COGNITIVE PROCESSES UNDERLYING MATHEMATICAL CONCEPT CONSTRUCTION: THE MISSING PROCESS OF STRUCTURAL ABSTRACTION

Thorsten Scheiner[1], Márcia M. F. Pinto[2]

[1]University of Hamburg, Germany, [2]Federal University of Rio de Janeiro, Brazil

The purpose of this paper is twofold: On the one hand, this work frames a variety of considerations on cognitive processes underlying mathematical concept construction in two research strands, namely an actions-first strand and an objects-first strand, that mainly shapes past and current approaches on abstraction in learning mathematics. This classification provides the identification of an often overlooked fundamental cognitive process, namely structural abstraction. On the other hand, this work shows a theory-driven and research-based approach illuminating the hidden architecture of cognitive processes involved in structural abstraction that gives new insights into an integrated framework on abstraction in learning mathematics. Based on our findings in empirical investigations, the paper outlines a theoretical framework on the cognitive processes taking place on mental (rather than physical) objects.

INTRODUCTION

Attributed as a crucial cognitive process in concept construction, abstraction has been the focus of many researchers in diverse research areas. Caused by both a confusion between abstraction and generalization and a characterization of abstraction aimed at *de*contextualization instead of *re*contextualization (see, van Oers, 1998), the term 'abstraction' has been almost "removed from the discourse of learning" (Sfard, 2008, p. 10). Though attention in research on abstraction has steadily declined since its peak in the pre-cognitive science era, some researchers still have advanced our all understanding on this issue by integrating 'modern' perspectives on past and current theories of learning in a broader theoretical frame. *The Nested RBC Model of Abstraction* originally described by Hershkowitz, Schwarz, and Dreyfus (e.g., 2001), for instance, provides an interesting conceptual undertaken in this area. The following pages present a further theoretical approach addressing the issue of abstraction in learning mathematics more broadly. In this work, the purpose is not to compete with other theories but to shed lights on a neglected cognitive process, namely on *structural abstraction*.

The proposed outline of the theoretical framework on structural abstraction results from (a) reconsidering Davydov's (1972/1990) *ascending from the abstract to the concrete* from a *dialectical* point of view as expressed by Ilyenkov (1982), (b) taking fundamental findings in cognitive science and psychology into consideration, (c) embedding the framework into philosophical grounds, and undertaking a reanalysis and presentation of data obtained in a previous study (Pinto, 1998).

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THEORETICAL BACKGROUND: TWO FUNDAMENTAL STRANDS IN RESEARCH ON ABSTRACTION IN LEARNING MATHEMATICS

Several approaches, partly distinct and partly overlapping, shape the theoretical landscape in mathematics education research on abstraction. Taking as poles of a wide spectrum, we can distinguish two strands of cognitive processes underlying concept construction, namely (1) an *actions-first* strand and an *objects-first* strand. The former has to do with processes of *focusing on the actions on objects*, in particular, individuals' reflections on actions on known objects, grounded in Piaget's work of 'genetic epistemology' that puts 'actions' in its heart with the underlying philosophy that knowledge is basically 'operative'. The latter has to do with processes of *focussing on the objects themselves*, in particular, paying attention to the properties and structures inherent in those objects. As shown in Fig. 1, in both strands, the focus of attention may take place on *physical* objects (referring to the real world) or *mental* objects (referring to the thought world). Both strands capture the bulk of theoretical and practical work in past and recent years, however, it seems that the mathematics education research literature has nearly limited its focus on actions-first theoretical approaches. Research within the actions-first strand has made considerable progress considering both physical and mental objects as a point of departure in abstraction processes, while the focus of attention within the objects-first strand is limited, with few exceptions, to physical (instead of mental) objects. The current study considers cognitive processes underlying concept construction that take *mental* objects as a point of departure. Based on philosophical grounds and findings in psychology and cognitive science, we argue that *structural abstraction* is the key cognitive process in this issue. Furthermore, the paper outlines how an integrative framework might conceptualize the functional interplay of cognitive processes building the architecture of structural abstraction.

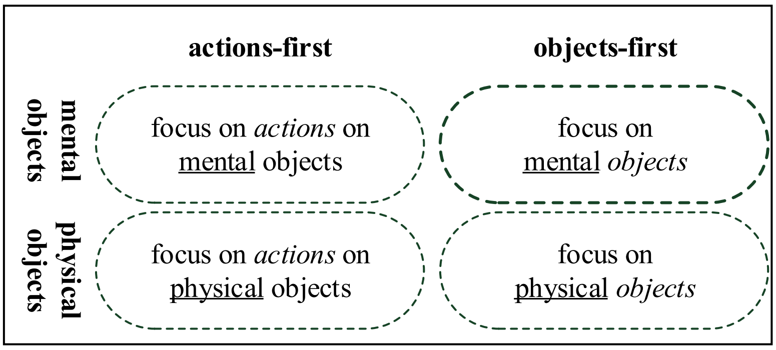

Figure 1: Actions-first and objects-first strand.

Actions-first Strand

Within this strand, two fundamental cognitive processes can be distinguished, namely (1) focussing on *actions* on physical objects and (2) focussing on *actions* on mental objects. The former refers to Piaget's *pseudo-empirical abstraction*, while the latter refers to Piaget's *reflective abstraction*. In his *Recherches sur l' abstraction réfléchissante*, Piaget (1977/2001) describes pseudo-empirical abstraction as a process by which individuals discover in objects the properties that have been introduced into them by their own activity. In other words, the results covered by pseudo-empirical abstraction are read off from material objects but the observed properties are actually introduced into the objects by the subject's activities. Yet, reflective abstraction is abstraction from the subject's actions on objects, mostly from the coordination between these actions. Abstracting properties of an individual's action coordinations is thought as the crucial function of Piaget's reflective abstraction. In mathematics education, its highest impact is considered in its process of *encapsulation* (or *reification*). From Piaget's reflective abstraction, Dubinsky et al (e.g., 1991) and his colleagues developed the APOS theory, describing the construction of concepts through the encapsulation of processes. Similar to the latter is *reification* – the main tenet of Sfard's (e.g., 1991) framework emphasizing the cognitive process of forming a (structural) concept from an (operational) process. In the same line, Gray and Tall (e.g., 1994) describe this issue in terms of an overall progress from procedural thinking to *proceptual* thinking.

Objects-first Strand

Symmetrical to the actions-first strand, two fundamental cognitive processes can be distinguished within this strand, namely (1) focussing on <u>physical</u> *objects* and (2) focussing on <u>mental</u> *objects*. The former refers to empiricist approaches in the sense of *seeing similarities* among objects that fall under a particular concept. *Empirical abstraction*, in the sense of Piaget, describes a process when an individual abstracts sensory-motor properties from experiential situations. In Piaget's (1977/2001) own words, empirical abstraction "draws its information from objects" (p. 317) but "is limited to recording the most obvious and global perceptual characteristics of objects" (p. 319). However, as argued by diSessa and Sherin (1998), though these abstraction processes (abstraction of dimensions that can be perceived) work well for *category-like* concepts, classical approaches (such as *classifying* or *categorizing* that are based on identifying commonalties from a set of specific exemplars) do not provide fertile insights into cognitive processes underlying concept construction in mathematics. An approach that goes beyond Piaget's empirical abstraction has been developed by Mitchelmore and White (e.g., 2007). Drawing on Skemp's (1986) conception on abstraction, their work on *empirical abstraction in learning elementary mathematics* describes abstraction in terms of the *underlying* structure rather than from superficial characteristics. This study of the *underlying* structure (of a mathematical concept) is considered as the heart of the objects-first strand in mathematics education research on abstraction. While Mitchelmore and White consider physical objects, the following subsection describes a cognitive process that takes *mental* objects as a point of departure.

STRUCTURAL ABSTRACTION

The notion of structural abstraction has been already used by Tall (2013) in the sense of a superordinate abstraction for empirical and platonic abstraction. Its "fundamental role […] throughout the full development of mathematical thinking" (ibid., p. 39) has been highlighted in Tall's (2013) work *How humans learn to think mathematically*. As described in earlier work (Scheiner, 2013) and argued in this paper, structural abstraction goes beyond Tall's conception of this particular kind of abstraction. The crucial puzzle lies in the observation that structural abstraction has a dual nature, namely (1) 'complementarizing' the aspects and structure underlying specific objects falling under a particular mathematical concept and (2) facilitating the growth of coherent and complex knowledge structures. From this point of view, structural abstraction takes place both on the objects-structure and on the knowledge-structure (see, Figure 2).

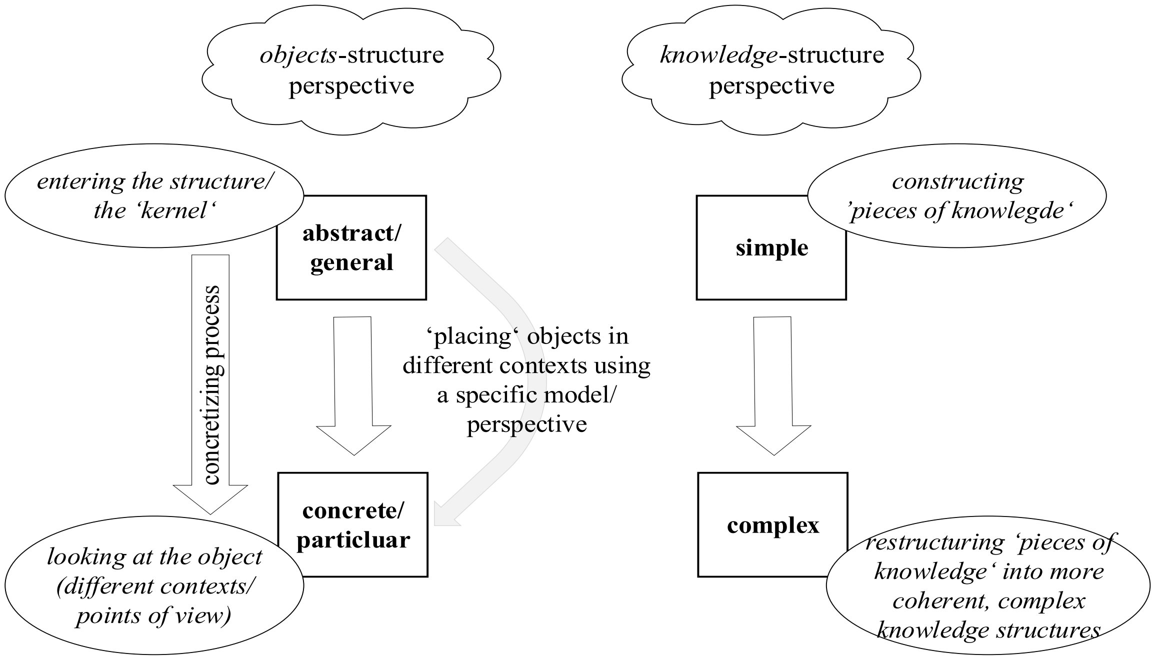

Figure 2: The dual nature of structural abstraction.

From the *objects*-structure perspective, structural abstraction means (mentally) structuring the diverse aspects and the underlying structure of specific objects that have been particularized through placing the objects in a variety of different contexts. However, structuring the diverse aspects and the underlying structure of objects falling under a particular concept requires a concretizing process where the mathematical structure of a specific object is entered by looking at the object in relation with itself or with other objects that fall under the particular concept. Through placing objects into different specific contexts using a realistic model or perspective that provides theoretical structure in constructing a concept the meaningful components of the object may be highlighted. Models are, in this sense, intermediate in abstractness between 'the abstract' and 'the concrete'. This means that at the start of a particular learning process a model is constituted that supports the 'ascending from the abstract to the concrete' as described by Davydov (e.g., 1972/1990). Davydov's strategy of ascending from the abstract to the concrete draws the transition from the general to the particular in the sense that learners initially seek out the primary general 'kernel' and, in further progress, deduce multiple particular features of the object using that 'kernel' as their mainstay. The crucial aspect in this approach is Ilyenkov's (1982) observation that "the concrete is realized in thinking through the abstract" (p. 37). Taking this view, models are embedded in goal structures and used by embodied agents. The key feature within the objects-structure perspective, however, lays in the idea that various specific objects falling under a particular concept mutually *complement* each other, so that the abstractness of each of them, taken separately, is overcome. From this perspective, structural abstraction is a movement towards *complementarity* of diverse aspects creating conceptual unity among objects. This is in line with a dialectical perspective described by Ilyenkov (1982) and differs from empiricist approaches in Skemp (1986).

From the *knowledge*-structure perspective, structural abstraction, on the other hand, implies a process of restructuring the 'pieces of knowledge' constructed through the mentioned processes. Further, it also implies restructuring knowledge structures coming from current concept images, essential for the new concept construction. The cognitive function of structural abstraction is to facilitate the assembly of larger, more complex knowledge structures. The guiding philosophy here is rooted in the assumption that learners initially acquire mathematical concepts on their backgrounds of existing domain-specific conceptual knowledge through progressive integration of previous concept images or by the insertion of a new discourse alongside them. The crucial aspect of structural abstraction, from the knowledge-structure perspective, is that structural abstraction moves *from simple to complex* knowledge structures, a movement with the aim of establishing highly coherent knowledge structures.

RESEARCH QUESTION AND METHOD

Which insights does the above outline on structural abstraction reveal for the analysis of an individual's striving for making sense of a mathematical concept and which aspects may be illuminated that have been hidden? These questions are addressed by returning to an earlier study (Pinto, 1998) that identified mathematics undergraduates' strategies of making sense of formal mathematics, which were not fully captured by "action-first" models of concept construction (e.g., Dubinsky, 1991). The original data collected undertook an inductive approach throughout two academic terms during students' first year at a university in England. It consists of classroom observation field notes and transcriptions of semi-structural individual interviews that took place every two weeks with eleven students. From a cross-sectional analysis of three pairs of students, two prototypical strategies of making sense could be identified, namely 'extracting meaning' and 'giving meaning'. Here the latter is our focus; through new lenses provided by the notion of 'structural abstraction' (Scheiner, 2013). Meanwhile, scrutinizing the old data contributes to the development of the very notion of structural abstraction itself. Due to the limited scope of the paper, we limit our focus on the case study of the learner Chris, who "consistently understood [the formal concepts] by just reconstructing it from the concept image" (Pinto, 1998, p. 301).

SELECTED FINDINGS

The above outline on structural abstraction provides indications to refine the characteristics of the 'giving meaning' strategy expressed by 'reconstructing a formal object from the concept image' (Pinto, 1998). If we return to examine the earlier study (Pinto, 1998), we find that several students take the formal definition of a mathematical concept as just one amongst other related representations built in earlier experiences at school and out of school – a full meaning for considering the concept definition inside the concept image cell. The formal concept definition does not necessarily have primacy over the other representations but has a *complementary power* to give deeper insights into the 'bigger picture' of the concept. Moreover, we could identify some learners who 'give meaning' but simply 'add' the formal definition to their concept image. By merely juxtaposing pieces of knowledge, occasionally conflicting, the structure underlying the different facets of the concept may stay inconsistent, hampering the structural abstraction process. On the other hand, there are modes to succeed. Reasons for our claim rely in part on the analysis of Chris' written formal definition of the limit of a sequence. We interpret that Chris firstly evokes a representation of a constructed object to start with, based upon his visual representation of a convergent sequence (see, Figure 3) and on his explanation of the meaning of the definition which starts as "... and you've got like the function there, and I think that ... ... it's got the limit there..." (Chris, first interview). Yet, his written discourse seems to recall a specific representation of a sequence tending to L, as he starts "if a_n tends to L" instead of "a_n tends to L if", as he was told in the lessons, self correcting and crossing out the first line. Chris' responses show that he developed and is guided by a *generic representation* of the limit concept. By taking a retrospective view, he described that he has developed this representation, looking at other sources than the lectures, through 'complementarization' of a variety of representations.

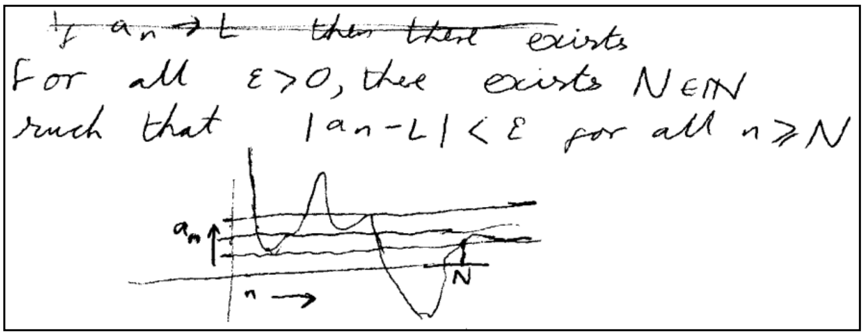

Figure 3: Chris' representation of the limit concept.

Chris expresses his doubts when responding whether the sequence 1,1,1,… has a limit:

> "(Laughter) I don't know really. It definitely it will ... it will always be one ... so I am not really sure (laughter) ... ... umm ... it's strange, because when something tends to a limit, you think of it as never reaching it ... so if it's ... 1 ... then by definition it has a limit but ... you don't really think of it as a limit (laughter) but just as a constant value."
>
> (Chris, first interview)

He evokes a dynamic view of the limit concept and an understanding (limit as unreachable) coexisting with the formal definition. His seriousness expressing his doubts suggest that, even immersed in the classroom culture at university, he will not simple let go 'old images' when faced with the formal definition, acknowledging that he is not making a complete sense of the concept in its overall structure, which at the time is composed by conflicting 'pieces of knowledge'. In a certain sense, there is no primacy of the formal definition in relation to other representations and he goes through a process of restructuring them into a coherent and complex whole proudly announcing in his last interview where he could express the formal definition of limit of a sequence "without making it formal" as follows:

A sequence has a limit if and only if as the sequence progresses, eventually, all values of the sequence gather around a certain value.

(Chris, last interview)

Modes to reconstruct earlier dynamical views of limit into the static version above, which seems unifying the various representations and we interpret as movements across levels of complexity, are only recovered through scrutinizing Chris' descriptions of his attempts to make sense of the formal definition. During the second interview, when Chris comments "[I could] see what the definition meant", may be referring to "... ... when you actually ... think that you can ... you make ε small." (Chris, seventh interview). Notice that "you can" suggests an experiment, which seems to be guided by his generic representation of a convergent sequence. He then self corrects, mentioning an action, "you make", in order to *define* a convergent sequence. Other instances from the first interview suggest that he experimented by giving N and finding a related ε, in a logical inversion of what is stated in the definition:

> ... you decide how far out ... and you can work out an epsilon from that ... or if you choose an epsilon you can work how far out.

However, moving N to the right and determining ε allows a dynamical feeling that the sequence is tending to a limit. Such thought experiments may have guided him to "... thinking about why you are doing it ... ... you find out why you are choosing *N* so they lie all there in, so ... it gradually tends towards the limit" (Chris, seventh interview). Finally, a central aspect in this reanalysis is related to modes of dealing with cognitive conflicts, which appear as a pivot issue during the process of structural abstraction. Since there are learners who are not aware of a cognitive conflict, as further findings indicate, a realistic model/perspective, as described in the outline of the framework, may be a helpful 'guide' in order to construct the right idea of the concept. Further, the impact of cognitive conflicts and learning through conceptual change in our approach on structural abstraction reflects crucial issues in cognitive science.

CONCLUDING REMARK

Structural abstraction, from our point of view, is considered as a movement 'from particular to unity' in terms of 'complementarizing' particularized meaningful components/structure into a whole, and, on the other hand, as a movement 'from simple to complex' in terms of restructuring already constructed 'pieces of knowledge' into coherent and complex knowledge structures. In synthesis, structural abstraction acknowledges abstraction as a movement across levels of complexity rather than levels of abstractions or generality. With this approach, we call to free of the term abstraction from connotations that have been associated with it through decades in many works.

References

Davydov, V. V. (1972/1990). Types of generalization in instruction: Logical and psychological problems in the structuring of school curricula. In *Soviet studies in mathematics education* (Vol. 2). (J. Teller, Trans.). Reston, VA: NCTM.

diSessa, A. A., & Sherin, B. L. (1998). What changes in conceptual change? *International Journal of Science Education*, *20*(10), 1155-1191.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 95-123). Dordrecht: Kluwer.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity and flexibility: a proceptual view of simple arithmetic. *Journal for Research in Mathematics Education*, *26*(2), 115-141.

Hershkowitz, R., Schwarz, B., & Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education*, *32*, 195-222.

Ilyenkov, E. V. (1982). *The dialectics of the abstract and the concrete in Marx's Capital*. Moscow: Progress.

Mitchelmore, M. C., & White, P. (2007). Abstraction in mathematics learning. *Mathematics Education Research Journal, 19*(2), 1-9.

Piaget, J. (1977/2001). *Studies in reflecting abstraction* [Recherches sur l' abstraction réfléchissante]. (R. Campbell, Trans.). Philadelphia: Psychology Press.

Pinto, M. M. F. (1998). *Student's understanding of real analysis.* UK: University of Warwick.

Scheiner, T. (2013, July). *Mathematical concept acquisition: Reflective, structural, and reflectural learners.* Paper presented at the working group 'Factors that foster or hinder mathematical thinking' of the 37th conference of the IGPME, Kiel, Germany.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*(1), 1-36.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. New York, NY: Cambridge University Press.

Skemp, R. R. (1986). *The psychology of learning mathematics* (2nd ed.). London: Penguin Group.

Tall, D. O. (2013). *How humans learn to think mathematically. Exploring the three worlds of mathematics*. Cambridge, UK: Cambridge University Press.

van Oers, B. (1998). From context to contextualizing. *Learning and Instruction*, *8*(6), 473-488.