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Research Article

Exploring students' thinking process in mathematical proof of abstract algebra based on Mason's framework

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Abstract

Mathematical proof is a logically formed argument based on students' thinking process. A mathematical proof is a formal process which needs the ability of analytical thinking to solve. However, researchers still find students who complete the mathematical proof process through intuitive thinking. Students who have studied mathematical proof in the early semester should not have completed abstract algebraic proof intuitively. Therefore, the aim of this research is to explore students' thinking process in conducting mathematical proof based on Mason's framework. The instrument used to collect data was mathematical proof problems test related to abstract algebra and interviews. There are three out of 25 students who did abstract algebra through intuitive thinking as they only used two stages of the Mason's thinking framework. Then, two out of three students were chosen as the subjects of the study. The selection of research subjects is based on the student's ability to express intuitive thinking verbally process which were conducted while completing the test. It is found that students can form structural-intuitive warrant that they use to complete the mathematical proof of abstract algebra. Structural-intuitive warrant formed by students at the stage of attack and review are in the form of: institutional warrant and evaluative warrant, while at the entry and attack stage are a priori warrant and empirical warrant.

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Introduction

Abstract algebra and mathematical proof are two interrelated things where arguments are needed to support a conjecture (Pedemonte, 2008). Argument is an important part of mathematical proof process, as it can use logic in the form of "if P then Q" which appears through the student's thinking process (Kosko & Singh, 2019). Mathematical proof is an argument to convince others about the truth of a certain claim (Panza, 2014; Varghese, 2009; Sekiguchi, 2002). An argument can be formed from students' thinking or reasoning construction based on logic, concepts, and problem-solving procedures (Selden & Selden, 2003; Imamoglu and Togrol, 2015). In an argument, there are several components that can be used to convince others. Simpson states that the analysis of arguments in mathematical proof can be performed using the Toulmin scheme (Simpson, 2015).

Toulmin states that there are six components in an argument, specifically data, claim, warrant, backing, rebuttal, and qualifier (Toulmin, 2003). A warrant has an important role in terms of making conclusions and reasoning in an argument because the truth of a claim can be recognized through warrant. Inglis suggests eliminating rebuttal and qualifiers in mathematical proof (Inglis, 2007). Therefore, the focus is on warrant and backing as guarantors to obtain

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the appropriate conclusions. Rebuttal is a denial argument or refutation raised by someone in making an argument. While the qualifier is a belief in an argument. The warrant is a guarantor used to draw a conclusion from mathematical proof. There are three types of warrant in mathematical, specifically inductive-warrant, deductive-warrant, and structural-intuitive warrant (Inglis, 2007; Tristanti, Sutawidjaja, As'ari, & Muksar, 2016). Inductive-warrant is used to ensure the truth of a conjecture that has been made by students. A structural-intuitive warrant is used to observe students' experience and mental structure in getting a conclusion. Whereas deductive-warrant is a formal mathematical justification for obtaining correct conclusions based on axioms, algebraic manipulation, or previous examples (Pedemonte, 2008; 2014).

Students who study Abstract Algebra subject in university should have understood the concept of modulo integers as it has been studied in Number Theory subject in the first semester. They need to know the concept of Number Theory first so they can understand modulo integer. As Toh stated that the subject of number theory can train students' thinking skill in learning mathematical proof (Toh, Leong, Toh, & Ho, 2014; Zetriuslita, Wahyudin & Jarnawi, 2017). However, some students were still found with difficulty in modulo integers operation to proof the close characteristics in group problems of abstract algebra. The students' difficulty in learning Number Theory is characteristics and theorems in the integer which is not applicable in both natural and numerical number (Nurrahma & Karim, 2018). Mistakes done by students happened intuitively as they didn't proof it thoroughly and subconsciously. This problem does not only happen in East Java but also in Israel University. There are 73 out of 133 students in Israel made mistakes in subgroup proofing as they done it intuitively by using Lagrange theorem and isomorphic theory (Leron, 2014; Leron & Hazzan, 2009).

As Kurniati stated in their research that thinking skills are significant in primary school, secondary school, even in a college level (Kurniati & Zayyadi, 2018). University students should have been able to solve mathematical proof related to abstract algebra correctly based on the conducted mental structure, as based on Piaget development stages which states that a person can do a formal or abstract thinking process after 12 years or above. Tall stated that the transition of students thinking process can do formal thinking to solve mathematical problems or mathematical proof problems (Tall, 2009; 2010). However, Piaget and Tall theories cannot be used to understand the students' thinking process in abstraction which make justification in proofing mathematical problems. Therefore, Mason's framework of thinking is needed to solve the problem.

The thinking process conducted by students in solving mathematical problems can be traced through three stages of mathematical thinking according to Mason (2010). The three stages are entry, attack, and review. At the entry stage, students identify problems by carefully reading questions, determining relevant ideas, focusing on their own findings, classifying short information, and representing it in the form of notations or symbols. The attack stage is marked by the appearance of conjecture or justification that shows confidence in a claim. The review stage is marked by the student's activity in checking the accuracy of the solution result, the solution which is in accordance with the question, giving reasons to guarantee an appropriate solution, and the implications of the conjecture or argument. Mathematical thinking is a way of learning mathematics, since students can be constructing conjecture by exploring the problem, formulating conjecture, justifying conjecture, and proving conjecture (Astawa, Budayasa, & Juniati, 2018; As'ari, Kurniati, & Subanji, 2019). Conjecture is a statement, argument, and conception system (Pedemonte, 2008), therefore the conjecture can be interpreted as an allegation of mathematical statements that need to be verified.

Table 1.
Mason's Framework

Phases	Rubric	Activities	
Entry	I know	Read the question carefully Specialize to discover what is involve What ideas/skill/facts seem relevant? Do I know any similar or analogous question?	
	I want	Classify and short information Be alert to ambiguities Specialize to discover what the real question is	
	Introduce	Images, diagrams, symbols Representation, notation, organization	
Attack	Try	Conjecturing	Cyclic process
	May be		Specialize systematically Analogy

	Why	Justification
Review	Check	Calculation Arguments to ensure that computations are appropriate Consequences of conclusions to see if they are reasonable That the resolution fits the question
	Reflect	On key ideas and moments On implications of conjectures and arguments On your resolution: can it be made clearer?
	Extend	The result to a wider context by generalizing By seeking a new path to the resolution By altering some of constrains

Applied the Toulmin scheme to analyze teacher's mathematical arguments in mathematics learning through warrant structures (Nardi, Biza, & Zachariades, 2012; Nardi, Biza, & Watson, 2014). Some researchers focused on analyzing students' arguments in solving mathematical proof problems based on the Toulmin scheme (Simpson, 2015; Metaxas, Potari, & Zachariades, 2016; Freeman & Freeman, 2005; Freeman, 2006; Pedemonte, 2008; 2014; Tristanti et al. 2016; 2017; Laamena, Nusantara, Irawan, & Muksar, 2018). Then, other researchers discussed about students' thinking processes in solving mathematical problems based on Mason's framework (Wardhani & Subanji, 2016) as well as other studies that focused on discussions about improving students' thinking abilities through Mason's theory (Tall, 2009; 2010; Mason, 2005). Experts conduct research on Toulmin's and Mason's theory separately, hence from previous studies it can be seen that no one has done research on structural-intuitive warrant formed by students in solving mathematical proof problems based on Mason's framework. Therefore, the aim of this study is to explore students mathematical thinking process in university level in conducting mathematical proof related with the material of abstract algebra based on Mason’s framework and Toulmin’s theory.

Method

Research Design

This research is classified as an exploratory type with a qualitative approach. The reason is because it needs to explore students' thinking skill which need qualitative data in expressing mathematical proof oral and written. It is written descriptively which is in accordance with As'ari statement that this research was qualitative in nature and employed an exploratory descriptive method (As’ari, Kurniati, Abdullah, Muksar, & Sudirman, 2019). The research design is also classified as an exploratory as Zayyadi argued that thinking is a form of communication between a person and himself which takes place in his brain, so that it cannot be accessed by others (Zayyadi, Nusantara, Hidayanto, Sulandra, & Sa’dijah, 2020).

Participants

This research was conducted at three different universities in East Java region of Indonesia. The subjects were students of the Mathematics Department who had taken Abstract Algebra course in their university. Of the 25 students of Mathematics Department, there were 22 students who did proofing by using thinking framework by Mason. Students are choosen to be research subjects because they solve the mathematical proof problem intuitively based on conjecture created and able to reveal the knowledge in their mind. The relationship between conjecture and proof is the most important thing to identify students' understanding of a mathematical theorem or statement (Pedemonte, 2014).

Data Collection

The data collection in this study was performed by looking at the subject's thinking process through the test of problem proofing and interviews. Validity and reliability in qualitative research were done through triangulation (Utami, Sa’dijah, Subanji, & Irawati, 2018; Cresswel, 2012). Triangulation used in this study was a method of triangulation by comparing data obtained from test results and interviews conducted to each subject. The proof of a problem test used in this study was the proof of Group in Abstract Algebraic adopted from Gilbert & Gilbert (2015). Those test is used as the instrument of the study which is shown as follow.

Solve the proof problems below:

- Prove or disprove that the set G = {- 1,1} is a closed property with respect to addition.
- Prove or disprove that the set of integer of multiple three is a group with respect to addition

Data Analysis

Data from test results and interviews were analyzed using qualitative data analysis. Data analysis was performed by looking at the written results of the subject's work and interview transcripts. The thinking process of the subject in solving the mathematical proof problem can be recognized from the words delivered by the subject during think-aloud and interview. Structural-intuitive warrant was recognized from the guarantor used by the subjects in identifying problems that will be proven as the basis for making conjectures and justifications. Conjectures and justifications formed by the subject occur intuitively because the subject identifies problems without going through rational and intellectual reasoning, therefore the warranties that arise occur quickly and automatically based on previous knowledge (Leron & Hazzan, 2009). Therefore, the data analysis in this study used the framework of thinking from Mason framework in Table 1 which includes identifying problems, making conjectures, making conclusions and justifications, therefore mental activities undertaken by subjects during the thinking process can be formed intuitively. Researchers conducted interviews with subjects related to the thinking process as the subject did not reveal completely about all mental activities which is occured at the time of proofing if it was not given in-depth questions by researchers.

Results

There were three students who conduct mathematical proof intuitively but only two students who were able to express verbally about the thinking process to make a conjecture based on the warranties. The selection of subjects in this study was based on the oral communication skills possessed by each subject to determine the mental activity conducted when performing proofing.

Tabel 2.

Selection of Research Subjects

University	Number of Students	Masons' Thinking Process	Intuitively Proving	Research Subject
A	8	8	0	0
B	12	10	2	1
C	5	4	1	1
Total	**25**	**22**	**3**	**2**

Table 2 shows that not all students conduct mathematical proof intuitively. According to the table, we can identify that at University A, there were no students who performed intuitive proofing. Then, at University B, there were two students who performed intuitive proof but only one was able to verbally reveal mental activities during the verification process, while at University C, there was only one student. Therefore, there were two students were selected to be research subjects. They were Miss Nay and Miss Selvy. Both were conducting mathematical proof about group problems in Abstract Algebra by using a warrant that appears intuitively.

Warrant that was raised by them in conducting mathematical proof formed an intuitive structural warrant. The structural-intuitive warrant is a guarantor used by students to obtain conclusions from a proof however the guarantor appears intuitively. The structural-intuitive warrant in stating the concept of color in everyday life can be classified into a priori warrant, empirical warrant, institutional intuition warrant, and evaluative warrant (Freeman, 2006). In this research, a premise that exists in the structural-intuitive warrant occurs based on cognitive meaning that appears intuitively. The process of forming an structural-intuitive warrant for each subject based on the thinking process is as follows:

First Subject

Miss Nay proved problem number 1 by identifying based on information available on the problem to be proven. Miss Nay identified the problem by making conjecture based on information that had been obtained before. Miss Nay said that G = {- 1,1} with the closed under addition operation without reasoning and proofing initially. Then she proved to check the truth of the allegations that have been made. The verification results show that G = {- 1,1} with the addition operation is not closed.

Researcher : *Do you think the problem number 1 is a closed property or not?*
Miss Nay : *Yes, Maam…*
Researcher : *Are you sure???*
Miss Nay : *Definitely*
Researcher : *Do you prove that* $G = \{-1,1\}$ *is closed under addition?*

Miss Nay : Not yet...
Researcher : Try to prove it

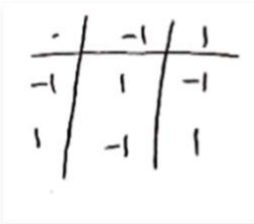


Figure 1.
It is a Multiplication Operation

From the proofing results of the subject, it shows that G = {- 1,1} is closed but the proof made by subjects was using multiplication operations, hence the conclusions obtained were wrong. Nay came to such a conclusion unconsciously because it followed a matter that had been previously proven that G = {- 1,1} was closed under multiplication operation, whereas the problem being proved was using the addition operation.

Researcher : How's the result?
Is it closed?
Miss Nay : Yes Maam,…
Researcher : Do you understand what is closed property?
Miss Nay : Yes, Maam. a number if operated with a number then the result is a member of the set. Because the results are (-1) and 1 it is closed to operations in G
Researcher : What operation do you use to prove? What operation does the problem demand?
Miss Nay : It is addition
Researcher : How about yours?
Miss Nay : Ah, I see. I used multiplication

Miss Nay performed the attack and review stages by proving twice to obtain the right conclusion. From the results of the proof, it shows that G = {- 1,1} is closed under addition operation. Then, she did the review stage by checking the conclusions that had been obtained. Apparently, the conclusions obtained were wrong because she used an operation that was not in accordance with that of the problem. After that, she constructed the knowledge possessed by re-verification using the addition operation. From the results of re-verification, it obtained the correct conclusion, namely G = {- 1.1} is not closed under addition operation.

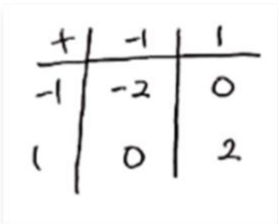


Figure 2.
It is an Addition Operation

Researcher : How do you conclude?
Identical?
Miss Nay : No Maam…
Actually $G = \{-1,1\}$ *with addition is not closed. After I proved once again it obtained* $\{-2, 0, 2\}$ *meanwhile* $\{-2, 0, 2\}$ *do not belong to* $G = \{-1,1\}$.

Furthermore, for problem number 2, Miss Nay identified the problem by mentioning the definition of the group that includes closed property, associative, identity elements, and inverse. Then she changed the triple number in the form of numbers {3,6,9,12}. Miss Nay performed an attack stage to prove closed, associative, identity elements, and inverse. The subject proved the closed property by making a Cayley table and then concluded that integers of multiples of three were closed to the addition operation. Furthermore, proof of the associative characteristic was conducted by taking any multiples of three. She took {3,6,9} to prove the associative characteristic, thus we obtained the conclusion that integers multiples of three with addition operations are associative because $(a°b)°c = a°(b°c)$.

Miss Nay found difficulty in determining the identity element and inverse. Intuitively she mentioned that the identity element for integer multiples of three with the addition operation is 3, therefore the inverse of 3 is 3, the inverse of 6 is 12, the inverse of 9 is 9, and the inverse of 12 is 6. From the proof of the identity element and the inverse, it can be seen that the steps taken by her to obtain a conclusion are wrong, but she felt that the proof is correct. Hence, she stated that the integer multiples of three with the addition operation is a group. Then, the researchers conducted interviews to determine the beliefs of Miss Nay based on the her thinking process that she had performed.

Researcher : *Are you sure that the integer of multiple of three is a group?*
Miss Nay : *Yes, Maam… But I got confused on how to prove its identity element and invers.*
Researcher : *See!… How do you obtain 3 for the identity element?*
Miss Nay : *The form of identity element is* $a + e = e + a = a$. *So the identity element is 3, I assumed that if the integer of multiple of three is multiplied by 3 then the result is the integer itself*
Researcher : *Really? Try to take randomly the integer of multiple of three and perform addition.*
Miss Nay : *Yes, Maam … I took 1+3=4*
Researcher : *Do you think 1 could be taken to determine the identity element?*
Miss Nay : *Yes, it can be…*
Researcher : *Why do you say so?*
Is 1 belong to the integer of multiple of three?
Miss Nay : *No, It is not…*
Researcher : *So, do you think 1?*
Miss Nay : *No, it cannot…*

Based on the verification result performed by Miss Nay in questions number 1 and number 2, it shows that she performed the intuition based on the previous knowledge. The warrant used to prove was closed property to solve problem number 1 and closed, associative, identity element, and inverse to prove problem number 2. The warrant raised to solve both problems appears intuitively. Therefore, the conclusion was wrong. This error was due to Miss Nay using the knowledge scheme about group verification for G = {-1,1} with the multiplication operation, even though the problem was using the addition operation. Whereas for problem number 2, Miss Nay used the number {3,6,9,12} as a form of multiple integers of 3. However, the integer multiples of 3 referred to in the problem are not only limited to {3,6,9,12}. This, Miss Nay could not determine identity element and inverse in group verification. Therefore, the structural-intuitive warrant formed by Miss Nay based on the thinking process can be seen in Figure 3.

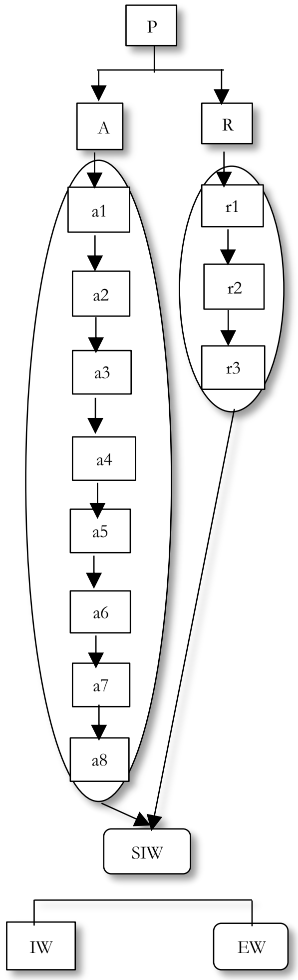


Figure 3.
Structural-intuitive Warrant by Miss Nay

Table 3.
Figure 3 Annotation

Codes	Thinking Pattern Description
P	The proved problem
A	Attack
R	Review
a1	Using a warrant in the form of group definition to prove problem number 1 and number 2.
a2	Making conjecture of $G = \{-1,1\}$ with multiplication is a group, then $G = \{-1,1\}$ is closed under addition.
a3	Making cayley table to prove
a4	Did not pay attention to performing the operations in the table. Nay used multiplication operations, whereas in the case of using addition operations.
a5	Proofing by changing the integer of multiple of three into $\{3,6,9,12\}$.
a6	From the results of proof, it obtained that $\{3,6,9,12\}$ is closed and associative
a7	Having difficulty when determining the identity element in question number 2. She mentioned that the identity element was 3.
a8	Provided the wrong answer when determining the identity element and inverse elements but drew the conclusion that integers multiples of 3 with the groups addition operation.
r1	Concluding that problem number 1 is closed property under addition and number 2 is a group.
r2	Justifying that the conclusion is correct since it was re-checked and re-proved.
r3	Unable to justify problem number 2 since the identity element and invers in the checking stage remain incorrect
SIW	Structural-intuitive Warrant
IW	Institutional Warrant
EW	Evaluative Warrant

Based on the thinking process performed by Miss Nay in conducting mathematical proofs related to Abstract Algebra problems, it produced structural-intuitive warrant in the form of institutional warrant and evaluative warrant. This type of structural-intuitive warrant occured because she did not prove based on the actual definition, but the subject employed only the basic knowledge. Miss Nay did not use the definition of multiples and group definitions correctly to perform the proof.

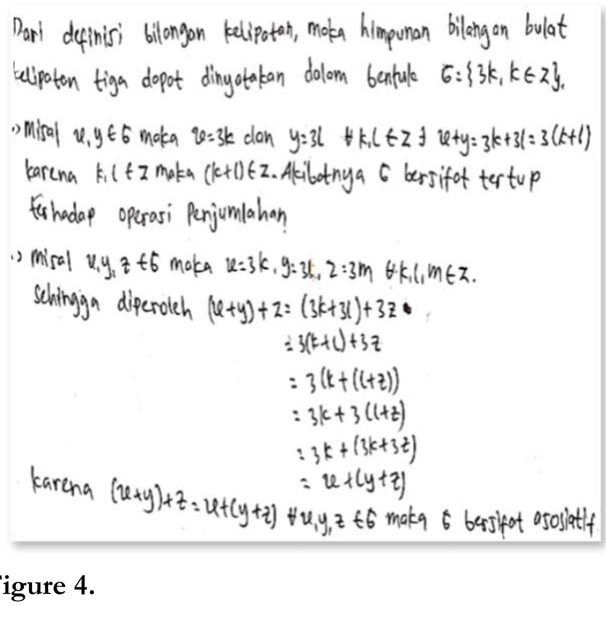
Second Subject

Miss Selvy proved problems number 1 and number 2 by entering the entry stage through the identification of the problem. Identification of the problem done in problem number 1 was to say verbally about what was written, as in the following quote:

Miss Selvy	:	*Ehmm…if* $G = \{-1,1\}$ *is operated by addition, the results are not a part of G Maam.*
Researcher	:	*What do you mean?*
Miss Selvy	:	*The results of the addition is* $(-1) + (-1) = -2$ *Maam. Meanwhile* (-2) *is not a part of* G.
Researcher	:	*Is that so?*
Miss Selvy	:	*Wait a moment Maam…* *I think I need to draw cayley table first.*
Researcher	:	*Ok…take your time…*
Miss Selvy	:	*The results from the table are* $\{-2, 0, 2\}$. *So, it is not closed Maam.*
Researcher	:	*Why do you think so?*
Miss Selvy	:	*Since* $\{-2, 0, 2\}$ *is not a part of* $G = \{-1,1\}$
Researcher	:	*I see… Then, what do you conclude?*
Miss Selvy	:	G *is not closed under addition Maam.*

The results of think-aloud quotation and the interview above show that the identification of the problem by Miss Selvy was taking one of the G members and then was operated. After that, the attack stage was performed by proving as a whole by operating one by one of each member of G. From the results of calculations performed by Miss Selvy in the cayley table, the results are {-2,0,2}, thus the subject drew the conclusion that G = {- 1 , 1} with is not closed under addition .

Then, for question number 2, Miss Selvy performed the entry stage by identifying problems based on the definition of multiples and group definitions. Then, she performed the attack stage by changing the set of integer multiples of three in the form of $G = (3k, k \in Z\}$. Miss Selvy mentioned that a non-empty set of G with binary operations is said to be a group if it meets closed, associative, identity elements, and inverse. The first step taken was proofing the closed property by assuming $x, y \epsilon G$ then $x = 3k$ and $y = 3l$ for each $k, l \epsilon Z$ in such a way $x + y = 3k + 3l = 3(k + l)$. Since $k, l \epsilon Z$ then (k + l) ϵZ, as a result, G is closed under addition operation. The second step was to prove the associative property by assuming $x, y, z \epsilon G$, then $x = 3k, y = 3l, z = 3m$ for each $k, l, m \epsilon Z$ thus it was obtained $(x + y) + z = x + (y + z)\ \forall\ x, y, z \epsilon G$.


Dari definisi bilangan kelipatan, maka himpunan bilangan bulat kelipatan tiga dapat dinyatakan dalam bentuk G:{3k, k∈Z}.
misal x,y∈G maka x=3k dan y=3l ∀k,l∈Z ∋ x+y=3k+3l=3(k+l) karena k,l∈Z maka (k+l)∈Z. Akibatnya G bersifat tertup terhadap operasi Penjumlahan
misal x,y,z∈G maka x=3k, y=3l, z=3m ∀k,l,m∈Z.
Sehingga diperoleh (x+y)+z = (3k+3l)+3z
= 3(k+l)+3z
= 3(k+(l+z))
= 3k+3(l+z)
= 3k+(3k+3z)
= x+(y+z)
karena (x+y)+z = x+(y+z) ∀x,y,z∈G maka G bersifat asosiatif

Translation:

From the definition of the integer of multiple, then the set of the integer of multiple of three could be stated as follows $G = \{3k, k \in Z\}$

- If $x, y \epsilon G$ then $x = 3k$ and $y = 3l, \forall k, l \epsilon Z \ni x + y = 3k + 3l = 3(k + l)$
  Since $k, l \epsilon Z$ then $(k + l)\epsilon Z$. Consequently G is closed under addition
- If $x, y, z \epsilon G$ then $x = 3k$ and $y = 3l, z = 3m\ \forall k, l, m \epsilon Z$
  Thus, it obtained:
  $(x + y) + z = (3k + 3l) + 3z = 3(k + l) + 3z = 3(k + (l + z)) = 3k + 3(l + z) = 3k + (3k + 3z) = x + (y + z)$

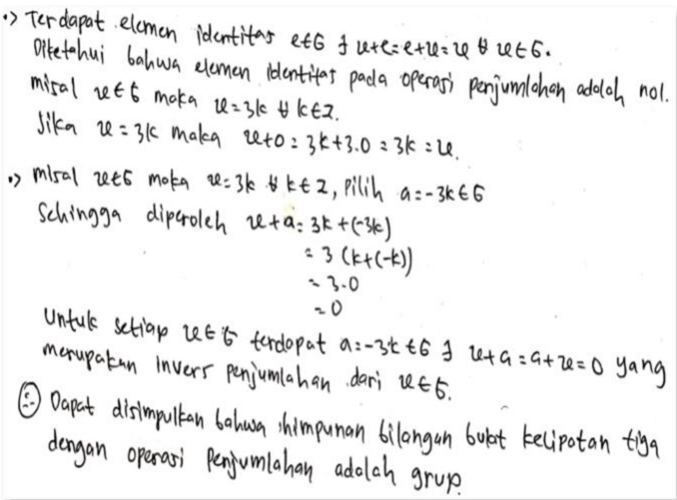
Since $(x + y) + z = x + (y + z), \forall k, l, m \epsilon Z$ then G is associative

Figure 4.

Closed and Associative Proving

The third step was to determine the identity element by writing for each $x \in G$ applies $x + 0 = 0 + x = x$, then $0 \in G$ was the identity element of the addition operation. Then, the fourth step was for each $x \in G$ then $x = 3k$ with $k \in Z$, there is $a = -3k \in G$ in such a way that $x + a = a + x = 0$ is the inverse of the addition of $x \in G$. Of the four proofing steps taken by Miss Selvy, she drew the conclusion that the set of integers multiples of three with the addition operation is a group.

Researcher : *How do you obtain* $G = \{3k, k \in Z\}$*?*
Miss Selvy : *From the integer of multiple of three definition Maam*
Researcher : *What do you mean?*
Miss Selvy : *So…. Based on the definition I know, the integer of multiple of three can be written as* $G = \{3k, k \in Z\}$ *with* k *is an element of the integer. So, I obtained a non-null set* $G = \{3k, k \in Z\}$ *to prove whether* G *is a group toward addition.*
Researcher : *What do you understand about group?*
Miss Selvy : *A non-null set which is binary operated Maam*
Researcher : *Only that?*
Miss Selvy : *No Maam… A non-null set with binary operation should meet closed property, associative, identity element, and inverse.*


•) Terdapat elemen identitas $e \in G \ni x+e=e+x=x \; \forall x \in G$.
Diketahui bahwa elemen identitas pada operasi penjumlahan adalah nol.
misal $x \in G$ maka $x=3k \; \forall k \in Z$.
Jika $x=3k$ maka $x+0=3k+3.0=3k=x$.
•) misal $x \in G$ maka $x=3k \; \forall k \in Z$, Pilih $a=-3k \in G$
Sehingga diperoleh $x+a=3k+(-3k)$
$=3(k+(-k))$
$=3.0$
$=0$
Untuk setiap $x \in G$ terdapat $a=-3k \in G \ni x+a=a+x=0$ yang merupakan invers penjumlahan dari $x \in G$.
∴ Dapat disimpulkan bahwa himpunan bilangan bulat kelipatan tiga dengan operasi penjumlahan adalah grup

Translation:

- The identity element of $e \in G \ni x + e = e + x = x, \forall x \in G$
  It is known that identity element for addition is 0.
  If $x \in G$ then $x = 3k, \forall k \in Z$
  If $x = 3k$ then $x + 0 = 3k + 3.0 = 3k = x$
- If $x \in G$ then $x = 3k, \forall k \in Z$ occurs $a = -3k \in G$
  It obtained $x + a = 3k + (-3k) = 3(k + (-k)) = 3.0 = 0$
  For each $x \in G$ occurs $a = -3k \in G \ni x + a = a + x = 0$ is an inverse from addition of $x \in G$
  Thus, the set of the integer of multiple of three with addition is a group.

Figure 5.
Determining Identity Element, Inverse, and Concluding

Based on the results of the proofing stage performed by Miss Selvy, it can be seen that the structural-intuitive warrant can be seen in Figure 6.

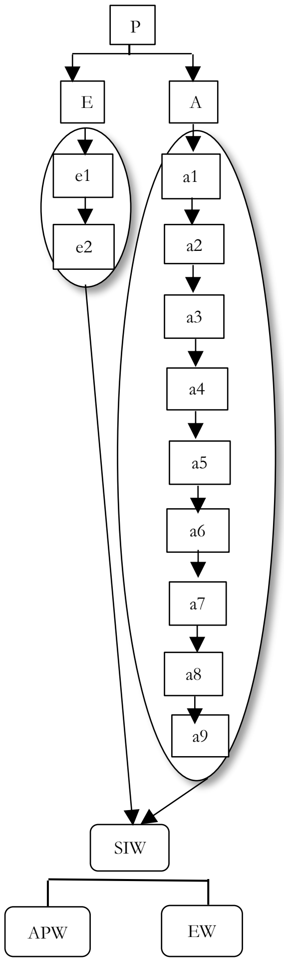


Figure 6.
Structural-intuitive Warrant by Miss Selvy

Table 4.
Figure 6 Annotation

Codes	Thinking Patterns Description
P	The proved problem
E	Entry
A	Attack
e1	Identifying problem by understanding the problem
e2	Employing warrant in the form of the integer of multiple of three definition, group definition with closed, associative, identity element, and inverse.
a1	Assuming that G is not closed since $(-1)+(-1)=-2 \notin G$
a2	Proofing the problem number 1 by operating each of G in cayley table.
a3	Proofing the problem number 2 by changing the set of the integer of multiple of three as $G=\{3k, k \in Z\}$.
a4	Proofing closed property
a5	Proofing associative property
a6	Determining identity element
a7	Determining inverse
a8	The proofing obtained that G is not closed under addition since $\{-2,0,2\} \notin G$
a9	The proofing obtained that $G=\{3k, k \in Z\}$ has met closed, associative, identity element, and inverse properties. And then make conclusion that the set of the integer of multiple of three with addition is a group
SIW	Structural-intuitive Warrant
APW	A priori Warrant
EW	Empirical Warrant

From the result of the proofing stage performed by Miss Selvy, it can be seen that the structural-intuitive warrant was formed based on the thinking process performed was a priori and empirical warrant. A priori arose because she performed proof using pre-existing definitions, namely in the form of multiples and group definitions. Then, from the definition, it was applied to the problem that will be proven thus it raises empirical warrant as a real form of a priori warrant.

Based on the results of exploration in this study, new findings can be obtained that the structural-intuitive warrant formed by students in conducting mathematical proof is presented in table 5.

Table 5.

Formed Structural-intuitive Warrant

Subjects	Types of Structural-intuitive Warrant	Description
Miss Nay	Institutional Warrant	The institutional warrant is formed because it proves unconsciously or without reasoning, Thus, the evidence is based on the findings themselves or appears externally based on the knowledge they have or the experience that has been done.
	Evaluative Warrant	the evaluative warrant is a process of evaluating the knowledge or experience to perform proof, hence it is not necessarily true.
Miss Selvy	A priori warrant	A priori warrant appears intuitively based on the previous concept in the form a set of multiple three of integers, group definitions, and the axioms that exist in the group definition.
	Empirical Warrant	The empirical warrant is a warrant that appears intuitively in the form of facts contained a priori. Then the definitions that exist in the a priori are applied to the proven problem. The fact given is changing the set of integers to a multiple of three in the form of $G = \{3k,\ k\epsilon Z\}$.

Discussion and Conclusion

This study shows that the thinking process undertaken by students in completing mathematical proof related to abstract algebra problems results in a structural-intuitive warrant in the form of institutional and evaluative warrant because she performed proofing in the form of numbers. The first subjects proved inaccurately because she did not pay attention to the operations in the problem, and changed the set of multiples of 3 in the form of $\{3,6,9,12\}$ whereas integers of multiples of three are not only limited to $\{3,6,9,12\}$, hence she experienced difficulty in determining the identity elements and inverse. Institutional warrant is not formed based on available facts but are formed from the findings themselves or appear externally based on the knowledge they have or the experience they have done. While the evaluative warrant is the process of evaluating the knowledge or experience to perform proof, therefore the evidence occurs intuitively and may not necessarily be true.

The second subject was in the form of a priori warrant and empirical warrant. A priori warrant appeared intuitively based on the previous concept in the form a set of multiples three of integer, group definitions, and the axioms that exist in the group definition. She used the definition of triple integers to change the statement in the problem in the form of a mathematical symbol $G = \{3k,\ k\epsilon Z\}$. Second subject also described one by one of the proof of the closed property, associative, identity element, and inverse in accordance with those in the group definition. The empirical warrant is a warrant that appears intuitively in the form of facts contained in the a priori definition. Second subject applied the definitions that correspond to the problem to do the proof. The fact given was to change the integers of multiples of three in the form of $3k$ with k is integer element.

The structural-intuitive warrant is obtained from the thinking process performed by each subject based on Mason's thinking framework (2010) in Table 1. Each subject conducted the process of thinking based on mental activities conducted intuitively, consequently the conclusions made can be true or false. This mental activity was in the form of cognitive processes which is performed to solve a mathematical problem. Analysis of cognitive activities that is performed can be identified from the difficulties experienced, the performed process of reasoning, the determined completion strategy, and the conducted evaluation (Giacomone, Beltrán-Pellicer, & Godino, 2019; Öztürk & Kaplan, 2019). A thinking process that occurs automatically, without going through rational and intellectual reasoning, unconsciousness, or lack of effort to solve a mathematical problem is called intuitive thinking (Leron & Hazzan, 2009; Ejersbo, Leron, & Arcavi, 2014). The guarantor used by the students to complete mathematical proof through thinking intuitively can appear implicitly and explicitly (Faizah, Nusantara, Sudirman & Rahardi, 2020).

sFirst, G is closed under $*$. That is $x \in G$ and $y \in G$ imply that $x * y$ is in G. Second, binary operation $*$ is associative. For all $x, y, z \in G$ berlaku $x * (y * z) = (x * y) * z$. Third, G has an identity element e. There is an e in

G such that $x * e = e * x = x$ for all $x \in G$. Fourth, G contains inverse. For each $a \in G$, there exists $b \in G$ such that $a * b = b * a = e$ (Gilbert & Gilbert, 2015).

The results of this study indicate that structural-intuitive warrant can be formed due to mental activities conducted by students in the form of thinking processes that occur in the brain. Mental activities performed by students produce structural-intuitive warrant in the form of a priori warrant, empirical warrant, institutional warrant, and evaluative warrant. These four warrants are guarantors used by students in proving mathematics related to abstract algebra problems that arise intuitively.

A priori warrant is a guarantor used by students in proving mathematics in the form of definitions and axioms. The empirical warrant can be interpreted as the application of definitions that exist in a priori into the mathematical statement that is being proven. A priori and empirical warrant are formed by students at the stage of entry and attack. While institutional and evaluative warrant are formed by students at the stage of attack and review. Institutional warrant is a guarantor used by students to prove mathematics in the form of numbers which is generally not applicable because it is not in the form of algebraic symbols. The evaluative warrant is the process of evaluating evidence carried out in an institutional warrant. The thinking process conducted by students during mathematical proof occured through identifying the problem, making a conjecture, testing the truth of the conjecture by outlining the evidence one by one, conclusions, re-checking the evidence that has been obtained, and making justification.

Recommendations

In this research, it was found that structural-intuitive warrant is the result of students' thinking process at the university level in the form of a priori warrant, empirical warrant, institutional warrant, and evaluative warrant. Recommendation from this research, should more research on teachers or students at secondary school as to find out the thinking process in completing mathematical proof formally.

Disclosure and Conflicts of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article. This research is original work and does not contain any libelous or unlawful statement or infringe on the rights or privacy of others.

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