Exploring the role of visual imagery in learning mathematics

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This paper reports on exploratory research into visual imagery in the learning of early mathematics. It aims to present a theoretical position in relation to perceptual perspectives of visual imagery. Students' visual imagery is explored in one example of a teacher's use of a protocol to encourage students to replicate dot patterns. This example indicated that some students were unable to accurately replicate the spatial and numerical features of the dot patterns. *Discernment* and *determinism* are considered as key elements impacting on the students' imagery, and consideration is given to a focus on these key elements in future research on the use of visual representations in the mathematics classroom.

Introduction

Duval (1999) claimed that "representation and visualization are at the core of understanding in mathematics" (p.3), and visual representations are prevalent in many mathematics classrooms. As such, there is almost an assumption that there is little or no difference between the teachers' use of visual materials and the visualisation of students. However, students may not "see what the teacher sees or believes that they will see" (Duval, 2014, p.160), and there may be a potential gap between a teacher's use of visual materials as teaching tools and their students' use as learning tools.

Recent studies have provided evidence that early number concepts are related to spatial reasoning skills. For example, subitising and comparison of numeracy magnitudes (Dahaene, 2011), and patterning and early algebraic skills (Carraher, Schliemann, Brizuela, & Earnest, 2006; Clements & Sarama, 2014; Mulligan & Mitchelmore, 2009; Papic, Mulligan, & Mitchelmore, 2011). Such spatial reasoning skills are seen to be important in early years mathematics and may impact on later mathematical development (Davis, 2015). This connection between number concepts and spatial reasoning skills is generally introduced to young students through visual representations. However, if students do not see what the teacher believes they will see, then students may not be engaging with the early number concepts as anticipated. In this paper, I explore one teacher's use of dot patterns as spatial representations to encourage early part-whole number concepts through conceptual subitising (Clements & Sarama, 2014) with her Grade 2 students, aged six and seven years old.

Many studies on visualisation relate to learning as a constructive process of creating more abstract mental representations or as abstract propositional theories that align imagery with logical verbal thought (Clements, 1981). The focus on visual imagery in this report relates to imagery as both a perceptual and non-perceptual experience (Schwartz & Heiser, 2006) and draws on theories such as dual coding theory (DCT) (Paivio, 1986) and *determinism* and *discernment* as elements of visual imagery. Analysis of examples from one teaching session are used to consider possible implications of this theoretical perspective for studying students' use of representations in learning mathematics and to explore why some students may not see visual representations in the way that the teacher intended.

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Visualisation and visual imagery in mathematics education

In this paper, I make a distinction between visualisation, visual imagery and visual representations. Arcavi (2003) defined visualisation as "both the product and the process of creating, interpretation and reflection upon pictures and images" (p.215). A visual image, as the product of visualisation, is defined as "a mental construct depicting visual or spatial information" (Presmeg, 2006, p.207). Visual representations, on the other hand, are the tools (diagrams or spatial arrangements) that may support visualisation (Duval, 2014). As a process, visualisation influences our thinking, but this thinking is informed by visual imagery that is held (almost) as a mental picture. From this perspective, visual imagery is a cognitive tool that bridges between the visual representations used in teaching and the process of visualisation that influences a student's thinking in learning mathematics.

Since the 1980s, research has explored the phenomena of visual imagery in the field of mathematics education. For example, there has been research on the classification of types of image schemata (Presmeg, 2006), and on the measurement of students' potential for imaging (Owens, 1999). Research has also suggested a relationship between students' use of visual imaging and performance on test scores (e.g. van Garderen, 2006). Research that has investigated how to promote students' use of visual imagery in mathematics classrooms is less prevalent. Presmeg (1986; 1991) identified the circumstances that existed in a classroom that might affect students' use of visualisation, and one small study by Yackel and Wheatley (1990) explored instructional activities to promote visual imagery with tangram pieces in learning geometric concepts.

Despite research pointing to the relationship between visualisation and learning in mathematics, so far there has been little research into the process of visualisation as students' creation, interpretation and reflection on the images that they (almost) hold as a mental picture. Such studies could help teachers use visual representations imagery more strategically in their mathematics classroom.

Visual imagery: Theoretical perspectives

Schwartz and Heiser (2006) postulated how spatial representations, that exist as both external drawings and internal images, exploit people's perceptual motor system as an embodiment of thought, as well as the non-perceptual. Relating to Paivio's (1986) dual coding theory (DCT), they considered how imagery yields both a perceptual and a verbal code. For example, when visualising a scene people can explain the content, as well as having an image. Hence, the image is not a mental photograph. There may be a sense of an image to depict a shape pattern or form, but they are not "mere echoes" of perception. Images integrate non-perceptual knowledge that allows people to imagine things that have not been perceived in the image. Images are not just "pictures in the head." They can be changed by other mental processes. This viewpoint indicates the potential of imagery in learning mathematics but also points to a complexity. Imagery inherits the structure or perception to complete computations that may not be so easy explained verbally. However, verbal linguistic codes, along with other non-verbal linguistic codes (e.g. gestures) also play a role.

This complexity can be used in the transformation processes of visualisation by integrating or separating parts, or by fading some and focusing on others (Schwartz & Heiser, 2006), but this complexity also has drawbacks in determining which way to orientate and use the image. Schwartz and Heiser identified *determinism* as a key element in visual imaging. Perception is deterministic, that is, people can only see one set of structures at any given moment. Schwartz and Heiser referred to a person's view of a cube as an example. Whilst there may be different ways to perceive the orientation of a cube, perception limits us to viewing one version at a time. Furthermore, whereas in language we can use vague terms such as 'the tree is next to the bush', in visualising this, the spatial relations need to be determined. In a visual image we need to see the tree in front or to the right and so on. Use of language means that people determine specific representational situations in their imagination.

A further element of perceptual learning is drawn from Gibson and Gibson's (1955) work on discernment, what is important to focus on in an image, and how one image differs from other images. Gibson and Gibson's work stems from ecological needs to distinguish differences in edible and non-edible objects (e.g. mushrooms) and they proposed that perceptual learning "involves the increased discernment or pick-up of information" (Schwartz & Heiser, 2006, p.4). From this perceptual perspective, learning is not a constructive process of creating more abstract mental representations. Learning is a process of improving abilities to perceive information that is already present.

Such theories are a shift from a Piagetian constructivist perspective of students' reflective abstraction and instead point towards an embodied perceptual view of learning and the retention of the properties of sensorimotor events in determining and discerning the complex detailed information contained in images (Schwartz & Heiser, 2006), and how these sensorimotor events integrate with language. In this paper, the key elements of determinism and discernment are considered in relation to students' replications of dot patterns to explore the integration or perceptual and non-perceptual. I consider how well such a theoretical perspective can help to understand why some students may not see what the teacher expects them to see.

One problem in such a study is that we can never know exactly what someone is imagining in their heads, and students' replications may not depict their mental imagery exactly. However, their replications, along with verbal explanations, can provide some evidence of how they are determining and discerning the key mathematical concepts from the spatial dot patterns.

An example from a study

An example of one teacher's lesson on dot patterns is taken from an action research project with three teachers and their students, aged 6 to 9 years, in New Zealand, that intended to promote the use of visual imagery in teaching mathematics. The teachers were experienced practitioners who were motivated by concerns for students' mathematics achievement, and for the effectiveness of strategies they were currently using. The research team worked with the teachers in carrying out action research cycles (Tripp, 2005) and in reflecting on the implementation and evaluation of strategic action plans. The cycles involved the production and analysis of video recordings of classroom teaching. Evaluations from research meetings were used to inform the next cycle of alterations, as the teachers refined their pedagogy and aligned changes with the purpose of supporting learners.

The research team worked with the teachers to devise protocols that encouraged the students to *hold an image* in their heads and to describe and replicate these images. The use of dot patterns as visual representations formed the basis of the first tasks that the teachers developed and trialled in their classrooms. Techniques based on imaging with dot patterns such as screening an image and flashing (briefly displaying a spatial arrangement with dots) have been employed elsewhere (Ellemor-Collins & Wright, 2009). However, the techniques do not focus on instruction that is explicit about students holding an image in their heads and replicating these images.

Analogies were used in the teaching to support visual imagery. The idea of moving eyes upwards was used analogously to draw the students' attention to holding an image in their heads, and the teachers focused on holding dot patterns above the students' heads to reinforce this analogy. Other analogies introduced were for the students to take a photo of the dot patterns, as if using a slow shutter (there was some discussion about old cameras), and then to project the image that they saw in their minds onto a frame (such as a blank rectangle on a whiteboard). These analogies were used in developing teaching protocols for use in classrooms for whole class teaching or paired work.

The example in this paper is taken from a whole class teaching session with one teacher and her Grade 2 students. The data are taken from video recordings. Patterns were devised by the teacher using magnetic dots on a whiteboard. The patterns were unfamiliar to the students but contained a structure that would enable the students to see small groups of dots All dots were the same colour with the intention that the students see groups of dots (conceptual subitising) in different ways to develop partitioning and part-whole thinking (Clements & Sarama, 2014).

Example: What did you see?

The students were sitting on the floor and the teacher was sitting on a chair in front of them. The teacher had a standing whiteboard next to her with a blank rectangle. Students each had a workbook open in front of them with blank rectangles drawn onto the page, ready for them to replicate the pattern. The transcript below shows the protocol used by the teacher.

Teacher holds dot pattern above students' eye line for one second and then removes it.

T: Talk to your partner, tell them what you saw.

Students turn and talk to a partner.

T: Turn to the front. I am going to show the pattern again. This is a chance to make changes if you need to. Then you will take a photo.

Teacher holds the same pattern up for three seconds.

T: Now take your photo.

Teacher removes the dot pattern and points to the blank rectangle on the white board.

T: Now project the pattern onto the board.

T: Now draw the pattern in your book.

Students work individually to draw the pattern into the blank rectangles in their books.

Teacher shows the pattern again and keeps it visible. Students check their pattern against the original.

T: What did you see? Tell me how you saw the pattern.

Students reply to the teacher and she models what they saw on her board.

Figures 1 and 2 show two of the dot patterns that the teacher used with this protocol. Examples of some of the students' descriptions in response to the "What did you see?" question are presented. As these explanations were given from the whole class it was not possible to systematically collect these data in relation to each of the students' drawings. The explanations are used to illustrate the students' visualisations.

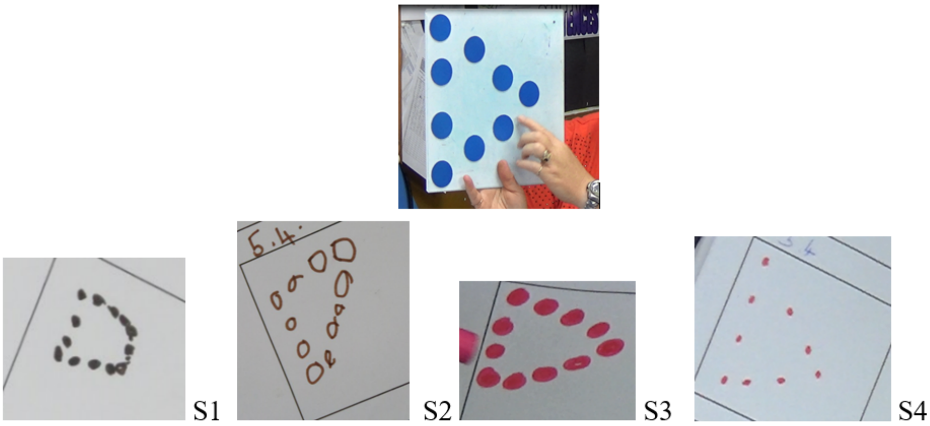

Figure 1: Dot pattern one: Teachers' original pattern and examples of students' replications.

Verbal and gestural explanations of what students S3 and S4 saw in Figure 1:

S3: Four going down there (gestures down), three going down the side (gestures diagonally down) and four going there (gestures diagonally up).

S4: Four over there, going down (gestures down); then two across (gestures diagonally up) and four there (gestures diagonally down).

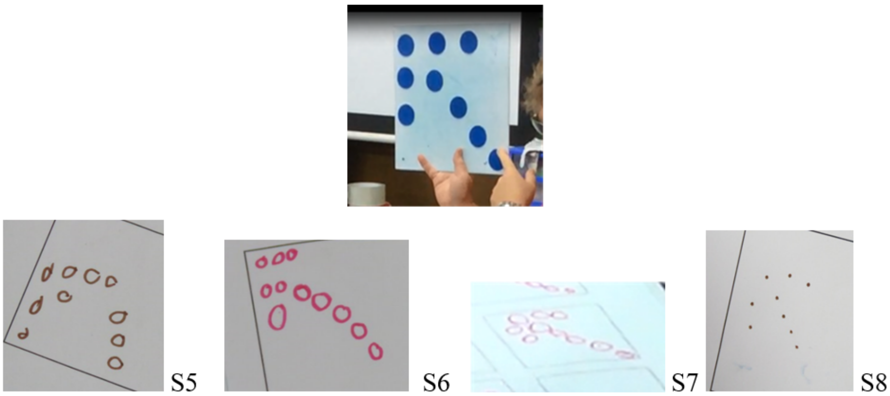

Figure 2: Dot pattern two: Teachers' original pattern and examples of students' replications.

Verbal and gestural explanations of what students S8 and S9 saw in Figure 2:

S8: I saw three going that way (gestures across); two going down (gestures down) and five going diagonal (gestures diagonally).

S9: I saw four going down (gestures diagonally); then three going that way (gestures down) and three going that way (gestures across). (The drawing of S9 was not available in the recording.)

For this pattern, the teacher also asked the students how many dots they saw altogether. Students answered eight, nine and eleven.

Analysis of students' replications and explanations of the dot patterns

Mulligan and Mitchelmore's (2009) construct of Awareness of Mathematical Pattern and Structure (AMPS) indicates the stages that students progress through in pattern and structure. Details of the stages used in AMPS research can be found in Mulligan and Mitchelmore's article, but the focus in this report is on the students' ability to replicate the pattern by integrating both numerical and spatial features, or where only one or the other of these features has been replicated. The interest is in how students' use of these features might be explained perceptually and non-perceptually in relation to determinism and discernment.

In the examples in Figure 1, S1 has replicated some element of the spatial features, and S2 and S3 replications represent the spatial features, but S1, S2, and S3 have not replicated the numerical features. S4 shows evidence of integrating the numerical and spatial features correctly. However, in the students' verbal explanations, both S3 and S4 included one dot in their groupings twice. Interestingly S4 maintained the numerical features in the replication, but the verbal explanation still referred to a repeated dot. This repetition is not incorrect if the students were describing a way to see the structure of the pattern, but the repetition does not promote partitioning to determine how many dots are in the pattern. This replication might explain why some students drew too many dots.

In the examples in Figure 2, S5 has not maintained either of the features (although the use of ten dots is close to the original). S6 and S7 maintain the spatial features but not the numerical features. Even though S8 integrated the numerical and spatial features, the student's verbal explanation suggested that one dot was represented in the groupings twice (at the apex). S9's explanation also replicated the dot at the apex.

In the examples in Figure 2, the students' responses to what they saw, suggested that at least one dot was represented in the groupings twice. This tended to be the dot at the apex, and maybe this was held as an important part of the spatial features. Again, these groupings may be one way of holding the structure of the pattern as a visual image, even though they do not maintain the numerical features of the original pattern. The temptation to see the pattern as a diagonal line of five dots and then the "arrowhead" with three and three dots might explain the response for all the dots as eleven (5 + 3 + 3).

Discussion

In the examples given here, the students' verbal explanations of the dot patterns, repeated at least one dot, even when their replication of the numerical features was correct (S4 and S8). It seemed that this repetition enabled the students to see the spatial features and, hence, the structure of the pattern, but the repetitions did not determine a way of seeing the image that would necessarily give an accurate way of partitioning the number of dots and, hence, supporting part-whole thinking. Whilst it was not possible to collect verbal explanations for all the students, it seemed, from their replications of the dot patterns, that they were not able to replicate the numerical features correctly. There is a possibility that these students were also repeating dots as they attempted to visual and then replicate the dot patterns (e.g. S3).

The students' explanations indicated that they were using verbal language and gesture to determine a visual image. Their use of language was helping them to determine a specific representational situation that might have helped them see the structure. However, this way of determining the structure introduced a complexity that may have influenced the way they replicated the dot patterns. Their determinism may have helped to identify the spatial features but detracted from the numerical features. Whilst we cannot know for certain what was happening in the students' minds as they attempted to visualise the dot patterns, the element of determinism may have played a role in yielding both perceptual and verbal codes (Paivio, 1986). Their replications were 'echoes' of perception that were integrating non-perceptual knowledge, that is, dots that were not there. As such students were focusing on some of the numerical and spatial features and fading the others (Schwartz & Heiser, 2006).

In this example lesson, the teacher was careful not to use language that might influence the way students saw the dot patterns. She also accepted the responses from the students on how they saw the dot patterns with no comment. Similarly, during the protocol, the students were given the opportunity to check their pattern, but there was little emphasis on discerning the differences between their replication and the original dot pattern. An explicit focus on discernment may have helped some students determine other ways of visualising and hence replicate a more accurate of partitioning the dot patterns.

The example presented here is based on a short extract of one lesson with dot patterns, but it suggests that students' visual imagery (as evidenced in their replication and explanations) may have been impacted by the complexity of perceptual and non-perceptual experiences. Whilst the students were attempting to replicate the images, the way they determined the patterns may not have a provided a way of visualising that was useful for the mathematics classroom. Further focus on discernment might help students with such problems.

Concluding comments

This study was positioned theoretically in relation to perceptual perspectives of learning. Learning is not positioned constructively as creating more abstract mental representations, but learning is about improving students' abilities to perceive information in a way that is useful for their learning. Mulligan and Mitchelmore's (2009) AMPS construct has provided evidence that students who are able to integrate numerical and spatial features correctly in replicating a pattern, are more likely to have a deeper conceptual understanding and to achieve better in mathematics. A focus on the integration of both perceptual and non-perceptual codes, and the recognition of the complexity involved in such an integration, may help to support students who are experiencing difficulties in learning mathematics to develop a deeper conceptual understanding.

This paper presents one small case in one classroom, but the findings highlight the problem for mathematics educators that, what they perceive as teachers, may not be the same thing as their students (Nathan & Koedinger, 2001). Further systematic research is needed to explore if other students have similar problems with dot patterns and other visual representations. A study of students' re-representations and verbal/gestural explanations could help to further understand how perceptual and non-perceptual codes impact on visualisation. Such studies could also help inform classroom instruction, by raising teachers' awareness of the complexities of visual imagery and developing steps to help students determine and discern visual representations in a way that is helpful to their learning in a specific mathematics topic. Some initial ideas for instruction might be to focus on contrasting cases to help students discern what is important in spatial representations, and to focus on teachers' language to take account of determinism.

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