RECONSTRUCTION OF ONE MATHEMATICAL INVENTION: FOCUS ON STRUCTURES OF ATTENTION

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The goal of the study was to reconstruct and dismantle a sequence of events that preceded an insight solution to a challenging problem by a ninth-grade student. A three-week long solution process was analysed by means of the theory of shifts of attention. We argue that concurrent focusing on what, how and why the student attends to when working on the problem can adequately explain his insight.

INTRODUCTION

The goal of the case study presented in this paper was to reconstruct a sequence of events that preceded an insight solution to a challenging problem by a 9th grade student, Ron, who worked on it with his classmate, Arik. Solving the problem required from the students to re-invent the Gauss' formula of the sum of the first *n* integers. The case of interest occurred in the framework of an on-going study that explores the affordances of a particular project-based learning instructional approach (Palatnik, in progress).

The study aims at contributing to research concerned with demystification of insight in mathematical problem solving. Cognitive psychologists frequently refer to an insight problem as one, which solution includes restructuring the initial representation of the problem followed by a sudden realization of the solution – so called *aha*-experience (e.g., Knoblich, Ohlsson & Raney, 2001). Cognitive mechanisms involved in restructuring the initial representation are still relatively uncertain (e.g., Cushen & Wiley, 2012). Furthermore, research on insight problem solving usually explores processes that last for minutes rather than weeks, as it happened in the case presented in this paper. In our study, the three-week-long solution process is analysed through the lenses provided by the Mason's (1989, 2008, 2010) theory of shifts of attention, which, as we argue below, can (partially) explain how the insight occurred.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

Mason (2010) defines learning as a transformation of attention that involves both "shifts in the form as well as in the focus of attention" (p. 24). To characterize attention, Mason considers not only *what* is attended to by an individual (i.e., what objects are in one's focus of attention), but also *how* the objects of attention are attended to. To address the *how*-question, Mason (2008) distinguishes five different structures of attention. Four of them have shown up in our data analysis.

According to Mason (2008), *discerning* details is a structure of attention, in which one's attention is caught by a particular detail that becomes distinguished from the rest of the elements of the attended object. Mason (2008) asserts that "discerning details is neither algorithmic nor logically sequential" (p. 37). *Recognizing relationships*

2014. In Liljedahl, P., Oesterle, S., Nicol, C., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 4, pp. 377-384. Vancouver, Canada: PME.

between the discerned elements is a development from discerned details that often occurs automatically; it refers to specific connection between specific elements. For instance, when attending to the string of numbers 6, 2 and 3 one can effortlessly recognize that they are connected by the relationship $6 \div 2 = 3$. Recognizing the same relationship, however, is more effortful when one looks at the string of numbers 1, 2, 3, 4, 5 and 6. *Perceiving properties* structure of attention is different from *recognizing relationships* structure in a subtle, but essential way. In words of Mason (2008), "When you are aware of a possible relationship and you are looking for elements to fit it, you are perceiving a property" (p. 38). To stretch the above example, when one searches the string 1, 2, 3, 4, 5 and 6 for the numbers that can fit a division relationship, one can effortlessly discern the numbers 6, 3 and 2. Finally, *reasoning on the basis of perceived properties* is a structure of attention, in which selected properties are attended as the only basis for further reasoning.

Since our study concerns the phenomena of insight problem solving, we choose to consider not only *what* is attended and *how*, but also *why* the solver's attention shifts. We found it useful to address a *why*-question by identifying obstacles embedded for the solver in attending to a particular object and discerning the possible "gains and losses" of the shift to a subsequent object. Three research questions guided the study:

1. What were some of the objects of attention for the pair of middle-school students in due course of re-inventing the Gauss formula in the context of coping, for three week, with an insight problem related to numerical sequences?
2. For each identified object of attention, what was the structure of attention?
3. Why did the students move from one object of attention to another?

METHOD

Context

The case of interest occurred in the framework of a project "Open-ended mathematical problems", which is conducted by the authors of this paper in 9th grade classes of one of schools in Israel. At the beginning of a yearly cycle of the project, a class is exposed to a set of about 10 challenging problems. The students choose a problem to pursue and then work on it in teams of two or three. The students work on the problem practically daily at home and during their enrichment classes. Weekly 20-minute meetings of each team with the instructor (the first author) take place during the enrichment classes. At the end of the project, the teams present their work at the workshop.

One of the mathematical problems proposed to the students was Pizza Problem (Figure 1). It is a variation of a problem of partitioning the plane by *n* lines (e.g., Pólya, 1954; Wetzel, 1978). When introduced to the problem, the students are briefly explained mathematical notation as well as the meaning of terms "recursive formula" and "explicit formula". It is of note that 9th graders in Israel, as a rule, do not possess any systematic knowledge on sequences; this topic is taught in 10th grade.

Every straight cut divides pizza into two separate pieces. What is the largest number of pieces that can be obtained by n straight cuts? A. Solve for n = 1, 2, 3, 4, 5, 6. B. Find a recursive formula for the case of n. C. Find an explicit formula. D. Find and investigate other interesting sequences.

Figure 1: Pizza Problem

The choice of the case, data sources and analysis

During the three years of the project, five groups of students choose to work on Pizza problem. One group out of five did not produce any explicit formula. Four groups did so, and in three of them the students were able to explain us how. In this paper, we focus on the remaining group, the team of Ron and Arik. This is for two reasons. First, it is a particularly illustrative case of successful learning (cf. Simon et al., 2010, for the rationale of focusing on successful learning cases). Second, Ron and Arik could hardly explain us, at least not straightforwardly, how they invented the formula. Moreover, the process of invention looked serendipitous to us. Thus, we found particularly interesting and important to attempt to dismantle this serendipity.

The data included the audiotaped and transcribed protocols of the weekly meetings, intermediate written reports that the students prepared for and updated during the meetings, and authentic drafts produced between the meetings. These data were juxtaposed to initially reconstruct the whole story. Pencil marks on the students' drafts were particularly informative for making suggestions about the occurrences of the shifts of attention. The initial reconstruction was shown to Ron, who took the leading role in the project, during a follow-up interview. (The interview was conducted six mounts after the events described.) In the interview, Ron provided us with additional information that supported most of our interpretations and rejected some of them. This information helped us to refine the initial reconstruction.

RECONSTRUCTION

At the beginning, the students produced about 30 drawings of circles representing a pizza, which were cut by straight lines. They counted the number of pieces on the drawings and observed that the maximum number of pieces is obtained if exactly two lines intersect within the circle. The answers for 1, 2, 3, and 4 cuts were found: 2, 4, 7 and 11 pieces, respectively. It was difficult for the students to find a number of pieces for 5 cuts from the drawings as they became overcrowded.

To overcome this difficulty, Ron created a *GeoGebra* sketch and found that the maximum number of pieces for 5 cuts is 16. The students recorded their results as a horizontal string of numbers. They noticed that the differences between the subsequent numbers in the string form a sequence 2, 3, 4, 5 and used this observation to solve the problem for 6 cuts. The next goal for the students was to find a recursive formula. After several unsuccessful attempts to think of the strings of numbers, the students organized their findings vertically and eventually drew a table (see Figure 2).

2a 2b 2c 2d 2e

Figure 2: The drafts produced during the first week

From this point, the students shifted their attention to exploring the tables. The students' way for so doing can be described as looking for the arithmetic relationships between the numbers in the tables and marking them. One of the first relationships that they attended to was a *zigzag pattern* (see Figure 2d). At this stage they introduced the notation: *P* for the number of pieces, *n* for place of P in the table (only later they noticed that *n* represents also the number of cuts) and, eventually, P_n. A formula $P_n = P_{n-1} + n$ was written as a symbolic representation of *zigzag* pattern.

Then the students began looking for an explicit formula, which would enable them, in words of Arik, "to find P_{100} without finding P_{99}"[1]. The students tried to find it on the Internet and did not succeed. They also considered finding the explicit formula in Excel since "there are a lot of formulas in Excel." When this plan did not work, they asked the instructor for help. The instructor only helped the students to build a spreadsheet based on their recursive formula and encouraged them to keep looking.

In a week, the students brought to the meeting five tables with marked patterns: a *diagonal pattern* corresponding to the previously obtained formula $P_n = P_{n-1} + n$ (Figure 3b), a *horizontal pattern* summarized by the formula $P_n = (P_{n-1} + n - 1) + 1$ (Figure 3c), a *mixed pattern* accompanied by (incorrect) formula $P_n = n + P_{n-1} + n - 1$, and a *vertical pattern* corresponding to the formula $P = \sum n + 1$.

The instructor noted that the first three formulas were algebraically identical; the students had not noticed it and were surprised. Surprisingly to the instructor, the students presented a *vertical pattern* and formula $P = \sum n + 1$ just as one of their results, and not as a milestone on the way to the explicit formula. He said:

Instructor: [Let's] focus on this way [*vertical pattern*]…Tell me, how do I get, for example, 22?

Ron: Twenty two without 16? It goes ... I make one plus zero and one and two and three and four and five and six.

[1] All the excerpts are our translations from Hebrew.

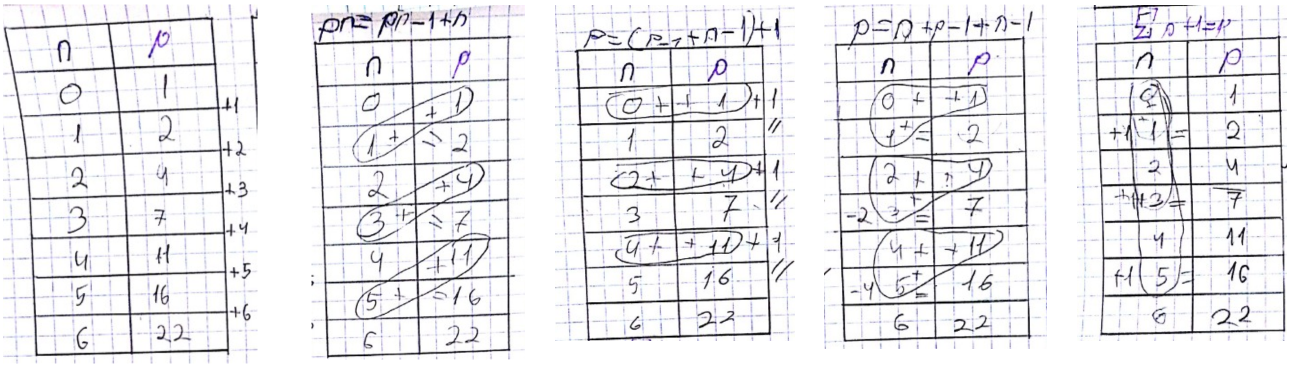
Instructor: One and two and three and four and five… There is some formula for calculating it.

…

Arik: So, [you ask] how to calculate it? Without summing the numbers?

Instructor: Yes, without summing the numbers. You know, there is a formula that can give you an answer [instantly]. Do you understand why this is important?

Arik: Because it takes time to calculate [by the formula $P=\sum n+1$].



3a 3b 3c 3d 3e

Figure 3: The drafts produced during the second week

At the next meeting, the students introduced the desired formula: $P_n=\frac{n}{2}(n+1)+1$. The instructor was astonished by the students' success and asked them to explain their invention in as much detail as possible. Ron took the lead. In his words: "I was stuck in one to six. And I just thought…six divided by two gives three. I just thought there's three here, but I could not find the exact connection [to 22]. I do not know why, but I multiplied it by seven, and voila – I got the result." This explanation along with the data from the follow-up interview enables us to offer, with some certainty, the following reconstruction of the events immediately preceding Ron's "voila".

Ron focused on the left column of a table similar to Table 3e. He experimented with the vertical string of numbers attempting to somehow, mostly by using the operations of addition and subtraction, create an arithmetic expression that would return a number from the right column. He asked his parents and the older sister for help; they tried and did not succeed. Then he came back to exploring the table, and this time he also tried to multiply and divide. One of these attempts began from computations $6\div 2=3$ and $3\times 7=21$. Ron realized that 7 in the second computation is not just a factor that turns 3 into 21, but also a number following 6 in the vertical pattern. He noticed (not exactly in these words) the following regularity: when a number from the left column is divided by 2 and the result of division is multiplied by the number following the initial number, the result differs from the number in the right column by one. He observed this regularity when trying to convert 6 into 22, and almost immediately saw that the procedure works also for converting 4 into 11 and 5 into 16. He observed that even when division by 2 returns a fractional result (5:2=2.5), the entire procedure still works. The *aha*-experience occurred at this moment. To verify the invention, he calculated P_{100} by the discerned procedure and compared the result with the corresponding number in his Excel spreadsheet. The last step was to convert the invented procedure into the formula. From the follow-up interview:

Instructor: How did you convert it [the observed regularity] into the formula?

Ron: It was a difficult part…I did it really in line with the arithmetic operations that I've used. I divided n by 2, and then I like multiplied by n+1, which is the next n, and then plus one.

SUMMARY OF FINDINGS

The answer the first research question straightforwardly steams from the above reconstruction. Namely, the students attended, among others[2], to the following objects: handmade sketches of a pizza, a GeoGebra sketch, strings of numbers, two-column tables, and a left column of a table similar to that in Figure 3e. For each of these objects, we now answer the second and third research questions. The answers for the first four objects are summarized in Table 1.

The last object of attention was identified as "The left column of the table similar to that in Figure 3e." The structures of attention for this object can be described as follows. Ron *discerned* sub-sets of the set of numbers 1, 2, 3, 4, 5 and 6, *recognized* various relationships in the sub-sets, *perceived* the division property and *discerned* a sub-set "2, 3, 6" that fits it. He *recognized* the relationship $6\div 2=3$, *discerned* a subset "3, 22", *recognized* the relationship $3\times 7+1=22$ and *perceived* numbers 6 and 7, which have been *discerned* in the above relationships, as numbers that belong to the vertical pattern. Ron then *perceived* the relationship "$3\times 7+1=22$" for additional triples of numbers, namely, $(4\div 2)\times 5+1=11$ and $(5\div 2)\times 6+1=16$. (This was his *aha*-experience). Solution to the problem was concluded by means of *symbolic reasoning* with the perceived property, that is, converting "$3\times 7+1=22$" into the formula $P_n=\frac{n}{2}(n+1)+1$.

Objects of attention	Structures of attention: How is the object attended to?	Why did the students move to the next object?
Handmade sketches	*Discerning* the bounded areas in order to count the pieces. *Perceiving* that the maximum number of pieces is obtained if exactly two lines intersect within the circle.	When there are more than four cuts, some areas become small and it is difficult to count them.
A GeoGebra sketch	*Discerning* the areas bounded by the circle and five cuts in order to count the pieces. Counting is supported by the easiness of moving the cuts so that small areas can be enlarged.	The drawings, even dynamic, are not convenient for the larger numbers of cuts; results of counting are not ordered.
Strings of numbers	*Discerning* the neighboring numbers of the string and r*ecognizing* the relationships between them: the differences of the neighboring numbers form a sequence 1, 2, 3,	The number of pieces (P_n) is visible in the string, but the number of cuts (n) is not; realization that producing an

[2] Additional objects of attention include an Excel spreadsheet and more. These objects were attended to, but turned to be secondary rather than primary objects of attention in due course of solving the problem.

	4, 5 etc.	explicit formula requires both n and P_n to be visible.
Two-column tables	*Recognizing* various numerical relationships between the numbers (including diagonal, horizontal, mixed and vertical patterns). *Symbolic reasoning* on the basis of the perceived properties: $P_n = P_{n-1} + n$, $P_n = (P_{n-1} + n - 1) + 1$, $P = \sum n + 1$	Realization, partially based on the instructor's prompt, that an explicit formula can be produced by looking at the vertical pattern, which is visually situated in the left column of the table.

Table 1: Structures of attention for the first four objects

DISCUSSION

Pizza Problem appeared to be extremely difficult for Ron and Arik, and one can wonder: why so? The research literature on algebraic reasoning provides us with some initial answers. In line with Radford (2000), we observe that the problem was difficult because it required from the students to shift from pattern recognition to algebraic generalization. In terms of Duval (2006), the problem required from the students to shift the representational registers for many times. In line with Zazkis and Liljedahl (2002), we conclude that the problem was difficult because in the course of its solution the recursive approach was dominant, and this approach is known to prevent the students from seeing more general regularities. Furthermore, Ron's *aha*-moment could usefully be analysed in terms of the representation theory of insight (e.g. Knoblich, Ohlsson & Raney, 2001): the insight occurred when a particular representation was put forward among many other representations.

However, considering the problem's difficulty due to the students' under-developed algebraic reasoning and explaining the insight by identification of shifts in representations is compatible only with one venue of the presented analysis, the one concerned with Mason's *what*-question (i.e., *what* objects are in the focus of attention?) An added value of our analysis is in putting forward also a *how*-question – this is in line with the Mason's theory – and a *why*-question. We argue that concurrent focus on these three questions is pivotal for explaining the observed phenomena. Specifically, focusing on the *how*-question enabled us to better understand the interplay of the structures of attention that lead Ron to his main insight. Focusing on the *why*-question enabled us to identify a pivotal sub-sequence of shifts of attention in a (seemingly) serendipitous chain of attempts.

Our last point is about possible pedagogical implications of the presented case study. Liljedahl (2005) found that *aha*-experiences have positive impact on students' attitude towards mathematics. He then raised a question of how to organize learning environments, in which such experiences might occur. An instructional format outlined in this paper can serve as an example of such an environment[3]. Let us point out

[3] We claim so based not only on the case of Ron and Arik, but on the fact that four out of five teams, who worked on the same problem, also experienced aha-moment when inventing the explicit formula.

its central characteristic. On one hand, the students had enough room for autonomous learning. On the other hand, the chosen format included opportunities for the instructor to focus the students' attention on the most promising idea from the pool of their ideas.

Acknowledgements: This research is supported, in part, by the Ministry of Science and Technology, Israel by the Israeli Science Foundation (grant 1596/13) and the Technion Graduate School.

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