ASSESSING TEACHERS' PROFOUND UNDERSTANDING OF EMERGENT MATHEMATICS IN A MASTERS COURSE

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Profound Understanding of Emergent Mathematics has been recently proposed as a means of conceptualizing the mathematics knowing of teachers as an open disposition that extends well-defined and fixed categories of knowledge proposed in the literature. The purpose of this paper is to explore how this disposition may be assessed through the assignments submitted by the eleven teachers enrolled in a master's-level course. The assignments included concept study and collaborative lesson design. An analysis of the assignments of one team of teachers is presented suggesting that while previous categories for mathematics knowledge were reflected, evidence of an open disposition was limited.

INTRODUCTION

Davis and Renert (2014) have recently proposed a conceptualization of mathematics knowledge for teachers, *Profound Understanding of Emergent Mathematics* (PUEM), based upon complexity sciences and extending the previous conceptualizations widely reported in the literature. Tracing historical approaches to mathematics knowledge for teachers, Davis and Renert claimed that PUEM includes elements of these conceptualization such as: a) formal mathematical knowledge in terms of postsecondary courses and formal mathematics (Begle, 1972); b) specialized mathematics for teachers, such as pedagogical content knowledge (Shulman, 1986) and didactics in mathematics (Freudenthal, 1983); and c) mathematical knowledge entailed for teaching mathematics, such as *unpacking* as a key process of teachers' practice (Ma, 1999)—this last approach includes a shift from knowing more to knowing different. Davis and Renert saw these prior conceptualizations as important but limited in two key elements. First they proposed that teachers' knowledge of mathematics is "vast, evolving and distributed" (p. 48)—similar to Mathematics as a body of knowledge. Second, teachers should embody an open disposition to emergent mathematics in the classroom, including the capacity to participate in a knowledge building community in which students' ideas, misconceptions and questionings play a major role in extending learning beyond formal mathematics. Davis and Renert provided an example in the form of a classroom episode. Grade eight students were asked to approach the following problem: "*Suppose that the earth is a perfect sphere and that we tie a rope tightly around the equator. How long will the rope be?*" (p. 104). Students calculated an approximation of the length of the rope. In class discussion they agreed that adding ten more meters to the rope would create a gap between the earth and the rope, which they calculated as 1.6 meters long, approximately. Teachers explained that the result surprised them as the slackening effect of adding 10 meters to

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the 40 000 kilometers long rope must be negligible. Without having an explanation for this result, teachers asked: "*How can it be that the gap is large enough to allow a child through? How would you explain this result to a person who cannot calculate it?*" (p. 105). One student provided the following explanation:

> For us, a gap of 1.6 meters looks big. But this gap 1.6 meters is added to the radius of the earth. If you compare 1.6 meters to the radius of the earth, which is 6391 kilometers, you can see that this it is not large at all. In fact, it's tiny (p. 105).

Teachers indicated that this problem puzzled them for a long time and considered the previous student's explanation as clear and sensible. The open disposition to this collective generation of knowledge was reflected in the planning of the class as teachers asked a question they had not yet answered.

As a means for nurturing teachers' PUEM, Davis and Renert (2014) proposed *concept* study, a mix of lesson study (Stigler & Hiebert, 1999) and concept analysis (Usiskin, Peressini, Marchisotto, & Stanley, 2003). A main focus in concept study is that: "Learning of mathematics should be more structured around *meanings* than *definitions*" (Davis & Renert, p. 38). Rather than providing a prescribed list of steps for concept study, Davis and Renert described four emphases for the collective study of mathematical concepts. The first emphasis is on *realizations* (Sfard, 2008), that is, the learners' possible ways of association used to make sense of a mathematical construct, including: formal definitions, algorithms, metaphors, images, applications and gestures. The second emphasis is on *landscapes*, which are visual ways for representing relations among realizations—usually in form of tables and maps. Grade level has been a very useful criterion for organizing landscapes. The third emphasis is on *entailments* of the different realizations of a concept, which refer to the logical implications of each realization. The fourth emphasis of concept study is on *blends,* which correspond to grander interpretations connecting the realization of a mathematical concept in a more formal fashion. The emphasis on conceptual blends is a deliberate move into a formal, axiomatic world—as described by Tall (2004).

Concept study has been enacted in several courses for mathematics teachers at both the master's and undergraduate levels in western Canadian universities for more than ten years. However, Davis and Renert (2014) still raised the question of "How might PUEM as an open disposition be assessed?" (p. 121). The purpose of this paper is to explore the potential evidence of PUEM, including this open disposition, in the assignments teachers submitted as part of a master's course in mathematics education, which included concept study and collaborative lesson design. Teachers' decisions within the lesson plans may serve to assess the open disposition toward emergent mathematics as enacted in the classroom.

THE COURSE

The master's program in mathematics education was designed exclusively for teachers at a particular school and consisted of four courses delivered on-site during one year. The course described here, Designing Tasks for the Math Classroom, was the second course of the program. In the first course teachers surveyed a variety of theories of learning in mathematics and questioned their current teaching practices. The school, ranging from grades 2 to 12, served the education of students with learning disabilities. Students were streamed in either the *collegiate program or the academy program*. The latter program focused on students coded with learning disabilities. The collegiate program was designated for students who reached academic skills at age and grade appropriate levels, allowing them to stay at the school instead of going to a regular school.

The course for this study included several goals for participants. First, they were expected to consult relevant literature on the social and historical context of selected topics and concepts in mathematics, as well as their related cognitive obstacles and alternative teaching approaches. Second, participants revisited literature regarding different forms of collaborative design—such as lesson study and learning study—in order to be able to design and enact it in their own context. Third, they were expected to develop capacities for the design, in collaboration with other teachers, of mathematical tasks aimed at student engagement in deep mathematical thinking. And finally, they engaged in ‘doing’ mathematics by solving diverse mathematical problems throughout the whole course—particularly, identifying stages of the problem solving process such as entry, conjecture, verification, specialization, and generalization, as per Mason, Burton and Stacey (2010). The assignments for the course are described in the following paragraphs.

Concept study. This assignment was a deep study of a major mathematical concept or topic from the curriculum comprising its: (a) historical development; (b) cognitive obstacles and students' common mistakes and misunderstandings; (c) images, analogies, metaphors and exemplars used for mathematics and mathematics education; (d) contemporary role/place outside school; and (e) development through the whole curriculum.

Lesson planning. This assignment consisted of planning/creating/selecting learning tasks and activities aimed at engaging students in mathematical thinking. This assignment elaborated from the concept study and extended it to anticipate students’ possible approaches and misunderstandings and appropriate teacher’s responses. This task was based on lesson study and teachers observed the enactment of the lessons.

Individual enactment report. This was an individual report of the enactment of the lessons, including: (a) a general description of the enacted lessons highlighting relevant moments; (b) proof of students’ mathematical thinking; and (c) conclusions.

Debriefing and refinement. A revisited version of the lesson planning including improvements and comments was submitted as a final assignment.

METHODOLOGY

This study took place in the context of a broader research aimed at studying changes in school culture when deliberate support is provided for the professional development of teachers, including the master's program. In order to explore the potential evidence of PUEM in the assignments of the course, I took a qualitative approach. I read these assignments repeatedly to make a general sense of the data as a whole, conducting an open coding and looking for emerging themes. Then, I decided to code for evidence of teachers' knowledge in terms of: *formal mathematics* (Begle, 1972), including knowledge about mathematics such as history and current applications; *pedagogical content knowledge* (Shulman, 1986); *knowledge of content and curriculum* (Ball, et al. 2008); and an *open disposition* (Davis & Renert, 2014).

There were two teams in the group (eleven participants in total). One team decided to focus on the concept of surface area and designed a sequence of lessons for grades four, five, seven and eight. The other group focused on Pythagorean theorem and designed a lesson for grade ten. Due to the length limit in this report, only data from the latter group is presented. Results were, however, similar in the group focused on surface area. In particular, there was a strong interest in promoting relational understanding (Skemp, 1978), as opposed to instructional understanding, in both groups. This was probably a result of the previous course of the master's program.

FINDINGS

The group of teachers focusing on Pythagorean theorem designed a sequence of three lessons for grade ten in both the academic and the collegiate programs. The first lesson focused on visual representations of how the sum of the areas to two squares could yield the area of a third square. In the second lesson students explored triplets of square, most of them Pythagorean triplets. Examples of when the sum of two squares did not yield the area of the third were presented as 'non-examples.' The Pythagorean relationship would be expected to emerge by the end of this class. In the last session students were expected to use the theorem to address a challenging problem in a three dimensional context consisting of finding the length of the diagonal in a closed box. Examples of the evidence found in the assignments that teachers in this team submitted are summarized in Table 1.

Evidence of disciplinary knowledge was clear from the concept study. This knowledge consisted of: references to literature reporting the use of the theorem across time and cultures, connections of related mathematical concepts, extensions such as Fermat's last theorem, and applications beyond school.

Knowledge	Examples
Knowledge of and about mathematics	Historical development including several cultures at different times Connections with other mathematical concepts such as: area, symmetry, square root and algebra Connections to Fermat last theorem Applications out of school mathematics: statistics in baseball, medicine, and microchip technology
Pedagogical Content knowledge	Examples and non examples of right triangles Representation of doubling the area of a square using wooden hinged toys Different images for proofs based on areas Images of the theorem using shapes other than squares Common students mistakes and learning obstacles such as: learning formula without understanding; identify right triangles before applying the theorem, and proper identification of legs and hypotenuse
Knowledge of content and curriculum	Landscapes based on grades as per Alberta's program of studies: K to 3, 4 to 6, and 7 to 9, 10, and 11 to 12 (Related topics for grade 10 included: linear measurement; trigonometric ratios; right triangles, perpendicular lines, right triangles; metric and imperial units; and area.)
Open disposition	Selection of a visual representation to ensure students understand that the sum of the areas of two squares can be the area of a third square. Design of an activity in which students have to figure out a relation between the areas of three squares.

Table 1: Examples of evidence for each type of knowledge

Pedagogical content knowledge could also be identified in the concept study and the decisions for the lesson plans. Common student obstacles, identified from both the literature and teachers' experience, were used in the design of the lessons. For instance, teachers identified that students have difficulties relating the square of a number with the area of a square having the length of its side equal to this number. The team of teachers decided to design an activity in which students would cut two squares into pieces and put them together into a third square: showing that the sum of the area of the two squares equaled the are of the third square. The rationale was that this kinesthetic activity would help students by providing recurring visual representations of squares. Teachers stressed the need for developing relational understanding, as opposed to only instrumental understanding. In particular teachers made the pedagogical decision of modifying the lesson plan format required by the school, which consisted of: 1) quick questions (5 min); 2) interesting idea intended to engage students in the topic, usually presented in a video, new story or other type of media (2 min); 3) a review covering concepts from the previous class (5 to 10 min); 4) introduction of a new concept in which students are led through guided examples and asked to work on independent examples (20 min); 5) seatwork in which students work independently in tasks that may be allocated as homework (10 min); and 6) a summary of the main topics covered in the lesson (2 min). In the concept study, teachers explained their decision for the modification in terms of instructional and relational understanding, as indicated in the following quotation.

> The above lesson plan [required by the school] allows for success with many of our students; however it is based on an instrumental approach to learning. … [This] lesson plan is a guided process, where students are led through the steps they need to successfully perform a required task. In the past, to teach Pythagorean Theorem, we would have introduced the students to the Pythagorean Theorem in the front end of the lesson. Students would have worked through several teacher–led examples on the board, and then completed independent examples. This format of lesson plan has its merits, such as students leave the classroom with a consistent level of understanding of the material. As well, the independent seatwork allows the teacher to discover misconceptions in understanding and immediately correct these errors.
>
> For our lesson study we will be using the general format of a [school's] lesson plan. However, we are excluding the introduction of a New Concept and instead jumping straight into Seatwork. We have decided to take a relational approach to teaching Pythagorean Theorem by creating an inquiry based experience. As a result, we will not be directly teaching the Pythagorean Theorem. In fact, we will not mention the Pythagorean Theorem until the end of the second lesson. Our goal is to have students conceive the Pythagorean Theorem through their own findings, without channelled teaching of the concept. (Concept Study on Pythagorean theorem)

The interest in promoting a relational understanding was explicitly addressed in every assignment, including all the individual enactment reports.

Knowledge of content and curriculum was evident in the landscapes of the concept study based on Alberta's program of studies. Figure 1 shows the landscape for grades 11 and 12. The connection to other topics helped to pay attention to mathematical concepts and skills required to understand Pythagorean theorem, such as: concept of area, square root; square numbers; and algebra.

While there was clear evidence of formal mathematical knowledge, pedagogical content knowledge, and knowledge of content and curriculum, the evidence for an open disposition to emergent mathematics was less obvious. The selection of images for the lessons and the emphasis on understanding that the theorem relates to the sum of areas of squares can be interpreted as attempts to create meaning with the theorem, instead of imposing the formula. Designing an activity, in the second day, in which students would discover the relationship between the areas of squares can be also interpreted as evidence of emerging mathematics in which students work collaboratively in class in order to find this connection. However, evidence of explorations beyond formal mathematics knowledge, as part of the open disposition proposed by Davis and Renert (2014), was lacking in the lesson plans and teachers' reflections. This result was similar for the other group of teachers that focused on surface area.

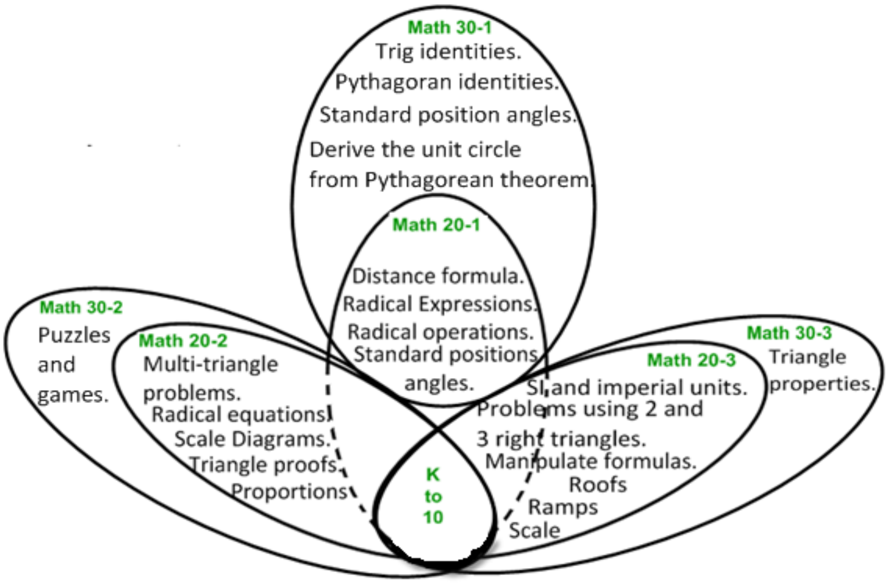

Figure 1: Grade 11-12 Landscape aligned with the Alberta Program of Studies in Mathematics

CONCLUSION

The assignments submitted by the teachers in the course described in this paper served to assess, at least partially, mathematics knowledge for teaching as per PUEM, including prior categories of knowledge as well as the open disposition toward mathematics. Concept study was useful for teachers to explore Pythagorean theorem including historical development and contemporary applications. This may help to extend teachers' understandings about the theorem, including different visual representations for the proofs. Pedagogical content knowledge and curricular content knowledge were also evident in both the concept study and the lesson plans. Teachers were informed by both the literature and their collective experience with this topic in anticipating students learning obstacles. In particular, teachers made a special effort to address relational understanding as they identified an exclusive teaching instrumental approach in the school's lesson format.

The open disposition to emergent mathematics was less obvious in the lesson plans. While common images were selected and students engaged in tasks aimed at the discovery of some relevant properties or relationships, these activities seemed to be more oriented for students to develop a better understanding of pre-existing knowledge. This approach is in contrast with the exploratory approach in the example of the rope and the earth provided by Davis and Renert (2014) and presented in the introduction of this paper. I believe that the combination of concept study and collaborative design is a sound means for teachers to develop an open disposition toward mathematics. However, this disposition may not be immediately reflected in teachers' lesson plans. A more deliberate effort to promote this disposition may enhance the effect of concept study in teachers' professional development.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389-407.

Begle, E. G. (1972). *SMSG reports No. 9: Teacher knowledge and student achievement in algebra.* Paolo Alto, CA: Stanford University.

Davis, B., & Renert, M. (2014). *The math teachers know: Profound understanding of emergent mathematics.* New York: Routledge.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Hingham, MA: Kluwer Academic Publishers.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Erlbaum.

Mason, J., Burton, L., & Stacey, K. (2010). *Thinking mathematically* (2nd ed.). Harlow, UK: Pearson.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge, UK: Cambridge University Press.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

Skemp, R. (1978). Relational understading and instrumental understanding. *Arithmetic Teacher*, *26*(3), 9-15.

Stiegler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.

Tall, D. O. (2004). Building theories: The three worlds of mathematics. *For the Learning of Mathematics*, *24*(1), 29-32.

Usiskin, Z., Peressini, A., Marchisotto, E. A., & Stanley, D. (2003). *Mathematics for high school teachers: An advanced perspective.* Upper Saddle River, NJ: Pearson.