TONY'S STORY: READING MATHEMATICS THROUGH PROBLEM SOLVING

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We describe an attempt by a former mathematics teacher to read an undergraduate mathematics proof aided by discussions with a mathematician using the language of mathematical problem solving. The literature on successful approaches to reading mathematics is scarce at the secondary and undergraduate levels. Shepherd, Selden and Selden (2012) offered three possible reasons why undergraduate students find it difficult to read passages from a mathematics textbook. From the ultimately fruitful attempt by the teacher, we postulate how a problem solving approach can successfully negotiate these three difficulties.

INTRODUCTION

Shepherd, Selden and Selden (2012) state that "it appears to be common knowledge that many, perhaps most, beginning university students do not read large parts of their mathematics textbooks in a way that is very useful in their learning". This concurs with our own various experiences as secondary school teachers and university lecturers with regards to our students' mathematics reading proclivities.

Tay(2001) citing Waywood (1992) who noted that the majority of reported work on writing to learn mathematics is focused at a primary level, lamented that little progress had been made at the higher levels and suggested some assessment modes which would encourage good mathematics reading. Others (Bratina & Lipkin, 2003; DeLong & Winter, 2003; Draper, 2002) have also made calls for students to be taught how to read mathematics. Wilkerson-Jerde and Wilensky (2011) investigated how mathematicians make sense of an unfamiliar proof that they read for the first time and try to elicit reading strategies for school students. On the whole however, Osterholm (2008), based on a survey of 199 articles having to do with the reading of word problems, reported that there was little about reading comprehension of more general mathematical text.

Difficulties in reading mathematics textbooks

Shepherd, Selden and Selden (2012) adapted the Constructively Responsive Reading framework (CRR) by Pressley and Afflerbach (1995) to understand the difficulties first-year university students had in reading their mathematics textbooks. They made some significant observations which we shall report in this section.

Their research (Shepherd, Selden, & Selden, 2012) involved eleven precalculus and calculus students who were each asked to read aloud a selected passage from their textbook. They were stopped at intervals during their reading and asked to attempt a

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task based on what they had read or a textbook example. Overall, the students performed poorly although they were considered good students based on their American College Test (ACT) Reading and Mathematics scores.

> The students' difficulties working tasks all seemed to arise from, or depend largely on, at least one of three main kinds of difficulty: (a) insufficient sensitivity to, or inappropriate response to, their own confusion or error; (b) inadequate or incorrect prior knowledge; and (c) insufficient attention to the detailed content of the textbook. The difficulties working tasks and their origins occurred throughout the passages read and were associated with exposition, definitions, theorems, worked examples, and explorations. Furthermore, most students exhibited all three of these difficulties usually several times. (p. 238)

In spite of the generally poor performance, there were a few students who when they failed to understand a passage, persisted in rereading the passage and reworking the task until they could do it correctly. Shepherd, Selden and Selden (2012) wondered if such students had an unusual feeling or belief that in persisting they could ultimately succeed and if such a feeling of the value of persistence can be engendered by providing supporting experiences.

Placed against the poor reading of most students and the nascent success attributable to persistence, Shepherd, Selden and Selden (2012) offered their observation of why mathematicians appear to be effective readers:

> For sufficiently important reading, we will work tasks or construct examples to check the correctness of our understanding and tend to look for errors. On finding an error, we rework the task, reread the appropriate passage, or construct an example. We each feel that we can benefit from such a process, and suspect that this feeling of self-efficacy arose from our past positive experiences with reworking tasks, rereading associated passages, and constructing examples. We suggest our students lacked the feeling that they can independently rework a task or reread a passage until they ultimately "get it right." (p. 243)

Reading mathematics through problem solving

In this paper, we propose that reading mathematics could be approached as a series of problem solving attempts. Recall that Pólya's (1945) model of problem solving requires firstly that one understands the problem. We conjecture that readers with a problem solving mindset will recognise their lack of understanding of a passage and so will be able to circumvent the difficulty (a) of the students of Shepherd, Selden and Selden (2012) with regard to "insufficient sensitivity to, or inappropriate response to, their own confusion or error". Every statement or phrase that is unclear can be stated as a problem and the relevant problem solving stages of Understand the Problem, Devise a Plan, Carry out the Plan and Look Back, can be employed to gain understanding. While carrying this out, it will not be surprising to find the reader exhibiting a mathematician's behavior of "construct[ing] examples to check the correctness of [his or her] understanding" (Shepherd, Selden, & Selden, 2012, p. 243).

In addition, a reader who is familiar with Schoenfeld's framework of mathematical problem solving (see Schoenfeld, 1985) will realize when his or her resources are inadequate for the passage. We conjecture that this constant realisation may be crucial to overcome the difficulty (b) of inadequate or incorrect prior knowledge.

Weber and Mejia-ramos (2014) contrasts the practice of a mathematician and that of the general undergraduate with respect to the amount of responsibility the reader bears in the comprehension of a proof.

> … the mathematician views his responsibility when reading the proof to be significant. Many new assertions in the proof require the construction of a sub-proof and sometimes understanding them also necessitates the drawing of pictures or the consideration of examples. Indeed, this mathematician suggests that the author of the proof *deliberately* left the responsibility of drawing the appropriate pictures to the reader of the proof, presumably because the reader would gain understanding from engaging in this process. (p. 91)

We conjecture that a problem solving mindset will guide the reader to break down the passage into a series of sub-problems to be worked out so as to gain a deeper understanding both through the stage of Understanding the Problem as well as the stage of Looking Back at the solution. Having a problem solving mindset will thus prepare the reader to scrutinize the passage and give sufficient attention to the detailed content of the textbook (difficulty (c) of the students of Shepherd, Selden and Selden (2012)).

Our research question in our pilot study of one reader is thus:

Can a problem solving approach to reading mathematics overcome the following difficulties encountered by readers who are not professional mathematicians?

a. Insufficient sensitivity to, or inappropriate response to, their own confusion or error.
b. Inadequate or incorrect prior knowledge.
c. Insufficient attention to the detailed content of the textbook.

PARTICIPANT AND METHOD

The reader, pseudonymously referred to as Tony, is a former school teacher who had taught Mathematics and Design and Technology for more than ten years in the secondary school environment. He is a mechanical engineering graduate with limited experience in undergraduate mathematics, having done only two mathematics modules in the university. Tony worked as a research associate with a problem solving project and enthusiastically embraced the Pólya model to problem solving. He found that the approach was very helpful to solving non-routine problems which were pitched at the secondary school level.

We gave Tony a proof that a particular number is transcendental (see Appendix) and asked him to read it. We also suggested that he should view any difficulties he had with the text as non-routine problems and approach them as such with the Pólya model of problem solving (see for example, Pólya, 1945; Toh, Quek, Leong, Dindyal & Tay, 2011).

Four sessions with a mathematician/researcher (the first author) to discuss his progress and difficulties were carried out. Field notes by the researcher and audio recording were taken for the sessions. Protocol analysis (see for example, Ericsson & Simon, 1993) was not used as Tony was not trained to say aloud his thoughts without disturbing his own cognition as he worked through some aspects of the text. Thus, the conversations in the sessions were mostly between the researcher and Tony while there were long stretches of silence as Tony worked on the reading. In addition, Tony was asked to write a reflection of his reading journey (see Tay, Quek, Dindyal, Leong & Toh, 2011) and this short journal was used to corroborate the field notes and audio recordings. Finally, an interview was conducted at the end of the sessions. The interview protocol was semi-structured with the three parts of the research question as the main foci.

TONY'S STORY

Tony was given the two-page text in early November 2013 and the researcher advised him to use a 'problem solving' approach to reading and understanding the text. Discussion I, lasting about 45 minutes, with the researcher was on 6 November. Tony then read the text again sporadically over the next few weeks. At a writing retreat over the period 4-7 December for the project team where there would be time to discuss, Tony continued to work on the text and had three further discussions (II – IV) with the researcher for a period of about an hour each. Tony's story will be told from his point of view through his written reflection interspersed with the researcher's comments on his interaction with Tony during the four discussions.

Tony's reflection of his reading journey

> <u>The first reaction</u> I started reading the entire proof once and I encountered many unfamiliar terms or definitions … "Foreign language!" was the first response … I sort of structure the reading into three main parts. Firstly, understand the definition, then Theorem 1 and its proof and finally Theorem 2 and its proof.
>
> <u>The main definition</u> The approach was to understand the definition line by line … I tried using simple examples to understand or refresh the meanings of the terms like 'complex number', 'root', … it seemed manageable. I … look[ed] for non-examples [unsuccessfully] … The assignment was then left aside for a while. Had the first discussion with [the researcher] to clarify. Explained the approach to him and realized that the non-example is actually defining transcendental number itself (… felt silly).

In Discussion I, the researcher suggested that Tony ignore the rest of the text and focus on the definitions at the beginning, but one at a time.

> … After spending some time trying to understand the definition again, there was some progress (i.e. managed to proceed to the second sentence). What was helpful was to use numbers and do some manipulations … Moving on, was the understanding of algebraic numbers with degree *n*. Again examples were used and it seems understanding this part was fine.

Within Discussion I, we successfully negotiated the definitions of an algebraic number and the degree of the number by making Tony produce concrete instances of each concept. For example, Tony picked out the word complex in the definition and considered if i was algebraic. The researcher suggested that it was the root of $x^2 + 1$, which was readily accepted. Then, he asked Tony to consider if $3+i$ was algebraic and Tony was able to work out the required polynomial starting from $x = 3+i$.

> We moved on with the definition and the next challenge came when I was asked to write out the statement for 'transcendental number'. I thought I understood and knowing 'transcendental number' is just 'not algebraic number' but I was not able to write out the statement. After some discussions, I was troubled by the concept of complement e.g. what is "not some" and what is "not any".

A problem that surfaced was Tony's difficulty with basic propositional logic – the researcher had to explain that the complement of "root of some polynomial" is "not root of any polynomial". Theorems 1 and 2 were tackled at the writing retreat.

> <u>Looking at Theorem 1</u> There were several terms in Theorem 1 that … I did not manage to understand them on my own … my plan was to proceed on to read the proof hoping that the proof might lead to the understanding of some terms … looking at it closer, I was convinced that [the proof] was just algebraic manipulation. So my plan was to break the long string into smaller parts to understand them. I was successful with the first part of the proof and it can be explained using long division … it appeared that things [in the next part of the proof] are linking back to the earlier part which I've skipped. I was not able to resolve it on my own. [Meanwhile] I [had instinctively] used some heuristics like working backwards and breaking into smaller parts to help myself to understand. In the end I still struggled to understand as [my ability to manipulate expressions with the] absolute [sign] was not strong. With some guidance and scaffolding [in Discussion III] … I was able to [understand] the rest of the proof for Theorem 1.

Although Tony struggled with the inequality

$$\frac{f(r_m)}{r_m - z} < n\,|a_n|\,(|z|+1)^{n-1} + (n-1)\,|a_{n-1}|\,(|z|+1)^{n-1} + \cdots + 2\,|a_2|\,(|z|+1) + |a_1|\,,$$

he was able to restate it as a sub-problem:

Show that $a_n(r_m^{n-1} + r_m^{n-2}z + \cdots + r_m z^{n-2} + z^{n-1}) + \cdots + a_3(r_m^2 + r_m z + z^2) + a_2(r_m + z) + a_1$

$$< n\,|a_n|\,(|z|+1)^{n-1} + \cdots + 3\,|a_3|\,(|z|+1)^2 + 2\,|a_2|\,(|z|+1) + |a_1|\,.$$

He worked on comparing term by term and was successful for the second and third last terms. But as he reported, he was unable to independently see the general term because of his lack of dexterity with inequalities and the absolute sign.

> [However] I was not able to see the link between the main objective, which is the transcendental number and Theorem 1. [The researcher] had to [explain] to me.

This link is not obvious until one finishes Theorem 2. Yet for Tony to have this realization that he still did not understand the whole picture although he had just understood a major piece, i.e. Theorem 1, is a sign of a good reader.

> Looking at Theorem 2 I proceeded on to understand Theorem 2 by myself. The plan was to use some numbers to help in the understanding. I was able to do that. I asked myself if there is a better way to understand it. I proceeded on with using algebraic manipulation. It also ended fine. The part which ended up challenging was the final statement. I tried algebraic manipulation and was still unable to break through. I discussed with [the researcher] again [Discussion IV].

Tony could not see the need for $m > n$ to falsify $\frac{1}{10^{(n+1)m!}} < \frac{1}{10^{(m+1)!-1}}$. Although he capably substituted suitable values of m and n to confirm the inequality, he knew that he did not understand the big picture. Later, he lamented that missing the big picture was a result of not understanding key phrases, in this case "sufficiently large".

> Then I realized, I was missing the big picture again. Reflecting on that, I realized that I actually did not have thorough understanding of the problem. The understanding of the phrase "sufficiently large" was neglected since the beginning. Little did I realize the importance of understanding this definition. After I was guided through to understand the phrase "sufficiently large", the understanding of the rest of the proof just fell into place. To stretch the understanding, we went on to 'check and expand' and I was able to give an example which I was happy about it.

Tony was keen to produce the 'next' transcendental number on his own and following the proof which he was now confident that he understood, he gave this number:

$$\frac{1}{100^{1!}} + \frac{1}{100^{2!}} + \cdots + \frac{1}{100^{n!}} + \cdots$$

DISCUSSION AND CONCLUSION

The proof for the 'first transcendental number' is not an easy one. Yet Tony who had only read two mathematics modules in his undergraduate studies was able to successfully understand it. The report in the section above showed that Tony was able to engage with the text for a very long time – in the interview, he estimated that he spent 15 hours working on understanding the proof. This may seem an inordinate amount of time but it pales compared to a colleague of ours who spent five months understanding the first page of a book during his doctoral studies. The report also shows that Tony consistently applied a problem solving approach to his reading – in the interview, he agreed that the first and last stages of Pólya were often applied to understanding definitions and statements in the theorems while the first three stages were useful in working out the algebraic manipulations.

Tony showed great sensitivity to his own confusion and errors – he felt "silly" that he was trying to find a non-example of an algebraic number when the whole proof was about doing that. He was acutely aware of his inadequate or incorrect prior knowledge and sought advice from the researcher. In the interview, he said that in the past (i.e.

without the challenge of a problem solving approach), he would just look for answers from the internet when he was stuck. Finally, Tony paid great attention to the detailed content of the text – in the interview, he agreed that almost every word was important. We think that Tony did not face the same three difficulties that the students of Shepherd, Selden and Selden (2012) had mainly because he had anticipated that the text would be difficult. Also, by adopting the problem solving approach, he had relished the challenge ahead of him as he was confident of making progress.

APPENDIX

The 'first' transcendental number

A complex number z is said to be **algebraic** if z is a root of some polynomial with all integral coefficients.

A complex number z is said to be an algebraic number of degree n if z is algebraic and it is a root of some polynomial of degree n with all integral coefficients but not of any polynomial of degree less than n with all integral coefficients.

A complex number z is said to be **transcendental** if z is not algebraic.

Theorem 1 (Liouville)

Let z be an algebraic number of degree $n > 1$ and let $r_m = \frac{p_m}{q_m}$ be a sequence of rational numbers converging to z. Then, for a sufficiently large M, $\left| z - \frac{p_m}{q_m} \right| > \frac{1}{q_m^{n+1}}$ for all $q_m > M$.

Proof Suppose that z is a solution to the polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0.$$

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$. Then

$$\begin{aligned} \frac{f(r_m)}{r_m - z} &= \frac{f(r_m) - f(z)}{r_m - z} \\ &= a_n (r_m^{n-1} + r_m^{n-2} z + \cdots + r_m z^{n-2} + z^{n-1}) + a_{n-1} (r_m^{n-2} + r_m^{n-3} z + \cdots + r_m z^{n-3} + z^{n-2}) + \cdots \\ &\quad + a_3 (r_m^2 + r_m z + z^2) + a_2 (r_m + z) + a_1. \end{aligned}$$

Letting m be such that $|z - r_m| < 1$, we may say that, for sufficiently large m, $\frac{f(r_m)}{r_m - z} <$

$$n|a_n|(|z|+1)^{n-1} + (n-1)|a_{n-1}|(|z|+1)^{n-1} + \cdots + 3|a_3|(|z|+1)^2 + 2|a_2|(|z|+1) + |a_1| = M.$$

Let $q_m > M$. Then $|z - r_m| > \frac{|f(r_m)|}{M} > \frac{|f(r_m)|}{q_m}$.

Now $$|f(r_m)| = \left| \frac{a_n p_m^n + a_{n-1} p_m^{n-1} q_m + \cdots + a_1 p_m q_m^{n-1} + a_0 q_m^n}{q_m^n} \right|.$$ □

Note that r_m cannot be a solution to $f(x) = 0$ because if it were, we could factor out $(x - r_m)$ and so z would necessarily be of lesser degree. Hence $f(r_m) \neq 0$. Furthermore, the numerator of this fraction is an integer so it must be at least 1. We conclude that

$$|z - r_m| > \frac{1}{q_m} \cdot \frac{1}{q_m^n} > \frac{1}{q_m^{n+1}}.$$

Theorem 2 (Liouville)

The number $z = \frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \cdots + \frac{1}{10^{n!}} + \cdots$ is transcendental.

Proof Let $r_m = \frac{p_m}{q_m} = \frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \cdots + \frac{1}{10^{m!}} = \frac{p_m}{10^{m!}}$. Then $|z - r_m| < 10 \cdot \frac{1}{10^{(m+1)!}}$. Now if z is an algebraic number of degree n, then Theorem 1 says that $|z - r_m| > \frac{1}{10^{(n+1)m!}}$ for sufficiently large m. So $\frac{1}{10^{(n+1)m!}} < 10 \cdot \frac{1}{10^{(m+1)!}} = \frac{1}{10^{(m+1)!-1}}$. But this is false for $m > n$, so z is transcendental. □

References

Ericsson, K. A., & Simon, H. A. (1993). *Protocol analysis: Verbal reports as data.* Cambridge, MA: MIT Press.

Osterholm, M. (2008). Do students need to learn how to use their mathematics textbooks? The case of reading comprehension. *Nordic Studies in Mathematics*, *13*(3), 7-27.

Pólya, G. (1945). *How to solve it*. Princeton: Princeton University Press.

Schoenfeld, A. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Shepherd, M. D., Selden, A., & Selden, J. (2012) University students' reading of their first-year mathematics textbooks. *Mathematical Thinking and Learning*, *14*(3), 226-256

Tay, E. G. (2001). Reading mathematics. *The Mathematics Educator*, *6*(1), 76-85.

Tay, E. G., Quek, K. S., Dindyal, J., Leong, Y. H., & Toh, T. L. (2011). Teachers solving mathematics problems: Lessons from their learning journeys. *Journal of the Korean Society of Mathematical Education Series D: Research in Mathematical Education*, *15*(2), 159-179.

Toh, T. L., Quek, K. S., Leong, Y. H., Dindyal, J., Tay, E. G. (2011). *Making mathematics practical: An approach to problem solving*. Singapore: World Scientific.

Waywood, A. (1992). Journal writing and learning mathematics. *For the Learning of Mathematics*, *12*(2), 34-43.

Weber, K., & Mejia-Ramos, J. P. (2014) Mathematics majors' beliefs about proof reading. *International Journal of Mathematical Education in Science and Technology*, *45*(1), 89-103.

Wilkerson-Jerde, M. H., & Wilensky, U. J. (2011). How do mathematicians learn math? Resources and acts for constructing and understanding mathematics. *Educational Studies in Mathematics*, *78*, 21-43.