TEACHERS' ABILITY TO EXPLAIN STUDENT REASONING IN PATTERN GENERALIZATION TASKS

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The purpose of this paper is to explore teachers' ability to explain student reasoning in linear and non-linear patterns and in different types of generalization tasks. A questionnaire consisting of student responses to different types of generalization tasks was developed and then given to a sample of 91 in-service mathematics teachers from 20 schools in Lebanon. Analysis of data shows that teachers' explanations exhibited variations in the extent to which they identified the elements and relationships found in students' responses. The results showed that teachers' ability to explain students' reasoning of linear tasks seemed to be higher than that of non-linear patterns. The findings also showed that teachers' explanations of students' reasoning for far generalization tasks exhibited a larger amount of data (elements and relationships) compared to the explanations for near generalization tasks.

BACKGROUND

Findings from previous studies reported that students' reasoning and strategies in pattern generalization is influenced by different factors. Of these factors is the function type of the pattern (linear and non-linear patterns). Krebs (2005) found out that while students are able to generalize linear patterns (constant difference between consecutive terms), they have difficulty in generalizing non-linear patterns (varying difference between consecutive terms). Another factor that has an impact on students' reasoning in patterns is the generalization type (near and far generalization tasks). Amit and Neria (2008) reported that while near generalization tasks (questions which can be solved by step-by-step drawing or counting) were accessible to the majority of students, those students faced difficulties in establishing and justifying a rule for the far generalizations (questions which are difficult to be solved by step-by-step drawing or counting).

On the other hand, few research studies aimed to explore teachers' ability to explain students' reasoning in pattern generalization. For example, El Mouhayar and Jurdak (2012) reported that teachers' ability to explain students' reasoning in far generalization tasks depend on their ability to explain students' reasoning in near generalization tasks. Other studies that focused on teachers' knowledge of pattern generalization used samples of prospective teachers. These studies indicated that prospective teachers recognize patterns in different ways. For example, some prospective teachers formulated rules from the sequence of numbers that are listed in a pattern, whereas others used relationships and cues that are established from the figural structure of a pattern (Chua & Hoyles, 2009; Rivera & Becker, 2007).

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The present study extends the previous research on teachers' knowledge of pattern generalization and it attempts to understand teachers' ability to explain students' reasoning in pattern generalization.

Teacher knowledge of linear and non-linear patterns

Findings in the literature report that the difference between teachers' identification of students' rules in linear and non-linear is not significant. For example, El Mouhayar and Jurdak (2012) focused on the ability of inservice mathematics teachers from grades 7-9 to explain students' reasoning in linear and non-linear patterns tasks. The findings showed that there was no significant difference in teachers' ability to identify symbolic rules that best corresponded to students' pattern generalization processing.

Other findings from previous research showed that teachers are capable of using a variety of strategies to generalize linear and non-linear patterns in different ways. Rivera and Becker (2007) found that teachers generalized linear patterns using numerical and figural strategies. Similarly, Chua and Hoyles (2009) reported that teachers were capable of using a variety of strategies to generalize non-linear patterns in different ways resulting in a range of equivalent rules.

Teacher knowledge of near and far generalization tasks

Previous literature indicates that teachers are able to successfully generalize patterns using different strategies (Chua & Hoyles, 2009; Rivera & Becker, 2007). However, findings of previous studies reveal that teachers' explanations of students' reasoning in near and far generalization tasks seem to be lacking in terms of the elements which constitute a complete explanation. El Mouhayar and Jurdak (2012) showed that more than half of the in-service school teachers (grades 7-9) who participated in the study were unable to provide complete explanations for students' reasoning in near and far generalization tasks. More specifically, while teachers' explanations focused on constant-related counting elements that are not dependent on the step number of the pattern, the teachers' explanations did not include the variable-related counting elements that are dependent on the step number by relating the growing parts of the pattern to the step number.

RATIONALE OF THE STUDY

This study extends previous research on teachers' knowledge of student reasoning in pattern generalization in four directions. First, the present study aims at exploring teachers' ability to explain students' reasoning of linear and non-linear patterns and of near and far generalization tasks whereas previous studies dealing with teachers' knowledge in pattern generalization did not address function type (linear and non-linear) and pattern generalization type (near and far) simultaneously. Second, this study attempts to confirm and extend the results of few previous studies that investigated teacher knowledge of students' reasoning in pattern generalization across a larger range of grade levels than previous studies. Third, the present study aims to explore in-service teachers' ability to explain students' reasoning using authentic student work whereas previous studies have used contrived illustrative models of students' reasoning taken from the literature. Fourth, this study uses the "SOLO model" as a theoretical construct to explore the extent to which teachers use elements of student response to generalize patterns. Although previous studies have used SOLO in the context of teachers' knowledge, these studies did not explore teachers' knowledge of students' reasoning in pattern generalization in particular.

RESEARCH QUESTIONS

The present study aims at exploring teachers' ability to explain student reasoning in linear and non-linear patterns and in immediate, near and far generalization tasks. In this paper we address the following research questions:

- How well are in-service teachers able to explain students' reasoning of linear and non-linear patterns?
- How well are in-service teachers able to explain students' reasoning of near and far generalization tasks?

THE SOLO TAXONONMY

The Structure of the Learned Outcomes (SOLO) taxonomy was developed by Biggs and Collis (1982) to describe a hierarchy of different levels of knowledge ranging from lack of ability to proficiency. The lowest level is called prestructural and it represents the use of no relevant aspect of knowledge in a task. Responses at this level show little understanding of the task. The second level, unistructural, represents the use of only one relevant aspect of the task and therefore indicates some understanding of the task. The third level is the multistructural level whereby responses contain more than one aspect of relevance to the given task; however, those aspects remain separated without being unified or integrated into a coherent structure. The fourth level of the SOLO taxonomy is relational. Relational responses include all the characteristics of the multistructural level in addition to the use of aspects that are related and integrated into a coherent structure. The fifth level and highest level of SOLO taxonomy is called extended-abstract which represents knowledge that goes beyond the task requirements and generalizes its structure. Several studies (e.g. Groth & Bergner, 2006) applied SOLO taxonomy to describe the knowledge of teachers in different contexts.

METHOD

Participants

Ninety one in-service school teachers from different grade levels were selected from 20 schools in Lebanon, particularly Beirut and its suburbs. The majority of the participants (75.8%) had five or more years of experience in teaching mathematics. Of the 91 participants, 79.8% were females and 20.2% were males. All of the participants had either a university teaching diploma or a BA/BS with 52.4% of them having a degree in math education and 29.8% having a degree in pure mathematics.

Instrument

The instrument consisted of two parts. The first part collected information about teachers' background including teaching experience, teacher's major and teaching certificate. The second part consisted of 10 items designed by the researchers to examine teachers' ability to explain student reasoning in pattern generalization. A sample of students' responses were taken from a survey used in a previous study (Jurdak & El Mouhayar, 2013) involving 1232 Lebanese students from grades 4 to 11. The survey included four tasks (two linear and two non-linear). Each of the items displayed the problem (a linear or non-linear task showing the first four figural steps) and students' responses to the (1) near generalization (predicting steps 5 and 9) and (2) far generalization (predicting step 100 or step n). Teachers were asked to analyze one of the two generalization types (near or far) for each of the ten items. For example, participants were asked to analyze student reasoning for step n for item 8 from the questionnaire (Figure 1). For each item, participants were asked "How did the student think to get the number of squares?

The internal reliability of the questionnaire was calculated and Cronbach's alpha was found to be 0.788. The questionnaire was piloted with in-service mathematics teachers from different grade levels to make sure that all the items were understood.

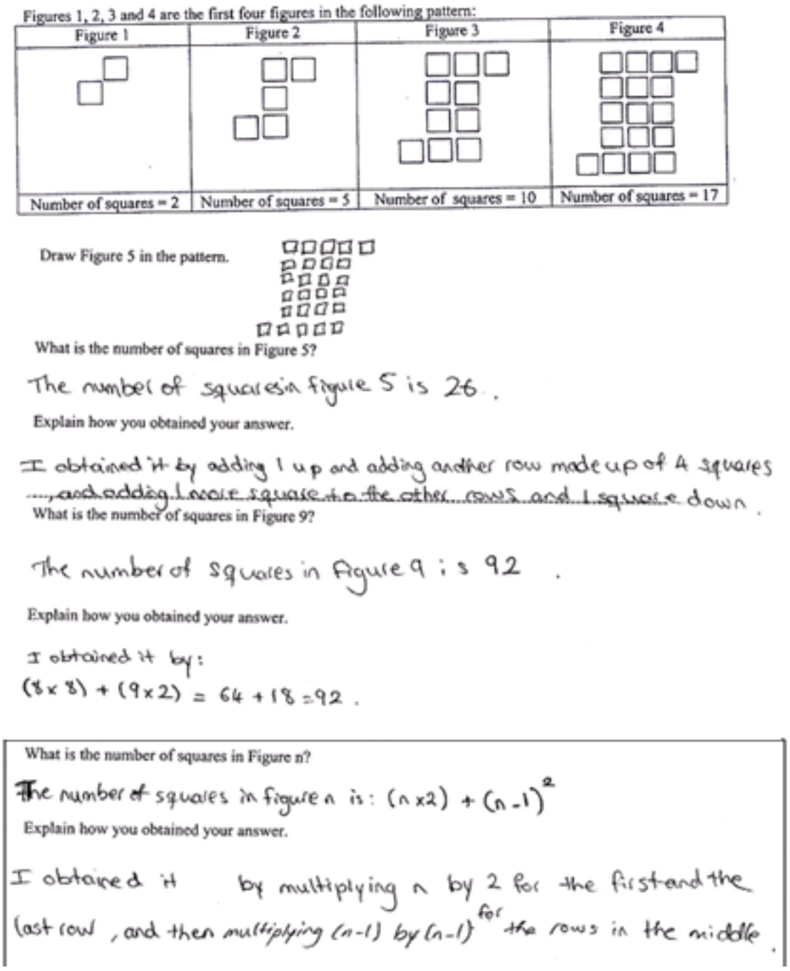

Figure 1: Item 8 from the questionnaire in a non-linear pattern

Data collection and analysis

In each of the 20 schools, participants filled out the questionnaire individually in the presence of the investigator. Filling out the questionnaire took around 90 minutes.

The data obtained were subjected to a series of analyses. First, for each item a rubric based on the SOLO taxonomy was constructed for the purpose of evaluating the teachers' responses to the questionnaire items. In particular, for each of the 10 items in the questionnaire, the investigator identified the elements and relationships that constituted a complete and coherent explanation of the students' responses. The scale points of a rubric were as follows: A teacher's explanation was given a score of 3 "relational" if the teacher identified all the elements of student's reasoning and connected them together; 2 "multistructural" if the teacher identified more than one element but did not address the relationships among these elements or did not address all the elements of student's reasoning; 1 "unistructural" if the teacher identified only one element and 0 "prestructural" if the teacher's explanation indicated a refusal or inability to become engaged in the problem. Two researchers coded the data independently and the discrepancies in coding were negotiated until consensus was reached. Second, cross-tabulations of teachers' ability to explain students' reasoning by (1) function type (linear and non-linear) and (2) pattern generalization type (near and far) were done to explore the possibility of significant differences. For this purpose, significant Chi-squared values and the adjusted residual values were examined. Third, a qualitative analysis was done on teachers' explanations in order to understand the differences in their ability to explain student reasoning by function type and pattern generalization type. Specifically, the qualitative analysis focused on the extent to which the elements and relationships were missing in teachers' explanation. For each item, the elements and relationships in each teacher explanation were identified based on the corresponding rubric. Consequently, the elements and relationships that were missing in teachers' explanations were identified from the rubric and the percentages for the missing elements and relationships were determined.

RESULTS

Teachers' ability to explain students' reasoning by function type

The cross tabulation of teachers' ability to explain students' reasoning by function type is shown in Table 1. Chi-squared was significant (χ^2 (3) = 22.424, p = 0.00) indicating significant differences in teachers' explanations.

For each of the linear and non-linear tasks, teachers showed different abilities to explain students' reasoning: pre-structural, uni-structural, multi-structural and relational with some variation in the relative frequencies across those levels (Table 1). Table 1 shows that teachers' explanations for linear tasks were mainly classified (based on the mode) as relational (29.7%) whereas teachers' explanations for non-linear patterns were mainly classified as either pre-structural (27.9%) or multi-structural (27.9%).

Function type	Prestructural	Unistructural	Multistructural	Relational	Total
Linear (%)	17.1*[1]	28.6	24.6	29.7*	100.0
Non-Linear (%)	27.9*	23.7	27.9	20.4*	100.0
Total (%)	22.5	26.2	26.3	25.1	100.0

Table 1: Cross tabulation of teachers' explanations by function type

A paired-samples t test was conducted to compare the mean of teachers' explanations of students' reasoning for the linear tasks (M=1.67, SD=1.077) with that of non-linear tasks (M=1.41, SD=1.101) and the difference was significant ($p<0.05$), which indicates that teachers' explanations of students' reasoning for linear tasks exhibited a larger amount of data (elements and relationships) than those for the non-linear tasks.

Teachers' ability to explain student reasoning by pattern type

The cross tabulation of teachers' ability to explain students' reasoning by pattern generalization type is shown in Table 2. For the four types of tasks, Chi-squared was significant (χ^2 (3) = 26.597, p = 0.00) indicating significant differences in teachers' ability to explain students' reasoning.

Table 2 shows that teachers' explanations for near tasks were mainly classified (based on the mode) at the multistructural level (30.1%) whereas teachers' explanations for far tasks were mainly classified at the relational level (32.3%).

Pattern type	Prestructural	Unistructural	Multistructural	Relational	Total
Near (%)	24.6	27.5	30.1*	17.8*	100.0
Far (%)	20.4	24.8	22.4*	32.3*	100.0
Total (%)	22.5	26.2	26.3	25.1	100.0

Table 2: Cross tabulation of teachers' explanations by pattern generalization type

A paired-samples t test was conducted to compare the mean of teachers' explanations of students' reasoning for near tasks (M= 1.41, SD= 1.045) with that for far tasks (M= 1.67, SD=1.131) and the difference was significant ($p<0.05$), which indicates that teachers' explanations of students' reasoning of far tasks exhibited a larger amount of data (elements and relationships) than those for near tasks.

A qualitative analysis of teachers' explanations focused on the amount of data (elements and relationships) that are used in explaining students' responses. The qualitative analysis suggests that teachers' explanations coded at the relational level used several elements of the students' responses and related them together. 50.2% (120 out of 239) of the teachers' explanations that were coded multistructural missed elements that were strategy specific whereas 49.8% (119 out of 239) of the teachers'

[1] * indicates that the adjusted residual was greater than |2|

explanations focused on several elements of student reasoning without relating them. Explanations that were coded unistructural missed at least two elements that were strategy specific. Exemplars were produced to illustrate different levels of teachers' explanations and to clarify discrepancies in explaining student reasoning in pattern generalization tasks.

The following excerpts are examples of teachers' explanations of a student reasoning in a far generalization task (step n) of a non-linear pattern (Figure 1). The pattern was perceived by the student as a large square in the middle with dimensions (n-1) by (n-1) and two additional rows at the top and bottom each of size equal to n.

Explanation at the relational level:

> The student separated the middle part of the figure from the upper and lower rows and noticed that the middle part is a square of dimensions equal to figure number minus 1. The area of the inside part is (n-1)2 such that n is the figure number. The number of squares in each of the upper and lower rows is equal to the figure number so it would be 2n for the two rows. For example, for figure 4 it would be 3×3 for the middle part and 2×4in the upper and lower rows.

Explanation at the multistructural level:

> The student related the figure number to the number of squares in the first and last row and to the number of rows in the middle.

The teacher's explanation explicitly referred to the different parts of the pattern and that there is a relationship with the figural step number, but did not explicitly point out the relationship of each part with the step number.

Explanation at the unistructural level:

> The student found a relationship between the figure number and the number of squares in each of the first four given examples. He/she applied the formula and found the number of squares in step 5. The same formula was applied in step n.

The teacher's explanation pointed out that there is a relationship between the number of squares forming the pattern and the step number; however, the teacher did not explicitly identify the different parts of the pattern and did not relate each part of the pattern with the step number.

DISCUSSION

One major finding in this study is that there were variations in teachers' ability to explain student reasoning in pattern generalization. Teachers' explanations fell into four levels: prestructural, unistructural, multistructural and relational. From a SOLO perspective, this finding indicates that teachers' explanations exhibited variation in the extent to which they used the elements and relationships found in student responses. This is supported by findings from other research studies that reported that teachers showed different abilities in analyzing students' reasoning (El Mouhayar & Jurdak, 2012) or in analyzing mathematical concepts and procedures (Groth & Bergner, 2006).

The findings in the present study indicate that teachers' ability to explain students' reasoning in linear patterns seems to be significantly higher than that of non-linear patterns. This finding does not parallel findings from previous research which indicate that differences between teachers' abilities to identify student rules for linear and non-linear patterns were not significant (EL Mouhayar & Jurdak, 2012). One plausible explanation for this finding is that linear patterns are less complex than non-linear patterns since the growth between the consecutive figural steps is constant whereas the growth in the latter varies.

The findings in the present study showed that teachers' ability to explain students' reasoning of far generalization tasks exhibited a larger amount of data (elements and relationships) compared to the explanations of near generalization tasks. This result does not parallel findings from other studies that indicate that while near generalization tasks were accessible to preservice teachers, they have difficulties in establishing and justifying a rule for the far generalization tasks (Rivera & Becker, 2007).

In conclusion, the findings in this study suggest that a move to help teachers in developing their abilities to analyze students' reasoning in pattern generalization is needed to ensure that teachers will have the ability to involve their students in pattern-based instruction as an approach for developing algebraic reasoning.

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