CONCEPTUAL AND BRAIN PROCESSING OF UNIT FRACTION COMPARISONS: A COGNEURO-MATHED STUDY

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This mixed-method, qualitative/quantitative study examined (a) how a constructivist-based intervention (CBI) effected adults' learning of unit fractions and performance on whole-number (WN) or unit fraction (FR) comparisons and (b) brain circuitry implicated (fMRI) when processing these comparisons. The CBI used unit-iteration based activities to foster a shift in participants' understanding of FR, from the prevalent, limiting "one-out-of-so-many-equal-parts" idea to a multiplicative relation conception and thus inverse magnitude relation among FR (1/n>1/m though m>n). Pre- and two post-intervention tests indicated CBI impact on decreased reaction time in comparing not just FR but also WN and differentiated brain regions implicated for each. Implications for theory testing and CBI impact on WN-FR links are discussed.

BACKGROUNG AND CONCEPTUAL FRAMEWORK

Alluding to President Obama's (White House, 2013) *BRAIN Initiative*, this study examined how task design for brain research and teaching unit fractions, rooted in a constructivist perspective (Piaget, 1985), may impact brain processing when adults compare numbers. It focused on a *milestone shift*—from direct comparison of whole numbers (e.g., 8>3) to the *inverse relationship* among unit fractions (1/3>1/8 while 8>3). At issue was (a) how a *conceptually driven* intervention, used for teaching adults who already knew the "inverse rule", may impact their performance and (b) what brain circuitry would be activated to process the numerical comparisons (i.e., identify the neuronal basis for operating on whole numbers (WN) vs. on unit fractions (FR)).

Cross-disciplinary work of neuroscientists and educators is a new trend. Initially, educators became interested in brain-based research (Westermann et al., 2007). Later, this unidirectional, neuroscience-to-education fertilization, has yielded collaboration and reciprocal scholarship (De Smedt et al., 2011). Five facets of brain research seem of interest to mathematics educators: (a) compare learning/thinking and brain functioning among different groups (e.g., child-adolescent-adult); (b) understand how learners perceive, process, and link symbolic (e.g., Arabic) and non-symbolic quantities; (c) develop/validate observation-based theoretical frameworks of thinking, learning, and teaching; and (d) test effectiveness of practices to promote learning (e.g., critical-yet-intractable domains like fractions). To-date, however, the differences in operating on WN to FR were studied in each discipline separately.

Much brain research has focused on how it represents and processes numerical information. Dehaene's seminal work (Dehaene, 1997; Dehaene et al., 2003) yielded a triple code model of human WN perception. In that model, Arabic numerals are

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processed and represented in low-level visual cortical regions, numeric words in more anterior and language related cortical areas (lingual gyrus, perysylvian cortex), and analog magnitudes (e.g., a “number-line”) involve the Intraparietal Sulcus (IPS). In contrast, only a few studies focused on how the brain processes fractions (Bonato et al., 2007; Ischebeck et al., 2009; Jacob & Nieder, 2009). One study demonstrated that when adults solve challenging tasks (e.g., 2/3-1/4), the WN triple code model seems to also pertain to FR (Schmithorst & Brown, 2004). However, research has not yet conjoined WN and FR into a single study, let alone used a MathEd conceptual framework to guide research questions and design. The present study addressed this lacuna, to advance knowledge that can explain difficulties and potential affordances provided by (a) common/different brain circuitry used for WN vs. FR and (b) how number recognition (“cue”) and comparison (operation) may impact processing, and hence learning, of FR.

Conceptual Framework

Von Glasersfeld’s (1995) scheme theory grounds this study. A scheme is considered a tripartite conceptual building block: a *situation* into which a person assimilates information (which triggers her goal), an *activity* for accomplishing that goal, and an expected *result*. Extending this work, Simon et al. (2004) proposed (a) *anticipation of activity-effect relationship* as a lens to delineate “conception”—a dyad comprising the last two parts of a scheme, and (b) *reflection* on this relationship (abbreviated as *Ref*AER*) as a mechanism underlying cognitive change. Ref*AER commences with assimilating a task into the situation part of an available scheme, which also sets one’s goal. The mental knowledge system recalls and executes the scheme’s activity. The learner’s goal regulates effects produced by the activity. This enables noticing of discrepancies between one’s goal and the actual effects. Via reflection on solutions to comparable tasks, the learner abstracts a new, invariant relationship between an activity and its anticipated effect(s). This central notion of anticipation, which was developed via observational studies, has been corroborated by recent neuroimaging studies (Schacter et al., 2012; Suddendorf & Corballis, 2007).

Importantly, this framework distinguishes objects on which the mind operates (e.g., number) from operations on those objects (e.g., ordering smaller to larger). This distinction informed task design for this study, so assimilation of cues would be triggered by only one of two possible symbols (number *or* operation) before an entire number-comparison task is presented. Cues that precede number comparisons were expected to differently effect performance due to the brain’s pre-task recognition and ‘pulling the cue’ from long-term into working memory. That is, we hypothesized that distinct patterns of brain activation and/or neuronal circuitry would be recruited when an object is presented before an operation or vice versa.

METHODOLOGY

Participants (N=21), ages 23-36, took a pre-intervention computerized (ePrime) test comprised of 4 runs, each including 90, four-step number comparisons (randomized).

In Step A of each task (1 sec) a symbol of number or operation appeared (e.g., 7, 1/7, >, or =). In Step B (1 Sec) another symbol accompanied the first (e.g., 7>, 1/7=). In Step C the comparison task appeared fully (e.g., 7>8?, 1/7>1/8?), providing up to 2.5 sec to respond by pressing a key on the right for "true" or the left for "false." Step D showed three dots (0.5 sec) to separate tasks (ITI).

A video recorded *teaching* episode (~50 minutes) followed pre-test immediately. First, participants provided, with drawn examples, their definition of fraction. Then, creating their perturbation was promoted via posing a problem for which that definition is inadequate (Figure 1). Next, they were engaged in the challenging task of equally sharing unmarked paper strips among 7 people (then, 11) without folding the paper or using a ruler. Instead, they were taught to use the Repeat Strategy (Tzur, 2000): estimating one person's piece, iterating that piece 7 times, comparing the resulting whole to the given one, adjusting the estimate, etc. Reflection on this activity, promoted by teacher probing into participants' reasons for those adjustments ("make the next shorter/longer? Why?"), aimed to foster a conception of the unique, multiplicative 'fit' between each unit fraction (1/n) and the whole (n times as much of 1/n), and of the inverse relationships among unit fractions (to fit more pieces—each must be smaller). Discussion of why a larger denominator implied a smaller unit fraction *for any FR*, but *no practice* of such comparisons, concluded the episode.

Sticks *A* and *B* are equal in length. *A* contains 4 equal parts. The shaded part on *B* is equal to the part above it on *A*. What fraction, if any, is the shaded part of *B*? Of *A*? Why?

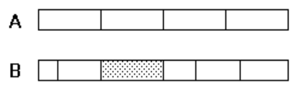

Figure 1

A first post-test as described above was conducted immediately after the intervention, and a post-test took place a few months later during fMRI scanning. To increase fMRI signal, runs were altered to include 140 two-step tasks (eliminating Step B above). Response time (RT) was recorded when subjects pressed a button in the right hand for "true" and the left for "false," but each task ended after exactly 2.5 seconds. Experimental tasks with a true ">" comparison included roughly 90% of all presented, while "=" and false ">" tasks served as control. Runs were organized in a hybrid-block design, including random-length sequence of like-comparisons (e.g., 1/3>1/8, 1/7>1/2, **8=8**, 5>3, 9>7, 4>3, 6>4, etc.).

ANOVA was calculated to determine the impact each independent variable (number type, Step A cue, testing occasion) has on the two dependent variables (RT, ER). Repeated observation and analysis of video recording helped inferring into participants' thinking about fractions before, during, and after instruction.

RESULTS

This section presents data and analysis of change in participants' conception of FR (qualitative), change in their performance of WN or FR comparisons (quantitative – behavioral), and differentiated brain circuitry activated (quantitative – fMRI).

Changing Adults' Conception of Unit Fractions

Upon completion of the pre-test in the computer room, each participant wrote down a definition for fractions (with example of 1/4). Then, s/he was asked to solve the Sticks Problem as a conceptual pre-test. All (100%) participants explained that a unit fraction is, "One out of so many equal parts of a whole," drew a circular figure ("pizza") partitioned into 4 parts and shaded one to show 1/4, and none was able to answer *both* questions about the shaded part on Stick B. Particularly prevalent (>50%) were responses such as, "The shaded part cannot be a fraction of Stick *A* because it is not a part of *A*" and "I cannot determine what fraction is the shaded part of Stick *B* because there are six unequal pieces on it."

Then, asked to equally share a given paper strip among 7 people without folding it or using a ruler, they initially had no solution. When prompted, "Could you estimate the share of one person and then find out?" each either generated the Repeat Strategy independently or was offered by Tzur to use it. Once iterating the first estimated part (say, too long), and asked if the next one had to be shorter/longer, they all knew the direction of change needed (here, shorter), explaining that more pieces had to be "squeezed" into the whole so each should be smaller. After making one piece that's too short and the other too long, they all also used a strategy of estimating the next piece's size between the closest short/long pieces already produced. Once the 7-piece iterated-whole seemed very close to the given whole, they were shown how to use JavaBars to produce an equally partitioned whole (with 7) and how to pull out one of these parts and measure it with the whole as a "Unit Bar" (1/7 shown on piece). Then, when asked if to share the whole among 11 people they would make the first estimate shorter/longer than the pulled-out 1/7-part, all (100%) knew to make it shorter, "because I have to squeeze even more parts into the same whole." At this point, each participant used the Repeat Strategy in JavaBars until the iterated whole was judged close enough to the given whole. Next, in reference to their activity, Tzur provided a definition (while they wrote it): "A unit fraction is a multiplicative relation to the whole; what makes 1/n what it is has to do with how many times it fits in the whole, or that the whole is n times as much of it. For example, your first estimated piece was 1/7 because the whole is 7 times as much of it." He also held one whole "fry" and asked if they could imagine the whole of which this single piece of paper would be 1/5. All explained they "saw" a strip that's 5 times longer.

At this point, Tzur returned to the Sticks Problem. All participants (100%) then explained that the shaded part is 1/4 of Stick *A* and 1/4 of Stick *B* for one and the same reason, namely, "the length of the whole is 4 times as much as the shaded piece's length." These data indicate that the CBI, via the Repeat Strategy, fostered each participant's reconceptualization of what a unit fraction is—not solely or mainly as a part of a whole but rather as a multiplicative relation between two magnitudes. They could thus "see" the shaded part on B as 1/4 in spite of the whole being marked into 6 unequal pieces, or as 1/4 of A although not part of A.

Improvements in Adults' Reaction Time (RT) for Processing WN and FR

Upon completion of each teaching episode, each participant re-took the computerized test (post). Analysis of test data showed that the average error rate in both occasions (pre/post) and for both number types (WN/FR) was very low (3-4%), while average reaction time (RT) significantly improved (p<.001). The latter included consideration of the cue that preceded each comparison trial: operation (>) or number (WN or FR). The chart below shows average RT (**in milliseconds**) for each type of task design, indicating statistically significant improvement (p<.001) from pre to post not only in comparing FR (as expected) but also, surprisingly, for WN. The data also show a cue X number-type interaction: non-significant impact of cue on RT for WN comparison vs. significant impact on RT for FR (p<.05). That is, RT when seeing FR before the comparison was shorter than when seeing ">" and this difference decreased in post-test. These results seem to lend support to the distinction among parts of a thinking process (scheme), as RT needed to recognize and process a mental object to be operated on is effected by how a "situation" is identified in the person's mind.

	Pre		Post	
	Cue: >	Cue: Number	Cue: >	Cue: Number
FR	1208	1144 (-64 = -5.3%)	923	901 (-22 = -2.4%)
WN	925	949 (+24 = 2.6%)	757	763 (+6 = 0.8%)

Table 1

Brain Circuitry Activated to Process Numbers (WN, FR) and Operation (>)

Figure 2 shows adult brain circuitry activated more for WN than FR comparisons (Figure 2a) and more for FR than WN comparisons (Figure 2b). The former shows WN implicated in: (A) the Hippocampus (LTM retrieval) and (B) the Medial Frontal and Anterior Pole (abstract retrieval). The latter shows *substantially greater activation for FR*, implicated in: (A) the bilateral IPS and Angular Gyrus (numerical judgments of denominators) and the Ventral Visual Processing Stream (object-based visual processing), (B) the Dorsal Fronto-Parietal control network (engaged in attention-demanding tasks, e.g., order inversion), (C) the Ventral-Frontal working memory network & Pulvinar (visual object attention/selection), and (D) the Supplementary Motor Area (SMA, preparing response). Combined, these analyses suggest that brain circuitry used by adults to compare FR involves higher activation in some areas used also for WN (e.g., IPS), along with a more widespread use of brain regions.

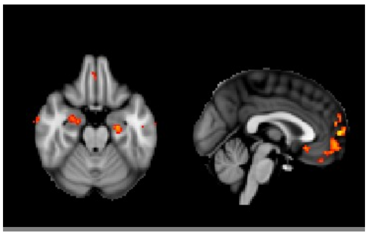

Figure 2a: WN > FR

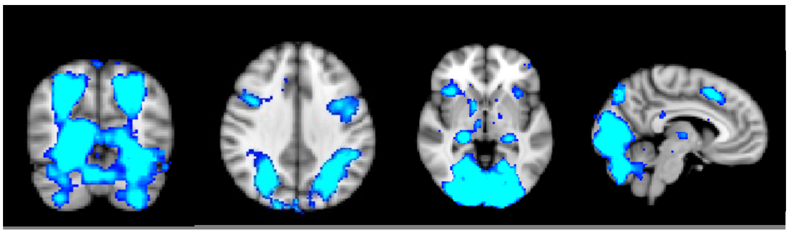

Figure 2b: FR > WN

Figure 3 shows adult brain circuitry activated more for numbers than for the ">" operation (yellow/red colors show this for WN and blue colors for FR). Essentially, when comparing activation of both types of numbers to the operation on these objects (directed by the goal of "find the larger of two numbers"), the same four regions seem to be recruited. The fMRI simulations show more activation for numbers (than ">") in: (A) the Ventral Visual processing stream/cortex (typical of object-based, visual processing mostly in the right hemisphere); (B) the IPS and Angular Gyrus (numerical judgments); (C) the SMA (preparing for response), and (D) Posterior Dorsolateral PFC (attention-demanding tasks). Combined, these analyses suggest that brain activation employed just for recognizing a "cue," before any comparison activity of the task is carried out, is markedly different (smaller) for the symbolized operation than for either type of symbolized numbers the brain processes. Not surprisingly, a remarkable overlap can be seen between these regions and those in which greater activation was found for FR than for WN. Both number types activate some similar circuitry much more than the symbolized operation, whereas processing FR comparisons does so to a much greater extent than WN.

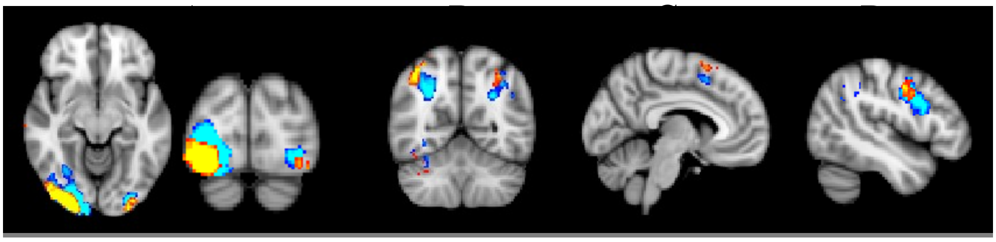

Figure 3: WN > FR

DISCUSSION

We presented three key findings about how a constructivist-based intervention (CBI) impacts adults' re-learning and performance of whole number (WN) and unit fraction (FR) comparisons, and of brain regions activated to process such comparisons. First, we found a change in participants' conception of unit fractions, from "part-of-whole" to a multiplicative relation. Second, we found a CBI's significant impact on their performance of numerical comparisons, not only for FR but *also for* WN. Third, we found significant differences in brain activation: Hippocampus activated more for WN comparisons (long-term memory), whereas IPS (numerical), PFC (task attention and control), Ventral-Frontal and Pulvinar (visual object attention) and SMA (motor response) were substantially more activated for FR comparisons. Combined, these findings entail three contributions to an emerging, cross-disciplinary field at the confluence of mathematics education and cognitive neuroscience.

A first contribution concerns the construction of differentiated brain circuitry to process different types of numerical objects, not identified in previous studies. The limited scope of the fMRI part of our pilot study precludes determining when and how have regions, specialized in recognizing FR and processing comparisons among them, evolved. Moreover, it is not possible to determine if the CBI changed these adults' previously constructed activation patterns, or the differentiated circuitry evolved when they first learned about FR (as children). While these two issues await future research, distinguishing these regions paves the way for (a) studying such an evolution, (b) figuring out if it depends on the nature of instructional methods, and most importantly (c) appreciating the implied, greater cognitive load involved in making sense of and solving FR comparison tasks. Simply put, FR is not just a simple extension of WN. The brain and mind need to construct circuitry that give rise to these numbers and, by way of extrapolation, likely also for other number types.

A second contribution is of a new way to test, and confirm or disconfirm, conceptual frameworks in mathematics education that were developed through observational studies. This pilot study provided an example of such a research pathway for the constructivist scheme theory (von Glasersfeld, 1995). Comparison tasks we designed capitalized on the distinction between the goal-directed activity and the object on which it operates, and showed differentiated impact on both brain circuitry (Fig. 3) and reaction time (see also, Tzur & Depue, 2014). Our findings seem to support the tripartite notion of a scheme, though more specific measures of brain circuitry that correspond to those parts are needed. Key here is that our study illustrates how a CogNeuro-MathEd collaboration can contribute to a two-way enrichment of research and knowledge, informing CogNeuro by MathEd frameworks and informing (curbing and/or expanding) MathEd by CogNeuro findings of the brain (De Smedt & Verschaffel, 2010).

A third contribution involves the CBI's impact on performance of WN comparisons. At issue is why, and how, would a conceptually driven method for teaching FR effect the comparison of WN—a long-established concept. We hypothesize that a person's focus on the multiplicative relation between a unit fraction and a whole into which it uniquely fits via unit iteration could bring forth reflecting on and re-conceptualizing WN as an iterable magnitude (Steffe, 2010) with direct relationship to other magnitudes. Future research can examine this hypothesis, and alternative ones, to better explain links between WN and FR at both the mind and the brain levels.

References

Bonato, M., Fabbri, S., Umiltà, C., & Zorzi, M. (2007). The mental representation of numerical fractions: Real or integer? *Journal of Experimental Psychology*, *33*(6), 1410-1419.

De Smedt, B., Holloway, I. D., & Ansari, D. (2011). Effects of problem size and arithmetic operation on brain activation during calculation in children with varying levels of arithmetical fluency. *NeuroImage*, *57*(3), 771-781.

De Smedt, B., & Verschaffel, L. (2010). Traveling down the road: From cognitive neuroscience to mathematics education ... and back. *ZDM – Mathematics Education*, *42*, 649-654.

Dehaene, S. (1997). *The number sense: How the mind creates mathematics*. NY: Oxford University.

Dehaene, S., Piazza, M., Pinel, P., & Cohen, L. (2003). Three parietal circuits for number processing. *Cognitive Neuropsychology*, *20*, 487-506.

Ischebeck, A., Schocke, M., & Delazer, M. (2009). The processing and representation of fractions within the brain: An fMRI investigation. *NeuroImage*, *47*, 403-413.

Jacob, S. N., & Nieder, A. (2009). Notation-independent representation of fractions in the human parietal cortex. *The Journal of Neuroscience*, *29*(14), 4652-4657.

Piaget, J. (1985). *The equilibration of cognitive structures: The central problem of intellectual development*. (T. Brown & K. J. Thampy, Trans.). Chicago: The University of Chicago. (Original work published 1975)

Schacter, D. L., Addis, D. R., Hassabis, D., Martin, V. C., Spreng, R. N., & Szpunar, K. K. (2012). The future of memory: Remembering, imagining, and the brain. *Neuron*, *76*, 677-694.

Schmithorst, V. J., & Brown, R. D. (2004). Empirical validation of the triple-code model of numerical processing for complex math operations using functional MRI and group Independent Component Analysis of the mental addition and subtraction of fractions. *NeuroImage*, *22*, 1414-1420.

Simon, M. A., Tzur, R., Heinz, K., & Kinzel, M. (2004). Explicating a mechanism for conceptual learning: Elaborating the construct of reflective abstraction. *Journal for Research in Mathematics Education*, *35*(3), 305-329.

Steffe, L. P. (2010). Operations that produce numerical counting schemes. In L. P. Steffe & J. Olive (Eds.), *Children's fractional knowledge* (pp. 27-47). New York: Springer.

Suddendorf, T., & Corballis, M. C. (2007). The evolution of foresight: What is mental time travel, and is it unique to humans? *Behavioral and Brain Sciences*, *30*, 299-351.

Tzur, R. (2000). An integrated research on children’s construction of meaningful, symbolic, partitioning-related conceptions, and the teacher’s role in fostering that learning. *Journal of Mathematical Behavior*, *18*(2), 123-147.

Tzur, R., & Depue, B. (2014). *Brain processing of whole-number vs. fraction comparisons: Impact of constructivist-based task design on reaction time and distance effect*. Paper presented at the Annual meeting of the American Educational Research Association, Philadelphia, PA.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning.* Washington, DC: Falmer.

Westermann, G., Mareschal, D., Johnson, M. H., Sirois, S., Spartling, M. W., & Thomas, M. S. C. (2007). Neuroconstructivism. *Developmental Science*, *10*(1), 75-83.

White House, Office of the Press Secretary. (2013). *Remarks by the President on the BRAIN Initiative and American Innovation*. Retrieved from http://www.whitehouse.gov/the-press-office/2013/04/02/remarks-president-brain-initiative-and-american-innovation