariational reasoning. With this assessment, we intend to produce a viable means of measuring students' mathematical reasoning using methods other than clinical, task-based interviews. The assessment is fully online, and readily accessible across different types of devices. We outline design aspects across and within the assessment items and provide three theoretically based design principles underlying the design of the assessment. Through this research, we contribute to the development of new theoretical approaches to investigate and assess complexities of students' mathematical reasoning.

Keywords: Assessment and Evaluation, Research Methods, Reasoning and Proof, Technology

We address the problem: How can a fully online assessment be developed, in place of clinical interviews, to make inferences into students' covariational reasoning? Building from the work of Norton, Tzur, and colleagues (Hodkowski, Hornbein, Gardner, Johnson, Jorgensen, & Tzur, 2016; Johnson, Tzur, Hodkowski, Jorgensen, Wei, Wang, & Davis, in press; Norton & Wilkins, 2009), we designed an assessment to measure undergraduate college algebra students' covariational reasoning. We extend existing research in two ways. First, we developed a fully online assessment, rather than a paper and pencil tool. Second, we designed our assessment to measure covariational reasoning, rather than multiplicative or fractional reasoning.

We aimed to not only assess students' covariational reasoning, but also to distinguish *gradations* in students' covariational reasoning (see also Johnson et al., in press). Building from the work of Johnson and colleagues (Johnson, McClintock, Hornbein, Gardner, & Greiser, 2017), we networked, or interweaved, theories to design assessment items. Interweaving Thompson's theory of quantitative reasoning (Thompson, 1994; Thompson & Carlson, 2017) and Marton's variation theory (Kullberg, Kempe, & Marton, 2017; Marton, 2015), we designed within and across assessment items. To distinguish gradations in students' covariational reasoning, we drew on Tzur's method of fine grain assessment (Tzur, 2007). Using the fine grain assessment method, we designed our assessment with the intent to investigate how students' opportunities to conceive of variation in individual attributes might foster their covariational reasoning. We outline design aspects within and across assessment items and provide three theoretically based design principles underlying the design of the assessment.

Theoretical and Conceptual Framework

Networking Theories to Design Within and Across Assessment Items

To design within and across assessment items, we networked Thompson's theory of quantitative reasoning (e.g., Thompson, 1994; Thompson & Carlson, 2017) and Marton's variation theory (e.g., Kullberg et al., 2017; Marton, 2015). Marton and colleagues (Kullberg et al., 2017; Marton, 2015) identify two key aspects of variation theory: Discernment and variation. For students to discern critical aspects of an object of learning, they need to experience variation (difference). Specifically, learners should experience variation in critical aspects across a background of invariance. Then learners should repeat experiences across different backgrounds. In our set of assessment items, we designed for variation (difference) within assessment items (a background of invariance), then across assessment items (different backgrounds).

To explain the object of learning—covariational reasoning—we appeal to Thompson's theory of quantitative reasoning (e.g., Thompson, 1994; Thompson & Carlson, 2017). In this theory, Thompson draws on students' conceptions of attributes to explain students' mathematical reasoning. In particular, some attribute is a quantity if an individual conceives of that attribute as possible to measure. By covariational reasoning, we mean students' conceptions of relationships between attributes that are capable of varying and possible to measure. For example, consider a toy car moving around a square track. A student engaging in covariational reasoning could conceive of a relationship between the toy car's total distance traveled and the toy car's distance from a center point on the track.

Across our assessment items, we included situations incorporating attributes having different kinds of variation (change). To clarify, we distinguish this use of variation (change) from Marton's use of variation (difference). Across the items, we varied not only the direction of change in attributes (e.g., increases or decreases); we also incorporated variation in unidirectional change in attributes (e.g., "increasing" increases or "decreasing" decreases).

The Fine Grain Assessment Method

To distinguish gradations in students' covariational reasoning, we adapted Tzur's (2007) method of fine grain assessment. When using fine grain assessment methods, designers begin with items that include no supports, then move to subsequent items including increasing amounts of supports. Because items including no supports appear before items with supports, assessment designers have the potential to investigate different levels, or gradations of students' reasoning (see also Hodkowski et al., 2016; Johnson et al., in press). In particular, we intend to use this assessment to investigate how opportunities to conceive of variation in individual attributes might foster students' covariational reasoning.

The Covariational Reasoning Assessment

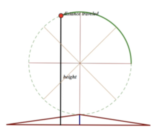
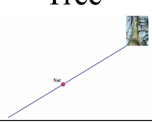
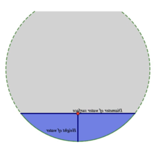
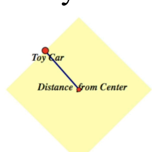
Table 1 provides a map of the covariation items. The covariational reasoning assessment contains four assessment items. Each assessment item contains four question groups.

Assessment Items and Question Groups

Assessment items. Each assessment item incorporates a situation involving changing attributes. We incorporated variation (difference) across the collection of assessment items. The collection incorporates different kinds of attributes (e.g., height from the ground, total distance traveled, diameter of the water surface of a fishbowl, and distance of a car from a center point). Furthermore, the collection incorporates different kinds of variation (change): Variation in the direction of change and variation in unidirectional change (e.g., an "increasing" increase).

Question groups. Each assessment item contains four question groups. Question group 1 serves as a member check, to assess if students comprehend the situation. Question group 2 serves as the first assessment of covariational reasoning: Reasoning without support. Question group 3 provides support for students' covariational reasoning. Question group 4 serves as a second assessment of covariational reasoning: Reasoning with support. We designed our question groups based on Tzur's (2007) method of fine grain assessment. If students respond correctly to Question group 2, they will move to a new assessment item. If students respond incorrectly to Question group 2, they will move to Question groups 3 and 4.

Table 1: Map of the covariation items in the covariational reasoning assessment

Assessment Item	Question Group 1: Member Check	Question Group 2: Reasoning without support	Question Group 3: Supports for Reasoning	Question Group 4: Reasoning with Support
Ferris Wheel  Nat, Path, Tree  Fish bowl  Toy Car 	Play Video Statement of Attributes Do you understand the situation? •Yes •No •If No, explain what makes this situation difficult to understand •If Yes, move to Q2	Play Video Select a graph that represents a relationship between the attributes •A •B •C •D Explain why you chose the graph you did •If Incorrect, move to Q3 •If Correct, move to next assessment item	Video of dynamic points moving along axes Describe how attribute 1 is changing Video of dynamic points moving along axes Describe how attribute 2 is changing	Play Video Select a graph that represents a relationship between the attributes •A •B •C •D Explain why you chose the graph you did

Design Principles

Assess for a Spectrum, Rather than a Switch

We use the analogy of a spectrum versus a switch to communicate our work to move beyond binaries in assessing students' covariational reasoning. Applying Tzur's (2007) method of fine grain assessment, we designed to distinguish gradations in students' covariational reasoning. We anticipate our gradations to be compatible with, yet perhaps not identical to, levels of covariational reasoning put forward in the framework of Thompson and Carlson (2017).

"Practically" Apply Theories to Design Assessment Items

We aim to interweave and apply theories to do practical work of assessment design. With Thompson's theory of quantitative reasoning, we designed assessment items in which students could have opportunities to conceive of attributes as possible to measure and capable of varying. Therefore, we interrogated the types of attributes and variation (change) included in the assessment. With Marton's variation theory, we designed for variation (difference) within and across assessment items. Within assessment items, we incorporated different types of graphs (e.g., linear/nonlinear). Across assessment items, we incorporated different backgrounds (e.g., a Ferris wheel, a toy car, etc.) and different types of variation (change) in attributes (e.g., variation in direction of change and variation in unidirectional change).

Leverage Technology to Promote Access and Opportunity

We leverage multiple technological affordances to promote greater student access and opportunity. Students can access the covariation assessment via a computer, a tablet, or a mobile phone. We created the animations following design factors for effective educational multimedia, including multiple representation types, pacing, cueing, and user manipulation (Plass, Homer, & Hayward, 2009). The assessment incorporates various multimedia, such as original video animations, which provided multimodal representations of dynamic graph attributes.

Concluding Remarks

Networking theories, we interweave variation (change) and variation (difference) in assessment design. We leverage technology to apply Tzur's method of fine grain assessment. Overall, we are working "practically" to network and apply theories to design accessible tools to investigate and assess gradations in undergraduate students' covariational reasoning.

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References

Hodkowski , N. M., Hornbein, P., Gardner, A., Johnson, H. L., Jorgensen, C., & Tzur, R. (2016, November). Designing a stage-sensitive written assessment of elementary students' scheme for multiplicative reasoning. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1581-1587). Tucson, AZ: The University of Arizona.

Johnson, H. L., McClintock, E., Hornbein, P., Gardner, A., & Grieser, D. (2017). When a critical aspect is a conception: Using multiple theories to design dynamic computer environments and tasks to foster students' discernment of covariation. In Dooley, T., & Gueudet, G. (Eds.). *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education* (CERME10, pp. 2738-2745). Dublin, Ireland: DCU Institute of Education and ERME.

Johnson, H. L., Tzur, R., Hodkowski, N., Jorgensen, C., Wei, B., Wang, X., & Davis, A. (in press). A written, large-scale assessment measuring gradations in students' multiplicative reasoning. *To appear in the Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education*.

Kullberg, A., Kempe, U. R., & Marton, F. (2017). What is made possible to learn when using the variation theory of learning in teaching mathematics? *ZDM: The International Journal on Mathematics Education*, *49*(4), 559–569.

Marton, F. (2015). *Necessary conditions of learning*. New York: Routledge.

Norton, A., & Wilkins, J. L. M. (2009). A quantitative analysis of children's splitting operations and fraction schemes. *The Journal of Mathematical Behavior*, *28*(2), 150–161.

Plass, J. L., Homer, B. D., & Hayward, E. O. (2009). Design factors for educationally effective animations and simulations. Journal of Computing in Higher Education, 21(1), 31-61.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY: State University of New York Press.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation and functions: Foundational ways of mathematical thinking. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.

Tzur, R. (2007). Fine grain assessment of students' mathematical understanding: participatory and anticipatory stages in learning a new mathematical conception. *Educational Studies in Mathematics*, *66*(3), 273–291.

INTERPRETIVE FRAMEWORK FOR ANALYZING COLLECTIVE LEARNING IN A MATHEMATICS CLASSROOM

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An interpretive framework for organizing and reducing data from a whole class teaching experiment for further analysis is described. Design research typically generates extensive data and can become potentially unmanageable for analysis. The framework described in this paper uses a situated cognitive perspective: Meaning making arises out of activity within the context of the classroom. Each layer affords and constrains each other. Data from a whole class teaching experiment illustrates how the framework was used to reduce and catalog the data for further analysis. The framework can be adapted by anyone who is engaging in design research.

Keywords: Design Experiments, Research Methods

Educational researchers are continually engaging in design research through whole class teaching experiments as a way to study and observe learning. Collective classroom learning is the normative understandings that emerge in a classroom community (Cobb & Yackel, 1996). Most whole class teaching experiments aim to analyze student learning by repeating the process of developing and testing instructional activities. The process of collecting and analyzing data in a design research project is messy as it embraces the complexity of the settings in which the learning occurs (Cobb, Zhao, & Dean, 2009). In order to study learning, researchers must have a variety of data sources to analyze and any source that relates to the broader phenomena being studied in the experiment should be collected (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). Data sources include notes from the field, video recordings, and documentation of anything that occurs in the classroom such as students' work, class discussions, and the teacher's reflections. Theories, forming from analysis, inform the design decisions that must be made throughout the teaching experiment. These ongoing interpretations of the activity in the classroom profoundly shape the process of experimenting to support learning (Cobb & Gravemeijer, 2008).

In teaching experiments using design research, the focus is on the whole class' mathematical learning as they practice classroom communal processes (e.g. Cobb, et al., 2009; Confrey & Smith, 1995; Lehrer & Schauble, 2004; Stephan, Bowers, & Cobb, 2003) and participate in learning as a whole group. Learning in this way is a social phenomenon in which mathematical ideas are established in a classroom community through patterns of interaction (Rasmussen & Stephan, 2008). Mathematical learning is both a process of constructing meaning as an individual and a process of acculturation of meanings in society (Eisenhart, 1988). This implies that learners of mathematics learn individually, but also through interaction, eventually constructing new knowledge.

Design researchers document the process of analysis and design in a form intended for public reflection and discussion (Edelson, 2002). We used a framework to make ongoing interpretations and conjectures about the collective learning that took place to guide our thinking process through the design. The framework relates to the conference theme of "Looking Back, and Looking Ahead" because it allows researchers to look back at what happened in a whole class teaching experiment and look ahead with a deeper understanding of student learning. Our purpose in this article is to describe the interpretive framework that we employed to analyze collective classroom learning and make design decisions. Practical applications of the layers of the framework contribute to the field of design research methodology in a whole class teaching experiment and empower teaching experiences.

The Interpretive Framework

The situated cognitive perspective provides a useful lens to study collective classroom learning. According to this perspective, all knowledge is considered situated and is regarded as a product of the activity, context, and culture in which it is produced and used (Brown, Collins, & Duguid, 1986). The situated cognitive perspective has made significant contributions over recent years as presenting knowledge not as a stable individual characteristic, but something that is distributed across activities, systems, and environments (Lave & Wenger, 1991). A challenge of studying the activity in a classroom from a situated perspective is to ensure that what is observed in the classroom is not reduced to single acts or products. Rather, the goal of developing such a framework is to make sense of classroom interactions as continuous, mutually constitutive, and evolving practices where students are engaged in making mathematical meaning. To frame collective learning, we adopt Davis' and Simmt's (2003) view that a classroom is a complex system, "not just the sum of its parts, but the product of the parts and their interactions" (p. 138). Our intention in this paper is to connect the conceptual and methodological theory of collective learning in a classroom to the pedagogical.

The interpretive framework used in this study (see Figure 1) was adapted from (de Silva, 2001) and accounts for individual and collective learning by considering the micro-culture of the classroom (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997). The students' mathematical understandings emerge as they engage in activity within the context of the classroom through social interaction and tools (Cobb, Stephan, McClain, & Gravemeijer, 2001). The interpretive framework reflects that sense making takes place through activity within the context of the classroom as illustrated in Figure 1. The arrows represent affordances and constraints influenced by each nested layer.

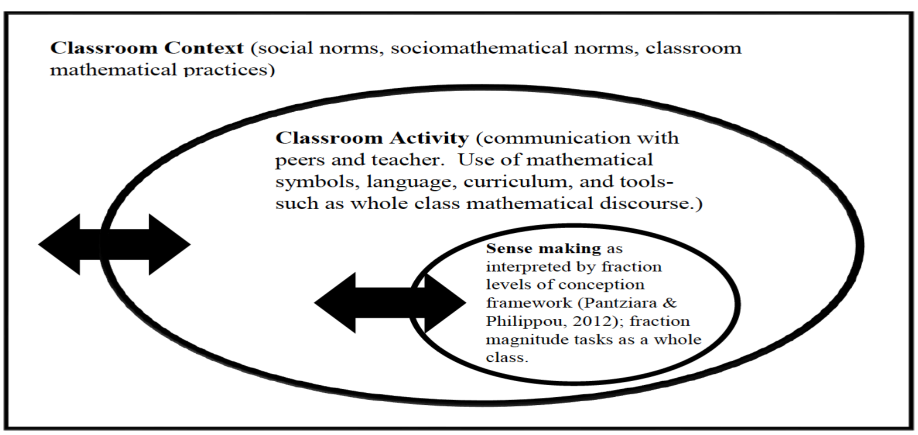

Figure 1. The Interpretive Framework used to analyze collective learning in a classroom (Bertolone-Smith, 2016).

Collective learning contains *sense making* at its core nested within activity that is nested in the context of a classroom. This model of collective learning specifies how sense making of mathematics is situated in a whole class setting. The core of the interpretive framework is the nested layer of sense making. This approach to sense making takes into account both the social perspective (the meanings the class is making as a whole) and the psychological perspective (the meanings and interpretations of individual students as they participate in the classroom community). This layer is critical when conducting a classroom design experiment with students because the purpose of the research is to study learning. The current study needed to address students' learning of fractions in the instructional unit to understand how they were making sense of the tasks and, subsequently, learning. These classroom mathematical practices, "taken-as-shared ways of reasoning, arguing, and symbolizing established while discussing particular mathematical ideas" (Cobb et al., 2001), provide a reference for sense making.

The second layer of the interpretive framework is *activity.* The role of activity in a social setting is to arrive at a common understanding. In the current study, activity accounted for the practices in the classroom and the assumption that students had their own way of communicating and interacting with each other, the teacher, and the content. The *Common Core State Standards for Mathematics* (NGA/CCSSO, 2010) and the *Principles and Standards for School Mathematics* (2000) emphasize communication as essential for how students learn mathematics. Activity occurs through student-student interactions, student-teacher interactions, and student-content interactions. Mathematical learning is both a process of constructing meaning as an individual as well as a process of acculturation of mathematical meanings in society (Eisenhart, 1988). Students learn mathematics individually but also through interaction with each other, as a process that occurs during social interactions, a circular sequence of events (Cobb, Yackel, & Wood, 1992). The theory of social constructivism (Vygotsky, 1978) reflects this perspective where the central tenet is that learners construct their own understanding by participating in meaningful shared discourse. Activity also includes the students' interactions with curriculum and representational tools.

The third layer of the interpretive framework is *context*. Student learning of mathematics is affected by experiences in the classrooms and these experiences are created by and depend on the context of the environment, including the teacher and other students. A context conducive to learning considers the physical environment and also the social environment that includes the interactions that occur among teachers, students, and content (Cohen & Ball, 1999).

In the overall framework, sense making is afforded and constrained by activity that in turn is afforded and constrained by the context. Hence, the framework can be thought of as three layers of nested concentric ovals that can be expanded and collapsed; *sense making* nested within *activity* further nested within *context*. Sense making is interpreted by how students think about and make meaning of the tasks that occur within the activity in light of the context of the classroom.

Using the Framework to Organize Data for Further Analysis

The following section illustrates how the framework was used to analyze and organize data from a whole class teaching experiment (Bertolone-Smith, 2016). We describe a methodological approach to organize and reduce data for further analysis. The data presented in this section represents a sample from a whole class teaching experiment conducted for illustrative purposes. The data collected as part of the whole class teaching experiment in this illustration included video of the teaching sessions, video of student discussions and comments, video and field notes of student interviews and discussions, video and field notes of teacher and researcher debriefing interview sessions, lesson plans, and student work. Analysis of the data occurred every day after each teaching episode. All lessons were transcribed and then analyzed through the interpretive framework. The raw transcript was organized to indicate classroom context, activity and sense making. Since mathematical discourse was a focus of the study, conversations were preserved to better understand how they impacted or represented the shifts in understanding.

Table 2: Emerging Categories through Context, Activity, and Sense Making

	Classroom Context	Classroom Activity	Sense Making
Emerging Categories	Teacher addressing student interruption. Practice expected behaviors. Prime class for "hard work".	Student generated justification of labeling includes distinguishing whole, gesturing to indicate distance by drawing hand along number line. Independence and accuracy celebrated	Student noticing that the whole "unit" and number of parts in whole generates fraction. Student counting by fractional pieces, gesturing across number line to indicate accuracy to justify to class
	Floating dialogue throughout lessons. *Adjusting mindset* of class when lesson increased cognitive demand	*Gesturing* by teacher adapted by students to *reinforce fraction distance from 0.*	Shift from *relying solely on part-whole construct* of fractions to *distance from 0.*

Implications and Discussion

The interpretive framework (Figure 1) is helpful for reducing and organizing data when conducting design research (Cobb et al., 2003) by illuminating aspects of the learning ecology to make design decisions during the experiment and retrospectively study the conditions that contributed to the sense making that took place.

References

De Silva (2001). Quotient construct, inscriptional practices and instructional design. Unpublished Doctoral dissertation, Arizona State University.

Bertolone-Smith (2016). A fourth-grade teaching experiment on fraction magnitude. Doctoral dissertation, University of Nevada, Reno.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, *32*(1), 9-13.

Cobb, P., & Gravemeijer, K. (2008). Experimenting to support and understand learning processes. In A. E. Kelly, R. A. Lesh, & J. Y. Baek (Eds.), *Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching* (pp. 68–95). New York, NY: Routledge.

Cobb, P., Gravemeijer, K., Yackel, E., McClain, K., & Whitenack, J. (1997). Mathematizing and symbolizing: The emergence of chains of signification in one first-grade classroom. In D. Kirshner, & J. Whitson (Eds.), *Situated cognition: social, semiotic, and psychological perspectives* (pp.151-234) Mahwah, NJ: Lawrence Erlbaum Associates.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*(3/4), 175-190.

Cobb, P., Zhao, Q., & Dean, C. (2009). Conducting design experiments to support teachers' learning: A reflection from the field. *Journal of the Learning Sciences*, *18*(2), 165-199.

Corbin, J., & Strauss, A. (2014). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (Fourth ed.). Thousand Oaks, CA: Sage.

Edelson, D. C. (2002). Design research: What we learn when we engage in design. *Journal of the Learning Sciences*, *11*(1), 105-121.

Eisenhart, M. A. (1988). The ethnographic research tradition and mathematics education research. *Journal for Research in Mathematics Education*, *19*(2), 99-114.

Lehrer, R., & Schauble, L. (2004). Modeling natural variation through distribution. *American Educational Research Journal*, 41, 635–679.

.Stephan, M., Bowers, J., & Cobb, P. (Eds.). (2003). Supporting students' development of measuring conceptions: *Analyzing students' learning in social context* (Journal for Research in Mathematics Education Monograph No. 12). Reston, VA: National Council of Teachers of Mathematics.

Vygotsky, L. (1978). Mind in society: The development of higher psychological processes. Cambridge, MA: Harvard University Press.

THEORIZING A TRANSLANGUAGING STANCE: ENVISIONING AN EMPOWERING PARTICIPATORY MATHEMATICS EDUCATION *JUNTOS CON* EMERGENT BILINGUAL STUDENTS

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We propose a translanguaging stance in the bilingual mathematics classroom defined by four elements: (1) Con respeto for other people's ideas, leading to positive intellectual relations, (2) Con cariño, a commitment to the learning of others, (3) Como familia, working for the good of the classroom collective and for the benefit of individuals, and (4) Con acompañamiento, where teachers actively do math with their students. We propose a focus for the bilingual mathematics classroom and emergent bilingual students that emphasizes the importance of rehumanizing mathematics classrooms for all students and valuing their full linguistic repertoires.

Keywords: Culturally Relevant Pedagogy, Teacher Affect, Beliefs, Identity

Mathematics education research advocates for practices that celebrate all students' mathematical reasoning and ways of knowing (Turner & Drake, 2016.) There remains a need to study what this means for emergent bilingual (EB) students, especially given the deficit-oriented historical context of U.S. education policy around language with an overemphasis on English mastery (Gandara, et al., 2010). We see an urgent connection between envisioning a different mathematics education for EB students and PMENA's enduring conference challenge of both preparing teachers of EB students and continuing to support *all* students through equitable teaching. Translanguaging, an evolving understanding of language practices (García & Wei, 2014), provides a different lens through which to view multilingual mathematics classrooms.

We briefly describe a translanguaging stance and make relevant connections to mathematics education research moving the field forward as we propose this new lens. We base our arguments on mathematical lessons observed in a 2nd grade dual language class and make an argument for how all EB students could participate in mathematics teaching and learning through ways that are rehumanizing to all people in the classroom space and to mathematics itself.

Theoretical framework

García and Kleifgen (2010) encourage moving past traditional language separation practices (i.e, English day and Spanish day) and instead consider the "dynamic bilingual practices" which emergent bilinguals enact and are context and person specific. García, Ibarra Johnson & Seltzer (2017) define a translanguaging classroom as a "space built collaboratively" with their own language practices, and teaching and learning in "deeply creative and critical ways" (pg.2). Due to a lack of translanguaging studies in elementary mathematics classrooms, we further argue that it is necessary for mathematics educators to cultivate a *translanguaging stance* and build classrooms in which teachers and students *juntos* (together).

We conceptualize a translanguaging stance in the mathematics classroom as the deliberate choice by teachers to use children's thinking while engaging in mathematics instruction that develops knowledge, dispositions and practices that not only support the development of children's mathematical thinking, but also build on students' cultural, linguistic and community-based knowledge (Turner et al., 2012). We posit that students can not only expand their sense of what they can do mathematically, but also develop a sense of what mathematics can be, and develop mathematical identities characterized by power and possibility (Adams, 2018). Building on an active anti-racist stance that first acknowledges EB student's racialized schooling and mathematics learning experiences in US schools (Martin, 2009), we theorize classroom practices in a translanguaging mathematics classroom. García, Ibarra Johnson & Seltzer (2017) identified four translanguaging classroom elements that support and value EB students: *con respeto, con cariño, como familia, and con acompañamiento*. We briefly present each of these elements and supporting mathematics education research.

Four Elements of a Mathematics Translanguaging Classroom

***Con respeto* (with respect)**. Two second grade EB students, Rubén and Hernán, shared their strategies, one in English and one in Spanish, for a subtraction problem, while the teacher, Ms. Amaris used mathematical notation to represent their mathematical thinking. Ms. Amaris elicited mathematical ideas in both languages and did not provide translation, but let the notation and sharing of the strategies remain the focus of the conversation. When teaching with respect, we observe how our EB students expand the entire class' ways of knowing both by honoring the languages in which students express themselves and the ideas and reasoning that they share. At the same time, by bringing students' knowledge and ideas to bear in discussions of mathematics, we are also showing other ways to *mathematize*, much like translanguaging expands notions of what it means to *language* (García, Ibarra Johnson & Seltzer 2017).

We also observed how the teacher understood children's multidigit subtraction strategies and selected powerful mathematical strategies to be shared with the entire class (Carpenter, et al., 2000). *Con respeto* is enacted when teachers make instructional decisions based on how their students understand mathematics (Franke, et al., 2001). Knowledge of children's mathematical thinking is an important component of how teachers can view their EB students' mathematical thinking in an empowering way, by positioning the students as mathematical thinkers and creating a shift in who "does" mathematics (Turner, Domínguez, Maldonado & Empson, 2013).

***Con cariño* (with fondness).** *Cariño* denotes the love that can bring together all members of a classroom community, along with their languages (García, Ibarra Johnson, & Seltzer, 2017). This includes the ways of doing math that students bring with them and how their mathematical reasoning intertwines with their languages and their experiences to weave ideas.

Ms. Amaris posed a Join Change Unknown problem, part of a unit in which the class explored how children were affected by the Flint water crisis. Her students proposed raising money for Flint children, providing a meaningful context for problems (Dominguez, 2011). The class shared strategies and Hernán shared his incorrect answer in Spanish. Ms. Amaris then explored a common computational error to promote Hernán's powerful subtraction base ten strategy. This was a common practice in Ms. Amaris' classroom: Hernán used his linguistic repertoire by sharing in Spanish, which provided a space where language was a resource for mathematical understanding. In this classroom, encouraging students to express their thinking in the language in which their minds happen to express the idea at that moment, shifted the focus from *form* (English vs. Spanish) to *content* (the mathematical idea being learned).

This corroborates with Moschkovich's (2015) definition of academic literacy in mathematics for English language learners which posits that academic mathematical literacy is not just about the cognitive aspects of mathematics, but also situated in experiences such as discussion and being able to express one's ideas in a known language. It is thus the participation in mathematical practices, in this case discussion of one's strategies, that leads to mathematical proficiency for EB students. In this interaction, Ms. Amaris makes a purposeful decision, *con* *cariño*, because of her knowledge of the student's academic literacy in mathematics, to use Spanish as the means through which to encourage a powerful strategy into the classroom space.

***Como familia* (like a family).** *Familia* as a translanguaging stance in the mathematics classroom takes the form of a group that collectively works to support and promote both the whole group and each individual member. García (2017) describes how, when a classroom is like a *familia*, classroom discussions address content and language along with "inner truths", with the goal of fostering meaningful learning and strong connections. The previous observed lesson showcased how Ms. Amaris' classroom behaves as a *familia* and shows mathematics being learned through a student-centered discussion and connected to the original ideas that students produced. Ms. Amaris often put the responsibility of making sense on students, and through this sense making the original idea shared shifted.

This element connects to vast research on discussion in mathematics education. Highlighted in Moschkovich's (2015) work as an important component of academic literacy in mathematics, other research has iterated how important discussion is for developing conceptual and procedural mathematics understanding (Stein, Engle, Smith & Hughes, 2008). Past studies of how teachers should include EB students in mathematical discussions (Maldonado, Turner, Dominguez & Empson, 2009) have not been viewed through a translanguaging lens, but the element of *como familia* aptly fits the type of classroom that this research advocates for.

***Con acompañamiento* (truly together).** This element goes beyond being together and empathizing for one another and involves actively doing mathematics with one another--in the classroom this means that instruction is planned so that teacher and students engage in learning, *languaging* (García, 2017) and *math-ing* together. Ms. Amaris' classroom shares a deep and profound understanding of *why* mathematics matters and are engaging in mathematics together. In learning mathematics *con acompañamiento* we see how the focus on the ways students learn to treat each other, and the respect they learn for people from circumstances different from their own, produce equitable relations that are encouraged and developed in the classroom. This supports the importance of using TEACH MATH modules, which encourage teachers to view the students and community as sources of mathematics learning (Aguirre, et al., 2012).

Conclusions

We have focused on the importance of a translanguaging stance for teaching mathematics to EB students, but feel that these elements serve the possibility of rehumanizing mathematics classrooms for *all* students who are seen from a deficit perspective. Moving forward, we also need to reconsider how we prepare future bilingual education teachers for this kind of mathematics teaching which shifts the focus from language use to mathematical reasoning. Given the tensions that bilingual teachers face in classrooms, we need to view this education as not only liberatory for the students, but for the teachers as well.

References

Adams, M. (2018). "I can solve all the problems": Latinx students (re)write their math stories. In I. Goffney & R. Gutierrez (Eds.), *Annual Perspectives in Mathematics Education (APME) 2018: Rehumanizing Mathematics for Students who are Black, Indigenous, and/or Latinx*. Reston, VA.: National Council of Teachers of Mathematics.

Aguirre, J., Turner, E., Bartell, T., Kalinec-Craig, C., Foote, M., Mcduffie, A., Drake, C. (2012). Making connections in practice: how prospective elementary teachers connect to children's mathematical thinking and community funds of knowledge in mathematics instruction. *Journal of Teacher Education*, 62(2), 178-192.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (2000). *Cognitively Guided Instruction: A Research-Based Teacher Professional Development Program for Elementary School Mathematics*. Research Report.

Domínguez, H. (2011). Using what matters to students in bilingual mathematics problems. *Educational Studies in Mathematics*, *76*(3), 305-328.
Franke, M. L., Carpenter, T. P., Levi, L., & Fennema, E. (2001). Capturing teachers' generative change: A follow-up study of professional development in mathematics. *American Educational Research Journal, 38*, 653–689.
Gándara, P., Losen, D., August, D., Uriarte, M., Gómez, M. C., & Hopkins, M. (2010). Forbidden language: A brief history of U.S. language policy. In P. Gándara & M. Hopkins (Eds.), *Forbidden language: English learners and restrictive language policies* (pp. 20–33). New York: Teachers College Press.
García, O., Ibarra Johnson, S.., & Seltzer, K. (2017). *The translanguaging classroom: Leveraging student bilingualism for learning*. Caslon.
García, O., & Kleifgen, J. A. (2010). *Education emergent bilinguals: Policies, programs, and practices for English language learners*. New York: Teachers College Press.
Garcia, O. & Wei, L. (2014). *Translanguaging: Language, bilingualism and education.* New York, NY: Palgrave Macmillan
Maldonado, L., Turner, E., Dominguez, H., & Empson, S. (2009). English language learners learning from, and contributing to mathematical discussions. In D. White & J. Silva Spitzer (Eds.), *Mathematics for every student: Responding to diversity, grades Pre-K-5* (pp. 7-22). Reston, VA: National Council of Teachers of Mathematics.
Martin, D. B. (2009). Researching race in mathematics education. *Teachers College Record*, *111*(2), 295-338.
Moschkovich, J. (2015). Academic literacy in mathematics for English learners. *Journal of Mathematical Behavior.* 40, 43-62.
Stein, M., Engle, R., Smith, M., & Hughes, E. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10(4)*, 313-340.
Turner, E., Domínguez, H., Maldonado, L., & Empson, S. (2013). English learners' participation in mathematical discussion: Shifting positionings and dynamic identities. *Journal for Research in Mathematical Education*, 44(1), 199-234.
Turner, E. & Drake, C. (2016). A review of research on prospective teachers' learning about children's mathematical thinking and cultural funds of knowledge. *Journal of Teacher Education*, 67(1), 32-46.
Turner, E., Drake, C., McDuffie, A., Aguirre, J., Bartell, T., & Foote, M. (2012). Promoting equity in mathematics teacher preparation: A framework for advancing teacher learning of children's multiple mathematics knowledge bases. *Journal of Mathematics Teacher Education*, 15(1), 67-82.

ADDRESSING TENSIONS IN MATHEMATICS EDUCATION THROUGH FORMATIVE INTERVENTION RESEARCH PARTNERSHIPS

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This paper describes an approach to cultivating formative intervention research partnerships in mathematics with school districts in the U.S. state of Missouri by researching the diagnosis and specification of problems as framed and experienced by those grappling daily with challenges related to learning and teaching mathematics. Through a mixed methods approach, we engage various stakeholders in identifying and describing important problems of practice, work to specify the inherent tensions that often exist in those challenges, and, by confronting those tensions, initiate a process of innovation rooted in the contexts in which students, teachers, leaders, and parents learn and work.

Keywords: researcher-practitioner partnership, formative intervention research

Changing the nature of mathematics teaching in the U.S. at any significant scale has proven to be extremely difficult (Hiebert, 2013). The predominant approach continues to be some variant of what Freire (1970/2000) criticized as the "banking" method, with teachers narrating largely procedurally-focused demonstrations and asking students to repeat it themselves. A common story in research in mathematics education has been the introduction of some conjectured driver of change (e.g., a professional development program, a new curriculum, different approaches to assessment, etc.), for which researchers work to find schools or districts willing to provide test-sites and subjects for their research, with the idea that an "effective" idea will eventually get "scaled up" (Institute of Education Sciences, 2016). While innovations in education are certainly needed (and worthy of support), and while there are potential benefits for the students, teachers, parents, principals, and district leaders who act as research participants, the problems and designs for solutions are typically identified at the outset by researchers, rarely originating in the communities with whom those researchers briefly partner. Consequently, the innovations that are put in place are often not sustained after the research effort has concluded (Resnick & Hall, 1998), and typically fail to change the "instructional core" of what happens between teachers and students around disciplinary ideas (Elmore, 2004; Hiebert, 2013).

Innovation Through Partnership

To address this shortcoming, different "improvement science" (Lewis, 2015) models for collaborating and conducting research in/with school districts have emerged in recent years, all of which share a commitment to taking up "problems of practice." For example, one of the more prominent models—and one that has been well received in mathematics education—has been *design-based implementation research* (DBIR). As described by Penuel, Fishman, Cheng, and Sabelli (2011), DBIR focuses on persistent problems of practice from multiple stakeholders' perspectives through iterative, collaborative design work. Consistent with the tradition of design research in education (Cobb, Confrey, DiSessa, Lehrer, & Schauble, 2003), the aim of DBIR is to develop theory related to both classroom learning and implementation through systematic inquiry, but it is also concerned with developing capacity for sustaining change in systems. Still, improvement science approaches such as DBIR implicitly require something to be implemented. But what if potential collaborators wish to begin at a less prescribed place, focusing first on understanding and defining the problem(s) and only then co-designing and pursuing potential solutions? As Lewis (2015) noted, "there is relatively little education research in the improvement science tradition, which emphasizes building organization members' understanding of the problem and its causes" (p. 2015).

One approach that holds promise for treating such problem specification as an object of interest (and not merely prerequisite work) is the cultural-historical activity theory tradition of *formative intervention research*. Drawing on Engeström (2011), Penuel (2014) suggested that the methodology includes three key commitments: (a) *focusing on a problem of practice*—a contradiction encountered by participants in their life or work activities (the "germ cell"); (b) *stimulating participants to produce innovations*—by first calling attention to a challenging situation or set of obstacles, and then triggering a process for overcoming those obstacles through design work; and (c) taking as the primary goal the *expansion of participants' agency*—to "enable new forms of collective activity to emerge through direct engagement with the contradictions embedded in practice" (p. 100). These contradictions are, at first, abstract. Through attempts to understand and model their relationship, they are made concrete, through which "learners learn something that is not yet there" (Engeström & Sannino, 2010, p. 2).

As an example in mathematics education, for teachers who strive to provide students with culturally relevant opportunities to engage in mathematical practice and to become proficient in standard ideas and skills, a contradiction in practice might be the inherent challenge of affording learning opportunities that are *emergent* in order to achieve learning goals that are largely *prescribed* (Gutiérrez, 2009; Munter, Stein, & Smith, 2015). Neither emergent nor prescribed learning goals can be eliminated; the transcendence of the contradiction is fundamental to the work (and agency expansion) of teachers and students. But, as the first commitment of formative intervention research above suggests, such "germ cells" cannot be identified a priori, but only as the result of investigating a problem of practice. Instead of pursuing a predetermined agenda or implementing a particular program, "the researcher aims at provoking and sustaining an expansive transformation process led and owned by the practitioners" (Engeström, 2011, p. 606). It is exactly there that our project is attempting to initiate partnerships with school districts—in the investigatory work of understanding problems that students, teachers, principals, parents, and leaders face, and identifying tensions for expansion through partnership. Ultimately, our aim is to change the learning and teaching of mathematics in the state of Missouri. Our starting point, however, is to enlist practitioner partners in co-investigating their current problems of practice.

Methods

Our work is guided conceptually by the three commitments of formative intervention research listed above. Our first step—and the focus of this paper—is to diagnose the problem(s) of practice that Missouri school districts are facing, and how leaders and other stakeholders frame those problems. To accomplish this, we employ a mixed methods approach—primarily through interviews, and supported by quantitative analyses of district data where applicable.

Setting and Sample

To date, our collaboration includes 9 K-12 school districts across the state of Missouri, ranging in size from large urban districts to very small rural districts. Our sample includes 32 district leaders, principals, teachers, and parents/community stakeholders.

Data Sources

Beginning with an initial, relevant contact in each district (e.g., curriculum director), we conducted semi-structured interviews, snowballing out to others from there (Spillane, 2000; Talbert & McLaughlin, 1999), including district leaders, principals, teachers, and other partners (e.g., parents). In each case, we wrote a summary of the interview, shared it with the participant and invited their feedback with respect to the summary's accuracy. Additionally, we made use of publicly available quantitative data for each participating district.

Analysis

After interviewing additional individuals suggested by interviewees and summarizing those interviews, we analyzed the summaries using qualitative analysis software (NVIVO) to identify the (a) problems, (b) underlying causes of those problems, and (c) responses to those problems described across all interviews in the district. In general, our interest was in understanding how individuals frame the problems that they identify, for which the three framing tasks described by Benford and Snow (2000) are applicable. The first, *diagnostic framing*, concerns the underlying cause(s) of a problem. The second, *prognostic framing*, pertains to the proposed solutions for (or responses to) a problem. The third, *motivational framing*, which "provides a 'call to arms' or rationale for engaging in ameliorative collective action" (Benford & Snow, 2000; p. 617) represents one task that we hope to work toward in collaboration with participating districts. We also conducted quantitative analyses of publically available data for the district that are relevant to the problems described by interviewees. We then wrote a synthesis report for participants to review and to initiate a follow-up discussion, in which we called attention to similarities and differences between participants with respect to not only what problems, causes, and responses were identified but also how those problems, etc. were framed—with the intention of sparking subsequent investigations of those problems and possible responses, which, in some cases, we might take up in continued partnership.

Results

Thus far, a number of patterns have emerged across multiple districts. For example, a key concern in a number of districts has been students' number sense in the early grades. However, the underlying causes of that problem that participants identify—both among and within school districts—vary widely. For example, some suggest the problem is a matter of teachers' mathematical knowledge (e.g., Hill, Schilling, & Ball, 2004) or views of students (Jackson, Gibbons, & Sharpe, 2017), while others view the cause as curricular in nature (e.g., an ineffective program, the lack of a program, or the lack of "vertical alignment" across years). A second finding concerns broader considerations of curriculum. By looking across multiple districts, we have pieced together a model of a "curriculum cycle," in which districts repeatedly gather and then relinquish control for determining curriculum. In such a cycle, districts often adopt a new program as a means for improvement. Initially, teachers are expected to strive for fidelity of implementation (i.e., using the program "as intended" by developers). However, this expectation can lead to over-scriptedness, which can limit teachers' professional autonomy and ability to respond to student needs and interests. In reaction, districts relax expectations, inviting teachers to adapt materials as they see fit. But, over time, this leads to the program being phased out, until the district has returned to a state of teachers drawing on different sources to develop their own instructional materials, with little coordination (or vertical alignment) across grades.

Discussion and Conclusion

Our efforts are dual in nature. First, by investing in understanding the problems of practice that district leaders face, including how individuals within the same community might frame those problems differently, we hope to initiate partnerships at the ground level—a co-investigation of problems before anything is "implemented." Second, through such a process, we hope to find inherent tensions on which we can focus sustained attention as a means for inducing innovation. For example, in initial presentations of our findings to potential partnering school districts, we have used a representation of the curriculum cycle to invite participants' reflection on whether a tension exists between teacher autonomy and centralized curricular efforts—which, as we view it, is a potential germ cell that could manifest at multiple levels.

As noted above, these are the very initial stages of what we hope will be prolonged partnerships. By starting a few "steps back" from where work typically has begun, we hope to make progress in confronting the stubborn challenge of changing mathematics teaching.

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References

Benford, R. D., & Snow, D. A. (2000). Framing processes and social movements: An overview and assessment. *Annual review of sociology*, *26*(1), 611–639.

Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational researcher*, *32*(1), 9–13.

Elmore, R. F. (2004). *School reform from the inside out*. Cambridge, MA: Harvard Education Press.

Engeström, Y. (2011). From design experiments to formative interventions. *Theory & Psychology*, *21*, 598–628.

Engeström, Y., & Sannino, A. (2010). Studies of expansive learning: Foundations, findings and future challenges. *Educational Research Review*, *5*, 1–24.

Freire, P. (2000). *Pedagogy of the oppressed* (M. B. Ramos, Trans., 30th ann. ed.). New York: Continuum. (Original work published 1970)

Gutiérrez, R. (2009). Embracing the inherent tensions in teaching mathematics from an equity stance. *Democracy and Education*, *18*(3), 9–16.

Hiebert, J. (2013). The constantly underestimated challenge of improving mathematics instruction. In K. R. Leatham (Ed.), *Vital directions for mathematics education research* (pp. 45–56). New York, NY: Springer.

Hill, H., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *Elementary School Journal 105* (1), 11–30.

Institute of Education Sciences (2016). *Request for applications: Partnerships and collaborations focused on problems of practice or policy*. Retrieved from https://ies.ed.gov/funding/pdf/2017_84305H.pdf.

Jackson, K., Gibbons, L., & Sharpe, C. J. (2017). Teachers' Views of Students' Mathematical Capabilities: Challenges and Possibilities for Ambitious Reform. *Teachers college record*, *119*(7).

Lewis, C. (2015). What is improvement science? Do we need it in education?. *Educational Researcher*, *44*(1), 54–61.

Munter, C., Stein, M. K., & Smith, M. S. (2015). Is there a common pedagogical core?: Examining instructional practices of competing models of mathematics teaching. *NCSM Journal of Mathematics Education Leadership, 16*(2), 3–13.

Penuel, W. R. (2014). Emerging forms of formative intervention research in education. *Mind, Culture, and Activity, 21*(2), 97–117.

Penuel, W. R., Fishman, B. J., Cheng, B. H., & Sabelli, N. (2011). Organizing research and development at the intersection of learning, implementation, and design. *Educational Researcher*, *40*(7), 331-337.

Resnick, L. B., & Hall, M. W. (1998). Learning organizations for sustainable education reform. *Daedalus, 127*, 89–118.

Spillane, J. P. (2000a). Cognition and policy implementation: District policy-makers and the reform of mathematics education. *Cognition and Instruction, 18*, 141–179.

Talbert, J. E., & McLaughlin, M. W. (1999). Assessing the school environment: Embedded contexts and bottom-up research strategy. In S. L. Friedman & T. D. Wachs (Eds.), *Measuring environment across the life span* (pp. 197-226). Washington, D.C.: American Psychological Association.

PSYCHOMETRIC ASSESSMENT OF THE MATH ANXIETY SCALE WITH THE RASCH MEASUREMENT MODEL

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The purpose of this study was to investigate the psychometric characteristics of the Math Anxiety Scale (MANX; Erol, 1989) with data collected from 952 middle school students in Turkey. The Rasch Rating Scale model was used to examine the MANX at the item level. The results reveal that although the MANX is sensitive to detect students with moderate levels of math anxiety and it is not targeted to identify those with very high and low math anxiety levels, it has high reliability and validity. Moreover, the majority of the MANX items are of good quality. The results provide strong evidence for the validation of the MANX despite the need for deletion of eight misfit items, and three redundant items.

Keywords: Affect, Emotion, Beliefs, and Attitudes, Middle School Education, Research Methods

Math anxiety can be defined as "feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations" (Richardson & Suinn, 1972, p. 551). Math anxiety has been consistently linked to several negative outcomes such as low academic performance (Ashcraft, 2002), and reduced working memory functioning (Young, Wu, & Menon, 2012). This, in turn, can cause many students to pursue career paths that do not involve mathematics.

In response to the need to identify math anxiety levels, researchers have developed self-report scales including the Mathematics Anxiety Rating Scale (MARS) by Richardson and Suinn (1972), and the Mathematics Anxiety Questionnaire (MAQ) by Wigfield and Meece (1988). In addition to these scales' Turkish translations, there also exist a few scales originally developed in Turkish such as the Math Anxiety Scale (MANX; Erol, 1989), which has been used to measure math anxiety levels of middle grades students (e.g., Erden & Akgül, 2010), and of high school students (e.g., Ader & Erktin, 2010). The only study, however, that examined the psychometric characteristics of the MANX was the one conducted by Erktin, Dönmez, and Özel (2006), which used factor analysis to determine the underlying factors of the scale. Erktin et al. identified four factors: (a) *test and evaluation anxiety*, which captures anxiety about exams and evaluation; (b) *apprehension of math lessons*, which captures anxiety related to mathematics lessons; (c) *use of mathematics in daily life*, which captures anxiety in daily situations involving mathematics; and (d) *self-efficacy for mathematics*, which captures self-perceptions on mathematics.

Regarding the negative long-term impacts of math anxiety, it is important to have scales that fully measure math anxiety levels of students for the sake of identifying those students and providing specific interventions based on the needs of each student. To achieve this aim, examination of the psychometric characteristics of the math anxiety scales is critical at the item level. To our knowledge, however, the studies which have examined the math anxiety scales predominantly rely on factor analysis for validation, which is at the total score level (e.g., Baloğlu & Zelhart, 2007; Beasley, Long, & Natali, 2001). Therefore, the purpose of this study was to examine the psychometric characteristics of the MANX for measuring 8th grade students' math anxiety at the item level. The following research questions were addressed:

1. Does the internal structure of the MANX represent gradations of item difficulty?

2. Do the MANX items demonstrate acceptable model-data fit supporting the validity of inferences in terms of students' math anxiety levels?

The present study is one of the few studies (e.g., Prieto & Delgado, 2007) that examine the validation of a math anxiety scale with the Rasch measurement model. Specifically, this study is the first effort to evaluate the item-level quality of the MANX using the Rasch Rating Scale model. Moreover, this study focuses on 8th grade students because this grade is a critical transition period from middle school to high school with potential high math anxiety levels.

Theoretical Framework

The theoretical framework is based on the Rasch Rating Scale model due to its advantages over other approaches such as factor analysis. First, if the data fit the Rasch model well, then person trait and item difficulty estimates can be obtained as independent from the sample (Engelhard, 2013). Second, the Rasch model transforms the ordinal raw data into interval measures as opposed to approaches such as factor analysis relying on correlations of sample dependent ordinal data (Bond & Fox, 2015). Third, the Rasch model provides a variable map that locates persons and items simultaneously on the latent trait.

Methods

The sample consists of 952 Turkish 8th grade students from nine schools located in different regions of Turkey. Students' range of age was between 14 and 16 years.
The *Math Anxiety Scale* (MANX), developed by Erol (1989), with 45 items, has four response categories for each item including "never," "sometimes," "usually" and "always." Coefficient alpha (i.e., α) was .91 in Erol (1989) and .92 in Erktin, Dönmez, and Özel (2006). By following the factors in Erktin et al., coefficient alpha in the present study is .95 for the MANX. Before performing the analyses, the 9 positively worded items of the MANX were reverse-coded so that higher scores demonstrate higher levels of math anxiety, and vice versa. Students completed the MANX during approximately 30 minutes of a class period.

To evaluate the psychometric characteristics of the MANX, we, first, evaluated the unidimensionality of the MANX by exploratory factor analysis as implemented in the SPSS 16.0 software (SPSS Inc., 2007). Second, we obtained reliability of person separation and reliability of item separation values. Third, the variable map was evaluated based on the appropriateness of the sample for the MANX. Fourth, we examined item quality of the MANX based on individual mean square (MS) error statistic. In this study, MS error statistic between 0.60 and 1.40 was considered reasonable fit for the MANX items (Linacre & Wright, 1994). All analyses except the analysis of unidimensionality were performed using the Facets program (Linacre, 20013).

Results

The unidimensionality of the MANX was confirmed with an exploratory factor analysis that reports eigenvalues of the first three factors as 14.3, 2.6, and 1.9. The total variance explained by the first factor was 31.7%, indicating that the MANX is essentially unidimensional (Reckase, 1979). Regarding the reliability values for person and item separation, the MANX has a high reliability of person separation with Rel = .93, χ^2 (951) = 11066.3, and $p < .01$, and of item separation with Rel = .99, χ^2 (44) = 6462.4, $p < .01$. Furthermore, Figure 1 displays a variable map showing item and person locations. In column 3, item locations are displayed with four sub-columns showing the underlying factors ranging from *test and evaluation anxiety* (i.e., sub-column 1) to *self-efficacy for mathematics* (i.e., sub-column 4) as left to right.

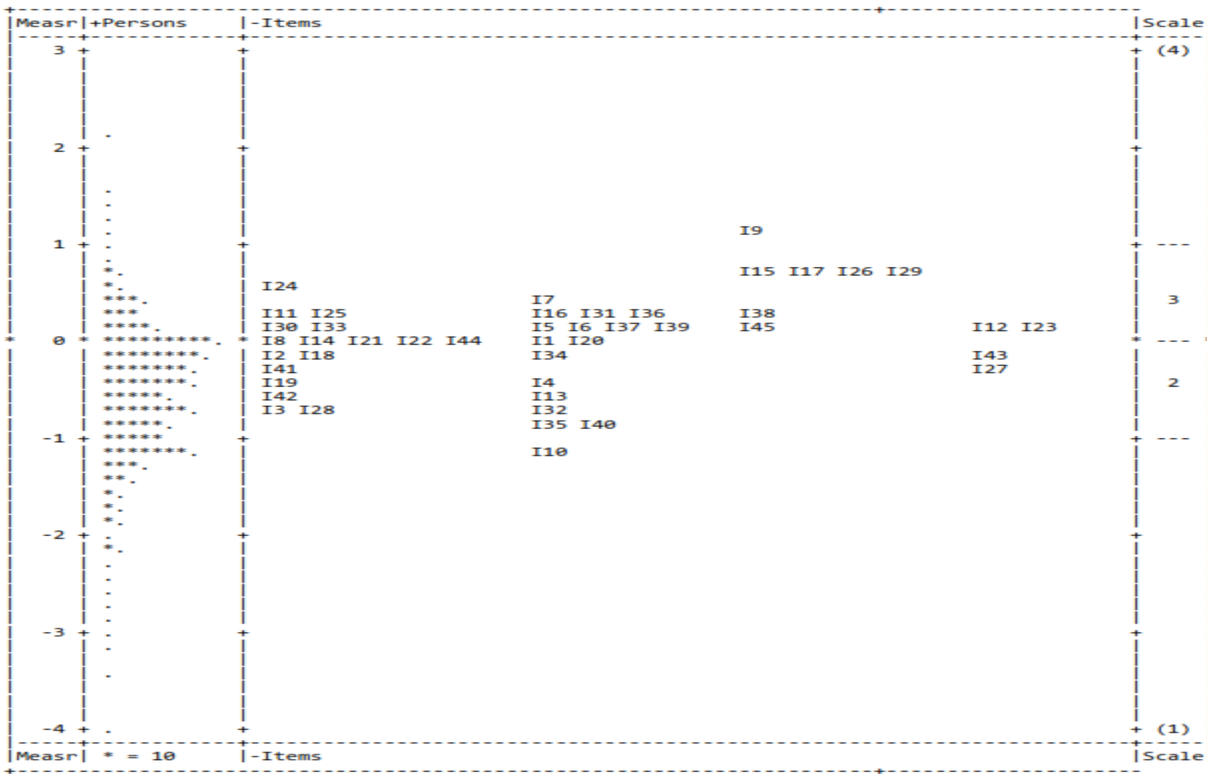

Figure 1. Variable map of students' math anxiety levels and item locations.

According to the variable map, there exists a good overlap between the person trait and item difficulty due to the match between means of the person trait (*M* = -0.58) and item difficulty (M = 0.00). Thus, the majority of the items were appropriate for the sample. While person trait measures range from 2.13 to -5.41 logits, the item difficulties range from 1.14 to -1.08 logits.

Regarding the variable map, the MANX appears to have few limitations. First, the MANX has lack of items that provide information about students who are located at the very high and low ends of the continuum (i.e., above 1.14 logits and below -0.85 logits). Second, regarding distributions of items with respect to their underlying factors (see Figure 1), most of the items are spread enough except those which belong to *use of mathematics in daily life* (i.e., items 9, 15, 17, 26, 29, 38, and 45). Five of these seven items' (i.e., items 9, 15, 17, 26, and 29) being located at the high end of the continuum indicates that these items do not provide adequate information to discriminate between students with very high levels of math anxiety related to the *use of mathematics in daily life*. Third, in terms of redundancy of the items, Item 17 and Item 26, represented by the *use of mathematics in daily life*; Item 6 and Item 37, represented by *apprehension of math lessons*; and Item 21 and Item 44, represented by *test and evaluation anxiety*, have same mean ratings, item difficulties and standard errors as within pairs. Therefore, Item 17, Item 37, and Item 21, which has slightly worse fit statistic (Infit and Outfit), can be eliminated from the scale.

Finally, according to item quality indices for each item, most of the items (i.e., 37 of the 45 items) have a good fit (Linacre & Wright, 1994). Eight of the 45 items, however, demonstrate misfit including items 5, 9, 10, 13, 20, 34, 35, and 43. Item 34 is the only item that exhibits less variation than expected (Infit = 0.59 and Outfit = 0.58). The remaining seven misfit items had more variation than expected due to unpredictability of students' responses in these items. Five of these items (i.e., items 10, 13, 20, 35, and 43) were positively worded (reverse-coded).

Discussion

The purpose of this study was to assess the psychometric characteristics of the MANX at the item level using the Rasch Rating Scale model. The results revealed that the MANX is essentially unidimensional and highly reliable for persons and items. Furthermore, the variable map indicated that the MANX, overall, captures a range of item difficulties reflecting order structure of math anxiety. However, we also found that items related to the *use of mathematics in daily life* are quite difficult to agree with for students, and there is shortage of items at the very high and low ends of the continuum, indicating that the MANX is not sensitive enough to discriminate among students with very high and low levels of math anxiety. Hence, researchers and teachers using the MANX in its current form should be cautious when interpreting their results for students who have very high and low math anxiety levels. Regarding item quality, 37 of the 45 items exhibited good psychometric quality, indicating that the items provide adequate information to measure moderate levels of math anxiety. Among the eight misfit items, five of them were reverse-coded. This might have confused the students and contributed to bias toward response styles. In addition, three redundant items can be removed from the scale to reduce the length of the administration. In conclusion, the results of this study provide strong evidence for the validation of the MANX despite the need for considering deletion of eight items that show misfit and of three items that are redundant. Future studies should consider including only the 34 items that have good item qualities and prioritize using the Rasch model for scale development and validation studies over other approaches.

References

Ader, E., & Erktin, E. (2010). Coping as self-regulation of anxiety: A model for math achievement in high-stakes tests. *Cognition, Brain, Behavior: An Interdisciplinary Journal, 14*(4), 311–332.

Ashcraft, M. H. (2002). Math anxiety: Personal, educational, and cognitive consequences. *Current Directions in Psychological Science, 11*(2), 181–185.

Baloğlu, M., & Zelhart, P. F. (2007). Psychometric properties of the revised mathematics anxiety rating scale. *The Psychological Record, 57*, 593–611.

Beasley, T. M., Long, J. D., & Natali, M. (2001). A confirmatory factor analysis of the Mathematics Anxiety Scale for Children. *Measurement and Evaluation in Counseling and Development, 34*, 14–26.

Bond, T. G., & Fox, C. M. (2015). *Applying the Rasch model: Fundamental measurement in the human sciences*. New York, NY: Routledge.

Engelhard, G. (2013). *Invariant measurement: Using Rasch models in the social, behavioral, and health sciences*. New York, NY: Routledge.

Erden, M., & Akgül, S. (2010). Predictive power of math anxiety and perceived social support from teacher for primary students' mathematics achievement. *Journal of Theory and Practice in Education, 6*(1), 3–16.

Erktin, E., Dönmez, G., & Özel, S. (2006). Psychometric characteristics of the mathematics anxiety scale. *Education and Science, 31*(140), 26–33.

Erol, E. (1989). *Prevalence and correlates of math anxiety in Turkish high school students*. Unpublished master thesis, Bogazici University.

Linacre, J. M. (2013). Facets Rasch Measurement (Version 3.71.3) [Computer Software]. Chicago, IL:Winsteps.com

Linacre, J. M., & Wright, B. D. (1994). Reasonable mean-square fit values. *Rasch Measurement Transactions, 8*(3), 370.

Prieto, G., & Delgado, A. R. (2007). Measuring math anxiety (in Spanish) with the Rasch Rating Scale Model. *Journal of Applied Measurement, 8*, 149–160.

Reckase, M. D. (1979). Unifactor latent trait models applied to multifactor tests: Results and implications. *Journal of Educational Statistics, 4*, 207–230.

Richardson F. C., & Suinn, R. M. (1972). The mathematics anxiety rating scale: Psychometric data. *Journal of Counseling Psychology, 19*(6), 551–554.

SPSS Inc. (2007). SPSS for Windows, Version 16.0. Chicago, SPSS Inc.

Young, C. B., Wu, S. S., Menon, V. (2012). The neurodevelopmental basis of math anxiety. *Psychological Science, 23*(5), 492–501

WHERE IS (IN)EQUITY? EXAMINING AN URBAN SCHOOL DISTRICT'S POLICIES FOR PURSUING RACIAL EQUITY IN MATHEMATICS

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In 2017, a National Council of Teachers of Mathematics research committee identified *inequity* as "a long-standing, thoroughly documented, and seemingly intractable problem in mathematics education," and called for more (and more *responsible*) research to help "solve this problem with all its facets" (Aguirre et al., 2017, p. 125). This multi-facetedness likely plays into how equity is defined and pursued. Gutiérrez (2012) defined it as access, achievement, identity, and power, all of which "are necessary if we are to have true equity" (p. 21).

But in practice, as schools and districts take up equity-focused policies and initiatives, to what extent do they attend to all of these dimensions? How are problems of inequity framed—in terms of both their sources ("diagnostic framing") and appropriate responses ("prognostic framing") (Benford & Snow, 2000), and are those framings well aligned? In short, *where* do educational leaders identify inequity and pursue equity?

With this poster, we describe an approach to answering such questions based on our work with one urban district. Our study originated in a multi-year, multi-institution partnership with an urban school district in the northeast U.S. focused on racial equity in secondary (grades 6-12) mathematics. In the course of that project we conducted annual interviews with district leaders as a way to understand the broader district policy context in which the partnership's primary efforts were situated. All interviews were audio-recorded and transcribed. We also collected relevant policy documents. In our analysis of both transcripts and artifacts, we characterized sources of inequity and each policy or initiative aimed at addressing those inequities.

Our analyses revealed that district leaders, at times, framed sources of inequity and corresponding pursuits of equity at different levels (individual, institutional, etc.) and according to different equity dimensions. As a consequence of such misalignments in frames, attempts to address inequities were sometimes misdirected and often undermined by disproportionate focus on achievement. With our poster we will discuss implications for research and policy.

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References

Aguirre, J., Herbel-Eisenmann, B., Celedón-Pattichis, S., Civil, M., Wilkerson, T., Stephan, M., … Clements, D. H. (2017). Equity within mathematics education research as a political act: Moving from choice to intentional collective professional responsibility. *Journal for Research in Mathematics Education, 48*(2),124–147.

Benford, R. D., & Snow, D. A. (2000). Framing processes and social movements: An overview and assessment. *Annual review of sociology*, *26*(1), 611–639.

Gutiérrez, R. (2012). Context matters: How should we conceptualize equity in mathematics education? In B. Herbel-Eisenmann, J. Choppin, D. Wagner, & D. Pimm (Eds.), *Equity in discourse for mathematics education: Theories, practices, and policies* (pp. 17–33). New York, NY: Springer.

COMPETING VIEWS ON PREPARING MATHEMATICS TEACHERS FOR CHANGE

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A longstanding challenge in U.S. mathematics education is that, despite research evidence supporting new approaches, conventional teacher-centered forms of instruction still predominate (Hiebert, 2013). Despite this problem's persistence, it has been difficult to achieve consensus education on the role novice teachers should play in effecting cultural change. This poster provides a theory-based distinction between two approaches to teacher preparation and different views on how preparation might carry forward into teachers' early career classrooms.

The *agent-of-change approach* is based on the notion that without a clear vision of a very different kind of mathematics education and the kind of political disposition necessary to pursue it, teachers will not look for or be prepared to act on opportunities for challenging and disrupting the status quo (Gutiérrez, 2013). It supports prospective teachers (PTs) in attempting to dismantle conventional practices by introducing a series of critical lenses—from access and opportunity to status and power, and then learning to enact more equitable and dialogic forms of teaching. The goal is that, upon graduation, PTs will have developed sufficiently robust visions and commitments to recognize and reject unproductive institutional routines and to find and create opportunities for pursuing a better mathematics education.

In contrast, the *incremental approach* starts in close proximity to conventional practices. It involves identifying aspects of conventional teaching that can be positive (e.g., conceptually-rich lectures) while also raising specific, small-scale critiques (e.g., lack of student involvement in particular lessons). In response to the critiques, PTs learn about and practice explicit, tangible recommendations for instruction. These recommendations should be small in number but developed deeply in conjunction with specific mathematical content, which is the type of preparation that has been found to carry forward (Jansen, Burk, & Meikle, 2017). Moreover, because PTs view this incremental preparation as realistic and relevant, they will gain trust in the preparation institutions and thus be amenable to additional incremental changes in the future.

The poster will review further literature and also propose research questions and potential study designs for assessing the effectiveness of the two approaches. This work will inform those who are revising teacher preparation programs and promote collaborations to make evidence-based decisions as our field moves to meet the longstanding challenge of instructional change.

References

Gutiérrez, R. (2013). Why (urban) mathematics teachers need political knowledge. Journal of Urban Mathematics Education, 6(2), 7–19.

Hiebert, J. (2013). The constantly underestimated challenge of improving mathematics instruction. In K. R. Leatham (Ed.), Vital directions for mathematics education research (pp. 45–56). New York, NY: Springer.

Jansen, A., Berk, D., & Meikle, E. (2017). Investigating alignment between elementary mathematics teacher education and graduates' teaching of mathematics for conceptual understanding. Harvard Educational Review, 87, 225–250.

USING ENGINEERING CONTEXTS TO TEACH MATHEMATICS

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Keywords: Design Experiments, Learning Progression, Geometry, Middle School Education

In the competitive world, America's' future scientists, technologists, engineers, and mathematicians must be prepared to remain effective and efficient through re-energized attention to Science, Technology, Engineering and Mathematics (STEM) education (Wang, Moore, Roehrig, & Park, 2011). A biggest challenge in K-12 STEM education is that few general guidelines or models exist for teachers to follow for how to teach using STEM (Kimmel, Carpnelli, & Rockland, 2007). The purpose of this study is to contribute to the ongoing dialogue regarding the efficacy of genuine STEM integration. Specifically, this study explores the process of design and implementation of a learning trajectory that attempts to integrate mathematics and engineering in an authentic manner to evoke STEM practices more generally. Our goal is to explore the feasibility of using engineering contexts to teach mathematics and develop a conceptual framework for genuine STEM integration.

The rationale behind choosing design based research approach is the ability to contribute to development of theory and means of that are designed to support that learning that is the educational practice (Plomp & Nieveen, 2007). The research aims to capture the emergence of STEM practices, which is the normative way of using STEM practices to interact with others in the classroom. The research was conducted in a 7th grade mathematics classroom of a STEM middle school in a sub-urban school district in the Southeast region of United States. The classroom teacher was part of the design team along with faculty specializing in mathematics education, chemistry, and engineering of a large urban university. The engineering context involved designing blueprints of a new housing development and building 3D houses. The instructional activities were designed to develop students' understanding of parallel lines as the same distance apart and measuring angles as the degree of turn.

From our preliminary analysis, a separate emphasis was intentionally placed on mathematics and engineering instructional sequences because, the goal of mathematician and engineer are different and hence the learning outcomes of mathematics and engineering are going to be different as well. The design constraints of the engineering problem was used to purposefully guide the exploration of mathematics context. For instance, students played the role of architect to design the layout of the phase II of a residential community and a design criteria was to create a new parallel road to an existing road in the phase I of the residential community. In the process, students' understanding of parallel lines moved to more sophisticated notions. The instructional sequences were designed to encourage students to switch back and forth between the engineering context and the mathematics embedded within that context.

References

Kimmel, H., Carpinelli, J., & Rockland, R. (2007). Bringing engineering into K-12 schools: A problem looking for solutions. *Proceedings of the International Conference on Engineering Education*, 1–5.

Plomp, T., & Nieveen, N. (Eds.) (2007). An introduction to educational design research. Enschede, the Netherlands: SLO

Wang, H.-H., Moore, T. J., Roehrig, G. H., & Park, M. S. (2011). STEM Integration: Teacher perceptions and practice STEM integration: Teacher perceptions and practice. J*ournal of Pre-College Engineering Education Research (J-PEER)*, 1(*2*), 1–13.

NATURE, CHALLENGES, AND STRATEGIES OF STEM RESEARCH TEAMS

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Concept and Research Questions

STEM education provides opportunities for mathematics educators to participate in research teams. However, research groups face challenges with communication and methodology (e.g., Lyall & Meagher, 2012). This poster examines the following questions: (a) What is the nature of STEM research collaborations? (b) What challenges exist for mathematics educators in research collaborations? and (c) What strategies enrich interactions in STEM research groups?

Methodology and Findings

Data consisted of research articles that explored STEM research collaborations published in one of the first ten mathematics education journals listed in Williams and Leatham (2017, p. 377). Search terms used to identify articles for the data were (a) interdisciplinary, transdisciplinary, multidisciplinary, monodisciplinary coupled with the term research and (b) STEM education. Based on the search, eight articles were identified as data for this study. A constant comparative analysis (Strauss & Corbin, 1994) was used to classify the articles based on their alignment with categories of integration (monodisciplinary, multidisciplinary, or interdisciplinary) as defined by Williams et al. (2016), articulated frameworks, reported challenges, and strategies for enriching research.

We found that the nature of the research teams described in the articles was multidisciplinary or interdisciplinary. In addition, six of the eight articles described a framework used by the research team. The frameworks in the research articles varied and included cognitive neuroscience, situated learning, and learning as a conceptual network. Challenges for the research groups included: (a) interacting and conducting research in two or more disciplines, (b) translating integrated findings into viable applications, and (c) having institutional/community support for research involving multiple disciplines. Strategies for enriching interdisciplinary research included: (a) focusing on communication and collaboration, (b) developing reconceptualized or more efficient models, (c) identifying shared research questions, and (d) understanding the role mathematics plays with other disciplines. Future studies should explore the role of mathematics education researchers in the context of STEM research collaboration.

References

Lyall, C., & Meagher, L. R. (2012). A masterclass in interdisciplinarity: Research into practice in training the next generation of interdisciplinary researchers. *Futures, 44*(6), 608-617.

Strauss, A., & Corbin, J. (1994). Grounded theory methodology: An overview. In Denzin, N. K., & Lincoln, Y. S. (Eds.), *Handbook of qualitative research*. Thousand Oaks: CA. Sage Publications.

Williams, S. R., & Leatham, K. R. (2017). Journal quality in mathematics education. *Journal for Research in Mathematics Education, 48*(4), 369-396.

Williams, J., Roth, M., Swanson, D., Doig, B., Groves, S., Omuvwie, M.,… Borromeo F. R., & Mousoulides, N. (2016). *Interdisciplinary mathematics education: A state of the art*. Springer Open. https://doi.org/10.1007/978-3-319-42267-1_1

Chapter 14

Working Groups

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.

ADDRESSING EQUITY AND DIVERSITY ISSUES IN MATHEMATICS EDUCATION: LOOKING BACK AND LOOKING AHEAD

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Embracing the theme of this year's meeting, we seek as a community to embrace the history of mathematics education, the history of PMENA, and the promise of the future of mathematics education. Following on the topics discussed at the Working Group between 2009-2017, this year the focus is continuing on the reestablished purpose of this group and supporting the development of new directions for equity-oriented research working groups. The sessions will focus on regrouping attendees interested in equity, generating and brainstorming new subtopics and potential projects, and working to establish standalone working groups dedicated to furthering research on equity. The purpose of this resetting is to encourage a move away from "big-tent" equity thinking and toward more productive working collectives.

Keywords: Equity and Diversity

Brief History

This Working Group originates from the Diversity in Mathematics Education (DiME) Group, one of the Centers for Learning and Teaching (CLT) funded by the National Science Foundation (NSF). DiME scholars graduated from one of three major universities (University of Wisconsin-Madison, University of California-Berkeley, and UCLA) that comprised the DiME Center. The Center was dedicated to creating a community of scholars poised to address critical problems facing mathematics education, specifically with respect to issues of equity (or, more accurately, issues of inequity).The DiME Group (as well as subsets of that group) has engaged in important scholarly activities, including the publication of a chapter in the *Handbook of Research on Mathematics Teaching and Learning* which examined issues of culture, race, and power in mathematics education (DiME Group, 2007), a one-day AERA Professional Development session examining equity and diversity issues in mathematics education (2008), a book on research of professional development that attends to both equity and mathematics issues with chapters by many DiME members and other scholars (Foote, 2010), and a book on teaching mathematics for social justice (Wager & Stinson, 2012) that also included contributions from several DiME members. In addition, several DiME members have published manuscripts in a myriad of leading mathematics education journals on equity in mathematics education. This working group provides a space for continued collaboration among DiME members and other colleagues interested in addressing the critical problems facing mathematics education.

It is important to acknowledge some of the people whose work in the field of diversity and equity in mathematics education has been important to our work. Over time, the Working Group has encouraged building on and featuring senior scholars' work, including Marta Civil (Civil, 2007; Civil & Bernier, 2006; González, Andrade, Civil, & Moll, 2001), Eric Gutstein (Gutstein, 2003, 2006; Gutstein & Peterson, 2013), Jacqueline Leonard (Leonard, 2007; Leonard & Martin, 2013), Danny Martin (Martin, 2000, 2009, 2013), Judit Moschkovitch (Moschkovich, 2002), Rochelle Gutiérrez (2002, 2003, 2008, 2012, 2013) and Na'ilah Nasir (Nasir, 2002, 2011, 2013; Nasir, Hand & Taylor, 2008; Nasir & Shah, 2011). We have as well been building on the work of our advisors, Tom Carpenter (Carpenter, Fennema, & Franke, 1996), Geoff Saxe (Saxe, 2002), Alan Schoenfeld (Schoenfeld, 2002), and Megan Franke (Kazemi & Franke, 2004), as well as many others outside of the field of mathematics education.

Previous iterations of this Working Group at PMENA 2009 – 2013, and 2015-2016 have provided opportunities for participants to continue working together as well as to expand the group to include other interested scholars with similar research interests. Experience has shown that collaboration is a critical component to this work. These efforts to expand participation and collaboration were well received; more than 40 scholars from a wide variety of universities and other educational organizations took part in the Working Group each of the past five years. Starting in 2017, an effort was made to "reset" the group toward providing opportunities for a new generation of scholars whose work intersects with issues of equity/inequity, diversity/inclusion, privilege/oppression, and justice in mathematics education research, practice, and development.

Focal Issues

Under the umbrella of attending to equity and diversity issues in mathematics education, researchers are currently focusing on such issues as teaching and classroom interactions, professional development, prospective teacher education (primarily in mathematics methods classes), factors impacting student learning (including the learning of particular sub-groups of students such as African American students or English learners), and parent perspectives. Much of the work attempts to contextualize the teaching and learning of mathematics within the local contexts in which it happens, as well as to examine the structures within which this teaching and learning occurs (e.g. large urban, suburban, or rural districts; under-resourced or well-resourced schools; and high-stakes testing environments). How the greater contexts and policies at the national, state, and district level impact the teaching and learning of mathematics at specific local sites is an important issue, as is how issues of culture, race, and power intersect with issues of student achievement and learning in mathematics. There continues to be too great a divide between research on mathematics teaching and learning and concerns for equity.

The Working Group has begun and will continue to focus on analyzing what counts as mathematics learning, in whose eyes (and for whose benefit), and how these culturally bound distinctions afford and constrain opportunities for traditionally marginalized students to have access to mathematical trajectories in school and beyond. Further, asking questions about systematic inequities leads to methodologies that allow the researcher to look at multiple levels simultaneously. This research begins to take a multifaceted approach, aimed at multiple levels from the classroom to broader social structures, within a variety of contexts both in and out of school, and at a broad span of relationships including researcher to study participants, teachers to schools, schools to districts, and districts to national policy.

Some of the research questions the Working Group will continue to consider are:

- What are the characteristics, dispositions, etc. of successful mathematics teachers for *all* students across a variety of local contexts and schools? How do they convey a sense of purpose for learning mathematical content through their instruction?
- How do beginning mathematics teachers perceive and negotiate the multiple challenges of the school context? How do they talk about the challenges and supports for their work in achieving equitable mathematics pedagogy?
- What impediments do teachers face in teaching mathematics for understanding?
- How can mathematics teachers learn to teach mathematics with a culturally relevant approach?

- What does teaching mathematics for social justice look like in a variety of local contexts?
- What are the complexities inherent in teacher learning about equity pedagogy when students come from a variety of cultural and/or linguistic backgrounds all of which may differ from the teacher's background?
- What are dominant discourses of mathematics teachers?
- What ways do we have (or can we develop) of measuring equitable mathematics instruction?
- How do students' out-of-school experiences influence their learning of school mathematics?
- What is the role of perceived/historical opportunity on student participation in mathematics?

Specific to the intent of this year's Working Group, we will organize around questions like the ones above in order to create specific, targeted working groups that are charged to address and act around such questions.

Plan for Working Group

Based on feedback from the previous year and the emergence of new working groups related broadly to "equity," this working group has shifted toward a renewed focus on facilitating "collaboration within the growing community of scholars and practitioners concerned with understanding and addressing the challenges of attending to issues of equity and diversity in mathematics education." We have reconfigured the working group toward being a catalyst for new spaces instead of a "destination" for the inclusion of equity discourse within the PME-NA organization. To put it differently, our vision for the working group is to bring together attendees toward developing their own agendas and specific working groups related to equity-oriented themes--or toward themes that push the field beyond traditional equity discourses yet adhere to the needs and challenges of inequity within mathematics education.

Our plans for PMENA 2018 will proceed as follows. Each session will build on previous sessions, beginning with a facilitated conversation around the previously stated purpose of the working group. The format for the sessions will include:

- DAY 1: Presenting the Vision, Norm-setting, and Brainstorming: On the first day, we will lead attendees through introductory activities, collective norm-setting, and a series of small- and whole-group brainstorming activities that will generate new ideas and directions for the working group more broadly.
- DAY 2: Agenda-setting: On the second day, the major focus will be the development and support of new smaller sub-specializing groups based on the reported interests of attendees. We will work with and encourage these subgroups to establish possible common topics of interests, potential products, and planning for the next year to support the growth of their group and topic.
- DAY 3: Working groups: On the third day, the newly established subgroups will "take flight" and initiate their yearly plan to support their chosen topics.

Previous Work of the Group

The Working Group met for productive sections since 2009. In 2009, participants identified areas of interest within the broad area of equity and diversity issues in mathematics education. Much fruitful discussion was had as areas were identified and examined. Over the past five years subgroups met to consider potential collaborative efforts and provide support. Within these subgroups, rich conversations ensued regarding theoretical and practical considerations of the topics. In addition, graduate students had the opportunity to share research plans and get feedback. The following were topics covered in the subgroups:

- Teacher Education that Frames Mathematics Education as a Social and Political Activity
- Culturally Relevant and Responsive Mathematics Education
- Creating Observation Protocols around Instructional Practices
- Language and Discourse Group: Issues around Supporting Mathematical Discourse in Linguistically Diverse Classrooms
- A Critical Examination of Student Experiences

As part of the work of these subgroups, scholars have been able to develop networks of colleagues with whom they have been able to collaborate on research, manuscripts and conference presentations.

As a result of the growing understanding of the interests of participants (with regard both to the time spent in the working group and to intersections with their research), we began to include focus topics for whole group discussion and consideration and continued to provide space for people to share their own questions, concerns, and struggles. With respect to the latter, participants have continually expressed their need for a space to talk about these issues with others facing similar dilemmas, often because they do not have colleagues at their institutions doing such work or, worse yet, because they are oppressed or marginalized for the work they are doing. These concerns, in part, informed the focus topics for whole group discussion and consideration. For example, in 2009 research protocols (e.g., protocols for classroom observation, video analysis and interviewing) were shared to foster discussions of possible cross-site collaboration. In 2012, the Working Group explicitly took up marginalization in the field of mathematics education with a discussion about the negotiation of equity language often necessary for getting published; this was done in the context of the 'Where's the mathematics in mathematics education' debate (see Heid, 2010; Martin, Gholson, & Leonard, 2010). Dr. Amy Parks was invited to join Working Group organizers to share reflections on their experiences. In 2013 the Working Group hosted its first panel in which scholars (Dr. Beatriz D'Ambrosio, Dr. Corey Drake, Dr. Danny Martin) shared their perspectives on the state of and new directions for mathematics education research with an equity focus. In 2017, we had collections around topics that have resulted in several proposals for sessions and working groups for this years convening of PMENA.

Pre-Conference and Follow-up Activities

In order to best plan for working group facilitation and prepare attendees for working group participation, we plan to send out pre-conference communication, including a survey, to former and potential participants in order to gauge the topics and kinds of work being done or sought, as well as the resources and forms of support desired by participants.

References

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal, 97*(1), 3-20.

Civil, M. (2007). Building on community knowledge: An avenue to equity in mathematics education. In N. Nassir & P. Cobb (Eds.), *Improving access to mathematics: Diversity and equity in the classroom.* (pp. 105-117). New York, NY: Teachers College Press.

Civil, M., & Bernier, E. (2006). Exploring images of parental participation in mathematics education: Challenges and possibilites. *Mathematical Thinking and Learning, 8*(3), 309-330.
DiME. (2007). Culture, race, power, and mathematics education. In F. Lester (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (Second ed., pp. 405-433). Charlotte, NC: Information Age Publishing.
Foote, M. Q. (Ed.). (2010). *Mathematics teaching & learning in K-12: Equity and professional development*. New York: Palgrave.
González, N., Andrade, R., Civil, M., & Moll, L. (2001). Bridging funds of distributed knowledge: Creating zones of practices in mathematics. *Journal of Education for Students Placed at Risk, 6*(1&2), 115-132.
Gutiérrez, R. (2002). Enabling the practice of mathematics teachers: Towards a new equity research agenda. *Mathematical Thinking and Learning, 4*(2&3), 145-187.
Gutiérrez, R. (2003). Beyond essentializing: The complex role of language in teaching Latina/o students mathematics. *American Educational Research Journal, 39*(4), 1047-1088.
Gutiérrez, R. (2008). A "gap gazing" fetish in mathematics education? Problematizing research on the achievement gap. *Journal for Research in Mathematics Education, 39*(4), 357-364.
Gutiérrez, R. (2012). Embracing "Nepantla:" Rethinking knowledge and its use in teaching. *REDIMAT-Journal of Research in Mathematics Education, 1*(1), 29-56.
Gutiérrez, R. (2013). Why (Urban) mathematics teachers need political knowledge. *Journal of Urban Mathematics Education, 6*(2), 7 – 19.
Gutstein, E. (2003). Teaching and learning mathematics for social justice in an urban, Latino school. *Journal for Research in Mathematics Education, 34*(1), 37-73.
Gutstein, E. (2006). Reading and writing the world with mathematics. New York: Routledge.
Gutstein, E., & Peterson, B. (Eds.). (2013). Rethinking mathematics: Teaching social justice by *the numbers* (2nd Ed.). Milwaukee, WI: Rethinking Schools, Ltd.
Heid, M. K. (2010). Where's the math (in mathematics education research)? *Journal for Research in Mathematics Education, 41*, 102–103.
Kazemi, E., & Franke, M. L. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education, 7*, 203-235.
Leonard, J. (2007). Culturally specific pedagogy in the mathematics classroom: strategies for *teachers and students.* New York, NY: Routledge.
Leonard, J., & Martin, D. B. (Eds.) (2013). The brilliance of Black children in mathematics: *Beyond the numbers and toward new discourse.* Charlotte, NC: Information Age Publishers.
Martin, D. B. (2000). Mathematics success and failure among African American youth: The *roles of sociohistorical context, community forces, school influence, and individual agency*. Mahwah, NJ: Lawrence Erlbaum Assoc. Inc.
Martin, D. B. (2009). *Mathematics teaching, learning, and liberation in the lives of Black children.* London: Routledge.
Martin, D. B. (2013). Race, racial projects, and mathematics education. *Journal for Research in Mathematics Education*, *44*(1), 316-333.
Martin, D. B., Gholson, M. L., & Leonard, J. (2010). Mathematics as gatekeeper: Power and privilege in the production of knowledge. *Journal of Urban Mathematics Education, 3*(2),12–24.
Moschkovich, J. (2002). A situated and sociocultural perspective on bilingual mathematics learners. *Mathematical Thinking and Learning, 4*(2&3), 189-212.
Nasir, N. (2011). Racialized Identities: Race and achievement for African-American youth. Stanford University Press.
Nasir, N. S. (2002). Identity, goals, and learning: Mathematics in cultural practice. *Mathematical Thinking and Learning, 4*(2/3), 213-247.
Nasir, N. S. (2013). Why should math educators care about race and culture? Plenary address at the 35th Annual Meeting of the North American Chapter of the Psychology of Mathematics Education. Chicago, Il.
Nasir, N., Hand, V., & Taylor, E. (2008). Relevant knowledge in school mathematics: Boundaries and intersections between cultural and domain knowledge in mathematics classrooms. *Review of Research in Education, 32*(1), 187-240.
Nasir, N. & Shah, N. (2011). On Defense: African American males making sense of racialized narratives in mathematics education. *Journal of African American Males in Education, 2*(1), 24-45.
Saxe, G. B. (2002). Children's developing mathematics in collective practices: A framework for analysis. *The Journal of the Learning Sciences, 1*(2&3), 275-300.

Schoenfeld, A. H. (2002). Making mathematics work for all children: Issues of standards, testing and equity. *Educational Researcher, 31*(1), 13-25.

Wager, A. A., & Stinson, D.W. (Eds.). (2012). *Teaching mathematics for social justice: Conversations with educators*. Reston, VA: National Council of Teachers of Mathematics.

MODELS AND MODELING WORKING GROUP

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The Models and Modeling Working Group at PME-NA has provided a forum for discussing and collaborating on research projects fundamental to this area since the first PME-NA conference in 1978. We propose to convene this Working Group at PME-NA 40 with a dual purpose: (1) to build on discussions at PME-NA 37-39, extending collaborative research directions formulated at the Indianapolis meeting, and (2) to continue to invite newcomers to the Models and Modeling Perspective (MMP), giving them an introduction to this design research tradition.

Keywords: Modeling, Problem Solving, Design-Based Research, Design Experiments

The Models and Modeling Working Group has been a significant presence in the PME-NA community since the inaugural year of the conference in 1978. Over the 40 years of its existence, working-group format has supported substantial research efforts, and also has helped to foster collaboration and mentoring relationships between researchers interested in what has come to be known as the "Models and Modeling Perspective" (MMP). Since its inception, the MMP has pursued fundamental pragmatic questions of effective knowledge, such as "What ... beyond having a mathematical idea ... enables students to use it in everyday problem-solving situations?" (Lesh, Landau, & Hamilton, 1983, qtd. in Lesh & Doerr, 2003, p. i.). An important historical purpose of the Working Group has been to pursue innovations in design-based research (cf, Kelly, Lesh, & Baek, 2008) – to discuss and extend the ways in which a focus on models and modeling can be used both to support learning in STEM, and to study such learning processes in action. Indeed, calls for STEM integration (English, 2016; English & King, 2015) and for attending to Engineering perspectives (Diefes-Dux, Moore, Zawojewski, Imbrie, & Follman, 2008; Roehrig, Moore, Wang, & Park, 2014) have only increased the potential of this long-standing tradition to contribute to efforts to innovate in teaching and research.

Early in its history, the Group focused heavily on the design and analysis of particular activities that enabled groups of learners to engage in a deep form of modeling and that produced an auditable trail of thinking, exposing their thought processes to teacher and researcher observers. In this phase of the field's development, a primary effort involved elaborating design principles for these activities as learning environments and documenting the idea development they promoted. Gradually over time, however, researchers associated with the Group have expanded their perspective to consider implementations and curricular sequences that have longer time-duration, and that integrate models and modeling into the experience of learning mathematics in more extensive ways. Several broad patterns in this more extensive and disseminated approach to modeling in the curriculum have emerged, and there is no sense that we have yet exhausted the space of possibilities. These broader perspectives open both exciting opportunities and significant challenges. On the one hand, new questions can be researched, opening the way for new forms of contact and interaction with classroom practice; on the other, the approach raises new challenges at the level of methodology, data analysis, and forms of evidence that are convincing backings for claims about learner activity.

We propose convening the Group at PME-NA 40 to continue a style of work that has characterized the Group's collaborations over the past several years. In particular, based on our experiences in 2015-2017, we propose a work-session structure that can serve two dual purposes: (a) making substantive progress in concrete projects of collaborative research – planning and framing shared implementation, analysis, and writing work; while also (b) integrating newcomers to Models and Modeling as a research area.

For this Working Group, these two goals are ***both*** essential: we do not propose to gather as a closed expert group. (Broadening exposure to this kind of learning design is important to the individuals of the Group as well as to the group as a collective whole.) And we also do not aim only at providing an initial introduction to the approaches characteristic of the MMP and the activity designs the tradition has developed. (There are urgent opportunities and problems of research and practice that the Working Group aims to address.)

In the following sections, we provide a very brief overview of the field of research represented by the Models and Modeling Perspective; we outline patterns in research efforts that have extended modeling activities over longer timescales; and we describe our plan of work in detail, illustrating how these goals are addressed as well as how we plan to productively integrate newcomers to the Group over the three working sessions offered in the Conference.

The Models and Modeling Perspective (MMP)

Since the 1970s, MMP researchers and educators have engaged in design research directed at understanding the development of mathematical ideas among groups of learners. A key principle behind this work has been that learners' ideas develop through, and in relation to conceptual entities called *models*. The core construct of a "model" and the activity of "modeling" are both central to the MMP; and they are also multidimensional, playing multiple roles in the MMP theory.

Models & Modeling Working Group founder Dick Lesh and Helen Doerr provide the following working definition of *models*:

> conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s)—perhaps so that the other system can be manipulated or predicted intelligently (Lesh & Doerr, 2003, p. 10)

In this spirit, "being a good modeler is in large part a matter of having a number of fruitful models in one's 'hip pocket.'" (Lesh, 1995, personal communication, qtd in Lehrer & Schauble, 2000).

But the term "model" not only applies to features of "target knowledge" that are built through learning: the interpretive systems that people *bring* to problems are also "models." When explicitly expressed through representational media, such models—personal interpretive systems—can provide illumination into how students, teachers, and researchers adapt, formulate, and apply relevant mathematical concepts in particular situations or contexts (Lesh, Doerr, Carmona, & Hjalmarson, 2003). In fact, "models" are the shape and vestment of most all knowledge--whether this knowledge appears as the patterns of perspectives and pre-conceptions that learners bring with them "in the door"; the shared ways of thinking that a group of learners build in solving a problem; or the systematic accounts of phenomena that represent the normative views of a scientific discipline at a given moment in its history.

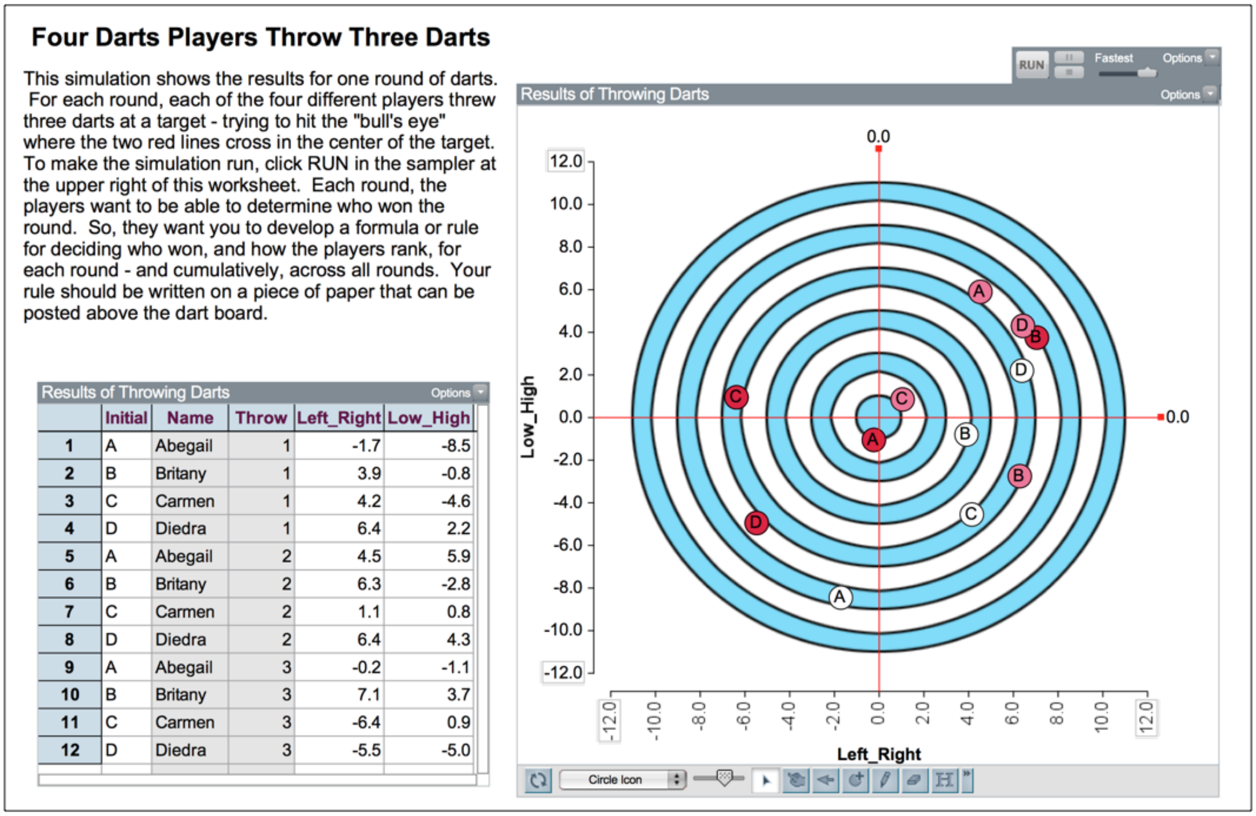
An early finding of research in the Models and Modeling Perspective (MMP) was that, under appropriate conditions, groups of learners can be supported in producing external representations of the models they bring to a situation, and that when these groups put their initial models into conversation with one another, they can be supported to revise, and refine them in rapid and iterative cycles, building toward a more robust model that reflects their achievement of a shared way of thinking. In particular, when individuals and groups encounter problem situations with specifications that demand a model-rich response, their models can be observed to grow through such relatively rapid cycles of development toward solutions that satisfy these specifications.

While the elicitation of initial ways of thinking is valuable, MMP researchers' interest quickly turned to this process of model refinement—that is, to the dynamics of model***ing*** (as opposed to the statics of mod***els***), and to the features of activity environments that foster modeling and make it visible for teachers and researchers. The dynamics of modeling represent, for the MMP, an account of *idea development*, as observed in the discourse and other representations produced by groups of learners as they iteratively work to mathematize and formulate a solution that meets the needs of a concrete *client* in a realistic setting.

Thus, the MMP tradition became focused squarely on local conceptual development (Lesh & Harel, 2003): that is, on investigating the micro-evolution of ideas in groups of students (and teachers). The resources and tools its researchers produced were first and foremost designed to study idea development and the range of possibility for this mode of learning activity. The results of this work include a body of *Model-Eliciting Activities* (MEAs), in which students are presented with authentic, real-world situations where they repeatedly express, test, and refine or revise their current ways of thinking as they endeavor to generate a structurally significant product—that is, again, a *model*, comprising conceptual structures for solving the given problem. These activities differ markedly from some other environments dedicated to applications of specific mathematical concepts and procedures. In contrast, MEAs give students the opportunity to create and adapt mathematical models in order to interpret, explain, and/or predict the behavior of real-world systems (Zawojewski, 2013). An extensive body of MMP research has produced accounts of learning in these MEA environments (Lesh, Hoover, Hole, Kelly, & Post 2000; Lesh & Doerr, 2003), design principles to guide MEA development (Hjalmarson & Lesh, 2007; Doerr & English, 2006; Lesh, et. al., 2000; Lesh, Hoover, & Kelly, 1992) and accounts and reflections on the design process of MEAs (Zawojewski, Hjalmarson, Bowman, & Lesh, 2008).

As an example of the activity-type of Model-Eliciting Activities (MEAs), consider the Darts problem (**Figure 1**). This problem provokes students to grapple with notions of centrality, spread, and distance. Students are often introduced to Darts problem after having engaged with several other MEAs that challenge them to develop operational definitions for intuitive but difficult-to-quantify constructs such as "worker productivity" or "volleyball-playing ability," and to use data to produce measures of such constructs. The Darts problem pushes them to extend this line of thinking, creating operational definitions both to evaluate *performances* (the darts games) and assess *performers* (the darts players).

Above is the statement of the Darts problem as presented to students. In this realization of the Darts problem, the printed sheet is accompanied by a TinkerPlots document, which allows learners to create simulated rounds of darts games by pressing the **Run** button shown in the diagram.



Results of Throwing Darts

	Initial	Name	Throw	Left_Right	Low_High
1	A	Abegail	1	-1.7	-8.5
2	B	Britany	1	3.9	-0.8
3	C	Carmen	1	4.2	-4.6
4	D	Diedra	1	6.4	2.2
5	A	Abegail	2	4.5	5.9
6	B	Britany	2	6.3	-2.8
7	C	Carmen	2	1.1	0.8
8	D	Diedra	2	6.4	4.3
9	A	Abegail	3	-0.2	-1.1
10	B	Britany	3	7.1	3.7
11	C	Carmen	3	-6.4	0.9
12	D	Diedra	3	-5.5	-5.0

Figure 1. Example MEA: The Darts problem

The second "phase" of the Darts problem asks students to evaluate *players* (**Figure 2**):

Darts, Part Two: *Going for the Championship*
The local pub is now preparing for a season of competitive darts, and you need to provide useful information to the captain, Laura, to help her field a team. Beyond the four players from Part One, a group of "regulars" at the pub will be trying out. Describe how results from several rounds of darts can be used to create a rating or description of players' darts throwing skills. Your procedure should help Laura to choose a team based on the tryouts.

Figure 2. The second phase of the Darts problem

Student groups iteratively develop solutions to this problem in the time allotted—usually 50-60 minutes for this MEA. Afterwards, the teacher may organize a "poster session" for the groups to share and learn from each others' solutions. In one version of this activity structure, one member of each group hosts a presentation of the poster showing their results. The other students circulate to learn about other groups' solutions, using a Quality Assurance Guide to assess the results produced by others in the class. These instruments are submitted to the teacher and contribute to assessment in various ways, providing evidence for the achievements of both individuals and groups.

MEAs like the Darts problem present learners with situations where familiar procedures, ways-of-operating, and constructs may be *applicable,* but where they are also *insufficient*. In this sense, they support a learning environment that has a "low threshold" but a "high ceiling" (cf

Papert, 1980). That is, on the one hand they are accessible to learners from a wide range of levels of ability, experiences, or knowledge (from upper elementary school through graduate school). On the other hand, learners encountering these problems find that they have no ready-made solution they can apply to address the client's needs. As a result, groups learners must engage in sense-making and solution-construction processes that position them as mathematical creators and also put them off balance in comparison to typical school-mathematics tasks. Indeed, this uncertainty is part of the design of MEAs, illuminating fundamental conceptual issues associated with the core mathematical structures involved.

MEA Design Principles

As individual MEAs emerged, an intense period of design research ensued to understand them as a genre of learning tasks that could (a) stimulate mathematical thinking representative of that which occurs in contexts outside of artificial school settings (Lesh, Caylor, & Gupta, 2007; Lesh & Caylor, 2007); (b) enable the growth of productive solutions through rapid modeling cycles; and (c) leave behind "auditable trails" - researchable traces of learners' ways of thinking during the process (Kelly & Lesh, 2000; English et al., 2008; Kelly, Lesh & Baek, 2008). The success of MEAs as an activity genre and as research tools was both enabled by and illustrated by the MMP's articulation of a set of six design principles (Lesh & Harel, 2003; Lesh et al., 2000; Hjalmarson & Lesh, 2007). These principles indicate essential elements of MEAs and their classroom implementations, enabling them to serve as rich contexts for student problem solving. Table 1, below, also indicates "touchstone" tests for whether each of these six principles has been realized in a given implementation setting.

Table 1: Six Design Principles for MEAs and Touchstones (see also Brady et al, 2017)

Principle	Touchstone Test for its Presence
Reality Principle	Students are able to make sense of the task and perceive it as meaningful, based on their own real-life experiences.
Model Construction Principle	To solve the problem, students must articulate an explicit and definite conceptual system (model).
Self-Evaluation Principle	Students are able to judge the adequacy of their in-process solution on their own, without recourse to the teacher or other "authority figure".
Model Generalizability Principle	Students' solutions are applicable to a whole range of problems, similar to the particular situation faced by the "client" in the MEA.
Model-Documentation Principle	Students generate external representations of their thinking during the problem-solving process.
Simplest Prototype Principle	The problem serves as a memorable representative of a kind of mathematical structure, which can be invoked by groups and by individuals in future problem solving.

Nested Levels of Modeling: Multi-Tiered Design Research

In parallel with learner-focused research using MEAs, researchers also have observed that *teachers'* efforts to understand their students' thinking involve yet another process of modeling: In this case, teachers engage in building models of student understanding. Although these teacher-level models are of a different category from student-level models, students' work while engaged in MEAs does provide a particularly rich context for teachers' modeling processes. Following this line of inquiry, the MMP community has also produced tools and frameworks that can be useful to teachers in making full use of MEAs in classroom settings, while also providing *researchers* with insights into *teachers'* thinking.

Finally, at a third level of inquiry, researchers' own understandings of the actions and interactions in *curricular activity systems* (Roschelle, Knudsen, & Hegedus, 2010) involving students, teachers, and other participants in the educational process can also be studied through the lens of model development. Multi-tier design experiments in the MMP tradition have done precisely this, involving researcher teams in self-reflection and iterative development as well (Lesh, 2002). Therefore, the MMP version of multi-tier design research can involve at least three levels of investigators— students, teachers, and researchers—all of whom are engaged in developing models that can be used to describe, explain, and evaluate their own situations, including real-life contexts, students' modeling activities, and teachers' and students' modeling behaviors, respectively. The situation can be further enriched by considering other educational stakeholders and learning settings, such as interactions between academic coaches and teachers (Baker & Galanti, 2017), and between schools and community organizations, and between students and parents.

Constructing Curricular Materials to Support Modeling at Larger Timescales

Over the past 15 years, MMP researchers have continued this direction of work in their own teaching and in partnerships with K-12 classroom teachers. Within the domain of statistical thinking in particular, this effort has produced resources and tools sufficient to support *entire courses* in several versions and including accompanying materials related to learning and assessment aimed at both student and teacher levels.

In their work on MEAs, students have rich but idiosyncratic mathematical experiences that need to be *unpacked* and placed into relationship with each other and with more canonical concepts, practices, and procedures from the discipline. To investigate such matters, MMP researchers attend to learner activity beyond the scope of single MEAs, formulating tools and designs for Model Development Sequences, or MDSs (Arleback, Doerr, and O'Neil, 2013; Doerr and English, 2003; Hjalmarson, Diefes-Dux, and Moore 2008; Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). Work here includes approaches for extending the modeling dynamics that MEAs foster and for unpacking and making explicit learners' ways of thinking, so that they are available to be reflected upon by the classroom group as a whole and systematized in relation to big ideas in the discipline (see also Brady, Eames, & Lesh, 2015 for tentative MDS design principles).

Because the courses supported by these materials were designed explicitly to be used as research settings, for investigating the interacting development of students' and teachers' ways of thinking, the materials were modularized so that important components could be easily modified or rearranged for a variety of purposes in different implementations. In particular, by selecting from and adapting *the same core collection of MEAs*, and surrounding them with MDS activities tailored to the learning goals and emergent ideas of different classrooms, parallel versions of the course have been developed for: (a) middle- or high-school students, (b) college-level elementary or secondary education students, and (c) workshops for in-service teachers. Existence-proof versions of these courses have produced impressive gains (see, e.g., Lesh, Carmona, & Moore, 2009).

As a result of the breadth of the MMP, other models for engaging with MEA-style modeling at larger timescales have also emerged. These larger structures reflect other professional and theoretical interests and concerns, such as a commitment to iterative design-based inquiry (Eames, et al, in press); a focus on socio-mathematical norms (Yackel & Cobb, 1996), or the dynamics of coaching (Baker & Galanti, 2017).

Research and Discussion Themes to Guide the Working Group

In all, our previous PME-NA working group meetings in East Lansing, Tucson, and Indianapolis have brought together over 30 participants from the US, Canada, and Mexico. Participants have described ongoing implementation and research across a wide range of grade levels and educational settings. In their work, attendees reported that they apply a variety of interpretive lenses and frameworks to modeling, and they situate their work in a variety of ways with respect to other current trends in mathematics education research. Moreover, in pursuing their practice, they developed definitions of core MMP constructs that were both broadly compatible (enabling productive discussion) and differently specialized and exemplified (enabling illuminating debate).

We have thus found Working Group meetings to offer unique opportunities to connect research voices and viewpoints, spurring conversations between research groups that have common inspiration and compatible interests, but very diverse local experiences and perspectives. The Working Group meetings have also consistently been a vehicle for connecting "old timers" with "newcomers." Some of the giants in the PME-NA community (e.g., Dick Lesh, Lyn English, Helen Doerr, Margret Hjalmarson, Jim Middleton, Tamara Moore, Eric Hamilton, and others) have also been leaders in the MMP, and in each of our sessions, we have invited participation from one or more of these leaders to offer perspectives on our thinking and on the field as a whole. We see these interactions as an important aspect of newcomers' (and old-timers') experience of the conference as a site for the exchange of wisdom, perspective, and enthusiasm among participants.

As one artifact of the meeting in Indianapolis, we identified several areas of prospective collaborative work. While we have pursued some of these ideas in the intervening five months, several of these remain to be done and can serve as indicators of the work that we will strategize and pursue in Greenville:

- Categorizing and classifying MEAs and other activities that make up MDS sequences. Providing a taxonomy and rationale. (This was a request by Lyn English, who described this as a gap in the literature and current discussion of modeling).
- Pursuing an idea of studying a particular MMP analytical perspective across multiple implementations of multiple MEAs (orchestrating multiple implementations of different MEAs in different settings, where a shared analytical frame was foregrounded and exercised).
- An article digging into the researcher level of the multi-tiered modeling research process, and investigating (as one does at teacher and student levels) what conceptual strategies and observation tools are needed to make these processes visible for reflection and analysis.
- An exercise in developing new MEA activities that are specifically designed to highlight themes in research on *learning dynamics* (as well as, or as opposed to only themes and big ideas in the discipline).
- A discussion and investigation of technology supports for MEAs and activities in Model Development Sequences.
- A concerted effort to bring the MMP to Latin America, with a focus on one or more Spanish-language journal articles that present the theory, differentiate it from other forms of activity, and present empirical results from implementations.
- A series of translations of canonical works in the MMP tradition into Spanish, with a space for commentary and relation to current work by researchers in Latin America.

(That is, moving far beyond "translation" to support a renaissance of inquiry into foundational MMP issues in the Latin American context.)

Session Outline: Advancing the Research Agenda while Building Community and Capacity

The working group will meet in three sessions over the course of the conference. As the organizers and facilitators do the preparatory work for the conference, these plans will be refined, but the broad outlines here reflect our current thinking.

As mentioned above, our experience of the working group over the past three PME-NA conferences has highlighted the value of these meetings for both (a) establishing and "hashing out" plans for innovative new research collaborations, and (b) inviting interested newcomers to the MMP and providing them opportunities to engage with its principles and practices, and interact with some of its founding members. Although it imposes an intense challenge for organizing and facilitating the working group, we aim to continue to support these two strands of activity. In part, we are committed to both because we recognize the importance to the MMP both to advance its agenda and rejuvenate its participant group. But in addition, we recognize that these two threads are in fact inseparable. Some of our most interesting theoretical discussions have come out of the friendly challenges from newcomers/outsiders, and we aim to cultivate and integrate rather than cordon off these voices and perspectives.

Session One. Accordingly, there are two tracks for the first day of the conference (one for newcomers, one for relatively experienced researchers who know the MMP). The rationale for initial separation is based on the need to produce sharable artifacts representing each group's interests and perspectives, so that we can integrate the participant group productively in Sessions Two and Three. We have tested and refined this broad structure in recent years, where it has enabled all attendees to participate meaningfully without producing a permanent separation between the Newcomer and Experienced groups.

The initial day's activity for the MMP Newcomer research group is oriented to introducing the MMP by directly experiencing the kinds of learning environment that it is committed to designing and studying. Participants will start by engaging with an MEA as though they were learners. We find that this "learning by doing" approach is as important for researchers as it is for teachers and students, and it creates a shared experience that can ground theoretical discussions that follow. Although Newcomer participants engage as learners, they also are encouraged to reflect on how they would want to study the kind of learning and group dynamics that they see and feel themselves enacting. This group ends the first day with a facilitated discussion of their "student" experience, reflecting on teacher and researcher perspectives of this kind of learning, and framing how they might formulate a research study around this kind of classroom activity. The group also prepares questions that they might want to ask the Experienced group participants in the General discussion that will start Session Two.

The Experienced MMP research group's initial activity will be oriented to tackling challenges for cross-institutional research in the MMP. This opening session is structured as a "research design charrette" in which we will identify, structure, and sketch a plan to tackle a shared problem of design research that the group could address after the conference. Participants will start by breaking into smaller groups to tackle one of the study topics generated in the Indianapolis meeting (see listing above). The small groups' goal for Session 1 is to frame an "elevator pitch" for a study or article, framing their pitch as a provocation for others in the group to identify relevant resources (data, un-identified connections in the literature, analytical frames, etc) as well as to critique and refine the pitch into a proposal. At the end of Session 1, one or two projects (depending on the participant group size) will be chosen as the focus of work for

Sessions 2 and 3. The particular shape of work for Sessions 2 and 3 will thus be somewhat dependent on the nature of the projects that emerge.

Sessions Two and Three. The collective goal of these sessions is to unify the group toward collective research action, identifying connection points between the Newcomer and Experienced groups and finding opportunities for all participants to contribute during the Conference and afterwards, as they wish. At the beginning of Session 2, Newcomer groups will present their solutions to the MEA to the larger group, and they will raise the questions they have generated in Session One. They will also have some opportunity to share their perspectives and impressions with the larger group, though moving to focused projects by the mid-point of Session 2 is critical for success of these projects.

The Experienced track will then present their "elevator pitches" for study projects, and the large group will have a brief discussion of the intrinsic interest of these projects and their significance for the MMP agenda.

Session Two will continue with options for members in the Newcomer track to join a project team based on the 'elevator pitches' of the Experienced track. The facilitator group is also considering an alternative option, in which newcomers engage in MDS activities with an eye to implementation at this scale. (If pursued, this activity would involve both engaging with and adapting/extending MDS activities.) Because a substantial number of PME-NA attendees depart before the final working group session, the end of Session Two must reach a clear “waypoint” in inquiry, and logistical issues must be addressed, such as establishing a group email list and collaboration tools.

Session Three takes the project (and possible MDS) tracks forward toward collaboration after the conference. The facilitator group typically also meets after the last session, to debrief and establish plans and to-do lists for follow up with the broader participant group.

References

Brady, C., Eames, C., & Lesh, R. (2015). Connecting real-world and in-school problem-solving experiences. *Quadrante: Revista de Investigaçião em Educaçião Matemática* [Special Issue, *Problem Solving*], *26*(2), 5-38.

Brady, C., Eames, C., Jung, H., Glancy, A., McLean, J., & Dominguez, A. (2017) Models and modeling working group. Proceedings of PME-NA 2017.

Baker, C. K., & Galanti, T. M. (2017). Integrating STEM in elementary classrooms using model-eliciting activities: responsive professional development for mathematics coaches and teachers. *International Journal of STEM Education*, *4*(1), 10.

Diefes-Dux, H. A., Moore, T., Zawojewski, J., Imbrie, P. K., & Follman, D. (2004, October). A framework for posing open-ended engineering problems: Model-eliciting activities. In *Frontiers in Education, 2004*. FIE 2004. 34th Annual (pp. F1A-3). IEEE.

Doerr, H. M., & English, L. D. (2006). Middle grade teachers’ learning through students’ engagement with modeling tasks *Journal of Mathematics Teacher Education*, *9*(1), 5-32.

Eames, C., Brady, C., Jung, H., & Glancy, A. (In press). Designing Powerful Environments for Examining and Supporting Teachers’ Models and Modeling. W. Blum & R. Borromeo Ferri (Eds), *Teacher Competencies for Mathematical Modeling*. Netherlands: Springer.

English, L. D. (2016). STEM education K-12: perspectives on integration*. International Journal of STEM Education*, *3*(1), 3.

English, L. D., Jones, G., Bussi, M., Tirosh, D., Lesh, R. & Sriraman, B. (2008) Moving forward in international mathematics education research. In L. D. English (Ed.). *Handbook of international research in mathematics education: Directions for the 21st century* (2nd edn). (pp. 872-905). New York: Routledge.

English, L. D., & King, D. T. (2015). STEM learning through engineering design: fourth-grade students’ investigations in aerospace. *International Journal of STEM Education*, *2*(1), 14.

Hjalmarson, M. & Lesh, R. (2007) Design research: Engineering, systems, products, and processes for innovation. In L. D. English (Ed.). *Handbook of international research in mathematics education: Directions for the 21st century* (2nd edn). (pp. 520-534). New York, NY: Routledge.

Katims, N., & Lesh, R. (1994). *PACKETS: A guidebook for inservice mathematics teacher development*. Lexington, MA: DC Heath.
Kelly, A. E., & Lesh, R. (Eds.) (2000). *The handbook of research design in mathematics and science education.* Hillsdale, NJ: Lawrence Erlbaum Associates.
Kelly, A. E., Lesh, R., & Baek, J. Y. (Eds.) (2008a). *Handbook of innovative design research in science, technology, engineering, mathematics (STEM) education*. New York, NY: Taylor & Francis.
Kelly, A. E., Lesh, R. A., & Baek, J. Y. (Eds.). (2008b). *Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching*. Routledge.
Lehrer, R., & Schauble, L. (2000). Developing model-based reasoning in mathematics and science. *Journal of Applied Developmental Psychology*, *21*(1), 39-48.
Lesh, R. (2002). Research design in mathematics education: Focusing on design experiments. In L. English (Ed.) *International handbook of research design in mathematics education* (pp. 27-50). Mahwah, NJ: Lawrence Erlbaum Associates.
Lesh, R., Carmona, G., & Moore, T. (2009). Six sigma learning gains and long term retention of understandings and attitudes related to models & modeling. *Mediterranean Journal for Research in Mathematics Education*, *9*(1), 19-54.
Lesh, R. & Caylor, E. (2007). Modeling as application vs modeling as a way to create mathematics. *International Journal of Computers for Mathematical Learning*, *12*(3), 173-194.
Lesh, R., Caylor, E. & Gupta, S. (2007). Data modeling and the infrastructural nature of conceptual tools. *International Journal of Computers for Mathematical Learning*, *12*(3), 231-254.
Lesh, R., Cramer, K., Doerr, H. M., Post T., & Zawojewski, J. S. (2003). Model development sequences. In R. Lesh & H. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning and teaching* (pp. 35-58). Mahwah, NJ: Erlbaum.
Lesh, R. & Doerr, H. (Eds.). *Beyond constructivism: A models and modeling perspective*. Mahwah, NJ: Lawrence Erlbaum Associates.
Lesh, R., & Doerr, H. M. (2003). In what ways does a models and modeling perspective move beyond constructivism? In R. Lesh & H. Doerr (Eds.), *Beyond constructivism: A models and modeling perspective* (pp. 519-556). Mahwah, NJ: Lawrence Erlbaum Associates.
Lesh, R., Doerr, H. M., Carmona, G., & Hjalmarson, M. (2003). Beyond constructivism. *Mathematical Thinking and Learning*, *5*(2-3), 211-233.
Lesh, R., & Harel, G. (2003). Problem solving, modeling, and local conceptual development. *Mathematical thinking and learning*, *5*(2-3), 157-189.
Lesh, R., Hoover, M., Hole, B., Kelly, A. E., & Post, T. (2000). Principles for developing thought-revealing activities for students and teachers. In A. Kelly, R. Lesh (Eds.), *Research Design in Mathematics and Science Education*. (pp. 591-646). Lawrence Erlbaum Associates, Mahwah, New Jersey.
Lesh, R., Hoover, M., & Kelly, A. E. (1992). Equity, technology, and teacher development. In I. Wirszup & R. Streit (Eds.), *Developments in school mathematics education around the world* (Vol. 3, 104-129). Reston, VA: National Council of Teachers of Mathematics.
Lesh, R., Landau, M., & Hamilton, E. (1983). Conceptual models and applied mathematical problem-solving research. *Acquisition of mathematics concepts and processes,* 263-343.
Roehrig, G. H., Moore, T. J., Wang, H. H., & Park, M. S. (2012). Is adding the E enough? Investigating the impact of K-12 engineering standards on the implementation of STEM integration. *School Science and Mathematics*, *112*(1), 31-44.
Roschelle, J., Knudsen, J., & Hegedus, S. (2010). From new technological infrastructures to curricular activity systems: Advanced designs for teaching and learning. In M. J. Jacobson & P. Reimann (Eds.), *Designs for learning environments of the future: International perspectives from the learning sciences* (pp. 233-262). New York, NY: Springer.
Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for research in mathematics education*, 458-477.
Zawojewski, J. S. (2013). Problem solving versus modeling. In R. Lesh, P. L. Galbraith, C. R. Haines, & A. Hurford (Eds.), *Modeling students' mathematical modeling competencies* (pp. 237-243). New York: Springer
Zawojewski, J. S., Hjalmarson, M. A., Bowman, K., & Lesh, R. (2008) A Modeling Perspective on Learning and Teaching in Engineering Education. In *Models and modeling in engineering education: Designing experiences for all students*. Rotterdam: Sense Publication (pp. 1-16).

FORGING NEW PATHS THROUGH CRITICAL PERSPECTIVES ON DISABILITY AND MATHEMATICS EDUCATION

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As mathematics education researchers and practitioners, we are poised to look ahead and to forge new narratives by challenging work traditionally relegated to special education. Our working group, which formed in 2016, is composed of researchers and educators who draw upon critical theories such as Disability Studies in Education, Critical Race Theory, and DisCrit, in order to offer an alternative vision of mathematics education based around a different conceptualization of disability and learning differences. Research on mathematics and disabilities traditionally has been conducted within a special education paradigm, which often implicitly or explicitly adopts a deficit model of the learner. The deficit model locates the "problem" within the individual student rather than in the social, discursive, political, or structural context. Instruction for these students tend to focus primarily on rote algorithms and calculation skills rather than the solving of rigorous, high cognitive demand problems. We consider both the framing of these students as well as how to reframe classroom practice utilizing models like Multiple Mathematics Knowledge Bases and Culturally Responsive and Sustaining Practices to create classrooms in which all students are able to access curriculum and have it represented in meaningful and rigorous ways.

Keywords: Equity and Diversity, Instructional activities and practices, Classroom Discourse, Learning Theories

Overview of the Working Group

As we think through how mathematics education has evolved over the last 40 years, this working group brings to light a challenge that has persisted with relation to issues of mathematics education and disability: How do we support the learning of all students, particularly those students with disabilities? The purpose of this working group is to create a community of scholars and educators who have share a critical lens to issues of mathematics education and disability. Historically, the consideration of disability in mathematics education has been examined from a special education lens, which focuses on identifying student deficits with the goal of "fixing" or remediation of these perceived deficits. While well-known epistemological differences exist between special education and mathematics education, we seek to explore new scholarly paths. As mathematics educators we believe a critical perspective on disability, drawing upon Disability Studies, Critical Race Theory, and DisCrit, can address these differences and offer an alternative vision of mathematics education and disability. Our working group is designed to create sustainable opportunities for researchers, teacher educators, and practitioners interested in bringing a critical lens to understanding disability, differences, and how we respond to differences in mathematics education and how these lenses can address persistent challenges in the education of each and every student. In this proposal we briefly present the history of this working group, the theoretical perspectives that this group draws upon, and our plans for collaboration.

History of the Working Group

Our PME-NA working group met for the first time in 2016. Fifteen researchers (faculty and graduate students) and two classroom teachers met during PME-NA 2016 in Tucson, Arizona. In our initial set of working group meetings, participating group members shared theoretical perspectives of disability they employed within their work in mathematics education, current projects they were engaged in related to disability and mathematics education, and critical issues members often found themselves confronting in taking up this complex work. Conversations that took place during this time were exciting. Many common interests were uncovered and shared among group members. At the conclusion of our time together at the conference, many of the group members were hopeful that the working group in subsequent years could serve as a foundation for continued work and conversation. The impact and work of this group continued through the year. We established an email list-serv which members have used regularly to share and solicit feedback on work and to organize our collective efforts. From this 2016 meeting we were able to garner a special issue in the journal Investigations in Mathematical Learning called Critical Approaches for Mathematics Learning of Students with Disabilities.

The second time the working group met was at PME-NA 2017 in Indianapolis, Indiana. During this time, we continued conversations of our identity as a working group, the goals we wanted to accomplish for the upcoming year, potential future collaborative work, and the extent to which we ought to seek partnership with a similar working group meeting at PME-NA that year. The group was made up of several returning members as well as new scholars interested in our work. One major goal we discussed for the upcoming year was a book project on critical mathematics education for practitioners. We identified NCTM as a potential venue to put forth our proposal and were thrilled to learn during the spring of 2018 that NCTM accepted our book proposal. In turn, this book project will be a major focus of this working group for the 2018 working group. During this meeting we also decided to open dialogue with the Special Education and Mathematics working group, yet scheduling conflicts during the conference prevented this from taking place. After this year’s meeting, we continued to communicate and to organize our collective efforts via our email list-serv. We also forged new partnerships between members including with doctoral student attendees who expressed interest in supporting and enacting this type of scholarship.

In this coming year we plan to continue and expand collaborations between members of this working group with the intention of looking ahead to how this working group will structure research through a critical lens to address the enduring challenge of reframing disability in mathematics education. This working group will create a space in which all involved can learn with and from each other with a purpose of establishing the foundation for collaborative work throughout the year. Members of the working group have submitted a conference proposal through the AERA conference grants. Even though we are actively pursuing opportunities to bring this working group together for a more extended period of time maintaining our PME-NA working group is essential. The space at PME-NA provides opportunities to connect with others, engage in ongoing projects, the possibilities of growing our community, especially in regards to bringing special education and mathematics education together. Each year brings this working group can come together, there is the potential to bring new members. This provides a synergistic place for an ever-growing number of scholars to pursue collaborations around these enduring challenges and hope this working group can provide a space for scholars who are taking a critical lens to explore disability related to mathematics education and create a conversation around the enduring challenge of supporting all students, with a specific focus on disability studies. This year, our group will also focus on strengthening connections with international scholars who are similarly interested in mathematics education and students with disabilities. Our efforts to connect with others who may have a similar interest have been ongoing since our first meeting.

Paradigm Shifts for Disabilities in Mathematics

Gathering together a group of researchers, teacher educators, graduate students, undergraduate students, and classroom teachers interested in developing an alternate paradigm around disability, we have begun to explore the different theoretical frameworks that we can use to analyze and change the status quo within mathematics education. This working group draws from a number of critical theories that have a common intersection around problematizing normativity. Our working group draws upon critical theories such as disability studies, critical race theory, and other poststructural theories (including queer studies and trans studies) in this analysis. In the following sections, we describe a sampling of these perspectives in order to frame the work of the group.

Critical Theories

Critical theory seeks to reshape reality, not just explain things as they are as evidenced by Karl Marx (1845) saying "The philosophers have only interpreted the world, in various ways; the point is to change it." Critical theory encompasses any theory that seeks to transform rather than explain society. The three main criteria for a critical theory as written by the Stanford Encyclopedia of Philosophy, are as follows, "it must explain what is wrong with current social reality, identify the actors to change it, and provide both clear norms for criticism and achievable practical goals for social transformation." As such, the theoretical perspectives that our working group draws upon are each in their own way "critical theories."

Disability Studies and Disability Studies in Education

Disability studies (DS) and disability studies in education (DSE) are modes of inquiry that question the taken-for-granted assumptions and practices in education and the intervention paradigm. Both DS and DSE provide frameworks for exploring questions such as: who is labeling, who is being labeled, whose voices we value, and how do we advance more equitable practices for all students. DS specifically questions the medical model of disability in which disability is seen as a deficit within an individual that requires intervention.

Many DS scholars and activists have adopted a social model of disability, which locates disability in an inaccessible environment and make a distinction between impairment, as any physical or mental limitation, and disability, as the "social exclusions based on, and social meanings attributed to, that impairment" (Kafer, 2013, p. 7). This distinction between impairment and disability is unhelpful because it "fails to recognize that both impairment and disability are social" (Kafer, 2013, p. 7). In the book Feminist Queer Crip, Kafer suggests the term "political/relational model" to refer to perspectives recognizing that both impairment and disability are socially constructed.

By framing disability as social, researchers acknowledge individuals with disabilities should be included within the research process itself. Traditionally researchers determine the questions asked, the methods of data collection, and the meaning made of the data, with no input from the individuals with disabilities, which is oppressive to individuals with disabilities (Oliver, 1992). By adopting a social model of disability, researchers acknowledge individuals with disabilities have unique insights into their lived experiences and empower them to engage directly in the research process (Barnes, 2003). Members of the working group recently published an emancipatory research study (Lewis & Lynn, 2018) where Lewis was positioned as the inquirer (the researcher) and Lynn as the expert (the participant). These research collaborations create powerful studies that illuminate students with disabilities lived experiences with mathematics and ensures the research is in service of those being studied.

In educational settings, this construction of disability manifests in the double education system that splits general education and special education. There is a divide in how students with disabilities are framed within special education research and mathematics education. One line of research pursued by working group members involves developing understanding and theorizing the research divide between special education and mathematics. Institutional practices such as writing Individual Education Plans (IEPs) construct certain students as having disabilities; however, from a disability studies perspective, "the label of students with IEPs [can be viewed] not as an inherent and static determinant of individual ability, but as a school-based designation which reflects and recreates differential ability within the classroom" (Foote & Lambert, 2011, p. 250; also see Dudley-Marling, 2004; McDermott, Goldman & Varenne, 2006; Skrtic, 2005). Particular students are chosen for this assessment and intervention, however the selection process is not objective and often singles out those students who are not from a dominant cultural background.

A DS perspective problematizes the taken-for-granted assumption that what is "wrong" with the situation requiring intervention is a pathology or deficit within students. Instead, the problem is located around the lack of access in the environment. The environment needs to be changed to allow access for students who differ from one another. As Reid and Valle (2004) assert, "the responsibility for 'fitting in' has more to do with changing public attitudes and the development of welcoming classroom communities and with compensatory and differentiated instructional approaches than with individual learners.

The focus of this group is on redesigning the context, not on remediating of individual's impairments. Different working group members address this in different ways. One scholar, draws upon a Vygotskian framing of disability and identifies the ways in which standard mediational tools (e.g., mathematical representations or symbols) are inaccessible to some learners. This scholar reframes the word "remediation" as "re-mediation" to move away from the deficit framing and toward a framing of disability in terms of access to mediational tools (Lewis, 2017).

A second scholar looks at interventions as increasing participation rather than specific skills and thinks through what types of interventions might create more equitable participation and deeper engagement across students in mathematics classrooms (Lambert & Sugita, 2016). This has been explored through empirical research focused on equitable participation in a Cognitively Guided Instruction algebra routine (Foote & Lambert, 2011). A political/relational model suggests inaccessibility is embedded in the context of power relations and in finding interventions to create a more accessible environment requires analyzing the power relations involved in maintaining inaccessibility.

A third scholar is exploring children's knowing and learning as cognitive diversity in the "Small Environment" (Hunt, 2018) of one-on-one and small group instruction. Specifically, the author studies children's learning trajectories - not as a means of "fixing" children but as a way to uncover the knowledge children DO have and leverage the diversity in conceptual understanding and strategic competence in instruction (Hunt, Westenskow, Silva, & Welch-Ptak, 2016; Robertson & Ne'eman, 2008). In this way, Hunt proposes research and instruction in Small Environments should begin by engaging in the mathematics of children and paying attention to their reasoning. Ultimately, teaching is leveraged to support children to explore, revise, and adapt both their own reasoning and the learning environment through a negotiation of meaning (Hunt & Tzur, 2017; Siebers, 2008).

Critical Race Theory and DisCrit

According to Critical Race Theory (CRT), racism is 'normal' rather than an anomaly in U.S. society (Delgado, 1995). CRT theorists claim the U.S. was founded on property rights, specifically that enslaved African Americans were considered property, rather than civil rights (Ladson-Billings & Tate, 1995). CRT reveals how race and racism continue to structure U.S. society and education. CRT theorists have addressed the issue of over-representation of students of color in special education. These analyses leave ableist assumptions in place and DS perspectives often fail to adequately consider race. DisCrit acknowledges that racism and ableism are both normalizing processes that are interconnected and collusive. Racism and ableism work in unspoken ways, yet, racism validates and reinforces ableism, and ableism validates and reinforces racism (Connor, Ferri, & Annamma, 2016). Studies of both administrators' and teachers' perceptions related to overrepresentation of students of color in special education reveal their perceptions tend to be rooted in "deficit thinking and infused with racial and cultural factors" (Connor, et al., 2016, ch. 1; also see Skiba et al, 2006). DisCrit perspectives identify the individual problematic attitudes of educators as one "accessible entry point for intervention" (Connor, et al., 2016, ch. 1).

Extending Previous Work

As a working group, we identify DisCrit as a crucial area of growth and for future scholarship. Several of our working group members have close connections with Dr. Subini Annamma, thus we are primed to integrate DisCrit into our work. Likewise, the charge of Waitoller and King Thorius (2016) to cross-pollinate Culturally Sustaining Pedagogies and Universal Design for Learning frameworks will be an important area to extend on our previous work. Indeed, transdisciplinary research work related to disability studies and mathematics education (Tan & Kastberg, 2017) is an emerging focus of our working group to extend our work.

Our working group's special issue in the journal *Investigations in Mathematics Learning* represents another area that we have extended on previous work. This special issue, which is scheduled to be published in the Fall of 2018, centers reframing disability in mathematics education and features papers addressing disability with a critical lens. One of the papers is from members of this working group who share their insider perspectives on experiences with a mathematics learning disability (Lewis & Lane, 2018). Lewis and Lane explore the compensation strategies used by Lane, who has been identified as having a mathematics learning disability (MLD), and how these compensation strategies could help other students with MLD gain access into mathematics.

A second paper explores middle grades d/Deaf and hard of hearing (DHH) students' perceptions of their identities of competence in mathematics classrooms (Goldstein, 2018). This paper is significant to the field and to our working group because this disability category is one that has been historically under-served and under-researched. Goldstein (2018) offers insight into two DHH students their experiences in learning mathematics in a DHH self-contained classroom. The students in the classroom experienced mathematics as only procedural in nature and viewed competence in mathematics as being able to complete problems. One student, Vivian, did not view herself as competent in mathematics, due to a variety of negative experiences in mathematics classes, including being moved from a mainstream classroom into a self-contained classroom. While in the mainstream classroom, she had a teacher who did not provide accommodations for Vivian, including not using an FM system, even though the classroom was equipped with one, or providing Vivian with someone who could sign for her. This illustrates issues of access for DHH students in mathematics classes and how these issues of access can result in students' lack of identities in competence in mathematics.

Another paper in this special issue highlights a conversation between the two authors, a special educator and a mathematics educator which was sparked by the question "How can I know what's possible in terms of engaging students with disabilities in mathematical thinking and reasoning?" (Greenstein & Baglieri, 2018). This piece is significant and extends our work because it offers ways in which synergistic activities across disciplinary fields can take happen. The conversation between Greenstien and Baglieri explores how mathematics educators and disability studies scholars could come together to support thinking about mathematics in ways yet to be imagined. This speaks to the work of this working group, as we come together to address critical issues in DS and mathematics education and continue to advance our work in new areas.

Summary of the Problem

This working group will examine issues related to disability in mathematics education and push for new understandings. Using multiple theoretical frameworks, the working group participants will analyze current practices in mathematics interventions, including the power relations involved, and develop and elaborate on alternatives in order to address enduring challenges in disability studies. The working group participants will also plan ways to evaluate these alternatives in various educational settings and contexts.

Plan for Active Engagement of Working Group Participants

Session 1

In the first session, the organizers will introduce the rationale for the working group and its two-year history. We will spend time hearing how each of the members is theoretically addressing disability from a critical perspective and collaboratively:

- Establish the groups' vision and mission for this coming year and beyond.
- Identify and refine the long- and short-term goals of the working group.
- Develop a strategic plan to meet our goals.

Session 2

In the second session, (based on input from Session 1) participants will either:

- Brainstorm potential collaborative endeavors.
- Brainstorm how to position this critical work to address enduring challenges in disability studies.
- Break up into subgroups to make progress on collective projects (e.g., NSF Conference grant, NCTM book project, collaborative work)
- Discuss how to develop projects around member's PMENA presentations or

- Discuss a shared reading (distributed to the list-serv before the conference and handed out during session 1) to push our thinking and shared understanding of critical perspectives.

Session 3

Session 3 will be devoted to planning our ongoing collaboration and distributing responsibilities for the group's shared endeavors (e.g., conference/book proposals, developing a shared online repository of research and teaching resources).

Plan for Sustainability: Anticipated Follow-up Activities

The working group sessions during the conference are designed to enable the participants to develop concrete plans for collaborative work beyond the end of the conference timeframe. We plan on continuing to communicate between working groups through our email list-serv. Specifically, the third session is allotted for developing specific plans for future collaborative work. Partnering with doctoral students will continue to be crucial to this group's sustainability as over the working group's history; they have expressed a great appreciation for this type of scholarship in pursuing their own work. In turn, members will continue to share information about this working group with others during professional meetings and in informal spaces. For example, several of us will also be attending the AERA conference in the spring 2019 and are members of the Disability Studies in Education and Research in Mathematics Education SIG. This will be an opportune time to both sustain our work and to expand in areas such as DisCrit. Lastly, we will continue to purse and apply for funding opportunities to meet as a group outside of PME-NA. This will allow us to meet for follow-up activities, sustain and advance our work, strengthen existing partnerships and forge new ones, and to produce collaborative scholarly works.

References

Barnes, C. (2003). What a difference a decade makes: reflections on doing 'emancipatory' disability research. *Disability and Society*, 18(1), 3–17.

Connor, D., Ferri, B., Annamma, S. (2016). *DisCrit: Disability studies and critical race theory in education*. New York, NY: Teacher's College Press.

Delgado, R. (Ed.) (1995). *Critical race theory: The cutting edge*. Philadelphia: Temple University Press.

Dudley-Marling, C. (2004). The social construction of learning disabilities. *Journal of Learning Disabilities*, 37(6), 482-489.

Foote, M. Q., and R. Lambert. (2011). I Have a Solution to Share: Learning through Equitable Participation in a Mathematics Classroom. *Canadian Journal of Science, Mathematics and Technology Education*, 11(3), 247-260. doi: 10.1080/14926156.2011.595882

Hunt, J.H. & Tzur, R. (2017). Where is difference? Processes of remediation through a constructivist lens. *The Journal of Mathematical Behavior*, 48, 62-76.

Hunt, J.H.; Westenskow, A.; Silva, J.; & Welch-Ptak, J. (2016). Levels of participatory conception of fractional quantity along a purposefully sequenced series of equal sharing tasks: Stu's trajectory. *Journal of Mathematical Behavior*, 41, 45–67.

Goldstein, D. (2018). Mathematics identities of competence in a middle grades D/Deaf and Hard of Hearing classroom. *Investigations in Mathematics Learning, 10 (3*), 145–154.

Greenstein S. & Baglieri, S. (2018). Imagining mathematical thinking for inclusive curriculum: a conversation. *Investigations in Mathematics Learning, 10* (3), 133–144.

Kafer, A. (2013). *Feminist Queer Crip*. Bloomington, IN: Indiana University Press.

Ladson-Billings, G. & Tate, W. (1995). *Toward a critical race theory of education*. Teachers College Record, 97(1), 47 – 68.

Lambert, R., & Sugita, T. (2016). Increasing engagement of students with learning disabilities in mathematical problem-solving and discussion. *Support for Learning*, *31*(4), 347–366. doi: 10.1111/1467-9604.12142

Lewis, K. E. (2017). Designing a bridging discourse: Re-mediation of a mathematical learning disability. *Journal of the Learning Sciences, 26*(2). 320-365. doi: 10.1080/10508406.2016.1256810

Lewis, K. E. & Lynn , D. M. (2018) An insider's view of a mathematics learning disability: Compensating to gain access to fractions, *Investigations in Mathematics Learning*, 10:3, 159-172

Marx, K. (1845). Theses on Feuerbach. Retrieved February 20, 2016 from <https://www.marxists.org/archive/marx/works/1845/theses/theses.htm>

McDermott, R., Goldman, S., & Varenne, H. (2006). The cultural work of learning disabilities. *Educational Researcher*, 35(6), 12-17.

Oliver, M. (1992). Changing the social relations of research production? *Disability, Handicap & Society*, 7(2), 101–114.

Reid, D. & Valle, J. (2004). The discursive practice of learning disability: Implications for instruction and parent-school relations. *Journal of Learning Disabilities*, 37(6), 466-481.

Robertson, S., & Ne'eman, D. (2008). Autistic acceptance, the college campus, and technology: Growth of neurodiversity in society and academia. *Disability Studies Quarterly*, 28(4). Available at http://www.dsq-sds.org/article/view/146/146.

Siebers, T. (2008). Disability Theory. Ann Arbor: University of Michigan Press.

Skiba, R. J., Simmons, A. B., Ritter, S., Kohler, K., Henderson, M, & Wu, T. 2006. The context of minority disproportionality: practitioner perspectives on special education referral. *Teachers College Record*, 108, 1424-1459.

Skrtic, C. (2005). A political economy of learning disabilities. Learning Disability Quarterly, 28(2), 149-155.

Tan, P., & Kastberg, S. (2017). Calling for research collaborations and the use of dis/ability studies in mathematics education. *Journal of Urban Mathematics Education*, 10(2). Retrieved from http://ed-osprey.gsu.edu/ojs/index.php/JUME/article/view/321

Waitoller, F. R., & King Thorius, K. A. (2016). Cross-pollinating culturally sustaining pedagogy and universal design for learning: toward an inclusive pedagogy that accounts for disability. *Harvard Educational Review*, 86(3), 366–389. https://doi.org/10.17763/1943-5045-86.3.366

DESIGNING AND RESEARCHING ONLINE PROFESSIONAL DEVELOPMENT

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In this working group, we continue with previous efforts to consider design and research methodologies related to teacher learning in online professional development contexts. We describe the extant literature on online professional development, including a review of digital technologies and the applicability of this practice to teacher learning in education. We then describe an innovative project designed to support the development of middle school mathematics teachers in rural contexts, with a focus on three distinct forms of online learning: digitally communicated teaching lab lessons, an online course, and online video coaching. Given recent technological advances and demands to support teachers in various contexts, we contend that researching and understanding these online models, as well as other online models is important for the broader field of mathematics education. As a result, Year Two of this proposed discussion group will combine whole-group and subgroup time to converse about: (a) the challenges of online professional learning experiences, (b) research tools, methods, and analyses, (c) the connections among different projects and studies, and (d) future collaborations and research.

Keywords: Teacher Education-Inservice/Professional, Research Methods, Learning Theory

All teachers need access to high quality professional development in order to meet the needs of students and to teach rigorous mathematics as outlined in college and career-ready standards (Marrongelle, Sztajn, & Smith, 2013). In order to accommodate the limited resources in some areas, including rural and urban school districts, online professional development has the potential to provide access to a wider range of teachers than what is possible face to face. Furthermore, given the propensity of millennials to seek online learning experiences, we feel that more attention needs to be given to the design, dissemination, and research of online professional development. Given the emerging importance and availability of online professional development, we propose the continuation of a working group that met at PMENA 2017 to continue focus on the design, dissemination, and research on online professional development. The working group participants will analyze current practices in online professional development, including the technology affordances and limitations. Major themes that will be addressed are:

- affordances of online platforms,
- affordances and constraints of synchronous vs. asynchronous experiences,
- challenges related to scaling up high-quality online professional development,
- methodologies used to research professional learning in online contexts.

As schools turn to digital learning contexts, it is inevitable that professional development will follow a similar trend. It is imperative to have research-based models that demonstrate how the features of high quality face-to-face professional development can be matched or augmented in online contexts. As an example of necessity, teachers in rural areas face constraints in terms of accessing the expertise and resources required for high-quality professional learning experiences, often because of lack of proximity to such resources as institutions of higher education and critical masses of teachers required to collectively reflect on problems of practice (Howley & Howley, 2005). Rural contexts are thus ideal sites for online professional development, which can be offered at a distance and can involve geographically dispersed participants (Francis & Jacobsen, 2013). At the same time, teachers in urban and suburban areas may have more regular access to professional development, but online formats afford conveniences and customized learning opportunities that may not be available in face-to-face settings. Digital learning contexts provide opportunities for connections and visual supports that may otherwise not be accessible in face to face professional development. As a result, we consider it necessary to research online professional development and to engage with other mathematics educators and researchers about online professional learning. This working group is intended to advance the practices of designing and researching online professional learning experiences by investigating the challenges of balancing high quality learning experiences and accessibility for teachers. The focus is also on reconnecting with those in attendance during the 2017 conference for updates on discussions about current happenings and experiences with online learning.

Below we provide an overview of the literature related to professional learning in online contexts. Then we revisit the NSF-funded model of online professional development discussed last year, describing what we have learned in the last year in terms of the learning environment and our efforts to research its impact. We will devote part of the first session providing updates on the project as a means of introducing possible models and methodologies to study online professional development, leaving opportunities over the next working sessions to incorporate discussion of other models and methodologies. We then conclude with aims for the 2019 working group.

Literature Related to Online Professional Learning

Digital Technologies

Online professional learning experiences combine longstanding and emerging digital technologies to provide high-quality, interactive, content-focused professional development. Longstanding digital technologies (e.g., electronic learning management systems) have been used to implement online courses to design and implement professional development for the past couple of decades. Emerging digital technologies involve an internet-based platform to implement online video coaching, or other online communications, in ways that augment the interactivity of face-to-face coaching. Online video coaching emerges from the content-focused face-to-face coaching that the project personnel have engaged in over the last ten years.

Research shows that while online communication lacks some of the modalities (e.g., gestures, facial expressions) and spontaneity of face-to-face communication (Tiene, 2000), there are also affordances unique to its asynchronous and text-based nature. In online discussions, communication tends to be more exact and organized (Garrison, Anderson, & Archer, 2001; McCreary, 1990), involve more formal and complex sentences (Sotillo, 2000; Warschauer, 1995) and incorporate critical thinking, reflection, and complex ideas (Davidson-Shivers, Muilenburg, & Tanner, 2001; Marra, Moore, & Klimczak, 2004). Research on synchronous online communication – which can include text chat windows and shared space in learning management systems – shows that it is experienced as more social than asynchronous spaces (Chou, 2002). Synchronous sessions induce *personal participation*, which Hrastinski (2008) compared to *cognitive participation* in that personal communication in synchronous spaces "involves more intense interaction … while cognitive participation is a more reflective type of participation supported by asynchronous communication" (p. 499). Furthermore, synchronous communication fosters multiple communication channels based on emerging networks within the larger group, including the use of chat boxes and personal email during synchronous sessions (Haythornthwaite, 2000, 2001). Researchers have reported positive outcomes from professional development involving synchronous exchanges via typing (e.g. Chen, Chen, & Tsai, 2009). However, synchronous verbal online discussions and group activities have not been a focus of research.

Online Professional Development in Education

Despite the growing popularity of online professional development, there is a continued need for empirical research regarding its quality and effectiveness (Dede, Ketelhut, Whitehouse, Breit, & McCloskey, 2009). Prior research has not demonstrated advantages for online professional development in terms of teacher outcomes (cf. Fishman et al., 2013), in part due to the lack of online professional development contexts that involve teachers in sustained, intensive reflection on their practices. Furthermore, teacher learning in online spaces can be challenging, especially related to complex forms of learning. Sing and Khine (2006) found that a number of factors make it difficult for teachers to engage in complex or difficult forms of learning in an online context, such as teachers' roles as implementers rather than producers, cultural norms where disagreement is seen as confrontational, and the cognitive demands relative to the available teacher time. Teacher learning in online contexts is discussed in more detail below.

In order to illustrate professional learning in an online context, we present a model that the authors are currently using in a project situated in rural contexts. We present the model in order to continue the discussion of this model and other potential models, as well as the learning platforms and other features, such as the synchronous or asynchronous nature of learning in online environments.

A Model of Online Professional Development

The innovative online professional learning experiences in the author's project focus on the development of teacher capacity to enact ambitious, responsive instruction aligned with the rigorous content and practice elements of the Common Core State Standards for Mathematics (CCSSM). We use the term *professional learning experiences* to denote that the professional development we employ differs from traditional workshop or other models that are too short or fragmented to be effective (Garet, Porter, Desimone, Birman, & Yoon, 2001).

In the project, we identified three primary research goals. To study and understand: (a) the ways online-based professional development can help teachers improve their instructional practices and their ability to notice and respond to student thinking; (b) the characteristics of the feedback cycles in the online coaching, the role of video feedback, and the asynchronous components of feedback cycle; and (c) the features of the professional development model that would inform efforts to scale up the model, including the resource commitments, the requisite capacity of the course instructors and coaches, and the logistical requirements of the courses and coaching. We are currently in year two of four years of the project. The following describes the three online components of our project. In the working group, we envision these and other components used by other researchers serving as the catalysts for dialogue around online professional learning and will engage particpatns with conversation around our learning in the past year while encouraging them to share their recent experiences.

Teaching Labs

In order to address the challenges of engaging teachers in learning complex practices in an online context, we include a component aimed at initiating and reinforcing relationships between participants and project personnel and at helping participants to understand the types of learning experiences and design and feedback cycles that will be the core of the project. Research on lesson study (e.g., Amador & Weiland, 2016; Stigler & Hiebert, 1999) has led to an emphasis on demonstration lessons where teams of teachers collectively plan, enact, and reflect on lessons in ways that make public the features of the lessons and teachers' instructional practices (Saphier & West, 2009). Consequently, one component of our project is a collaborative classroom activity, a Teaching Lab, that builds from the *studio classroom* model developed by the Teachers Development Group (2010), with features consistent with content-focused coaching (West & Staub, 2003). For each lesson, a group led by project personnel plans a lesson around a cognitively demanding task. On the day of the lesson, project participants review the lesson plan and explore the task, the mathematics embedded in the task and anticipated student approaches to solving the task, and the related CCSSM practice and content standards, making necessary revisions to facilitate the lesson. The project personnel then enact the lesson while the rest of the group observes and collects evidence of student thinking and learning. The group then collectively reflects on the experience, with a focus on describing evidence for student understanding using the data gathered by the teachers and observers. This process is repeated regularly with participants.

In the beginning of the project all demonstration lesson activities were face-to-face. However, at this point in our project, this component has moved to an online format of synchronous and asynchronous activities. The planning and debriefing portions are held via a video conferencing platform, Zoom, allowing for synchronous engagement in both whole group and small group discussions of the lessons. Asynchronous activities include watching and reflecting on the video recording of the demonstration lesson between the planning and debriefing sessions. Discussion in the working group will center on the affordances and constraints of the online model and possible modifications to ensure the intended professional development goals are met.

Online course - Orchestrating Mathematical Discussions

The second component of our project is the two online course modules, *Orchestrating Mathematical Discussions Parts One and Two*, aimed at orienting the participants toward high-leverage discourse practices that facilitate mathematically productive classroom discussions (Smith & Stein, 2011). In this course, the participants solve and discuss a series of high cognitive demand tasks, activities that will be accompanied by synchronous and asynchronous discussions around the *5 Practices for Orchestrating Productive Mathematics Discussions* (i.e. anticipating, monitoring, selecting, sequencing, connecting; Smith & Stein, 2011). The courses are designed to develop awareness of specific teacher and student discourse moves that facilitate productive mathematical discussions, to understand the role of high cognitive demand tasks in eliciting a variety of approaches worthy of group discussions, and to further develop participants' mathematical knowledge, particularly the rich connections around big mathematical ideas that are helpful to teach with understanding (Ball, 1991; Boaler & Staples, 2008; Chapin, O'Connor, & Anderson, 2003; Choppin, 2007a; 2007b; 2014; Herbel-Eisenmann, Steele, & Cirillo, 2013; Ma, 1999; O'Connor & Michaels, 1993).

In order to take advantage of the affordances of both asynchronous and synchronous characteristics of online communication, the course is embedded in a learning management system (LMS) that: allows for synchronous whole class and small group interaction; the sharing of artifacts, including those collectively developed in the LMS; and asynchronous discussion threads. In the online course modules in the LMS, the facilitator verbally presents a challenging task to the participants, which is viewed in the shared work space. The course instructor then assigns participants to virtual breakout rooms, in which the participants work synchronously in a common workspace, creating virtual white boards to share with the other groups. They can talk to each other, work simultaneously in the virtual space, and use the chat window to communicate. The course instructor can listen to and participate in these group discussions to determine when the groups are ready to present their solutions. The course instructor then closes the virtual breakout rooms, which automatically returns all participants to the main room to conduct a summary discussion of the different strategies, in effect modeling the practices in the *5 Practices* book. Asynchronously, the group can continue to go back and reflect and comment on the task and related solutions, as well as on the readings from the *5 Practices* book using discussions threads in the LMS. Participants are also encouraged to share resources, lesson plans, and student work as appropriate. The working group will discuss this format for online professional learning as well as other formats and tools that have proven beneficial for users.

Online Video Coaching

The third – and most innovative – component of our project's professional development program is the online video coaching that builds from models of content-focused coaching (West & Staub, 2003). More recently, thanks to the advent of improved internet-based software aimed at increasing collaboration around video data, the project personnel have begun conducting online video coaching with teachers. The coaching cycles are focused on identifying and unpacking the mathematics with the teacher, while anticipating likely student strategies, conceptions, and misconceptions. The coach helps the teacher identify evidence for demonstrating how students are thinking (from the video as well as from student artifacts) and make connections between different student approaches in order to help the teacher structure the summary discussion of the lesson.

The online coaching experiences involve synchronous and asynchronous components, with the goal of engaging participants in reflective or deliberative practice. The online coaching has features similar to face-to-face coaching, such as video conferencing conversations via Zoom, in which the coach and participant collaborate to plan lessons and reflect on the qualities of lessons. However, the online coaching includes an innovative component that involves asynchronous collaboration and feedback that structures the post-lesson collaborative reflection, features that augment or surpass the kind of feedback that can be given face-to-face. Teachers video-record themselves using Swivl, which allows them to place a camera (iPhone or other device) on a robot that tracks them around the room, allowing for teacher-focused video without the necessity of someone operating the camera. The video is automatically uploaded into a password-protected site and processed, and is immediately accessible to view and notate. The notation feature in Swivl allows the coach and the candidate to separately view and annotate the video. For example, a teacher can stop the video by hitting the pause button and type in a comment or question that is synced with the video, so that when the coach watches the video, she can read the comment during the point in the video referenced by the comment. The coach can do the same. The video can be viewed repeatedly, which allows for more thorough reflection and analysis. The notation provides for more in-depth and substantive feedback, pointing to specific instances of practice. The discussion group will focus on this model for professional coaching as well as other models or avenues for supporting individual teachers in online professional learning.

Researching Online Professional Learning Experiences

There is a dearth of research on online professional development, especially online professional development that is sustained and intensive. Similarly, while there have been 16 years of intensive efforts to implement coaching in schools, much of the research have revolved around the role and impact of coaches (Coburn & Russell, 2008; Penuel, Riel, Krause, & Frank, 2009), and less around the impact on reflective or deliberative practice. Although coaching has now been around for over ten years, there is limited research on the effectiveness of coaching in terms of improving teacher quality (Matsumura, Garnier & Spybrook, 2012). The greatest dearth of research involves online video coaching in education, as opposed to face-to-face video coaching, which has no peer-reviewed research yet associated with it.

Structure of the Working Group Sessions

Within this working group we propose to explore the following questions related to researching online professional learning experiences:

1. What are various platforms and models for online professional development?
2. What theoretical framework and methodologies are salient for researching online digital technologies and online professional learning experiences?
3. What data analysis methods are suited to the data captured in online environments?
4. In what ways can online professional learning experiences help teachers improve their instructional practices and their ability to notice and respond to student thinking?
5. In what ways can the characteristics of the feedback cycles in online coaching, the role of video feedback, and the asynchronous components of feedback cycle inform the design on online professional learning experiences to maximize teacher learning?
6. What features of the professional development model would inform efforts to scale up the model, including the resource commitments, the requisite capacity of the course instructors and coaches, and the logistical requirements of the courses and coaching?

Plan for Working Group:

In Session 1, the organizers will present brief update reports on the Author's project and research design, as well as a recap of the 2017 working group discussions. Subgroups will be formed to continue conversations around design and implementation efforts with online professional learning experiences from their own research and current efforts in the field; attendees from this working group at PMENA 2017 will provide updates on their respective projects in the small-group setting.

During Sessions 2 and 3 we will provide the subgroups time to continue collaborating on themes identified in the PMENA 2017 working group: a) identifying the challenges of online professional learning experiences that are the most challenging and why—this will include a specific look across the projects presented and with a focus on activities on the last year, b) refining research tools, methods, and analyses, c) exploring connections among different projects and studies for former and new attendees, and d) discussing future collaborations and research. We will close Session 3 with time to review group progress and discuss next steps for our work as shown in Table 1. Meeting notes, work, and documents will continue to be shared and distributed via our Google Folder (set up for this Working Group). The use of Google documents allows members to create an institutional memory of activities during the working group that we will continue to use and add to following the 2017 and 2018 working groups. This shared folder will also provide a shared space for future collaborations and writing projects related to online professional learning experiences within the 2017 and 2018 working group members.

	Overview of Proposed Working Subgroup Sessions	
Session 1	**Activities:** 1. Introductions and Agenda 2. Brief Presentations of Authors' Project and Research Questions 3. Brief Presentations of former Attendees' and New Attendees' Projects and Research Questions 4. Subgroup formation and initial work time - designing Online PD experiences	**Guiding Questions:** 1. What are the different forms of online professional development? 2. What research is being done related to online professional development? 3. Which aspects of online professional learning experiences are the most challenging to implement or research? 4. What are new questions that have arisen within the last year?
Session 2	1. Overview of subgroup's work from previous day 2. Subgroup work time - engagement in online professional learning experiences 3. Brief sharing of work in subgroups	1. How can online learning support teacher learning? 2. What are the affordances and constraints of various platforms? 3. What are the affordances and constraints of synchronous and asynchronous experiences?
Session 3	1. Overview of subgroup's work from previous day 2. Subgroup work time - researching online professional learning experiences 3. Brief sharing of work in subgroups 4. Final reflections – future collaborations and research	1. What theories and theoretical frameworks have informed the design of your research project(s)? 2. How might your work inform theory in researching online professional learning experiences? 3. What issues and challenges have you faced in designing studies in this area? 4. What challenges may exist for scaling up high-quality online professional development?

References

Amador, J., & Weiland, I. (2016). Conversational affordances and constraints of professional noticing during preservice teacher lesson study. *Journal of Mathematics Teacher Education,* online first.

Ball, D. L. (1991). Teaching mathematics for understanding: What do teachers need to know about subject matter? In M. M. Kennedy (Ed.), *Teaching academic subjects to diverse learners*. New York: Teachers College Press.
Boaler, J., & Staples, M. (2008). Creating mathematical futures through an equitable teaching approach: The case of Railside School. *Teachers College Record, 110*(3), 608-645.
Chapin, S. H., O'Connor, M. C., & Anderson, N. C. (2003). *Classroom discussions: Using math talk to help students learn*. Sausalito, CA: Math Solutions Publications.
Chen, Y., Chen, N., & Tsai, C. (2009). The use of online synchronous discussion for web-based professional development for teachers. *Computer & Education, 53*(4), 1155-1166.
Choppin, J. (2007a). Engaging students in collaborative discussions: Developing teachers' expertise. In M. E. Strutchens & G. Martin (Eds.), *The Learning of Mathematics: 69th NCTM Yearbook* (pp. 129-140). Reston: National Council of Teachers of Mathematics.
Choppin, J. (2007b). Teacher orchestrated classroom arguments. *Mathematics Teacher, 101*(4), 306-310.
Choppin, J. (2014). Situating expansions of students' explanations in discourse contexts. In A. Roth McDuffie & K. Karp (Eds.), *Annual Perspectives in Mathematics Education (APME) 2014: Using Research to Improve Instruction* (pp. 119-128). Reston, VA: National Council of Teachers of Mathematics.
Chou, C. C. (2002). A comparative content analysis of student interaction in synchronous and asynchronous learning networks. Paper presented at the 35th International Conference on System Sciences, Hawaii.
Coburn, C. E., & Russell, J. L. (2008). District policy and teachers' social networks. *Educational Evaluation and Policy Analysis, 30*(3), 203-235.
Davidson-Shivers, G. V., Muilenburg, L. Y., & Tanner, E. J. (2001). How do students participate in synchronous and asynchronous online discussions? *Journal of Educational Computing Research, 25*(4), 351-366.
Dede, C., Ketelhut, D. J., Whitehouse, P., Breit, L., & McCloskey, E. (2009). A research agenda for online teacher professional development. *Journal of Teacher Education, 60*(1), 8-19.
Fishman, B. J., Konstantopoulos, S., Kubitskey, B. W., Vath, R., Park, G., Johnson, H., & Edelson, D. C. (2013). Comparing the impact of online and face-to-face professional development in the context of curriculum implementation. *Journal of Teacher Education, 64*(5), 426-438.
Francis, K., & Jacobsen, M. (2013). Synchronous online collaborative professional development for elementary mathematics teachers. *International Review of Research in Open and Distance Learning, 14*(3), 319–343.
Garet, M. S., Porter, A. C., Desimone, L., Birman, B. F., & Yoon, K. S. (2001). What makes professional development effective? Results from a national sample of teachers. *American Educational Research Journal, 38*(4), 915-945.
Garrison , D. R., Anderson, T., & Archer, W. (2001). Critical thinking, cognitive presence, and computer conferencing in distance education. *American Journal of Distance Education, 15*(1), 7-23. doi:10.1080/08923640109527071
Haythornthwaite, C. (2000). Online personal networks: Size, composition, and media use among distance learners. *New Media & Society, 2*(2), 195-226.
Haythornthwaite, C. (2001). Exploring multiplexity: Social network structures in a computer-supported distance learning class. *The Information Society, 17*, 211-226.
Herbel-Eisenmann, B., Steele, M., & Cirillo, M. (2013). (Developing) teacher discourse moves: A framework for professional development. *Mathematics Teacher Educator, 1*(2), 181-196.
Howley, A., & Howley, C. (2005). High-quality teaching: Providing for rural teachers' professional development. *The Rural Educator, 26*(2), 1-5.
Hrastinski, S. (2008). The potential of synchronous communication to enhance participation in online discussions: A case study of two e-learning courses. *Information and Management, 45*, 499-506. doi:10.1016/j.im.2008.07.005
Ma, L. (1999). *Knowing and teaching elementary mathematics*. Mahwah, NJ: Lawrence Erlbaum.
Marra, R. M., Moore, J. L., & Klimczak, A. K. (2004). Content analysis of online discussion forums: A comparative analysis of protocols. *Educational Technology Research and Development, 52*(2), 23-40.
Marrongelle, K., Sztajn, & Smith (2013). Scaling up professional development in an era of Common Core State Standards. *Journal of Teacher Education, 64*(3), 202-211.
Matsumura, L., Garnier, H., & Spybrook, J. (2012). The effect of content-focused coaching on the quality of classroom text discussions. *Journal of Teacher Education, 63*(3), 214-228.
McCreary, E. K. (1990). Three behavioral models for computer-mediated communication. In L. M. Harasim (Ed.), *Online education: perspectives on a new environment* (pp. 117-130). New York: Praeger.
O'Connor, M. C., & Michaels, S. (1993). Aligning academic task and participation status through revoicing: Analysis of a classroom discourse strategy. *Anthropology and Education Quarterly, 24*(4), 318-355.

Penuel, W., Riel, M., Krause, A., & Frank, K. (2009). Analyzing teachers' professional interactions in a school as social capital: A social network approach. *Teachers College Record, 111*(1), 124-163.
Saphier, J., & West, L. (2009). How coaches can maximize student learning. *Phi Delta Kappan*, 46-50.
Sing, C. C., & Khine, M. S. (2006). An Analysis of Interaction and Participation Patterns in Online Community. *Educational Technology & Society, 9*(1), 250-261.
Smith, M. S., & Stein, M. K. (2011). *5 practices for orchestrating productive mathematical discussions*. Reston, VA: National Council of Teachers of Mathematics.
Sotillo, S. M. (2000). Discourse functions and syntactic complexity in synchronous and asynchronous communication. *Language Learning & Technology, 4*(1), 82–119.
Stigler, J. W., & Hiebert, J. (1999). The teaching gap: Best ideas from the world's teachers for improving education in the classroom. New York: Free Press.
Teachers Development Group. (2010). About the mathematics studio program: Transforming a school's culture of mathematics professional learning. West Linn, Oregon: Teachers Development Group
Tiene, D. (2000). Online discussions: A survey of advantages and disadvantages compared to face-to-face discussions. *Journal of Educational Multimedia and Hypermedia, 9*(4), 371-384.
West, L., & Staub, F. C. (2003). *Content-focused coaching: Transforming mathematics lessons*. Portsmouth, NH: Heinemann.

CONCEPTIONS AND CONSEQUENCES OF WHAT WE CALL ARGUMENTATION, JUSTIFICATION, AND PROOF

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Argumentation, justification, and proof are conceptualized in many ways in extant mathematics education literature. At times, the descriptions of these objects and processes are compatible or complementary; at other times, they are inconsistent and even contradictory. The inconsistencies in definitions and usages of the terms argumentation, justification, and proof highlight the need for scholarly conversations addressing these (and other related) constructs. Collaboration is needed to move toward, not one-size-fits-all definitions, but rather a framework that highlights connections among them and exploits ways in which they may be used in tandem to address overarching research questions. Working group leaders aim to facilitate discussions and collaborations among researchers to advance our collective understanding and conveyed use of argumentation, justification and proof, particularly the relationships among these important mathematical constructs. The 2018 working group sessions will extend previous conversations into issues of teacher preparation with respect to argumentation, justification, and proof. The group will also discuss how issues of equity, social justice, and marginalized populations intersect with the teaching and learning of argumentation, justification, and proof.

Keywords: Reasoning and Proof; Advanced Mathematical Thinking

Brief History of the Working Group

The *Conceptions and Consequences of What We Call Argumentation, Justification, and Proof* Working Group (AJP-WG) has met for three consecutive years, beginning in 2015. The working group's primary focus is on the field's conceptualization of the interrelated objects and processes of argumentation, justification, and proof. Previous working groups at PME and PME-NA had focused on either proof or argumentation, but the present working group is the first to attend specifically to the connections among these three constructs.

During the working group's initial meeting (2015), attendees made progress on considering the interrelationships among argumentation, justification and proof, and we deepened our understandings of our own perspectives and the range of perspectives held by others in the group. The goal during this meeting was not consensus or deciding a best approach. Rather, we sought to better understand the complexity and diversity of individuals' perspectives with respect to their research agendas and professional practice. We were encouraged that our efforts were well received, with 46 scholars, including at least 10 graduate students, participating in the working group the first year.

The second gathering of the AJP-WG was held in 2016 at the annual meeting for PME-NA in Tuscon, AZ. Thirty-six scholars attended the sessions, which continued the group's focus on the interrelationship between and among the concepts and terms related to argumentation, justification, and proof. Specifically, participants considered their definitions for these concepts and how such definitions may change given different contexts and foci.

The AJP-WG met for the third time in 2017 in Indianapolis at the 39th Annual Meeting of PME-NA. This meeting, attended by 28 scholars, involved examining a set of middle grades student artifacts from different perspectives on argumentation, justification, and proof. Analyses were presented by members of the working group, and these analyses were critiqued and compared by the working group during our meetings. Additional information about these meetings of the AJP-WG is included later in the proposal.

Summary of Focal Issues

Engagement in disciplinary practices of constructing viable arguments, justifying conclusions, critiquing the reasoning of others, and constructing proofs for mathematical assertions appear in educational policy documents of K-12 students. (National Council of Teachers of Mathematics [NCTM], 2000; National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010). Most recently policy documents have begun to focus on the teaching practices and the preparation and development of teachers in facilitating argumentation, justification, and proof. For example, according to *Principles to Actions: Ensuring Mathematical Success for All* (NCTM, 2014): "Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments" (p. 29). An example of teacher preparation and development is the *Standards for Preparing Teachers of Mathematics* from the Association of Mathematics Teacher Educators (AMTE, 2017) "Well-prepared beginners encourage students to communicate their reasoning, critique the reasoning of others, and develop arguments through discourse and mathematical writing" (p. 77). Due to these recent calls, the 2018 AJP-WG will engage participants in sessions to extend previous conversations of argumentation, justification, and proof into issues of teacher education.

There is a growing body of research in mathematics education focused on teachers' argumentation, justification, and proof. The research on proof, for example, have included studies on teachers' conceptions (Knuth, 2002) and perceptions (Boyle, Bleiler, Yee, & Ko, 2015) of proof, how teachers enact proof-related tasks (Bieda, 2010; Bostic, 2016), how teachers engage with students in proving (Herbst, 2002; Martin, McCrone, Bower, & Dindyal, 2005), teachers' evaluation of student-generated proofs (Tsamir, Tirosh, Dreyfus, Barkai, & Tabach, 2009), and how coursework impacts teachers' reasoning and proving (Karunakaran, Freeburn, Konuk, & Arbaugh, 2014). While proof has received more attention, argumentation, as a concept, is garnering new prominence with increasing attention in the mathematics education research. Research on argumentation has included examining new methodologies to understand teachers' argumentation (Metaxas, Potari, & Zachariades, 2016), teachers' contextualization of argumentation in the classroom (Staples & Newton, 2016), and teachers' discursive actions during argumentation (Conner, Singletary, Smith, Francisco, & Wagner, 2014; Kim & Hand, 2015). In regards to justification, researchers have investigated justification in mathematics courses for prospective elementary teachers (Simon & Blume, 1996), teachers' perceptions of justification (Staples, Bartlo, & Thanheiser, 2012), and professional development to support justification (Lesseig, 2016).

Highlights from the Year 1 Working Group Discussions

A focal activity during the initial working group sessions in 2015 was the development of Diagrams/Concept Maps in which each participant generated a representation of the relationships among argumentation, justification, and proof from his or her perspective. Discussion surrounding the relationships between constructs was supported by an initial presentation by Keith Weber on Day 1, followed by a panel presentation, moderated by Samuel Otten, with Kristen Bieda, AnnaMarie Conner, and Pablo Mejía-Ramos serving as expert panelists (i.e., Bieda – justification; Conner – argumentation; and Mejía-Ramos – proof). The panelists shared how they conceptualized the interrelationships among argumentation, justification and proof; they explicated how they came to use the central construct they use in their research and why they felt that choice was productive for their work; and they offered their thoughts on the current state of the field and what we might need to tackle next in relation to these constructs. The final day of the 2015 working group sessions offered the opportunity to revisit participants' Diagrams/Concept Maps, though now potentially informed by additional perspectives and questions gained from the prior two sessions.

Three products were generated from the 2015 meeting. First, a pair of podcasts generated by Samuel Otten are available worldwide: The first podcast is of Weber's talk (http://mathed.podomatic.com/entry/2015-11-16T07_01_19-08_00), and the second podcast is the moderated panel discussion (http://mathed.podomatic.com/entry/2015-11-19T07_19_37-08_00). The second product was a white paper that was developed by the working group organizers, the panelists, and several other participants from the working group who volunteered to participate in the online publication (Cirillo et al., 2016). The white paper summarized the working group activities and discussions and also includes the set of 44 Diagrams/Concept Maps that were generated as well as annotations and analyses. The final product was a poster presentation for PME-NA 2016 that was based on the analyses of the Diagrams/Concept Maps (Strachota et al., 2016).

Weber's presentation highlighted different traditions and points of disagreement, for example, citing Reid's (2001) observation that research simultaneously suggests that secondary students struggle to construct proofs, while at the same time suggesting that primary children are capable of engaging in proof. Weber prompted the group to consider how different traditions may inform each other in order to advance the field collectively. In particular, he outlined three broad traditions in proving: *proving as problem solving, proving as convincing,* and *proving as socially embedded activity.* Each corresponds to a different focus for research and/or instruction. Weber offered two thought-provoking suggestions, both of which may help us understand the lack of convergence in results and definitions. One suggestion was that proof may not be a singular, easily defined concept, but rather a *cluster concept,* as used by Lakoff (1987). In this sense, there is no list or decision procedure to identify a proof, but rather there is a set of features associated with the concept, and many - but not all - apply in any one instance.

The second suggestion was that the features or properties associated with proof may be closely interrelated for mathematicians, but not for students. In particular, for mathematicians, a convincing argument and socially sanctioned argument are often one-in-the-same. For students, however, those are not tightly connected and may describe very different types of arguments. He suggested, "Perhaps much of the disagreement amongst mathematics educators is that they are using proof as a shorthand to denote things that are different to students but similar for mathematicians" (Cirillo et al., 2016, p. 7).

The panel discussion on our second day further raised awareness of how crucial it is to not only define one's terms, but also to specify the context of one's work. For example, Bieda and Conner both work closely with students and teachers in secondary settings, and in that context, they have found proof and proving to be terms that distance or invoke conceptions of end product and formality. Consequently in their work, they have chosen different focal constructs. Both Bieda and Conner articulated an explicit link of their work to students' proof-producing capacities at the tertiary levels, but they do not centralize that term in their research in school mathematics.

Our discussions and subsequent analyses of the Concept Diagrams/Maps offered additional information about the variety of ways these three important constructs are understood in relation to one another. During the gallery walk and subsequent analyses across the Concept Diagrams/Maps, we could discern little to no agreement about the relationship between justification and argumentation. It seemed that participants held these two constructs (justification and argumentation) either fully distinct from the set of things we call proof, or that proof was a very specific version of each of these. Some considered justification a subset of arguments; others positioned arguments as a subset of justifications; and still others had them as overlapping, but not concurrent, sets.

Proof (and proving) seemed to be of a different nature than argumentation and justification for our participants, and proof participants either positioned it at the far end of a continuum of the constructs, a plane above, or as a separate entity. Others considered proof to be a specific subset of arguments and justifications. Additional details can be found in the White Paper (Cirillo et al., 2016). A question raised in the discussion on Day 3 was whether proof was so valorized that we position it as the "desired end product" for all arguments, even when that might not be a productive or educative goal. This lack of not only convergence but general clarity provides an important opportunity for further exploration and raises questions about the consequences of these different concepts. It implores us to continue to work to develop a framework to connect these constructs and clarify not only our commitments and definitions but the interrelationships among these important ideas.

Highlights from the Year 2 Working Group Discussions

During the first session of the second meeting of the AJP-WG, Samuel Otten moderated panel presentations on differing applications of definitions provided by Eric Knuth, David Yopp, and Orit Zaslavsky. On the second day, participants examined data artifacts from different grade levels and were asked to consider whether and in what ways they would define features in the data with respect to argumentation, justification, and proof. All participants were provided two common artifacts (a transcript of a high school math class discussion & five samples of grades K-3 writing samples) as well as one additional artifact of choice (middle school, high school, or tertiary level artifact). Discussion surrounding the artifacts provided an opportunity for participants to consider how their definitions were affected by application to the differing data. It provided an arena in which to see how each definition served as a lens for viewing the artifacts and how these different lenses might lead one to position a work sample in different ways, depending on the type of mathematical activity represent in the work sample based on the definition. This discussion continued through to the final session. Further, the discussion surrounding the artifacts facilitated initial organization of three networking groups–one focused on reading and sharing journal articles and two focused on producing written products. One product available from this second AJP-WG meeting is a podcast of the panel discussion produced by Samuel Otten (https://www.podomatic.com/podcasts/mathed/episodes/2016-11-

14T07_10_36-08_00). A second product is a white paper (available at https://www.researchgate.net/publication/317267228) summarizing the working group activities and discussions of the artifacts from year two (Staples et al., 2017). The final product is the organization of networking groups for continued collaboration related to, but independent from, the working group. Thus, activity across both years of the working group has facilitated the continued development of a community of mathematics education researchers who we anticipate will continue these discussions over several years.

During the panel discussion, our panelists shared how they used and defined justification, argumentation, and proof and how their definitions and applications of these constructs influenced, and were influenced by, their research questions or contexts. A key point arose from David Yopp's juxtaposition of his definition of proof with Stylianides' (2007) definition of proof in relation to a particular 8th-grade work sample. Yopp argued that by Stylianides' definition, the student's work was not a proof (violating two of three conditions). By Yopp's definition, however, the student's work was a proof. Yopp stated that a proof eliminates the possibility of counterexamples. Thus, from the student's perspective—and perhaps class's perspective—he had eliminated the possibility of counterexamples, thus providing proof of the claim (see Figure 1).

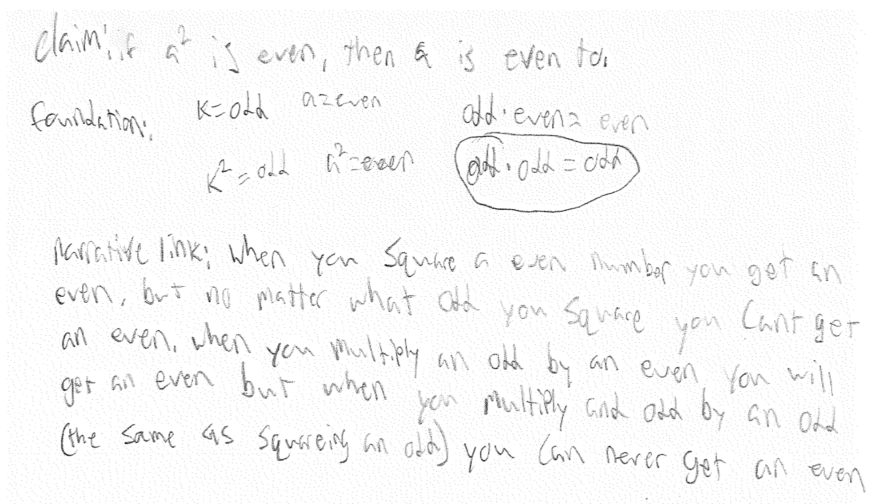

Figure 1. Student work samples shared by David Yopp.

We then turned our attention to reviewing artifacts of K-16 classrooms, to provide a concrete opportunity to see how definitions might interact with context. Participants were offered two artifacts in common, from elementary grades work samples and a high school transcript, and then could choose a third artefact to review, from middle school, high school or tertiary level. Although it became apparent was that we offered too much in too short a time frame, our discussion was productive, and we recap a few of the key ideas offered here.

One question raised was how much context we needed to know in order to engage the question, is this a proof? Or is this an argument? This question was posed based on recognizing that both Yopp and Stylianides offered definitions of proof that dependent to a degree on a child's/class's conceptual sphere. In Stylianides' definition, this context element comes through with the criteria that all must be "known by" or "within the conceptual reach" of the community. In Yopp's, this context element comes through with the idea that one must eliminate all possible counterexamples, so if a student is not aware of, say, complex numbers, s/he does not have to offer a proof that accounts for all complex numbers. The student can only attend to and eliminate the possibility of counterexamples (actively) from his or her realm of possibility. Continuing with the general idea of context, questions were raised about how grade-level dependent a proof was, and whether one needed to know more about what was taken-as-shared in a class to evaluate whether an argument did or did not comprise a proof. One of the final points raised was that a proof is not a stand-alone entity; we need to know what the norms and assumptions are in order to understand if something is operating as a proof in that context.

In our final session, we sought to continue the discussion from the previous session and also organize toward "networking groups" for those interested in sustaining or advancing conversations with respect to argumentation, justification and proof during the year. Participants each wrote brief questions they were interested in pursuing and indicated a level of commitment they might have for the pursuit (e.g., discussion groups, reading groups, research group). Our time then was devoted to participants having conversations to connect with one another around shared areas of interest and similar levels of desired future commitment.

Highlights from the Year 3 Working Group Discussions

In Year 3, the working group explored the consequences of examining data from differing perspectives on the definitions of and relationships among argumentation, justification, and proof (see Cirillo et al., 2015; Staples at al., 2016; Conner et al., 2017). During the first session, working group participants examined a set of data that included both a transcript and students' written artifacts from a middle grades classroom (shared by Megan Staples from the JAGUAR project), using perspectives negotiated among themselves. Conversations revealed working group participants used a wide range of perspectives as they looked at the data. Working group participants raised a number of questions as they looked at the data, including the importance of knowing the teacher's goals for the lesson; emphasizing how a task was set up would impact the students' productions of argumentation, justification, or proof; and how different definitions of argumentation, justification, and proof might yield different conclusions about the conversation and written work.

During the second session, the working group heard presentations from two researchers who had deliberately taken different perspectives on the data presented in session 1. Kristen Bieda examined the data using Harel and Sowder's (2007) definition of proof, and Megan Staples examined the data using a community-of-practice lens (Lave & Wenger, 1991; Wenger, 1998) involving justification. Harel and Sowder's definition of proof involves the constructs of ascertaining and persuading. Bieda raised the question of collective ascertaining, asking if ascertaining has to be done by one person or if it could be done by or within a group. She also noted that the definition of proving as ascertaining and persuading omits specifying forms of reasoning that are valid. Staples pointed out that her analysis showed different kinds of language use during small-group and whole-class discussion. Significantly, she did not find any teacher use of the word justification, but she found numerous instances of justification within the transcript, particularly in the context of "explain your thinking." Discussion after the presentations included the importance of positioning definitions of argumentation, justification, and proof within a larger perspective. Also discussed were issues of disciplinary norms, including a tendency to examine proof from a disciplinary perspective but justification without assigning a disciplinarily specific definition.

In session 3, Karl Kosko introduced a third perspective on the data, using Pierce's (1903/1998) semiotic view coupled with Toulmin's (1958/2003) diagrams of argumentation to examine how grammar in the mathematics register functioned to convey meaning. He examined how use of warrants influences the co-construction of mathematical argumentation. In the subsequent discussion of the three presentations, working group participants emphasized that definitions and methodology were intertwined and difficult to separate. Essentially, in examining the data, the three researchers were looking at different things: justification as a means of concept development, understanding whether doubt is removed, and what students are doing as they use or omit warrants. All of these perspectives resonated with participants, and the conversation validated the importance of specifying both the perspective and the definition being used in research about argumentation, justification, and proof.

Focus for Year 4

Our focus for Year 4 is how mathematics teacher educators prepare K-16 teachers for enacting argumentation, proof, and justification in their own classrooms. We will engage in discussion on how focusing on argumentation, proof, and justification may result in different outputs from teachers, in teacher preparation programs and professional development. As learned in Year 1 and 2, the members of this community have a variety of definitions and perspectives of argumentation, proof, and justification (see Cirillo et al., 2015; Staples et al., 2016; Conner et al., 2017). The community's perspectives on the relationship between argumentation, justification, and proof also varied. We will explore how these differing perspectives inform the ways prospective and in-service teachers conceptualize argumentation, justification, and proof. Guiding this year's focus are the following questions: How have our group members helped teachers conceptualize argumentation, justification, and proof? What conjectures do our community members have about the consequences of the teachers' conceptualizations of argumentation, justification, and proof in their teaching and learning of mathematics?

In Year 3, the relationship between context and the individual's definitions of argumentation, justification, and proof was further explored. We plan to build on these previous conversations by including how different members have attempted to teach prospective and inservice teachers about argumentation, justification, and proof. We hypothesize one's conceptualization of argumentation, justification, and proof will inform the activities and tasks given to prospective and inservice teachers. Time will be allotted for the participants of the community to work together to discuss, brainstorm, and share different activities they have enacted or could enact with preservice and inservice teachers to highlight important aspects of argumentation, justification, and proof. Moreover, ways in which argumentation, justification, and proof could be emphasized in already enacted activities (e.g. high cognitive demanding tasks) will also be discussed. We will use these opportunities to build collaboration among the members of the working group. Facilitating these conversations on developing potential activities for prospective and inservice teachers, we will construct an online space to share the different activities, tasks, and other interventions the members of the working group develop or have enacted previously. Continued collaboration will be promoted to design and investigate the enactment and learning results of these activities on argumentation, proof, and justification.

Plan for the Working Group

Session 1: Exploring the teaching and learning of AJP with prospective teachers

In Session 1, we will begin with introductions and review our progress from the past three years. Then, we will start a panel discussion. The specific focus of this panel discussion is: How can we, as mathematics teacher educators, support prospective teachers to learn and teach argumentation, proof and justification from elementary to tertiary education?

In past discussions, working group participants shared how they defined and conceptualized the interrelationships among argumentation, proof, and justification. This year, we will follow up to explore how their different perspectives influence their preparation of prospective teachers to teach argumentation, justification, and proof in mathematics classrooms. During the session, we will invite working group participants to share how they prepare prospective teachers to engage in argumentation, justification, and proof in their classrooms.

Session 2: Issues of equity and social justice in classrooms involving AJP

In Session 2, we will have a series of presentations exploring the intersection of equity and social justice and teaching and learning AJP. We will focus on teaching involving AJP in areas with marginalized populations. Researchers will be invited and will share their work, particularly about how they conceptualize and use AJP in their studies and implications for marginalized students. With working group participants, we will discuss what AJP might look like in mathematics classrooms with marginalized students and what the role of equity and social justice is and how they can be incorporated into the learning of AJP in those contexts. We will also discuss what else a teacher should consider in the intersection of conceptualizations of AJP and equitable instruction.

Session 3: Brainstorming recommendations for the preparation of teachers

In Session 3, working group participants will brainstorm recommendations for teacher educators who are preparing prospective and practicing teachers to teach AJP in the context of mathematics classrooms. We will discuss what activities or resources might be helpful for the teachers learning to teach the AJP constructs in classrooms and what the teachers could learn from participating in these activities. Building on session 2, we will also discuss what the teachers should pay attention to when implementing AJP in their classrooms and what relevant researchable questions are important to the field.

Anticipated Follow-up Activities

We anticipate that the products and follow-up activities from Year 4 will build on the activities and products from our three previous years. A major follow-up activity for this year will be a white paper exploring recommendations for exploring argumentation, justification, and proof in teacher education. We will continue to encourage networking groups, which will likely shift and expand. Additional networking groups with other related interests will also be encouraged.

References

Association of Mathematics Teacher Educators. (2017). *Standards for Preparing Teachers of Mathematics.* Available online at amte.net/standards.

Bieda, K. N. (2010). Enacting proof-related tasks in middle school mathematics: Challenges and opportunities. *Journal for Research in Mathematics Education*, *41*(4), 351–382.

Bostic, J. D. (2016). Fostering justification: A case study of preservice teachers, proof-related tasks, and manipulatives. *Journal of Mathematics Education at Teachers College*, *7*(1). Retrieved from http://journals.tc-library.org/index.php/matheducation/article/view/1156

Boyle, J. D., Bleiler, S. K., Yee, S. P., & Ko, Y.-Y. (Winnie). (2015). Transforming perceptions of proof: A four-part instructional sequence. *Mathematics Teacher Educator*, *4*(1), 32–70. https://doi.org/10.5951/mathteaceduc.4.1.0032

Cirillo, M., Kosko, C., Newton, J., Staples, M. & Weber, K. (2015). Conceptions and consequences of what we call argumentation, justification and proof. In Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp 1343-1350). East Lansing, MI: Michigan State University.

Cirillo, M., Kosko, K. W., Newton, J., Staples, M., Weber, K., Bieda, K., Conner, A., Mejía-Ramos, P., Otten, S., Hummer, J., & Strachota, S. (2016). *Conceptions and consequences of what we call argumentation, justification, and proof* (white paper). PME-NA Working Group for Argumentation, Justification, and Proof.

Conner, A., Kosko, K. W., Staples, M., Cirillo, M., Bieda, K., & Newton, J. (2017). Conceptions and consequences of what we call argumentation, justification, and proof. In E. Galindo & J. Newton (Eds.). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1464-1473). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Conner, A., Singletary, L. M., Smith, R. C., Wagner, P. A., & Francisco, R. T. (2014). Teacher support for collective argumentation: A framework for examining how teachers support students' engagement in mathematical activities. *Educational Studies in Mathematics, 86*(3), 401-429. doi: 10.1007/s10649-014-9532-8

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 805-842). Charlotte, NC: Information Age Publishing.

Herbst, P. G. (2002). Engaging students in proving: A double bind on the teacher. *Journal for Research in Mathematics Education*, *33*(3), 176–203. https://doi.org/10.2307/749724

Karunakaran, S., Freeburn, B., Konuk, N., & Arbaugh, F. (2014). Improving preservice secondary mathematics teachers' capability with generic example proofs. *Mathematics Teacher Educator*, *2*(2), 158–170. https://doi.org/10.5951/mathteaceduc.2.2.0158

Kim, S., & Hand, B. (2015). An analysis of argumentation discourse patterns in elementary teachers' science classroom discussions. *Journal of Science Teacher Education*, *26*(3), 221–236. https://doi.org/10.1007/s10972-014-9416-x

Knuth, E. J. (2002). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education*, *5*(1), 61–88. https://doi.org/10.1023/A:1013838713648

Lakoff, G. (1987). *Women, fire, and dangerous things.* Chicago: University of Chicago Press.

Lave, J., & Wenger, E. (1991). Situated learning: Legitimate peripheral participation. New York: Cambridge.

Lesseig, K. (2016). Fostering teacher learning of conjecturing, generalizing, and justifying through Mathematics Studio. *Mathematics Teacher Education and Development, 18*(1), 100-119.

Martin, T. S., McCrone, S. M. S., Bower, M. L. W., & Dindyal, J. (2005). The interplay of teacher and student actions in the teaching and learning of geometric proof. *Educational Studies in Mathematics*, *60*(1), 95–124. https://doi.org/10.1007/s10649-005-6698-0

Metaxas, N., Potari, D., & Zachariades, T. (2016). Analysis of a teacher's pedagogical arguments using Toulmin's model and argumentation schemes. *Educational Studies in Mathematics*, *93*(3), 383–397. https://doi.org/10.1007/s10649-016-9701-z

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all.* Reston, VA: Author.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics*. Washington, DC: Authors.

Peirce (1998) A syllabus of certain topics of logic. In: Houser N, De Tienne A, Eller JR, et al. (eds) *The Essential Peirce: Selected Philosophical Writings*, Vol. 2. Bloomington, IN: Indiana University Press (written originally in 1903), pp.258–299.

Reid, D. (2001). Proof, proofs, proving and probing: Research related to proof. In M. van den Heuvel-Panhuizen (Ed.), Proceedings of the *Twentieth-Fifth Annual Conference of the International Group for the Psychology of Mathematics Education,* (Vol. I, p. 360). Utrecht, Netherlands. Retrieved from http://files.eric.ed.gov/fulltext/ED466950.pdf

Simon, M. A., & Blume, G. W. (1996). Justification in the mathematics classroom: A study of prospective elementary teachers. *The Journal of Mathematical Behavior*, *15*(1), 3–31. https://doi.org/10.1016/S0732-3123(96)90036-X

Strachota, S., Hummer, J., Cirillo, M., Newton, J., Kosko, K. W., Staples, M., & Weber, K. (2016). Conceptions of proof, argumentation, and justification. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter for the Psychology of Mathematics Education* (p. 712). Tucson, AZ: The University of Arizona.

Staples, M. E., Bartlo, J., & Thanheiser, E. (2012). Justification as a teaching and learning practice: Its (potential) multifacted role in middle grades mathematics classrooms. *The Journal of Mathematical Behavior*, *31*(4), 447–462. https://doi.org/10.1016/j.jmathb.2012.07.001

Staples, M., & Newton, J. (2016). Teachers' contextualization of argumentation in the mathematics classroom. *Theory Into Practice, 55*, 294-301.

Staples, M., Newton, J., Kosko, K. W., Conner, A., Cirillo, M., Bieda, K. (2016). Conceptions and consequences of what we call argumentation, justification, and proof. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.). *Proceedings of the 38th Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1704-1712). Tucson, AZ: The University of Arizona.

Staples, M., Newton, J., Kosko, K. W., Conner, A., Cirillo, M., Bieda, K., Yopp, D., Hummer, J., Strachota, S. Singh, R., An, T., Going, T., & Zhuang, Y. (2017). *Using Artifacts to Explore Conceptions and Consequences of Argumentation, Justification, and Proof* (white paper) PME-NA Working Group for Argumentation, Justification, and Proof. (https://www.researchgate.net/publication/317267228)

Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education, 38*(3), 289-321.

Toulmin, S. E. (2003). *The uses of argument* (updated ed.). New York, NY: Cambridge University Press. (Original work published 1958)

Tsamir, P., Tirosh, D., Dreyfus, T., Barkai, R., & Tabach, M. (2009). Should proof be minimal? Ms T's evaluation of secondary school students' proofs. *The Journal of Mathematical Behavior*, *28*(1), 58–67. https://doi.org/10.1016/j.jmathb.2009.04.002

Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. Cambridge: Cambridge University Press.

THE MATHEMATICS EDUCATION OF ENGLISH LEARNERS

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This Working Group builds on the work of Working Groups in 2015 and 2016. We will continue considering multiple aspects of research and practice related to mathematics learning and teaching with English Learners. Our goals for the 2018 Working Group include: (1) sharing current perspectives on extant research related to mathematics and English learners, (2) understanding venues for dissemination of research on ELs, including novel outlets that connect research to practice, and (3) fostering new collaborations and supporting further connections among researchers. In Session 1, the organizers will present brief reports on the work that will be used to catalyze subsequent sessions' work. During Session 2, we will engage in a shared data analysis activity drawing on excerpts of student interview data. The activity will center on comparing and contrasting methods for analyzing the learning and teaching of ELs. In the final session, we will have four breakout groups from which participants can choose to join. Each group will revisit a key aspect of the prior days' work. We will close with time to review group progress and discuss next steps for our work.

Keywords: Equity and Diversity; Teacher Education-Pre-service; Teacher Education-In-service/Professional Development; Research Methods

Focal Issues

English Learners (ELs) are the fastest growing group of U.S. students (Verplaetse & Migliacci, 2008). U.S. schools have seen an increase of 152% in the population of EL students over the past 20 years (National Clearinghouse for English Language Acquisition, 2008). This increase in the population of ELs has created a need for schools and teachers to create inclusive and equitable mathematics classrooms. No longer is supporting ELs a concern for only for educators in states like Arizona, Texas and California with traditionally high numbers of EL students. Instead, with all but 15 states across the country seeing increases in their EL populations between 2004-05 and 2014-15 (National Center for Educational Statistics (NCES), 2017), there is increasing nationwide pressure for support in addressing the needs of these students.

The implementation of the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices, Council of Chief State School Officers, & Officers, 2010) has put additional pressure on mathematics teachers of students who are still gaining proficiency in English. Teaching aligned with the CCSSM's content standards and the standards for mathematical practice must address the language demands required to engage in mathematical discourse. Providing ELs with quality educational experiences is no longer relegated only to language specialists; it is a "mainstream" issue for which all mathematics teachers must be prepared (Bunch, 2013). In light of these demographic and curricular shifts, there is a growing body of research on the teaching and learning of ELs in school mathematics.

In the following we briefly summarize research on mathematics teacher preparation for working with ELs and mathematics curriculum design issues for ELs.

Despite the rise in the number of ELs and changes to school mathematics standards, there has not been an increase in teacher preparation geared toward supporting ELs, and this is especially the case for content-specific preparation (e.g., Kaufman & Stein, 2010). Although the consensus among mathematics educators that teachers need focused, content-specific preparation to teach ELs effectively, most inservice teachers have had few, if any, professional learning experiences around working with ELs in mathematics classrooms (Ballantyne, Sanderman, & Levy, 2008). In fact, many mathematics teachers struggle to understand their role in supporting ELs' mathematics language development (Willey, 2013). In 2008, Ballantyne et al. found that it was "likely that a majority of teachers have at least one English language learner in their classroom," although "only 29.5% of teachers with ELs in their classes have the training to do so effectively" (p. 9). This misalignment of the realities of today's classrooms and teacher preparation has necessitated examination into effective means of supporting current teachers and preparing prospective teachers to meet the needs of linguistically diverse learners and studying such work.

In addition to the need for inservice teacher training to teach mathematics with ELs, preservice teachers are also underprepared to teach mathematics with ELs. Preservice teachers typically have few opportunities to think specifically about how they will work with ELs in their mathematics classrooms. Preparation programs often include coursework on teaching ELs that is not specific to the content areas in which preservice teachers will work. Meanwhile, the content courses for preservice teacher often focus solely on content, devoid of exploring how to support ELs in mathematics classrooms. Recently researchers have shared strategies for engaging preservice teachers in working with ELs in mathematics classrooms. For example, Fernandes (2012) suggested a series of mathematics task-based interviews to engage preservice teachers in the process of noticing the linguistic challenges that ELs face and the resources these students bring to their mathematical communication. In another example of such work, the TEACH MATH research team is using tasks in their content courses to support preservice teachers in drawing on students' funds of knowledge (Turner et al., 2012).

Beyond using tasks in content courses, teachers use tasks in their classrooms in the form of curriculum. Such curriculum plays a key role in the teaching and learning of mathematics and teachers play a pivotal role in selecting and enacting curriculum materials for students. The choice of curriculum materials impacts students' opportunities for learning in the mathematics classroom (Kloosterman & Walcott, 2010). Early work on curriculum and English learners focused specifically on the challenges ELs encounter when completing or interpreting word problems (Téllez, Moschkovich, & Civil, 2011). Since that time, the field has transitioned to focus on curriculum that draws on and connects to students' cultural resources (e.g., Barton, 1996; D'Ambrosio, 2006), the development of curriculum materials specifically for ELs (e.g., Freeman & Crawford, 2008), and the evaluation of a curriculum's appropriateness for ELs (e.g., Khisty & Radosavljevic, 2010). More recently, work related to ELs and curriculum has begun to look at teachers' use of curriculum (e.g., de Araujo, 2017).

This year, our Working Group aims to look backward to synthesize research related to the mathematics education of ELs broadly, and in relation to this Working Group specifically, while also looking forward to consider new research in the field. In looking back at research that the field has completed, the organizers have completed a literature review of research on ELs and mathematics education (de Araujo, Roberts, Willey, & Zahner, under review), and we found 99 empirical articles on this topic since 2000. This represents an increase in attention toward the mathematics education of ELs, with most recent work being framed using sociocultural and sociopolitical frameworks. We note that this research is occurring in concert with the political landscape, acknowledging the sociocultural and political dimensions of mathematics and of school mathematics (Gutiérrez, 2013). There is still much work to be done; some contexts and critical dimensions remain unexamined. Moreover, teachers still need more support, and we still need to acknowledge and build on the resources that ELs bring to classrooms.

In addition to synthesizing the literature, the 2018 Working Group will look forward as we foster new collaborations and help define future directions for research. To do this work, we will bring together researchers from diverse contexts to examine current and past research on ELs in mathematics education while also supporting burgeoning collaborations to establish future, imperative research directions.

Brief History of the Working Group

The facilitators of this Working Group initially came to work together through the NSF-funded Center for the Mathematics Education of Latinas/os (CEMELA). CEMELA brought together researchers from across the country to collaborate on research focused specifically on critical issues related to Latinos/as in mathematics. Prior to CEMELA, researchers interested in such a focus worked mostly in isolation. In considering issues related to Latinos/as in US schools, issues of language and culture were central, which often had direct implications for English Learners (ELs) more broadly. While not all Latinos/as are English Learners, and not all ELs are Latinos/as, these two groups have significant overlap. For example, about 80% of ELs speak Spanish as a first language, and Spanish-speaking ELs appear to struggle on measures of academic achievement (Goldenberg, 2008).

CEMELA expanded the field's knowledge of ELs in mathematics through conducting interdisciplinary studies that helped researchers and practitioners better understand the reality of Latinas/os and mathematics teaching and learning. CEMELA's research focused on teacher education, research with parents, and research on student learning, resulting in over 50 publications and presentations. Again, several of these studies involved the investigation of questions related to the interplay of language, culture, and mathematics education. CEMELA also had the goal of connecting the network of scholars focused on these issues, as a means to build capacity and sustain the work.

Following the conclusion of CEMELA's funding, Zandra de Araujo, Sarah Roberts, Craig Willey, and William Zahner continued to meet regularly. These meetings focused on examining intersections among these early career scholars' work related to the mathematics education of ELs. To date these meetings have resulted in a number of national presentations at the annual meetings of the National Council of Teachers of Mathematics, the American Educational Research Association, and PME-NA. Currently, this group is working on several manuscripts and follow-up studies related to the preparation of teachers to work with ELs. Perhaps most notable is an extensive review of the international literature focused on mathematics and ELs.

The Mathematics Education and ELs Working Group met at PME-NA in 2015 (de Araujo et al., 2015) and 2016 (de Araujo et al., 2016). At those meetings, we brought together a diverse group of about 30 researchers who started working together on several projects related to the mathematics education of ELs (see descriptions of these projects in the section below titled *Previous Work of the Group*). Our aim for the 2018 Working Group is to provide a space for these scholars to continue their work and bring new scholars into the fold.

Previous Work of the Group

The 2015 and 2016 Working Groups (de Araujo et al., 2015, 2016) began with whole group discussions aimed at examining five aspects related to the mathematics education of ELs including: a) Student Learning; b) Family and Community Resources; c) Language Perspectives; d) Teacher Education; and e) Curriculum. In the following sections we briefly summarize our discussions and subsequent work in each of these areas.

Student Learning

Building upon situated and sociocultural perspectives (Moschkovich, 2002), the student learning group started from the premise that ELs, like all students, learn mathematics through a process of appropriating discourse practices, tool use, and perspectives of mathematics. In reviewing the literature on the mathematics learning of ELs, we have identified a need to understand better how research in mathematics education at large is connected with research on the mathematics learning of ELs. Much of the content-focused work in mathematics education is isolated from research on how ELs develop specific mathematical understandings. Previous discussions at this working group have supported ongoing research and development efforts focused on bridging the literature on supporting ELs in mathematics classrooms and the literature on students' mathematics learning through focused instruction.

Family and Community Resources

Families and communities can serve as resources for ELs in their mathematics learning in myriad ways. Families can advocate for their children and provide and support learning experiences both in and out of the classroom. Communities can also provide a wealth of support mechanisms and learning possibilities. Moll et al. (1992) described how students studied candy making and selling within their neighborhood to explore mathematics within this context, such as discussing and analyzing production and consumption. In doing so, the teachers and students acknowledged the value of these community experiences. Additionally, Civil and Bernier (2006) explored the challenges and possibilities of involving parents in facilitating workshops for other parents around key math topics. These studies and others like them illustrate the promise of family and community resources in fostering ELs' mathematics learning. At our 2015 meeting, Civil shared her recent work on how school language policies impact ELs' engagement and how teacher educators can draw on family and cultural resources in support of ELs. We followed up on this work during our 2016 with small group discussions.

Language Perspectives

Teachers' and researchers' conceptions of language, second language acquisition, and bilingualism affect teaching and learning mathematics for ELs. In 2015, Judit Moschkovich shared her work that highlighted how perspectives of language, second language acquisition, and bilingualism appear in both theory and practice. We also discussed, in particular, how work focused on ELs can draw on current work on language and communication in mathematics classrooms, classroom discourse, and linguistics. Looking for these intersections and connections was crucial because it ensures that work in mathematics education is both theoretically and empirically grounded in relevant research, and it will prevent researchers from reinventing wheels.

Teacher Education

Much of the prior work on teacher education related to ELs has focused on more general strategies (e.g., sheltered instruction, as in Echevarria, Vogt, & Short, 2007), such as using visuals, modifying texts or assignments, and using slower speech. We argue there is a need for content-specific support for mathematics teachers of ELs. At our previous meetings, we explored ways to support teachers, both pre-service and in-service, in better understanding the strengths, and meeting the needs, of ELs in the mathematics classroom.

During the 2015 and 2016 Working Groups, the teacher education subgroup focused on the primary issues that arise in the preparation of teachers to teach ELs at the various institutions. As a group we recognized that there were few attempts to include the teaching and learning of mathematics to ELs beyond the states where there was a high population of ELs. Given that some of the group members were meeting for the first time, a significant portion of the allotted time was spent sharing the details of the research we did and our interest in being part of this particular subgroup. One of the members shared a survey about examining preservice teachers' conceptions about teaching mathematics to ELs and the other members agreed to administer the survey at their locations. Together, the responses will provide us with some insight about possible conceptions that need change and the steps we can take to make that happen. The group has stayed in touch online and continue the discussions about potential collaborations.

Curriculum

In 2015, the curriculum subgroup focused on the role of textbooks, specifically teachers' guides and student work pages, in demonstrating how one might approach supporting ELs in building mathematical understanding and developing mathematics language. We inquired about the process publishers undergo to incorporate and offer support to teachers. What assumptions do they make? Who do they consult? What motivates them to invest in serving ELs better? What is/are their end goal(s)? The group decided to conduct an analysis of various middle grades curriculum to ascertain what supports and guidance are offered to teachers. It was suggested that we might build on the work of Pitvorec, Willey, and Khisty (2010), who explored the features of Finding Out/Descubrimiento (FO/D) that proved to be successful with bilingual children of migrant families in the 1980's and partially contributed to the development of complex instruction (Cohen, Lotan, Scarloss, & Arellano, 1999).

At the 2016 meeting, this group continued to examine textbooks to understand better the supports they provide for ELs. Participants considered language issues in mathematics texts for ELs, especially as related to word problems and assessment items. We shared a short literature review of relevant research on linguistic complexity and vocabulary for mathematics word problems. Based on that research, we summarized recommendations for addressing language complexity and vocabulary in designing word problems for instruction, curriculum, or assessment. We then used examples of released sample Smarter Balanced Assessment Consortium items to illustrate how to apply those recommendations to designing word problems and to designing supports for ELs to work with word problems. Several of the participants have continued this work examining curriculum accommodations for ELs and are completing a research study based on the work started at the 2016 Working Group.

Aims for the 2018 Working Group

Previous meetings of this Working Group brought together a large and diverse group of attendees. For example, in 2016 there were approximately 30 attendees including teachers, preservice teachers, researchers, graduate students, and teacher educators from a range of institutions. In light of the variety of contexts from which the attendees came and the attendees' various career stages, we have worked to reconceptualize the Working Group in order to offer substantial benefits to all in attendance. The result is a highly interactive, three-day series of meetings focused on bridging past research findings and looking forward to new avenues for research and practice guided by the following objectives: *(1) Sharing current perspectives on extant research related to mathematics and English learners, (2) Understanding venues for* *dissemination of research on ELs, including novel outlets that connect research to practice, and (3) Fostering new collaborations and supporting further connections among researchers.* In the following section we provide an overview of each of the sessions in relation to these aims.

Plan for the 2018 Working Group Sessions

During the three sessions in 2018, participants will continue the work of the subgroups and bring new members up to date on the group's prior activities. In participating in the three sessions, participants will work to: a) clarify research questions, b) refine research tools, methods, and analyses, c) explore connections among different projects and studies, and d) discuss further collaborations and research on learning and teaching mathematics in classroom with English learners.

Session 1: Introductions and Brief Reports

The goals for the initial session are to allow participants to meet fellow attendees and to share current perspectives on extant research related to mathematics and English learners. The organizers will present brief reports on the work that resulted from the 2015 and 2016 Working Groups. Each of these brief presentations (5-10 minutes) will focus on the conference theme of "Looking Back, Looking Ahead." We will preview the schedule for this year's Working Group sessions.

Activities

- Introduction and overview of the Working Group including introduction to the Google Community.
- Brief presentations by panel members from each of the subgroups providing overviews of research projects with specific examples of how researchers have designed the studies. The purpose is to provide an overview and update of scholarly activities that were initiated at the 2015 and 2016 Working Group.
- Following the presentations, participants will discuss specific questions posed by the subgroups and aims for the subgroups' work at the 2018 meeting.

Guiding Questions

1. What research has been done in relation to the Working Group's prior meetings and avenues exist for future work?
2. Which aspects of studies focusing on English learners do you find most puzzling? Most useful? Most misunderstood?
3. What are the goals for the 2018 Working Group?

Session 2: Interactive Data Analysis

During Session 2 we will engage in a joint exercise focused on data analysis, using data from Zahner's ongoing research in linguistically diverse high schools to motivate a discussion of research methods, tools, and questions.

Activities

- Introduce a focused set of data for discussion (videos, transcript, and written work from two 9th grade ELs' problem-based interviews).
- Consider the interview tasks and discuss potential foci.

- Break into small groups. Immerse selves in data for 45 minutes, discuss guiding questions using the data as touchstone for discussion.
- Share results of small group discussions with larger group.
- Facilitated discussion focused on the following set of questions.

Guiding Questions

1. What theories and theoretical frameworks have informed the design of your research project(s)?
2. How do sociocultural theories of learning mathematics and language allow for a non-deficit view of ELs' mathematics reasoning?
3. How might your work inform theory building in studies of mathematics learning and teaching for ELs? How can work on this student population expand our theoretical lenses?
4. What issues and challenges have you faced in designing studies and analyzing data?

Session 3: Breakout Groups and Next Steps

Given the diverse backgrounds and experiences of attendees of our prior working groups, our final session will allow attendees to engage in small group discussions aligned with their particular interests. The organizers will lead subgroups as described below.

- **Subgroup 1: Diving deeper into the data**. This subgroup will focus on continuing the work from the prior session. Zahner will continue to guide the participants in further analysis and open space for further discussion of the data and avenues for future research.
- **Subgroup 2: Diving into the translation of research for non-researchers**. This subgroup will focus on ways in which to researchers can communicate their findings with the general public. Willey will share his experiences writing policy briefs and open access papers for practitioners, policy makers, and community members.
- **Subgroup 3: Diving into teaching.** In this subgroup, de Araujo will lead discussion on how to take what we have learned from research into the preservice mathematics methods classroom. The group's discussions will focus on studying our teaching practice in relation to ELs and ways of sharing lessons and activities with other teacher educators.
- **Subgroup 4: Diving into dissemination.** Roberts will lead this small group in a discussion of publication venues that are open to scholarship related to the mathematics education of ELs. She will frame the discussion in what was learned from the organizers' review of literature on the mathematics education of ELs. She will also discuss funding opportunities for research related to this work and questions raised by our literature review.

We will close Session 3 with time to review group progress and discuss next steps for our work. Meeting notes, work, and documents will continue to be shared and distributed via our Google Community (https://plus.google.com/communities/104376842129206334879) and corresponding Google Drive folder (https://goo.gl/EXhUVm). The use of Google Community allows members to create an institutional memory of activities during the Working Group that we have continued to use and add to following the 2015 and 2016 Working Groups. In addition to the Google Community and Drive resources, anticipated follow-up activities include planning for a continuation of the Working Group at PME-NA in 2019, an informal group meeting at

TODOS' 2020Conference, and organizing one or more collaborative writing projects on this topic.

Activities

- We will engage in a brief welcome activity focused on the prior session's data analysis activity and to recap session 2 and to ensure attendees have a chance to meet those in attendance.
- The organizers will briefly introduce the four subgroups and invite attendees to select a group to join for continued collaboration.
- The subgroups will work to discuss their given focus and possible directions for continued collaboration.
- The organizers will bring the groups back together to debrief the subgroup discussions.
- We will end with a discussion of next steps for continued collaboration, particularly via the Google Community

Guiding Questions

1. How might we take what we learned from the data analysis activity and apply it to your work researching and teaching the mathematics education of English learners?
2. How will we disseminate our work?
3. What are the next steps in continuing this work?

Follow-up Activities

We anticipate that this Working Group will again attract other researchers interested in issues related to the mathematics education of ELs. Therefore, an important component of this third meeting of the Working Group will be to maintain current relationships while also continuing to establish connections with other interested researchers in order to build opportunities for future collaborations. We will provide space for new researchers to contribute to our collective work, to suggest new directions, and to add to the growing body of research on mathematics and ELs. At the first session of our Working Group, we will share our ongoing online Google Community, which uses Google applications (Hangout, Groups, Drive, etc.). Google's applications are freely available and allow for a number of collaborative opportunities, including video conferencing, group messaging, collaborative document development, and shared web and social media space. Through this collaborative Google Community, we have organized follow up meetings both virtually and at conferences such as TODOS and the NCTM Research Conference. These meetings, both face-to-face and virtual, allow us to set concrete goals in preparation for the creation of a Working Group proposal for PME-NA 2018 to continue our work.

In addition to these short-term goals, we have several longer-term goals for this Working Group, depending on which groups individuals join during their time on days 2 and 3. First, for those who *dive into data* will consider further collaborations will extend with a Google group where individuals can connect with one another as they consider their own data dives around mathematics and ELs and where to go next with this work. Second, those who *dive into translating research* will work together beyond the conference to consider white papers and publications with broader audiences, particularly teachers. This will also be an opportunity to engage in efforts to transform the way mathematics curriculum is developed and enacted with respect to enhancing access and engagement of ELs. Third, those who *dive into teaching* will consider collaborations around enhancing PSTs' work with ELs and the development of actual lessons and activities beyond the conference for doing this work. Finally, those who *dive into research* will consider friendly publication venues, set-up writing groups, and identify future research partnerships.

References

Ballantyne, K. G., Sanderman, A. R., & Levy, J. (2008). *Educating English language learners: Building teacher capacity*. Washington, DC. Retrieved from http://www.ncela.gwu.edu/practice/mainstream_teachers.htm

Barton, B. (1996). Making sense of ethnomathematics: Ethnomathematics is making sense. *Educational Studies in Mathematics*, *31*, 201–233.

Bunch, G. C. (2013). Pedagogical Language Knowledge: Preparing Mainstream Teachers for English Learners in the New Standards Era. *Review of Research in Education*, *37*(March 2013), 298–341. http://doi.org/10.3102/0091732X12461772

Civil, M., & Bernier, E. (2006). Exploring images of parental participation in mathematics education: Challenges and possibilities. *Mathematical Thinking and Learning*, *8*, 309–330.

Cohen, E. G., Lotan, R. A., Scarloss, B. A., & Arellano, A. R. (1999). Complex instruction: Equity in cooperative learning classrooms. *Theory into Practicey*, *38*(2), 80–86.

D'Ambrosio, U. (2006). *Ethnomathematics: Link between traditions and modernity*. Rotterdam, the Netherlands: Sense Publishing.

de Araujo, Z. (2017). Connections between secondary mathematics teachers' beliefs and their selection of tasks for English language learners. *Curriculum Inquiry*. http://doi.org/10.1080/03626784.2017.1368351

de Araujo, Z., Civil, M., Fernandes, A., Moschkovich, J. N., Roberts, S. A., Willey, C., & Zahner, W. (2016). Mathematics and English learners. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting for the North American Chapter for the Psychology of Mathematics Education* (pp. 1673–1681). Tucson, AZ.

de Araujo, Z., Roberts, S. A., Anhalt, C. ., Civil, M., Fernandes, A., Moschkovich, J. N., & Willey, C. (2015). Mathematics education and English learners. In M. V. Martinez & A. C. Superfine (Eds.), *Proceedings of the 37th annual meeting for the North American Chapter for the Psychology of Mathematics Education* (pp. 1384–1393). Chicago, IL.

de Araujo, Z., Roberts, S. A., Willey, C., & Zahner, W. (under review). Mathematics education of English learners.

Echevarria, J., Vogt, M. E., & Short, D. (2007). *Making content comprehensible for English learners: The SIOP model* (3rd ed.). Boston, MA: Pearson Allyn & Bacon.

Fernandes, A. (2012). Mathematics preservice teachers learning about English language learners through task-based interviews and noticing. *Mathematics Teacher Educator*, *1*(1), 10–22. http://doi.org/10.5951/mathteaceduc.1.1.0010

Freeman, B., & Crawford, L. (2008). Creating a middle school mathematics curriculum for English-language learners. *Remedial and Special Education*, *29*(1), 9–19. http://doi.org/10.1177/0741932507309717

Goldenberg, C. (2008). Teaching English language learners: What the research does and does not say. *American Educator*, *32*, 8–44.

Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education*, *44*(1), 37–68. http://doi.org/10.5951/jresematheduc.44.1.0037

Kaufman, J. H., & Stein, M. K. (2010). Teacher learning opportunities in a shifting policy environment for instruction. *Educational Policy*, *24*(4), 563–601. http://doi.org/10.1177/0895904809335106

Khisty, L. L., & Radosavljevic, A. (2010). A descriptive analysis of Math Pathways and Pitfalls in a Latina/s bilingual classroom.

Kloosterman, P., & Walcott, C. (2010). What we teach is what students learn: Evidence from national assessment. In B. Reys & R. E. Reys (Eds.), *Mathematics curriculum: Issues, trends, and future directions* (pp. 89–102). Reston, VA: National Council of Teachers of Mathematics.

Moll, L. C., Amanti, C., Neff, D., & Gonzalez, N. (1992). Funds of knowledge for teaching: Using a qualitative approach to connect homes and classrooms. *Theory into Practice*, *31*(2), 132–141. http://doi.org/10.1080/00405849209543534

Moschkovich, J. N. (2002). A situated and sociocultural perspective on bilingual mathematics learners. *Mathematical Thinking and Learning*, *4*(2–3), 189–212. http://doi.org/10.1207/S15327833MTL04023_5

National Center for Educational Statistics (NCES). (2017). English language learners in public schools. Retrieved from https://nces.ed.gov/programs/coe/pdf/coe_cgf.pdf

National Clearinghouse for English Language Acquisition. (2008). *How has the limited English proficient student population changed in recent years?* Washington, DC.

National Governors Association Center for Best Practices, Council of Chief State School Officers, & Officers, N. G. A. C. for B. P. & C. of C. S. S. (2010). *Common core standards for mathematics*. Washington, DC: Author.

Pitvorec, K., Willey, C., & Khisty, L. L. (2010). Toward a framework of principles for ensuring effective mathematics instruction for bilingual learners through curricula. In B. Atweh, M. Graven, W. Secada, & P. Valero (Eds.), *Mapping equity and quality in mathematics education* (pp. 407–422). Netherlands: Springer.

Téllez, K., Moschkovich, J. N., & Civil, M. (2011). *Latinos/as and mathematics education: Research on learning and teaching in classrooms and communities*. Charlotte, NC: Information Age Publishing.

Turner, E. E., Drake, C., McDuffie, A. R., Aguirre, J., Bartell, T. G., & Foote, M. Q. (2012). Promoting equity in mathematics teacher preparation: A framework for advancing teacher learning of children's multiple mathematics knowledge bases. *Journal of Mathematics Teacher Education*, *15*(1), 67–82. http://doi.org/10.1007/s10857-011-9196-6

Verplaetse, L. S., & Migliacci, N. (Eds.). (2008). *Inclusive pedagogy for English language learners*. New York, NY: Lawrence Erlbaum Associates.

Willey, C. (2013). *A case study of two teachers attempting to create active mathematics discourse communities with Latinos [Unpublished doctoral dissertation]*. University of Illinois at Chicago, Chicago, IL.

MATHEMATICAL MAKING IN IMMERSIVE VIRTUAL ENVIRONMENTS WORKING GROUP

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We propose a working group that will bring mathematics education researchers together to explore how immersive spatial display technologies (e.g., room scale virtual reality) can be used for the teaching and learning of mathematics. The sessions will provide participants with the opportunity to explore a range of virtual worlds that are accessible via the HTC Vive, including mathematics-specifics environments that were designed by session organizers. The working group will provide a venue where participants can work together to plan research studies, discuss instructional activities, and design virtual environments for making and exploring mathematical figures. Our group aims to look ahead to a time in the near future–when immersive spatial displays are as commonplace as mobile phones–and anticipate the research, design, and development that can guide how these technologies will be integrated into schools.

Keywords: Virtual reality, augmented reality, dynamic mathematics, spatial thinking

Motivation

The commercial release of the HTC Vive (April, 2016) marked the beginning of a new era for representing and interacting with information. The Vive is an example of a technology that can host immersive, room-scale virtual reality experiences. The experiences are immersive because one's visual and aural senses are saturated by the sights and sounds of a virtual world. Room scale virtual reality experiences are distinct from stationary experiences, such as those that can be supported by phone-based viewers (e.g., Google cardboard, Gear VR). In room scale virtual reality, one's physical location within a physical room is used to determine one's position in a virtual world. This location tracking allows users to have direct control over where they are in a scene, and local movement in a virtual world is linked to one's movements in the physical world. Room scale virtual reality also affords users more options for interacting with virtual environments than VR viewers, such as the ability to create virtual objects.

Room scale virtual reality is an example of an *immersive spatial display*—a technology that enables *space itself* to be the canvas for making inscriptions (Dimmel & Bock, 2018). We use the term *immersive spatial display* to bring a range of related technologies under the same heading, including augmented reality (e.g., the Hololens), holographic projection, and volumetric video. The capacity for making three-dimensional inscriptions in virtual space will open up new modes for making and exploring mathematical figures. An example of a mathematical activity that is possible with spatial inscriptions that would be infeasible without them is plane-and-sphere constructions (i.e., the three-dimensional analogs of compass and straightedge constructions; Dimmel & Bock, 2018).

The mathematical questions that pertain to generalizations of geometric constructions have been answered (Franklin, 1919). But before the advent of immersive spatial displays, it would have been difficult to engage in the activity of constructing spatial figures from the intersections of planes and spheres, such as the tetrahedron that is shown in Figure 1. The construction is the spatial analog of the compass-and-straightedge construction, from two congruent, intersecting circles, of an equilateral triangle: the three centers of the spheres determine the vertices of the equilateral triangle that forms the base of the tetrahedron, and a point of concurrency for the three spheres determines its apex and other equilateral triangular faces. The image shown in Figure 1 was captured from *HandWaver* (Dimmel & Bock, 2017a), a gesture-based virtual environment for making and exploring mathematical figures. Plane-and-sphere constructions are just one example of new types of mathematical activities that will be possible with immersive spatial displays.

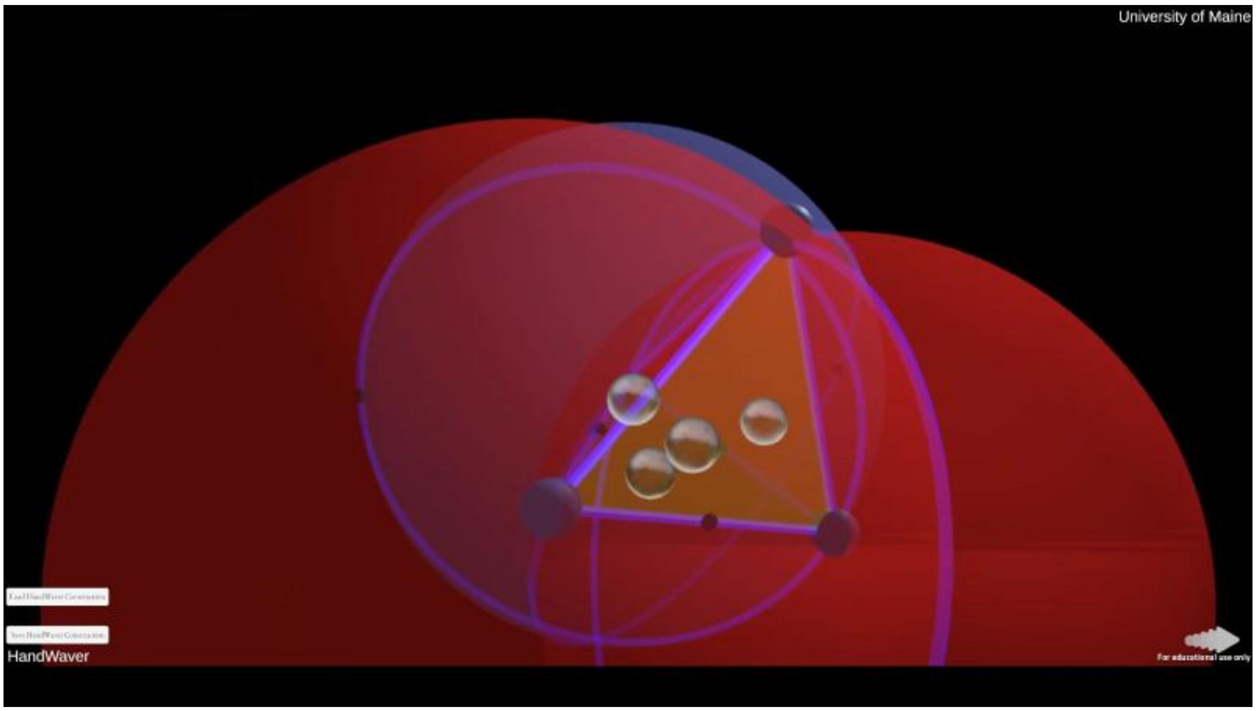

Figure 1. A tetrahedron constructed from three spheres.

That inscriptions have been confined to two-dimensional surfaces has constrained how mathematical concepts are canonically represented and influenced what subjects are taught, especially in the case of geometry (Franklin, 1919; Herbst et al., 2017). The fine-motor skill coordination that is necessary to create two-dimensional inscriptions has also limited who has the ability to inscribe and engage in mathematical representations. Spatial display technologies will make it possible to view and interact with dynamic, virtual, three-dimensional figures and allow learners to share a virtual space with the mathematical objects they are investigating. They have the potential to introduce new subjects of study into the school mathematics curriculum, such as inquiries into the mathematical properties of polyhedrons that go further than calculating the Euler characteristic of the platonic solids or explorations of higher-dimensional polytopes more generally.

Because spatial displays make available an additional dimension for representing mathematical figures, students could explore four-dimensional objects by investigating their three-dimensional projections. Limited versions of such experiences are possible with two-dimensional dynamic diagrams or physical models, but spatial displays make it possible to have the best of both worlds: They combine the graspability of physical things with the variability of dynamic diagrams.

Immersive spatial displays also present the possibility of allowing rich mathematical participation to students who are limited by geography, disease (hospital/homebound), or any other socio-economic or physiological deterrents. Two-dimensional surfaces have prevented some vulnerable populations from meaningful engagement in mathematics, and in some cases (e.g., children with cerebral palsy), the ability to create mathematical inscriptions all together (Nickels, Barrett, & Fralish, 2018). Immersive spatial displays allow users to circumvent or overcome their physical disability and create inscriptions that reveal, for the first time, their mathematical thinking and reasoning. The working group we propose is motivated by a broad question: How can the new modes of representing and interacting with information that are possible in spatial displays transform mathematics education?

Relevance to PME-NA Conference theme: Looking back, looking ahead

Mathematics has always been linked to display technology, from the sand flats used in the ancient world (Davis, 1972), to the chalkboards that are ubiquitous in mathematics departments (Greiffenhagen, 2014), to the digital displays that emerged in the latter part of the 20th century. And historically, new representations of mathematical figures have lead to the generation of new knowledge, such as when 19th century mineralogical collections that featured crystals-within-crystals helped Lhuilier and Hessel broaden the definition of three-dimensional figures that could be considered polyhedral (Lakatos, 1976, p. 15). Already, mathematicians have used the visual immersion possible with virtual reality viewers to model and gain new insights into non-Euclidean geometry (Hart et al., 2017a, 2017b). Whereas mathematicians will be the principals who use immersive spatial displays for generating mathematical knowledge, the mathematics education research community has an essential role to play in describing and researching the opportunities for mathematics education that are possible with these new technologies.

The working group we propose will bring together mathematics education researchers to explore what is currently possible with immersive spatial displays, collaborate to plan research studies and imagine mathematics-specific virtual environments, and consider the opportunities and challenges of using immersive spatial displays in mathematics classrooms. With current technology, it is still impracticable for every student in a class to have access to his or her own immersive spatial display (i.e., they are not yet a one-to-one technology); but as the state of the art for immersive spatial displays improves, the technologies will become smaller, less expensive, and will soon be as ubiquitous as mobile phones. When immersive spatial displays are commonplace, how will they be used for mathematics education? We are thus looking ahead to a time in the near future and attempting to anticipate the research, design, and development that can guide how immersive spatial display technologies will be integrated into schools.

Focal Issues

Research has suggested effective uses of technology in the teaching and learning of mathematics. We should rely on these findings as we consider the affordances and limitations of using immersive spatial displays. We should not use a new technology simply because it is available, rather we should be asking how we can best use this technology and if the benefits outweigh the costs. We note several key suggestions for effective technology use which can guide our initial attempts to integrate immersive spatial displays into mathematics education. We note researchers suggest using technology to engage students in exploring and conjecturing (e.g., Ball & Stacey, 2005, Cuoco & Goldenberg, 1996; Doerr & Zangor, 2002; Laborde, 2007; Wilson, 2008), to present and connect multiple representations (e.g., Cuoco & Goldenberg, 1996;

Doerr & Zangor, 2002; Laborde, 2007; Lee & Hollenbrands, 2008; Wilson, 2008), and to support students as they work with data or reason from cases (e.g., Cuoco & Goldenberg, 1996; Doerr & Zangor, 2002; Lee & Hollenbrands, 2008).

We should point out that we do not see these three uses of technology as mutually exclusive and in fact imagine ways in which there can and should be overlaps or interaction among the three. We do not suggest these three uses of technology as the only possible effective uses, but rather as examples and starting points as we set out to explore the potential role of immersive spatial displays in the teaching and learning of mathematics. This work will be guided by essential questions that pertain to student cognition, instructional design, teacher professional development, and assessment: How do immersive spatial displays affect student mathematical conceptions? What new mathematical learning experiences are possible with immersive spatial displays? How can immersive spatial displays be used as an instructional technology? What supports do mathematics teachers require to meaningfully incorporate immersive spatial displays into their classes? How can we design assessments that will measure what students learn by exploring immersive mathematical worlds?

An additional focal issue concerns the use of theory in research related to immersive spatial displays. Until this point in time, theories in mathematics education have acted as multiple lenses for examining our data. They have answered the question: *How does the theory make meaning for the data?* Immersive spatial displays, however, have the potential to significantly disrupt what is known about mathematical thinking, learning, and teaching. Inherently, they pose textual, lexical, and semantic questions as to what counts as inscriptions—moving beyond analysis of word-based text and 2-d graphical representations. Immersive spatial displays provide, at minimum, data that is not normally or easily accessible—e.g., real-time tracking on the user's head, body, and gestures, and the user's first-person point-of-view of their activities in the environment. Thus, the question, *How does the theory make meaning for the data?* is, on its own, insufficient. We must now ask, *What does an immersive spatial display do to the theory?* To address this focal issue, we will begin to consider: What substantive theories within mathematics education are appropriate to immersive spatial displays regarding students' schemes and reasoning? What effects do immersive spatial displays have on students' progression through hypothetical learning trajectories and other known hierarchical theories of mathematical learning?

Session Organization and Participant Engagement

Session goals

One goal of our proposed working group is to provide a setting where participants can explore the virtual worlds that are possible with immersive spatial displays. While spatial display technologies have been commercially available since April, 2016, they are still relatively novel. We believe there will be value to the PME-NA community if we provide a setting where mathematics education researchers can learn first hand about the kinds of experiences that are now possible in virtual worlds. Our second goal is to provide a venue where participants can work together to plan research studies, discuss instructional activities, and design virtual environments for making and exploring mathematical figures. Throughout each session, participants will be engaged in the working group by exploring virtual environments, working collaboratively to examine the potential of immersive spatial displays for mathematics education practice and research, and discussing how the potential of immersive spatial display technologies can be used by teachers in schools.

Each session will feature some time when participants can work in small groups to explore different virtual environments, followed by small group and whole group discussions. During the discussions, participants will describe what they did in the virtual environments, what they would have liked to do, and how they see spatial display technologies as relevant to mathematics education. The organization of each session will be open and flexible so that participants will have the opportunity to shape the work that we do during the three days of the conference. The session plans that we describe below are intended to give some idea of the different types of activities participants' would be engaged in throughout their participation in the working group. Session organizers will draw on their experience conducting focus group studies of secondary mathematics teachers' encounters with immersive virtual worlds to facilitate small group discussions.

Session 1: Introduction to Immersive Spatial Displays

The session will begin with a presentation (< 10 minutes) that describes the technological state of the art and considers the affordances for representing mathematical concepts that are possible with currently available immersive spatial display technologies. Following the presentation, participants will break out into four groups (6-8 participants per group) and take turns exploring virtual environments (<60 minutes). The environments that will be available to participants will include commercially available products (e.g., *CalcFlow*, *TiltBrush*, *Blocks*) as well as products that are under active development by session organizers (e.g., *HandWaver, LatticeLand*). Each group of participants will have access to a head-mounted display, and each session organizer will work with one group of participants. The goal for the first session will be to ensure that every participant that wants to explore the virtual environments has the opportunity to do so. Each of the session organizers has experience introducing new users to immersive virtual reality, and the session organizers will work with participants to make sure that they are comfortable. Once every participant has explored the virtual environments, we will convene as a large group and participants will discuss their experiences (remainder of session).

Session 2: Research and Design with Immersive Spatial Displays

For the second session, participants will start in their breakout groups. Each group will have access to a head mounted display that participants can use at their discretion to explore virtual environments. The first activity (<30 minutes) will be small group discussions at each table, each of which will be facilitated by one of the session organizers. The purpose of the opening discussion will be to brainstorm ideas about how the virtual environments that were surveyed during the first session could be used for mathematics education research. After the breakout discussions, participants will share their ideas with the whole group (<20 minutes). Session organizers will then present to the whole group about mathematics education research that has been conducted with immersive spatial displays. Nickels and Cullen will present about their work using *TiltBrush* to investigate children's conceptions of units of measure (<10 minutes). Dimmel and Bock will present their work designing and developing mathematics-specific environments for use in room scale virtual reality (<10 minutes). A discussion across both presentations will follow (<10 minutes). For the final activity of the second session, participants will break out in small groups again and work together to design virtual environments that could be supported by the immersive spatial display technologies (remainder of session).

Session 3: Immersive spatial displays as instructional technologies

The third session will focus on how immersive spatial display technologies could be used in mathematics classrooms. The session will begin with a short presentation (<10 minutes) during which session organizers (Dimmel and Bock) will describe their work designing and installing a virtual makerspace in a rural Maine school. Participants will then breakout into small groups and discuss how immersive spatial displays could be used by teachers in schools (<30 minutes). Participants could work together to design instructional activities, discuss strategies for how the technologies could be equipped in schools, and consider the pedagogical challenges of teaching with a technology that, for the near future, will remain impracticable to implement one-to-one. During the breakout session, each group will be able to use an HTC Vive as needed to test ideas or ground their discussion in the specifics of the virtual environments. The remaining time will be used for each group to share their ideas about using spatial displays as instructional technologies (<15 minutes) and a final, whole group discussion that reviews the work we did together across all three sessions (<35 minutes). The final group discussion will also provide an opportunity to plan for further collaboration among participants in the group.

Anticipated Follow Up Activities

The work that we begin at PME-NA 40 will be sustained by several anticipated follow up activities. Each working group session will feature ample time for participants to work together in breakout groups to explore and discuss existing virtual environments or to imagine virtual environments they would like to be developed. Following the conference, the connections participants make through working together during the breakout sessions will create opportunities for participants to plan collaborative projects that are linked to immersive spatial displays. For example, participants could work together to plan collaborative projects, design instructional activities that could be used as the basis for teaching experiments, or propose studies that could be conducted by a network of researchers at different universities.

The session organizers have experience equipping spaces with the hardware that is necessary to conduct research with immersive spatial displays, and we could advise interested participants about how to set up such equipment in their own research labs. For participants that are interested in designing and developing immersive virtual environments, we could point them to open-source projects and to resources for getting started using the Unity development engine. To facilitate these ongoing discussions, we will create an immersive spatial displays *slack* channel, use folders in Google drive to share resources (e.g., literature, proposals, manuscripts in progress), and, if there is sufficient interest from participants, we could create a mathematics in immersive spatial displays website that could serve as an information clearing house. We also anticipate that we would continue to convene the working group at future meetings of PME-NA.

References

Ball, L., & Stacey, K. (2005). Teaching strategies for developing judicious technology use. In W. J. Masalski & P. C. Elliott (Eds.), *Technology-supported mathematics learning environments* (pp. 3–16). Reston, VA: National Council of Teachers of Mathematics.

Cuoco, A. A., & Goldenberg, E. P. (1996). A role for technology in mathematics education. *The Journal of Education*, *178*(2), 15–32. doi:10.1177/002205749617800202

Davis, P. J. (1972). Fidelity in mathematical discourse: Is one and one really two? *American Mathematical Monthly*, *79*(3), 252–263. doi:10.2307/2316620

Dimmel, J.K., and Bock, C (2017a). HandWaver [computer software].

Dimmel, J.K., & Bock, C. (2017). HandWaver: A Gesture-Based Virtual Mathematical Making Environment. In G. Aldon & J. Trgalová (Eds.), Proceedings of the 13th international conference on technology in mathematics teaching (pp. 323 – 330). Lyon, France: École Normale Supérieure de Lyon.

Dimmel, J.K., & Bock, C.G. (2018). Dynamic mathematical figures with immersive spatial displays: The case of *HandWaver.* To appear in G. Aldon and J. Trgalova (eds.), *Selected papers of the 13th international conference on technology in mathematics teaching.* Berlin: Springer

Doerr, H., & Zangor, R. (2000). Creating meaning for and with the graphing calculator. *Educational Studies in Mathematics, 41*, 143–163. doi:10.1023/A:1003905929557

Franklin, P. (1919). Some Geometrical Relations of the Plane, Sphere, and Tetrahedron. *The American Mathematical Monthly*, *26*(4), 146-151.

Greiffenhagen, C. (2014). The materiality of mathematics: Presenting mathematics at the blackboard. The British Journal of Sociology, 65(3), 502e529. http://dx.doi.org/10.1111/1468-4446.12037.

Hart, V., Hawksley, A., Matsumoto, E. A., & Segerman, H. (2017a). Non-euclidean virtual reality I: explorations of H^3. *arXiv preprint arXiv:1702.04004*.

Hart, V., Hawksley, A., Matsumoto, E. A., & Segerman, H. (2017b). Non-euclidean virtual reality II: explorations of H^2 X E. *arXiv preprint arXiv:1702.04862*.

Herbst, P., Fujita, T., Halverscheid, S., & Weiss, M. (2017). The Learning and Teaching of Geometry in Secondary Schools: A Modeling Perspective. Taylor & Francis.

Lee, H., & Hollebrands, K. (2008). Preparing to teach mathematics with technology: An integrated approach to developing technological pedagogical content knowledge. *Contemporary Issues in Technology and Teacher Education*, *8*(4), 326–341.

Laborde, C. (2007) The role and uses of technologies in mathematics classrooms: Between challenge and modus vivendi. *Canadian Journal of Science, Mathematics and Technology Education*, *7*(1), 68–92. doi:10.1080/14926150709556721

Lakatos, I.: 1976, *Proofs and Refutations: The Logic of Mathematical Discovery*, J. Worrall and E. Zahar (eds.), Cambridge University Press, Cambridge.

Nickels, M., Barrett, J., & Fralish, B. (2018). Measurement concepts in virtual reality for children with cerebral palsy. Paper presented at the 2018 annual meeting of the American Educational Research Association. Retrieved [Retrieval Date], from the *AERA Online Paper Repository*. http://www.aera.net/Publications/Online-Paper-Repository/AERA-Online-Paper-Repository

Nickels, M. & Fralish, B. (2017). *Mathematical immersion through virtual reality for children with cancer*. Paper presented at the Research Conference of the National Council of Teachers of Mathematics, San Antonio, TX, April 4, 2017.

Wilson, P. (2008). Teacher education: A conduit to the classroom. In G. Blume & K. Heid (Eds.) *Research on Technology and the Teaching and Learning of Mathematics: Volume 2 Cases and Perspectives* (pp. 415–426). Charlotte, NC: Information Age Publishing.

DEVELOPING A RESEARCH AGENDA OF K–8 TEACHERS' IMPLEMENTATION OF DIGITAL CURRICULA

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The purpose of this working group is to understand how teachers are implementing digital curricula to teach elementary mathematics, to explore best teaching practices for using digital curricula, and to understand how students are learning from digital curricula. We propose a working group on digital curricula that focuses on the themes of: access, equity, and empowerment; data generation/use; mathematical content; motivation and mindset; and nurturing NCTM Mathematics Teaching Practices and Common Core State Standards for Mathematical Practices. The overarching goals are to:

1. *Understand and develop ways to support teaching and learning with digital curricula,*
2. *Create a community that will continue to grow and learn and initiate long-range projects to improve teaching and learning with digital curricula, and*
3. *Serve as a forum to spark conversation and establish a research agenda on teaching and learning mathematics with digital curricula.*

During our time at the conference, each theme group within the working group will work toward: (a) identification of the issues that surround the theme, (b) recommendations for a research agenda, and (c) initiation of work on potential data collection tools.

Keywords: Curriculum Analysis, Technology, Elementary School Education, Middle School Education

Across the nation, schools are moving toward one-to-one capability with technology in every classroom. As districts seek ways to make best use of newly purchased technology, school spending on digital curricula is rapidly increasing: from $1.8 billion in 2013 to $4.8 billion in 2014 (Cauthen, 2017). Pepin, Choppin, Ruthven, and Sinclair (2017) propose a definition for "digital curriculum materials/resources/programmes" (DCR):

> Curriculum materials that are programmatic in concern: specifically, organised systems of digital resources in electronic formats that articulate a scope and sequence of curricular content. It is the attention to sequencing—of grade- or age-level learning topics, or of content associated with a particular course of study (e.g., algebra)—so as to cover (all or part of) a curriculum specification, which differentiates DCR from other types of digital instructional tools or educational software programmes (p. 657).

Pepin et al. (2017) contends that this definition is "distinct from the broader literature on instructional technology, though clearly there is some overlap" (p. 647). For the purpose of this working group, we plan to adopt this definition under the umbrella of "digital curricula." We contend that a broader definition allows participants to think more openly about digital curricula.

In 2016, spending on digital curricula increased by 25% over 2015, so that spending on digital curricula "now exceeds all K12 spending by $3.5 Billion" (Kafitz, 2017, n.p.). The Bill and Melinda Gates Foundation study (2015) found that 60% of elementary teachers believe digital tools are effective in teaching mathematics. While some educators may view a one-to-one classroom—where each student has access to one technology device (e.g., laptop, tablet, smartphone)—as a means to increase academic outcomes (Sauers & McLeod, 2012), other teachers may view a one-to-one classroom as detrimental to students' creativity, self-concept, or socio-emotional skills (Gardner & Davis, 2013).

With the surge of digital curricula, it is critical that enactment of that curricula enables rich mathematics teaching and learning within the classroom. We believe that such rich mathematics teaching and learning is best understood through the lens of the NCTM (2014) Mathematics Teaching Practices and the Common Core State Standards for Mathematical Practice (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), and that these practices and standards (see Table 1) are seamless within the context of digital curricula.

Table 1: NCTM Mathematics Teaching Practices and CCSS for Mathematical Practices

NCTM Mathematics Teaching Practices	CCSS for Mathematical Practices
1. Establish mathematics goals to focus learning. 2. Implement tasks that promote reasoning and problem solving. 3. Use and connect mathematical representations. 4. Facilitate meaningful mathematical discourse. 5. Pose purposeful questions. 6. Build procedural fluency from conceptual understanding. 7. Support productive struggle in learning mathematics. 8. Elicit and use evidence of student thinking.	MP1: Make sense of problems and persevere in solving them. MP2: Reason abstractly and quantitatively. MP3: Construct viable arguments and critique the reasoning of others. MP4: Model with mathematics. MP5: Use appropriate tools strategically. MP6: Attend to precision. MP7: Look for and make use of structure. MP8: Look for and express regularity in repeated reasoning.

Given the emerging availability of digital curricula, we believe that more attention must be given to supporting teaching and learning with digital curricula and to creating a research community that will discuss and further establish a research agenda on teaching and learning with digital curricula. Therefore, we propose a working group on digital curricula that focuses on the themes of: access, equity, and empowerment; data generation/use; mathematical content; motivation and mindset; and nurturing NCTM's Mathematics Teaching Practices and the CCSS Standards for Mathematical Practice.

Overview of Literature Related to Teachers' Instructional Choices

Research has reported that the teacher's role in implementing new curriculum is consequential for student outcomes (Remillard, 1999; Sherin & Drake, 2009; Son & Kim, 2015, 2016). Some teachers embrace new curricula to meet the goals of reform, while other teachers consciously or unconsciously make changes and bypass the goals of reform (Sherin & Drake, 2009; Smith, 2000). Thus, the decisions teachers make about curriculum can enhance or hinder students' understanding of mathematics (Nguyen & Kulm, 2005).

Teachers develop curriculum strategy frameworks to implement new curriculum in the classroom. Implemented curricula can take three forms: written, intended, and enacted (Stein, Remillard, & Smith, 2007). The *written curriculum* is sterile in itself. Therefore, teachers receive guidance or "strategy frameworks" from professional development about the *intended curriculum*. Curriculum authors also foreground and highlight particular approaches and mathematical ideas. Curriculum materials mediate teachers' decision making while, at the same time, a teacher's orientation toward such materials reflects that teacher's characteristics, prior experiences, perception of the intended goals, and personal goals that influence instructional decision making and ultimately, the *enacted curriculum*, or what actually takes place. District- and school-level support, professional development, and communicated perspectives can also influence teachers' instructional choices and effectiveness in implementing the curriculum.

Gender can also be a factor in digital curricula implementation, although research is mixed. Some research indicates that female teachers tend to be less confident with technology and integrate it into their instruction less (Markauskaite, 2006; Wozney, Venkatesh, & Abrami, 2006), while other studies indicate the gap has reduced (Yukselturk & Bulut, 2009). Duration of teaching experience has also been found to influence use of technology in classrooms (Hernandez-Ramos, 2005).

The SAMR model can be used to classify teachers' instructional choices in leveraging digital technologies (Puentedura, 2013). In the *Substitution* level, digital technology is simply substituted for analog technology. For example, the elementary teacher substitutes paper worksheets for digitized PDFs, but everything else remains the same. *Augmentation* uses technology in place of a previous task, and the use of technology improves the task. In a third-grade classroom, for instance, instead of a teacher-led, whole-class lesson in which the teacher explains a mathematical task, students use tablets to read or listen to an explanation, while English language learners listen to the explanation in their first language. At the *Modification* level, technology integration requires significant redesign of a task. For example, instead of using worksheets with the same problems for every student, students complete a set of problems using a digital curriculum that adapts to where students are in their thinking. Finally, the *Redefinition* level uses technology to create novel tasks. For example, instead of teaching students a skill and having them practice it and then apply it to solve word problems, the teacher poses a problem, each student decides what skills they need to solve that problem, and then they individually access web-based instruction on those skills. The technology enables students to drive the learning process in a way that was not possible before; thus, students' activity and engagement with mathematics is redefined.

Overview of Literature Related to Digital Curricula

The digital curriculum program iXL found through self-funded research that although use of their program in California ranged from 1 minute to 100 minutes per week in the classroom, there was a strong positive correlation between iXL usage and state test scores (Empirical Education, 2013). A randomized control study of the Odyssey program with fourth-grade students found that use of the program for 60 minutes per week did not significantly increase students' achievement (Wijekumar, Hitchcock, Turner, Lei, & Peck, 2009). Salman Khan, founder and executive director of Khan Academy (KA), recommended using KA to personalize instruction, freeing up class time for engaging, high-yield activities like student discourse and meaningful collaborative projects (Khan, 2012). Khan added, "ironically, the technology makes the classrooms more human for the teachers and students. It has also made the teachers that much more valuable" (Weltner, 2012, n.p.). Contrary to Khan's assertion however, a small study by Cargile and Harkness (2015) found that KA was not used to foster more active learning in the classroom; nor was instruction customized to students' progress and achievement levels.

Other researchers contend that digital curricula have the potential to transform teaching and learning and allow for frequent updating, use of multimedia, increased interactivity, connection to virtual communities, varied socialization, reduced costs in production and distribution, customized instruction, ongoing assessment, and reporting of student progress (Abell, 2006; Fletcher, Scaffhauser, & Levin, 2012; Mayer, 2003; Selwyn, 2007; Zhao, Zhang, & Lai, 2010). Digital curricula designs can cover a wide spectrum; however, Confrey (2016) identifies specific design decisions and design principles that should be the foundation for all digital curricula. When addressed, these principles (e.g., "*A workspace for deploying and using tools separated from lesson flow allows students to become independent and proficient tool users*." (p. 15)) lead to powerful resources that enable the NCTM Mathematics Teaching Practices and the CCSS Standards for Mathematical Practice.

Choppin, Carson, Borys, Cerosaletti, and Gillis (2014) analyzed how six digital curriculum programs leverage student learning experiences; support differentiation/individualization; and support socialization or communication through discussion boards, Google Docs, or other virtual spaces. Through this analysis, they identified two distinct curriculum types. Individualized learning programs (e.g., *ALEKS*, *Dreambox*) were programs "designed for use by individual students . . . with minimal teacher intervention with respect to the design of lessons or assessment" (p. 15). The digitized traditional textbook (i.e., *ConnectED*) included multimedia files, virtual tools, management and assessment systems, and communication features. Choppin (2016) expanded the number of programs to eight and extended to four types of programs: a) individualized learning programs; b) collection of digital lessons (e.g., *LearnZillion*); c) digitized versions of traditional textbooks; and d) a microworld environment (i.e., *Algebra in Action*).

Yerushalmy (2016) provides a more expansive idea of e-textbooks and includes three e-textbook models: interactive, integrative, and evolving. In identifying three models of e-textbooks, he warns that the models "are not distinct but are presented this way for clarity of the discussion, as each marks a key rationale or aspect of functionality" (p. 88). The authors of these e-textbooks can be teachers or other experts, and each of the three types of e-textbooks "can be either used as is or changed by its users (teachers or students)" (p. 88).

Overview of Literature Related to Teachers' Use of Assessment Data

Effective assessment data analysis is one tool to inform instruction, yet schools often lack the capacity to execute what research suggests (Marsh et al., 2005; Petrides & Nodine, 2005). To improve this issue, some school districts have allocated resources toward management information systems and professional development (see, e.g., Borja, 2006). However, research suggests that teachers are not using the resultant data to inform their classroom planning or instruction (Earl & Katz, 2006).

Teachers can use data to determine if students are responding to the digital instructional program, to identify students who are not making adequate progress, and to help in designing effective intervention programs for students who are not profiting from typical instruction (Fuchs & Stecker, 2003). Use of progress monitoring for individualized instruction has shown increases in learning achievement (Stecker, Fuchs, & Fuchs, 2005), yet less is known about progress-monitoring practices in mathematics, as compared to reading (Foegen, Jiban, & Deno, 2007). Research has focused on curriculum-based measurement (CBM) and computer adaptive tests (CATs). Specifically, CBM in mathematics for elementary and middle school consists only of computation and concepts/application measures (Fuchs, Hamlett, & Fuchs, 1999).

CATs are an emerging method for monitoring learning progress. One benefit of CATs is that the administration of items on a computer allows for subsequent items to be generated, based on the accuracy of a student's response. A program such as Dreambox can acquire about 50,000 data points per student per hour, tracking "a student's every click, correct answer, hesitation, and error" (Singer, 2017, n.p.). CATs can also provide a student or class profile of items learned or not learned. As Stecker, Fuchs, and Fuchs (2005) acknowledge, teachers need support in deciphering, and then effectively using progress monitoring results.

SRI Education (2014) conducted a two-year study of KA with 20 elementary, middle, and high schools in California, including more than 70 teachers and 2000 students in grades 5 through 10. Over half of the teachers (56%) reported reviewing student performance data at least once a week. With approximately 40% of teachers reporting that they reviewed student performance data less than once a month or not at all, questions remain about how teachers use real-time data to differentiate instruction and address specific individual, small-group, or whole-class learning needs.

Since formative assessment strategies such as communicating learning expectations, providing descriptive feedback, and adapting instruction based on feedback have positively influenced student learning (Hattie & Jaeger, 1998; Stiggins, 2005), it is imperative to investigate whether this holds true when teachers use digital curriculum assessments. Identifying the presence or absence of these assessment issues will help professional development focus on improved digital curriculum assessment strategies and inform teacher education programs on how to better prepare preservice teachers to use digital curriculum assessment.

Overview of Literature Related to Teachers' and Students' Disposition

The Association of Mathematics Teacher Educators' (2016) *Standards for Preparing Teachers of Mathematics* emphasizes that a positive disposition (i.e., attitudes, beliefs, mindset, motivation) toward mathematics is essential for successful mathematics teaching. Yeşilyurt, Ulaş, and Akan (2016) contend that "academic self-efficacy, teacher self-efficacy, and computer self-efficacy are important predictors of the attitude toward computer-assisted learning" (p. 592). A concern in regard to academic self-efficacy is that many elementary teachers bring their own mathematics anxiety with them into the classroom (Bursal & Paznokas, 2006; Haciomeroglu, 2014). Overall, about one-third of elementary teachers may feel anxiety about mathematics (McAnallen, 2010). On the other hand, teachers with confidence in their mathematics tend to better transform lower-cognitive-demand problems into higher-demand problems (Son & Kim, 2016). Furthermore, curricula, professional development, administrators, etc. can communicate conflicting perspectives about the nature of mathematics. For instance, mathematics can be seen as the development of isolated skills and knowledge, as a vehicle for interacting with one's world, or as the means of fostering citizenship and responsibility (Winter, 2001).

One way to positively influence disposition is to positively influence one's mindset. Dweck (2006) summarized decades of research showing that when students develop a "growth mindset," they believe the brain and intelligence can grow. On the other hand, those with a "fixed mindset" believe that you are born with a certain level of intelligence: either you are smart or you are not smart. Dweck suggests that the type of feedback students receive can influence students' mindsets. When feedback communicates that mistakes are learning opportunities and that learning takes time and effort, students are more likely to develop growth mindsets. Incentives have been shown to increase student motivation (Nguyen, Hsiech, & Allen, 2006), yet research also indicates that generic praise reduces persistence in children (Zentall & Morris, 2010). Digital curriculum includes different types of feedback and motivational strategies for students, such as badges and personalized avatars. When students receive personalized feedback on what they are and are not understanding, they can target their learning more efficiently.

Teachers' dispositions toward mathematics and toward using technology to teach mathematics will affect how they integrate digital curriculum into their instruction—which will, in turn, directly influence student outcomes. Similarly, students' dispositions toward mathematics and toward using technology to learn mathematics will affect their motivation toward mathematics, mindset about mathematics, and how they learn—or fail to learn—through digital curriculum. Learning about teachers' dispositions toward digital curriculum may help in understanding what influences their decisions to adopt, modify, or reject it. Furthermore, research on the types of feedback and motivational strategies digital curriculum provides students, and its effect on their dispositions toward mathematics, is needed.

Structure of the Working Group

To understand how teachers are implementing digital curricula to teach elementary mathematics, to explore best teaching practices for using digital curricula, and to understand how students are learning from digital curricula, the overarching goals of this working group are to:

1. Understand and develop ways to support teaching and learning with digital curricula,
2. Create a community that will continue to grow and learn and initiate long-range projects to improve teaching and learning with digital curricula, and
3. Serve as a forum to spark conversation and establish a research agenda on teaching and learning mathematics with digital curricula.

Theme Groups

The working group will achieve these goals by forming five theme groups:

- Access, Equity, and Empowerment: Digital curricula often enters schools with the promise of handling remedial issues in mathematics. The intention of this working group is to examine issues with the teaching and learning of mathematics involving students with learning disabilities, learning differences, or difficulties in mathematics; talented and gifted students; and concerns about disproportionate access to learning. This group will discuss who benefits from digital curricula use and if using digital curricula is influencing broader participation in STEM.
- Data Generation/Use: Participants in this theme group will discuss how to make sense of the abundance of data that can be generated by digital curricula to maximize student learning, the nature of that data, who should use such data and for what purposes, which data is most valuable for informing learning, and how teachers can be assisted in making data-driven instructional decisions to meet the needs of all learners.
- Mathematical Content: This theme group will discuss how digital curricula can support conceptual understanding, problem solving, procedural fluency, and strategic reasoning of mathematics. Participants will consider how to examine which features of digital curricula are most effective in aiding students' development of deep and flexible mathematics content knowledge.
- Motivation and Mindset: This theme group's focus will be issues and concerns with how digital curricula use influences students' motivation for learning mathematics and affects students' mathematical mindsets toward either "growth" or "fixed". This theme group will also consider how digital curricula use influences K–8 teachers' attitudes about mathematics and the teaching of mathematics.
- Nurturing the Standards for Mathematical Practices: The Mathematical Practices must be a central feature in the teaching and learning of mathematics. Participants in this theme group will discuss the impact of digital curricula on students' development of the Standards for Mathematical Practices through the Mathematics Teaching Practices. Discussion will also emphasize teachers' actions during non-digital curricula activities to enhance students' mathematical practices.

During our time at the conference, each theme group will work toward: a) identification of the issues that surround the theme, b) recommendations for a research agenda, and c) initiation of work on potential data collection tools.

Working Group Outline

The outline of each session is described in Table 2:

Table 2: Outline of Each Session

Session 1: *What do we know already?*		
	Activities	Guiding Questions
Whole Group (30 min)	Discuss proposed working definition of digital curricula. Discuss goals for the working group. Discuss proposed themes.	Examine digital curricula implementation through the five themes named above. Are these the right themes? Are there others?
Theme Groups (45 min)	Describe the nature and importance of the theme. Identify and categorize issues with digital curricula based on the group theme.	How is digital curricula being implemented in K–8 classrooms? What are we seeing in classrooms regarding the identified themes? What are components of the theme? Implications?
Whole Group (15 min)	Briefly report on Theme Group work. Identify next steps.	
Session 2: *What do we need to know and understand?*		
	Activities	Guiding Questions
Whole Group (30 min)	Set goals for the session.	What do we need to know and understand?
Theme Groups (45 min)	Begin writing a research agenda.	What are potential areas of research pertaining to this theme? What might that research look like? What type of data would be useful?
Whole Group (15 min)	Briefly report on Theme Group work. Identify next steps.	
Session 3: *How will we find out what we need to know and understand?*		
	Activities	Guiding Questions
Whole Group (30 min)	Set goals for the session.	How will we find out what we need to know and understand?
Theme Groups (45 min)	Establish a research plan. Initiate identification and design of data collection tools. Write group report.	What potential tools might be used to examine implementation of digital curricula through this theme? Write 5–10 questions you would want to see on a survey, 5–10 situations you would want to observe, and 3–5 interview questions for both teachers and students. What would be the structure and methods of the research?

		Write research plan and timeline.
Whole Group (15 min)	Report on Theme Group work. Identify next steps.	Groups share their action plan for further work. Who is willing to collect this data? Determine organization for the coming year/timeline.

Anticipated Follow-Up Activities

As a result of our working group discussion and research plan development, we anticipate several follow-up activities:

- Ongoing work with Google Docs for collaborative development of data collection tools for observations, surveys, and interviews;
- Online webinar in early January 2019 to finalize data collection tools and initiate data collection for the Spring 2019 semester;
- Online webinar in April 2019 to check in regarding data collection;
- Online webinar in July 2019 to initiate state data analysis; and
- PME-NA 2019 to reunite the working group and to bring together data across states. Analysis of the data across states will be used to initiate discussions on how to improve both professional development for inservice teachers and teacher preparation for preservice teachers.

References

Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics.* Retrieved from https://amte.net/standards

Abell, M. (2006). Individualizing learning using intelligent technology and universally designed curriculum. *Journal of Technology, Learning, and Assessment*, *5*(3), 4-20.

Bill & Melinda Gates Foundation. (2015). Teachers know best what educators want from digital instructional tools 2.0. *Bill & Melinda Gates Foundation*. Retrieved from https://s3.amazonaws.com/edtech-production/reports/Teachers-Know-Best-2.0.pdf

Borja, R. R. (2006). District initiative. *Education Week*, *25*(35), 24-41.

Bursal, M., & Paznokas, L. (2006). Mathematics anxiety and preservice elementary teachers' confidence to teach mathematics and science. *School Science and Mathematics*, *106*(4), 173-180. doi:10.1111/j.1949-8594.2006.tb18073.x

Cargile, L. A., & Harkness, S. S. (2015). Flip or flop: Are math teachers using Khan Academy as envisioned by Sal Khan? *Tech Trends*, *59*(6), 21-27.

Cauthen, L. (2017). How BIG is the K-12 digital curriculum market? *The Learning Counsel*. Retrieved from http://thelearningcounsel.com/article/how-big-k-12-digital-curriculum-market

Choppin, J. (2016). Analysis of eight digital curriculum programs. In M. Bates & Z. Usiskin (Eds.), *Digital curricula in school mathematics* (pp. 161-176). Charlotte, NC: Information Age.

Choppin, J., Carson, C., Borys, Z., Cerosaletti, C., & Gillis, R. (2014). A typology for analyzing digital curricula in mathematics education. *International Journal of Education in Mathematics, Science, and Technology, 2*(1), 11-25.

Confrey, J. (2016). Designing curriculum for digital middle grades mathematics: Personalized learning ecologies. In M. Bates & Z. Usiskin (Eds.), *Digital curricula in school mathematics* (pp. 7-33). Charlotte, NC: Information Age.

Council of Chief State School Officers. (2011). Common core state standards (CCSS) mathematics curriculum materials analysis project. Retrieved from https://math.la.asu.edu/~milner/PDF/CCSSOMathematicsCurriculumAnalysisProject.pdf

Dweck, C. S. (2006). *Mindset: The new psychology of success*. New York: Random House, Inc.

Earl, L., & Katz, S. (2006). *Leading schools in a data rich world*. Thousand Oaks, CA: Corwin Press.

Empirical Education. (2013). A study of student achievement, teacher perceptions, and IXL math. *IXL.com*. Retrieved from https://www.ixl.com/research/IXL-Research-Study-2013.pdf
Fletcher, G., Scaffhauser, D., & Levin, D. (2012). *Out of print: Reimagining the K-12 textbook in a digital age*. State Educational Technology Directors Association.
Foegen, A., Jiban, C., & Deno, S. (2007). Progress monitoring measures in mathematics. *The Journal of Special Education*, *41*(2), 121-139.
Fuchs, L. S., Hamlett, C. L., & Fuchs, D. (1999). *Monitoring basic skills progress: Basic math concepts and applications*. Austin, TX: Pro-Ed.
Fuchs, L. S., & Stecker, P. M. (2003). *Scientifically based progress monitoring*. Washington, DC: National Center on Student Progress Monitoring.
Gardner, H., & Davis, K. (2013). *The app generation: How today's youth navigate identity, intimacy, and imagination in a digital world*. New Haven, CT: Yale University Press.
Haciomeroglu, G. (2014). Elementary pre-service teachers' mathematics anxiety and mathematics teaching anxiety. *International Journal for Mathematics Teaching and Learning*, 1-10.
Hattie, J., & Jaeger, R. (1998). Assessment and classroom learning: A deductive approach. *Assessment in Education: Principles, Policy & Practice*, *5*(1), 111-122.
Hernandez-Ramos, P. (2005). If not here, where? Understanding teachers use of technology in silicon valley schools. *Journal of Research on Technology in Education*, *38*(1), 39-64.
Kafitz, D. (2017). Digital curriculum strategy survey and assessment tool. Retrieved from http://learningcounselgathering.com/2017-survey
Khan, S. (2012). *The one world schoolhouse: Education reimagined*. New York: Hachette Book Group.
Markauskaite, L. (2006). Gender issues in preservice teachers' training: ICT literacy and online learning. *Australasian Journal of Educational Technology*, *22*(1), 1-20.
Marsh, J., Kerr, K. A., Ikemoto, G. S., Darilek, H., Suttorp, M., Zimmer, R., et al. (2005). *The role of districts in fostering instructional improvement: Lessons from three urban districts partnered with the institute for learning*. Santa Monica, CA: Rand Corporation.
Mayer, R. (2003). The promise of multimedia learning: Using the same instructional design methods across different media. *Learning and Instruction*, *13*, 125-139.
McAnallen, R. R. (2010). *Examining mathematics anxiety in elementary classroom teachers* (Doctoral dissertation). Retrieved from http://search.ebscohost.com/login.aspx?direct=true&db=eric&AN= ED530770 &site=ehost-live
National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: Author.
National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics*. Washington, DC: Authors.
Nguyen, D. M., Hsieh, Y. J., & Allen, G. D. (2006). The impact of web-based assessment and practice on students' mathematics learning attitudes. *The Journal of Computers in Mathematics and Science Teaching*, *24*(1), 5-25.
Nguyen, D., & Kulm, G. (2005). Using web-based practice to enhance mathematics learning and achievement. *Journal of Interactive Online Learning, 3*(3). Retrieved from https://www.researchgate.net/profile/Gerald_Kulm/publication/228805333_Using_Web-based_practice_to_enhance_mathematics_learning_and_achievement/links/00463528c11358b157000000/Using-Web-based-practice-to-enhance-mathematics-learning-and-achievement.pdf
Pepin, B., Choppin, J., Ruthven, K., & Sinclair, N. (2017). Digital curriculum resources in mathematics education: Foundations for change. *ZDM Mathematics Education*, *49*(5), 645-661. doi: 10.1007/s11858-017-0879-z
Petrides, L., & Nodine, T. (2005). *Anatomy of school system improvement: Performance-driven practices in urban school district*. San Francisco, CA: New Schools Venture Fund.
Puentedura, R. (2013, May 29). SAMR: Moving from enhancement to transformation [Web log post]. Retrieved from http://www.hippasus.com/rrpweblog/archives/000095.html
Remillard, J. (1999). Curriculum materials in mathematics education reform: A framework for examining teachers' curriculum development, *Curriculum Inquiry, 29*(3), 315-342.
Sauers, N. J., & Mcleod, S. (2012). What does the research say about school one-to-one computing initiatives? *Center for the Advanced Study of Technology Leadership in Education,* Lexington, KY: University of Kentucky.
Selwyn, N. (2007). Curriculum online? Exploring the political and commercial construction of the UK digital learning marketplace. *British Journal of Sociology of Education, 28*(2), 223-240.

Sherin, M. G., & Drake, C. (2009). Curriculum strategy framework: Investigating patterns in teachers' use of a reform-based elementary mathematics curriculum. *Journal of Curriculum Studies*, *41*(4), 467-500. doi:10.1080/00220270802696115

Singer, J. (2017, June 6). The Silicon Valley billionaires remaking America's schools. *New York Times.* Retrieved from https://www.nytimes.com/2017/06/06/technology/tech-billionaires-education-zuckerberg-facebook-hastings.html

Smith, M. S. (2000). Balancing old and new: An experienced middle school teacher's learning in the context of mathematics instructional reform. *Elementary School Journal, 100*(4), 351-375.

Son, J. W., & Kim, O. K. (2015). Teachers' selection and enactment of mathematical problems from textbooks. *Mathematics Education Research Journal, 27*(4), 491-518.

Son, J. W., & Kim, O. K. (2016). Curriculum enactment patterns and associated factors from teachers' perspectives. *Mathematics Education Research Journal, 28*(4), 585-614.

SRI Education. (2014). Research on the use of Khan Academy in schools: Implementation report. *SRI International.* Retrieved from http://www.sri.com/work/publications/research-use-khan-academy-schools

Stecker, P. M., Fuchs, L. S., & Fuchs, D. (2005). Using curriculum-based measurement to improve student achievement: Review of research. *Psychology in the Schools*, *42*(8), 795-819. doi:10.1002/pits.20113

Stein, M. K., Remillard, J., & Smith, M. S. (2007). How curriculum influences student learning. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 319-370). Charlotte, NC: Information Age Publishing.

Stiggins, R. (2005). From formative assessment to assessment for learning: A path to success in standards-based schools. *Phi Delta Kappan*, *87*(4), 324-328.

Weltner, J. (2012). Pilot project with Khan Academy seeks to revolutionize learning in Idaho. *Khan Academy*. Retrieved from http://www.khanidaho.org/resources/Khan-Idaho-Pilot-PR.pdf

Wijekumar, K., Hitchcock, J., Turner, H., Lei, P. W., & Peck, K. (2009). *A multisite cluster randomized trial of the effects of Compass Learning Odyssey® Math on the math achievement of selected grade 4 students in the Mid-Atlantic region*. Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education. Retrieved from https://eric.ed.gov/?id=ED507314

Winter, J. (2001). Personal, spiritual, moral, social and cultural issues in teaching mathematics. In P. Gates (Ed.), *Issues in Mathematics Teaching* (pp. 197-213). London: Routledge Falmer.

Wozney, L., Venkatesh, V., & Abrami, P. (2006). Implementing computer technologies: Teachers' perceptions and practices. *Journal of Technology and Teacher Education, 14*(1), 173-207.

Yerushalmy, M. (2016). Inquiry curriculum and e-textbooks: Technological changes that challenge the representation of mathematics pedagogy. In M. Bates & Z. Usiskin (Eds.), *Digital curricula in school mathematics* (pp. 87-106). Charlotte, NC: Information Age.

Yeşilyurt, E., Ulaş, A. H., & Akan, D. (2016). Teacher self-efficacy, academic self-efficacy, and computer self-efficacy as predictors of attitude toward applying computer-supported education. *Computers in Human Behavior*, *64*, 591-601. https://doi.org/10.1016/j.chb.2016.07.038

Yukselturk, E., & Bulut, S. (2009). Gender differences in self-regulated online learning environment. *Journal of Educational Technology & Society*, *12*(3), 12-22.

Zentall, S. R., & Morris, B. J. (2010). "Good job, you're so smart": The effects of inconsistency of praise type on young children's motivation. *Journal of Experimental Child Psychology*, *107*(2), 155-163.

Zhao, Y., Zhang, G., & Lai, C. (2010). Curriculum, digital resources and deliver. In P. Peterson, E. Baker & B. McGaw (Eds.), *International Encyclopedia of Education* (pp. 390-396). Oxford, United Kingdom: Elsevier, Ltd.

COMPLEX CONNECTIONS: REIMAGINING UNITS COORDINATION

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Students' construction, coordination, and abstraction of units underlie success across multiple mathematics domains. Structures for coordinating units underscore notions of numbers as composite units (e.g., five is a unit of five and five units of one). In this working group, we seek to facilitate collaboration amongst researchers and educators concerned with composite unit formation and units coordination. The aim for this working group is two-fold: (1) to extend our research around units coordination to new grade levels; and (2) to collaborate with researchers who investigate students with learning differences in school settings to determine diverse students' mathematics learning trajectories.

Introduction

The phrase "units coordination" has many connotations. One meaning of the phrase refers to operations related to understanding place value, such as exchanging ten 1s for one 10 when executing arithmetic algorithms (e.g., Thanheiser, 2009). Other meanings involve relationships between measurement units, such as extensive measure equivalencies (e.g., 1 foot = 12 inches) or intensive measure definitions (e.g., speed = distance/time) (Smith & Barrett, 2017). In Steffe and colleagues' research program, the term "units coordination" refers to the mental operations of composing and distributing units involved in a numerical conception (Steffe, 1992). Units coordination is fundamental for distinguishing children's ways of counting - specifically the ways that students unitize, iterate, partition, disembed, and compose units. Researchers have investigated how ways of coordinating units are connected to students' learning within and across mathematics content domains, from additive and multiplicative reasoning to algebra (i.e., Hackenberg & Lee, 2015; Norton & Boyce, 2015; Steffe, 2017; Ulrich, 2015). For instance, Hackenberg and Lee (2015) described similarities and differences between units coordination with whole number units, fractional units, and variable units (e.g., linear algebraic equations); Ulrich (2015) described units coordination with signed integer addition.

In Steffe's 2017 plenary for PME-NA, he substantiated particular needs for investigating how children develop operations when coordinating units. The aim of this working group is to facilitate collaboration amongst researchers and educators concerned with units coordination for all learners. Our main goals are to extend our research around units coordination to new grade levels and to collaborate with researchers who investigate students with learning differences in school settings to determine diverse students' mathematics learning trajectories. Outcomes from these collaborative efforts could inform the development/revision of theoretical frameworks, learning trajectories for diverse populations, and professional development for educators.

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.

Background and Theoretical Perspective: Composite Units – Old and New

We provide a theoretical background for units coordination (Norton & Boyce, 2015) focusing on units coordination in students' number sequences (Steffe, 1992), multiplicative concepts (Hackenberg & Tillema, 2009), and fractions schemes (Steffe & Olive, 2010).

Units Coordination

The construction of mathematical objects forms the basis for units coordination, as students rely on concrete, pictorial, fingers, symbolic numerals, and language to evidence internalized (being able to mentally re-imagine contextual actions) or interiorized (being able to draw on de-contextualized actions) actions. Unitizing, or setting an object or a unit aside for further action or activity (Steffe, 1992), is the basis for units coordination. Consider the construction of additive reasoning as an example. When considering how to add eight and seven, students might not yet see the cardinality of eight, counting a set of objects to create eight, then another set of objects to create seven, and finally combining the two sets, beginning at one to quantify the total. Should students see the cardinality of eight, they might use it as an input for solving the problem. They will count on from eight, using objects (cubes, fingers) to keep track of the addition (e.g., 8…9 [raises a finger], 10 [raises another finger]…). The double counting involved in this activity (e.g., **8**…, 9 (1), 10 (2), 11 (3), 12 (4), 13 (5), 14 (6), **15** (7)) promotes a coordination of the start value and the stop value. That is, eight, seven, and 15 are taking on some meaning as composites (8 and 7) and a unitized whole (15). Evidence of this meaning includes the breaking apart of one or both of the numbers to arrive at the total (e.g., 8 is 5 and 3; 7 is 5 and 2; 8 + 5 is the same as 5 + 5 + 3 + 2, or 15). This type of units coordination can be explained through the type of numerical sequence students produce and rely upon.

Number Sequence Types (INS, TNS, ENS, GNS)

Steffe and Olive (2010) described four counting sequences that children develop and evidence: (1) Initial Number Sequence (INS), (2) Tacitly-Nested Number Sequence (TNS), (3) Explicitly-Nested Number Sequence (ENS), and (4) Generalized Number Sequence (GNS). Each illustrates stages of children's units development and coordination.

Initial Number Sequence. Steffe (1992) explained that children who segment and interiorize number sequences have developed *Initial Number Sequence (INS).* Children who develop an INS are characterized by their counting of single units and then their segmenting of a numerical sequence (evidenced through "counting on" activity). When children segment numerical sequences, they are interiorizing patterned templates for counting, which allows them the ability to count on from a composite unit (e.g., developing one composite unit to use when counting on, 4…5-6-7-8). Thus, through counting actions, numerical patterns are developed and become interiorized (evidenced through less reliance on sensory-motor experiences; e.g., verbalizing counts, using fingers, or tapping) before being segmented into a composite unit.

Tacitly-Nested Number Sequence. Once children have developed composite units through their INS activity, they can begin to *coordinate* these composite units, treating the result of counting activity as <u>both</u> a unit to count on from and one to keep track of when counting. These actions evidence children's development of a *Tacitly-Nested Number Sequence (TNS).* Steffe (1992) explains that when children have a numerical sequence interiorized and segmented they can use their segmented numerical sequence as material for making new composite units within these numerical sequences. The awareness of one number sequence contained inside another, or *double-counting,* is an indication of TNS, as is a skip count (i.e., 4, 8, 16,…) to solve early multiplicative kinds of problems, such as how many 3's are contained in 12.

Explicitly-Nested Number Sequence. Children who are described as having part-whole number reasoning in place are capable of disembedding parts from wholes and developing iterable units of one. These children are described as reasoning with an *Explicitly-Nested Number Sequence* (ENS) (Olive, 1999; Steffe, 1992; Ulrich & Wilkins, 2017). Children capable of multiplicatively understanding a single unit and a composite unit without disrupting the whole are said to be operating with an ENS (Ulrich & Wilkins, 2017). The two given composite units (e.g., parts and whole) provide children material to coordinate while constructing a third composite unit; e.g., a unit of units of units (Steffe, 1992). This part-whole reasoning with abstract units provides children multiplicative number structures.

Generalized Number Sequence. Children capable of developing iterable composite units where units of units of units can be coordinated, are described as operating with a GNS. For example, Olive (1999) explained that when children are asked to find common multiples, they are required to keep track of two series of composite units (e.g., 3, 6, 9, 12; 4, 8, 12; 12. The LCM of 3 and 4 is 12). Children evidencing successful completion of tasks like this are described as using two iterable composite units (e.g., 3 and 4), while keeping track of the common composite unit in each sequence. At the root of much of this number sequence development, units development and coordination explain how and why children are capable of transitioning from additive/subtractive operative structures towards multiplicative/division operative structures towards rational number understanding.

Multiplicative Levels of Units Coordination

A student is said to assimilate with one level of units when she conceives of multiplication situations, such as seven iterations of four, by counting on from the first or second set of four by ones and double-counting the number of fours to reach a stop value (*e.g., **4**, **8**,…9-10-11-**12**; 13-14-15-**16**; 17-18-19-**20**; 21-22-23-**24;** 25-26-27-**28***). Here, the child has to model or carry out the situation by using internal (e.g., subvocal counting) or external (e.g., fingers or objects) representations. This is referred to as coordinating two levels of units *in activity*. Units coordination in activity is ephemeral: in a follow-up task, such as how many ones are in eight iterations of four, the student would likely repeat a similar process rather than count-on.

A student's use of strategic reasoning in such situations may be evidence that she *assimilates* the situation with two levels of units. For example, a student assimilating with two levels of units might conceive of seven iterations of four as five iterations of four plus two iterations of four (e.g., five 4s is **20**; 21-22-23-**24**; 25-26-27-**28**). As opposed to modeling the entire coordination, the child can *anticipate* breaking apart the composite unit of seven into five and two and use each of those parts to solve the problem. For a student assimilating with two levels of units, the result of operating is simultaneously 28 ones and 7 fours; hence a follow-up task of finding the number of 1s in 8 fours would not require building up from 5 fours again.

A student is said to assimilate with three levels of units when she can conceive of a situation such as seven iterations of four as resulting in three distinct yet coordinated units: (a) one unit of 28 that contains (b) seven units of four, each of which contains (c) two units of two. Students assimilating with three levels of units have flexibility to reason strategically with each of the units. For instance, a student assimilating with three levels of units might solve the task, "How many more twos are in 32 than in 28?" by reasoning that 32 is one more 4, which is thus two more 2s. This reasoning involves assimilating three levels of units, multiplicatively.

Norton, Boyce, Ulrich and Phillips (2015) conducted a cross-sectional analysis of 47 sixth-grade students' reasoning in whole number multiplicative settings in clinical interviews. Figure 1 displays descriptors of attributions of students' activities Norton and colleagues identified as corresponding with students' reasoning with one, two, or three levels of units. For instance, they describe students' activity when transitioning from reasoning with one level of units to reasoning with two levels of units with descriptors G-K (Norton et al., 2015, p. 62). Though there are commonalities with descriptions of students' counting schemes (e.g., descriptors C and J), the focus of the descriptors are more generally about how and whether students are able to flexibly reverse and reflect on their multiplicative reasoning.

Descriptors			
A. Iterates units of 1	E. Works with at least one level of units within a figurative three-level structure	I. Works with at least two levels of units within a figurative three-level structure	M. Assimilates three levels of units within a task
B. Builds composite units	F. Indicates a reciprocal relationship between the size and number of parts in a whole	J. Distributes composite units	N. Reflects on all levels within a three-level structure
C. Iterates composite units	G. Reverses multiplicative relationships	K. Immediacy of response	O. Conflates or confuses units
D. Uses multiplicative relationships	H. Builds a three-level structure	L. Flexibly works with a three-level structure	P. References the structure of units

Figure 1. Descriptors of counting schemes.

These stages of units coordination, as delineated in Norton et al.'s (2015) findings, comprehensively explains transitions children make from additive operations to multiplicative operations. These findings also explain fractional unit development, as recursive units coordination in which to act upon and explain how students develop fractions as mental objects.

Fractional Units

Reasoning with fractional units requires unitizing a fractional size, $1/n$th. When children first conceptualize fractional units, Steffe (2001) posited that they would reorganize natural number schemes to develop fractional schemes. Olive (1999) and Steffe (2001) argued that they would have to re-interiorize their units coordination operations, to consider a fractional unit as a result of *equi-partitioning* a unit whole into a size that, when iterated n times, would result in the size of 1. This re-interiorization of schema requires students to recursively develop and coordinate new composite units relationships (Olive, 1999). Thus, children's production of numerical sequences and their associated units coordination provide children necessary operations for their fractional units coordination.

To conceive of a fraction m/n as a number, one must understand m/n as equivalent to m $1/n$ths, n of which are equivalent to 1. In the case of $m > n,$ the meaning of $1/n$ must transform from thinking of $1/n$ as one out of n total pieces (a *parts-out-of-wholes scheme*) to thinking of $1/n$ as an amount that could be iterated more than n times without changing its relationship with the size of 1 (an *iterative fraction scheme*). This measurement conception of fraction (Lamon, 2008) involves coordinating three levels of units of nested units: 8/3 is 8 times (1/3), 1=3/3 is 3 times (1/3), thus an 8/3 unit contains both a unit of 1 and a unit of 1/3 within 1 (Hackenberg, 2010). Students in the intermediate stages of constructing such a measurement conception may reason about the size of proper fractions of form m/n by counting the number of parts of size $1/n$ within a whole of n/n. Such students are not yet iterating the amount of $1/n$, which limits their ability to iterate unit fractions beyond the size of the whole (Tzur, 1999).

Issues Related to the Psychology of Mathematics Education

Steffe's 2017 plenary included both a summary of contributions and important extant problems for mathematics educators pertaining to units coordination. Given these advances in research surrounding K-8 students' units coordination, our mathematics education field is still limited by the context in which students' units coordination develops and how preschool and high school students' units coordination may inform these trajectories. Further, Steffe (2017) proposes that about 40% of first grade students have very different learning trajectories than their peers, suggesting a need to develop alternative means in which units coordination may be developed by children. Finally, Steffe suggests that our field would benefit by investigating children's transitions in scheme development. For instance:

> It is especially crucial to investigate possible changes that indicate *fundamental* transitions between reasoning with two levels of units and three levels of units induced in the construction of quantitative measuring schemes and their use in the construction of multiplicative and additive measuring schemes (Steffe, 2017, p. 46).

One of our intentions for this working group is for researchers from different backgrounds to collaborate to work toward solving such problems. Consider in Smith and Barrett's review of research on quantitative reasoning, the following:

> [We] found it striking how often the same conceptual principles and associated learning challenges appear in the measurement of different quantities… Despite the clear focus in research on equipartitioning, units and their iteration, units and subunits… curricula (and arguably most classroom teaching) focus students' attention on particular quantities and the correct use of tools, as if each was a new topic and challenge. (p. 377).

Consistent with Arbaugh, Herbel-Eisenmann, Ramirez, Knuth, Kranendonk, and Quander's (2010) call to "develop mathematics proficiency in various school, cultural, and societal contexts" (p. 13), we aim is to connect units coordination research with other learning theories.

Unfortunately, many students with mathematics learning difficulties do not transition from two levels of units to three levels of units at the same pace as their more successful peers. In fact, the construction of ENS is one pervasive mathematical impediment for students with mathematics difficulties (Landerl, Bevan, & Butterworth, 2004). Compared to students without mathematics difficulties, students with mathematics difficulties develop less sophisticated strategies for number computation problems over time, suggesting a lack of a conceptual basis for ENS engagement *that actually plays a part in their later disability identification* (Butterworth, Varma, & Laurillard, 2011). Therefore, particular research programs with these foci are desperately needed to nurture multiplicative and rational number conceptions (Boyce & Norton, 2017; Grobecker, 1997; Kosko, 2017) and operations (Grobecker, 1997; Grobecker, 2000; Kosko, 2017; Norton & Boyce, 2015) for students who need our support.

By designing interventions with children's mathematics and their units coordination, the field has grown over the years, yet there is much opportunity for improvement. Relations between cognitive factors and test performance may be important, yet these relationships are only one way to conceptualize "cognition." Instead of focusing on aspects of students' working memory or processing, researchers can revolutionize access for these students by intervening on the *malleable cognitive factors that can be improved upon* through students' own development.

Research Designs and Methodologies

The primary methodology for investigating units coordination has been the radical constructivist teaching experiment (Steffe & Thompson, 2000). A main role of these teaching experiments is to generate (and refine) epistemic models of students' mathematics - models for how students with common underlying conceptual operations learn within a particular mathematics domain (Steffe & Norton, 2014). Such teaching experiments involve close interactions with a teacher-researcher modeling the dynamics of students' ways of operating longitudinally. Teaching experiment methodology is also used as part of design research (Cobb, Confrey, diSessa, Leher, & Schauble, 2003), to inform instructional approaches or interventions that could be "scaled up" to heterogeneous classroom settings.

Results from analyzing teaching experiments have also informed methods for assessing a child's ways of coordinating units at a particular moment. In addition to task-based clinical interviews (Clement, 2000), Norton and Wilkins (2009) created written instruments for assessing middle-grades' fractions schemes and operations associated with units coordination. These instruments have been used to validate conjectured learning trajectories for children's construction of schemes for coordinating fractional units (e.g., Norton & Wilkins, 2012). These instruments currently serve as tools for participant selection in teaching experiments with middle-grades students (e.g., Hackenberg & Lee, 2015). These instruments also serve as valuable opportunities for connecting results to classroom teaching and learning.

Consideration of these methodologies for researching units coordination suggests areas for collaborative work to build our understanding not only of the research programs Steffe (2017) described, but also opportunities and needs regarding related research domains. For instance, Norton and Wilkins' (2009) instruments have been modified to assess prospective teachers' units coordination with fractions (Lovin, Stevens, Siegfied, Wilkins, & Norton, 2016). Thus, research development with these methodologies are better served in collaborative designs to allow more perspectives in the design and analyses when determining students' mathematics.

First Aim: Extending Units Coordination Research

The first aim of this working group proposal is to extend units coordination research to investigations that include both older students and younger students. For instance, questions regarding whether differences in secondary students' units coordination persist beyond eighth-grade, and, if so, how these differences manifest in older students' learning, remain underexplored. In interview and teaching experiment settings, Grabhorn, Boyce, and Byerley (2018) have found that students enrolled in calculus in university do not necessarily coordinate three levels of units. Further expanding understanding of students' units coordination beyond eighth grade would contribute to the development of "coherent frameworks for characterizing the development of student thinking" (Arbaugh et al., 2010, p. 15).

In addition to studies focusing on relationships between units coordination and older students' mathematics, studies of pre-kindergarten children are also warranted. For instance, Wright (1991) found that kindergarten students had a wide variance in their number knowledge, suggesting critical mathematics learning occurs prior to the elementary school. With a significant dearth of research studies in the early childhood years (De Smedt, Noel, Gilmore, & Ansari, 2013), we posit that studies of young children are a critical area of research. Another potential need regards equity and access in mathematics education. Researchers might investigate whether analyses of students' units coordination provides insight to the mathematical reasoning of diverse and underserved populations. We envision collaborations around similar conceptual principles and learning challenges, to investigate units coordination of children enrolled in different grade levels and representing different population groups (i.e., special education, early childhood, secondary education, teacher education), which would allow more coherent mathematical learning theory and practical means in which to link research to mathematics classrooms.

Steffe (2017) estimated that about 40% of first grade students rely solely on perceptual material when counting all items (Counters of Perceptual Unit Items - CPUI) and that 45% of first grade students are capable of counting figurative unit items (CFUI), a particularly necessary step when interiorizing counting actions and developing "counting-on" activity (e.g., INS) (see

Figure 2). These estimates indicate that children entering early elementary school grades have not yet been able to internalize their actions and rely on relatively more abstract representations for number. Steffe further posits that these two distinct groups of children would require different learning trajectories, as they have different operational means and different conceptual material in which to coordinate for TNS and ENS activity.

With more U.S. students attending preschool programs and an increase of 48% in national funding towards preschool programs, it would be advantageous to develop research that could directly inform early childhood mathematics curricula (Diffey, Parker, & Atchison, 2017; Sarama & Clements, 2009). Also, given that these curricula should be coherently aligned with elementary grade mathematics curricula initiatives, it would serve early childhood curriculum designers to bridge research programs around number development in early elementary grades to preschool grade levels. For instance, MacDonald and Wilkins (under review) found in a case study that one preschool student may be using early subitizing activity to promote her composite unit development and units coordination. Investigating how early, perceptual actions may relate to students operations development around units coordination would serve these foci. Thus, our first aim is to extend our research around units coordination to new grade levels.

Grade/N Seq.	CFUI or INS	ENS	GNS
First	≈ 45 Percent	≈ 10 to 15 Percent	≈ 0 to 5 Percent
Second	≈ 30 Percent	≈ 25 to 30 Percent	≈ 0 to 5 Percent
Third	≈ 5 Percent	≈ 45 to 50 Percent	≈ 0 to 10 Percent

Figure 2. Steffe's (2017) estimated percentage of students capable of counting figurative unit items (CFUI), engaging with initial number sequence (INS), engaging with explicitly-nested number sequence (ENS), or engaging with generalized number sequence (GNS) (p. 41).

Second Aim: Widening Units Coordination Research

Our second aim is to widen units coordination research by collaborating with researchers who investigate students with learning differences, from diverse cultures, and from low-socioeconomic households to determine diverse students' mathematics learning trajectories. For instance, Hunt and MacDonald (in press), who work closely with students with learning differences in case study research, uncovered three important challenges students experience as they work toward more sophisticated coordination of units. First, when engaging in tasks that support the construction of composite unit, the use of memorized fact combinations or teacher taught strategies eclipsed the use of one students' natural reasoning (in press) yet supported another's (Hunt, Silva, & Lambert, 2017). All three students evidenced tacit ways of reasoning (e.g., 2 is contained in 3, 3 is contained in 4); two reverted to pseudo-empirical abstractions, tricks, or algorithms that they could not explain to solve the tasks. These ways of engaging in the tasks were counterproductive to one student's reflection on their own actions upon units such that adaptations in their reasoning could occur. Conversely, the second student leveraged his knowledge of number facts and alternative representations to advance his fractional reasoning and compensate for his perceptual motor differences (2017). For both students, teacher encouragement and support to engage in each student's own ways of reasoning was imperative. Second, Hunt & Silva (forthcoming) found evidence that confirms previous research (Geary,

2010) that one of the students sometimes lost track of counting during a count on, possibly due to working memory. We conjecture that this learning difference interferes with the move from counting on to more sophisticated additive reasoning (e.g., breaking apart tens). Yet, in this case, the problem is alleviated through opportunities for within-problem reflection through experiences that the child has to construct addends involved in number problems through counting, sweeping small numbers as lengths, and improving the usability of small composites like 2, 4, 3, 5, and 6 (forthcoming). Lastly, both students evidenced an affinity towards symmetric notions of units coordination across learning situations and across natural and fractional number reasoning. Hunt is currently conducting cross-case analysis to discern whether the symmetry is indicative of unique trajectories in number and/or fractional reasoning.

We argue that equitable instruction for all students begins with increased opportunities to adapt *their own thinking* grounded in a construction within their own mathematical realities. When well-intentioned educators provide children interventions that promote procedures and actions, not only are they not serving their children's mathematics learning needs, they may be preventing them from engaging in learning situations that support the children to adapt their thinking structures and advance their learning.

Specific Goals and Aims

To extend and widen units coordination research this working group intends to accomplish the following: (a) develop a research agenda for the group, (b) brainstorm specific research questions and unpack theoretical frameworks that will address that agenda, (c) explore research methodologies that can answer the potential research questions, (d) discuss the logistics of collaborations to carrying out these studies, and (e) embark upon collaborations leading to publication and funding opportunities.

Plan for Working Group

Session 1: Concept Formation

<u>GOAL: Generation of research questions that are important to the group and/or sub-groups</u>. *Introduce focus for the working group* by asking "what types of problems would members like to explore?", by viewing/discussing short video clips of students working through various mathematical concepts to better understand the students' thinking, and developing potential research foci (e.g., overall purpose/goals of this working group, ties between composite units, coordination of units, and particular mathematics content). Finally, subgroups will be developed to form research questions that can cross-domains and use questions to form collaborations based on each members' area of interest and expertise.

Session 2: Theoretical Frameworks and Methodologies

<u>GOAL: Explore appropriate research methodologies</u>. *Formulate plans for research and collaboration across group members by* examining a variety of methodologies. Means for these examinations would include the following: (a) view videos of work already conducted to highlight possible methodologies for future studies; (b) discuss other potential methodologies not highlighted during the video viewing; (c) discuss how to design robust collaborative studies. *Small group would entail:* (1) work already done; (2) research agenda development; *Large group would entail*: (1) share small group discussions; (2) delineate session 3 goals

Session 3: Planning and Writing

<u>GOAL: Embark on collaborations</u>. *Small group will entail*: (1) work on written product of research agenda; (2) develop shared conceptual framework and the relationship of our framework to what is currently being done; (3) identify target journals and outlets or grants and funding sources. *Large group will entail*: (1) share progress and commitments from small group

discussion; (2) finalize a plan for individual groups to continue updating progress to the larger group; (3) creation of working group website or blog.

Anticipated Follow-up Activities

Throughout the year, the members of this working group will continue working on research problems of common interest. They will contribute to a common website in which they will update other members of the working group about the progress of the various research collaborations. In the future, this working group will propose a special issue to a leading journal in the field and/or construct a grant proposal to a nationally recognized funder.

References

Arbaugh, F., Herbel-Eisenmann, B., Ramirez, N., Kranendonk, H., Knuth, E., & Quander, J. R. (2010). Linking research and practice: Practitioner community priorities for research in mathematics education. In *Reston, VA: Report for the National Council of Teacher of Mathematics, Research Agenda Conference.*

Byerley, Cameron O. (2016). *Secondary Teachers' and Calculus Students' Meanings for Fraction, Measure and Rate of Change* Unpublished Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University.

Boyce, S., & Norton, A. (2017). Dylan's units coordinating across contexts. Journal of Mathematical Behavior, 45, 121-136.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. Lesh (Eds.), *Handbook of research data design in mathematics and science education* (pp. 547-589). Mahwah, NJ: Lawrence Erlbaum Associates.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32* (1) 9-13.

De Smedt, B., Noël, M. P., Gilmore, C., & Ansari, D. (2013). How do symbolic and non-symbolic numerical magnitude processing skills relate to individual differences in children's mathematical skills? A review of evidence from brain and behavior. *Trends in Neuroscience and Education, 2*(2), 48-55.

Diffey, L., Parker, E., & Atchison, B. (2017). State Pre-K Funding 2016-17 Fiscal Year: Trends and opportunities. 50-State Review. *Education Commission of the States.* Retrieved from https://www.ecs.org/ec-content/uploads/State-Pre-K-Funding-2016-17-Fiscal-Year-Trendsandopportunities-1.pdf

Grabhorn, J., Boyce, S., & Byerley, C. (2018, February). Relationships between calculus students' ways of coordinating units and their ways of understanding integration. Poster presented at SIGMAA on RUME's Conference on Research in Undergraduate Mathematics Education, San Diego, CA.

Grobecker, B. A. (1997). Partitioning and unitizing in children with learning differences. *Learning Disability Quarterly, 20*(4), 317-335.

Grobecker, B. (2000). Imagery and fractions in students classified as learning disabled. *Learning Disability Quarterly, 23*(2), 157-168.

Hackenberg, A. J., & Lee, M. (2015). Relationships between students' fractional knowledge and equation writing. *Journal for Research in Mathematics Education, 46*(2), 196-243.

Hackenberg, A. J., & Tillema, E. S. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fraction composition schemes. *The Journal of Mathematical Behavior*, *28*(1), 1 – 18.

Hunt., J.H., MacDonald, B.L., & Silva, J., (in press). Tacit, trick, or teach: What is Gina's mathematical reality? Paper presented at the 40th Conference of the International Group for the Psychology of Mathematics Education North America (PME-NA), Greenville, SC.

Hunt., J.H., Silva, J., & Lambert, R. (2017). From Trajectories, deficit, and differences to neurodiversity: The case of Jim. Paper presented at the 39th Conference of the International Group for the Psychology of Mathematics Education North America (PME-NA), Indianapolis, IN.

Karl W. Kosko (2017) Reconsidering the role of disembedding in multiplicative concepts: Extending theory from the process of developing a quantitative measure, Investigations in Mathematics Learning, 10:1, 54-65, DOI: 10.1080/19477503.2017.1375358

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester (Ed.) *Second handbook of research on mathematics teaching and learning: a project of theNational Council of Teachers of Mathematics* (pp. 629-667). Charlotte, NC : Information Age Publishing

Landerl, K., Bevan, A., & Butterworth, B. (2004). Developmental dyscalculia and basic numerical capacities: A study of 8–9-year-old students. *Cognition, 93*(2), 99-125.

Lovin, L. H., Stevens, A. L., Siegfried, J., Wilkins, J. L. M., & Norton, A. (2016). Pre-K-8 prospective teachersunderstanding of fractions: An extension of fractions schemes and operations research. *Journal of Mathematics Teacher Education.* https://doi.org/10.1007/s10857-016-9357-8
MacDonald, B. L. & Wilkins, J. L. M. (under review). Subitising activity relative to units construction and coordination: A case study. *Research in Mathematics Education Journal.*
Norton, A., & Boyce, S. (2015). Provoking the construction of a structure for coordinating n+ 1 levels of units. *The Journal of Mathematical Behavior, 40*, 211-232.
Norton, A., Boyce, S., Phillips, N., Anwyll, T., Ulrich, C., & Wilkins, J. L. (2015). A written instrument for assessing students' units coordination structures. *Mathematics Education Journal 10*(2), 111-136.
Norton, A., Boyce, S., Ulrich, C., & Phillips, N. (2015). Students' units coordination activity: A cross-sectional analysis. *The Journal of Mathematical Behavior, 39*, 51-66.
Norton, A., & Wilkins, J. L. M. (2010). Students' partitive reasoning. *Journal of Mathematical Behavior, 29,* 181-194.
Norton, A., & Wilkins, J. L. M. (2012). The splitting group. *Journal for Research in Mathematics Education, 43*(5),557-583.
Olive, J. (1999). From fractions to rational numbers of arithmetic: A reorganization hypothesis. *Mathematical Thinking and Learning, 1*(4), 279-314.
Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children.* New York: Routledge.
Smith, J., P., & Barrett, J. E. (2017). Learning and teaching measurement: Coordinating quantity and number. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 355-385). Reston, VA: National Council of Teachers of Mathematics.
Steffe, L. P. (1991). The constructivist teaching experiment: Illustrations and implications. In E. von Glasersfeld (Ed.), Radical constructivism in mathematics education (pp. 177-194). New York: Kluwer Academic Publishers.
Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences, 4*(3), 259-309.
Steffe, L. P. (2001). A new hypothesis concerning children's fractional knowledge. *The Journal of Mathematical Behavior*, *20*(3), 267 – 307.
Steffe, L. P. (2017). Psychology in mathematics education: Past, present, and future. In E. Galindo & J. Newton (Eds). *Proceedings of the 39th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 27-56). Indianapolis, IN.
Steffe, L. P., & Cobb, P. (1988). Construction of arithmetical meanings and strategies. New York: SUNY Press.
Steffe, L. P., & Norton, A. (2014). Perspectives on epistemic algebraic students. In L. Steffe, K. Moore, & L. Hatfied (Eds.), *Epistemic algebraic students: Emerging models of students' algebraic knowing* (pp. 317 – 321). Laramie, WY: University of Wyoming Press.
Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge*. NY: Springer.
Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education*, 267 – 306. Hillsdale, NJ: Lawrence Erlbaum Associates.
Thanheiser, E. (2009). Preservice elementary teachers' conceptions of multidigit whole numbers. *Journal Research in Mathematics Education, 40*(3), 251-281.
Ulrich, C. (2015). Stages in constructing and coordinating units additively and multiplicatively (Part 1). *For the Learning of Mathematics, 35*(3), 2-7.
Ulrich, C., & Wilkins, J. L. (2017). Using written work to investigate stages in sixth-grade students' construction and coordination of units. *International Journal of STEM Education, 4*(1), 23.
Wright, B. (1991). What number knowledge is possessed by children beginning the kindergarten year of school?. *Mathematics education research journal, 3*(1), 1-16.

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Embodied cognition is growing in theoretical importance and as a driving set of design principles for curriculum activities and technology innovations for mathematics education. The central aim of the EMIC (Embodied Mathematical Imagination and Cognition) Working Group is to attract engaged and inspired colleagues into a growing community of discourse around theoretical, technological, and methodological developments for advancing the study of embodied cognition for mathematics education. A thriving, informed, and interconnected community of scholars organized around embodied mathematical cognition will broaden the range of activities, practices, and emerging technologies that count as mathematical. EMIC builds upon our prior working groups with a specific focus on how we can leverage emerging technologies to study embodied cognition and mathematics learning. In particular, we aim to develop new theories and extend existing frameworks and perspectives from which EMIC collaboration and activities can emerge.

Keywords: Technology; Cognition; Informal Education; Learning Theory

Motivations for This Working Group

Recent empirical, theoretical and methodological developments in embodied cognition and gesture studies provide a solid and generative foundation for the establishment of **a regularly held Embodied Mathematical Imagination and Cognition (EMIC) Working Group** for PME-NA. The central aim of EMIC is to attract engaged and inspired colleagues into a growing community of discourse around theoretical, technological, and methodological developments for advancing the study of embodied cognition for mathematics education, including, but not limited to, studies of mathematical reasoning, instruction, the design and use of technological innovations, learning in and outside of formal educational settings, and across the lifespan.

The interplay of multiple perspectives and intellectual trajectories is vital for the study of embodied mathematical cognition to flourish. While there is significant convergence of theoretical, technological, and methodological developments in embodied cognition, there is also a trove of technological, methodological, and theoretical questions that must be addressed before we can formulate and implement effective design principles. As a group, we aim to address basic theoretical questions such as, What is grounding? And practical ones such as, How can we reliably engineer the grounding of specific mathematical ideas? We need to understand how variations in actions and perceptions influence mathematical reasoning, including self-initiated vs. prescribed actions, and actions that take place in intrapersonal versus interpersonal interactions; how gestural point-of-view when enacting phenomena from a first- versus third-person perspective, including how gestures move through space, influences reasoning and communication; how actions enacted by oneself, observed in others, or imagined influence cognition; and how gestures connect with external visual representations (Alibali & Nathan, 2012; in press).

A thriving, informed, and interconnected community of scholars organized around embodied mathematical cognition will broaden the range of activities and emerging technologies that count as mathematical, and envision alternative forms of engagement with mathematical ideas and practices (e.g., De Freitas & Sinclair, 2014). This broadening is particularly important at a time when schools and communities in North America face persistent achievement gaps between groups of students from many ethnic backgrounds, geographic regions, and socioeconomic circumstances (Ladson-Billings, 1995; Moses & Cobb, 2001; Rosebery, Warren, Ballenger & Ogonowski, 2005). There also is a need to articulate evidence-based findings and principles of embodied cognition to the research and development communities that are looking to generate and disseminate innovative programs for promoting mathematics learning through movement (e.g., Ottmar & Landy, 2016; Smith, King, & Hoyte, 2014). Generating, evaluating, and curating empirically validated and reliable methods for promoting mathematical development and effective instruction through embodied activities that are engaging and curricularly relevant is an urgent societal goal. As embodied cognition gains prominence in education, so, too, does new ways of using technology to support teaching and learning (Lee, 2015). These new uses of technology, in turn, offer novel opportunities for students and scientists to engage in math visualization, symbolization, intuition, and reasoning. In order for these designs to successfully scale up, they must be informed by research that demonstrates both ecological and internal validity.

Past Meetings and Achievements of the EMIC Working Group

"Mathematics Learning and Embodied Cognition," the first PME-NA meeting of the EMIC working group, took place in East Lansing, MI in 2015. Our group has been growing ever since. In addition to the PME-NA meeting each year, there are a number of ongoing activities that our members engage in. We have built an active website which provides updates on projects, and hosts resources. On this website, we have a list of members with their emails and bios, information about our PME-NA presence, and short personal introduction videos. We've also created a space for members to share information about their research activities – particularly for videos of the complex gesture and action-based interactions that are difficult to express in text format. In addition, we have a common publications repository to share files or links (including to ResearchGate or Academia.edu publication profiles, so members don't have to upload their files in multiple places). Our members collaborate on ongoing projects, and have presented at other pre-conference workshop events such as the Computer Supported Collaborative Learning conference (Williams-Pierce et al., 2017), and the APA Technology, Mind, and Society conference (Harrison et. al, 2018). Several research programs have formed to investigate the embodied nature of mathematics (e.g., Abrahamson 2014; Alibali & Nathan, 2012; Arzarello et al., 2009; De Freitas & Sinclair, 2014; Edwards, Ferrara, & Moore-Russo, 2014; Lakoff & Núñez, 2000; Melcer & Isbister, 2016; Ottmar & Landy, 2016; Radford 2009; Nathan, Walkington, Boncoddo, Pier, Williams, & Alibali, 2014; Soto-Johnson & Troup, 2014; Soto-Johnson, Hancock, & Oehrtman, 2016), demonstrating a "critical mass" of projects, findings, senior and junior investigators, and conceptual frameworks to support an ongoing community of like minded scholars within the mathematics education research community.

Some of **our collaborative accomplishments** since 2015 include:

1. Developing a group website using the Google Sites platform to connect scholars, support ongoing interactions throughout the year, and regularly adding additional resources/activities https://sites.google.com/site/emicpmena/home
2. Joint submission of several NSF grants by members who first met during the 2015 EMIC sessions
3. Some senior members joining junior members' NSF grant proposals as Co-PIs and advisors
4. Submission to the IES CASL program to study the role of action in pre-college proof performance in geometry (Funded 2016-2020 for Nathan & Walkington), as well of the use of perceptual learning on algebra learning (Ottmar & Landy, 2018)
5. Submission of 2 NSF Proposals to host Workshops on Embodied Cognition (one for researchers and one for K-16 math educators)
6. A joint symposium on Embodied Cognition with 6 members at the 2018 APA Technology, Mind & Society Conference
7. Examining the potential for an NSF Research Coordination Network (RCN)
8. Application for a grant from Association for Psychological Science (APS) to develop a better website and offer stipends for contributors
9. Presenting a pre-conference workshop to CSCL 2017 on the embodied tools to promote STEM education (Williams-Pierce et al., 2017)
10. A Conference at Berkeley's Graduate School of Education to examine the relevance of coordination dynamics -- the non-linear perspective on kinesiological development -- to individuals' sensorimotor construction of perceptual structures underlying mathematical concepts. https://edrl.berkeley.edu/cdme2018

Current Working Group Organizers

As the Working Group has matured and expanded, we have a broadening set of organizers that represent a range of institutions and theoretical perspectives (and is beyond the limit of six authors in the submission system). This, we believe, enriches the Working Group experience and the long-term viability of the scholarly community. The current organizers for 2018 are (alphabetical by first name):

- Candace Walkington, Southern Methodist University
- Carmen J. Petrick Smith, University of Vermont
- Caro Williams-Pierce, University at Albany, SUNY
- David Landy, Indiana University
- Dor Abrahamson, University of California, Berkeley
- Edward Melcer, University of California, Santa Cruz
- Emily Fyfe, Indiana University
- Erin Ottmar, Worcester Polytechnic Institute
- Hortensia Soto–Johnson, University of Northern Colorado
- Ivon Arroyo, Worcester Polytechnic Institute
- Martha W. Alibali, University of Wisconsin-Madison
- Mitchell J. Nathan, University of Wisconsin-Madison

Focal Issues in the Psychology of Mathematics Education

Emerging, yet influential, views of thinking and learning as embodied experiences have grown from several major intellectual developments in philosophy, psychology, anthropology, education, and the learning sciences that frame human communication as multimodal interaction, and human thinking as multi-modal simulation of sensory-motor activity (Clark, 2008; Hostetter & Alibali, 2008; Lave, 1988; Nathan, 2014; Varela et al., 1992; Wilson, 2002). As Stevens (2012, p. 346) argues in his introduction to the *JLS* special issue on embodiment of mathematical reasoning, "it will be hard to consign the body to the sidelines of mathematical cognition ever again if our goal is to make sense of how people make sense and take action with mathematical ideas, tools, and forms".

Four major ideas exemplify the plurality of ways that embodied cognition perspectives are relevant for the study of mathematical understanding: (1) *Grounding of abstraction in perceptuo-motor activity as one alternative to representing concepts as purely amodal, abstract, arbitrary, and self-referential symbol systems.* This conception shifts the locus of "thinking" from a central processor to a distributed web of perceptuo-motor activity situated within a physical and social setting. (2) *Cognition emerges from perceptually guided action* (Varela, Thompson, & Rosch, 1991). This tenet implies that things, including mathematical symbols and representations, are understood by the actions and practices we can perform with them, and by mentally simulating and imagining the actions and practices that underlie or constitute them. (3) *Mathematics learning is always affective*: There are no purely procedural or "neutral" forms of reasoning detached from the circulation of bodily-based feelings and interpretations surrounding our encounters with them. (4) *Mathematical ideas are conveyed using rich, multimodal forms of communication, including gestures and tangible objects in the world.*

In addition to theoretical and empirical advances, new technical advances in multi-modal and spatial analysis have allowed scholars to collect new sources of evidence and subject them to powerful analytic procedures, from which they may propose new theories of embodied mathematical cognition and learning. Growth of interest in multi-modal aspects of communication have been enabled by high quality video recording of human activity (e.g., Alibali et al., 2014; Levine & Scollon, 2004), motion capture technology (Hall, Ma, & Nemirovsky, 2014; Sinclair, 2014), developments in brain imaging (e.g., Barsalou, 2008; Gallese & Lakoff, 2005), multimodal learning analytics (Worsley & Blikstein, 2014), and data logs generated from embodied math learning technologies that interacts with touch and mouse-based interfaces (Manzo, Ottmar, & Landy, 2016).

Theme: Looking Back, Looking Ahead: Celebrating 40 Years of PME-NA

Inspired by the PME-NA 2018 theme, we will specifically focus on the ways in which the field of embodied cognition has developed and how new emerging technologies and innovative pedagogies can influence mathematics teaching and learning. This effort will be crafted to align with recent developments in the embodiment literature, and new theoretical frameworks tying various perspectives on embodiment to different forms of physicality in educational technology (Melcer & Isbister, 2016; see Figure 1 below).

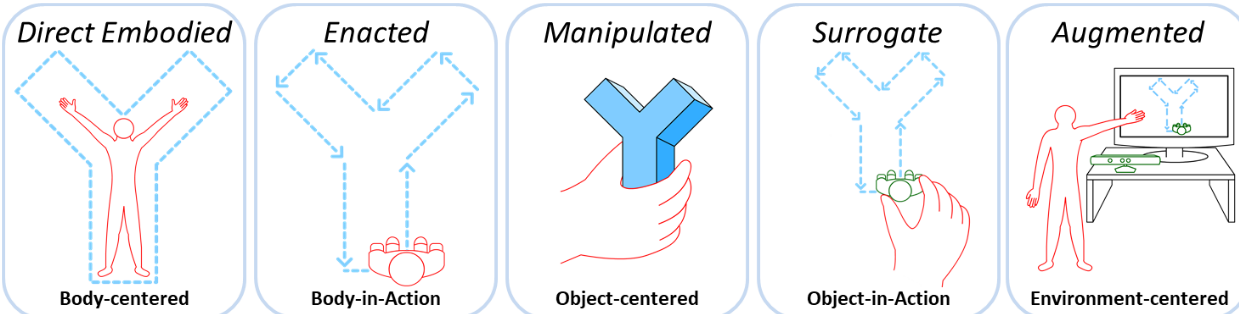

Figure 1. Five distinct approaches to facilitating embodiment through bodily action, objects, and the surrounding environment in educational technology.

Examples include: coding videos of pre-service teachers' distributed gestures to explore a mathematical conjecture (Walkington et al., 2018); exploring mathematical transformations while using a dynamic technology tool (Ottmar & Landy, 2016), having students and teachers play and create embodied technology games to teach mathematics and computational thinking (Arroyo et al., 2017; Melcer & Isbister, 2018); usng dual eye tracking (Shvarts & Abrahamson, 2018), and a teacher guiding the movements of a learner exploring ratios (Abrahamson & Sánchez-García, 2016). Through these examples, we will explore questions such as: what role does technology play on supporting the connections between the mind, body, and action? During the conference, participants in our EMIC workshop will engage in dedicated activities and guided reflections as a basis for exploring the role of technology, action, and embodiment in the emergence of mathematics learning.

Plan for Active Engagement of Participants

Our formula from prior PME-NA working groups proved to be effective: By inviting participants into open ended math activities at the beginning of each session, we were rapidly drawn into those very aspects of mathematics that we find most rewarding. We plan to facilitate collaborative EMIC activities, followed by group discussions (and we now have many activities and members who can trade off in these roles!) that will help us all to "pull back" to the theoretical and methodological issues that are central to advancing math education research. Within this structure of beginning with mathematical activities and facilitated discussions, on **Day 1** we plan to begin with four different group activities that highlight the interplay of mathematics content, cognition, physicality, and action. These activities will serve as the foundation for a broad group discussion about the varied roles of technology in EMIC. See Figure 2 below for examples of collaborative activities from PME-NA 2017.

Figure 2. Collaborative activities. Participants explore geometric rotation and reflection(left); two groups act out and prove mathematical conjectures (middle and right).

The full first session will generally be taken up by introductions and a round of open-ended activities followed by discussion. On **Day 2**, we will begin the session with technology-based collaborative activities, with four stations that pairs of participants rotate through. Examples of three of those stations are in Figure 3. Continuing with the routine established in Day 1, a full group discussion will follow, with a particular focus on designing EMIC digital contexts to support ongoing collaboration.

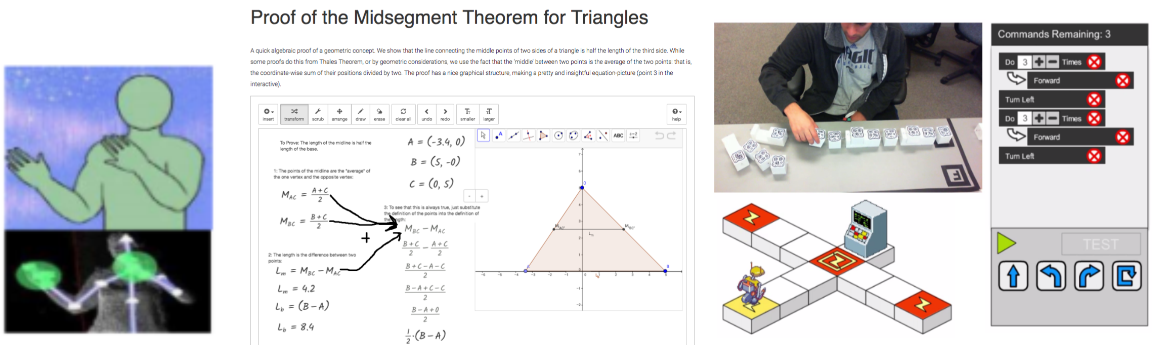

Figure 3. Technology activities. *The Hidden Village* (left); *Graspable Math* (middle); Bots & (Main)Frames (right).

After the discussion, we will discuss different EMIC activity ideas, with the goal of developing additional collaborative activities that can be used in various research and learning contexts. The final activities will be shared on the EMIC website.

Day 3 is agenda-setting day, where we all discuss how we will keep the momentum going, such as developing an NSF Research Coordination Network (RCN) to build the networked community of international scholars from which many fruitful lines of inquiry can emerge. A second group may draft a proposal for a special issue of the *Journal of Research in Mathematics Education* that focuses on creating an integrated theoretical framework or sharing the different theoretical perspectives, research activities, and operationalization of EMIC by the working group members.

In order to find common ground for the RCN submission and the JRME special issue, we will perform a live concept mapping activity that is displayed for all participants to explore the range of EMIC topics and identify common conceptual structure. We will discuss different general foci, such as teacher professional development with EMIC, designing EMIC games or museum exhibits, etc. Building on the four major ideas that we developed earlier, possible topics for organizing this activity will be explored, such as:

1. Grounding Abstractions
    a. Conceptual blending (Tunner & Fauconnier, 1995) & metaphor (Lakoff & Núñez, 2000)
    b. Perceptuo-motor grounding of abstractions (Barsalou, 2008; Glenberg, 1997; Ottmar & Landy, 2016; Landy, Allen, & Zednik, 2014)
    c. Progressive formalization (Nathan, 2012; Romberg, 2001) & concreteness fading (Fyfe, McNeil, Son, & Goldstone, 2014)
    d. Use of manipulatives (Martin & Schwartz, 2005)
2. Cognition emerges from perceptually guided action: Designing interactive learning environments for EMIC

   a. Development of spatial reasoning (Uttal et al., 2009)
   b. Math cognition through action (Abrahamson, 2014; Nathan et al., 2014)
   c. Perceptual boundedness (Bieda & Nathan, 2009)
   d. Perceptuomotor integration (Ottmar, Landy, Goldstone, & Weitnauer, 2015; Nemirovsky, Kelton, & Rhodehamel, 2013)
   e. Attentional anchors and the emergence of mathematical objects (Abrahamson & Bakker, 2016; Abrahamson & Sánchez–García, 2016; Abrahamson et al., 2016; Duijzer et al., 2017)
   f. Mathematical imagination (Nemirovsky, Kelton, & Rhodehamel, 2012)
   g. Students' integer arithmetic learning depends on their actions (Nurnberger-Haag, 2015).
3. Affective Mathematics
   a. Modal engagements (Hall & Nemirovsky, 2012; Nathan et al., 2013)
   b. Sensuous cognition (Radford, 2009)
4. Gesture and Multimodality
   a. Gesture & multimodal instruction (Alibali & Nathan 2012; Cook et al., 2008; Edwards, 2009)
   b. Bodily activity of professional mathematicians (Nemirovsky & Smith, 2013; Soto-Johnson, Hancock, & Oehrtman, 2016)
   c. Simulation of sensory-motor activity (Hostetter & Alibali, 2008; Nemirovsky & Ferrara, 2009)

We will also discuss the implications of this work and the different areas of the concept map for teaching, and discuss ideas for bridging the gap between research and practice.

Follow-up Activities

We envision an emergent process for the specific follow-up activities based on participant input and our multi-day discussions. At a minimum, we will continue to develop a list of interested participants and grant them all access to our common discussion forum and literature compilation. Those that are interested in the NSF RCN plan will work to form the international set of collaborations and articulate the intellectual topics that will knit the network together; and those that are interested in the JRME special issue proposal will outline a specific timeline for progressing. One additional set of activities we hope to explore is to create a series of instructional activities that can be used to introduce educational practitioners at all levels of administration and across the lifespan to the power and utility of the EMIC perspective.

In the past several years, we have seen a great deal of progress. This is perhaps best exemplified by coming together of the EMIC website, the ongoing collaborations between members, and the annual workshops, which each draws across multiple institutions. We thus will strive to explore ways to reach farther outside of our young group to continually make our work relevant, while also seeking to bolster and refine the theoretical underpinnings of an embodied view of mathematical thinking and teaching.

References

Abrahamson, D. (2014). The monster in the machine, or why educational technology needs embodied design. In V. Lee (Ed.), *Learning technologies and the body: Integration and implementation* (pp. 21-38). New York: Routledge (Taylor & Francis Group).

Abrahamson, D., & Bakker, A. (2016). Making sense of movement in embodied design for mathematics learning. *Cognitive Research: Principles and Implications, 1*(1), article 33. doi:10.1186/s41235-016-0034-3

Abrahamson, D., & Sánchez-García, R. (2016). Learning is moving in new ways: The ecological dynamics of mathematics education. *Journal of the Learning Sciences*, *25*, 203–239.

Abrahamson, D., Shayan, S., Bakker, A., & Van der Schaaf, M. F. (2016). Eye-tracking Piaget: Capturing the emergence of attentional anchors in the coordination of proportional motor action. *Human Development*, *58*(4-5), 218-244.

Alibali, M. W. & Nathan, M. J. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *Journal of the Learning Sciences.* (Special Issue on Embodiment in Mathematics.), *21*(2), 247-286. DOI: 10.1080/10508406.2011.611446.

Alibali, M. W., Nathan, M. J., Wolfgram, M. S., Church, R. B., Jacobs, S. A., Martinez, C. J., & Knuth, E. J. (2014). How teachers link ideas in mathematics instruction using speech and gesture: A corpus analysis. *Cognition and Instruction. 32*(1), 65-100, doi: 10.1080/07370008.2013.858161.

Arroyo, I, & Ottmar, E. (2016). *Developing Computational Thinking Through Embodied Math Games.* National Science Foundation.

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics*, *70*(2), 97-109.

Barsalou, L. W. (2008). Grounded cognition. *Annual Review of Psychology, 59*, 617-645.

Bieda, K. N. & Nathan, M. J. (2009). Representational disfluency in algebra: Evidence from student gestures and speech. *ZDM - The International Journal on Mathematics Education, 41*(5), 637- 650. [DOI 10.1007/S11858-009-0198-0.]

Clark, A. (2008). Supersizing the Mind: Embodiment, Action, and Cognitive Extension: Embodiment, Action, and Cognitive Extension. Oxford University Press, 2008.

Cook, S. W., Mitchell, Z., & Goldin-Meadow, S. (2008). Gesturing makes learning last. *Cognition, 106*(2), 1047-1058.

De Freitas, E., & Sinclair, N. (2014). Mathematics and the body: Material entanglements in the classroom. Cambridge University Press.

Duijzer, A. C. G., Shayan, S., Bakker, A., Van der Schaaf, M. F., & Abrahamson, D. (2017). Touchscreen tablets: Coordinating action and perception for mathematical cognition. *Frontiers in Psychology, 8*(144). doi:10.3389/fpsyg.2017.00144

Edwards, L. D. (2009). Gestures and conceptual integration in mathematical talk. *Educational Studies in Mathematics, 70*(2), 127-141.

Edwards, L., Ferrara, F., & Moore-Russo, D. (Eds). (2014). Emerging perspectives on gesture and embodiment in mathematics. Charlotte, N.C.: Information Age.

ESM. *Educational Studies of Mathematics* (2009): PME Special Issue: Bodily Activity and Imagination in Mathematics Learning.

Fyfe, E. R., McNeil, N. M., Son, J. Y., & Goldstone, R. L. (2014). Concreteness fading in mathematics and science instruction: a systematic review. *Educational Psychology Review*, *26*(1), 9-25.

Gallese & Lakoff, 2005)

Glenberg, A. M. (1997). What memory is for: Creating meaning in the service of action. *Behavioral and brain sciences, 20*(01), 41-50.

Glenberg, A. M. (2010). Embodiment as a unifying perspective for psychology. *Wiley Interdisciplinary Reviews: Cognitive Science, 1*(4), 586-596.

Goldstone, R. L., & Son, J. Y. (2005). The transfer of scientific principles using concrete and idealized simulations. *The Journal of the Learning Sciences*, *14*(1), 69-110.

Hall, R., & Nemirovsky, R. (2012). Introduction to the special issue: Modalities of body engagement in mathematical activity and learning. *Journal of the Learning Sciences, 21*(2), 207-215.

Hall, R., Ma, J. Y., & Nemirovsky, R. (2014). Rescaling Bodies in/as Representational Instruments in GPS Drawing. Learning Technologies and the Body: Integration and Implementation In Formal and Informal Learning Environments, 112.

Harrison A., Ottmar, E., Arroyo, I., Rosenbaum, L., Bakker, A., Abrahamson, D., Hulse, T., Manzo, D., & Landy, D. (2018, April). *Embodiment and action in mathematics games*. Symposium conducted at the American Psychological Association's conference on Technology, Mind & Society, April 5-7, 2018, Washington, DC.

Hostetter, A. B., & Alibali, M. W. (2008). Visible embodiment: Gestures as simulated action. *Psychonomic Bulletin & Review, 15*(3), 495-514.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic books.

Landy, D., Allen, C., & Zednik, C. (2014). A perceptual account of symbolic reasoning. Frontiers in psychology, 5, 275.
Lave, J. (1988). *Cognition in practice: Mind, mathematics and culture in everyday life*. Cambridge University Press.
LeVine, P., & Scollon, R. (Eds.). (2004). *Discourse and technology: Multimodal discourse analysis*. Georgetown University Press.
Ma, J. Y., Hall, R., & Leander, K. M. (2010, June). Shifting between person, structure and settlement scales in anthropological field work. In *Proceedings of the 9th International Conference of the Learning Sciences-Volume 2* (pp. 158-159). International Society of the Learning Sciences.
Martin, T., & Schwartz, D. L. (2005). Physically distributed learning: Adapting and reinterpreting physical environments in the development of fraction concepts. *Cognitive science, 29*(4), 587-625.
Melcer, E. F., & Isbister, K. (2016). Bridging the Physical Learning Divides: A Design Framework for Embodied Learning Games and Simulations. In *Proceedings of the 1st International Joint Conference of DiGRA and FDG*.
Melcer, E. F., & Isbister, K. (2018). Bots & (Main)Frames: Exploring the Impact of Tangible Blocks and Collaborative Play in an Educational Programming Game. In *Proceedings of the 2018 CHI Conference on Human Factors in Computing Systems*. DOI: 10.1145/3173574.3173840
Nathan, M. J. (2014). Grounded Mathematical Reasoning. In L. Shapiro (Ed.). *The Routledge Handbook of Embodied Cognition* (pp. 171-183). Routledge: New York.
Nathan, M. J., Srisurchan, R., Walkington, C., Wolfgram, M., Williams, C., & Alibali, M. (2013). Cohesion as a mechanism of STEM integration. *Journal of Engineering Education, 102*(1), 77-116.
Nathan, M. J., Walkington, C., Boncoddo, R., Pier, E. L., Williams, C. C., & Alibali, M. W. (2014). Actions speak louder with words: The roles of action and pedagogical language for grounding mathematical proof. *Learning and Instruction, 33*, 182-193. DOI: 10.1016/j.learninstruc.2014.07.001
Nemirovsky, R., (2007). AERA Symposium, “Mathematics Learning and Embodied Cognition” presented to the American Educational Research Association annual meeting. Chicago, IL.
Nemirovsky, R., & Smith, M. (2013). Diagram-Use and the Emergence of Mathematical Objects. In B. M. Brizuela & B. E. Gravel (Eds.), ‘*Show me what you know' Exploring representations across STEM disciplines* (pp. 143-162). New York, NY. Teachers College Press.
Nemirovsky, R., & Ferrara, F. (2009). Mathematical imagination and embodied cognition. *Educational Studies in Mathematics, 70*(2), 159-174.
Nemirovsky, R., Kelton, M. L., & Rhodehamel, B. (2012). Gesture and imagination: On the constitution and uses of phantasms. *Gesture, 12*(2), 130-165.
Nemirovsky, R., Kelton, M. L., & Rhodehamel, B. (2013). Playing mathematical instruments: Emerging perceptuomotor integration with an interactive mathematics exhibit. *Journal for Research in Mathematics Education, 44*(2), 372-415.
Núñez, R. (2004). Do real numbers really move? Language, thought, and gesture: The embodied cognitive foundations of mathematics. In *Embodied artificial intelligence* (pp. 54-73). Springer Berlin Heidelberg.
Nurnberger-Haag, J. (2015). How students’ integer arithmetic learning depends on whether they walk a path or collect chips. In Bartell, T. G., Bieda, K. N., Putnam, R.T., Bradfield, K.,& Dominguez, H.(Eds.).(pp. 165-172). Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. East Lansing, MI: Michigan State University.
Ottmar, E.R. & Landy. D. (2016). Concreteness fading of algebraic instruction: Effects on mathematics learning. *Journal of the Learning Sciences.* http://dx.doi.org/10.1080/10508406.2016.1250212
Ottmar, E. R., Landy, D., Goldstone, R. L., & Weitnauer, E. (2015). Getting from here to there: Testing the effectiveness of an interactive mathematics intervention embedding perceptual learning. *Proceedings of the Thirty-Seventh Annual Conference of the Cognitive Science Society. (pp. 1793-1798)*. Pasadena, CA: Cognitive Science Society.
Petrick Smith, Carmen, King, Barbara, & Hoyte, Jennifer (2014). Learning angles through movement: Critical actions for developing understanding in an embodied activity. *The Journal of Mathematical Behavior*, 2014; 36: 95 DOI: 10.1016/j.jmathb.2014.09.001
Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics, 70*(2), 111-126.
Romberg, T. A., (2001). Designing Middle School Mathematics Materials Using Problems Created to Help Students Progress from Informal to Formal Mathematical Reasoning. In L.P. Leutzinger, & S.P. Smith (Eds.) *Mathematics in the Middle* (pp. 107-119). Reston, VA: National Council of Teachers of Mathematics.

Shvarts, A., & Abrahamson, D. (2018, April). *Towards a complex systems model of enculturation: A dual eye-tracking study*. Paper presented at the annual conference of the American Educational Research Association (Special Interest Group: Learning Sciences), NYC, April 13-17.

Sinclair, N. (2014). Generations of research on new technologies in mathematics education. *Teaching Mathematics and its Applications, 33*(3), 166-178.

Soto-Johnson, H., Hancock, B., & Oehrtman, M. (2016). The Interplay Between Mathematicians' Conceptual and Ideational Mathematics about Continuity of Complex-Valued Functions. *International Journal of Research in Undergraduate Mathematics Education, 2*(3), 362-389.

Smith, C., King, B., González, D., Garon, T., Barr, K., Craft, J., & Scrivani, S. (April, 2016). *Patterns of collaborative actions in a body-based mathematics task.* Paper presented at the annual meeting of the American Educational Research Association, Washington, DC.

Stevens, R. (2012). The missing bodies of mathematical thinking and learning have been found. *Journal of the Learning Sciences, 21*(2), 337-346.

Tunner, M., & Fauconnier, G. (1995). Conceptual integration and formal expression. *Metaphor and Symbol, 10*(3), 183-204.

Uttal, D. H., Meadow, N. G., Hand, L. L., Lewis, A. R., Warren, C., & Newcombe, N. S. (2009). Training spatial skills: What works, for whom, and for how long. *Psychological Bulletin*.

Varela, F. J., Rosch, E., & Thompson, E. (1992). *The embodied mind: Cognitive science and human experience*. MIT press.

Varela, F. J., Thompson, E., & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience*. Cambridge, MA: M.I.T. Press.

Walkington, C., Nathan, M., Wolfgram, M., Alibali, M., & Srisurichan, R. (2014). Bridges and barriers to constructing conceptual cohesion across modalities and temporalities: Challenges of STEM integration in the precollege engineering classroom. In S. Purzer, J. Strobel, & M. Cardella (Eds.). *Engineering in pre-college Settings: Research into Practice* (pp.183-210). West Lafayette, Indiana: Purdue University Press.

Williams-Pierce, C., Walkington, C., Landy, D., Lindgren, R., Levy, S., Nathan, M.J., & Abrahamson, D. (accepted). *Enabling and understanding embodied STEM learning.* Pre-conference workshop submitted to the 2017 International Conference on Computer-Supported Collaborative Learning.

Williams-Pierce, C. (2016). Provoking mathematical play through hidden deep structures. In Looi, C. K., Polman, J. L., Cress, U., and Reimann, P. (Eds.), *Transforming Learning, Empowering Learners: The International Conference of the Learning Sciences, Vol. 2* (pp. 1241-1242). Singapore: National Institute of Education, Nanyang Technical University.

Wilson, M. (2002). Six views of embodied cognition. *Psychonomic bulletin & review, 9(*4), 625-636.

Worsley, M., & Blikstein, P. (2014). Using multimodal learning analytics to study learning mechanisms. In J. Stamper, Z. Pardos, M. Mavrikis, & B. M. McLaren (Eds.), *Proceedings of the 7th International Conference on Educational Data Mining* (pp. 431-432). London, UK: Institute of Education.

WORKING GROUP ON GENDER AND SEXUALITY IN MATHEMATICS EDUCATION: LINKING THE PAST TO THE FUTURE

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After not having met for a number of years, the Gender and Sexuality in Mathematics Education Working Group will meet again this year with a new name, new members, and a new focus. During this Working Group, we will discuss current research being conducted on these topics and identify themes for discussion and possible research collaborations. We plan to include a discussion about how research assumptions, such as language assuming that gender is binary, can affect research findings. All PME-NA attendees with an interest in gender and sexuality in mathematics education are invited to attend.

Keywords: Gender; Equity and Diversity; Affect, Emotion, Beliefs, and Attitudes

Introduction

The Gender and Sexuality in Mathematics Education Working Group will reconvene this year with a new focus and a new name. After not having met for many years, there is a renewed interest in this topic and new leadership for this Working Group. During last year's PME-NA, in Indianapolis, a group of scholars met to discuss their interests in gender and sexuality in mathematics education. As a result of those October 2017 meetings, we set up a Google Docs shared folder so that we could share relevant research and other resources. At that time, we also agreed that having a formal Working Group at PME-NA in 2018 would allow the work of this group to coalesce and would be a productive way to invite interested others to join us. We have also chosen to rename the Working Group to include the word "sexuality" in order to be more inclusive and to remain current with the expanding themes within this area of research.

In this proposal, we share a brief history of the past work of this group and discuss connections of our work to the psychology of mathematics education and to the theme of this year's conference. We also discuss some relevant research being conducted by current members of the group and discuss initial themes noted within this work. Finally, we describe our plans for the active engagement of participants during the conference and conclude with a discussion of anticipated follow-up activities and future directions for the Working Group.

Brief History of the Working Group

The Gender and Mathematics Working Group (GMWG) began 20 years ago at PME-NA XX in Raleigh, North Carolina. At this conference, mathematics education scholars came together "in order to weave together the findings of various strands in research and understanding of issues in gender and mathematics" (Damarin et al., 1998). This Working Group continued to meet between 1998 and 2007, with the exception of the 2003 joint conference between PME and PME-NA in Honolulu, Hawaii. The group also reconvened for the 2011 PME-NA conference in Reno, Nevada.

The Working Group's past discussions have involved reviewing the scholarship surrounding gender and mathematics, defining research strands, determining gaps in the literature, and establishing directions for future work. Some notable accomplishments of this group have been creating a visual representation of the field of gender and mathematics and the social and psychological complexities of the topic (Erchick, Condron, & Appelbaum, 2000) and publishing a monograph on gender and mathematics research as a joint project with scholars from the International Group for PME (Forgasz, Becker, Lee, & Steinthorsdottir, 2010).

Connections to PME and the Conference Theme

The conference theme this year, "Looking Back, Looking Ahead," nicely ties in with the resurgence of this Working Group. Research on gender and sexuality in mathematics education has drastically shifted focus throughout the past few decades. In the 1970s and 1980s, much of the work on gender and mathematics focused on sex differences in mathematics achievement, often with a focus on biological differences. Through the years, given that sex differences in achievement have been found to be relatively small and have decreased over time (Hyde, Fennema, & Lamon, 1990), gender research has shifted its focus more to participation, with an emphasis on social/cultural influences, as opposed to biological ones. In other words, the research questions tend to center around why girls/women are not pursuing mathematics or mathematical careers at the same rate as boys/men and what can be done to change this imbalance. This work has revolved around many topics, such as agency and voice, self-efficacy and self-concept, stereotypes and stereotype threat, curriculum and pedagogy, and differential treatment and institutionalized discrimination.

In more recent years, the field has also begun discussing the need for more research on queer, transgender, and gender non-conforming students. Recent researchers have noted that work done on gender and mathematics tends to utilize binary wording – dividing participants into males/boys/men and females/girls/women, without clearly defining these categories and without allowing for students who might not identify with one of these categories (Damarin & Erchick, 2010; Glasser & Smith, 2008; Hall, 2014). In looking ahead, it is important for researchers in this field to develop and define appropriate and inclusive terminology and research methodologies that can address issues that arise when conducting gender and sexuality research.

Current Research in Gender and Sexuality in Mathematics Education

To provide more details on the type of work currently being done in the field with respect to gender and sexuality in mathematics education, the following is a short summary of each author's recent work in this area. More detailed discussions and connections across this work will take place during the Working Group sessions.

Laurie Rubel, Brooklyn College of the City University of New York/University of Haifa

Laurie Rubel's current research focuses on the educational experiences of women Palestinian citizens of Israel (Arab Israelis, who represent 25% of the population of Israel). Arab Israelis are a minority group with limited access to opportunity and resources and lower socioeconomic standing in Israel (Bar-Tal & Teichmann, 2005; Zuzovsky, 2010). Arab Israeli girls outperform boys in mathematics (Rapp, 2015) and have been found to have beliefs about success in mathematics different from Western or Westernized peers (Forgasz & Mittelberg, 2008). It has been hypothesized that girls appropriate academic success to overcome the lower status that they hold as a minority in Israel as Palestinians, which is further exacerbated by their role as women in a patriarchal society (Rapp, 2015), and as an unintended consequence of less diversity among course offerings in Arab high schools (Ayalon, 2002).

Rubel's research methodology consists of life-story interviews with Arab Israeli women in undergraduate and graduate programs in mathematics education, from diverse Arab ethnic groups and geographies, in which they are asked how they understand and interpret the effects of local racialized narratives in relation to learning, how they interpret their achievement and others' lack of achievement, and which factors they identify as supporting or inhibiting their success. Analysis will pursue the educational experiences of these women, as told in their own stories and counterstories, with particular attention to how these stories converge or contrast with Western narratives. The commentary will include analysis of researcher positionality (Milner, 2007) and will discuss methodological tensions inherent to research about equity in mathematics in relation to social groups known to espouse conservative, patriarchal, and homophobic ideologies.

Lynda Wiest, University of Nevada, Reno

Lynda Wiest's scholarly interests involve understanding factors that influence gender differences in mathematics, including those that relate to dispositions and beliefs, and strategies and opportunities for supporting and encouraging females in mathematics. Wiest has particularly focused on the role that out-of-school-time (OST) learning can play in this regard because it is an area of rising scholarly interest in education, especially STEM education (e.g., McCombs et al., 2012; Slates, Alexander, Entwisle, & Olson, 2012). Wiest developed and has directed a residential summer math program for middle school girls for 20 years, conducting research in association with the program and recently publishing the first book of its kind on OST STEM programs for females (Wiest, Sanchez, & Crawford-Ferre, 2017).

In researching factors that participants consider most important in a one-week residential summer mathematics and technology program for middle school girls, Wiest and colleagues (2017) have determined that the girls find the academic learning most important, indicating that they can willingly engage in rigorous mathematics even outside of school. Although social opportunities (sometimes embedded in recreational activities) were reported as a distant second among the most important program features, the academic and social opportunities interacted in important ways in the context of a mathematics experience to influence dispositions and beliefs (Lavy & Sand, 2012; Szucs, 2014). In terms of most effective instructional approaches, the girls most prominently listed active engagement (e.g., hands-on, collaborative, and communicative learning) and a high-quality staff (based on individual attention given and perceived effectiveness).

Currently, Wiest is completing a study on the influence of the summer program on motivation (internal and external). She and her co-researchers concur with existing findings that dispositions can influence STEM achievement and participation (Ellis, Fosdick, & Rasmussen, 2016; Goetz, Bieg, Lüdtke, Pekrun, & Hall, 2013). They find the fact that females grapple with less favorable dispositions toward STEM than males, such as weaker self-beliefs and confidence as well as lower interest (Ellis et al., 2016; Lubienski, Robinson, Crane, & Ganley, 2013), problematic and thus investigate ways OST programs might favorably influence females' dispositions in mathematics overall and in relation to race/ethnicity and socioeconomic status.

Rebecca McGraw, University of Arizona

Rebecca McGraw's work related to equity, gender, and mathematics includes co-facilitating a residential summer math camp for middle grades girls (led by another co-author of this proposal), studying teacher classroom practice from an equity perspective (e.g., distribution of images by gender in classroom resources, patterns of participation, course-taking, and achievement), and investigating pre-service secondary teacher preparation (Bay-Williams & McGraw, 2008; Eli, McGraw, Anhalt, & Civil, in press; McGraw & Lubienski, 2007; McGraw, Romero, & Krueger, 2009; Rubinstein-Avila et al., 2014). Currently, McGraw is particularly interested in the development of teacher and student beliefs about mathematics and learning, and the development of middle/high school students' mathematical identities.

Very recently, McGraw began serving as PI on a multi-institutional NSF research project (#1758401) focusing on the development of equitable teacher practices in middle/high school pre-service and beginning teachers. The study is framed by the theories of equity literacy (Gorski, 2014; Gorski & Swalwell, 2015) and culturally relevant teaching (Gay, 2002), and the researchers seek to understand the ways that teacher education programs can productively engage future teachers in identifying and confronting bias and discrimination (as expressed by oneself, others, and institutions), rejecting deficit views and fixed-ability mindsets, and engaging with and building upon the resources that students bring to the classroom. This project will include five teacher education programs from across the U.S. and approximately 100 future/beginning mathematics teachers. The Working Group will provide an important venue for the bidirectional sharing of ideas – from project research to working group members, and from the varied expertise of group members back to the project.

Angela Hodge, Northern Arizona University

Angela Hodge's research interests lie at the intersection of active learning (Ernst, Hodge, & Yoshinobu, 2017) and gender equity in the STEM disciplines. Most recently, Hodge conducted research on a four-week summer camp for underrepresented middle school girls (Hodge, Matthews, & Squires, 2017). The camp was based on empowering the young women by engaging them in hands-on, minds-on activities in STEM. The girls participated in physical activities each day as well as explored various STEM careers. Hodge was one of the co-founders of the camp, which has now been active for six years. During the Working Group, Hodge will offer insights gained by research on starting a summer camp for girls, maintaining the camp, and selecting activities that are both engaging and empowering for young women. Future research ideas for the camp can also be explored during the Working Group.

In addition, Hodge conducted a research study on why some women choose STEM fields and what has made them successful in such majors (Weber & Hodge, 2012). It was found that childhood experiences greatly influenced these women in choosing a STEM major. Support groups and mentors helped influence their choices to stay in the STEM major, even when their coursework became challenging.

One of the current research projects in which Hodge is involved relates to perceptions of women related to their mathematics history. Data were collected in university calculus classes (a course with few women enrolled, but all students enrolled are in STEM disciplines) and in a math for elementary education course (a course mostly populated by women not in STEM). The data first being analyzed come from a pre-course writing assignment called a math autobiography. The students in both classes were asked the same questions/prompts. Women's perceptions of themselves as students of mathematics will be explored with both of these groups of students. This project will be further discussed during the Working Group.

Kathy Stoehr, Santa Clara University

Kathy Stoehr's research focuses on preservice and early-career teachers' experiences with mathematics anxiety. She utilizes different theoretical lenses to add a deeper understanding of how and why mathematics anxiety is so prevalent among women (Penafiel, Stoehr, & Martinez, 2016; Stoehr & Olson, 2015; Stoehr, 2017a, 2017b). Stoehr argues that this gender issue is important to study so that teacher educators may address this issue and be better equipped to prepare competent and confident women to teach elementary mathematics.

Stoehr is also a co-researcher on two large grants. As Co-Principal Investigator on a two-year research project with Dr. Marta Civil from the University of Arizona, she is working with in-service women elementary teachers and mothers from an underserved community to focus on their understanding of each other's roles in supporting children's mathematics education. In addition, she works with the teachers and mothers to develop deeper confidence and competence in mathematics.

For the other grant, Stoehr is a co-researcher on a five-year U.S. Department of Education Grant with Co-Principal Investigators Marco Bravo and Claudia Rodriguez-Mojica. The aim of the grant is to enhance teacher preparation for dual-language environments. Stoehr's responsibilities include co-designing two elementary mathematics courses for bilingual preservice elementary teachers (mainly women) that focus on the role language plays in learning mathematics. She also develops and facilitates workshops for parents (mainly mothers) of bilingual students on ways to promote children's co-development of mathematical knowledge and bilingual language proficiency at home.

Katrina Piatek-Jimenez, Central Michigan University

Katrina Piatek-Jimenez's research interests focus on what motivates women to study mathematics at the undergraduate level and what influences their decisions whether or not to continue in mathematical careers. In order to begin addressing these questions, Piatek-Jimenez has explored many potential factors, such as the development and role of one's mathematics identity (Cribbs, Piatek-Jimenez, & Mantone, 2015; Piatek-Jimenez, 2015), images of mathematicians (Piatek-Jimenez, 2008a), knowledge of mathematical careers (Piatek-Jimenez, 2008b), and equity within mathematics textbooks (Piatek-Jimenez, Madison, & Pzybyla-Kuchek, 2014).

In her more recent work, Piatek-Jimenez has been focusing on the role of stereotypes held by college students. In one study, Piatek-Jimenez explored whether college students believe that society sees particular personality attributes as gendered. Findings suggest that personality attributes often associated with the field of mathematics (such as "thinks logically" and "analytical") were seen as gender neutral by participants, but those associated with caring for others (such as "shows concern for people's well-being" and "puts others' needs above one's own") were ranked as female dominant. Furthermore, women planning on entering STEM careers rated these caring attributes as female dominant even more so than the men in the study or the women who were not planning on STEM careers. Piatek-Jimenez and colleagues (Cribbs, Piatek-Jimenez, & Gill, in press) raise concerns about the influence that these pressures may have on women interested in STEM fields, especially since mathematically-based careers are often viewed as being less people- or service-oriented than many other careers (Morgan, Isaac, & Sansone, 2001).

In another recent study, Piatek-Jimenez explored college students' stereotypes of mathematicians using two different data collection methods. One method was the "Draw a Mathematician Test." The second involved focus groups in which students were asked to consider photos of real people and determine whether or not they thought each was of a mathematician. Initial results suggest that the distinct methods employed allowed the researchers to identify different stereotypes and beliefs of the participants. As such, Piatek-Jimenez cautions how choice of methodology in gender research may affect results.

Jennifer Hall, Monash University

Jennifer Hall's interest in gender research in mathematics education began when she was a master's student. In her master's research, she collected data on the experiences of women who were upper-year undergraduate mathematics majors, investigating the supports and challenges they faced as they selected and persevered in this field of study, which remains dominated by men in Canada, where the research was conducted (Hall, 2010).

During her doctoral studies, Hall's research focused on elementary students' experiences with and views of mathematics and mathematicians, and the ways that their views are impacted by parents' views, teachers' views, and popular media representations. Throughout her doctoral journey, Hall became increasingly aware of how to conduct research in a gender-sensitive manner, and ensured that all aspects of her research, from conception to presentation of findings, were conducted in such a manner (Hall, 2014).

More recently, Hall was part of an international research project led by Helen Forgasz and Gilah Leder. This project involved asking the general public about their views on gender and mathematics, as well as related fields, via street-level interviews. While reporting on her findings from the Canadian dataset (Hall, 2017), she raised concerns about the binary wording ("girls or boys") of the survey questions. As a result, she designed a more recent project that involved replicating the aforementioned project, but with an alteration of the data collection instrument (1) to make all the questions non-binary (e.g., "For which gender…") and (2) to add questions that explicitly queried the participants' views on gender. Data collection took place in Canada and Australia, and initial analyses are currently underway.

Hall has also begun a new project in which she and two colleagues will investigate the supports and challenges faced by undergraduate students in two Australian universities. This project will also have a gendered focus, as the researchers will investigate the ways that gender impacts students' experiences.

Elizabeth Kersey, Purdue University

Elizabeth Kersey also notes concerns about utilizing binary wording in gender research. She expresses concern about dividing participants into males and females without clearly defining those categories and without acknowledging transgender, nonbinary, and intersex identities. Kersey is working to address this oversight by using a narrative methodology to examine how gender affects the experiences of transgender students in mathematics and other STEM fields. She draws from research and theory in (a) feminism, (b) queer theory and intersectionality, (c) general education research on transgender students, (d) science education research on LGBTQ issues or queer theory, and (e) research on gender and equity in mathematics education. Drawing from both feminism and queer theory facilitates a dual focus on gender category privilege/oppression and on gender conformity privilege/gender transgression oppression, as suggested by Rands (2009).

The transgender participants in her recent study were more attuned to differences in gendered treatment both before and after transitioning, from which we can learn about how mathematics and other STEM fields are gendered for all students, not just those who are gender variant. Participants in this study regularly described the culture of their fields as masculine. One exception is mathematics education, which was described as much less masculine than mathematics. Participants also reported being taken less seriously when they transitioned from male to female. For instance, they might make a suggestion that would be ignored, only to have the same suggestion be taken up when suggested by a male colleague. For those participants who were transgender women, however, the improvement in their quality of life when their presentation corresponded with their identity far outweighed any disadvantages they faced when presenting as female. All participants in this study were either transgender women or nonbinary; more research is required that includes transgender men.

One strong pattern that emerged in this study is that all participants waited until during or after college to transition. While this is not true of everyone who is transgender, it is a common phenomenon. Based on this observation, Kersey proposes a model of gender refraction. In this model, gender often appears to be a binary because conservative societal norms enforce that gender binary. However, when one experiences an open and accepting environment where it is safe to explore one's identity, this environment acts as a prism and reveals gender to be a spectrum with a multitude of possible identities. College often acts as this prism, but this is not always the case, and different people experience the same environment in different ways. The most important aspect of this model is that the spectrum of identities was always present, but it was not visible until the conditions were favorable. Kersey argues that our goal as a society and as mathematics educators should be to create more environments that can act as a prism and allow our students to explore their identities in an environment of safety and acceptance.

Initial Themes

The study of gender and sexuality in mathematics education is a complex field and is approached from many angles. During the Working Group sessions, we plan to discuss in depth themes and commonalities found in our work and to identify ways that we can continue to support each other's work and move the field forward.

Based on an initial review of our respective work, some preliminary themes begin to arise. One important theme is language and methodology, which involves not only the choice of language that we as scholars use when conducting research on gender and sexuality in order to be inclusive of all individuals and to reflect a multi-dimensional understanding of gender and sexuality, but also a reflection on how our choice of language and our choice of methods may affect our research results and interpretations. Another theme found in our research is interactions between gender/sexuality and students' perceptions of themselves as mathematical learners. While some of our work focuses on mathematics autobiographies and mathematics identity and other work specifically explores math anxiety, these are all topics that consider the learners' experiences related to gender/sexuality and mathematics. A third theme found in our work is the role of curriculum, pedagogy, and teacher education in the study of gender and sexuality in mathematics education. From the role of classroom materials to that of teacher practices, the experiences and messages to which students are exposed within the mathematics classroom affects their achievement and participation. The final theme involves summer camps for middle school girls intended to engage and empower them in the pursuit of mathematics.

Plans for Active Engagement During the Conference

Session 1:

Because this Working Group has not met for many years, and since the group is composed of mostly new leadership and new participants, it is important for the participants to become familiar with one another's work. Therefore, we will begin this session with introductions and a brief overview of the plan for this Working Group at the conference. Each author of the proposal will then present a short synopsis of their research relevant to the topic of gender and sexuality in mathematics education, each followed by a brief discussion with the group at large. Finally, as time permits, other participants will be invited to introduce their related research to the group, continuing into the next session as needed.

Session 2:

We will begin this session with a discussion of language and methodology in gender and sexuality research. Given that the language used within the field changes so quickly, and that certain fields, such as sociology, have worked at refining this language, we consider it important to discuss and develop some consistent language to be used within the group. During this session, we will also facilitate a broader discussion that focuses on connections identified between one another's work, and directions the participants see the field of gender and sexuality in mathematics education moving. Through these conversations, we anticipate that themes will arise that will naturally allow us to divide into smaller, theme-based groups for the third session.

Session 3:

During the third session, breakout groups organized the previous day will meet and discuss potential collaborations and other ways in which they can support each other's work on gender and sexuality in mathematics education.

Anticipated Follow-up Activities and Future Directions

After last year's PME-NA conference, we created a Google Docs shared folder, which has been used to share related articles and information about conferences and funding opportunities. We plan to add all interested attendees of this year's Working Group to the Google Docs shared folder and intend to continue the discussions and shared resources begun during the Working Group throughout the rest of the year.

Also, given that it has been nearly a decade since a monograph has been published on work in gender and sexuality in mathematics education, and because the field changes so quickly, we will begin thinking about putting together an edited volume with current work being done in the field.

References

Ayalon, H. (2002). Mathematics and science course taking among Arab students in Israel: A case of unexpected gender equality. *Educational Evaluation and Policy Analysis*, 24(1), 63–80.

Bar-Tal, D., & Teichmann, Y. (2005). *Stereotypes and prejudice in conflict: Representations of Arabs in Israeli-Jewish society*. Cambridge, England: Cambridge University Press.

Bay-Williams, J., & McGraw, R. (2008). Issues of implementation: Making mathematics accessible to all learners. In M. Meyer, C. Langrall, F. Arbaugh, D. Webb, & M. Hoover (Eds.), *A decade of middle school mathematics curriculum implementation: Lessons learned from the Show-Me Project* (pp. 125-137). Charlotte, NC: Information Age.

Cribbs, J., Piatek-Jimenez, K., & Mantone, J. (2015). The relationship between mathematics identity and personality attributes with students' career goals. In T. G. Bartell, K. N. Bieda, R. T. Putnam, K. Bradfield, & H. Dominguez (Eds.), *Proceedings of the 37th Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 502-509). Eugene, OR: All Academic.

Damarin, S. K., & Erchick, D. B. (2010). Toward clarifying the meanings of gender in mathematics education research. Journal for Research in Mathematics Education, 41(4), 310-323. Retrieved from http://www.nctm.org/publications/article.aspx?id=27738

Damarin, S. K., Erchick, D. B., Buerk, D., Condron, L., Confrey, J., Cossey, R., …Brosnan, P. (1998). Gender and mathematics: Integrating research strands. In S. Berenson, K. Dawkins, M. Blanton, W. Coulombe, J. Kolb, K. Norwood, & L. Stiff (Eds.), *Proceedings of the 20th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 77-78). Raleigh, NC: ERIC.

Eli, J. A., McGraw, R. H., Anhalt, C. O., & Civil, M. (in press). *Stronger together: The AZ mathematics teaching (MaTh) Noyce program's collaborative model for secondary teacher preparation.*

Ellis, J., Fosdick, B. K., & Rasmussen, C. (2016). Women 1.5 times more likely to leave STEM pipeline after calculus compared to men: Lack of mathematical confidence a potential factor. *PLOS One, 11*(7), 1-14. doi:10.1371/journal.pone.0157447

Erchick, D. B., Condron, L., & Appelbaum, P. (2000). Gender and mathematics working group: Emergent themes. In M. Fernandez (Ed.), *Proceedings of the 22nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 53-57). Tucson, AZ: ERIC.

Ernst, D. C., Hodge, A., & Yoshinobu, S. (2017). Doceamus: What Is Inquiry-Based Learning? *Notices of the AMS,* 64(6), 570-574.

Forgasz, H. J., Becker, J. R., Lee, K., & Steinthorsdottir, O. (Eds.). (2010). *International perspectives on gender and mathematics education*. Charlotte, NC: Information Age.

Forgasz, H., & Mittelberg, D. (2008). Israeli Jewish and Arab students' gendering of mathematics. *ZDM Mathematics Education, 40*(4), 545-558.

Gay, G. (2002). Preparing for culturally responsive teaching. *Journal of Teacher Education*, *53*(2), 106-116.

Glasser, H. M., & Smith, J. P. III (2008). On the vague meaning of "gender" in education research: The problem, its sources, and recommendations for practice. Educational Researcher, 37(6), 343-350. doi:10.3102/0013189X08323718

Goetz, T., Bieg, M., Lüdtke, O., Pekrun, R., & Hall, N. C. (2013). Do girls really experience more anxiety in mathematics? *Psychological Science, 24*(10), 2079-2087. doi:10.1177/0956797613486989

Gorski, P. C. (2014). *Reaching and teaching students in poverty: Strategies for erasing the opportunity gap*. New York: Teachers College Press.

Gorski, P. C., & Swalwell, K. (2015). Equity literacy for all. *Educational Leadership*, *7*(6), 34-40.

Hall, J. (2010). The influence of high school and university experiences on women's pursuit of undergraduate mathematics degrees in Canada. In H. Forgasz, J. R. Becker, K. Lee, & O. B. Steinthorsdottir (Eds.), *International perspectives on gender and mathematics education* (pp. 365-390). Charlotte, NC: Information Age.

Hall, J. (2014). Unpacking "gender issues" research. *Philosophy of Mathematics Education Journal, 28*, 1-10. Retrieved from http://people.exeter.ac.uk/PErnest/pome28/index.html

Hall, J. (2017). Gendered? Gender-neutral? Views of gender and mathematics held by the Canadian general public. In L. Jao & N. Radakovic (Eds.), *Transdisciplinarity in mathematics education: Blurring disciplinary boundaries* (pp. 193-212). Cham, Switzerland: Springer International.

Hodge, A., Matthews, M., & Squires, A. (2017). Hands-on, minds-on STEM for at-risk middle school girls. In L. R. Wiest, J. E. Sanchez, & H. G. Crawford-Ferre (Eds.), *Out-of-school-time STEM programs for females: Implications for education research and practice, Volume I: Longer-Term Programs* (Vol. 1, pp. 81-100). Charlotte, NC: Information Age Publishing.

Hyde, J. S., Fennema, E., & Lamon, S. J. (1990). Gender differences in mathematics performance: A meta-analysis. *Psychological Bulletin*, *107*(2), 139-155.

Lavy, V., & Sand, E. (2012). *The friends factor: How students' social networks affect their academic achievement and well-being?* Cambridge, MA: National Bureau of Economic Research. Retrieved from http://www.nber.org/papers/w18430

Lubienski, S. T., Robinson, J. P., Crane, C. C., & Ganley, C. M. (2013). Girls' and boys' mathematics achievement, affect, and experiences: Findings from ECLS-K. *Journal for Research in Mathematics Education*, *44*(4), 634-645.

McCombs, J. S., Augustine, C., Schwartz, H., Bodilly, S., McInnis, B., Lichter, D., & Cross, A. B. (2012). Making summer count: How summer programs can boost children's learning. *The Education Digest*, *77*(6), 47-52.

McGraw, R., & Lubienski, S. T. (2007). NAEP findings related to gender: Achievement, student affect, and learning experiences. In P. Kloosterman & F. K. Lester, Jr. (Eds.), *Results and interpretations of the 2003 mathematics assessment of the National Assessment of Educational Progress* (pp. 261-287). Reston, VA: National Council of Teaches of Mathematics.

McGraw, R., Romero, D., & Krueger, R. (2009). Facilitating whole-class discussions in diverse classrooms: Strategies for engaging all students. In A. Flores (Ed.), *Mathematics for every student: Responding to diversity, grades 9-12* (pp. 17-27). Reston, VA: National Council of Teachers of Mathematics.

Milner, R. (2007). Race, culture, and researcher positionality: working through dangers seen, unseen, and unforeseen. *Educational Researcher 36*(7), 388-400.

Morgan, C., Isaac, J. D., & Sansone, C. (2001). The role of interest in understanding the career choices of female and male college students. *Sex Roles*, *44*(5-6), 295-320.

Penafiel, B., Stoehr, K., & Martinez, S. (2016). In-service teachers narrative experiences of mathematics anxiety. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 395–401). Tucson, AZ: The University of Arizona.

Piatek-Jimenez, K. (2008a). Images of mathematicians: A new perspective on the shortage of women in mathematical careers. *ZDM – The International Journal on Mathematics Education*, 40(4), 633-646.

Piatek-Jimenez, K. (2008b). "So I've chosen to major in math. Now what?": Mathematics students' knowledge of future career options. In *Proceedings of the 11th Conference on Research in Undergraduate Mathematics Education*. Retrieved from http://sigmaa.maa.org/rume/crume2008/Proceedings/Proceedings.html.

Piatek-Jimenez, K., Madison, M., & Pzybyla-Kuchek, J. (2014). Equity in mathematics textbooks: A new look at an old issue, *Journal of Women and Minorities in Science and Engineering*. 20(1), 55-74.

Piatek-Jimenez, K. (2015). On the persistence and attrition of women in mathematics, *Journal of Humanistic Mathematics*, 5(1), 3-54.

Piatek-Jimenez, K., Cribbs, J., & Gill, N. (in press). College students' perceptions of gender stereotypes: Making connections to the underrepresentation of women in STEM fields. *International Journal of Science Education.* doi: 10.1080/09500693.2018.1482027

Rands, K. E. (2009). Considering transgender people in education: A gender-complex approach. *Journal of Teacher Education*, 60(4), 419–431.

Rapp, J. (2015). *Gender gaps in mathematics and language in Israel: What can be learned from the Israeli case?* Retrieved from cms.education.gov.il/EducationCMS/Units/Rama/MaagareyYeda/

Rubinstein-Avila, E., Sox, A., Kaplan, S., & McGraw, R. (2014). Does biliteracy + mathematical discourse = binumeracy? A close look at a dual-language middle-school mathematics classroom. dual-language mathematics classroom. *Urban Education, 58*(8), 899-937.

Slates, S. L., Alexander, K. L., Entwisle, D. R., & Olson, L. S. (2012). Counteracting summer slide: Social capital resources within socioeconomically disadvantaged families. *Journal of Education for Students Placed at Risk*, *17*(3), 165-185.

Stoehr, K. (2017a). Building the wall brick by brick: One woman prospective teacher's experiences with mathematics anxiety. *Journal of Mathematics Teacher Education. 20*(2), 119–139. doi:10.1007/s10857-015-9322-y.

Stoehr, K. (2017b). Mathematics anxiety: One size does not fit all. *Journal of Teacher Education, 68*(1), 69–84. doi: 10.1177/0022487116676316

Stoehr, K., & Olson, A. (2015). Can I teach mathematics? A study of preservice teachers' self-efficacy and mathematics anxiety. In T. G. Bartell, K. N. Bieda, R. T. Putnam, K. Bradfield, & H. Dominguez (Eds.), *Proceedings of the 37th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 948–951). East Lansing, MI: Michigan State University.

Szucs, L. (2014). Impact of positive school connectedness on adolescent mental health and academic performance. *College Stations, TX: Texas A&M University*, 1207.

Weber, C., & Hodge, A. (2012). Navigating the gendered math path understanding women's experiences in university mathematics classes. *International Review of Qualitative Research, 5*(2), 153-174.

Wiest, L. R., & Crawford-Ferre, H. G. (2017). *Influence of a summer mathematics program on middle school girls' perceptions of mathematics as a gendered domain.* Manuscript submitted for publication.

Wiest, L. R., Sanchez, J. E., & Crawford-Ferre, H. G. (2017). *Out-of-school-time STEM programs for females: Implications for education research and practice.* Charlotte, NC: Information Age Publishing.

Zuzovsky, R. (2010). The impact of socioeconomic versus linguistic factors on achievement gaps between Hebrew speaking and Arabic-speaking students in Israel in reading literacy and in mathematics and science achievements. *Studies in Educational Evaluation, 36*(4), 153-161.

MATHEMATICS TEACHER EDUCATORS' INQUIRY INTO THEIR PRACTICE: UNPACKING METHODOLOGIES FOR PROFESSIONAL AND PERSONAL GROWTH

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The field of mathematics education is embracing a diversity of research paradigms, which have allowed mathematics teacher educators (MTEs) to research their academic experiences. This has begun to unearth the diversity and complexity of MTEs' work. Yet, there are few spaces for MTEs to explore and reflect on those experiences. In this working group, we introduce narrative inquiry, self-study, and autoethnography as methodologies that focus on the study of the self. Through posing research questions and creation of research texts from existing field texts, participants will explore these methodologies. Field texts used will be videos, personal journals, and transcripts of conversations that the leaders of the working group have constructed in their different research studies. The creation of a nurturing space to conduct and review research using these methodologies will be promoted in this working group.

Keywords: Mathematics Teacher Educators, Research Methods, Narrative Inquiry, Self-study, Autoethnography

Significance and Historical Context of the Working Group

PME-NA has historically created spaces for mathematics teacher educators' (MTEs') professional growth (e.g., Bastable & Shifter, 1997; Kastberg et al., 2012). As the mathematics education field starts refuting closure and embracing a diversity of possibilities for mathematics teaching and learning (Stinson & Walshaw, 2017), MTEs' sharing of their experiences has disclosed the complexity of their academic work (e.g., Chauvot, 2009; Jaworski & Wood, 2008; Tzur, 2001). For example, Chauvot (2009), building on Shulman (1986), documented the nature of knowledge she used in her different roles as MTE. Tzur (2001), building on his personal experiences as a mathematics learner and becoming an MTE, documented the importance of promoting MTEs' appreciation for reflecting about their practice. The first International Handbook of Mathematics Teacher Education (vol. 4) was dedicated to the work of MTEs as professionals (Jaworski & Wood, 2008). In 2013, the Journal for Research in Mathematics Education Equity Special Issue documented a conversation among MTEs where they shared their experiences, struggles, questions, and tensions about how they position themselves in their work (D' Ambrosio et al., 2013). Back in 2009, Gutierrez claimed that politics is always a present aspect of MTEs' work whether issues of power are acknowledged or avoided.

As research is one important part of MTEs' work, we acknowledge that there seems to be a consensus that the main goal of doing research is to develop new knowledge or understand a phenomenon (Lincoln, Lynham, & Guba, 2011). However, the kind of research that is legitimate and what knowledge is valued varies according to external contexts and overarching paradigms held by the researcher. MTEs' personal worldviews or paradigms inform the decisions we make in designing and implementing research. Teppo (1997) defined paradigms as "overarching worldviews that represent particular belief systems about the nature of knowledge and how that knowledge is acquired" (p. 3). Ernest (1997) explained that each research paradigm has its own expectations about the nature of reality (ontology), knowledge and how do we know (epistemology), and the best way to gain that knowledge (methodology). To build perspective on a research paradigm, Ernest (1997) advised researchers to consider their personal mission for knowledge, whether they want "to predict and control the phenomena under study (technical interest), to understand and make sense of them (the practical interest), and to achieve social justice through this understanding (the emancipatory interest)" (p. 33). All of these are valuable missions that will determine the methodology of a study and contribute to making the field of mathematics teacher education rich in diversity of researchers and research approaches (e.g., Stinson & Walshaw, 2017).

At the same time, MTEs' daily work may be influenced by external factors, such as funding and/or peer acceptance, as they try to decide what research questions to pursue and select methods for exploring these questions. Negotiating external factors and internal passions to conduct inquiry may constrain MTEs' work. Historically, political organizations in the education community have privileged the practice of scientific methodologies in research (e.g., National Mathematics Advisory Panel, 2008; National Research Council, 2002). However, recently researchers in general teacher education (e.g., Borko, Liston, & Whitcomb, 2007; Lather, 2004), and particularly in mathematics education (e.g., Gutstein, 2008; Stinson & Walshaw, 2017), have called for more diversity in research. In this present moment there is opportunity for MTEs to develop spaces according to their passions and explorations of their identities as writers and researchers. Such as work would fulfill the emancipatory interests described by Ernest (1997), while develop new insights into MTEs' practices.

Energizing MTEs has been identified as critical to sustaining the field of mathematics teacher education (e.g., Whitcomb, Liston, & Borko, 2009). As Wilson (2006) described, being a "teacher educator-researcher requires understanding the practice of teacher education and the practice of teacher education research" (p. 316). MTEs' experiences working in institutions, teaching courses, providing service, conducting research, and wrestling with the competing demands of stakeholders as well as within their own practice could make them feel that academic life is unsustainable. We argue that MTEs can create spaces to share human experiences of their work, and their work with teachers, by conducting research that explores their lived histories and practices. Currently this is possible because our field is embracing the exploration of identity and power using methodologies "that privilege the self" (Hamilton, Smith, & Worthington, 2008, p. 17). We envision this working group as a space for those who have used or are interested in the methodologies of narrative inquiry (Clandinin & Connelly, 2000), self-study (LaBoskey, 2004), and autoethnography (Ellis & Bochner, 2000). Using these methodologies, MTEs could unearth new research questions and communicate the complexity of their work. These methodologies have allowed the authors to empathetically and respectfully collaborate with students and teachers, while also giving us an opportunity to develop self-awareness of our identity, experiences, and bias.

History of the Working Group

Before becoming part of this group of MTEs, we carried out studies independently using the methodologies of narrative inquiry (Clandinin & Connelly, 2000), self-study (LaBoskey, 2004), and autoethnography (Ellis & Bochner, 2000). As conferences and committee meetings brought us together, we developed a mutual understanding of the potential of these methodologies as tools for MTEs' professional growth. We also came to a greater understanding of that potential only after mingling our work and examining the relationship of one study to another. We came to the understanding that our collective research and practices were not only situated in the context of time and place, but also in the social (Clandinin & Connelly, 2000), and that our work is enriched by the sharing of personal history and experience. Furthermore, as Ellis (2004) described, we came to understand human experiences as being connected to ones' culture, consisting of both the beliefs and values one holds. These understandings help us to be mindful of how difficult it could be to incorporate these methodologies when studying mathematics teacher education, and that a space for sharing experience and ideas is necessary to support MTEs who strive to use them. Our collective experience is the strength we bring to the working group and at the same time drives the need for it.

Signe started using self-study as a methodology in 2009 to identify her core beliefs and explicitly align these with her MTE practice. This collaborative self-study provided a mechanism for aligning Signe's research, teaching, and service. Different happenings in her life as a MTE motivated Signe to see self-study methodology as personally rewarding, but not necessarily integrated her roles as MTE. In 2014, to understand the experiences of teacher-learners, Signe explored the creation of moments of intrigue in her methods courses (Kastberg & D'Ambrosio, 2012). Over the years, Signe started understanding that self-study was not just a methodology that was personally rewarding, but that could be used to improve her practice as MTE. For example, in her current work on written feedback, Signe and her colleagues reported the tensions that make some "problems of practice" unsolvable and accepted the notion of wrestling with tensions as a process of growth (Kastberg, Lischka, & Hillman, 2016a, 2016b). Signe's passion for teaching teacher-learners motivated her to engage with other MTEs to gain insight into the experiences of designing and exploring of mathematics methods courses (Kastberg, Tyminski, Lischka, & Sanchez, 2017).

Dana's research journey shifted in 2014 after working with a group of colleagues on a research paper where they explored their beliefs and practices as teacher educators and education researchers (Cox, D'Ambrosio, Keiser, & Naresh, 2014). This was the first time that Dana used the self-study methodology and wrote the results in the form of a narrative. Becoming aware of contradictions in her practice brought Dana to start understanding how qualitative research can act to subvert teacher voices and identity. Later, she challenged the notion that researchers are distinct from participant in Cox and D'Ambrosio (2015) and called for new methodologies that allowed for a blurring of that distinction in D'Ambrosio and Cox (2015). Empathy is a theme that plays out across these three manuscripts. Empathy for others in that work is supported by the use of narrative inquiry as a methodology. This positions her participants, be they teachers or students, as partners and knowledgeable people from whom she learn, and together construct new understandings of mathematics and its teaching. This is reflected in her recent pieces (Cox & Harper, 2017; Simon & Cox, under review) where she has used narrative inquiry as a methodology to explore and communicate experiences working with her students on problem solving and posing in technological environments, and mathematical design.

Olive, who studied with Michael Connelly, has used narrative inquiry as a research methodology in studies of her teaching and mathematics teachers, a research tool to obtain phenomenological data in studies with mathematics teachers, and an approach to in-service and prospective mathematics teachers' learning. With the goal of understanding mathematics teachers from a holistic context that considers their perspectives, Olive has been communicating mathematics teachers' personal meanings (Chapman, 1994; Chapman, 1997) and teachers' agency on their practice change (Chapman & Heater, 2010; Chapman, 2011; Chapman, 2013).

Olive has also used narratives as a research tool and pedagogical tool. When working with prospective teachers (Chapman, 2008a), Olive has asked them to write and share narratives of teaching mathematics and re-write them later in the course. This pedagogical tool has triggered prospective teachers' reflection of mathematics teaching practices. Olive's research and teaching experience using narrative inquiry and narratives to approach teachers in different stages of their career had brought her to lead one of the PME 2017 plenary session, chapters in handbooks (e.g., Chapman, 2008b; Chapman, 2009), and discussion groups at PMENA and ICME (Beswick, Chapman, Goos, & Zaslavsky, 2012; Beswick & Chapman, 2013) that focused directly or indirectly on the mathematics teacher educator knowledge and learning that included narratives and self-study.

As an early childhood mathematics educator, Jennifer seeks to engage teachers on exploring and validating children's mathematics. She learned about this by being involved in the community that surrounds the school and engaging in conversations with children. In her methods courses, Jennifer has been working on different ways to encourage teacher learners to explore children's mathematics and their cultural backgrounds. Part of her journey creating spaces for teacher learners to explore children' mathematics was described in Ward (2017). In this chapter, Jennifer described her experience adapting an assignment so that it provides entries for teacher learners to explore children's ways of knowing, which includes sociocultural background information. Seeking to add the voice of a mathematics teacher to the existing literature on teaching math for social justice, Jennifer chronicled her journey as a MTE working in an early childhood setting with four and five year old children. During this time, Jennifer was focused on merging mathematics content with social justice goals as described by Gutstein (2008) as she sought to explore her own experience in teaching mathematics for social justice with young children. It was through the use of autoethnographic methods that Jennifer was able to convey her story, seeking to connect with other who had similar experiences with a level of honest and raw narratives about the experience.

As part of her dissertation study, Elizabeth explored the concept of personal practical knowledge ([PPK], Elbaz, 1981, 1983) while working with an eighth-grade mathematics teacher for three years. Dewey's (1938) theory of experience and Schwab's (1969, 1983) conceptualization of curriculum served as theoretical lenses in her study. Different histories and experiences encouraged Elizabeth to understand teachers' ways of knowing through the methodology of narrative inquiry. The teaching of secondary mathematics in Chile; learning about research in education; having a collaborative relationship with a mathematics teacher (Suazo, 2016); and encountering Jean Clandinin at an AERA 2016 satellite seminar are some of them. Following this methodology, Elizabeth documented her experiences planning and implementing a mathematics lesson with the teacher. Once outside of the school, when working on the research text, Elizabeth realized that she learned about the teacher's PPK, but also about herself as MTE. The research text relates both mathematics teachers' PPK before coming together and during the planning and implementation of the mathematics lesson. Insights into their personal experiences, teaching practices, and conceptualization of the concept area are interrelated in her narrative inquiry study.

Through our collaboration writing a proposal for and presenting at the Association of Mathematics Teacher Educators (AMTE) 2018, we realized that our methodologies had commonalities and differences. As Hamilton, Smith, and Worthington (2008) stated, these methodologies have in common the "I" and the goal of understanding participants' ways of being or thinking. Self-study and autoethnography share the goal of understanding the self, but autoethnography pays attention to the cultural surroundings of the participants and self-study has the goal of improving the teacher/researcher's practice. Narrative inquiry focuses on acknowledging research as collaboration where researchers focus on the knowledge that is constructed when interacting with participants. Something that is common to all these methodologies is the rewarding feeling we have experienced. Reflecting on and discussing our research made us conscious of lenses and biases involved in our design and implementation of research. This contributed to our learning. In our representations of our work and methods, we also began to consider ways we might invite other MTEs to engage with us and with one another on the development of researchable questions and the use of these methodologies in mathematics education.

Background Information

To consider what sorts of activities might support the emergence of research questions and understandings of the methodologies, we began by summarizing our views of these methodologies that have nurtured and energized our practices.

Narrative Inquiry. The purpose of this methodology is to study lived experiences (Clandinin & Connelly, 2000) and the way we create meaning in our lives. In narrative inquiry, it is understood that the only way to construct knowledge of participants' experiences is to be involved with them in their environments and move to a place of empathy (i.e., making emotional and cognitive connections). The benefit of using narrative inquiry (Clandinin & Connelly, 2000) is to capture more personal and human dimensions of experience. Records of experiences are taken as they occur, but are then layered with reflection upon those experiences. While the experiences are temporal and situated in a place and social context, their meaning expands through reflection. More than just providing access to the lived experiences of individuals who may or may not be represented in the literature, narrative inquiry is a way to uncover things that we as a field, or as individuals are not even aware that we do not know. As practitioners of narrative inquiry, we can attest to the transformative nature of this work; we, as researchers, cannot remain passive and unchanged. We are active within the study and the study is only possible because of our proximity.

Self-study. This methodology involves studying self as teacher in context (LaBoskey, 2004). MTEs' lived experiences as learners and teachers in relation to learners are sources of insight about knowing and knowledge. As a form of practitioner research (Borko, Liston, & Whitcomb, 2007), self-study is self-initiated and aimed at improving one's practice (LaBoskey, 2004). We view self-study as characterized by openness, collaboration, and reframing (Samaras & Freese, 2009). Self-studies further situate questions in existing research literature and suggest implications for "the larger audience of teacher-educators" (Borko, Liston, & Whitcomb, p. 9). We view MTEs' self-awareness and commitment to self-improvement in the context of working with others as the initial steps in self-study. In addition, we view self-study as a methodology that can be used to improve practice and support sustainable scholarly inquiry.

Autoethnography. Situating the writer as both researcher and participant, autoethnography allows MTEs systematically analyze their experiences to understand a culture or group in which they are members (Ellis & Bochner, 2000). Presenting the story of experience, we aim to connect with other MTEs that have had similar experiences through elements of coherence, verisimilitude, and interest. As such, we see autoethnography as a means to nurture and energize our own practice through critical examination of our own personal beliefs and values. By employing autoethnography as a methodology we have been able to develop a sense of connection with other MTEs engaged in similar work. These connections manifest into feelings of empathy as we share about experiences and constructed beliefs with other MTEs.

Beyond Representations to Developing Research Spaces

Participants in the AMTE session, were curious about and interested in incorporating these methodologies in their work. Many of them were looking for spaces that support research under such methodologies. This encouraged us to think about proposing the construction of a similar space at PMENA 2018. AMTE 2018 was the physical place where we connected for the first time with Olive Chapman and invited her to join our group. Meeting our audience and Olive energized us in continuing creating spaces for MTEs who are interested on, or could use, these methodologies. The construction of such space as AMTE 2018 also brought us to the writing a chapter called Mathematics Teacher Educators Inquiry into Their Practice for the International Handbook of Mathematics Teacher Education. This chapter will be part of a volume that focuses on MTEs as a developing professional. We envision having a working group at PMENA 2018 as another evidence that our field is embracing diversity in research methodologies and exploring new research questions related to MTEs' work.

Outline of Working Group Session

Our first session will include introductions and overviews of the participants, and the leaders', and an exploration of the overviews of the methodologies of narrative inquiry, self-study, and autoethnography described in the background information of this proposal. We anticipate that this will generate questions from the audience about the nuances of these methodologies. To build from participants' interests and experiences, we will provide opportunities to brainstorm research questions that have emerged for participants from their work as MTEs. These questions will be compiled and discussed to gain insight from the group about how to study these questions using the stated methodologies. In the second session, participants will select one methodology to explore the study of a research question that was previously identified as being answered by a certain methodology (i.e., narrative inquiry, self-study, or autoethnography). Part of the second and third day, participants will engage in the creation of research texts from existing field texts in the forms of video, personal journals, and transcripts of conversations using the different methodologies. By having the experience of analyzing existing field texts and creating narratives from them that align with the methodologies, we hope participants will engage in a reflection about the "use" of these methodologies and their interactions with their personal worldviews. We anticipate this activity will generate a discussion about the challenges, tensions, dilemmas, and joys of implementing studies under these methodologies. In the third session, we also plan to discuss ways to continue and extend our spaces for learning and support of the writing and publication of studies begun or fostered through the working group. Some of the activities we plan to implement to generate ideas about follow up activities are the following:

1. Participants grouped according to the methodology they are interested in will share their reflections about their experiences exploring such methodologies during the working group session and plan for scholarly inquiry.
2. Small groups will share ideas with the whole group and together we will develop potential collaborations and plans to write, lead sessions in conferences, or publish manuscripts.
3. Participants will discuss ways (e.g., social media, emails, Google Drive, Skype) to continue communicating throughout the year.

References

Bastable & Shifter (1997). Practicing teachers becoming teacher educators: What issues arise as teachers conduct seminars for their peers? In *J., Dossey, J., Swafford, M. Parmantie, & A. Dossey (Eds.), Proceedings of the 19th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Bloomington/Normal, Illinois: Illinois State University.

Beswick, K., Chapman, O., Goos, M., & Zaslavsky, O. (2012). Discussion group 12: Mathematics teacher educators' knowledge for teaching. Conducted at the *12th International Congress on Mathematics Education* (ICME-12), Seoul, South Korea.

Beswick, K. & Chapman, O. (2013). Mathematics Teacher Educators' Knowledge. Discussion Group 3. In A. M. Lindmeier & A. Heinze (Eds.), *Proc. 37th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1, pp. 215). Kiel, Germany: PME.

Borko, H., Liston, D., & Whitcomb, J. A. (2007). Genres of empirical research in teacher education. *Journal of Teacher Education, 58*(1), 3-11.

Chapman, O. (1994). Teaching problem solving. A teachers' perspective. In J. P. Ponte & J. F. Matos (Eds), *Proceedings of the 18th PME International Conference, 2,* 168-175.

Chapman, O. (1997). Metaphors in the teaching of mathematical problem solving. *Educational Studies in Mathematics, 32*(3), 201-228.

Chapman, O. (2008a). Narratives in mathematics teacher education. In D. Tirosh & T. Wood (Eds.), *International handbook of mathematics teacher education: Vol. 2: Tools and Processes in Mathematics Teacher Education* (pp. 15 - 38). Rotterdam, The Netherlands: Sense Publishers.

Chapman, O. (2008b). Mathematics teacher educators' learning from research on their instructional practices: A cognitive perspective. In B. Jaworski & T. Wood (Eds.) *International handbook of mathematics teacher education: Vol. 4: The Mathematics Teacher Educator as a Developing Professional* (pp. 110 - 129). Rotterdam, The Netherlands: Sense Publishers.

Chapman, O. (2009). Self-study as a Basis of Prospective Mathematics Teachers' Learning of Problem Solving for Teaching. In S. Lerman & B. Davis (Eds.), *Mathematical Action and Structures of Noticing (pp. 163-174).* Rotterdam: Sense Publishers.

Chapman, O., & Heater, B. (2010). Understanding change through a high school mathematics teacher's journey to inquiry-based teaching. *Journal of Mathematics Teacher Education*, *13*(6), 445-458.

Chapman, O. (2011). Elementary school teachers' growth in inquiry-based teaching of mathematics. *ZDM*, *43*(6-7), 951-963.

Chapman, O. (2013). Mathematics teachers' learning through inquiry. *Sisyphus-Journal of Education*, 1, 3, 122-150.

Chauvot, J. B. (2009). Grounding practice in scholarship, grounding scholarship in practice: Knowledge of a mathematics teacher educator–researcher. *Teaching and Teacher Education*, *25*(2), 357-370.

Clandinin, J. & Connelly, M. (2000). *Narrative Inquiry: Experience and story in qualitative research*. CA: Jossey-Bass.

Cox, D., D'Ambrosio, B.S., Keiser, J., & Naresh, N. (2014). Repositioning ourselves: acknowledging contradiction. *Bolema: Boletim de Educação Matemática*, *28*(49), 990-1011.

Cox, D. C., & D'Ambrosio, B. S. (2015). Finding Voice: Teacher Agency and Mathematics Leadership Development. In T. G. Bartell, K. N. Bieda, R. T. Putnam, K. Bradfield, & H. Dominguez (Eds.), *Proceedings of the 37th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 640–647). East Lansing, MI: Michigan State University.

Cox, D. C., & Harper, S. R. (2017). Using narratives to articulate mathematical problem solving and posing in a technological environment. In *E. Galindo & J. Newton, (Eds.), Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 985–988). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

D'Ambrosio, B. S., & Kastberg, S. (2012). Giving reason to prospective mathematics teachers. *For the Learning of Mathematics*, *32*(3), 22-27.

D'Ambrosio, Frankenstein, Gutiérrez, Kastberg, Martin, Moschkovich, Taylor, & Barnes (2013). Positioning oneself in mathematics education research. *Journal for Research in Mathematics Education*, *44*(1), 11-22.

D'Ambrosio, B. S., & Cox, D. (2015). An examination of current methodologies in Mathematics education through the lenses of purpose, participation, and privilege. *Perspectivas Da Educaçao Matemàtica*, *8*(18), 687-708.

Dewey, J. (1938). *Experience and education*. New York: Kappa Delta Pi.

Elbaz, F. (1981). The teacher's "practical knowledge": Report of a case study. *Curriculum Inquiry, 11*(1), 43-71.

Elbaz, F. (1983). *Teacher thinking: A study of practical knowledge*. Great Britain: Croom Helm.

Ellis, C. (2004). The ethnographic I: A methodological novel about autoethnography. Walnut Creek:

Rowman and Littlefield Publishers.
Ellis, C. & Bochner, A. P. (2000). Autoethnography, personal narrative, reflexivity: Researcher as subject. In Denzin, K., & Lincoln, Y. S. (Eds.), *Handbook of qualitative research* (pp. 733- 768). Thousand Oaks, CA: Sage Publications.
Ernest, P. (1997). Chapter 3: The epistemological basis of qualitative research in mathematics education: A postmodern perspective. *Journal for Research in Mathematics Education. Monograph, 9*, 22-39.
Guba, E. (1990). *The paradigm dialog*. California: Sage Publications.
Gutiérrez, R. (2009). Embracing the inherent tensions in teaching mathematics from an equity stance. *Democracy and Education, 18*(3), 9-16.
Gutstein, E. R. (2008). The political context of the national mathematics advisory panel. *The Mathematics Enthusiast*, *5*(2), 415-422.
Hamilton, M. L., Smith, L., & Worthington, K. (2008). Fitting the methodology with the research: An exploration of narrative, self-study and auto-ethnography. *Studying Teacher Education*, *4*(1), 17-28.
Jaworski, B., & Wood, T. (2008) *International handbook of mathematics teacher education: Vol. 4: The Mathematics Teacher Educator as a Developing Professional*. Rotterdam, The Netherlands: Sense Publishers.
Kastberg, S. (August, 2012). Building an understanding of learning from moments in teaching. In L. B. Erickson, J. R. Young & S. Pinnegar (Eds.), *Extending Inquiry Communities: Illuminating Teacher Education through Self-Study*. Proceedings of the Ninth International Conference on the Self-Study of Teacher Education Practices (pp. 167-170). [Herstmonceux Castle, UK]. Provo, Utah: Brigham Young University.
Kastberg, S., Sanchez, W., Edenfield, K., Tyminski, A., & Stump, S. (November, 2012). What is the content of methods? Building an understanding of frameworks for mathematics methods courses. In Van Zoest, L. R., Lo, J.-J., & Kratky, J. L. (Eds), *Proceedings of the 34th annual meeting of the North American Chapter of the International Groups for the Psychology of Mathematics Education* (pp. 1259-1267). Kalamazoo, MI: Westerns Michigan University.
Kastberg, S. E., Lischka, A. E., & Hillman, S. L. (2016a). Exploring prospective teachers' written feedback on mathematics tasks. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.) *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 783-790). Tucson, AZ.: The University of Arizona.
Kastberg, S. E., Lischka, A. E., & Hillman, S. L. (2016b). Exploring written feedback as relational practice. In D. Garbett, & A. Ovens (Eds.), *Enacting self-study as methodology for professional inquiry* (pp. 387-394). Herstmonceux, UK: S-STEP.
Kastberg, S. E., Tyminski, A. M., Lischka, A. E., & Sanchez, W. B. (Eds.). (2017). *Building Support for Scholarly Practices in Mathematics Methods*. IAP.
LaBoskey, V. K. (2004). The methodology of self-study and its theoretical underpinnings. In J. Loughran, M. L. Hamilton, V. K. LaBoskey & T. Russell (Eds.), *International handbook of self-study of teaching and teacher education practices* (pp. 817-869). Dordrecht, The Netherlands: Springer.
Lather, P. (2004). This IS your father's paradigm: Government intrusion and the case of qualitative research in education. *Qualitative Inquiry, 10*(1), 15-35.
Lincoln, Y., Lynham, S., & Guba, E. (2011). Paradigmatic controversies, contradictions, and emerging confluences, revisited. In Denzin, N., & Lincoln, Y. (Eds.), *The SAGE handbook of qualitative research* (pp. 97-128). California: Sage Publications.
National Research Council. Committee on Scientific Principles for Education, R. (2002). *Scientific research in education*. Washington, DC: National Academy Press.
Pirie, S. (1997). Chapter 6: Working toward a Design for Qualitative Research. *Journal for Research in Mathematics Education, Monograph, 9*, 79-97.
Samaras, A. P., & Freese, A. R. (2009). Looking back and looking forward: An historical overview of the self-study school. *Self-study Research Methodologies for Teacher Educators*, *7*, 3-20.
Schwab, J. J. (1969). The practical: A language for curriculum. *The School Review*, *78*(1), 1-23.
Schwab, J. J. (1983). The practical 4: Something for curriculum professors to do. *Curriculum Inquiry, 13*(3), 239-265.
Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.
Simon, L. & Cox, D.C. (Under Review). The Role of Prototyping in Mathematical Design Thinking. *Journal of Mathematical Behavior*.
Stinson, D., & Walshaw, M. (2017). "Theory at the crossroads:" Mapping moments of mathematics education research onto paradigms of inquiry. In Galindo, E., & Newton, J., (Eds.), *Proceedings of the 39th annual*

meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 1407-1414). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.
Suazo F., Elizabeth (2016). Working Together: A Caring Relation Between a Teacher and a Mathematics Educator, *Purdue Journal of Service-Learning and International Engagement, 3*(1), 34-37.
Teppo, A. R. (1997). Chapter 1: Diverse ways of knowing. *Journal for Research in Mathematics Education, Monograph, 9,* 1-16.
Tzur, R. (2001). Becoming a mathematics teacher-educator: Conceptualizing the terrain through self-reflective analysis. *Journal of Mathematics Teacher Education*, *4*(4), 259-283.
United States. National Mathematics Advisory, P. (2008). *Foundations for success findings and recommendations from the National Mathematics Advisory Panel*. Washington, D.C.: National Mathematics Advisory Panel.
Ward, J. (2017). Shifting Focus: Exploring the Evolution of the Learner Analysis. In Kastberg, S. E., Tyminski, A. M., Lischka, A. E., & Sanchez, W. B. (Eds.), *Building Support for Scholarly Practices in Mathematics Methods* (pp. 215-232). IAP.
Wilson, S. M. (2006). Finding a canon and core: Meditations on the preparation of teacher educator-researchers. *Journal of Teacher Education*, *57*(3), 315-325.
Whitcomb, J., Liston, D., & Borko, H. (2009). Searching for vitality in teacher education. *Journal Teacher Educator, 60*(5), 439-442.
Whitehead, J. (1989). Creating a living educational theory from questions of the kind, 'How do I improve my practice?' *Cambridge Journal of Education*, *19*(1), 41-52.

EXPLORING THE NATURE OF MATHEMATICAL MODELING IN THE EARLY GRADES

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This working group will engage PMENA members to better understanding the nature of mathematical modeling in the early grades while considering the student perspective and recognizing the importance of teachers knowing their students and the contexts that are meaningful to their students. We will investigate how K-6 teachers demonstrate the interdisciplinary nature of mathematical modeling, the diversity of mathematical approaches taken by student modelers, and the multiple pathways the teacher can use to elicit students' mathematical thinking. We will explore how mathematical modeling bridges equity and social community in teaching and learning mathematics for all students. Exemplar tasks that emphasized local contexts and tapped into students' funds of knowledge and student artifacts will be shared to illustrate the child's perspective and the developmental progression. These topics will facilitate group discussions exploring the learning progression for mathematical modeling thinking and habits of mind that can develop for emergent mathematical modelers from an early grade. We will map out productive learning pathways for mathematical modeling and task design for K-6 mathematics education and beyond.. Finally, based on the interests of the participants, we will devote work time to finding synergistic collaborative topics to pursue for future research and practice.

Keywords: Mathematical Modeling, Elementary Education, Teaching Practices, Professional Development, Learning Progressions, Knowledge of Content and Pedagogy

Overview of the Working Group

This working group had its first meeting at the 2017 PMENA in Indianapolis, IN. We wanted to build on PMENA's long tradition of working groups on *Models and Modeling* with a special focus on early mathematical modeling. Our goal continues to focus on broadening the access of mathematical modeling to diverse learners in the elementary grades and advance the field's collective understanding of the interrelated processes of mathematical modeling in the elementary grades and beyond. Although there has been a long history of mathematical modeling at PME and PMENA, the focus has primarily been on middle, high school and university levels. We believe it is critically important to understand the learning progression of mathematical modeling from elementary to secondary grades to ensure coherence and rigor in the mathematics curriculum.

Our goal from last working group leaders was to propose an edited handbook or a special issues journal venue for mathematical modeling where participants interested in submitting manuscripts can work together to provide a comprehensive research trajectory documenting the progression of mathematical modeling from emergent levels to more sophisticated levels of modeling. We are excited to share that this working group was able to secure a contract with Springer to publish an edited handbook on this very topic! We hope to continue this working group so that we can invite more mathematics educators in the important research of mathematical modeling in the early grades. Since our first working group meeting where we had leaders three university collaborating working with school districts, we have now expanded just within our facilitator leadership over six different universities working with diverse populations to understand the nature of mathematical modeling in the elementary grades. In our design-based implementation research, each university site worked with the collaborating district's teacher leaders to co-plan the professional development. Teachers became co-designers of the mathematical modeling curriculum for the elementary classrooms. In our project, we engaged elementary teachers in considering mathematical modeling using real world tasks that contained several of the following attributes: (a) Openness; (b) Problem-posing; (c) Creativity and choices; d) Iteration and revisions.

Implementing mathematical modeling in the elementary grades is not just going "light" with the high school math modeling curriculum. Instead we advocate integrating aspects of mathematical modeling in the early grades effectively to enhance student learning and to help build their competency in real-world problem solving using their current mathematical knowledge. The latter content knowledge is expected to evolve as students continue to learn new mathematics as they progress towards high school and beyond. So what does mathematical modeling look like in the elementary grades? Why focus on early grades? In addition to the direct benefits of modeling, the elementary school environment affords many advantages that complement work in mathematical modeling. Elementary students often rely on using concrete referents such as objects, drawings, diagrams, and actions or pictures to help conceptualize and to construct carefully formulated arguments to solve a problem. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades (CCSSO 2010). Young students have high potential to become fluent – native speakers, thinkers and dreamers of mathematics. Thinking creatively may come more easily to children first learning and exploring mathematical concepts. Kindergarten students can use manipulatives to independently solve traditional multiplication or division problems they have never seen before, which is evidence that students come with knowledge--we don't have to wait to incorporate modeling activities until we have "shown them how" to do everything. Because early grade teachers are generalists, they can address several subjects simultaneously through modeling activities. Mathematical Modeling is of interest and relevance to the mathematics education community especially because it connects to the need for professional development focused on mathematical modeling in the elementary grades.

Our researchers used Design-Based Implementation Research methodology, DBIR (Fishman, Penuel, Allen, Cheng, & Sabelli, 2013) to examine the design of our professional development and to study and enhance our design through feedback from our iterative implementation cycles. DBIR was a method of choice for our study because it has (1) a focus on problems of practice from multiple stakeholders' perspectives; (2) a commitment to iterative, collaborative design; (3)

a concern with developing theory and knowledge related to both classroom learning and capacity for sustaining change in systems (Fishman, Penuel, Allen, Cheng, & Sabelli, 2013, p. 136).

Through our work, we are gaining a better sense of teaching practices and classroom routines that support modeling. We are contributing to the understanding of what is possible in early elementary grades and how these processes support the development of critical 21st century skills. As we continue in our research to consider what constitutes the practice of Mathematical Modeling (MM) and how it could be implemented in classrooms at different grain size, we invite the larger PMENA community to build on this knowledge. Over the past decades, working group leaders have individually and in subgroups, been theorizing about as well as collecting, analyzing, and reporting on data relating to mathematics modeling. This Working Group builds on and extends the work of previous Model and Modeling tradition by discussing current work from leading scholars from diverse perspectives.

Relevance to Psychology of Mathematics Education

The purpose of this working group is to invite individuals across the research community interested in synthesizing the literature and collaborating on research focused on mathematical modeling along the developmental continuum. Our goal of mapping a learning progression of mathematical modeling from K-12 education, particularly starting from elementary to middle grades is critically important to provide coherence in the mathematics curriculum.

The primary focus for this working group will be around the following three goals:

1. Examine current research and discuss the nature of mathematics modeling and detailing the development of teachers' content knowledge, teaching practices and students' modeling competencies.
2. Map the learning pathways for mathematical modeling and task design for K-6 mathematics education and explore how mathematical modeling can bridge equity and social community in teaching and learning mathematics for all students.
3. Analyze dialogue and collaboration among individuals and groups conducting research on student- and teacher-related outcomes related to implementing mathematical modeling, ways mathematical modeling promotes 21st century skills, and ways in which early modeling can develop interdisciplinary skills in STEM.

Related Research

Mathematical proficiency, in today's world, moves beyond computational ability. It includes the development of 21st century skills (i.e., critical thinking, creativity, communication, and collaboration), conceptual understanding of mathematics (NCTM, 2014), and mathematics that has practical relevance outside of the classroom (Gravemeijer, Stephan, Julie, Lin, & Ohtani, 2017). Mathematical modeling (MM) is a powerful tool for developing students' 21st century skills (Suh, Matson, & Seshaiyer, 2017), advancing their conceptual understanding of mathematics, and for developing their appreciation of mathematics as a tool for analyzing critical issues in the world outside of the mathematics classroom (Greer & Mukhopadhyay, 2012). It provides the opportunity for students to solve genuine problems and to construct significant mathematical ideas and processes instead of simply executing previously taught procedures and is important in helping students understand the real world (English, 2010).

There is broad agreement among mathematics educators on the relevance of mathematical modeling in schools, however, the field has yet to come to a consensus on the definition of mathematical modeling or on how it might be taught and learned in schools (Kaiser, 2017). Although mathematical modeling has been reserved for secondary and college students, its enactment in schools contributes to broad educational goals that are relevant to learners of all ages (Ferri, 2018). In addition, scholars have argued that engaging in mathematical modeling is important for elementary school students (Carlson, Wickstrom, Burroughs, Fulton, 2016).

Mathematical modeling has received increased attention in the United States since the release of the Common Core Standards in Mathematics (the Common Core hereafter) in 2010. Modeling is incorporated as a specific area of expertise that teachers should cultivate in students across Grades K–12. The Common Core's Standards for Mathematical Practice, SMP4 is called *Model with Mathematics.* Although SMP4, as a mathematical practice, cuts across Grades K–12, mathematical-modeling opportunities are not highlighted in connection with the K–8 content standards presenting an implementation challenge for teachers (Cirillo, Pelesko, Felton-Koestler, & Rubel, 2016). Modeling with mathematics, the topic of SMP4, refers to both modeling mathematics and mathematical modeling. The distinction between modeling mathematics and mathematical modeling is not clear to many teachers (Meyer, 2015), nor is it clear in Common Core documents or in mathematics education literature (Cirillo et al., 2016). The key difference between mathematical modeling and modeling mathematics is where the mathematical activity begins. Modeling mathematics begins in the mathematical world (Van de Walle, Karp, & Bay-Williams, 2016), whereas mathematical modeling begins in the unedited real world (Pollak, 2007). The explicit focus on getting a problem outside of mathematics into a mathematical formulation and explicitly translating the mathematical solution back into the real world is what differentiates mathematical modeling from modeling mathematics. The real-world focus also distinguishes mathematical modeling from problem solving and application problems (Lesh & Caylor, 2007; Schukajlow et al., 2012).

One of the ways, the researchers in this working group approached MM in the elementary grades was to immerse students in a real world situation within their local context that was relatable and personally meaningful. In bringing mathematics closer to the social community spaces, mathematical modeling became a vehicle that bridged teaching and learning mathematics closer to all students. Reform mathematics have advocated for mathematics to be more related to students' lives by building on community and cultural knowledge and practices with issues that matter to them, which then helps students view mathematics as a vehicle through which they learn to be active change agents for social justice (Bartell et al. 2017; Civil 2007).

To keep the initial problem open, students were encouraged to develop the habit of mind of being problem posers by identifying the many questions around the real phenomenon, then defining a mathematical problem that can be solved by way of mathematics. After the identification process of the problem, the modeler makes assumptions, eliminates unnecessary information, and identifies important quantities in order to develop a solution. This mathematical solution focuses on the usefulness of mathematics to solve a real-world problem. It should be noted that there can be several mathematical solutions for a given real-world situation. After solving the problem the results are translated back to the real-world and interpreted in the original context. The problem-solver then validates the solution by checking whether it is appropriate or reasonable for the purpose. This process of making assumptions, identifying variables, formulating a solution, interpreting the result, and validating the usefulness of the solution is iterative in nature and is modified or changed and repeated until a satisfactory solution has been obtained and communicated (Blum, 2002).

It is important to note that teachers play a crucial role in MM. The teacher must be able to: (1) find appropriate questions to move students through the modeling cycle, (2) handle discussions in nondirective but supportive ways, (3) allow students time for productive struggle, and (4) provide scaffolding without directing the problem (Burkhardt, 2006). Teachers also need to develop problem-posing expertise (Suh et al., 2017) and to base their instructional decisions on responses to students' work (Bleiler-Baxter et al., 2016). Thus, learning teach mathematical modeling requires teachers to develop multiple knowledge bases. For example, teachers must understand modeling processes and tasks, including the potential mathematical content embedded within tasks; learn about students' mathematical and personal experiences to predict the strategies they might use when responding to modeling tasks; know what content is on the mathematical horizon in order to anticipate what mathematical ideas students might construct and learn to engage individual and groups of students in the modeling process. (e.g. Blum, 2011; Ferri, 2018).

Previous work with elementary school children demonstrated it is feasible for them to develop a disposition towards realistic mathematical modeling (Lieven & De Corte, 1997). One of the issues in implementing MM at the elementary level is that MM can be difficult for both teachers and students to implement (Blum & Ferri, 2009). MM can be difficult for teachers to implement as they must be able to merge mathematical content and real-world applications while teaching in a more open and less predictable way (Blum & Ferri, 2009). Mathematical modeling can be a challenge for students because each step of the modeling process presents a possible cognitive barrier (Blum & Ferri, 2009). As stated in the Common Core Standards for Mathematical Modeling, *"Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process. These real-world problems tend to be messy and require multiple math concepts, a creative approach to math, and involves a cyclical process of revising and analyzing the model"* (Carter et. al., 2009). To model, students need support to develop mathematical modeling competency i.e., the ability to independently carry out the various phases of the modeling process (Vorhölter & Kaiser, 2016) and its related sub-competencies (Schukajlow et al., 2015). Based on the Standard for Mathematical Practice 4 in the Common Core: Model with Mathematics, Bleiler-Baxter, Barlow, and Stephens (2016) designated the mathematical modeling sub-competencies needed by students as: simplification (e.g., making assumptions), relationship mapping (e.g., identifying important quantities and their relationships), and situation analysis (e.g. interpreting results in the context of the situation). In addition, students need to develop a metacognitive modeling competency as this is "indispensable in order to enable students to solve complex modeling problems independently, which is an indispensable part of true modeling activities" (Vorhölter & Kaiser, 2016, p. 279).

In a previous PMENA report, Suh, Matson, Williams and Seshaiyer (2016) reported the challenges and affordances of mathematical modeling in the early grades.

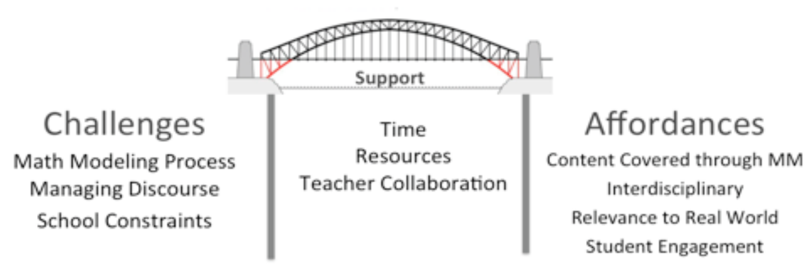

Figure 1. Challenges and affordances of mathematical modeling.

Teacher Challenges. The challenges teachers faced when implementing mathematics modeling in the elementary grades included: a) Novelty and ambitious nature of the *modeling process*-When implementing MM in the classrooms for the first time, teachers found it was difficult to move students through the full process as it was a novel approach and students had never been introduced to creating and validating their mathematical models; b) *Managing discourse*- Another difficulty encountered by teachers in the MM process was in defining their role as facilitators. The teachers commented that "...it is really difficult as a teacher to help students find a direction to go with their solution but not direct or guide them toward a teacher goal."; and c) *Constraints around mandated standards*-Participants acknowledged that MM takes time to implement in the classroom and that additional class time to implement these tasks would be helpful. An additional challenge noted by teachers was that mathematical modeling didn't go the way they expected it to and they wrestled with the need to meet state standards.

Affordances of Mathematical Modeling. The main affordances our teacher-designers mentioned were that mathematical modeling provided opportunity for content to be covered without direct instruction, had interdisciplinary connections, and provided mathematical relevance, and student engagement: a) *Content covered without direct instruction-* When teachers implemented MM in their classrooms for the first time they were amazed at the amount of content that could be covered without direct instruction. Students could see how the mathematics could serve their needs as they used the mathematics they learned while other times, the mathematics related to future learning objectives which allowed them to revisit their model as their learning progressed; b) *Interdisciplinary opportunities*-Another positive take-away from implementing MM in these teachers' classrooms for the first time was how MM created a space where content covered was interdisciplinary connecting to social studies, STEM and language arts; c) *Relevance*-By providing authentic tasks for students to grapple with through the MM process, mathematics became relevant to the students; d) *Student engagement*-A number of our teachers indicated how engaged their students were in their MM tasks. Mathematical modeling inspired these teachers' endeavors and provided pictures of practices that served as the proof of concept they needed to sustain their professional commitment to mathematical modeling.

Support Teachers Need. The three main areas of support teacher-designers requested were access to MM resources, pictures of practice, time and collaboration with like-minded teachers: a) *Resources and pictures of practice*-Teachers indicated a desire to use MM in their classrooms but indicated a need for a bank of open-ended MM lessons and new ideas for continuing to create these lessons; b) *Time*-Teachers expressed the need for more time to work through and become comfortable with implementing the modeling process in their classrooms. Teachers noted it was only in working through the MM cycle several times that they felt comfortable with the process and felt their students were able to understand the whole MM process; and c) *Teacher collaboration*-Teachers indicated a desire to continue to work with a cohort to build MM lessons; to observe other teachers implement MM in their classrooms; and to work alongside a colleague who valued MM and with whom they could share ideas.

Other related research the team will share include, Carlson, Wickstrom, Burroughs & Fulton's (2016) work, *A Case for Mathematical Modeling in the Elementary School Classroom*, where they provide a teaching framework for MM using the "organize - monitor - regroup" cycle to support the teachers' work in engaging young students in modeling. Wickstrom, Carr and Lackey (2007) will showcase an engaging article using mathematical modeling to explore Yellowstone National Park. Suh, Matson, & Seshaiyer (2017) will also share ways in which mathematical modeling enhanced students creativity, collaboration, critical thinking and communication skills and exposed students to interdisciplinary themes of service learning and STEM integration.

Plan for Active Engagement of Participants

The working group will meet three times during the conference and virtually during the course of one year. In each session, PMENA members will engage in mathematical modeling while sharing their perspectives in teaching and learning mathematics, considering synergistic areas fruitful for future research and practice, and finding collaborators within our group.

Session 1: Exploring the Nature of Mathematical Modeling in the Early Grades

The first session will focus on better understanding the nature of mathematical modeling in the elementary grades while considering the student perspective and recognizing the importance of teachers knowing their students and the contexts that are meaningful to their students. We will investigate how K-6 teachers demonstrate the interdisciplinary nature of mathematical modeling, the diversity of mathematical approaches taken by student modelers, and the multiple pathways the teacher can use to elicit students' mathematical thinking. We will explore how mathematical modeling bridges equity and social community in teaching and learning mathematics for all students. Exemplar tasks that emphasized local contexts and tapped into students' funds of knowledge and student artifacts will be shared to illustrate the child's perspective and the developmental progression. These topics will facilitate group discussions exploring the learning progression for mathematical modeling thinking and habits of mind that can develop for emergent mathematical modelers from an early grade. We will map out productive learning pathways for mathematical modeling and task design for K-6 mathematics education and beyond.

Session 2: Identifying the Knowledge of Content and Pedagogy Needed for Mathematical Modeling in the Elementary Grades

In our second session, we will focus on clearly defining modeling teaching practices and competencies needed for mathematical modeling and outlining research goals and objectives to monitor the enactment of these practices. We will detail classroom routines, such as the "organize - monitor - regroup" cycle (Carlson, et al. 2017), and the Pedagogical Practices for Mathematical Modeling (Suh, Matson, & Seshaiyer, in press) as we share designed activities and lesson vignettes to solicit more ideas around high leverage MM teaching practices. We will explore what mathematical knowledge is needed to "successfully" facilitate mathematical modeling tasks in elementary grades. We will examine current research and discuss the nature of mathematics modeling and detailing the development of teachers' content knowledge, teaching practices and students' modeling competencies.

Session 3: Finding the Synergy between Mathematical Modeling and the 21st Century Skills Frameworks and PBLs in STEM

The third session will outline several 21st century skill frameworks and teaching approaches and how mathematics educators, researchers and practitioners can find a synergistic way to bring important process skills without overwhelming teachers and students. We will discuss the ways elementary teachers can make connections between the problem-based ways they have engaged students in mathematical modeling and STEM. The teachers are able to take advantage of interdisciplinary opportunities across the subjects they teach and find complementary connections between subjects and common classroom practices that support MM. We will analyze dialogue and collaboration among individuals and groups conducting research on student- and teacher-related outcomes related to implementing mathematical modeling, ways mathematical modeling promotes 21st century skills, and ways in which early modeling can develop interdisciplinary skills in STEM

Anticipated Follow-up Activities and Goals of Working Group

In the spirit of exploring the theme of 2018 PMENA on changing mathematical and pedagogical demands for PK-6 education and supporting *all* students through a concerted focus on equitable teaching practices. In addition, this working group will attend to the interdisciplinary nature of mathematics and how it connects to social justice, STEM and civic responsibility for our citizen in our country. Each session will engage participants to share their research interests related to mathematical modeling and form groups that might pursue research collaboratively based on the interests of the participants. We will also work on book chapters for the handbook. Some of the questions that we will engage in include:

- What defines successful mathematical modeling at different grade levels?
- How does mathematical modeling support each and every learner?
- How does mathematical modeling connect one to issues of social justice, STEM and civic responsibility of citizens?
- What can we learn from teachers who implement MM regularly in their classrooms?
- How is mathematical modeling ambitious teaching and how can we support teachers enacting MM through lesson plans and other resources?
- How can we map out the learning pathways of MM across grade levels?
- How and what can we learn about models elicited from student artifacts from MM tasks?
- What do “successful” modeling practices look like in our elementary mathematics classrooms? How are they similar or different from practices in secondary classrooms?
- What does it mean to “see the math” in the components of mathematical modeling?
- How do teachers select and/or develop modeling problems? How can Professional Learning Communities or Teacher Study Groups help teachers anticipating how students will answer the MM questions?

Our goal is for the working group leaders to propose an edited handbook or a special issues journal venue for mathematical modeling where participants interested in submitting manuscripts can work together to provide a comprehensive research trajectory documenting the progression of mathematical modeling from emergent levels to more sophisticated levels of modeling.

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References

Bartell, Tonya, Anita Wager, Ann Edwards, Dan Battey, Mary Foote, and Joi Spencer. “Toward a Framework for Research Linking Equitable Teaching with the Standards for Mathematical Practice.” *Journal for Research in Mathematics Education* 48, no. 1 (2017): 7–21.

Bleiler-Baxter, S. K., Barlow, A. T., & Stephens, D. C. (2016). Moving beyond context: Challenges in modeling instruction. In C. R. Hirsch & A. R. Roth McDuffie (Eds.), *Mathematical modeling and modeling mathematics* (pp. 53–64). Reston, VA: National Council of Teachers of Mathematics.

Blum, W. (2002). Icmi Study 14: Applications and Modelling in Mathematics Education — Discussion Document. *Educational Studies in Mathematics*, *51*(1/2), 149–171.

Blum, W., & Ferri, R. B. (2009). Mathematical modelling: Can it be taught and learnt? *Journal of Mathematical Modelling and Application*, *1*(1), 45–58.

Blum, W., Henn, H. W., Galbraith, P., & Niss, M. (Eds.). (2006). *Modelling and applications in mathematics education: the 14th ICMI study* (1st ed). New York: Springer.

Carlson, M.A., Wickstrom, M.H., Burroughs, B. & Fulton, E. (2016). *A Case for Mathematical Modeling in the Elementary School Classroom*. Annual Perspectives in Mathematics Education (APME) 2016: Mathematical Modeling and Modeling Mathematics, ed., Christian R. Hirsch, pp. 121–29. Reston, Va.: National Council of Teachers of Mathematics.

Carter, J., Forster, S., Howe, R., Kader, G., Kepner, H., Quander, J. R., ... & Valdez, P. (2009). *Focus in high school mathematics: Reasoning and sense making*. Reston, VA: National Council of Teachers of Mathematics.

Cirillo, M., Pelesko, J., Felton-Koestler, M., & Rubel, L. (2016). Perspectives on modeling in school mathematics. In C. R. Hirsch & A. R. Roth McDuffie (Eds.), *Mathematical modeling and modeling mathematics* (pp. 3–16). Reston, Va: National Council of Teachers of Mathematics.

Civil, Marta. "Building on Community Knowledge: An Avenue to Equity in Mathematics Education." *Improving Access to Mathematics: Diversity and Equity in the Classroom* (2007): 105-117.

Council of Chief State School Officers. Common Core State Standards (Mathematics): Washington D.C.: National Governors Association Center for Best Practices, 2010.

Consortium for Mathematics and Its Applications [COMAP,Inc.]; Society for Industrial and Applied Mathematics [SIAM]. (2016). *GAIMME: Guidelines for assessment & instruction in mathematical modeling education.* Garfunkel S. A.; Montgomery, M., Eds.; 2016.Retrieved from http://www.siam.org/reports/

English, L. D. (2010). Young children's early modelling with data. *Mathematics Education Research Journal*, *22*(2), 24–47.

English, L. D. (2013). Complex modelling in the primary and middle school years: An interdisciplinary approach. In G. A. Stillman, G. Kaiser, W. Blum, & J. P. Brown (Eds.), *Teaching Mathematical Modelling: Connecting to Research and Practice* (pp. 491–505). Dordrecht: Springer Netherlands. Retrieved from http://link.springer.com/10.1007/978-94-007-6540-5_42

English, L.D.,, Fox, J.L., & Watters, J.J. (2005). Problem posing and solving with mathematical modeling. *Teaching Children Mathematics*, 12(3), 156-163.

Gravemeijer, K., Stephan, M., Julie, C., Lin, F.-L., & Ohtani, M. (2017). What mathematics education may prepare students for the society of the future? *International Journal of Science and Mathematics Education*, *15*(S1), 105–123. https://doi.org/10.1007/s10763-017-9814-6

Greer, B., & Mukhopadhyay, S. (2012). The Hegemony of Mathematics. In O. Skovsmose, O. Skovsmose, & B. Greer (Eds.), *Opening the Cage* (pp. 229–248). Sense Publishers.

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of Representation in the teaching and learning of mathematics* (pp. 33–40). Hillsdale, N.J.: Erlbaum.

Lesh, R. A., & Doerr, H. M. (2003). Foundations of a Models and Modeling Perspective on Mathematics Teaching, Learning, and Problem Solving. In R. A. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 3–34). Mahwah, N.J: Lawrence Erlbaum Associates.

Lieven, V., & De Corte, E. (1997). Teaching realistic mathematical modeling in the elementary school: A teaching experiment with fifth graders. *Journal for Research in Mathematics Education*, *28*(5), 577–601.

Suh, J. M., Matson, K., Willliams, M. & Seshaiyer, P. (2016). Immersing elementary teachers in mathematical modeling as co-designers through Lesson Study. Presented at the conference for the International Group for the Psychology of Mathematics Education.Tucson, AZ: The University of Arizona.

Suh, J.M., Matson, K. & Seshaiyer, P (2017). Engaging Elementary Students in the Creative Process of Mathematizing their World through Mathematical Modeling. *Educational Sciences. 7(*62). doi:10.3390/educsci7020062.

Vorhölter, K., & Kaiser, G. (2016). Theoretical and pedagogical considerations in promoting students' metcognitive modeling competencies. In C. R. Hirsch & A. R. Roth McDuffie (Eds.), *Mathematical modeling and modeling mathematics* (pp. 273–280). Reston, VA: National Council of Teachers of Mathematics.

Wickstrom, M. H., Carr, R., & Lackey D. (2017). Exploring yellowstone park with mathematical modeling. *Mathematics Teaching in the Middle School*. NCTM.

WORKING TO UNDERSTAND MEDIATED FIELD EXPERIENCES AND STUDY THEIR IMPACT

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Though teacher preparation programs include university coursework and fieldwork in K-12 school settings, they are often disparate experiences. This group of mathematics teacher educators (MTEs) has been working to integrate these preparation efforts by purposefully designing coursework around McDonald et al.'s (2013) learning cycle to include mediated field experiences. Such experiences are embedded within public school classrooms and are organized around core teaching practices so that pre-service teachers (PSTs) engage in authentic learning; they learn mathematics and ways to teach mathematics through the act of teaching. Our goals as a working group are to establish cross-cutting and essential features of these mediated field experiences and identify future research questions and ways to standardize data collection to create clear lines of investigation to explore the impacts of such experiences.

Keywords: Teacher Education-Preservice, Teacher Education Programs, Field Experiences; Practice-based; Mediated Field Experiences; School-Embedded Courses

Motivations for This Working Group

Mathematics teaching is a complex professional practice; relational in nature and requiring a deliberate, coached effort to learn and do well (Grossman, Hammerness, & McDonald, 2009). Teacher preparation programs include both university course work and clinical field experiences, yet decades of research indicate that the two are often disparate experiences for pre-service teachers (PSTs)(Grossman et al., 2009; Korthagen & Wubbels, 2001; Lampert & Graziani, 2009; McDonald, Kazemi, & Schneider-Kavanagh, 2013). The clinical component of mathematics teacher preparation—in the form of PSTs observing and taking increased responsibility in PK-12 classrooms—can be influential in the development of teachers' practice (National Research Council, 2010) and current standards for the preparation of mathematics teachers call for increased opportunities for PSTs to learn via clinical experiences (Association of Mathematics Teacher Educators, 2017).

Embedding university courses within K-12 schools where PSTs have opportunities for interactional clinical experiences with the support of a mathematics teacher educator (MTE) is a promising model to support the development of skilled practice (Virmani et al., 2017). Horn and Campbell (2015) present a hybrid space situating coursework and field work together as "mediated field experiences." Within a mediated field experience, PSTs are provided opportunities, structure, and support to apply knowledge of teaching by confronting specific situations and determining ways to integrate realities of the classroom with teaching ideals. For example, PSTs may be asked to create a lesson plan, teach a lesson, or reflect about a teaching experience; however, without an MTE to "mediate" the implementation, the PSTs may apply their university knowledge problematically, or they may not notice or make sense of particular features of the teaching practice that produce unexpected results (Campbell & Dunleavy, 2016). Because of the unpredictable nature of teaching and learning, flexibility is key within the mediated field experience. Horn and Campbell (2015) found MTEs often modified the focus of what was mediated according to what happened during their observations of PSTs.

Despite a growing body of literature that focuses on the need to more deeply connect field experiences to teacher preparation coursework, there is little evidence for what aspects contribute to PSTs' development of the knowledge, skills, and dispositions that help them meet the realities of the classroom. The central aim of our Mediated Field Experience (MFE) Working Group is to explore the essential features of our school-embedded mediated field experiences that take place during PSTs' preparation, describe ways mediated field experiences can transfer to a variety of contexts (different types of institutions and courses), and establish future research questions leading to identification of ways mediated field experiences impact PSTs learning.

Background of Our Work

During a NSF-funded conference held in 2015, ten MTEs, five on this current proposal, began collaborating about ways to use MacDonald et al. (2013) learning cycle of enactment and investigation (see Figure below) to create mediated field experiences in various math methods and content courses to prepare PSTs at our respective institutions. The "pedagogies of practice in professional education" (Grossman et al., 2009) framework underpins this cycle; PSTs with mentors decompose and approximate practice in increasingly authentic settings. Instructional activities act as containers in which PSTs engage in the act of teaching by preparing activities, including rehearsing during university class time, enacting activities with children in authentic classroom settings, and then reflecting and analyzing their learning about content, student thinking, and pedagogy (Kazemi, Franke, & Lampert, 2009). Our working group, was particularly interested in investigating ways embedding our courses into the schools while using McDonald et al.'s (2013) learning cycle could support PSTs' development of core teaching practices. By situating the learning within practice, we aimed to develop a community of practice in which PSTs, MTEs, classroom teachers, and elementary students learned from each other as they engaged with core teaching practices within carefully selected instructional activities.

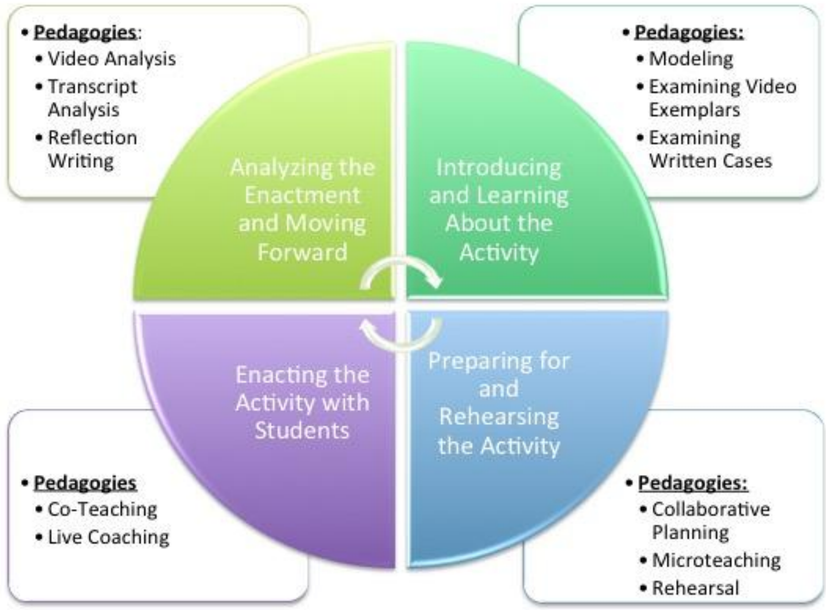

Figure 1. Cycle for collectively learning to engage in an authentic and ambitious instructional activity (McDonald, Kazemi, & Kavanagh, 2013, p. 382).

We used Dr. Elham Kazemi's work at the University of Washington to guide our pilot study. Since their teacher preparation work is a graduate program focus with the support of graduate student and funding for cooperating teacher involvement, we wanted to examine ways to implement McDonald et al.'s (2013) cycle of enactment in different contexts teacher preparation courses without additional university classroom supports, varying in intensity of integration of methods and content courses, within a K-12 school setting. From our work together, five key themes emerged (Virmani et al., 2017):

1. PSTs work in a P-12 classroom with students and at least one teacher educator in a face-to- face setting.
2. MTEs use time-outs or planned/unplanned pauses in ways that (a) provide opportunities for all teachers to think in the moment (e.g., make sense of student thinking, discuss a next teacher move), and (b) position students as powerful and productive thinkers.
3. Participants, including PSTs, MTEs, & classroom teachers, approach a lesson with a shared curiosity and/or plan to learn something particular about teaching and learning.
4. Classroom visits focus on observing, studying, and/or trying something related to eliciting student thinking in ways that can't be as authentically explored in the university classroom without students.
5. All participants debrief, analyze, and reflect on the experience individually and collectively. This includes conversations with a teacher educator centered on making meaning of classroom events in terms of student thinking.

Since this initial pilot study, a subgroup has continued to work together in developing and studying the affordances of school-embedded methods and content courses. Recently, at the Association of Mathematics Teacher Educators (AMTE) conference in February 2018, we presented our initial analysis on debriefs, or reflective conversations, occurring at various stages within the learning cycle. Debriefs took place between the whole group of PSTs, MTEs, and the classroom teacher and provided opportunities for PSTs to immediately talk about what happened during their interactions with students and share common experiences. These debriefs produced more authentic conversations about teaching, learning, and students and provided opportunities for MTEs to name or "label" what PSTs saw in the classroom and connect it to course materials (readings, theory, etc.). Further, there were many areas of growth made visible via debriefs including: core teaching practices, PSTs identity development, and the attention to student thinking and viewing students as competent sense makers.

At the AMTE conference, we connected with other MTEs engaged in similar work at their institutions. We found there is a need in the math education field to better understand how to support the work of embedding teacher preparation coursework in schools via mediated field experiences while using McDonald et al.'s (2013) learning cycle to support the development of core teaching practices. Together, as a working group, we aim to study our different models of these mediated field experiences to identify essential features which would allow these experiences to transfer into a variety of contexts and begin initial efforts of investigating the impacts of these experiences on PSTs' development.

Review of the Literature and Background Information

Recently, AMTE published the *Standards for Preparing Teachers of Mathematics* (2017) outlining standards and related indicators for effective programs for preparing beginning teachers of mathematics. The standards list five key domains including (1) well designed and integrated partnerships with all partners productively engaged (2) opportunities for PSTs to develop content knowledge and mathematical practices through sustained, quality experiences, (3) opportunities for PSTs to learn to teach mathematics which includes knowledge of content, students, and contexts in which they teach (4) opportunities for PSTs to learn through clinical experiences where they learn from their own teaching and the teaching of others, and (5) recruitment and retention. Research across decades has shown, however, that there are numerous barriers to accomplishing these standards with traditional models of field experiences that are separate from university-based teacher preparation coursework (Akerson, Morrison, & McDuffie, 2006; Borko & Mayfield, 1995; Fryholm, 1996; Gainsburg, 2012; Zeichner & Tabachnick, 1981).

One such barrier lies in the separation of theory and practice. "What remains to be developed is… pedagogies of enactment through which novice teachers are supported in actually doing the practice of teaching" within their teacher preparation programs (Kazemi, Franke, & Lampert, 2009, p. 12). In many teacher preparation programs, instruction focuses on learning *about* strategies and practices for teaching rather than directly enacting and honing these skills (Grossman et al., 2009). This separation of theory and practice is problematic; as Lampert (2005) describes,

> …learning about a method or learning to justify a method is not the same thing as learning to do the method with a class of students, just as learning about piano playing and musical theory is not learning to play the piano (p. 36).

One way to bridge the theory/application gap, is to engage in practice-based learning, where PSTs directly learn via the *act of teaching*. The term *practice* conveys a variety of perspectives in the research literature related to teacher preparation (Lampert, 2010). *Practice- based learning* describes types of field experiences or programs situating PSTs' learning in K-12 classrooms coupled with instruction as well as training or coursework focusing explicitly on the work of teaching (Forzani, 2014). Grossman et al. (2009) describe how a core-practice approach in teacher education necessitates organizing coursework and fieldwork around *practices* of the teaching profession while simultaneously providing PSTs ample opportunities to "practice" and enact these teaching practices. Other research within the teacher education community has identified "core" or "high-leverage" teaching practices that effective teachers use while teaching (i.e., Ball & Forzani, 2009; McDonald, Kazemi, & Kavanagh, 2013; NCTM, 2014).

The *Standards for Preparing Teachers of Mathematics* (2017) describe standards for effective teacher preparation programs that are practice-based. They call for PSTs' development of teaching practices based on the National Council of Teachers of Mathematics (2014) identification of eight effective mathematics-teaching practices. These standards require teacher preparation programs to provide scaffolded and multiple opportunities for PSTs to integrate their learning and incorporate practice-based experiences in authentic classrooms throughout the program, not just in student teaching. Moreover, these standards call for teacher preparation programs to be actively involved *in partnership with* the schools that their candidates are placed in, working toward shared goals for instructional improvement and student learning.

> Only when preparation programs purposefully engage with schools, not just in schools, will their clinical preparation become truly robust in ways that maximize candidates' skill development and therefore their abilities to support the mathematics learning of students (p.37).

These practice-based experiences must provide PSTs opportunities to develop skills with teaching practices, gain mathematics content knowledge and insight into students as mathematical learners, and reflect critically and regularly about mathematics teaching and learning. Responding to this call, the question of how we might structure learning experiences to integrate content knowledge, theory, and practice is central for practice-based teacher education and the focus of our working group through studying our mediated field experiences.

Approximations of Practice to Develop Adaptive Expertise

Additionally, Grossman et al. (2009) argue PSTs need opportunities to engage in "approximations of practice" in both the university classroom and through authentic K-12 classroom settings. Lampert & Graziani (2009) highlight the importance of instructional activities or "enact-ables" that PSTs could rehearse, enact, and receive feedback on in this work. McDonald et al.(2013) propose a *cycle for learning* (Figure above) that is flexible in a variety of settings where PSTs learn to enact teaching practices and involves: learning about the instructional activity (including envisioning the practice), preparing for enacting these practices (including rehearsing), having opportunities to enact the practices with real students in actual K-12 classrooms, and analyzing and reflecting upon these enactments. Horn and Campbell's (2015) work emphasize the need for mediation of these learning experiences. By purposefully designing teacher preparation coursework to include the pedagogies of enactment through our mediated field experiences in K-12 school settings, we are working to develop PSTs ability to develop core teaching practices. However, further research is needed to provide empirical evidence that these efforts produce well-prepared beginning teachers of mathematics.

Some argue that too much specification could lead to "deskilling" teachers, others argue that the ability to specify and codify common practices is a precondition of a profession (Ball & Forzani, 2009). With this commitment to specification and building core practices or instructional routines, there has been less attention given to what it would take to help teachers develop *adaptive expertise* (Darling-Hammond & Bransford, 2005; Hatano, 1988). Adaptive experts are not only highly skilled in the performance of a limited number of tasks, but are also able to adapt and extend their routines, by accepting new possibilities or attempting to solve old problems in different ways (Chi, 2011). Given that teaching requires such adaptive expertise, as the complex interactions of a classroom can never be fully specified, Janssen (2015) has argued that research on practice-based teacher education will also need to attend to the more adaptive and improvisational dimension of practice. The field has started to see emerging research that supports using enactable activities such as instructional routines to help novices start to develop adaptive expertise. However, we need more empirical evidence that field-based teacher education organized around enactable instructional activities, like our mediated field experiences, significantly impact PSTs' practice in ways *that are not possible without those experiences*.

Connection to PME-NA Goals

The new AMTE (2017) standards urge teacher educators to develop innovative, practice-based models for supporting PST learning through the purposeful integration of coursework and field experiences. Though the McDonald et al. (2013) learning cycle is well-grounded, we can not expect teacher preparation programs to change until there is robust, empirical evidence that the learning cycle is effective in preparing PSTs and is best achieved via mediated field experiences in authentic school settings. Across our various projects, we have found mediating and entering authentic settings alongside PSTs provides incredible opportunities to learn with and from children. But this shift also inserts MTEs into the central conflict PSTs have long navigated alone: how to support a range of learners to meaningfully engage in high-quality mathematics instruction. It is not surprising that in challenging classroom settings, PSTs often forfeit the principled vision of high-quality instruction described in their teacher preparation programs for lower expectations for traditionally-underserved students (Frykholm, 1996). By entering into real classrooms with PSTs, we as MTEs have committed to support PSTs in navigating the realities of the classroom through these mediated field experiences.

The research proposed by this working group exemplifies PME-NA's commitment to equitable teaching practices and addressing the needs of all students. We want to investigate how these mediated field experiences help PSTs learn to build off of student thinking instead of directing or leading the students to follow their thinking. Our initial findings indicate that PSTs are able to understand the need for and develop equitable teaching practices to meet the needs of all learners by providing preservice teachers the opportunities to practice such core teaching practices like eliciting and responding to student thinking in a real elementary or secondary classroom. While working with the K-12 students, receiving feedback on their teaching, and reflecting on their experiences, PSTs are able to recognize the importance of providing all students with the opportunities to engage in challenging mathematics and practice pedagogical strategies that allow all students to participate (e.g., think pair share, or turn and talk). These mediated field experiences are an example of a transformation in the preparation of teachers of mathematics at all levels with hopes to make changes to the mathematical and pedagogical demands in the K-12 school setting through better preparation efforts for PSTs.

Goals of the Working Group

This working group has three main goals that it hopes to achieve during its time at PME-NA. Our first goal is that we would like to come together to identify features of each MTE's implementation of these mediated field experiences that are similar across institutions. We hope to identify these cross-cutting features so that we may start to establish a consensus for which ones might be essential to include when creating these experiences for our PSTs. Our second goal is to describe ways mediated field experiences can transfer to a variety of contexts (different types of institutions and courses), and our third is to develop clear lines of investigation by posing research questions, identifying the data to be collected, and standardizing the collection process across researchers to address the questions posed. By developing a clear research agenda, we hope to explore the impacts of these field experiences for PSTs.

Identifying Cross-Cutting Features to Develop "Transfer"

Across the working group, a variety of models for mediated field experiences have emerged (Virmani et al., 2017). These models range from exploratory integration of pedagogy and content courses within classrooms to those that have seized the opportunity to fully integrate a majority of coursework at school sites. Although these models and levels of integration differ, two main themes emerged across the models: (a) composing coursework around the practice of teaching, and (b) developing partnerships with teachers and schools. It is our hope that during these working group sessions, we will be able to dive deeper into these two themes to identify essential elements of these mediated field experiences that provide the PSTs the opportunity to develop adaptive expertise.

A strength of our working group is that we have MTEs from a variety of institution types, (e.g., large state schools, small liberal arts schools), programs (elementary, secondary, undergraduate and/or graduate), and courses (methods/pedagogy courses and content courses). The diversity in our group members affords us the opportunity to identify the essential features that can be utilized by virtually any MTE to support PST learning through teaching and their development of adaptive expertise. We would like to show other MTEs how McDonald et al.'s (2013) learning cycle can transfer into a number of different contexts with differing levels of support and that the University of Washington's model, while it is productive and researched, is not the only way this work can be done.

Creating Clear Lines of Investigation

The AMTE standards promote the idea of improving the connections between preparation coursework and clinical experiences, however there is a lack of empirical evidence or practical guidance for *what works* and *how to do it.* Our third goal of this working group is to collectively develop a clear research trajectory for studying and improving our work across institutions using evidence of learning to inform the revisions of (a) our collective theory of action, and (b) our individual program models. Our process for outlining a research trajectory for designing, implementing, studying, and revising these *mediated field experiences* in teacher education will be informed by educational design research methodology (Cobb, Jackson & Sharpe, 2017) and/or case study research design, based on the research questions that emerge from our working group discussions. Anderson and Shattuck (2012) describe design-based research (DBR) as,

> A methodology designed by and for educators that seeks to increase the impact, transfer, and translation of education research into improved practice. In addition, it stresses the need for theory building and the development of design principles that guide, inform, and improve both practice and research in educational contexts" (p.16).

In addition to DBR, some contributors are interested in case-study methodology. Case study lends itself to providing a "how" and "why" description of the mediated field experiences from variety of settings (Merriam, 2009; Yin, 2008). Once we collaboratively create our research questions during a working group session, some contributors plan on developing cross comparative case studies that address our research questions.

Given the multiple learners involved in these mediated field experiences, empirically documenting the results of our designs means tracing several lines of investigation simultaneously as we study how each of the following groups engages in *and develops as result of* learning opportunities in mediated field experiences:

A. K-12 students in the partner classrooms
B. Pre-service teachers (PSTs)
C. Cooperating classroom teachers (CTs)
D. Teacher educators (TEs)

The third goal of this working group requires us to work together in small groups to develop a clear research trajectory for each of the above lines of investigation based on researcher interest.

Plans for Active Engagement of the Working Group

Three working group sessions are planned to address the goals detailed above.

Session 1: Sharing our Experiences Implementing Mediated Field Experiences

In the first session, each of the eight working group members will share their current structure for implementing mediated field experiences at their institution, including current findings and research practices; each presentation will last approximately ten minutes. We will use Google Slides so that all presentation slides are in one slide deck to minimize transition time and a Google spreadsheet to organize the details of the different aspects of each implementation. Participants currently engaging in similar work will be invited to include their implementation data into the spreadsheet and included in the analysis. We end with a whole group discussion where participants begin initial identification of common features across institutions and settings.

Session 2: Co-Developing a Vision for Mediated Field Experiences

In our second session together, we will continue to collectively analyze the different features of the mediated field experiences across institutions. All participants will break into small groups, based on interest, where each group will analyze a different feature across institutions. Using open-coding techniques, each breakout group will work to classify a certain feature as essential or not across institutions. As a whole group, we will share findings and discuss disagreements to identify essential features of these mediated field experiences. We will work to organize these features to help others see how these ideas can "transfer" into different contexts. Our goal is to establish what features are "essential" to these experiences or which features lead to specific learning outcomes so that other MTEs may begin to see how they could incorporate these ideas into their own teacher preparation coursework or programs.

Session 3: Standardizing Our Data Collection to Create Clear Lines of Investigation

For our last session, we will identify the research questions that stem from our analysis and the data we want to collect to investigate how this is impacting PSTs' learning along with the learning of the others involved (K-12 students, classroom teachers, and even the teacher educator). Again, participants will break out into different groups based on their research interests--which aspect of their implementation would they like to discuss and investigate? For instance, some working group members are interested in creating observational tools to measure current PST teaching practices and provide a framework for giving feedback to PSTs both in the moment of teaching and after the fact. Some MTEs are interested in investigating the impact of these mediated field experiences on PSTs' content knowledge, and others are interested in exploring which instructional routines are helpful in supporting PSTs develop adaptive expertise. We also want to standardize data collection so that researchers can pool data and make claims across institutions, which will take place while working in the break out groups based on researcher interest. We will end the session with a whole-group share out of research questions, data collection procedures, and next steps the small groups want to take.

Next Steps and Future Directions

Members of this group have been working together since 2015 and all on this proposal plan to continue this work after our proposed meeting sessions. Our first steps are to publish our current findings. We are in the process of completing a manuscript for the *Mathematics Teacher Educator* that details our findings from our initial efforts mediating field experiences, and we are submitting a proposal to AMTE's 2019 Annual Conference where we plan to present on our observations of PST learning against the AMTE Standards. At this 2019 conference, we plan to meet again as a group to share our experiences and any further insights we have made. Each breakout group from Session 3 will continue pursuing the research agenda they set up for themselves based on their research questions and goals. Some members of the group are planning to apply for an NSF DRK-12 Grant to support an investigation of following our PSTs into the field to investigate any lasting impacts of the mediated field experiences on their teaching practices or preparedness to teach in their own classrooms. Finally, we will continue to share resources and pool data using our shared Google folder and work to disseminate new findings to the field.

The early results of our work has been so positive that a long-term goal is to motivate programmatic shifts at our individual institutions for different areas of preparation and encourage other MTEs to reconsider the structure of their coursework and fieldwork in their preparation programs.

References

Anderson, T. & Shattuck, J. (2012). Design-based research: A decade of progress in education research. *Educational Researcher*, *41(1)*, 16-25.

Akerson, V. L., Morrison, J. A. & McDuffie, A. R. (2006). One course is not enough: Preservice elementary teachers' retention of improved views of nature of science. *Journal of Research in Science Teaching*, *43*(2), pp.194–213.

Association of Mathematics Teacher Educators (2017). *Standards for Preparing Teachers of Mathematics*. Available online at amte.net/standards.

Ball, D. & Forzani, F. (2009). The work of teaching and the challenge for teacher education. *Journal of Teacher Education*, *60*(5), 497-511.

Borko, H. & Mayfield, V. (1995). The roles of the cooperating teacher and university supervisor in learning to teach. *Teaching and Teacher Education*, *11*(5), pp.501–518.

Campbell, S.S. & Dunleavy, T.K., (2016). Connecting University Course Work and Practitioner Knowledge through Mediated Field Experiences. Teacher Education Quarterly, 43(3), p.49.

Chi M.T.H. (2011) Theoretical Perspectives, Methodological Approaches, and Trends in the Study of Expertise. In: Li Y., Kaiser G. (eds) *Expertise in Mathematics Instruction*. Springer, Boston, MA.

Cobb, P., Jackson, K. & Sharpe, C., 2017. Compendium for Research in Mathematics Education. In Cai, Jinfa, ed. National Council of Teachers of Mathematics, pp. 208–236.

Forzani, F. (2014). Understanding "Core Practices" and "Practice-Based" Teacher Education: Learning from the Past. *Journal of Teacher Education*, 65(4), 357-368.

Frykholm, J. A. (1996). Pre-service teachers in mathematics: Struggling with the Standards. *Teaching and Teacher Education*, *12*(6), pp.665–681.

Gainsburg, J. (2012). Why new mathematics teachers do or don't use practices emphasized in their credential program. *Journal of Mathematics Teacher Education*, *15*(5), pp.359–379.

Grossman, P., Hammerness, K. & McDonald, M. (2009). Redefining teaching, re-imagining teacher education. *Teachers and Teaching: Theory and Practice*, *15*(2), 273-289.

Horn, I.S. (2007). Fast kids, slow kids, lazy kids: Framing the mismatch problem in mathematics teachers' conversations. *The Journal of the Learning Sciences, 16*(1), pp. 37–79.

Horn, I. S., & Campbell, S. S. (2015). Developing pedagogical judgment in novice teachers: Mediated field experience as a pedagogy in teacher education. *Pedagogies*, 10, 149–176.

Janssen, F., Grossman, P., and H. Westbroek. (2015). Facilitating decomposition and recomposition in practice-based teacher education: The power of modularity. *Teaching and Teacher Education*, 51: 137-146.

Kazemi, E., Franke, M. L., & Lampert, M. (2009). *Developing pedagogies in teacher education to support novice teachers' ability to enact ambitious instruction.* Paper presented at the annual meeting of the Mathematics Education Research Group of Australasia, Wellington, New Zealand.

Korthagen, F., & Wubbels, T. (2001). Learning from practice. In F. Korthagen (Ed). *Linking practice and theory: The pedagogy of realistic teacher education* (pp. 32-50). Mahwah, NJ: Lawrence Erlbaum Associates.

Lampert, M. (2010). Learning Teaching in, from, and for Practice: What do we mean? *Journal of Teacher Education*, *61*(1-2), 21-34.

Lampert, M. & Graziani, F. (2009). Instructional activities as a tool for teachers and teacher educators' learning. *The Elementary School Journal, 109*(5), 491-509.

McDonald, M., Kazemi, E. & Kavanagh, S.S., (2013). Core Practices and Pedagogies of Teacher Education A Call for a Common Language and Collective Activity. *Journal of Teacher Education, 64*(5), pp.378–386.

Merriam, S. B., 2009. *Qualitative research: A guide to design and implementation approach.* San Francisco, CA: Jossey Bass.

National Council for the Accreditation of Teacher Education (NCATE). (2010). *Transforming teacher education through clinical practice: A national strategy to prepare effective teachers*. Washington, DC: NCATE.

National Council of Teachers of Mathematics (2014). *Principles to actions: Ensuring mathematical success for all.* Reston, VA: NCTM.

Schoenfeld, A.H. (2006). Design experiments. Handbook of complementary methods in education research, pp.193–205.

Yin, R. K. (2008). *Case study research: Design and methods (*4th ed.). Thousand Oaks, CA: Sage.

Virmani R. Taylor, M., Rumsey, C., Box, T., Kazemi, E., Knapp, M., Lynch, S., … Woods, D. (2017). Bringing mathematics methods into classrooms. In S. E. Kastberg, A. M. Tyminski, A. E. Lischka, & W. B. Sanchez (Eds.), *Building Support for Scholarly Practices in Mathematics Methods* (pp. 233-248). Charlotte, NC: Information Age Publishing.

Zeichner, K. M. & Tabachnick, B. R. (1981). Are the effects of university teacher education 'washed out' by school experience? *Journal of teacher education*, *32*(3), pp.7–11.

DEVELOPING THEORY, RESEARCH, AND TOOLS FOR EFFECTIVE LAUNCHING

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Launching cognitively demanding tasks is a much valued, but little understood aspect of ambitious mathematics teaching. This working group supports the development of shared descriptions, theories, and tools to support teachers, teacher educators, and researchers in understanding and developing effective launching. We will begin to develop a shared understanding of the purposes of effective launches, as well as common challenges teachers face, and structures and skills that teachers can call upon when launching. The working group will provide a set of common artifacts that will ground discussion of purpose, structure, and skill. These artifacts will include demonstrations and video of launches and launching tools for teachers and teacher educators. The working group will support ongoing discussion and collaboration between researchers, teacher educators, and practitioners in the field that will contribute to our knowledge of effective launches.

Keywords: Classroom Discourse, Instructional Activities and Practices, Curriculum, Problem Solving

Introduction

The Problem

Mathematics education researchers and practitioners have long understood how cognitively demanding tasks can provide essential opportunities for student reasoning, problem solving, and conceptual learning. Almost as soon as researchers began touting the importance of demanding tasks, they have recognized that how teachers introduce, or launch, those tasks to students can fundamentally affect the quality of learning opportunities for students (Stein & Lane, 1996). Research has documented how teachers can reduce the demand of the task (and the quality of the learning opportunities) by introducing a specific method or procedure during the launch or by placing too much emphasis on the answer (Stein & Smith, 1998). On the other hand, when teachers help students make sense of the context, focus on important mathematical relationships, and clarify important vocabulary, students are able to have richer, more productive discussions about their solutions (Jackson, Garrison, Wilson, Gibbons, & Shahan, 2013). Clearly, skillful launching is an important element of a problem-solving lesson.

Despite the apparent widespread agreement on the importance of effective launching, it remains undertheorized and under-researched. Teachers looking for support in launching have few resources, and mathematics teacher educators and researchers do not have a widely shared detailed understanding of the purpose and structure of an effective launch, nor do they necessarily agree on the kinds of challenges that teachers face while launching demanding tasks.

This Working Group will provide a forum for researchers, teacher educators, and practitioners to share localized knowledge in an effort to build a shared understanding of launching. In particular, participants will identify and share understandings of launching through experiencing, analyzing, and observing a variety of launches and other activities. Participants will also examine a variety of tools designed to research, evaluate, plan, and teach launching. Finally, participants will have opportunities to identify potential projects for research, development and dissemination, while seeking out collaborators for these projects. In particular, researchers and practitioners will be encouraged to pool their resources and expertise to describe and share localized knowledge and develop theory and tools that will reflect and inform practice.

Literature Review

The importance of launching stems from an emphasis on tasks and their implementation in supporting ambitious learning goals for mathematics students. Almost 30 years ago, the National Council of Teachers of Mathematics (NCTM) publicly advocated for mathematics instruction that supported students in gaining deep, conceptual understanding of important mathematical ideas, and developing this understanding in the context of reasoning and problem solving (NCTM, 1989). During the 1990's researchers documented essential features of classrooms that supported these new, ambitious goals (see, for example, Hiebert et al., 1997; Yackel & Cobb, 1996).

One especially important aspect of teaching for understanding was cognitively demanding tasks that would present students with opportunities to use what they knew to solve non-routine problems, and, in the process, reason and justify their thinking to their peers. In short, if teachers wanted their students to learn how to reason and solve problems, they needed to give their students problems that they did not already know how to solve (Stein & Lane, 1996). Researchers began by describing the characteristics of demanding tasks themselves, helping teachers identify and adapt tasks for their students (Stein, Grover, & Henningsen, 1996; Stein & Lane, 1996). Such tasks proved challenging to enact, however. Stein and Smith (1998) identified maintaining the cognitive demand of these tasks as a major challenge for teachers, describing six factors associated with the maintenance of the demand, and seven factors associated with the decline of these demands.

Others have sought to support teachers in maintaining demand through describing specific effective teaching practices. O'Connor, Anderson, and Chapin (2003) documented and described specific moves that supported discourse around demanding tasks. Stein, English, Smith and Hughes (2008) described five practices that teachers use to focus discussions on important student thinking and structure the content of those discussions so that students are more likely to make connections between solutions and connect those solutions to important underlying mathematical ideas.

One crucial aspect of enacting demanding tasks is how a teacher introduces, or "launches" the task. Early research indicated that although tasks may be designed to provide opportunities for problem solving and reasoning, the way teachers launched the task could lower the demand as experienced by students (Stein & Lane, 1996). Two of the seven factors associated with lowering the cognitive demand (Stein & Smith, 1998, cited above) seem especially germane to launching. First, teachers could indicate or even demonstrate a specific solution path, thus transforming the task for students from problem solving to replicating the teacher's solution. For instance, if students are told that a 30-pound bag of sand sells for $6 at the hardware store, and a 50-pound bag of sand sells for $9 at the lumberyard and asked which is the better deal, the teacher might suggest that they choose an amount of money that they can spend at each store and then see how much sand they get for that amount. Such a suggestion would take away the opportunity for students, themselves, to come up with this strategy, but would also make it less likely that students might come up with a variety of different strategies which would then provide opportunities for reasoning and making connections.

Second, teachers could also clarify specific aspects of the task, removing ambiguity and foreclosing opportunities to make sense of problematic situations. For instance, if students were asked to describe what was the best deal, a video streaming plan that costs $40 per month with unlimited streaming, or a plan that costs 50 cents per hour of streaming, the teacher could add that the customer's family streamed 75 hours of material each month. This would change the problem from one in which students would have to explain how each of the two plans might be better, depending on how much streaming the customer planned to do, to a problem with one clear answer, where students would not need to figure out that there could be two different answers depending on the anticipated streaming time.

Despite the clear evidence that ineffective launching can undermine students' opportunities to learn from demanding tasks, there has been little research on effective launching that could provide guidance for teachers and teacher educators. Early guidance was clearly directed towards helping teachers new to problem-based lessons develop a culture of inquiry in their classrooms. Effective launching included setting expectations for student collaboration, as well as managing the logistical challenges of extended group work (Stein, Smith, Henningsen, & Silver, 2009). Effective launches included clear descriptions of the final product, clear instructions for logistical details such as who sat where, and what tools were available for student use, and discussions of what good cooperative work looked and sounded like.

Jackson, Garrison, Wilson, Gibbons, and Shahan (2013) described several effective launching practices that led to deeper and more productive concluding discussions. The effective launches they studied shared these four features:

- Clarified context
- Highlighted important mathematical relationships
- Developed common language
- Maintained the cognitive demand

This framework certainly advanced our understanding of launching. However, applying the framework gives rise to difficult questions. How does one highlight the important mathematical relationships, for instance, without indicating potential solution paths? How does one clarify the context without removing ambiguity? How might effective teacher moves differ depending on the task or the students? A greater understanding of what effective teachers *actually do* when they launch, along with more of a shared understanding of the purpose of a launch might inform these questions.

A second article for practitioners described a specific technique for launching tasks, the think-aloud strategy borrowed from literacy research (Trocki, Taylor, Starling, Sztajn, & Heck, 2014). The authors described teachers engaging in thinking aloud when first presenting a problem, providing a model of how one might interrogate the problem, make sense of it, and begin to think about how to solve it. This technique, if properly applied, supported productive discourse and led to students' success on similar problems without lowering the cognitive demand of the tasks. However, the authors also warned against lowering the demand by introducing a procedure or method for solving it, and the teachers who used this specific technique worried about "crossing the line" into doing too much of the work for their students.

More recent research on teachers and their understanding of launching documented this dilemma quite clearly. González and Eli (2017) described how future and current teachers conceived of launching. They found that practicing teachers were keenly aware of the need to support students in making sense of the problem and providing access to the context, while at the same time not giving the problem away or telling them too much. More specifically, constraining a problem might make it more likely for the students to attend to the mathematical goal of the lesson but might also foreclose opportunities for modeling and problem-solving. González and Eli also found their participants saw launching as accomplishing a wide range of goals, including activating prior knowledge, motivating and engaging students, previewing and reviewing work necessary for the solution, providing hints, and clarifying important terms. Pre-service teachers were, on the whole, more concerned with motivation and engagement, while practicing teachers were more concerned with providing mathematical support through activating prior knowledge or clarifying important aspects of the problem.

Clearly, launching is an important aspect of enacting demanding tasks, fraught with possible dilemmas. However, there is little research to inform teachers and teacher educators. There is no published research on how curriculum materials support teachers in launching, but a cursory examination of a number of materials demonstrate a lack of alignment between and within curricula. Methods texts do describe the importance of involving students in the context, activating prior knowledge, being sure the problem is understood, and establishing clear expectations (see, for instance, van de Walle, Karp, & Bay-Williams, 2013 and Markworth, McCool, & Kosiak, 2015). However, there remains the difficult balance between some of the suggestions (solve a simpler version of the task, brainstorm possible strategies) and the warning to not lower the cognitive demand.

Despite this lack of theoretical background or specific research guidance, teachers have been launching demanding tasks for decades. Researchers have yet to tap what could be a potentially rich source of knowledge. Indeed, this gap between research and practice is a perennial problem for mathematics education and education at large. 20 years ago, Kennedy described the difference between the "expert knowledge" of researchers and the "craft knowledge" of practitioners (Kennedy, 1999). Earlier work also described this gap (see, for instance, Cuban, 1993; Lampert, 1985). This has led to calls for a new knowledge base for teaching that connects expert and practitioner knowledge, in which researchers collaborate more closely with teachers and practicing teachers have a greater say in identifying research problems and contexts (Hiebert, Gallimore, & Stigler, 2002). Launching is a potentially rich field for such collaborations.

Another potential source of knowledge about launches comes from other content domains. Science education researchers and teacher educators have also worked to introduce demanding tasks that support ambitious science teaching. Windschitl, Thompson, and Braaten (2008) write that science investigations often begin with "readings, video-clips demonstrations or even 'pre-modeling' laboratory activities" which will "motivate interest in some aspect of the natural world" and provide resources for students to "develop a tentative representation of the phenomenon." In addition, they suggest a number of "starter questions" that can engage students in inquiry. This description of how science teachers might launch inquiries are similar to videos used frequently in mathematics classes as a way to introduce problems. (See Meyer, 2011 for examples of math problems launched with videos).

Genesis of this Working Group

The desire for a working group focused on launching has grown out of several recent experiences shared by some of the group's organizers. This work began with a group of mathematics methods instructors designing a module to support pre-service teachers in learning to launch. The lack of research and theory on launching was evident to this group as they worked together to depict a launch. There were few common images of effective launches in the research literature, nor were there descriptions of the kinds of problems that students and teachers experienced during launches. As a result, the content and design of the module were based on designers' best guesses, representing untested hypotheses based on their experience and judgement.

Analyzing data and sharing results from pilots of this module further demonstrated the need for greater shared understanding of launches. In working to determine what students may have learned from the module, designer-researchers uncovered their own unstated assumptions about launching and found that they were quite different. For instance, while one researcher assumed that any discussion of solution strategies to the problem should not occur during the launch, another researcher saw such sharing as a common launching move that could be accomplished without lowering the cognitive demand of the task.

In 2016 these designer-researchers found both a high level of interest and a marked lack of consensus when they presented an early version of their work at PME-NA (Wieman & Jansen, 2016). The session was extremely well-attended and prompted animated discussion about how the launch depicted in the module did, or did not, represent good launching practice and how it compared with launches that attendees showed as models for their pre-service teachers. After this session, the presenters continued the conversation with participants and expanded it to include others not in attendance. This discussion resulted in a symposium at the 2018 AMTE annual meeting (Wieman, Jackson, Kelemanik, Land, & Tyminski, 2018). This symposium brought together researchers who had studied launches, teacher educators who were teaching teachers to launch effectively, and professional developers with extensive experience in schools. Again, this symposium was well attended, and generated extensive discussion, as well as a clear diversity of thought and experience. This working group seeks to expand the work begun at the symposium. Many of the activities and discussions planned for the three sessions of the working group will allow the participants delve deeper into issues raised at the symposium, and explore some of the ideas planned, but not addressed, in that session.

Goals of the Working Group

The leaders of this working group have the following long-term goal:

- Support mathematics teachers in launching tasks effectively, mathematics teacher educators in supporting teachers learning to launch, mathematics education researchers in generating knowledge about launching, and curriculum writers in supporting teachers launching by creating empirically-based, shared knowledge about launching cognitively demanding tasks.

In the service of this long-term goal, the organizers' aims during the three sessions at this conference are to:

- Facilitate an ongoing discussion about launching, surfacing assumptions about effective and ineffective launching practices.
- Provide a set of common artifacts to initiate this discussion and prompt participants to develop and share other artifacts to facilitate ongoing discussion and study.

- Foster collaboration and exchange between researchers, teachers and teacher educators, supporting productive partnerships that will add to the shared knowledge base about launching.
- Identify key research questions about launching that can help guide the field, and these partnerships.

Questions We Will Be Examining

The sessions will provide participants an opportunity to examine the following questions:

- What is the purpose of an effective launch?
- What challenges do teachers face when planning, enacting and evaluating launches?
- What are typical experiences for students in launches?
- Do effective launches look different depending on the task?
- How do we support teachers and pre-service teachers in developing skill in planning, enacting and reflecting on launching (and how might we improve these efforts)?
- How do we support teacher educators and professional developers in helping others get better at launching (and how might we improve these efforts)?

Connection to the Conference Theme

One of the central questions stemming from the conference theme of "Looking Back, Looking Ahead: 40 years of PME-NA" is how do we support ALL students through equitable teaching? At its heart, launching is a part of ambitious teaching that provides access and opportunity for ALL students to grapple with challenging mathematics. The small amount of extant research on launching begins to describe some of the conditions that must be present in order for all students to learn from engaging in demanding tasks and how those conditions might, or might not, be present for diverse groups of students. Beginning research has shown that how a teacher enacts a launch can either open up access to a wider range of students through establishing those conditions for all students, or how it can close off access to opportunities to learn for groups of students by failing to establish, or actively undermining those conditions. In addition, effective launches have the potential to support students in developing mathematical authority and agency as they make sense of the task at hand. The lack of research and work on launching represents an ongoing challenge to the field, and advancing this knowledge provides possible ways to address fundamental problems of access, authority and agency in an arena in which teachers actually have power and control. Effective launches have the potential to contribute greatly to equitable opportunities to learn from demanding tasks, and ineffective launches can really negatively affect students' access to those opportunities.

Plan for Sessions/Participation

The sessions for this working group will ground discussion and work in a series of shared artifacts. These artifacts will serve as a set of common points of reference for participants as they examine their own assumptions and work to answer the central questions of the working group. The sessions will begin by having participants experience a variety of launches, and then discuss their features and the assumptions about purposes, and challenges that lay beneath them. Participants will then look at a variety of tools that teachers, teacher educators and researchers use to support and study effective launching, connecting those tools to the central questions of purpose, structure and challenge. Finally, the group will begin to identify specific research projects that will add to the shared knowledge base for effective launching.

Session One

The main goal of this session will be to introduce the central questions of the working group and to begin sharing assumptions and knowledge about launching. The session will feature three actual launches that differ in important and interesting ways (i.e. content, grade level, length, extent to which students reason and talk about the problem or the context, extent to which the teacher introduces representations or tools, etc.) Participants will begin by writing down some thoughts about launches then watch a video of a launch and engage in a discussion in which they compare the launch they just watched with the purposes, structures, and challenges they envision for launches. Then, participants will experience two more launches facilitated by organizers of the working group, both of which are clear routines with specific structures, and both of which differ in significant ways from each other and from the initial example. Again, these will provide common experiences which will allow participants to engage in more conversation about the purpose, structure and challenges of launches. By the end of the first session, the group will have developed some general areas of agreement, some ongoing open questions, and have some common images and shared vocabulary that they can use as touchstones for further discussion.

Session Two

The main goal of this session will be to continue the discussion of purpose, structure and challenge of launches, with a particular focus on tools for teachers, teacher educators and researchers. As in the first session, participants will engage with a common set of artifacts, which will then act as the basis for further discussion.

The session will begin with participants planning a launch using a set of common curriculum materials. Groups will share the kinds of issues they thought about, questions they asked themselves, how their understanding of purpose, structure and challenge informed their planning, and how they used the materials to support their planning. Participants will then watch a video of a teacher engaging in a think-aloud while planning a launch for the lesson using the same materials. They will talk about how their own planning compared with that of the teacher in the video, and what issues they face in supporting teachers in planning and enacting launches.

Participants will then examine and analyze a variety of launching tools. Organizers will have available a set of planning tools, a set of observation/evaluation tools, and a set of curriculum materials. Participants will examine and analyze the tools with the following questions in mind:

- What assumptions do these tools seem to have about the purpose, structure and possible challenges of an effective launch?
- What specific scaffolds or supports do the materials offer?
- How do participants envision teachers engaging with these tools? How do the tools align (or not) with how participants think teachers (or teacher educators, or researchers) work, think and make decisions?
- How might we revise these tools in light of the conversations we have been having?
- What questions do we have about these tools and their use?

Depending on the feeling of the group, we might look at one or two of the tools as a large group, deciding which tools and in what order based on the conversation from the first session and the first half of the second session. Alternatively, we might create separate groups, each of which concentrate on specific tools. Participants could engage with whatever tool they find most interesting or compelling. There would be time at the end of the session for each group to share their findings with the whole.

Session Three

The main goal of the third session is to identify, support and stimulate further work. This could take several different forms, but in general will involve brainstorming, identifying and exploring possible projects and/or partners and making commitments for future work. Depending on the emerging interests and dynamics of the group, each of these processes could take place in the context of a whole group session, smaller group sessions, or some combination of the two. There will also be an opportunity for participants to share work that they have already begun, and possible opportunities for them to pilot materials and instruments or ask for help analyzing data or providing feedback. Finally, the organizers will bring some specific projects to the table for those interested. These may include initial planning for a conference devoted to the topic, researching curricular supports for launching, developing and research specific launching routines or researching teacher planning and enactment of launches. The session will end with participants writing and sharing specific commitments to future work.

Possible Future Work

We are hopeful that this working group will facilitate collaborations and identify possible avenues of further work that could result in a wide variety of products and future work together. We see this work unfolding in three interconnected arenas, theoretical and empirical research on launching, materials and tools for teacher education, and materials and tools for classroom teachers.

Theoretical and Empirical Research on Launching

One goal for this working group is to encourage and facilitate work that leads to more detailed theoretical frameworks for enacting and studying launching. We anticipate that discussions at the working group will stimulate theoretical pieces that clearly define the purpose of a launch, as well as hypothesize potential challenges that teachers and students face during the launch. This work would both inform and be informed by empirical work that could validate and refine this theory.

Materials and Tools for Teacher Education

Another goal of the working group is to stimulate the development and dissemination of materials that could support teacher educators in supporting pre-service and in-service teachers in learning to launch more effectively. These could include protocols and templates for planning, evaluating, and enacting launches, similar to those shared in session two described above. They could also include representations, approximations and decompositions of practice (Grossman, et al., 2009), which could support teachers in focusing on specific aspects and issues in launching.

Materials and Tools for Classroom Teachers

A final goal of the working group is to stimulate the development of tools for teachers in launching complex tasks. In addition to the tools described above for teacher educators, the working group will support curriculum developers and professional developers as they create and refine materials for teachers.

Possible Products

Ideally, each of these three areas will support each other. Tools for teacher educators could also work as supports for teachers, while also providing data collection tools for researchers. Theory could support the design of such materials.

Specific products growing out of this working group could take a variety of forms, forms that span the three areas above, or concentrate in one single area. Some possible outcomes we envision are:

- Groups of people working to develop, refine and disseminate specific routines or structures for launching tasks
- Research projects examining launches.
- A conference on launching
- An edited book on launching, with articles that might, in part, stem from work catalyzed by this working group
- A practitioner book about launching
- A series of research articles, perhaps in a special issue of a research journal, on launching
- Exploring funding opportunities to help support any of the projects above
- Curriculum development or research

References

Cuban, L. (1993). *How teachers taught: Constancy and change in American classrooms, 1890-1990* (2nd ed.). New York: Teachers College Press.

González, G., & Eli, J. A. (2017). Prospective and in-service teachers' perspectives about launching a problem. *Journal of Mathematics Teacher Education*, *20*(2), 159-201.

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. (2009). Teaching practice: A cross-professional perspective. *Teachers College Record*, *111*(9), 2055-2100.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K. C., Wearne, D., Murray, H., . . . Human, P. (1997). *Making sense: Teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann.

Hiebert, J., Gallimore, R., & Stigler, J. W. (2002). A knowledge base for the teaching profession: What would it look like and how can we get one? *Educational Researcher*, *31*(5), 3-15.

Jackson, K., Garrison, A., Wilson, J., Gibbons, L., & Shahan, E. (2013). Exploring relationships between setting up complex tasks and opportunities to learn in concluding whole-class discussions in middle-grades mathematics instruction. *Journal for Research in Mathematics Education*, *44*(4), 646-682.

Kennedy, M. M. (1999). Ed schools and the problem of knowledge. In J. D. Raths & A. C. McAninch (Eds.), *What counts as knowledge in teacher education?* Stamford, CT: Ablex.

Lampert, M. (1985). How do teachers manage to teach? Perspectives on problems in practice. *Harvard Educational Review*, *55*(2), 178-194.

Markworth, K., McCool, J., & Kosiak, J. (2015). *Problem solving in all seasons: Prekindergarten - grade 2*. Reston, VA: National Council of Teachers of Mathematics.

Meyer, D. (2011). The three acts of a mathematical story. Retrieved from http://blog.mrmeyer.com/2011/the-three-acts-of-a-mathematical-story/

National Council of Teachers of Mathematics. (1989). *Principles and standards for school mathematics*. Reston, VA: Author.

O'Connor, C., Anderson, N. C., & Chapin, S. H. (2003). *Classroom discussions: Using math talk to help students learn*. Sausalito: Math Solutions Publications.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, *10*(4), 313-340.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, *33*(2), 455-488.

Stein, M. K., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. *Educational Research and Evaluation*, *2*(1), 50-80.

Stein, M. K., & Smith, M. S. (1998). Mathematical tasks as a framework for reflection. *Mathematics Teaching in the Middle School*, *3*(4), 268-275.

Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2009). *Implementing standards-based mathematics instruction: A casebook for professional development* (Second ed.). Reston, VA: National Council of Teachers of Mathematics.

Trocki, A., Taylor, C., Starling, T., Sztajn, P., & Heck, D. (2014). Launching a discourse-rich mathematics lesson. *Teaching Children Mathematics, 21*(5), 277-281.
van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2013). *Elementary and middle school mathematics: Teaching developmentally* (Eighth ed.). Boston: Allyn & Bacon.
Wieman, R., Jackson, K., Kelemanik, G., Land, T., & Tyminski, A. (2018). *Learning to launch, launching to learn: Shared images of effective launches*. Symposium presented at the Annual Meeting for the Association of Mathematics Teacher Educators (AMTE), Houston, TX.
Wieman, R., & Jansen, A. (2016). Improving pre-service teachers' noticing while learning to launch. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 837-844). Tucson, AZ: The University of Arizona.
Windschitl, M., Thompson, J., & Braaten, M. (2008). Beyond the scientific method: Model-based inquiry as a new paradigm of preference for school science investigators. *Science Education*, *92*(5), 941-967.
Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27*(4), 458-477.

MATHEMATICAL PLAY: ACROSS AGES, CONTEXT, AND CONTENT

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Mathematical play has a fairly short history, with strong roots further back in time (e.g., Papert, Montessori), and understanding the role of mathematical play from early childhood to adulthood is, as yet, unmapped. This working group will provide a community space to explore and discuss mathematical play broadly, ranging from early childhood to undergraduate learners, with an emphasis on physical and digital interactions designed specifically to support mathematical play. Each day will focus specifically on a different approach to and definition of mathematical play. Day 1 will focus on early childhood mathematical play with pattern blocks; Day 2 will focus on mathematical play in videogames; and Day 3 will focus on using Rubik's Cubes to investigate undergraduate mathematics, namely algebraic groups. Throughout the sessions, we will be examining threads of common ground that will assist in developing a more flexible and appropriate model of mathematical play that can inform design of environments and activities across age groups, content, and context.

Keywords: Instructional activities and practices; Design experiments; Technology

Mathematical play has a fairly short history, with strong roots further back in time (e.g., Papert, Montessori). The majority of research on this topic stems from early childhood research on play, and researchers have begun to map out the phenomenon of mathematical play in young children (see Wager & Parks, 2014, for a review). This approach identifies the mathematical play children naturally engage in during open-ended play activities, and explores how to further mathematize that play and the consequent learning. In addition, researchers have explored how mathematicians in the course of their work engage in mathematical play (e.g., Holton, Ahmed, Williams, & Hill, 2001). Given that young children and mathematicians both engage naturally in mathematical play, there is an intriguingly underexplored area of promise between those two populations. A small number of researchers have examined how to support students in approaching mathematics problems with a playful bent (e.g., Steffe & Wiegel, 1994; Holton et al., 2001), but understanding the role of mathematical play from early childhood to adulthood is, as yet, unmapped.

This working group will provide a community space to explore and discuss mathematical play broadly, ranging from early childhood to undergraduate learners, with an emphasis on physical and digital interactions designed specifically to support mathematical play.

The Origin of this Working Group

The authors have been engaging in researching mathematical play in their own arena of interest for some time and are beginning to engage in conversations across age groups (e.g., early childhood, undergraduate), content (e.g., patterns, fractions, algebraic groups), and contexts (e.g., Rubik's cubes, videogames). This working group proposal is the first collaborative effort from this diverse group of researchers.

Our goal for this working group is to facilitate mathematical play experiences and discussions around open research questions that transcend our individual lines of research, such as:

- What is the nature of mathematical play across the age/grade bands?
- What are the features, characteristics, and affordances of mathematical play?
- How might context (e.g., physical, digital) influence mathematical play?
- How might content (e.g., fractions, group theory) influence mathematical play?
- How might factors such as gender, race and ethnicity, and parental income/education level influence experience of and access to mathematical play?
- How are mathematical play and mathematical learning related?
- How does mathematical play influence problem solving?
- How do mathematicians (experts) engage in mathematical play, and how might that mindset be fostered for learners (novices)?
- When might a didactical introduction to the content support more productive mathematical play?
- How might mathematical play support or influence learning in other disciplines (e.g., a broader STEM perspective)?

Although answering all of these is beyond the scope of possibility for our working group, we will use these questions to facilitate and orient discussions during each of the three days, and as potential topics for future collaborative investigations.

Definitions of Mathematical Play

There are a variety of definitions of mathematical play, each emerging from different contexts and with different ages. In this section, we briefly review a selection of those definitions in order to contextualize the activities we have planned for the working group sessions.

Steffe and Wiegel (1994) conducted a teaching experiment with a digital microworld designed to support operations with quantities, and the role of the instructor was directly to shift students from engaging in independent mathematical activity into engaging in mathematical play. Appropriately, they then defined mathematical play as “independent mathematical activity with a playful orientation” (p. 27). Sarama and Clements (2009) took a different approach: instead of focusing on supporting the shift from independent mathematical activity to mathematical play, they distinguish between *play that involves mathematics* and *mathematical play*. They give examples of the former, including a store in the classroom where one student plays the shopkeeper, and another purchases toy dinosaurs for a dollar apiece, and the students engage in play while also engaging in counting, place value, and practicing arithmetic. For mathematical play, on the other hand, children play with the mathematics itself, and the authors describe a four-year-old who is playing with three of five train engines, and first describes those three train engines as being numbers 1, 2, and 3 – thus missing engines numbered 4 and 5. Then she decides that the trains she has are actually numbered 1, 3, and 5, leaving trains 2 and 4 missing – as Sarama and Clements (2009) note, “she was playing with the idea that counting words themselves could be counted” (p. 327), that is, she was playing directly with mathematics.

Holton et al. (2001) define mathematical play as “that part of the process used to solve mathematical problems, which involves both experimentation and creativity to generate ideas, and using the formal rules of mathematics to follow any ideas to some sort of a conclusion” (p. 403). They identify six components of mathematical play:

(1) It is a solver-centred activity with the solver in charge of the process;
(2) It uses the solver's current knowledge;
(3) It develops links between the solver's current schemata while the play is occurring;
(4) It will, via 3, reinforce current knowledge;
(5) It will, via 3, assist future problem solving/mathematical activity as it enhances future access to knowledge;
(6) It is irrespective of age. (p. 404)

They additionally note that mathematical play will not necessarily directly lead to a correct solution of the problem being played with – in fact, one of the important components of mathematical play is that it "provides a non-threatening environment where incorrect solutions are not read as mistakes and may lead to a better understanding of the problem and/or the confrontation of misconceptions" (p. 404).

Another framework for mathematical play (Williams-Pierce, 2016, 2017) also highlights the importance of a non-threatening environment where mistakes are perceived as natural and appropriate. This framework emerges from a blend of scholarship on mathematical learning and videogames research, the latter of which has embraced the conceptualization of *failure* as an important and often enjoyable experience within the realm of gameplay (e.g., Juul, 2009; Litts & Ramirez, 2014). Williams-Pierce (2016, 2017) defined mathematical play as *voluntary engagement in cycles of mathematical hypotheses with occurrences of failure*, and describes five features of digital contexts that support such mathematical play:

(1) consistent and useful feedback;
(2) high enough levels of difficulty and ambiguity that players experience frequent failure that is closely paired with the feedback;
(3) non-standard mathematical representations and interactions;
(4) mathematical notation introduced late or not at all; and
(5) the legitimate possibility of alternative conceptual paths for successful progression.

Each of the definitions of mathematical play described above emerge from different contexts: some digital (Steffe & Wiegel, 1994; Williams-Pierce, 2016), some physical (Sarama & Clements, 2009), some paper and pencil-based (Holton et al., 2001). One of the crucial open questions that we highlighted above is how the definitions may vary in the features that they prioritize, due to the differing contexts. For example, might Holton et al. (2001) and Williams-Pierce (2016) find common ground if they examined similar contexts? Or are their approaches too fundamentally different to ever come to agreement? Each session of the working group is oriented around a specific definition and operationalization of mathematical play, so that attendees have concrete experiences grounded in different definitions to facilitate discussion across these different frameworks.

Working Group Schedule and Activities

For each of the three Working Group sessions, we plan to begin with a hands-on mathematical play activity designed from a different mathematical play approach. The first author has been involved in conducting PME-NA working groups in the past (Nathan et al., 2016, 2017), and found that beginning with a relevant activity quickly develops fruitful discussions between participants. In particular, working groups tend to have a variety of attendees with a wide-ranging level of expertise in the topic, and grounding the discussion in shared experiences serves to support a sense of community. Consequently, we will facilitate a mathematical play experience each day, then guide the discussion towards the mathematics at play (pun intended) and the specific characteristics of that mathematical play. During these group discussions, we will regularly orient the conversation specifically towards the open research questions listed above. We will take notes during these conversations, and conclude each session by collecting names and emails of working group attendees.

Each day will focus specifically on a different approach to and definition of mathematical play. Day 1 will focus on early childhood mathematical play with pattern blocks; Day 2 will focus on mathematical play in videogames; and Day 3 will focus on using Rubik's Cubes to investigate undergraduate mathematics, namely algebraic groups. While each day's session moves up in age (from early childhood to undergraduate), we will also be moving down in terms of established research. In particular, Day 1 showcases the most researched area of mathematical play and will be primarily about introducing attendees to the rich history of early childhood mathematical ply; Day 2 focuses on comparing two instantiations of mathematical play, and seeks to identify similarities and differences across the digital designed contexts; and Day 3 involves the initial design of an undergraduate mathematical play experience, and will focus on gathering input from the attendees on potential areas of redesign.

Day 1 Session – following Wager & Parks (2014)

The first session will focus on mathematical play in early childhood. Mr. Reimer will engage participants in a form of pattern block puzzle play that involves spatial reasoning and the composition and decomposition of shapes. While mathematical play is a generally new area of investigation, Wager and Parks (2014) found that "studies involving block play are the most well documented and longest established" (p. 217), although spatial learning with puzzle play is still under-examined. Furthermore, most geometry and spatial learning studies have taken place in artificial contexts (e.g., research labs), rather than natural contexts. Consequently, this activity relies upon early childhood play frameworks that emphasize child agency through spontaneous interaction, choice, and opportunities for repeated trials (Wager & Parks, 2014), with a focus on exploration with physical objects to support the development of mental transformations.

Mr. Reimer will bring sets of pattern block puzzles for the participants (see Figure 1) – for use in solo activity or small groups, depending upon the number of attendees. He will highlight the importance of spontaneous interaction, choice, and repeated trials, and throughout the session activity, he will provide orienting questions and comments–such as initiated constraints and challenges to extend or deepen the activity–to help participants explore possibilities and mathematize their play.

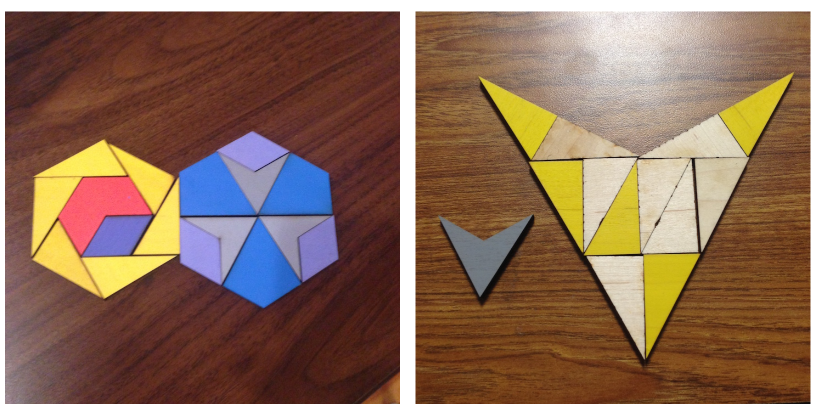

Figure 1. A sample of puzzle designs participants may create.

The activity will conclude with an open discussion that focuses on the affordances of the physical materials in encouraging mathematical play, the importance of orienting questions, and the role of narratives in describing and supporting participants' play activity. Then we will guide the discussion towards the open research questions listed above.

Day 2 Session – following Williams-Pierce (2017)

The second activity will focus on mathematical play in digital contexts. Dr. Williams-Pierce will coordinate this activity by bringing two games that were designed to support mathematical play: *Dragonbox 12+* by WeWantToKnow (a commercial learning game that focuses on balancing equations), and her in-revision game, *Rolly's Adventure* (which focuses on the development of understanding multiplying fractions). Both of these games support mathematical play and learning in distinctly different ways: *Dragonbox 12+* through faithfully enacting mathematical phenomena with novel representations and interactions; and *Rolly's Adventure* through secretly modeling the operation of multiplication and requiring players to engage repeatedly in interpreting feedback through failure in order to unpack that secret operation. See Figure 2 for screenshots of both games.

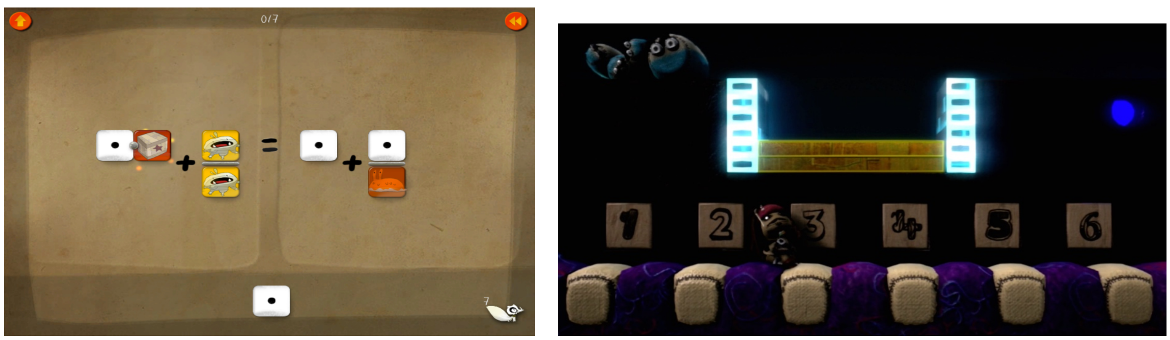

Figure 2. A screenshot each of *Dragonbox 12+* (left) and the original *Rolly's Adventure* (right).

We chose those particular two games for a variety of reasons. First, having the designer of *Rolly's Adventure* present means that she can unpack any of the hidden design decisions behind the game that may emerge as participant questions during play and discussion. Second, while the *Dragonbox* series is highly popular (with over a million downloads worldwide, and a *Wired* article that touted the original game's release; Liu, 2012), co-authors of this working group have strongly divergent views about how mathematical the play in *Dragonbox 12+* actually is. Consequently, we anticipate that these two games will provoke a lively discussion!

Both games are iPad games, and Dr. Williams-Pierce will bring ten iPads to share with participants, as well as instructions for installing her game for free on their own touchpad devices. Depending upon the number of attendees, this activity may be in pairs or small groups. Half the participants will play *Dragonbox 12+*, and half *Rolly's Adventure*, while taking guided notes on their experience of mathematical play with the games. These guided notes will be based upon the five features described above (Williams-Pierce, 2016, 2017). For example, the guided notes will include questions like: how does the game give you feedback on what you are doing well or poorly? How does the experience of *failure* influence how you attend to feedback? What, if any, is the role of mathematical notation? After playing, Dr. Williams-Pierce and Dr. Ellis – as the two authors who disagree about mathematical play in *Dragonbox 12+* – will co-facilitate a discussion about how the games instantiate those five features differently, and how the participants felt their experiences were or were not mathematical play.

Day 3 Session – following Holton et al. (2001)

First, Dr. Plaxco will discuss his preliminary work in designing activities to engage undergraduate students in mathematical play with Rubik's Cubes and connections to the cube's associated algebraic group structure. This will entail an introduction to several algorithms that can be useful for manipulating specific pieces on the cubes as well as a discussion of the usefulness of some group theoretical constructs (e.g. conjugation and commutators) in generating designs on the cubes (see Figure 3). Holton et al. (2001) note that "Mathematical play involves pushing the limits of the situation and following thoughts and ideas wherever they may lead. Hence there are no obvious short-term goals for mathematical play; it is designed to allow complete freedom on the part of the solver to wander over the mathematical landscape" (p. 403). Rubik's Cubes allow for such mathematical play, though players are typically less familiar with the rules and operations of the cubes because the setting is noncommutative, which allows for surprising results when trying to generalize from experiences, and because the "landscape" is rather large – a 3x3x3 cube has over $4.3\text{x}10^{17}$ positions.

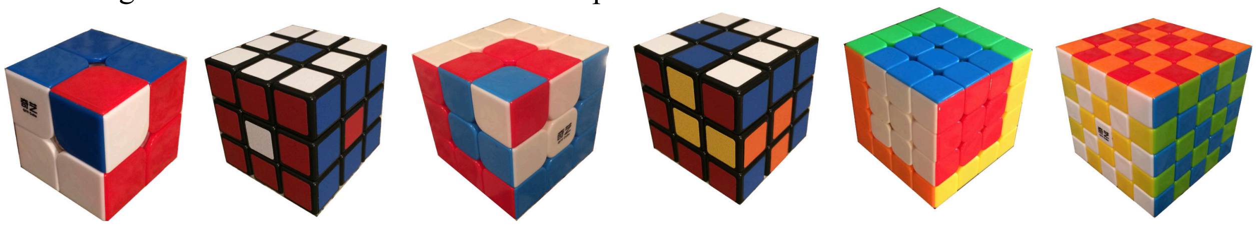

Figure 3. A Sample of Cube Designs That Working Group Participants May Attempt

Dr. Plaxco will distribute a class set of solved 2x2x2 and 3x3x3 Rubik's Cubes so that each member of the working group will have access to a cube. Members will also receive a laminated sheet with instructions for each of the algorithms discussed and a second sheet with possible designs that members can set out to re-create. One difficulty Dr. Plaxco has encountered with introducing Rubik's Cubes to students is the need for prolonged orientation to gain a sense of how to use algorithms to produce desired outcomes. Because of this, we will sequence the tasks so that participants will work to change a solved cube into increasingly difficult designs, incorporating longer algorithms and combining algorithms in new and different ways. Participants will also be invited to explore the possibilities of what they can create with the algorithms and generate their own designs on the cubes. We will give participants iPads to videotape their manipulations of their cubes in order to help them remember the sequences of moves they make. We will ask the participants to try and develop notations to help record these moves as well. This activity will culminate with a "Match Game" in which groups of participants will create designs, recording and notating the required moves and challenge other teams to re-create their designs. We will then close this session with a discussion of potential directions for activities with undergraduate students, including avenues for collaborations in researching mathematical play with the Rubik's Cubes.

After Dr. Plaxco's group activity, in order to conclude the working group, we will discuss our experiences across each of the three days. In particular, we will focus on the three mathematical play approaches emphasized on different days, and how the differing contexts and content may have influenced the development of these approaches, following the open research questions identified at the beginning of the document. We will seek to find threads of common ground that will assist in developing a more flexible and appropriate model of mathematical play that can inform design of such environments and activities across age groups, content, and tools. Finally, we will conclude by identifying next steps for the working group members, as outlined in the following section.

Future Plans

This working group is the first step in establishing a network of support for designing and examining mathematical play at all ages. First, we will develop a plan for a white paper about mathematical play that draws out common themes across the differing ages and content that have been researched thus far, and identifies aspects of the frameworks and definitions that may be irreconcilable. For example, might unstructured mathematical play and structured mathematical play be fundamentally different? Are there commonalities about mathematical play that can be established across paper-and-pencil activities, physical manipulatives, and digital contexts? How might audience (e.g., voluntary players who can stop playing at any time, students who have no choice but to participate, research participants who chose to be part of the research) influence the experience of mathematical play? In this paper, we will explore those questions, and identify the particular next steps that should be undertaken to further understand and unpack the phenomenon of mathematical play. Second, we will investigate future potential conference proposals or journal articles that draws across different areas of mathematical play expertise, in order to begin productively synthesizing the phenomenon. For example, one of the discussions that developed during the crafting of this proposal revolved around the use of Holton et al.'s (2001) components of mathematical play for the Rubik's cube activity, and how certain aspects of the Williams-Pierce (2016, 2017) framework might apply better. One potential line of investigation then is to examine the possibility of synthesizing the two frameworks in a way that accounts for the very different contexts they initially emerged from.

Third, we will set up a listserv for the attendees, so that they find it convenient and simple to stay in touch with others interested in mathematical play. Immediately after the conference, we will share this listserv so that participants may continue discussing any potential collaboration plans developed through discussions during the working group. Finally, we plan to submit for an NSF Advancing Informal STEM Learning: Conferences grant in order to conduct yearly meetings that focus solely on mathematical play throughout the lifespan. The participants at the Mathematical Play Working Group at PME-NA 2018 will be the first list we contact to invite to apply for this opportunity.

References

Holton, D., Ahmed, A., Williams, H., and Hill, C. (2001). On the importance of mathematical play. *International Journal of Mathematical Education in Science and Technology*, 32, 401-415.

Juul, J. (2009) Fear of Failing? The Many Meanings of Difficulty in Video Games. In Mark J. P. Wolf & Bernard Perron (Eds.), *The Video Game Theory Reader 2* (pp.237-252). New York: Routledge.

Litts, B., & Ramirez, D. (2014). *Making people fail: Failing to learn through games and making.* Proceedings of the Games Learning Society (GLS) Conference. Madison, Wisconsin.

Liu, J.H. (2012) https://www.wired.com/2012/06/dragonbox/

Nathan, M.J., Ottmar, E.R., Abrahamson, D., Williams-Pierce, C., Walkington, C., & Nemirovsky, R. (2016). Embodied Mathematical Imagination and Cognition (EMIC) Working Group. In M.B. Wood, E.E. Turner, M. Civil, M., & J.A. Eli (Eds.), *Proceedings of the Psychology of Mathematics Education – North American chapter conference* (pp. 1545-1548). Tucson, AZ: The University of Arizona.

Nathan, M.J., Williams-Pierce, C., Abrahamson, D., Ottmar, E., Landy, D., Smith, C., Walkington, C., DeLiema, D., Soto-Johnson, H., Alibali, M.W., & Boncoddo, R. (2017). *Embodied Mathematical Imagination and Cognition (EMIC) Working Group.* In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th annual conference of the North American chapter of the International Group for the Psychology of Mathematics Education* (pp. 1497-1506). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Sarama, J., & Clements, D. H. (2009). Building blocks and cognitive building blocks: Playing to know the world mathematically. *American Journal of Play*, *1*(3), 313–337.

Steffe, L.P., & Wiegel, H.G. (1994). Cognitive play and mathematical learning in computer microworlds. *Journal of Research in Childhood Education, 8*(2), 117-131.

Wager, A. A., & Parks, A. N. (2014). Learning mathematics through play. In L. Brooker, M. Blaise, & S. Edwards (Eds.), *The SAGE Handbook of Play and Learning in Early Childhood* (pp. 216–227). London: SAGE Publications. http://doi.org/10.4135/9781473907850.n19

Williams-Pierce, C. (2016). *Provoking mathematical play through hidden deep structures.* In C.K. Looi, J.L. Polman, U. Cress, and P. Reimann (Eds.), Transforming Learning, Empowering Learners: The International Conference of the Learning Sciences, Vol. 2 (pp. 1241-1242). Singapore: International Society of the Learning Sciences.

Williams-Pierce, C. (2017, April). *Designing provocative objects for mathematical play.* Paper presented at the 2017 American Educational Research Association Annual Meeting and Exhibition. San Antonio, TX.

SPECIAL EDUCATION AND MATHEMATICS WORKING GROUP

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This Working Group will continue the conversation started since PME-NA-34, with a goal to move forward the teaching and learning of mathematics involving students with learning disabilities or difficulties in mathematics (LDM). This Working Group is rooted in following premises: (1) students with LDM are capable of and need to develop conceptual understanding of mathematics, and (2) special education as well as inclusive classroom instruction need to transition toward this focus. Participants will continue (a) the collaborative research agenda for the group, and (b) the dissemination effort through publications that reflect the cross-disciplinary collaborative work of this group within the context of international perspectives.

According to US National Report Card (National Assessment of Educational Progress [NAEP], 2015), 85% of 4th grade students with disabilities performed below the proficiency in mathematics, and 100% of 8th grade students with disabilities performed below the basic level in mathematics. In fact, about five to ten percent of school-age children are identified as having mathematics disabilities and students whose math performance was ranked at or below the 35 percentile are often considered at risk for learning disabilities or for having learning difficulties in mathematics (LDM). In spite of substantial work on learning disabilities or difficulties in mathematics conducted and disseminated by special educators, it seems current practices remain highly ineffective in promoting desired changes.

That being said, emerging collaborations among special education and mathematics education researchers have been showing promising results. Whether this emerging research will change the current math instructional practice involving students with disabilities remain uncertain. "To most teachers, math is just about calculation" and calculation math is "drill and kill!" (Janell Uerkwitz, NSF-I-Corps program interview note, March 2nd, 2018**).** The underlying common approach the Working Group has been developing revolves around paradigm shift away from the focus on behavioral and procedural oriented teaching to guided constructivist approach that focus on conceptual learning and teaching.

The purpose of our Working Group is to not only facilitate the collaboration between math education and special education researchers with an intention to move forward the teaching and learning of mathematics involving students with LDM, but also to disseminate the work resulting from such collaboration to inform the research and practice in the field.

This Working Group is formed to create sustainable opportunities for collaborative work between researchers and practitioners from both the fields of math education and special education, who are working with students with LDM. Understanding how students with LDM develop mathematics concepts and skills and how teaching and instructional environment can promote this process is an important dimension of the psychology of mathematics education, has several implications for both research and practice. First, practitioners in both general and special education can benefit from diverse perspectives as well as empirical evidence and gain a

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.

richer understanding of how students with LDM learn mathematical concepts. Secondly, active study of the development of mathematics concepts and skills for students with disabilities provides both researchers and practitioners with mechanisms for moving toward a methodological focus on pedagogy rooted in assessment of what students with disabilities are capable of learning.

For the purposes of continuing the conversation around mathematics in special education, this group is concerned with students who have significant issues with mathematics, including:

- students with learning disabilities specific to mathematics (MD)
- students with cognitive differences in how they understand and process number
- students who are placed in special education and have difficulties with mathematics

We refer to these students as having learning disabilities or difficulties in mathematics (LDM) in the remainder of this paper.

History of the Working Group

Our PME-NA/PME Working Group has met six times; each year our group had good participation of both returning and new members. In 2012, 15 researchers (faculty and graduate students) and 2 practitioners met during PME-NA in Kalamazoo, MI. This first meeting specifically focused on better understanding mathematical learning disabilities (MD). The Working Group began with a discussion of the issues around identification and definition of MD. In particular, the group discussed the unique characteristics of students with MD (e.g., slow speed of processing despite average reasoning; fundamental issues with number sense; over learning of procedural knowledge at the expense of mathematical reasoning) and implications for instruction and assessment. We took up a theoretical stance that positioned disability as an issue of diversity and considered the origin of the disability as the inaccessibility of instruction rather than a defect within the individual. Members shared videotapes of various students with MD solving problems in assessment and teaching situations and discussed the need for teachers to target and teach toward the specific mathematical strengths and weaknesses demonstrated by the student. We further discussed at what point(s) the learning paths of students with MD may differ from what is documented among students in general education, how existing developmental trajectories may or may not fit the population of students with MD, and the need to expand or further document current trajectories to include students with MD. Moreover, discussions focused on issues surrounding motivation related to the design and use of instruction, mathematical tools, and mathematical tasks. A rich discussion was held concerning the nature and sequencing of mathematical tasks, the use of concrete and pictorial representations and the extent to which they are and are not supportive of the abstraction of mathematical concepts for this population, and the need for increased research to inform the creation of practitioner tools and resources.

During the first year of our Working Group, our focus was specifically on MD- those students with a biological and cognitively-based difference in how their brain processes numerical information. Based on our discussions during the first year of our Working Group we decided to expand from a narrow focus on MD to a more inclusive focus on students in special education who struggle with mathematics (i.e., learning disabilities/difficulties in mathematics, LDM). This not only avoids the definitional issues at the forefront of the field (i.e., the lack of assessments to accurately identify students with MD and the resulting conflation of low achievement and MD), but also more accurately reflects the diversity of interests of the members of this group.

In the second year of our Working Group (2013) fourteen participants focused on a collaboration that was a result of the contacts that were made during the first Working Group. In this collaboration two faculty members worked together on a teaching experiment about fraction knowledge, a compliment to a 2013 funded National Science Foundation (NSF) CAREER project (Hunt, 2013). Their collaboration resulted in each bringing unique expertise; the mathematics education scholar brought insight into the mathematical thinking of the student, while the special education scholar brought insight into learning differences. The goal of the teaching experiment was to document how the foundation scheme of unit fractions (1/n) evolves in the mathematical activity of two cases of students with learning disabilities. The students' evolving conceptions were supported by constructivist-oriented pedagogy. Video data segments (i.e., each girl's conceptualization of the multiplicative nature of and inverse relation ($1/m > 1/n$ if $m < n$) among unit fractions; the girls' solutions to novel problems) from this project served as starting points for discussions in the subsequent PME-NA Working Group meeting. Specifically, Working Group members used the video segments and descriptions of the collaboration as a springboard for discussing possible research questions and methods of data analysis to employ in future, collaborative work. It is collaborations like these that this Working Group is designed to foster.

In 2014, we met as a Working Group at the joint PME and PME-NA conference in Vancouver. There our Working Group continued to expand to 25 members, now including international members from outside of North America. During the meeting, two Working Group members (one from math education and one from special education) shared a multiplicative reasoning assessment tool resulting from their NSF-funded research project (Xin, Tzur, & Si, 2008). Upon examining this instrument, the group discussed alternative ways for assessing students with LDM and implications for intervention development. As a result of our collaboration in Vancouver, we had two main accomplishments. First, as a group we identified three research subgroups: (a) cognitive characteristics of students with LDM, (b) interventions for students with LDM, and (c) teacher preparation or professional development, that represented the interests of the members. Each research subgroup identified pertinent research questions and an agenda for further collaboration.

Second, as a group, we proposed an idea for developing a proposal for a special issue to be published in an influential special education journal to address the research around the intersection of math and special education. Later in the year, members of the Working Group developed the proposal for the special issue and worked extensively with the editors of a special education journal *Learning Disabilities Quarterly* (LDQ). The quality of our proposal led to the acceptance of this special issue proposal by the co-Editors of LDQ. Two members from the Working Group served as co-Guest Editors of this special issue and identified potential contributing works from the Working Group members. In addition, we invited a well-known scholar from the field of special education for co-authoring a commentary paper as part of this special issue.

In 2015 PME-NA, we continued and expanded collaborations between members of this Working Group, by focusing discussions around two central themes: (a) math concept development and corresponding methodologies for studying its emergence in students with special needs, and (b) designing research questions and writing a research plan around this topic. We invited interested researchers and educators to the Working Group sessions. We had several new members joining this Working Group including scholars from countries other than U.S. During the Working Group sessions, equity was brought up as a new discussion point. Following 2015 PME-NA, two members of this Working Group, along with a new member, formed a new collaboration; and they worked on writing a practitioner piece pertinent to differentiating instruction for diverse students—that article was accepted to MTMS.

In 2016, the Math and Special Education Working Session met twice during PME 40 held in Australia. Nine people from five continents attended these sessions, while another three people actively participate in our projects throughout the year. On the first day, there were several new members to the group, so we introduced each other and explained the history and goal of the working session: to promote cross-disciplinary work that supports the learning of math by students with mathematical learning difficulties. We then discussed a project that we started at PME 38 and have recently concluded. That is the special series of *Learning Disabilities Quarterly* (LDQ) on the subject of the intersection between mathematics and special education. As a result of this discussion, we came up with a number of new questions that we would like to explore (e.g., challenges and strategies of collaborative work between professionals/scholars from the field of math education and the field of special education) and consider questions from an international perspective.

On the second day of the Working Group session we discussed our next big project to work on as a group. We decided that we would like to produce a book that teacher educators can use for teaching undergraduate and graduate students about the intersection between mathematics education and special education. The group members committed to the collaborative work on this book project.

In 2017 during PME-NA39, a total of 26 people (including three members via Skype from UK and Canada) attended the Working Group sessions across three conference days. Day one, the group went over the history of the Working Group and reported the publication of the special series that this Working Group initiated in 2014 at the joint PME and PME-NA conference in Vancouver. All five papers of this special series plus an introduction paper (Xin & Tzur, 2016) have been published in the journal of LDQ (impact factor =1.028) across four issues (from LDQ Vol 39 (4) published in November 2016, to LDQ Vol 40 (3) published in August 2017). These papers address characteristics of individuals with LDM and interventions from different theoretical frameworks. From a socio-cultural framework, one paper addressed "mediational tools" that supported the development of math or language abilities of one individual with MD (Lewis, 2016). From a constructivist framework, another paper examines the partitioning and units coordination of 44 students with MD or at-risk for MD in terms of initial numeric and fractional reasoning (Hunt, Welch-Ptak, & Silva, 2016). From an integration of constructivist pedagogy and explicit strategy instruction, one intervention study paper reported the outcome of a RCT study that evaluated the acquisition and generalization effect of the PGBM-COMPS intelligent tutor program (Xin, Tzur, Si, Hord, Liu, & Park, 2017) on enhancing multiplicative reasoning and problem solving of students with LDM. Lastly, building on the framework of *Conversational Repair* from the field of linguistics, the second intervention study paper reported the effectiveness of a discourse-based intervention on math problem solving and reasoning (Liu & Xin, 2017).

During the second and third day of the Working Group sessions, the group engaged in heated discussion of the book proposal—the new project initiated by this group following the special series project. The discussion revolved around the purpose of the potential book, potential title of the book, potential audience and market of the book, unique selling point of the book, potential competitors of the book, and potential table of content. Between the Working Group sessions during the conference, the team conducted an initial search for similar books in the current market. Then, based on the review of these existing "similar" books, the Working Group had further discussion about the significance and unique feature of our potential book. Following PME-NA 39, the Working Group members have continued their work on the book proposal and produced an initial draft of the book proposal.

Issues Relating to Psychology of Mathematics Education

Historically, special education researchers and teachers focused almost exclusively on students' mastery of procedural skills, such as basic number combinations and ability to execute mathematical algorithms (Jackson & Neel, 2006; Fuchs et al., 2005; Geary, 2010; Swanson, 2007; Kameenui & Carnine, 1998). A recent literature review comparing instructional domains for students with disabilities found that the majority of research conducted in the field of special education addressed basic computation and problem solving, with the primary focus placed on mnemonics, cognitive strategy instruction (e.g., general heuristic four-step strategy: read, plan, solve, and check), or curriculum-based measurement (Van Garderen, Scheuermann, Jackson, & Hampton, 2009). Instructional practices mainly focused on task analysis (breaking up skills into decontextualized steps that need to be memorized and followed), flash cards, or general heuristics that do not help with domain knowledge learning and concept development (Cole & Washburn-Moses, 2010). In particular, the focus on primarily procedures-driven instruction and rote memorization of skills seems to result in students' incomplete and inaccurate understanding of fundamental mathematics concepts as well as a lack of retention and/or transfer (Baroody, 2011).

Importance of Both Conceptual and Procedural Knowledge

Crucial for rich mathematical understandings that enable retention and transfer of fundamental concepts is the iterative development of conceptual understanding along with procedural proficiency (Rittle-Johnson, Siegler, & Alibali, 2001; Rittle-Johnson & Koedinger, 2005). Rittle-Johnson and Alibali (1999) noted that conceptual knowledge supports procedural generalization. In particular, conceptual knowledge could aid children in mindfully avoiding the use of procedures that fail to work in novel situations. Additionally, an ability to understand and manipulate different mathematical representations to navigate conceptually a mathematical context contributes to conceptual understanding and procedural skill (Ball, 1993; Kaput, 1987; Rittle-Johnson et al., 2001). It seems that any investigation into mathematical cognition, whether related to disability or not, must fundamentally engage with issues of conceptual understanding (Hunt & Empson, 2015).

A focus on procedural skills limits students with disabilities' access to the general education curriculum, which is a requirement of the Individuals with Disabilities Educational Improvement Act (Maccini & Gagnon, 2002). In mathematics, access to the general education curriculum means addressing problem-solving, mathematical modeling, higher order thinking and reasoning, and algebra readiness as required by the new Common Core Standards (CCSSI, 2012). To accomplish these Standards, mathematics educators need to engage students actively in making conjectures, justifying and questioning each other's ideas, and operating in ways that lead to deep levels of mathematical understanding (Kazemi & Stipek, 2001; Lampert, 1990; Martino & Maher, 1999; Yackel, 2002).

Conceptual Diagnosis Based Pedagogy

A pedagogical approach to be explored and advanced during this Working Group's meetings is one that focuses on promoting conceptual learning in students with LDM. This approach is rooted in a constructivist stance (Piaget, 1985; von Glasersfeld, 1995), particularly the notion of assimilation, which stresses the need to build instruction on what students already know and are able to think/do. That is, teaching needs to be sensitive, relevant, and adaptive to students' available ways of operating mathematically (Steffe, 1990). To this end, teachers must learn how to: (a) diagnose students' available conceptions, and (b) design and use learning situations that both reactivate these conceptions and lead to intended transformations in these conceptions.

Building on Simon (2006)'s core idea of hypothetical learning trajectories, Tzur (2008) has articulated such an adaptive pedagogy, which revolves around the Teaching Triad notion: (a) students' current conceptions, (b) goals for students' learning (intended math), and (c) tasks/activities to promote progression from the former to the latter. Key here is that in designing every lesson one proceeds from conceptual diagnosis of the mathematics students are capable of thinking/doing. That is, assessment methods need to focus on dynamic (formative) inquiry into student understandings, as opposed to on testing correct and incorrect answers per se. This day-to-day diagnosis, obtained via engaging students in solving tasks and probing for their reasoning processes, gives way to selecting goals for students' intended learning. Building on this diagnosis, a mathematics lesson begins with problems that students can successfully solve on their own, which Vygotsky (1978) referred as the Zone of Actual Development (see also Tzur and Lambert, 2011). Recent studies of mathematics teaching in China (Jin, 2012; Tzur & Lambert, 2011) revealed a strategic, targeted method, bridging, which is geared specifically toward both: (a) reactivating mathematical conceptions the teacher supposes all students know, and (b) directing their thinking to the new, intended ideas.

Exemplar Research Activities with Students with LDM

Multiplicative Reasoning Project: From 2008 to 2015, two members of this Working Group (one from math education and one from special education) have been working collaboratively on a federal funded grant project (Xin, Tzur, and Si, 2008). This project integrated research-based practices from mathematics education and special education and was aimed to promote multiplicative reasoning and problem solving of elementary students with LDM. As an outcome of this collaborative project, the research team has developed an intelligent tutor, PGBM-COMPS. The PGBM-COMPS intelligent tutor draws on three research-based frameworks: a constructivist view of learning from mathematics education (Steffe & D'Ambrosio, 1995), data (or statistical) learning from computer sciences (Sebastiani, 2002), and Conceptual Model-based Problem Solving (COMPS) (Xin, 2012) that generalizes word-problem underlying structures from special education.

Rooted in a constructivist perspective on learning (Piaget, 1985; von Glasersfeld, 1995), the PGBM part of the intelligent tutor focused on how a student-adaptive teaching approach (Steffe, 1990), which tailors goals and activities for students' learning to their available conceptions, can foster advances in multiplicative reasoning. This approach eschews a deficit view of students with learning disabilities. Rather, it focuses on and begins from what they do know and uses task-based activities to foster transformation into advanced, more powerful ways of knowing. On the other hand, intelligent computer systems can play an important role in students' learning by effectively modeling their thinking and dynamically recommending tasks tailored to their conceptual profiles. Going hand-in-hand, the COMPS part of the program (Xin, 2012) generalizes students' understanding of multiplicative reasoning to the level of mathematical models. At this stage, students no longer rely on concrete or semi-concrete models for problem solving; rather, the mathematical models directly drive the solution plan.

The collaborative research team has conducted several piloting studies to field test the PGBM-COMPS intelligent tutor with elementary students with LDM. The preliminary studies have shown promising results—participating students with LDM who interacted with this intelligent tutor not only enhanced their problem-solving skills on a researcher-designed criterion test but also a norm-reference standardized test (Xin et al., 2017). In addition, the results of these piloting studies have shown success in promoting students' substantial conceptual advances (e.g., concept of number, multiplicative reasoning). In fact, a paper resulting from a randomized control trial (RCT) study of this project has been published (Xin et al., 2017) as part of the special series produced by this Working Group. As a follow up, the research team has continued on this line of work and recently embarked on a new NSF supported project (Xin, Kastberg, and Chen, 2015), which focuses on additive reasoning.

Most recently, building the data gleaned from Multiplicative Reasoning Project, the team members of this Working Group formed further collaboration and conducted a quantitative analysis of teacher-student interactions during discourse–oriented instruction involving elementary students with LDM in solving multiplicative word problem (under revision). The findings from this discourse analysis study indicate that with appropriate opportunities /prompting as well as guidance and support, students with LDM demonstrated various degrees of problem solving and reasoning in a discourse–oriented mathematics instructional environment. Findings of this study support the notion that quality classroom discourse and teachers' responsiveness to students' thinking contribute to students' learning and achievement.

Fraction Project. Another Working Group member is documenting learning trajectories of elementary school children with LDM as they come to understand fractions as quantities (Hunt, 2013). In its first year, this work produced models of children's key developmental understandings from semi-structured interviews with 44 second, third, fourth, and fifth graders with LDM (Hunt et al., 2016). Interviews followed a protocol that established a basis for questioning but also allowed for maximum researcher flexibility to fully examine student thinking and examine trends across the students' unique ways of reasoning. Two design experiments also took place during Year 1 to test an expanded group of tasks from which the trajectory is based. The design experiment exemplified Hunt's (2018) evolving characterization of knowing, learning, and teaching in terms of the mathematical knowledge students with learning disabilities (LDs) *do* possess. Utilizing an iterative, three phase qualitative analysis, Hunt (2018) illuminated two children's perceptions and meaning-making for the unit fraction concept observable through work in learning situations that supported partitioning and unit coordinating activity. The analysis revealed each child's evolving knowledge as a *complexity* consisting of the negotiation and construction of meaning among the child's mind, her participation in a learning environment, and the teacher's responsiveness to the child's ways of reasoning and making sense.

In Years 2 and 3, the researcher took up a teaching experiment methodology, much like those used in the collaborative pilot that resulted from this Working Group, to document how eight children with LDM constructed conceptions of number and fractions alongside the factors that supported, extended, or constrained their understanding. Data analysis within and across each of the eight cases is ongoing, yet initial findings from three of the cases can be shared. First, when engaging in tasks that support the construction of composite unit, the use of memorized fact combinations or teacher taught strategies eclipsed the use of two students' natural reasoning (Hunt, MacDonald, & Silva, in press) yet supported the third's (Hunt, Silva, & Lambert, 2017). All three students evidenced tacit ways of reasoning (e.g., 2 is contained in 3, 3 is contained in

4); two reverted to pseudo-empirical abstractions, tricks, or algorithms that they could not explain to solve the tasks. These ways of engaging in the tasks were counterproductive to these two students' reflection on their own actions upon units such that adaptations in their reasoning could occur. Conversely, the third student leveraged his knowledge of number facts and alternative representations to advance his fractional reasoning and compensate for his perceptual motor differences (2017). For all three students, teacher encouragement and support to engage in each student's own ways of reasoning was imperative. Second, Hunt and colleagues (under review) found evidence that confirms previous research (Geary, 2010) that two students sometimes lost track of counting during a count on, possibly due to working memory. Author conjectures that this learning difference interferes with the move from counting on to more sophisticated additive reasoning (e.g., breaking apart tens). Yet, in one case, the problem is alleviated through opportunities for within-problem reflection through experiences that the child has to construct addends involved in number problems through counting, sweeping small numbers as lengths, and improving the usability of small composites like 2, 4, 3, 5, and 6 (under review). Lastly, all three students evidenced an affinity towards symmetric notions of units coordination across learning situations and across natural and fractional number reasoning. Hunt is currently conducting cross-case analysis to discern whether the symmetry is indicative of unique trajectories in number and/or fractional reasoning.

As forgoing, this Working Group participants will use artifacts from projects described above as possible starting points to illuminate and further explore possible applications of student-adaptive pedagogy (conceptual diagnosis-based) as well as conceptual development trajectories in the design of effective/efficient assessment and intervention programs for students with LDM. We believe such approaches are complimentary and have the potential to become core methodological approaches for teaching and studying the conceptual understandings of students with LDM. In a similar way, this Working Group provides a venue to give and receive feedback on ongoing cutting-edge empirical work, which is reshaping how students with LDM are researched.

Plan for Working Group

The aim of this Working Group is to continue the conversation surround the intersection of mathematics education and special education. In particular, one of the major goals of this year's Working Group sessions is to revised and finalize the proposal for a new book that university professors and teacher educators can use for teaching graduate students about the intersection between mathematics education and special education.

This Working Group intends to accomplish the following:

- Review the draft of the book proposal
- Make addition or modification to current book proposal draft
- Finalize the content of the book (Table of Content)
- Identify potential team members who will work on each of the proposed chapters
- Each Team will brainstorm the outline of the chapter.
- Discuss the logistics of collaborations to carrying out the identified tasks following the PME-NA conference, and
- Discuss further collaborations leading to additional publication and funding opportunities.

These goals are further outlined across sessions as follows:

Session 1: Introduction and Progress-to-date
GOAL: **To identify participant's affinity for establishing potential sub-groups for collaborative work pertinent to math and special education.**

- Prior members will briefly introduce the Working Group's history and describe the collaborations that have emerged in prior years.
- Participants will each introduce themselves and their current research and interest in students with LDM.
- Form sub-groups or collaborative teams among participants based on their common (research) interests.
- Group(s) will be engaged in discussing common research agenda and potential collaboration opportunities.

Session 2: Book Proposal Development
GOAL**: Revised and finalize book proposal**

- Revise the content of the book proposal
- Revise the draft Table of Content; come up key words we think readers may use.
- Revise the list of book titles with which our book will compete and revisit /revise how our book will differ from these titles.
- Revise the selling points of our book based on draft book proposal

Session 3: Continue the ongoing collaboration
GOAL: **Establish next steps for both the sub-groups and the whole Working Group.**

- Identify potential team members who will work on each of the chapters, so that after this PME-NA conference, the team will continue their collaborative work in making the abstract for the chapter and even drafting the chapter.
- Team members work together and exchange ideas of how to approach each of the chapters, and perhaps discuss the logistics of collaborations to carrying out the identified tasks, and
- Discuss further collaborations leading to additional publication and funding opportunities.
- Determine what our next whole group meeting will entail (e.g., PMENA Working Group for the following year)

Anticipated Follow-up Activities

Throughout the year, the members of this Working Group will work collaboratively creating the outline of the chapter assigned, drafting the abstract and then the content for the chapter. In addition, the members of this working group will continue their effort in forming collaborative research agenda and conducting collaborative research that is relevant to the goal of this Working Group.

References

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal, 93* (4), 373-397.

Baroody, A. J. (2011). Learning: A framework. In Fennell, F. M., & National Council of Teachers of Mathematics. (Eds.) *Achieving fluency: Special education and mathematics*. Reston, VA: National Council of Teachers of Mathematics.
Cole, J. E., & Wasburn-Moses, L. H. (2010). Going beyond "the math wars". *Teaching Exceptional Children, 42* (4), 14-20.
Common Core State Standards Initiative (2012). Introduction: Standards for mathematical practice. Retrieved on July 5, 2012 from http://www.corestandards.org/the-standards/mathematics.
Daro, P., Mosher, F. A., & Corcoran, T. (2011). *Learning trajectories in mathematics: A foundation for standards, curriculum, assessment, and instruction*. Philadelphia, PA: Consortium for Policy Research in Education.
Fuchs, L. S., Compton, D. L., Fuchs, D., Paulsen, K., Bryant, J. D., & Hamlett, C. L. (2005). The prevention, identification, and cognitive determinant of math difficulty, *Journal of Educational Psychology, 97 (*3), 493-513.
Geary, D. C. (2010). Mathematical learning disabilities, *Advances in Child Development and Behavior, 39*, 45-79.
Hunt,J.H. (2013*). Fraction activities and assessments for conceptual teaching*. National Science Foundation.
Hunt, J.H. (June 2018). *Navigating disability in intensive instruction: Learning complexity and small environments.* International Society of the Learning Sciences, London, United Kingdom.
Hunt, J. H., & Empson, S. B. (2015). Exploratory study of informal strategies for equal sharing problems of students with learning disabilities. *Learning Disability Quarterly*, *38*(4), 208-220.
Hunt, J.H., MacDonald, B., & Silva, J. (in press). *Tacit, trick, or teach: What is Gina' mathematical reality*? North American Chapter of the Psychology for Mathematics Education (PME-NA), Greenville, S.C.
Hunt, J.H., Silva, J., & Lambert, R. (November 2017). *From Trajectories, deficit, and differences to neurodiversity: The case of Jim.* Paper presented at the 39th Conference of the International Group for the Psychology of Mathematics Education North America (PME-NA), Indianapolis, IN.
Hunt, J.H., Welch-Ptak, J., & Silva, J. (2016a). Initial understandings of fraction concepts evidenced by students with learning disabilities and difficulties: A framework. *Learning Disabilities Quarterly*, 39(4), 213-225.
Hunt, J. H., Westenskow, A., Silva, J., & Welch-Ptak, J. (2016b). Levels of participatory conception of fractional quantity along a purposefully sequenced series of equal sharing tasks: Stu's trajectory. *The Journal of Mathematical Behavior*, *41*, 45-67.
Jackson, H. G., & Neel, R. S. (2006). Observing mathematics: Do students with EBD have access to standards-based mathematics instruction? *Education and Treatment of Children, 29*(4), 593-614.
Jin, X. (2012). Chinese middle school mathematics teachers' practices and perspectives viewed through a Western lens. Monash, Melbourne.
Kameenui, E. J., & Carnine, D. W. (1998). Effective teaching strategies that accommodate diverse learners. Upper Saddle River, NJ: Merrille.
Kaput, J. J., & Educational Technology Center (Cambridge, Mass.). (1987). *A concrete-to-abstract software ramp: Environments for learning multiplication, division, and intensive quantity*. Cambridge, MA: Educational Technology Center, Harvard Graduate.
Kazemi, E. & Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *The Elementary School Journal, 102*(1), 59-80.
Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal, 27*(1), 29-63.
Lewis, K. E. (2016). Beyond Error Patterns: A Sociocultural View of Fraction Comparison Errors in Students With Mathematical Learning Disabilities. *Learning Disability Quarterly, 39(4)* 199-212. DOI: 10.1177/0731948716658063
Liu J. & Xin, Y. P. (2017). Eliciting Repair of Mathematics Explanations of Students with Learning Disabilities. *Learning Disability Quarterly, 40(3). 132-145.*
Maccini, P., & Gagnon, J. C. (2002). Perceptions and Applications of NCTM Standards by Special and General Education Teachers. *Exceptional Children, 68,* 3, 325-44.
Martino, A.M. & Maher, C.A. (1999). Teacher questioning to promote justification and generalization in mathematics: What research practice has taught us. *Journal of Mathematical Behavior, 18*(1), 53-78.
Piaget, J. (1985). The equilibration of cognitive structures: The central problem of intellectual development (T. Brown & K. J. Thampy, Trans.). Chicago: The University of Chicago.
Rittle-Johnson, B. & Alibali, M.W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology*, 91(1), 175-189.
Rittle-Johnson, B., Siegler, R., & Alibali, M.W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, 93(2), 346-362.

Rittle-Johnson, B. & Koedinger, K.R. (2005). Desigining knowledge scaffolds to support mathematics problem solving. *Cognition and Instruction*, 23(3), 313-349.
Sebastiani, F. (2002). Machine learning in automated text categorization, *ACM Computing Surveys, 34*, 1-47.
Siegler, R. S. (2006). Microgenetic analyses of learning. In W. Damon & R. M. Lerner (Series Eds.) & D. Kuhn & R. S. Siegler (Vol. Eds.), *Handbook of child psychology: Volume 2: Cognition, perception, and language* (6th ed., pp. 464-510). Hoboken, NJ: Wiley.
Simon, M. A. (2006). Key Developmental understandings in mathematics: A direction for investigating and establishing learning goals. *Mathematical Thinking and Learning*, *8*(4), 359–371
Steffe, L. P. (1990). Adaptive mathematics teaching. In T. J. Cooney & C. R. Hirsch (Eds.), *Teaching and learning mathematics in the 1990s* (pp. 41-51). Reston, VA: National Council of Teachers of Mathematics.
Steffe, L. P., & D'Ambrosio, B. S. (1995). Toward a working model of constructivist teaching: A reaction to Simon. *Journal for Research in Mathematics Education, 26,* 146-159. doi:10.2307/749206
Swanson, H. L. (2007). Cognitive aspects of math disabilities. In Berch, D. B., & Mazzocco, M. M. M. (Eds). *Why is math so hard for some children?: The nature and origins of mathematical learning difficulties and disabilities*. Baltimore, MD: Paul H. Brookes Pub. Co.
Tzur, R. (2008). Profound Awareness of the Learning Paradox (PALP): a journey towards epistemologically regulated pedagogy in mathematics teaching and teacher education. In T. Woods, & B. Jaworski (Eds.), Handbook of Research in Mathematics Teacher Education (vol. 4, pp. 1–2). Rotterdam: Sense.
Tzur, R., & Lambert, M. A. (2011). Intermediate participatory stages as zone of proximal development correlate in constructing counting-on: A Plausible Conceptual Source for Children's Transitory "Regress" to Counting-All. *Journal for Research in Mathematics Education*, *42*(5), 418-450.
Van Garderen, D., Jackson, C., Hampton, D., & Scheuermann, A. (2009). Supporting the collaboration of special educators and general educators to teach students who struggle with mathematics: An overview of the research. *Psychology in the Schools, 46,* 1, 56-77.
von Glasersfeld, E. (1995). Radical constructivism: A way of knowing and learning. Washington, D.C.: Falmer.
Vygotsky, L. S. (1978). Mind in society: The development of higher psychological processes. Cambridge, MA: Harvard University Press.
Woodward, J. & Tzur, R. (2017). Final Commentary to the Special Series. *Learning Disability Quarterly, 40(3)* 146-151.
Xin, Y. P. (2012). *Conceptual model-based problem solving: Teach students with learning difficulties to solve math problems*. The Netherlands: *Sense Publishers.*
Xin, Y. P. (PI), Kastberg, S., & Chen, V. (2015). *Conceptual Model-based Problem Solving (COMPS): A Response to Intervention Program for Students with LDM.* National Science Foundation (NSF) funded project, $450,000.
Xin, Y. P. & Tzur, R. (2016). Cross-disciplinary Thematic Special Series: Special Education and Mathematics Education. *Learning Disability Quarterly*, *39*(4), 196-198.
Xin Y. P., Tzur, R., & Si, L. (2008). Nurturing Multiplicative Reasoning in Students with Learning Disabilities in a Computerized Conceptual-Modeling Environment. National Science Foundation (NSF) funded project.
Xin, Y. P., Tzur, Si, L. Hord, C., Liu, J., Park, J. Y. (2017). An intelligent tutor-assisted math problem-solving intervention program for students with learning difficulties. *Learning Disability Quarterly, 40(1), 4-16.*
Yackel, E. (2002). What we can learn from analyzing the teacher's role in collective argumentation. *Journal of Mathematical Behavior, 21*, 423-440.