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Semi-parametric item response functions in the context of guessing

Abstract

We present a logistic function of a monotonic polynomial with a lower asymptote, allowing additional flexibility beyond the three-parameter logistic model. We develop a maximum marginal likelihood based approach to estimate the item parameters. The new item response model is demonstrated on math assessment data from a state, and a computationally efficient strategy for choosing the order of the polynomial is demonstrated. Finally, our approach is tested through simulations and compared to response function estimation using smoothed isotonic regression. Results indicate that our approach can result in small gains in item response function recovery and latent trait estimation.

Keywords: filtered polynomial, guessing, item response theory

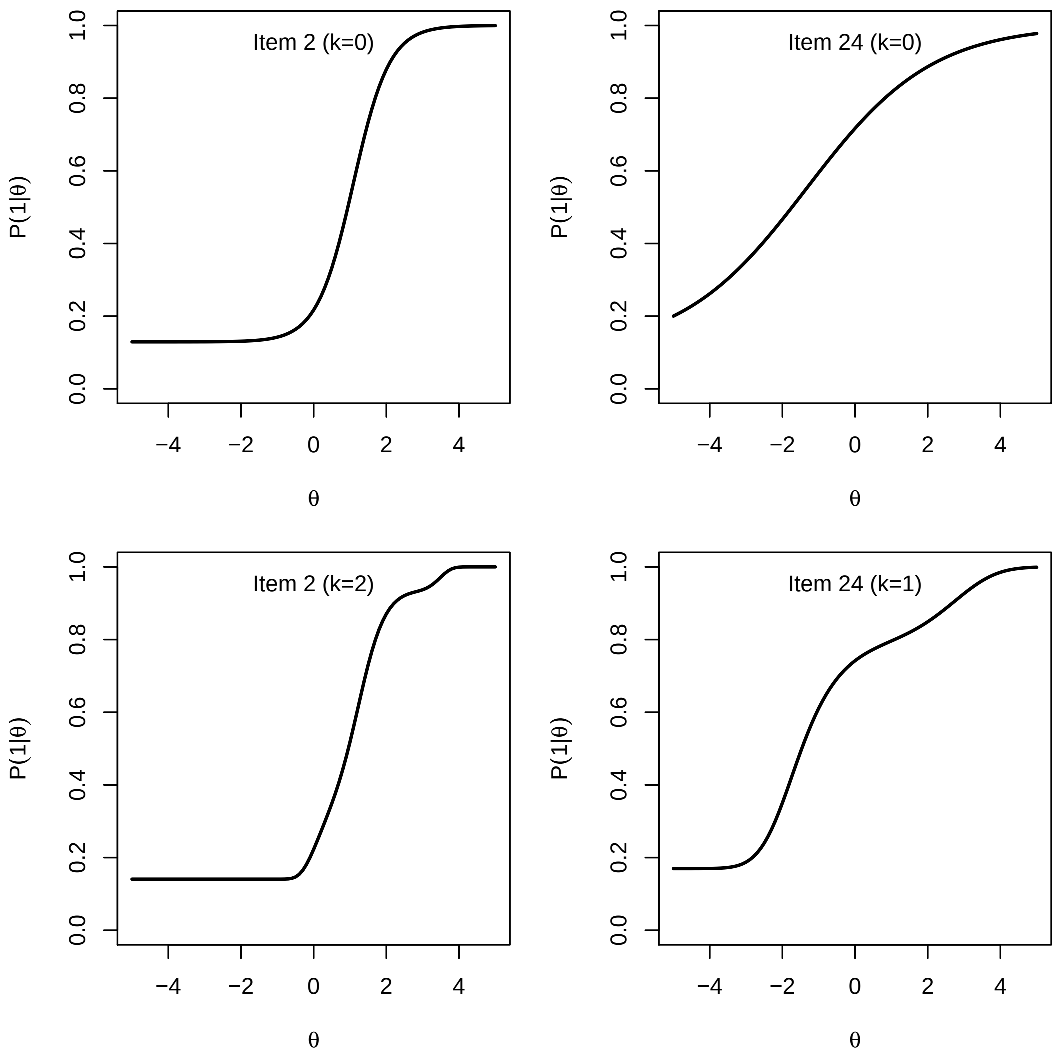
1 Introduction

The three-parameter logistic (3PL; Birnbaum, 1968) model is a commonly used item response model that includes a lower asymptote (top panel in Figure 1). This item model can be particularly useful in educational assessments when the lower asymptote represents the probability of correctly “guessing” the answer to a multiple choice question. But, what happens when there is a non-zero probability of guessing and the item response function (IRF) that generated the data does not neatly follow this functional form (bottom panel in Figure 1)?

When the true IRF follows an atypical shape, use of standard item response models such as the two-parameter logistic (2PL; Birnbaum, 1968) and generalized partial credit models (GPC; Muraki, 1992) can result in poor recovery of the response function, item or model misfit, and suboptimal scoring of individuals’ proficiency (e.g., Falk & Cai, in press; L. Liang, 2007; L. Liang & Browne, 2015; Ramsay & Abrahamowicz, 1989). We expect these same problems to occur when forcing a 3PL model on item response data that could have been generated by an IRF with a different functional form. Remedies to this problem in general include approaches that allow for more flexibility in the estimated IRF. Non-parametric methods may quickly estimate IRFs with good recovery, including smoothed isotonic regression (Lee, 2002, 2007) and kernel smoothing (Ramsay, 1991) - the latter of which can result in IRFs that are not monotonically increasing. Bayesian non-parametric methods may also be employed (Duncan & MacEachern, 2013; Miyazaki & Hoshino, 2009; Qin, 1998), but may be slow to estimate (L. Liang, 2007).

The present research builds upon recent semi- (or quasi-) parametric item response functions that are built by replacing the linear predictor of standard response functions with a monotonic polynomial (Falk & Cai, in press; L. Liang, 2007; L. Liang & Browne, 2015). So far, these approaches have been demonstrated to allow more flexibility to the 2PL and GPC item models, can result in better recovery of IRFs and latent proficiency, and perform comparably to kernel smoothing. Specifically, we will show how a logis-

Figure 1: Example item response functions



Note. The top two panels are response functions fit to math assessment data using the 3PL item model; the bottom panels are the same items fit with the LMPA model.

tic function of a monotonic polynomial with a lower asymptote (LMPA) can allow for greater flexibility than the standard 3PL. The LMPA model reduces to the 3PL at the lowest-order polynomial.

We propose that use of the LMPA is possible in conjunction with other standard item models (e.g., 2PL, 3PL, GPC) and focus on examples from large-scale educational testing. We consider this approach to be potentially appealing and familiar versus a fully nonparametric approach in which all IRFs may follow a non-standard shape. In addition, we assume that monotonicity of the IRFs is often a desired feature when estimating students' factor scores. Since IRF estimation utilizing a monotonic polynomial has apparently not yet been compared with other nonparametric monotonic approaches, through simulations we will compare our modeling approach against smoothed isotonic regression (SISO; Lee, 2002, 2007).

An additional issue we address is the selection of the order of the polynomial for each item. Previous research has so far relied mostly on using information criteria (e.g., AIC or BIC) for selecting the order of the polynomial, which may require refitting the model. For example, Falk and Cai (in press) used an AIC step-wise approach where at each iteration, each item in turn was considered as a candidate for being modeled as a higher order polynomial. The item that improved AIC the most was selected as having a higher-order polynomial, and the process then repeated until no progress could be made. When used in conjunction with the EM algorithm (e.g., Bock & Aitkin, 1981), this process can be computationally prohibitive with a long test. Thus, the most items used in simulations by Falk and Cai (in press) was 20. Use of a different estimation approach may be faster computationally (L. Liang, 2007; L. Liang & Browne, 2015), but has not been demonstrated for cases where multiple group analyses are employed or the dataset contains missing data. As an alternative, we suggest that candidate items may be identified without additional model fitting by considering item fit statistics such as $S - X^2$ (Orlando & Thissen, 2000, 2003). Since $S - X^2$ has been successfully used in previous research to identify atypical IRFs, its use has potential in item screening and may result in refitting the model a fewer number of times. That is, only items that fit poorly according to $S - X^2$ may be good candidates for modeling with higher-order polynomials. Since it is possible that as part of a testing program poor item fit may be used as a flag for deactivating items, this approach also has an advantage if it is able to reduce the number of poorly fitting items.

The remainder of this manuscript is organized as follows. In Section 2, we present the proposed item model, LMPA. Section 3 presents an illustration using a large-scale state math assessment that uses $S - X^2$ to screen items before modeling with the LMPA with higher-order polynomials. Section 4 then presents simulation results illustrating the ability of the LMPA item model to improve IRF recovery in conjunction with $S - X^2$ guided polynomial order selection, and its performance against SISO. Finally, Section 5 contains concluding remarks.

2 The proposed item response model

2.1 Logistic function of a monotonic polynomial with asymptote (LMPA)

To introduce notation, consider $i = 1, 2, \ldots, N$ examinees complete $j = 1, 2, \ldots, n$ dichotomously scored items, with observed item responses $y_{ij} \in [\begin{array}{cc} 0 & 1 \end{array}]$. One way of writing the 3PL model for the "correct" response is as follows:

$$P(1|\theta_i, \kappa_j, \delta_j, \gamma_j) = c(\kappa_j) + \frac{1 - c(\kappa_j)}{1 + \exp(-(\delta_j + \gamma_j \theta_i))} \tag{1}$$

where δ_j and γ_j are the intercept and slopes, respectively. The pseudo-guessing parameter, $c(\kappa_j)$, determines the lower asymptote, which is an estimate of the proportion of examinees in the latent class adopting the "guessing" strategy for this item. This pseudo-guessing parameter is reparameterized as a function of κ_j to allow unconstrained estimation:

$$c(\kappa_j) = \frac{1}{1 + \exp(-\kappa_j)}. \tag{2}$$

To allow one or more "bends" in the 3PL model, we can replace $\gamma_j \theta_i$ with a monotonic polynomial function for item j:

$$P(1|\theta_i, \delta_j, \omega_j, \boldsymbol{\alpha}_j, \boldsymbol{\tau}_j) = c(\kappa_j) + \frac{1 - c(\kappa_j)}{1 + \exp(-(\delta_j + m_j(\theta_i, \omega_j, \boldsymbol{\alpha}_j, \boldsymbol{\tau}_j)))} \quad (3)$$

Here the polynomial (omitting the intercept and item subscripts) is of order $2k + 1$, and its derivative with respect to θ is of order $2k$,

$$m(\theta, \omega, \boldsymbol{\alpha}, \boldsymbol{\tau}) = m(\theta, \mathbf{b}) = b_1\theta + b_2\theta^2 + \cdots + b_{2k+1}\theta^{2k+1} \quad (4)$$

$$m'(\theta, \mathbf{a}) = a_0 + a_1\theta + a_2\theta^2 + \cdots + a_{2k}\theta^{2k} \quad (5)$$

where k is a user-specified non-negative integer, which may vary across items. Thus, if $k = 0$, the model reduces to the 3PL. To ensure monotonicity of the polynomial (i.e., the function increases as θ increases), the derivative is parameterized such that it is always positive, which entails the coefficients $\mathbf{b} = \begin{bmatrix} b_1 & \cdots & b_{2k+1} \end{bmatrix}$ are a complicated function of the parameters ω, $\boldsymbol{\alpha}$, and $\boldsymbol{\tau}$. That the polynomial is monotonically increasing leads to a monotonically increasing response function for the correct response. Details of this parameterization are given by Falk and Cai (in press), are due much to the hard work of others (L. Liang, 2007; Elphinstone, 1985), and appear in Supplementary Materials.

2.2 Example response functions

The example response functions in the bottom panel of Figure 1 were based on the LMPA model, with $k = 1$ and $k = 2$, and their item parameters reported in Table 1. These response functions were obtained by analyses conducted on a state math assessment discussed in Section 3. Note that higher-order polynomials tend to result in more flexibility, but could be prone to overfitting noise in the data.

Non-standard response functions such as these can be the result of a number of different reasons. For instance, responses to self-report personality or psychopathology rating scales may not follow commonly used parametric forms (Meijer & Baneke, 2004).

Figure 2: Mixture of item response functions

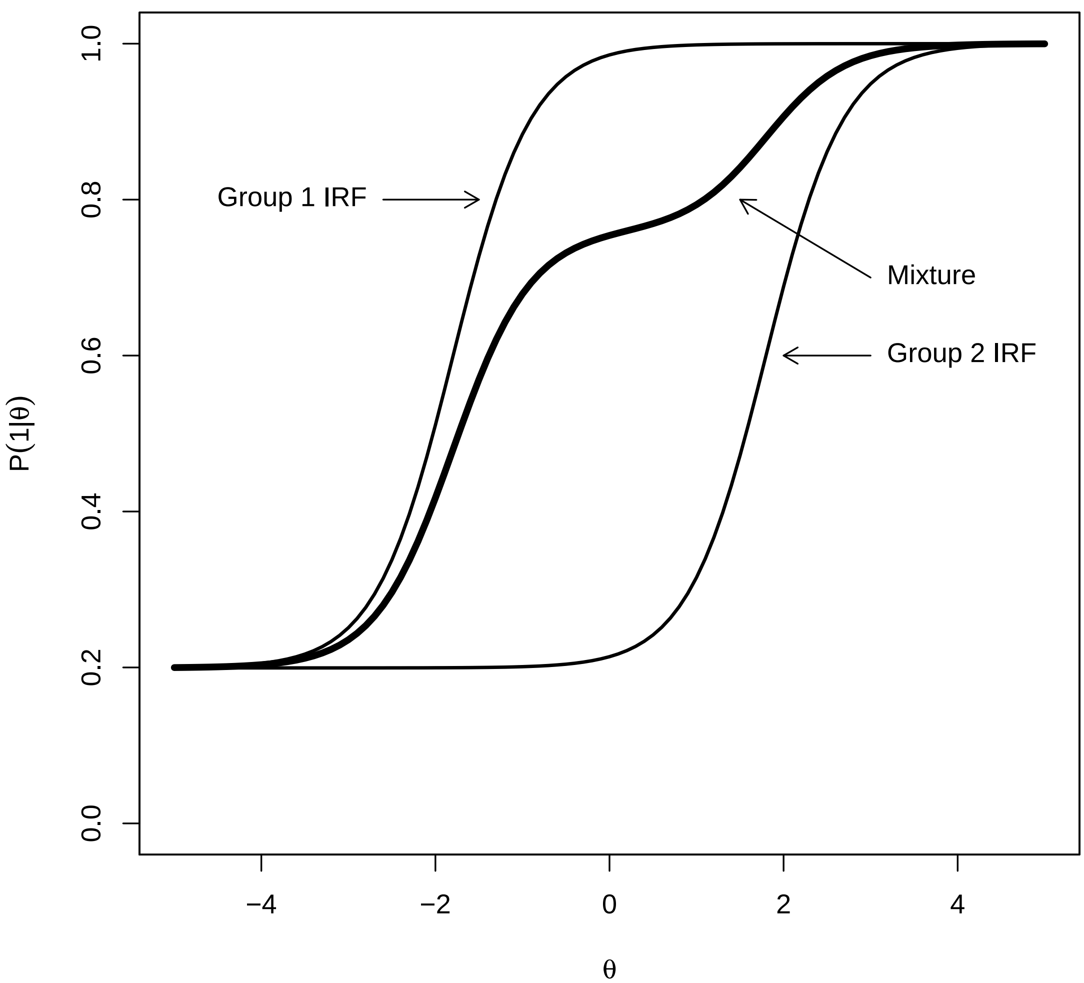


Table 1: Item parameters from example item response functions

Parameter	Item 2 ($k = 0$)	Item 24 ($k = 0$)	Item 2 ($k = 2$)	Item 24 ($k = 1$)
κ	-1.91	-2.14	-1.81	-1.59
δ	-2.19	.77	-2.25	.80
ω	.70	-.54	1.14	-.84
α_1			.59	.40
α_2			.35	
τ_1			.25	-1.20
τ_2			-6.47	

In the context of educational assessments, we propose that data from a heterogeneous population could result in non-standard response functions. Suppose data with respect to a particular item come from two different unidentified groups that do not differ in overall proficiency, but differential item functioning exists for the item. That is, the item is easier for Group 1 than Group 2 (see Figure 2). Such a case could happen in practice if, for example, the item has unique/specific content that some students were exposed to through instruction, whereas other students were not. The data generating model for the correct response to this item is then a mix of the two group's response functions:

$$P(1|\theta) = p_1 P_1(1|\theta) + p_2 P_2(1|\theta) \tag{6}$$

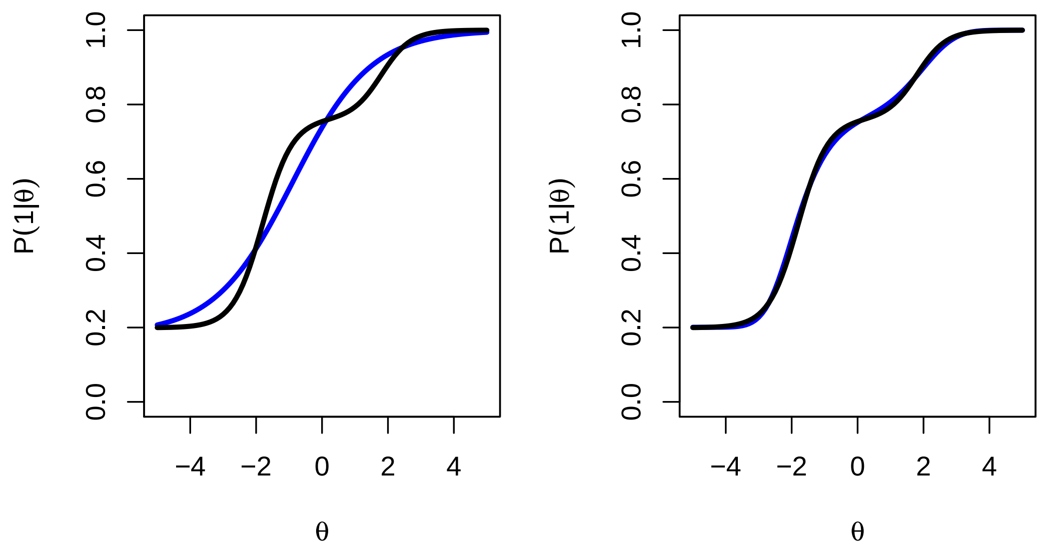
where p_1 and $p_2 = 1 - p_1$ are the proportion of respondents in each group, and P_1 and P_2 are short-hand for the response functions for Group 1 and 2, respectively. Suppose item parameters of $\kappa = -1.39$ and $\omega = .8$ for both groups, but $\delta = -4$ for Group 1 and $\delta = 4$ for Group 2. If we use mixing proportions of 70% and 30% for Groups 1 and 2, then this results in a response function with a non-standard bend (see Figure 2). Although this response function does not come from the LMPA model, the LMPA can still provide a good approximation to it. To illustrate, Figure 3 displays the same mixture IRF (black line) with the best-fitting 3PL line (blue line; left panel) and LMPA line (blue line; right panel) with only a 3rd-order polynomial ($k = 1$). Although the LMPA does not completely overlap with the mixture IRF, the two are almost indistinguishable. Thus, while some approaches do more closely represent the mixing of IRFs or their parameters to achieve flexible IRF shapes (e.g., Duncan & MacEachern, 2013; Miyazaki & Hoshino, 2009) or explicit identification of latent classes (e.g., Rost, 1990), the LMPA approach may provide a reasonable alternative solution that can accommodate non-standard IRF shapes. Importantly, it may also serve as a leading indicator followed by more detailed and computationally-demanding analysis.

Finally, the same two groups may not necessarily be responsible for non-standard IRFs on other items. For example, two other unidentified groups (call these Group A and B) whose group membership is not mutually exclusive with Group 1 and 2 could be responsible for a non-standard IRF on a second item. Thus, the latent classes involved in constructing each non-standard IRF may be different across items, and group membership may overlap in a non-trivial way. This could arise in practice if we consider students' exposure to item content due in part to many teachers whose similarities in implementation of curriculum overlap in a non-trivial way.

2.3 Estimation and the use of stabilizing prior distributions

As Falk and Cai (in press) did for other models including the monotonic polynomial, we used the Bock and Aitkin (1981) machinery of maximizing the marginal likelihood using the EM algorithm for use with the LMPA approach. Full first- and second-order derivatives of the LMPA model necessary for estimation are given in Supplementary Materials. Since the LMPA model adds additional parameters to the 3PL, which is known to be difficult to estimate, prior distributions were placed on some model parameters to provide additional stability. Technically, our approach is Bayesian. It constitutes finding the mode of the posterior distribution for model parameters, though we note this approach is common in estimation of the 3PL (Bock, Gibbons, & Muraki, 1988; Cai, Yang, & Hansen, 2011; Mislevy, 1986). It may be best to understand the effects of these pri-

Figure 3: Best fitting 3PL and LMPA functions to mixture IRF



Note. Black line is mixture IRF. Left panel includes best fitting 3PL (blue), and right panel includes best fitting LMPA line (blue) using $k = 1$.

ors as stabilizing soft constraints on gradients and ridging/conditioning constants on the Hessian, without completely abandoning the operational efficiencies of maximum marginal likelihood.

Specifically, we used diffuse normal priors on all α and τ parameters as also done by Falk and Cai (in press), such as $\pi(\alpha) \sim \mathcal{N}(0, 50)$ and $\pi(\tau) \sim \mathcal{N}(-1, 50)$. To provide stability to the lower asymptote, we use a prior such as $\pi(\kappa) \sim \mathcal{N}(-1.39, .25)$, where $c(-1.39) \approx .20$, or the rate of guessing we might expect from a five option multiple choice question (see also Cai et al., 2011). The final prior used involves that analogous to placing a Beta prior on item uniqueness to prevent Heywood cases in item factor analysis (e.g., Bock et al., 1988; Mislevy, 1986). We show how to adapt this prior developed under multidimensional IRT to the current setting.[1]

[1] A version of the LMPA model that omitted the Beta prior was presented at the 2013 International Meeting of the Psychometric Society. Omission of the Beta prior, a weaker prior on κ, and little troubleshooting of starting values tended to yield mediocre performance of the LMPA in simulations.

Suppose we derived the LMPA item model using a probit rather than a logistic link function. For example, underlying the observed response, y_{ij}, is a variable that is a function of a monotonic polynomial,

$$y_j^* = m(\theta, \boldsymbol{\lambda}_j) + \varepsilon_j = \sum_{q=1}^{2k+1} \lambda_{jq}\theta^q + \varepsilon_j \tag{7}$$

We may further assume that $\varepsilon_j \sim \mathcal{N}(0, \psi_j^2)$, and that $\theta \sim \mathcal{N}(0, \sigma^2)$, usually with $\sigma^2 = 1$. Dichotomous responses are produced from y_j^* via:

$$y_{ij} = \begin{cases} 1, & \text{if } y_{ij}^* \geq r_j \\ 0, & \text{if } y_{ij}^* < r_j \end{cases} \tag{8}$$

where r_j is the threshold for item j. The probability of a correct response under this model may be written as:

$$P^*(1|\theta) = c(\kappa_j) + \frac{1 - c(\kappa_j)}{\psi_j\sqrt{2\pi}} \int_{r_j}^{\infty} \exp\left\{-(1/2)\left(\frac{r_j - m(\theta, \boldsymbol{\lambda}_j)}{\psi_j}\right)^2\right\} dy_j^* \tag{9}$$

$$= c(\kappa_j) + (1 - c(\kappa_j))\Phi\left(-\frac{r_j - m(\theta, \boldsymbol{\lambda}_j)}{\psi_j}\right) \tag{10}$$

where Φ is the standard cumulative normal distribution function. This resembles a standardized version of the normal ogive model with guessing (e.g., Bock et al., 1988), but with the linear predictor replaced by a monotonic polynomial. The goal in remedying Heywood cases in this situation involves preventing the unique variance ψ_j^2 from getting too small, or alternatively the steepness of the function implied by $m(\theta, \boldsymbol{\lambda}_j)$ from getting too large. If we assume a standard normal distribution for y_j^* and uncorrelated θ and ε_j, it follows that

$$\psi_j^2 = 1 - \mathrm{var}(m(\theta, \boldsymbol{\lambda}_j)) \tag{11}$$

Thus, ψ_j^2 is not directly estimated, but is a function of $m(\theta, \boldsymbol{\lambda}_j)$. We may also approximate ψ_j^2 as a function of the parameters implied by the LMPA:

$$\psi_j^2 \approx \frac{1}{1 + (1/D^2)\text{var}(m(\theta, \mathbf{b}_j))} \tag{12}$$

where $D = 1.702$ is the usual scaling constant. Placing a Beta prior on ψ_j^2, such as $\pi(\psi_j^2) \sim B(p,q)^{-1}(\psi_j^2)^{p-1}(1-\psi_j^2)^{q-1}$, where B is the Beta function and with $q = 1$, effectively prevents ψ_j^2 from being too small (e.g., Bock et al., 1988), and results in changes to the posterior mode for the item parameters in $m(\theta, \mathbf{b}_j) = m(\theta, \omega_j, \boldsymbol{\alpha}_j, \boldsymbol{\tau}_j)$. This strategy of developing weakly informative priors is not atypical in Bayesian inference, where one may choose to motivate the choice by imposing a prior on an alternative parameterization of the model (the item unique variance in this case), which then *induces* the desired prior on parameters of interest (the slopes). A typical choice for p is 1.5 (Cai et al., 2011), which we employ throughout this paper, and further details as to how the Beta prior changes the log-likelihood and derivatives are given in Supplementary Materials.

3 Example: Large-scale state math assessment

To illustrate the LMPA item model, we utilized data from 10,000 Grade 4 students who provided responses to 44 dichotomously scored items on a state math assessment. The 3PL model and all LMPA models described in this section used priors as described in the previous section. As a preliminary check, use of a 3PL model for all items (AIC = 48,5581, BIC = 48,6533) indicated better fit than using a two-parameter logistic model for all items (AIC = 48,7290, BIC = 48,7924), and the mean pseudo-guessing parameter, $c(\kappa)$, for the 3PL model was approximately .17. We therefore concentrated on the 3PL as our baseline model.

3.1 Screening of item fit

The next step was to determine whether the LMPA model may be useful in improving model and/or item fit. For this task, we utilized $S - X^2$ using the version outlined by Orlando and Thissen (2000). This approach was chosen as it is a common way of assessing item fit and does not require fitting additional item models to the data. In brief, by conditioning on sum-score based groups, the test statistic detects whether observed counts of correct/incorrect are congruent with the expected counts based on the model. Specifically for item j,

$$S - X_j^2 = \sum_{t=1}^{v-1} N_t \frac{(O_{jt} - E_{jt})^2}{E_{jt}(1 - E_{jt})} \tag{13}$$

where the sum-score groups range from $0, 1, \ldots, v$ with v being the maximum observed sum-score, N_t is the number of respondents in sum-score group t, O_{jt} is the observed proportion of correct responses, and E_{jt} is the expected proportion of correct responses. E_{jt} in turn can be computed using the Lord-Wingersky algorithm (Lord & Wingersky, 1984) as described by Orlando and Thissen (2000). The statistic is compared to a central chi-square distribution with degrees of freedom of $df = t - 1 - z_j$ where z_j is the number of estimated parameters for item j. If adjacent sum score groups are collapsed due to low expected counts (usually less than 1), additional df adjustments are made.

The procedure for screening items for misfit for the empirical example, and in simulations, was as follows, using the baseline (3PL) model as the starting model.

1. Compute $S - X^2$ for all items in the current model.

2. Flag all items with misfit below a threshold (e.g., $p < .05$).

3. For all flagged items (if any), increase the order of the polynomial (e.g., if $k = 0$, change to $k = 1$; if $k = 1$ increase to $k = 2$).

4. If any items were changed as a result of Step 3, re-fit the model.

The Steps 1 through 4 may be repeated as many iterations as desired. For the current example and in simulations, we repeated this process only twice, meaning that at most the model was fit 3 times (3PL and two cycles of the above steps) and the maximum polynomial order for any fitted item was $k = 2$ (5th order). In addition, for Step 2, it is possible to employ either very liberal criteria in screening out items (e.g., no control of Type I error or false discovery rate), or to employ a variety of corrective techniques. We experimented with using no correction (referred to hereafter as NC), and p-value adjustments using the Benjamini-Hochberg procedure (referred to hereafter as BH; Benjamini & Hochberg, 1995), which is sometimes advocated for use in IRT contexts as a high-power alternative to the Bonferroni method (Thissen, Steinberg, & Kuang, 2002). While we may expect the NC approach to lead to overfitting, it will have higher power and we later examine in simulations whether such overfitting has averse consequences.

3.2 LMPA Results

$S - X^2$ initially flagged 11 items as poorly fitting under NC and 7 items when using the BH correction. Thus, roughly 16% to 25% of items may have poor fit. Flagging and fitting items twice under NC resulted in 6 items modeled with $k = 1$ and 6 items with $k = 2$ using the LMPA model. Under BH, these numbers were predictably lower, with 5 items as $k = 1$ and 2 items as $k = 2$. Both approaches led to fewer items having poor fit according to $S - X^2$. For instance, the final NC model had 6 items with poor fit and the final BH model had only 2. As shown in Table 2, both final NC and BH models also improved AIC, with the NC model indicating slightly better fit. However, BIC actually preferred the 3PL model.

We interpret these results as meaning that there is some promise for the LMPA in reducing the number of poorly fitting items according to $S - X^2$. The information criterion results are mixed in terms of whether the overall final model is an improvement, with AIC suggesting an improvement, but a higher penalization for more model parameters under BIC suggesting overfitting. It is therefore difficult to guess whether the LMPA is

Table 2: Model overview of empirical example

	3PL	NC Final Model	BH Final Model
# Parameters	132	168	150
$-2\log L$	485317	485111	485199
AIC	485581	485447	485499
BIC	486533	486659	486580

Note. 3PL = three parameter logistic; NC = no correction to $S-X^2$ p-values; BH = Benjamini-Hochberg correction to $S-X^2$ p-values.

substantially improving IRF recovery. We next turn to simulation results to further test the potential impact of the LMPA and our modeling procedure.

4 Simulation

4.1 Method

We performed a small simulation study to test the performance of the LMPA IRT model and the procedure we employed in using $S-X^2$ to flag candidate items. Whereas the primary goal was to assess IRF recovery of the model versus standard approaches (3PL), secondary goals included a comparison of the model versus smoothed isotonic regression and recovery of factor scores. Simulations were conducted in R (R Core Team, 2015) based on programming by the corresponding author.

4.1.1 Data generation

For data generation, we used a 2 (% non-standard IRF: 20% vs. 40% of items) $\times$ 2 (N: 1,000 vs. 5,000) overall design. In all cases, we used 40 items, generated latent proficiencies, θ, from a standard normal distribution, and performed 100 replications per cell of the design. Studied conditions were thus chosen to partially overlap with conditions found with the empirical example.

Most items were 3PL items (LMPA with $k=0$) with item parameters (κ, ω, and δ) randomly drawn in each replication from a multivariate normal distribution that matched the mean and covariance of the all 3PL model in the previous section. To avoid too many extreme items, parameter draws for any item greater than 1.65 SD away from their mean were discarded and a new set was drawn. The remaining items (8 or 16 items, depending on condition), were randomly constructed from a mixture of normal cumulative distribution functions (CDF) and lower asymptotes. Specifically, $p_1(g_1 + (1 - g_1)\Phi(\theta|\mu_1, \sigma_1^2)) + p_2(g_2 + (1 - g_2)\Phi(\theta|\mu_2, \sigma_2^2)) + p_3(g_3 + (1 - g_3)\Phi(\theta|\mu_3, \sigma_3^2))$, with p_1 and $p_2 \sim \text{unif}(.2, .4)$, $p_3 = 1 - p_1 - p_2$, $\mu_1 \sim \mathcal{N}(-1.5, .4^2)$, $\mu_2 \sim \mathcal{N}(1.5, .4^2)$, $\mu_3 \sim \mathcal{N}(0, .4^2)$, all g parameters drawn from $\text{unif}(.1, .3)$, and σ_1, σ_2, and σ_3 independently drawn from a log-normal distribution, $\ln \mathcal{N}(-1.03, .22)$. Here we use Φ to denote the cumulative distribution function along θ for a given mean and variance.

4.1.2 Fitted models and factor score estimation

To each generated dataset, we both mimicked the steps taken in our empirical example, and estimated IRFs using smoothed isotonic regression. That is, we initially fit a 3PL model, followed by flagging items using NC or BH, and fitting flagged items with higher-order polynomials. We repeated re-fitting twice for both NC and BH, resulting in final NC and BH models. Thus, we are interested in comparing the 3PL to the final NC and BH models. Numerical integration necessary for estimation was done with 49 equally spaced quadrature points between -5 and 5 across θ. The maximum number of M-step iterations was set at 50, and the maximum number of EM cycles was set at 500. Model estimation terminated if item parameters from one EM cycle to the next differed by .001 or less.

To further mimic a real data analysis situation, we never started estimation at the true model parameters, but rather at $\kappa = -1$, $\delta = 0$, $\omega = 0$, $\alpha = 0$, and $\tau = -1$ for all items. However, to automate troubleshooting of estimation problems, we employed the following ad-hoc procedure for all replications. Initial starting values for item parameters were further tweaked by using the final estimates of a model run with very few maximum M-step iterations and EM cycles (2 and 5, respectively). We then commenced with estimation as outlined in the previous paragraph. If the maximum number of EM iterations was reached, or if ridging the Hessian ever failed to yield an invertible matrix, new starting values were attempted by using normalized sum scores in the complete-data likelihood for each item. The item parameter estimates based on this complete-data analysis were then used as starting values.

For a comparison with our proposed approach, estimation of IRFs via smoothed isotonic regression (SISO) was conducted by adapting S-Plus code provided by Lee (2002). In brief, isotonic regression is a least squares approach to regression analysis with the important property that the resulting function is monotonically increasing. In IRF estimation, given ordered values of the latent trait, $\theta_1 \leq \theta_2 \leq \cdots \leq \theta_N$, the isotonic regression function $P^*(\theta)$ is increasing as a function of the latent traits, $P^*(\theta_1) \leq P^*(\theta_2) \leq \cdots \leq P^*(\theta_N)$. In addition, $P^*(\theta)$ qualifies as the isotonic regression function if and only if it minimizes $\sum_{i=1}^{N} \left[\tilde{P}(\theta_i) - \hat{P}(\theta_i)\right]^2$, where $\hat{P}(\theta)$ could be any isotonic function of θ. In applications of isotonic regression with observed variables, $\tilde{P}(\theta_i)$ may take the place of some known value associated with θ_i. Although the latent traits are unknown, Lee (2002) determines the percentile ranks of θ from observed sum scores with ties broken randomly, which may be further transformed onto the hypothesized distribution for θ (standard normal) using the quantile function. $\tilde{P}(\theta_i)$ then becomes the observed response (0/1) for respondent i and the function $P^*(\theta)$ that minimizes the sum of squares is estimable.

Since the isotonic regression function, $P^*(\theta)$, often resembles a step function, kernel smoothing may additionally be used to provide a smoother IRF shape, which can result in better response function recovery (Lee, 2002, 2007). Following Lee (2002, 2007) we used a Guassian kernel and based in part on examining the performance of bandwidths between .05 and .3 (in .05 increments), we report results based on bandwidths of .2 and .1 for $N = 1000$ and $N = 5000$ data generating conditions, respectively, as these arguably yielded the best performance. Once the SISO function was estimated, linear interpolation was used if evaluation of the function at a point along θ not already available was required.[2] For instance, estimation of factor scores under our proposed approach and under SISO was conducted using the *expected a posteriori* (EAP; Bock & Aitkin, 1981; Bock & Mislevy, 1982) method using rectangular quadrature across θ from -5 to 5 in .1 increments.

4.2 Results

Overall it was our hope and expectation that the procedure of using $S-X^2$ would tend to correctly flag non-standard IRFs (using either NC or BH), which could in turn be modelled by the LMPA with higher-order polynomials. This may result in better recovery of individual IRFs.

4.2.1 Ability of $S-X^2$ to detect non-standard item response functions

In general, power to detect non-standard IRFs was slightly less than anticipated. Table 3 displays power and false positive rates based on the final NC and BH models for all cells in the design. Power is the proportion of items with non-standard IRFs that were fitted with the LMPA model with $k>0$, whereas the false positive rate is the proportion of items with a true 3PL shape fitted by the LMPA model with $k>0$. Both were computed across all replications in each cell. Thus, power to detect non-standard IRFs only reaches a maximum of .36 or .33 for NC and .14 for BH (both when $N=5,000$). This amounts to correctly detecting approximately 3/8 items for NC and 1/8 items for BH when 20% of items have non-standard IRFs, or 5 to 6/16 items for NC and 2/16 items for BH when 40% of items have non-standard IRFs.

While we do not immediately have a full explanation, we note that (Orlando & Thissen, 2003) found the lowest power for $S-X^2$ for items that had a non-standard bend or a plateau in the middle of the IRF - visually similar to the items we generated - versus items that had non-monotonicity or an omitted asymptote. Since non-standard IRFs were generated via a random process, it is not necessarily the case that all items

[2] Linear interpolation was also used by Lee (2002). In addition, smoothing with the shown bandwidths occurs across the percentile ranks for the provisional θ estimates *before* mapping the smoothed isotonic regression functions onto the quantiles for θ.

Table 3: $S - X^2$ detection rates for the simulation study

	20% non-standard IRFs		40% non-standard IRFs	
	$N = 1,000$	$N = 5,000$	$N = 1,000$	$N = 5,000$
Power				
NC	.101	.361	.111	.336
BH	.011	.144	.009	.144
False positive				
NC	.052	.055	.059	.082
BH	.003	.004	.002	.011

Note. IRF = item response function; NC = no correction to $S - X^2$ p-values; BH = Benjamini-Hochberg correction to $S - X^2$ p-values.

would in fact look very extreme. This low power rate may not necessarily be problematic, however, to the extent that both approaches may flag the worst-fitting items and flag non-standard items at a higher rate than 3PL items.

4.2.2 Recovery of item response functions

We assessed recovery of IRFs by using Root Integrated Mean Square Error (RIMSE; e.g., L. Liang, 2007; Ramsay, 1991), defined as the following for item j:

$$\text{RIMSE}_j = \left(\frac{\sum_{q=1}^{Q} (\hat{P}_j(1|\theta_q) - P_j(1|\theta_q))^2 \phi(\theta_q)}{\sum_{q=1}^{Q} \phi(\theta_q)} \right)^{1/2} \times 100 \tag{14}$$

where $\hat{P}(1|\theta)$ and $P(1|\theta)$ are short-hand for the estimated and true response function for the correct response, ϕ is a standard normal density function, and the sum is across a series of Q equally spaced quadrature points along θ (-5 to 5 in .1 increments).[3] RIMSE thus represents the root of a weighted average squared discrepancy between the true and estimated response functions, with more weight given to discrepancies towards the middle of the θ distribution. Lower values indicate better IRF recovery. This index

[3]Since under our estimation procedure θ was fixed to a standard normal distribution (the same as the data generating model), no additional linking procedure was required for assessment of IRF and factor score recovery.

Table 4: Item response function recovery for replications with items flagged under NC

	# Rep.	Overall RIMSE	3PL items	Non-standard items
$N = 1,000$				
20% non-standard IRFs				
3PL Model	89	2.67	2.17	4.64
NC Model	89	2.69	2.24	4.48
SISO	89	3.54	3.31	4.46
40% non-standard IRFs				
3PL Model	95	3.16	2.14	4.67
NC Model	95	3.12	2.21	4.48
SISO	95	3.71	3.26	4.38
$N = 5,000$				
20% non-standard IRFs				
3PL Model	97	1.62	1.03	4.01
NC Model	97	1.50	1.06	3.23
SISO	97	2.44	2.22	3.34
40% non-standard IRFs				
3PL Model	100	2.22	1.04	3.99
NC Model	100	1.96	1.09	3.26
SISO	100	2.62	2.19	3.27

Note. # Rep. = number of replications; RIMSE = Root integrated mean square error; 3PL = three parameter logistic; NC = no correction to $S - X^2$ p-values; SISO = Smoothed isotonic regression.

was computed for all estimated items and was aggregated across replications and items where necessary.

Since our goal was to compare the LMPA approaches (NC and BH) to the 3PL and SISO, we focus only on those replications where $S - X^2$ flagged ill-fitting items under NC (Table 4) and BH (Table 5). Both of these tables utilized the same simulated data. Overall, the NC model slightly improved RIMSE versus the 3PL model across most conditions (see Table 4), with gains more prominent at the larger sample size ($N = 5000$), perhaps in part to higher power to flag non-standard IRFs. Examining items where the true underlying IRF was either a 3PL or a non-standard item revealed that all of these gains were because of improvement in recovering non-standard items. In fact, recovery of 3PL items was slightly worse for the NC model, though perhaps negligibly so. Overall, SISO performed relatively poorly even compared to the all 3PL model. However, closer inspection reveals that SISO did well at recovering non-standard items on par with NC, but did not recover 3PL items as well. This is not particularly surprising given that the 3PL and NC models are likely fitting the true item model to the data for many of the 3PL items.

Overall, the BH model also slightly improved RIMSE versus the 3PL model across all studied conditions (see Table 5), in a similar pattern to NC. Gains in RIMSE were also due to better modeling of non-standard items. Albeit based on a much smaller number of replications, use of BH appeared to have no averse impact on the recovery of 3PL item IRFs, in slight contrast to using NC. Overall recovery for SISO was not as good as the 3PL and BH models. The pattern was such that SISO did not recover 3PL items as well, but slightly outperformed the other approaches in recovery of non-standard items.

4.3 Recovery of factor scores

Recovery of factor scores was assessed using root mean square error (RMSE):

$$\text{RMSE} = \left(N^{-1} \sum_{i=1}^{N} (\hat{\theta}_i - \theta_i)^2 \right)^{1/2} \times 100 \tag{15}$$

Table 5: Item response function recovery for replications with items flagged under BH

	# Rep.	Overall RIMSE	3PL items	Non-standard items
$N = 1,000$				
20% non-standard IRFs				
3PL Model	15	2.71	2.18	4.84
BH Model	15	2.66	2.19	4.54
SISO	15	3.49	3.26	4.40
40% non-standard IRFs				
3PL Model	16	3.18	2.16	4.71
BH Model	16	3.13	2.18	4.56
SISO	16	3.73	3.25	4.45
$N = 5,000$				
20% non-standard IRFs				
3PL Model	67	1.65	1.02	4.14
BH Model	67	1.53	1.02	3.58
SISO	67	2.46	2.22	3.39
40% non-standard IRFs				
3PL Model	89	2.23	1.04	4.01
BH Model	89	2.04	1.03	3.55
SISO	89	2.62	2.19	3.27

Note. # Rep. = number of replications; RIMSE = Root integrated mean square error; 3PL = three parameter logistic; BH = Benjamini-Hochberg correction to $S - X^2$ p-values; SISO = Smoothed isotonic regression.

where $\hat{\theta}_i$ is the EAP score for simulee i and θ is the actual simulated value.

As with IRF recovery, factor score recovery is reported only for replications involving ill-fitting items according to NC and BH in Tables 6. Across both sets of results, NC and BH tended to result in improved factor score recovery, with the lone exception with the smallest sample size ($N = 1000$) and fewest non-standard items (20%) where NC (36.93) was nearly identical to the all 3PL model (36.91). However, it could be argued that such gains by NC and BH were very small. For instance, NC and BH improved recovery over the 3PL by at most .28 and .2, respectively, both at the largest sample size ($N = 5000$) and with the most non-standard items (40%). The performance of SISO in factor score recovery was relatively worse than other methods in all cases, except at the largest sample size ($N = 5000$) and number of non-standard items (40%) where it slightly outperformed the 3PL model.

5 Discussion

We have presented a new IRT model, the logistic function of a monotonic polynomial with asymptote (LMPA), that can account for guessing as does the 3PL, but has a more flexible IRF shape. We proposed a possible data generating mechanism that may yield such non-standard IRFs - heterogeneous populations whose IRFs are mixed together in generating the item responses. We tested use of $S - X^2$ to flag candidate items for use with the LMPA model in an empirical example and through simulations. Finally, our approach was also compared against both the 3PL model and smoothed isotonic regression.

Our empirical example suggests that use of $S - X^2$ and LMPA can improve the number of well-fitting items without it being necessary to employ a computationally intensive step-wise approach utilizing AIC, BIC, or likelihood ratio tests. For instance, the number of poorly fitting items was reduced by approximately half or better, depending on whether NC or BH p-values under $S - X^2$ were examined. Such initially poor-fitting items may be those that show instructional sensitivity in cases where curriculum and

Table 6: Factor score recovery for replications with items flagged under NC and BH

	NC		BH	
	# Rep.	RMSE	# Rep.	RMSE
$N = 1,000$				
20% non-standard IRFs				
3PL Model	89	36.91	15	37.44
NC Model	89	36.93		
BH Model			15	37.36
SISO	89	37.56	15	37.94
40% non-standard IRFs				
3PL Model	95	36.64	16	36.41
NC Model	95	36.61		
BH Model			16	36.34
SISO	95	37.00	16	36.67
$N = 5,000$				
20% non-standard IRFs				
3PL Model	97	36.62	67	36.59
NC Model	97	36.46		
BH Model			67	36.45
SISO	97	36.92	67	36.88
40% non-standard IRFs				
3PL Model	100	36.29	89	36.26
NC Model	100	36.01		
BH Model			89	36.06
SISO	100	36.23	89	36.20

Note. # Rep. = number of replications; RMSE = Root mean square error; 3PL = three parameter logistic; BH = Benjamini-Hochberg correction to $S - X^2$ p-values; NC = no correction to $S - X^2$ p-values; SISO = Smoothed isotonic regression.

instruction is not implemented in a fully standard way across the entire population. Our procedure thus serves the dual purpose of identifying potential items, and allowing retention of such items (which can be expensive to develop) in a large-scale assessment if practice dictates that poor fitting items are flagged for possible deactivation.

Our simulation results suggest that the procedure utilized on our empirical example is sound and can lead to better IRF recovery, though only very small gains in factor score recovery. The performance of our approach was especially better with a higher sample size, and where more items had true non-standard IRFs. Allowing the LMPA item model to be mixed with other item types such as the 3PL allows for parsimony and resulted in better overall IRF and factor score recovery than SISO in our simulations. However, we note that SISO performed on par and sometimes better at recovering non-standard items, perhaps in part to the low power of $S - X^2$ in detecting candidate items for use with the LMPA item model. It remains possible that the presence of a higher percentage of non-standard items on a test could lead to more comparable overall performance between both LMPA approaches and SISO.

It is worth noting that although we assume that monotonic IRFs are desired as this is the most likely case in large-scale testing programs, estimation of IRFs can be performed by both nonparametric and semi-parametric methods that allow for nonmonotonicity. For example, kernel smoothing (Ramsay, 1991) could be employed, or the constraints enforcing monotonicity of the LMPA could be released by reparameterizing (e.g., see Falk & Cai, in press). In retrospect, it may also be worthwhile to explore alternative ways of screening for poorly fitting items as some of these may have higher power than $S - X^2$ and involve fitting nonparametric approaches that can reveal nonmonotonicity (Douglas & Cohen, 2001; T. Liang & Wells, 2015; Wells & Bolt, 2008).

Finally, although use of $S - X^2$ for identifying candidate items for the LMPA model may not result in the best IRF recovery possible, we argue that our goal is not to necessarily find the best solution, but to merely improve item fit and IRF recovery in a computationally efficient way. Our examples focus on tests with less than fifty items, yet we note that many testing programs may use hundreds or thousands of items, which can make finding the most optimal solution unfeasible if coupled with any step-wise approach. Thus, we close with the observation that our approach can be used in such situations (unlike a step-wise approach), but that further room for improvement is possible in determining the order of the polynomial for the LMPA and other monotonic polynomial item models.

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