CARDINALITY AND CARDINAL NUMBER OF AN INFINITE SET: A NUANCED RELATIONSHIP

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This case study examines the salient features of two individuals' reasoning when confronted with a task concerning the cardinality and associated cardinal number of equinumerous infinite sets. The APOS Theory was used as a framework to interpret their efforts to resolve the "infinite balls paradox" and one of its variants. These cases shed new light on the nuances involved in encapsulating, and de-encapsulating, a set theoretic concept of infinity. Implications for further research are discussed.

This research explores the intricacies of reasoning about, and with, concepts of infinity as they appear in set theory – i.e., as infinite sets and their associated transfinite cardinal numbers. The APOS (Action, Process, Object, Schema) Theory (Dubinsky and McDonald, 2001) is used as a lens to interpret participants' responses to variations of a well-known paradox which invite a playful approach to two distinct ideas of infinity: *potential infinity* and *actual infinity*. According to Fischbein (2001), *potential infinity* can be thought of as a process which at every moment in time is finite, but which goes on forever. In contrast, *actual infinity* can be described as a completed entity that envelops what was previously potential. These two notions are identified with process and object conceptions of infinity, respectively (Dubinsky et al., 2005), with the latter emerging through the encapsulation of the former. Borrowing APOS language, this study explores the question of "*how to act*?" – a question which speaks to the mental course of action an individual might go through when reasoning with concepts of infinity, as well as to how an action in the APOS sense may be applied.

PARADOXES OF INFINITY

In this study, two versions (P1 and P2) of the infinite balls paradox are considered. P1:

> Imagine an infinite set of ping pong balls numbered 1, 2, 3, …, and a very large barrel; you will embark on an experiment that will last for exactly 60 seconds. In the first 30s, you will place balls 1 – 10 into the barrel and then remove ball 1. In half the remaining time, you place balls 11 – 20 into the barrel, and remove ball 2. Next, in half the remaining time (and working more quickly), you place balls 21 – 30 into the barrel, and remove ball 3. You continue this task ad infinitum. At the end of the 60s, how many balls remain in the barrel?

Briefly, the normative resolution to P1 compares three infinite sets: the in-going balls, the out-going balls, and the intervals of time. This resolution relies on two facts: (1) A set is infinite if and only if it can be put into a bijection (or one-to-one correspondence) with one of its proper subsets; and (2) Two infinite sets have the same cardinality (or 'size') if and only if there exists a bijection between them (Cantor, 1915). (The cardinalities of infinite sets are identified by transfinite cardinal numbers – a class of

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numbers that extends the set of natural numbers. While many of the properties of transfinite cardinal numbers are analogous to properties of natural numbers, there are some important exceptions, illustrated below.) Using facts (1) and (2), one may show that although there are more in-going balls than out-going balls at each time interval, at the end of the experiment the barrel will be empty – all of the sets are infinite, the cardinalities for all sets are the same, and since the balls were removed in order, there is a specific time interval during which each of the in-going balls was removed.

A variation to the paradox can easily be imagined. Consider the following, P2:

> Rather than removing the balls in order, at the first time interval remove ball 1; at the second time interval, remove ball 11; at the third time interval, remove ball 21; and so on… At the end of this experiment, how many balls remain in the barrel?

The difference between P1 and P2 is a subtle matter of which balls get removed – balls 1, 2, 3, … in P1, and balls 1, 11, 21, … in P2. The consequence is that although both experiments involve the same task (subtracting a transfinite number from itself), the results are quite different: P1 ends with an empty barrel; P2 ends with infinitely many balls in the barrel (balls 2-10, 12-20, etc.). Taken together, the two paradoxes illustrate an anomaly of transfinite arithmetic – the lack of well-defined differences.

BACKGROUND

Classic research into learners' understanding of infinity has centred predominantly on strategies of comparing infinite sets (e.g., Fischbein, et al., 1979; Tsamir & Tirosh, 1999; Tsamir, 2003). While a more recent trend has looked toward infinite iterative processes (e.g. Radu & Weber, 2011), power set equivalences (e.g. Brown, et al., 2010), and paradox resolution (e.g. Dubinsky, et al., 2008; Mamolo & Zazkis, 2008).

In the classic studies, participants were given pairs of sets and asked to compare their cardinalities. A common approach by participants was to reflect on knowledge of related finite concepts and extend these properties to the infinite case. For example, students were observed to compare sets in ways that are acceptable for finite sets, such as reasoning that a subset must be smaller than its containing set, but which result in contradictions in the infinite case (see fact (1) above). Furthermore, students were observed to rely on different and incompatible methods of comparison depending on the presentation of sets. If (e.g.) two sets were presented side-by-side, students were more likely to conclude the sets were of different cardinality than if the same sets were presented one above the other. Radu and Weber (2011) similarly found that students reasoned differently depending on the context of the problem – when infinite iterative processes were presented via geometric tasks, students reasoned about "reaching the limit", while an abstract vector task "evoked object-based reasoning" (p. 172).

In their work on power set equivalences, Brown and colleagues observed that while students "demonstrated knowledge of the definitions of the objects involved, all of the students tried to make sense of the infinite union by constructing one or more infinite processes" (McDonald & Brown, 2008, p. 61). These attempts were made despite the

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problem being stated in terms of static objects. Dubinsky et al. (2008) explored the process-object duality in a variant of P1, and observed a common strategy of "trying to apply conceptual metaphors" but noted that "the state at infinity of iterative processes may require more than metaphorical thinking" (2008, p. 119).

This study extends on prior research by using paradoxes to explore the nuances involved in reasoning with and about transfinite cardinal numbers. With APOS as a lens, this study offers a first look at participants' understanding of "acting" on transfinite cardinal numbers via arithmetic operations, focusing in particular on the challenges associated with the indeterminacy of transfinite subtraction.

THEORETICAL PERSPECTIVES

Due to space limitations, familiarity with the APOS Theory is taken for granted (see Dubinsky and McDonald, 2001 for details), and we focus on aspects which most closely relate to conceptualizing infinity. Dubinsky et al. (2005) suggest that "potential and actual [infinity] represent two different cognitive conceptions that are related by the mental mechanism of encapsulation" (2005, p. 346). Specifically, potential infinity corresponds to the imagined Process of performing an endless action, though without imagining every step. They associate potential infinity with the unattainable, and propose that "through encapsulation, the infinite becomes cognitively attainable" (ibid). That is, through encapsulation, infinity may be conceived of an Object – a completed totality which can be acted upon and which exists at a moment in time.

Brown et al. (2010) elaborate on what it means to have an encapsulated idea of infinity. Such an object is complete in the sense that the individual is able to imagine that all steps of the process have been carried out despite there not being any 'last step'. To resolve the issue of a complete yet endless process, Brown et al. introduce the idea of a transcendent object – one which is the result of encapsulation yet which is understood to be "outside of the process" (p. 136), that is, the object or "state at infinity is not directly produced by the process" (p. 137). Recalling P1, the empty barrel at the end of the experiment corresponds to what Brown et al. refer to as the state at infinity. As an object, it is transcendent since it is not produced by the steps of the process itself, but instead through encapsulation of the process. In accordance with APOS, Brown et al. consider encapsulation to be catalysed and characterized by an individual's attempt to apply actions to a completed entity. They offer an example to support Dubinsky et al.'s claim that while actual infinity results from encapsulation, the "underlying infinite process that led to the mental object is still available and many mathematical situations require one to de-encapsulate an object back to the process that led to it" (2005, p. 346). Brown et al. (2010) observed that the de-encapsulation of an infinite union back to a process was helpful in applying evaluative actions to an infinite union of power sets.

Weller, Arnon, and Dubinsky recently suggested a refinement to the APOS Theory, which includes a new stage they term *totality*. This refinement emerged from their analysis of students' understanding of 0.999… = 1. They observed differences among participants who "reached the Process stage but not the Object stage", and suggested

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an intermediate stage, wherein individuals would progress from process to totality and then to object. They noted that: "Because an infinite process has no final step, and hence no obvious indication of completion, the ability to think of an infinite process as mentally complete is a crucial step in moving beyond a purely potential view" (2009, p. 10). The authors suggested that while some of their participants could imagine 0.999… as a totality (e.g. with all of the 9's existing at once), they were not able to see 0.999… as a number that could be acted upon. They suggest that the totality stage may be necessary for encapsulation of repeating decimals.

In this study, the paradoxes P1 and P2, when taken together, offer a situation similar to, but different in important ways from, previous lenses through which to interpret individuals' understanding of infinity. As mentioned, prior research indicates that de-encapsulation has been connected to learners' successes in applying actions to the object of infinity. The studies address different contexts of infinity, but share a common feature: they examine instances in which de-encapsulation makes use of properties of a process that extend naturally to the object. In contrast, P1 and P2 offer a way to explore transfinite subtraction, whose indeterminacy suggests a potential problem with de-encapsulation. The extent to which properties of the process may be relevant to properties of the object of infinity and the question of what other situations may or may not require de-encapsulation of an object in order to facilitate its manipulation are still open. This study takes an important step in that direction by exploring the following related questions: (1) How does one "act on infinity"? And (2) What can the "how" tell us about an individual's understanding of infinity? As indicated above, the "how" refers to both the mental course of action an individual might go through when attempting to reason with actual infinity, as well as to how in the APOS sense an action (in this case transfinite subtraction) may be applied.

PARTICIPANTS AND DATA COLLECTION

For the purpose of this proposal, data from two participants with sophisticated mathematics backgrounds, Jan and Dion, will be considered. Jan was a high-attaining fourth year mathematics major who had formal instruction on comparing infinite sets via bijections. Dion was a university lecturer who taught prospective teachers in mathematics and didactics, the curriculum for which included comparing cardinalities of infinite sets. Neither participant had experience with transfinite subtraction.

Data was collected from one-on-one interviews, where participants were asked to respond to the paradox P1. Following their responses and justifications to P1, participants were asked to address the variant P2. A discussion of the normative resolution to P2 ensued, after which participants were encouraged to reflect on the two thought experiments and their outcomes. Jan and Dion were chosen for this study because they both resolved P1 correctly within the normative standards mentioned above, and because of their object-based reasoning which emerged in contrast to prior research (e.g. Mamolo & Zazkis, 2008; Dubinsky et al., 2008). As such, results and analyses will focus on their responses to P2 in comparison to their approaches to P1.

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RESULTS

As mentioned, both Dion and Jan resolved P1 with appropriate bijections and language which referred to the sets as completed objects. When addressing the comparison between sets of balls and time intervals, both participants explained that the cardinalities were the same, and that "every ball that is put into the barrel is removed."

Jan's response to P2 was consistent with her approach to P1 – that is, she reasoned abstractly and deductively with the form of set elements, with sets as completed totalities, and with formal properties and definitions. She observed that "transfinite cardinal arithmetic doesn't work exactly like finite cardinal arithmetic" and connected her understanding of correspondences between infinite sets to explain the indeterminacy of transfinite subtraction. She elaborated:

> By assumption, only the balls 1, 11, 21, 31,.... are removed, (i.e. All balls of the form 10n+1 for n=0,1,2,3...). Now f(n) = 10n + 1 is not a bijection from the set of naturals to itself, since for example, there is no natural n such that f(n) = 2, so f(n) is not onto. So at first, one might guess that "the infinity of balls put in is somehow greater than the infinity of the balls removed". However! here we get into the indeterminacy of the "quantity" infinity minus infinity… The set of balls put into the barrel DOES have the same cardinality as the set of balls removed from the barrel, since there is a bijection between the set **N** of all naturals and the set [*writes*] S = {10n+1 | n is a natural number}, namely f(n) = 10n+1, which IS a bijection from **N** to S, but NOT from **N** to **N**. But even though there is a bijection… there are still an infinite number of balls left in the barrel after the minute is up! This is because **N** … is equinumerous with a proper subset of itself.

Thus, Jan realized that although the quantity of balls taken out of the barrel was the same as the quantity put in, this was not sufficient to conclude that *all* of the balls had been removed. She observed that remaining in the barrel was the set of balls numbered

> {10n+2 | n=0,1,2,...}. This set is clearly infinite, and represents a subset of the balls left... Since the set of balls left contains an infinite subset, it too must be infinite… we have changed the remaining balls from zero to infinity!

In contrast, Dion's response to P2 showed a shift in attention from describing cardinalities of sets to enumerating their elements. He used language consistent with a process conception of infinity, and overlooked the specific form of the set. While Dion commented on the similarities between P1 and P2 as well as the relevance of Cantor's work to his solutions, he reasoned with P2 informally, rather than deductively. When addressing P2, Dion noted that, as in P1, there existed bijections between pairs of sets of in-going and out-going balls and time intervals. He concluded that the variant and the "ordered case" should yield the same result: an empty barrel. When asked to elaborate, Dion argued for an empty barrel because "after you go [remove] 1, 11, 21, 31, …, 91, etc, you go back to 2" – language that describes a process of moving balls. During the interview, Dion struggled with the idea of a nonempty barrel. He stated:

> If ball number 2 is there, so is ball 2 to 10, etc… so, infinite balls there? I have trouble with that. (long pause) I have a strong leaning to Cantor's theorem (*sic*) and to use one-to-one… I want to subtract, but I can't.

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Eventually, Dion conceded he was "convinced" of the normative solution to P2 since "you can't reason on infinity like you do on numbers", and he observed that while "on one hand infinite minus infinite equals zero, on the other it's infinite" – a property of transfinite arithmetic that was new to him.

DISCUSSION

This study delves into the conceptions of two individuals with sophisticated mathematics backgrounds, as elicited by variations of the infinite balls paradox, with the intent to shed new light on the intricacies of accommodating the idea of actual infinity. Dubinsky et al. (2005) proposed that the idea of actual infinity emerges from the encapsulation of potential infinity, and is recognised by an individual's ability to apply actions and processes to completed infinite sets. This study is a first look at individuals' understanding of 'action' given the nuanced relationship between an infinite set and its associated transfinite cardinal number. The issue of transfinite subtraction is explored and a first attempt is made to address the relationship amongst encapsulation of infinite sets and transfinite cardinal numbers, and the manner in which an individual applies actions to those entities.

How does a learner act on infinity?

In the context of set theory, actual infinity can be conceptualized in two ways – as the encapsulated object of a completed infinite set (to which bijections can be applied), and as the encapsulated object of a transfinite number representing the cardinality of an infinite set (to which arithmetic operations can be applied). Focusing on arithmetic operations, the data reveal two ways an individual may attempt to "act on infinity": (i) by deducing properties through coordinating sets with their cardinalities and element form, and through the existence of bijections between sets; and (ii) by de-encapsulating the object of an infinite set to extend properties of finite cardinals (elements of its conceptualization as a process) to the transfinite case. Exemplifying the former was Jan's reasoning with and resolution of the P2. Jan's ability to deduce consequences of a set being equinumerous with one of its proper subsets was showcased throughout her response. She consistently used language that referred to sets as completed totalities, reasoning with the form of elements (e.g. 10n+1) and bijections, rather than relying on enumerating elements (e.g. 1, 11, 21,…) to describe sets and relationships. Jan's response indicates that she consistently reasoned with the encapsulated object of an infinite set, using its properties to make sense of the paradoxes. Her approach allowed her to transition from acting on sets to acting on cardinals and contributed to her understanding of the indeterminacy of transfinite subtraction, allowing her to "act" – both by comparing sets and by applying arithmetic operations – in ways that are consistent with the normative standards of Cantorian set theory.

In contrast, Dion, who revealed a normative understanding of infinite set comparison in his resolution of P1, struggled during his engagement with P2. His attention to the process of removing balls ("go back to 2") suggests that Dion had de-encapsulated infinity (conceptualized as an infinite set) and tried to reason with properties of the

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process in order to make sense of applying the action of transfinite subtraction to the object of infinity (conceptualized as a transfinite cardinal number). This approach is consistent with other attempts to reason with infinity (e.g., Brown et al., 2010), however, in Dion's case, this lead to considerable frustration and self-described "trouble". Dion's struggle may be attributed to attempts to make use of properties of a process of *infinitely many finite entities* rather than make use of properties of an object of *one infinite entity*. In the case of subtraction, properties of the former are inconsistent with properties of the latter, whereas this is not necessarily the case with other actions. When Dion was faced with a non-routine problem regarding transfinite subtraction, he "acted" by de-encapsulating infinity, making use of the process and generalizing his intuition of subtracting finite cardinal numbers, and thus experienced difficulty with the indeterminacy of subtracting transfinite cardinals.

What can the "how" tell us about an individual's understanding of infinity?

Dion's difficulty and Jan's success with P2 suggest that:

- It is possible to conceptualize an infinite set as a completed object without conceiving of a transfinite cardinal number as one;
- De-encapsulation of an infinite set in order to help make sense of an encapsulated transfinite cardinal number is problematic; and
- In set theory, accommodating infinity involves more than being able to act on infinite sets, and includes knowledge of how to act on transfinite cardinals.

Further, Dion's difficulty highlights the importance of acknowledging the distinction between how actions or processes behave differently when applied to transfinite versus finite entities as an integral part of accommodating the idea of actual infinity. Through Dion's frustration that "I want to subtract, but I can't", and his insistence that "at some point we'll get back to 2" a conflict emerged that was difficult for him to resolve. Dion's realization that "you can't reason on infinity like you do on numbers", was important: it helped convince him of the normative resolution to P2.

Dion's struggle to re-encapsulate infinity in order to appropriately apply transfinite subtraction indicates that an understanding of how actions ought to be applied is relevant to the encapsulation of a cognitive object. Although Dion seemed able to consider the infinite sets of ping pong balls as completed entities which could be compared, his understanding of infinity nevertheless lacked one of the fundamental features that contributed to Jan's profound understanding: the knowledge of how to act on transfinite cardinal numbers. In Jan's words, "it is nearly impossible to talk about it [infinity] informally for too long without running into entirely too much weirdness".

An important contribution of this study distinguishes between the object of an infinite set and the object of a transfinite cardinal number, and identifies the significance of understanding properties of transfinite arithmetic in order to accommodate the idea of actual infinity. While there is substantial research focused on individuals' reasoning with cardinality comparisons, how individuals conceptualize transfinite subtraction has not previously been addressed. Jan and Dion illustrate two ways to try and make

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sense of transfinite subtraction: via deduction that coordinated completed sets and their cardinalities or via the use of properties of an infinite process through de-encapsulation. Taking into account the newly identified stage of *totality* in a genetic decomposition of infinity (e.g., Weller et al., 2009), questions also arise about the relationship and tensions between object, process, totality, and the de-encapsulation of an object to make use of properties of its conception as a process.

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