AN EMERGING THEORY FOR DESIGN OF MATHEMATICAL TASK SEQUENCES: PROMOTING REFLECTIVE ABSTRACTION OF MATHEMATICAL CONCEPTS[1]

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This paper describes an emerging approach to the design of task sequences to promote reflective abstraction. The approach aims at promoting particular mathematical understandings. Central to this approach is the identification of available student activities from which students can abstract the intended ideas. The approach differs from a problem-solving approach. The paper illustrates the approach through data from a teaching experiment on learning of fraction concepts with fourth and fifth graders.

INTRODUCTION

Mathematical tasks are designed for a variety of reasons. The instructional design approach discussed here has the specific focus of promoting particular changes in students' conceptual understanding and, as such, offers an approach to addressing difficult to learn concepts and to working with students who are struggling to learn specific concepts. This task design approach does *not* address other important areas of learning mathematics, particularly the important area of mathematical problem solving. Therefore, the approach is meant to complement existing approaches, not replace them. The emerging task design theory is a product of a research program, Learning Through Activity (LTA, Simon et al, 2010; Simon, 2013), aimed at understanding conceptual learning, particularly the development of abstraction from one's own mathematical activity (activity that occurs in the context of designed sequences of mathematical tasks). Thus our research program involves a spiral approach in which we design task sequences to study learning through student activity, and we use what we come to understand about learning to improve our understanding of task design, and so forth.

What is unique about our instructional approach is that it involves students actively in the developing of new concepts, yet it does *not* depend on the uncertain breakthroughs required in authentic problem solving lessons. Task sequences are designed to elicit the specific activity that will lead to the new conceptualization. I discuss an example of this below.

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THEORETICAL BASIS

The theoretical basis of our research program derives from Piaget's (2001) work on reflective abstraction. DiSessa and Cobb (2004) pointed out,

> Piaget's theory is powerful and continues to be an important source of insight. However, it was not developed with the intention of informing design and is inadequate, by itself, to do so deeply and effectively. (p. 81)

Our research program is aimed at building theory that can inform instructional design.

Three characteristics of reflective abstraction are foundational to our work. First, reflective abstraction is not abstraction of properties of objects, but rather abstraction based on the learner's activity and results in a learned anticipation. Second, "activity" refers to *goal-directed* activity, which includes both physical and mental activity. The notion of goal-directed is important, because the learners' goals partially determine both what knowledge they call upon and what they pay attention to and can notice. Third, Piaget (2001) described reflective abstraction as a coordination of actions. We understand the coordination of actions in the following way. Each action is called upon for a particular purpose (i.e., with anticipation of its results). Thus, the actions called upon are each part of existing schemes (the result of prior reflective abstractions). Thus the coordination of actions really is a coordination of schemes. Thinking about coordination of actions as coordination of schemes allows a way of understanding how new knowledge to be constructed from prior knowledge. In our research and theoretical work, consistent with Piaget and others, (c.f., Hershkowitz, Schwarz, & Dreyfus, 2001; Mitchelmore & White, 2008), we consider mathematics conceptual learning as the process of developing new and more powerful abstractions, specifically reflective abstractions.

Our task design approach builds on this theoretical base and involves specifying hypothetical learning trajectories (Simon, 1995) at multiple levels. In this paper, I focus on the level of design for the learning of particular mathematical understandings, not the planning of trajectories for larger mathematical topics. A hypothetical learning trajectory consists of three components (Simon, 1995), (1) a learning goal, (2) a set of mathematical tasks, and (3) a hypothesized learning process. Whereas the specification of the learning goal generally precedes the specification of the tasks and hypothesized learning process, these latter two components necessarily co-emerge. The learning process is at least partially determined by the tasks used and the tasks used must reflect conjectures about the possible learning processes. The design approach outlined here provides a conceptualization of the design process with respect to these two components.

The Design Approach

The first two steps in our design approach are the first two steps in most instructional design that is aimed at conceptual learning. We assess student understanding and articulate a learning goal[2] for the students relative to their current knowledge. It is after these first two steps that our approach diverges.

Our third step is to specify an *activity that students can call on* that can be the basis for the abstraction specified in the learning goal.[3] The consideration of what activity we might elicit begins in a way that is similar to Realistic Mathematics Education (Gravemeijer, 1994), that is a consideration of students' informal strategies. Whereas RME focuses on developing progressively more formal solution strategies, our approach is focused on developing concepts by developing anticipations from those activities.[4] The fourth step is to complete the hypothetical learning trajectory, that is, to design a task sequence and related hypothesized learning process. The task sequence must both elicit the intended student activity and lead to the intended anticipation on the part of the students. The hypothesized learning process must account for not only the overt activity of the students, but also the mental activities that are expected to accompany those overt activities. I will not focus on steps beyond step 4 (e.g., symbolizing, introducing vocabulary, discussing justification), because again they are common to many approaches.

I will now use an example from our current project that focuses on the learning of fraction concepts. Kylie was a fourth grade student (9–10-years old) with whom we worked in a one-on-one teaching experiment. We were using the computer application JavaBars (Biddlecomb & Olive, 2000). In Java Bars, quantities can be represented by rectangles of different lengths. The bars can be partitioned and bars and parts of bars can be iterated. In the example, Kylie is developing a concept of recursive partitioning, the understanding of the size of a particular part of a part.

Task 1: "This is one-third of a unit [pointing to a rectangular bar on the screen], make one-sixth of a unit."

Kylie made clear that the only way she knew how to do the task was by first making the unit. She did not know how to just "cut up" the bar on the screen. She made the whole by iterating the third three times and then cut the first third in half. She indicated that one of the small pieces was one-sixth. Her explanation indicated that she was thinking about the number of subparts that would be created if she subdivided each of the three thirds (i.e., mentally iterating the two subparts three times).

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[2] Articulation of conceptual learning goals is a problematic issue not covered here. It is a theoretical and empirical challenge to specify learning goals in a way and level of specificity that adequately guides instructional design (as well as instruction and assessment).

[3] The learning goal is a researcher/educator construct. There is no assumption that the students, after a successful lesson, will have an identical understanding to that of the instructor. Rather, formative and summative assessment will reveal whether students have a compatible understanding.

[4] Although there are often overlaps in what is learned by students using these two approaches, I emphasize here the differences in the primary aim and the theory built to achieve that aim. We definitely build on aspects of RME, particularly their use of *model of becoming model for* (Gravemeijer, 1994).

Task 2: "This is one-fifth of a unit, make one-tenth of a unit."

Again, Kylie iterated the part to make the whole and then subdivided one of the parts, "Here, you have one-tenth of a unit."

After working three tasks in this way, she spontaneously showed a change in the following task.

Task 3: "This is one-third of a unit, make one-ninth of a unit."

This time Kylie *immediately* divided the third bar into three pieces (without iterating to make the whole).

K: One of those is one-ninth.

R: How do you know?

K: Because, um. How many times does three go into nine? ... Three times. And it's one third! So. Three times three is nine, and one of -- if you cut [the thirds] up into thirds again. That is, um. ... And you take one, it would be ... one-third. … But that's really one-ninth of a unit.

Task 4: "This is one-fifth of a unit, make one-twentieth of a unit."

She immediately cut the fifth into four. She went on to complete two more tasks in this way in this session.

In this example, Kylie learned that she could produce 1/mn from 1/n by partitioning 1/n into m parts. She developed an anticipation that partitioning 1/n into m parts creates a fraction of the unit that is n times smaller than the 1/m (the fraction of the part), that is, a subpart that iterates n times more in the unit than it does in the original 1/n part. Let us look more closely at this transition.

Kylie's partitive fraction scheme (Steffe & Olive, 2010), available at the outset of this set of tasks, included an understanding that 1/p is a part that can be iterated p times to make a unit. Initially, this allowed her to iterate the original part, one-third (Task 1), three times to make the whole. She knew she needed to partition each part into two subparts to make sixths, using her multiplication scheme (# items/group x # groups = # items) along with her partitive fraction scheme.

In Task 3, Kylie was no longer employing the sequence of actions she used in the first two tasks. Rather, she had developed a *new* action that was at a higher level than the sequence from which it was built and allowed her *knowledge at once*. The new abstraction moved Kylie from thinking about iterating a composite unit to thinking about an m split in a part, 1/n, as resulting in a subpart that is n times smaller in relation to the unit than it is in relation to the part. In other words, she knew that the part increased by n times the number of times the subpart would iterate to the unit compared to the number of times it iterated to the part. She now had an anticipation about the relationship of a part of a part to the unit.

What we see from this example is that the learning process began with Kylie setting a goal (e.g., to complete the task) and bringing to bear available schemes (actions) to accomplish the goal. Initially, she used these actions in sequential fashion. However, she came to a *coordination of* those actions. As a result of this coordination, she no longer needed to go through the sequence of actions used previously. The result of the coordination was a structure that was at a higher level than the component schemes. (The reader is referred to Simon et al, 2010 for another example of this instructional design approach with a more in-depth analysis of the learning that took place.)

I now return to the instructional design approach which I summarize as follows:

Step 1: Assess relevant student understanding.

Step 2: Identify learning goal.

Step 3: Specify an activity, which the students can call on, that could be the basis for the intended abstraction.

Step 4: Design a sequence of tasks in conjunction with a hypothesized learning process that accounts for how the students will progress from the initial activity to the intended abstraction.

In the example, the task sequence was quite simple. Of course, this did not represent the full treatment of recursive partitioning in our teaching experiment, but it gives us a straightforward example for discussion. The tasks elicited an initial activity (action sequence):

1. Iterate the part to make the whole.
2. Divide the number of needed subparts by the number of parts to find how many subparts will be in each part.
3. Carry out the appropriate subdivision.

This activity afforded the opportunity for Kylie to make an abstraction. The key was that Kylie's abstraction required no leap of insight or problem-solving breakthrough. Further, it required no input from the teacher or other students. The abstraction emerged from Kylie's activity. Kylie was able to do every one of the tasks without assistance. However, an understanding of recursive partitioning emerged in the course of solving the tasks with her available activity. This seemed to indicate that Kylie was able to use her extant knowledge to develop the intended new understanding.

CONCLUSIONS

Our task design approach is an emerging one. I highlight here two of its features that can be seen in the example above. First, the approach provides a strategy for promoting specific mathematical understandings. It contrasts with strategies in which students must solve novel problems to progress (or hear the solutions of more able peers). Although mathematics teaching cannot cause learning, this is an approach that involves engineering task sequences so that participating students predictably make the new abstraction. (Of course, successful understanding of prerequisite concepts is required.) Second, the learning goal is *not* to learn to solve the tasks, as it is in many approaches. The tasks are made to initially elicit activities that the students already are capable of engaging in. Kylie was able to solve all of the tasks prior to making the intended abstraction. Further, she *was not* consciously trying to find an easier way. Her learning was a product of coordination of actions across a sequence of tasks.

Let us examine some of the possible implications of this approach to task design.

1. This approach potentially provides a way to design task sequences for concepts that students tend to not learn well. These are concepts that many students do not spontaneously reinvent in problem solving situations and of which they do not develop deep understanding by being part of a class discussion with more knowledgeable students. The approach focuses the instructional designers on identifying key activities that are likely to afford the intended abstraction.
2. Small group work using task sequences, of the kind discussed here, can lead to somewhat different class discussions. If students are making the new abstraction as a result of their engagement with the task sequence, discussions can focus more on articulation of the new idea, justification, and establishing the idea as taken-as-shared knowledge.
3. The approach has potential to address issues of equity in two ways. First, many students who have conceptual gaps early on seem to never recover. This design approach provides a general methodology for building up the specific experience, based on students' currently available activities, needed to make particular abstractions. Second, if during small group work, students are generally successful in deriving the new abstractions from their activity with the task sequence, a greater number of students will be able to participate in and benefit from the class discussions that follow. The underlying hypothesis here is that students who abstract ideas through their activity, based on their work with the mathematical tasks, tend to learn the concepts in a more powerful way than those who only *follow* the explanation of their more able peers offered in class discussions.

One final point that was discussed briefly at the beginning of this paper is the relationship of our approach with mathematical problem solving. The approach that I have described and exemplified does not focus on students developing their problem solving abilities. Rather it focuses only on the development of mathematical concepts. Developing problem solving abilities is a key part of mathematics education. One could argue that conceptual understanding and problem solving are the two wings of mathematics education – students cannot fly without effective use of the two together. Students can learn concepts through problem solving lessons. Our approach is in no way intended to minimize the importance of lessons in which that is the case. Rather, our approach provides an additional tool that has the potential for success in areas where mathematics education has been less successful. One open question is how to use this tool in conjunction with the powerful tool of problem solving lessons to maximize the learning of students.

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