Monitoring and Prompting Emergent Algebraic Reasoning in the Middle Years: Using Reverse Fraction Tasks

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To succeed in mathematics middle-years' students must move from additive to multiplicative thinking and from arithmetic calculations to generalised algebraic strategies. If we ask the right questions this progression can be monitored and prompted through fraction tasks. Students' solution strategies for fraction tasks vary from a dependence on diagrams, to methods that demonstrate algebraic reasoning. Based on testing and interviews two frameworks have been developed. The first is used to classify strategies students use to find an unknown whole, when given a known fractional part of the whole, and its equivalent quantity. The second framework monitors the extent to which algebraic reasoning is apparent when opportunities for generalised responses are prompted.

One of the challenges of teaching middle years students' mathematics is moving students' thinking from the specific, concrete and additive strategies to generalised, abstract and multiplicative strategies. Early algebra, with an emphasis on generalising, is important because "our students deserve the chance to develop to the best of their potential" (Lins & Kaput, p. 64). Some topics in the curriculum offer specific opportunities to address this challenge. In this paper we report on research aimed to monitor and prompt emerging algebraic thinking through well-chosen fraction items. First, we review the background literature. Then we outline the study and two frameworks whose development was informed by the students' data. Finally, we consider the implications for monitoring and prompting students' progress.

Many researchers, such as Jacobs, Franke, Carpenter, Levi, and Battey, (2007) Empson, Levi, and Carpenter, (2011) and Siegler and colleagues (2012), have investigated the links between fractional competence and algebraic thinking or reasoning. Wu (2001) suggested that the ability to efficiently manipulate fractions is: "vital to a dynamic understanding of algebra" (p. 17). Lamon (1999) and Wu (2001) argued that the basis for algebra rests on a clear understanding of both equivalence and rational number concepts. Lee and Hackenberg (2014) investigated students' quantitative and algebraic reasoning. Their research involved 18 students in middle school and senior high school and showed that for these students, fractional knowledge appeared to be closely related to establishing algebra knowledge in the domains of writing and solving linear equations.

Previous, related research investigated the types of strategies primary students use to solve fraction tasks and the connections between fraction knowledge and whole number knowledge (Hunting, Davis & Pearn, 1996). Other research (Pearn & Stephens, 2007) highlighted the misconceptions revealed by middle-years students and their reliance on rules and procedures even when these were deemed to be inefficient and incorrect.

Building on these earlier findings we investigated the links between fractional competence and algebraic reasoning. The research questions we address are: "How does middle years students' fractional competence and reasoning show evidence of emerging algebraic reasoning?" and "How can this evidence help teachers monitor and prompt students' progress towards reasoning algebraically?" Fractional competence for this research is deemed to be understanding fraction size and relationships, and basic arithmetic

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competence with simple fractions (Pearn & Stephens, 2015; 2016) while algebraic reasoning is based on the definition from Kaput and Blanton (2005), that is:

> a process in which students generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways. (p. 99)

The Study

This was a mixed method study. Six hundred and seven Victorian middle-years students attempted two paper and pencil assessments: The Fraction Screening Test (FST) (Pearn & Stephens, 2015) and the Algebraic Thinking Questionnaire (ATQ) (Pearn & Stephens, 2016). Table 1 shows the number of students at each year level. Eleven primary school students* did not indicate their year level. The Year 9 students were part of an advanced group who volunteered to participate. This was a convenience sample and there were no Year 7 students as no Year 7 teacher volunteered to participate in this research. Following the testing, 45 students (19 primary, 26 Year 8) were interviewed using the Structured Interview. These students had all successfully answered two of three reverse fraction tasks (Figure 1) and their solution strategies represented the range shown in Figure 7.

Table 1
Number and year level of students who attempted the two paper and pencil tests

Year 5	Year 6	Year 5/6*	Year 8	Year 9	Total
190	269	11	122	15	607

The Paper and Pencil Instruments

The FST is divided into three parts. Part A contains simple comparisons and calculations similar to the students' text book examples, Part B items include number line tasks and Part C include contextualised tasks as well as the three reverse fraction tasks shown in Figure 1. These tasks are referred to as 'reverse fraction tasks' as they require students to find an unknown whole when presented with one known quantity representing a known fraction of the whole. Students are more accustomed to finding the number of objects representing a given part of a whole rather than finding the whole when given the part and the number of objects representing that part. Trials of the FST had already shown that students use a range of strategies to solve these reverse fraction tasks. As shown in Figure 1, only two of the reverse fraction tasks include a diagram.

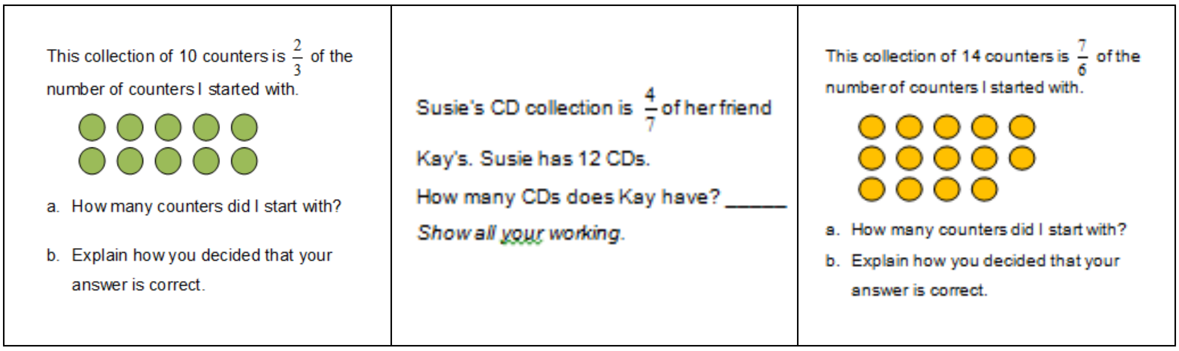


Figure 1: The three reverse thinking fraction questions (Fraction Screening Test C5 – C7)

The ATQ is divided into two sections. Part M focussed on multiplication tasks using both fractions and whole numbers while Part D focussed on division of both whole numbers and fractions. Figure 2 has examples of the types of questions involving both whole number and fraction tasks from Question 1 of both parts of the ATQ.

	Part M: Multiplication focus	Part D: Division focus
Question 1 Task 1	$36 \times 25 = 9 \times \square$	$3 \div 4 = 15 \div \square$
Question 1 Task 4	$\frac{2}{5} \times - = 1$	$\frac{7}{6} \div - = 1$

Figure 2: Examples of Question 1 tasks from the Algebraic Thinking Questionnaire

Question 2 of both Part M and Part D of the ATQ focuses on students' understanding of equivalence relationships with two unknown numbers represented by Box A and Box B, and by symbolic representations of two unknowns (see Figure 3). Students are also required to explain the relationships between the respective unknown numbers or the given symbolic representations (*c* and *d* or *a* and *b*). Students found the ATQ (Pearn & Stephens, 2016) much more difficult than the FST (Pearn & Stephens, 2015) as evidenced by the number of unanswered questions, question marks and comments such as "I don't know".

Task 2	Task 4	Task 5
When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?	What can you say about *c* and *d* in this mathematical sentence?	What can you say about *a* and *b* in this mathematical sentence?
$5 \times \square_{\text{Box A}} = 10 \times \square_{\text{Box B}}$	$c \times 2 = d \times 14$	$a \times \frac{3}{4} = b \times 1\frac{1}{2}$

Figure 3: Examples of Question 2 tasks from the Algebraic Thinking Questionnaire

The Structured Interview

A Structured Interview protocol was developed to enable the researchers to observe more closely the types of strategies students used to solve reverse fraction tasks. This interview included reverse fraction tasks similar to those of the written test but with progressive levels of abstraction to capture students' ability to generalise (see Pearn, Stephens & Pierce, 2018). Tasks started from particular instances of fractions and quantities and became progressively more generalised. The first three questions of the Structured Interview are shown in Figure 4, using the same three fractions as for the FST, but without diagrams and with different quantities representing each fraction.

1. Imagine that I gave you 12 counters which is $^2/_3$ of the number of counters I started with. How many counters did I start with? Explain your thinking.	2. Susie has 8 CDs. Her CD collection is $^4/_7$ of her friend Kay's. How many CDs does Kay have? _____ Explain your thinking.	3. Imagine that I gave you 21 counters which is $^7/_6$ of the number of counters I started with How many counters did I start with? Explain your thinking.

Figure 4: The Structured Interview: Question 1 - 3

In a second set of three questions (4, 5, and 6), the first part used a new quantity with the same fraction; and the second part started with: "If I gave you *any* number of counters which is also a (given fraction) of the number I started with, what would you need to do to find the number of counters I started with?" Question 4, shown in Figure 5, focuses on the fraction two-thirds. Question 5 focuses on four-sevenths and Question 6 on seven-sixths.

4a. If I gave you 18 counters, which is $^2/_3$ of the number of counters I started with, how would you find the number of counters I started with?	4b. If I gave you *any* number of counters, which is also $^2/_3$ of the number I started with, what would you need to do to find the number of counters I started with?

Figure 5: The Structured Interview: Questions 4a and 4b

Students who satisfactorily completed the first six questions of the Structured Interview were asked Question 7 (Figure 6), which required them to use a generalisable method.

What if I gave you *any* number of counters, and they represented *any* fraction of the number of counters I started with, how would you work out the number of counters I started with? Can you tell me what you would do? Please write your explanation in your own words.

Figure 6: The Structured Interview: Question 7

Results from the Paper and Pencil Instruments

Scores for the FST ranged from a minimum of zero to a maximum of 35, with a mean of 21.9 and standard deviation (SD) of 7.1. Scores for the ATQ ranged from a minimum of zero to a maximum of 26, with a mean of 6.9 and a SD of 6.5. Table 2 shows the mean for the total scores by year level with the SD in brackets. Students scored higher results on the FST than the ATQ.

Table 2
Results for the two paper and pencil tests by year level

	Year 5 (n = 190)	Year 6 (n = 269)	Year 5/6* (n = 11)	Year 8 (n = 122)	Year 9* (n = 15)
FST scores	19.6 (7.0)	21.7 (7.0)	24.0 (7.4)	24.8 (6.2)	30.6 (3.4)
ATQ scores	4.3 (4.6)	6.6 (6.2)	8.4 (7.5)	9.5 (6.5)	21.0 (4.1)

To determine whether scores for specific parts of the FST had an association with the scores for the ATQ correlations were checked; all results are statistically significant at the $\alpha = 0.01$ level. Scatterplots were drawn for each part of the FST against the scores for the ATQ. The relationships modelled by positive linear trend-lines indicate that 25% of the variation in the ATQ scores may be explained by FST: Part A scores (R^2=0.25) and 28% by FST: Part B scores (R^2=0.28). While students with a low score for Part A and Part B also have a low score for the ATQ, a high score for each of these parts from the FST does not necessarily correspond with a high score on the ATQ.

The left-hand side of Figure 6 shows a scatterplot of the students' ATQ scores plotted against their scores for FST: Part C, while the right-hand side graph plots the students' ATQ scores against their total FST scores. The relationships are modelled by positive linear trend-lines indicating that, for both, about 45% of the variation in ATQ scores may be explained by FST: Part C scores (R^2=0.46), and by FST total scores (R^2=0.45). In both cases, typically, students with low scores on FST: Part C and the overall scores for the FST achieved low scores on the ATQ, and students who achieved high scores on the FST achieved high scores on the ATQ.

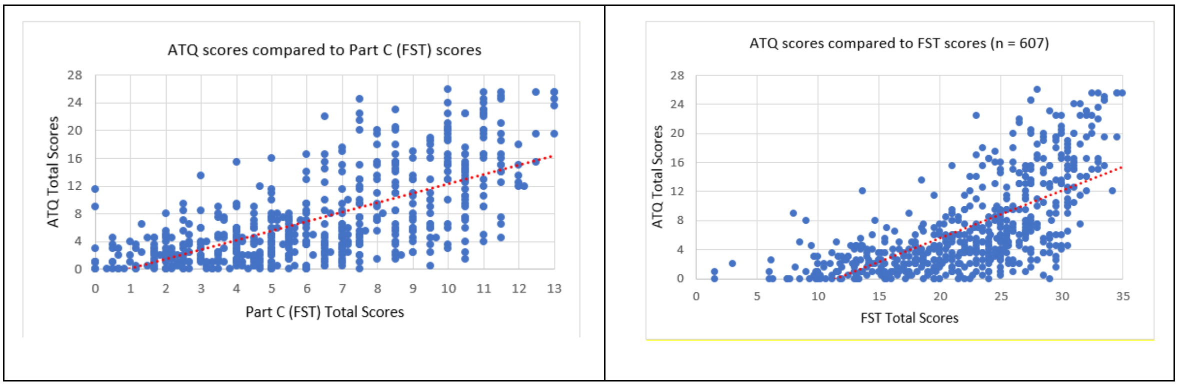


Figure 6: Comparison of results Algebraic Thinking Questionnaire results with Part C and total scores

Comparison of strategies used to solve reverse fraction tasks

The three reverse fraction tasks (see Figure 1) provided an initial lens into students' solution strategies. Students' written responses for these fraction tasks have been classified according to the strategies used for each task. Students' responses have shown that the successful strategies varied from the concrete (diagram dependent), strictly arithmetic (additive, partly multiplicative) to the generalisable (multiplicative) and algebraic (advanced multiplicative). Students successfully responded to unfamiliar situations by employing a range of mathematical strategies as expected in the current curriculum.

Figure 7 shows the *Framework for Reverse Fraction Task Strategies* with explanations illustrated by examples of strategies used to solve the task shown in the middle column of Figure 1. Students using multiplicative methods as well as additive methods can successfully solve all three reverse fraction tasks shown in Figure 1. However, we were unsure whether the students had used the only strategy they knew, or whether they used a strategy that they found easiest, or thought the teacher would prefer. To further investigate the depth of students' knowledge and their use of strategies with such fraction tasks the Structured Interview was developed.

Students using fully multiplicative strategies typically used division by the numerator to find the quantity representing the unit fraction and then multiplied this quantity by the denominator to calculate the whole. These students tended to use the same method regardless of the fraction presented in the task. Some students compressed these two steps into a single step by either dividing by the fraction or multiplying by its reciprocal. These students have correctly interpreted the task and used an appropriate rule or procedure which may or may not indicate algebraic reasoning. It may represent simply a learned rule or it may represent a deeper understanding of the structure of fractions. This could not be decided on the basis of their written responses to FST alone.

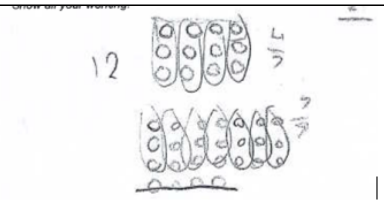
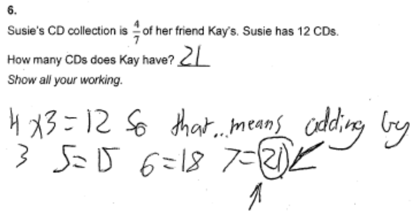
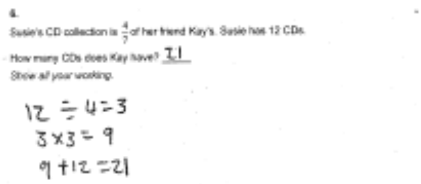
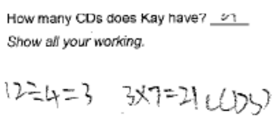
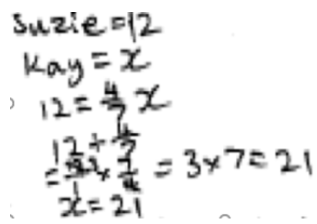
Classification	Explanation	Example
Diagram dependent	Students use explicit partitioning of diagrams before using additive or subtractive strategies	
Additive/ subtractive	Students find the number of objects needed to represent the unit fraction and then use counting or repeated addition or subtraction to find the number of objects needed to find the whole.	
Partially multiplicative	Students use both multiplicative and additive methods. For example, they calculate the missing fractional part ($^3/_7$) and then add it onto the original quantity.	
Fully multiplicative	Students find the quantity represented by the unit fraction using division and then multiply the quantity of the unit fraction to find the whole.	
Advanced multiplicative	Students use appropriate algebraic notation to find the whole, or a one-step method by dividing the given quantity by the known fraction.	

Figure 7: Framework for Reverse Fraction Task Strategies

Results from the Structured Interview

The Structured Interview was designed to encourage students to demonstrate and expand their repertoire of algebraic reasoning with fraction tasks. While responses to each individual task from the Structured Interview could be classified according to the *Framework for Reverse Fraction Task Strategies* (Figure 7) the overall results for each interview also needed to be classified in terms of the development of algebraic reasoning. Students' overall responses to the Structured Interview were analysed using a thematic analysis approach (Braun & Clarke, 2006). These responses varied from computational reasoning to fully generalised algebraic reasoning. *The Emerging Algebraic Reasoning Framework* with levels and descriptions based on the interview data is shown in Figure 8.

	Level	Description
1	Computational fluency Partial	Solved only some questions with method restricted to given fractions and quantities.
2	Computational fluency Complete	Solved all questions with given fractions and quantities but unable to answer more than one question with ‘any quantity’.
3	Generalising - Additive	Solved all questions with given fractions and quantities. Used additive or mixed methods to solve questions with ‘any quantity’. No appropriate generalized multiplicative response for ‘any fraction’ and ‘any quantity’.
4	Generalising-Multiplicative	Solved all questions with given fraction and ‘any quantity’ using multiplicative methods. No appropriate generalised response to ‘any fraction’ and ‘any quantity’.
5	Algebraic generalisation - Verbal	Solved all questions with known fractions and ‘any quantity’ using consistent multiplicative methods. Students verbalised but did not symbolise full generalisation to ‘any fraction’ and ‘any quantity’.
6	Algebraic generalisation - Symbolic	Solved all questions with known fractions and ‘any quantity’ and generalised using consistent multiplicative methods. Appropriate algebraic notation used to solve ‘any fraction’ and ‘any quantity’ task.

Figure 8: The Emerging Algebraic Reasoning Framework

Conclusion

This study aimed to address the questions of how middle-years students' fractional competence and reasoning shows evidence and guides monitoring of emerging algebraic reasoning. Based on the data from the FST, ATQ and Structured Interview, two frameworks have been developed: The *Framework for Reverse Fraction Task Strategies* (Figure 7) and *The Emerging Algebraic Reasoning Framework* (Figure 8).

Analysis of students’ solution strategies using these frameworks has shown that many students did not attempt any or all of these reverse fraction tasks, or gave incorrect responses to those they attempted, or could not explain their solution methods, or needed a diagram in order to use a ‘guess and check’ method to solve one or more of the items. This might be expected at Year 5, but according to the *Australian Curriculum: Mathematics* (ACARA, 2016) Year 8 students should be solving rational number tasks and simplifying equations written in algebraic form.

Students relying on additive or partially multiplicative strategies were unable to solve the last task in the Structured Interview asking them to work with ‘*any fraction*’ and ‘*any number’*. Some students’ solution methods changed in the course of the interview when a fraction, and the corresponding number of objects, changed. Some students who appeared to rely on concrete or additive strategies were able to move confidently to using multiplicative methods. Some students were unable to successfully complete the tasks when presented with ‘*any number*’ instead of a given quantity during the interview.

Although students used a variety of methods to solve reverse fraction tasks on the written test, the frameworks identified those students who were computationally proficient but unable to generalise, as distinct from those who were beginning to generalise, and those who could fully generalise their solutions. This allowed researchers to identify students whose proficient generalisations showed clear evidence of algebraic reasoning.

The two frameworks, used in this study, highlight the connection between middle-years students' fractional competence and reasoning and emerging evidence of their algebraic reasoning. For teachers, these frameworks serve a double purpose. First by providing indicators that enable teachers to identify the stage where students are at, and second to monitor students' progress by giving clear suggestions for how students can and need to be prompted to make the next steps.

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