STUDENTS' MANIPULATION OF ALGEBRAIC EXPRESSIONS AS 'RECOGNIZING BASIC STRUCTURES' AND 'GIVING RELEVANCE'

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Manipulating algebraic expressions is guided by students' structure sense and the individual process of relating subexpressions to each other. This paper presents a framework for identifying the underlying cognitive processes of manipulating algebraic expressions, based on 'basic structures' and 'giving relevance'. A design-research case study illustrates how different cognitive processes, related to each of these two constructs, lead to different activities of manipulating algebraic expressions.

MANIPULATION OF ALGEBRAIC EXPRESSIONS AS THE APPLICATION OF RULES

For several reasons, students need to learn to manipulate algebraic expressions in line with the appropriate rules. Not only does this help students to change the form of an expression in order to determine if two algebraic expressions are equal to one another, but it may also be a source for students' meaning making in algebra (Kieran, 2004). However, several studies show that it is hard for students to learn how to manipulate algebraic expressions, as it poses several, non-trivial challenges (e.g. Linchevski & Livneh, 1999).

In order for students to manipulate algebraic expressions in line with the appropriate rules, students must be able to see structures in an algebraic expression. Such a Structure Sense allows students to see, if and in what way a rule for algebraic manipulation can be applied to a given algebraic expression (Hoch & Dreyfus, 2005). However, there may be different ways in which an algebraic expression can be manipulated based on its structures. For example, $ab+ab$ can be transformed into $2ab$, but also into $a(b+c)$. Thus, not only is the manipulation of algebraic expressions guided by structure sense, but the rule-based manipulation of algebraic expression is also guided by cognitive processes, that might, more appropriately, be described as an amalgam of structure sense and of focusing on certain aspects of an expression.

This leads to the question, what characterizes such cognitive processes that lead students to focus on certain structures of an algebraic expression, while neglecting other structures? This paper attempts to characterize these cognitive processes.

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MODEL FOR APPLYING RULES TO ALGEBRAIC EXPRESSIONS

Structure Sense and the manipulation of algebraic expressions

Studies about how students manipulate algebraic or even arithmetic expressions suggest that the rule-based transformation of such expressions is a complex interplay between the structure of an expression and the students' ability to see structures in the expression. Linchevski & Livneh (1999) introduced the concept of structure sense in order to grasp students' problems with the structure of algebraic expressions.

Hoch & Dreyfus refined the definition of structure sense and found that structure sense is a compound of students' abilities to

- "Deal with a compound literal term as a single entity. (SS1)
- Recognise equivalence to familiar structures. (SS2)
- Choose appropriate manipulations to make best use of the structure. (SS3)" (Hoch & Dreyfus, 2005, p. 146)

The starting point of Hoch and Dreyfus' model of structure sense is the structure of algebraic expressions, as the above list illustrates. It is the structure of a given algebraic expression that shapes the students activities to manipulate the expression (Fig. 1, right side).

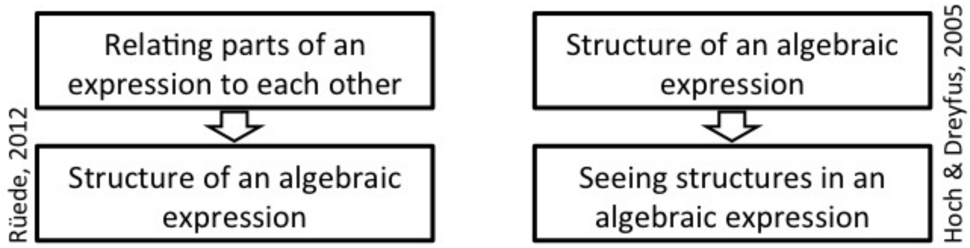


Figure 1: Comparison of Hoch & Dreyfus's and Rüede's models of structure sense

Rüede's (2012) notion of structure sense emphasizes that structure sense is the *individual* process of seeing structures and that the structure of an expression is constituted by the individual student. Rüede defines structure sense as the students' ability to see "different parts of the expression in relation to each other" (Rüede, 2012, p. 113). He empirically specifies four levels for allocating students' structure sense, with increasing degrees of elaboration for relating expressions to each other, e.g., on the first level, finding graphical similarities between subexpressions. Rüede concludes that "[choosing] the appropriate manipulation" for manipulating an algebraic expression is based on the students' ability

- to see subexpressions in an algebraic expression...
- …and to relate these subexpressions to each other and to the whole expression (Rüede, 2012) (see Figure 1, left side).

Synthesized model of cognitive activities of manipulating algebraic expressions

In this paper, it is assumed that the manipulation of algebraic expressions involves both processes of dealing with the structure of algebraic expressions, shown in Figure 1.

The first process is reconstructing the structure of an expression through relating its subexpression, the second process is identifying structures in an expression by "familiar structures". These are regarded as different processes, and are modelled with the notions of "giving relevance" and of "basic structures" respectively.

First, the role of a student's ability to relate the parts or subexpressions of an algebraic expression with each other (Figure 1, left side) is conceptualized as 'Giving Relevance'. Giving Relevance describes a student's individual ways of focusing on certain subexpressions or parts of an expression, while neglecting other parts. This definition follows Rüede's arguments of the importance of relating subexpressions; however, in contrast to him it is here proposed, that seeing a subexpression and relating it to others or to the whole expression is a matter of Giving Relevance to a subexpression. That is, for example, seeing the importance of a subexpression for applying an algebraic rule or for manipulating an expression with a certain aim.

Second, the role of familiar structures in an algebraic expression (Figure 1, right side) is conceptualized with the notion 'Basic Structures' and recognizing Basic Structures. Basic Structures are a student's individual knowledge of structures together with their symbolic manifestation. For example, $ab+ab$ may be a basic structure for a student, and may be associated with "the sum of two products with equal factors". Furthermore, a basic structure can, for a student, be associated with an algebraic rule, in the sense that it can establish the domain of applicability of a rule. For example, the application of the rule $ab+ac=a(b+c)$ might be guided by recognizing the basic structure $ab+ac$. Thus, there are basic structures that guide the application of already learned and conventionalized rules like $ac+ac=2ac$ or $a(b+c)=ab+ac$. However, there may also be basic structures that might lead to spontaneously invented and un-conventional transformations, as Demby (1997) suggests.

The manipulation of algebraic expressions is guided by both processes (also described below in Figure 2). For example, in an expression like $ab+ac+ab$ it might be that a student gives relevance to the two ab's as a basic structure (sum of equal subexpressions), which might lead him to perceive the applicability of the rule $ab+ab=2ab$, and thus, might lead him to transform the expression into $2ab+ac$. On the other hand, if the student foregrounds the basic structure that underlies $ab+ac$, he might give relevance to all subexpressions (sum of subexpressions with one equal factor), which might then lead him to the transformation $a(b+c+b)$.

That is why in the theoretical approach of this article, both basic structures and giving relevance are conceptualized to moderate the students' activities of manipulating an algebraic expression. They allow framing the underlying cognitive processes of manipulating algebraic expressions (Figure 2). For that, the term cognitive activities is introduced. Cognitive activities describe the whole of cognitive processes underlying a manipulation and the activity of manipulating itself. Different cognitive activities can be distinguished by the way cognitive processes (upper line of Figure 2) and actual manipulations (bottom line of Figure 2) are related and interconnected. For example, there may be a cognitive activity “classifying”. In such a cognitive activity, relevance is given to two or more subexpressions, according to some shared characteristic. A basic structure like $ab+ab$ might then lead to group equal subexpressions, in order to apply the underlying rule. It is the aim of this paper, to reconstruct and characterize the students’ cognitive activities of manipulating algebraic expressions.

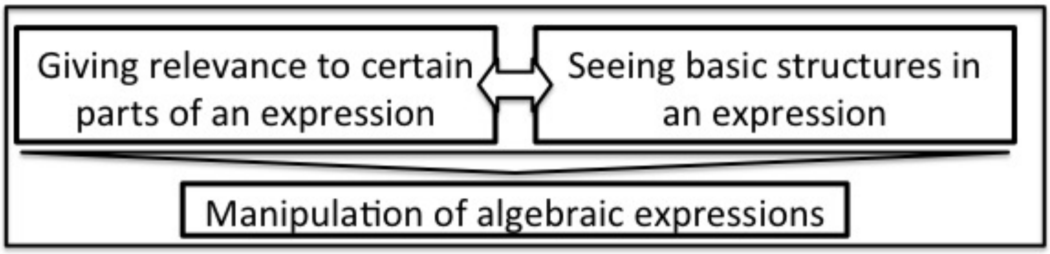


Figure 2: Synthesized model of cognitive activities of manipulating expressions

In the next chapter, the empirical part of a design-research study is presented, which is specifically designed to support such cognitive activities of manipulation.

METHODOLOGY

Design research as a methodology

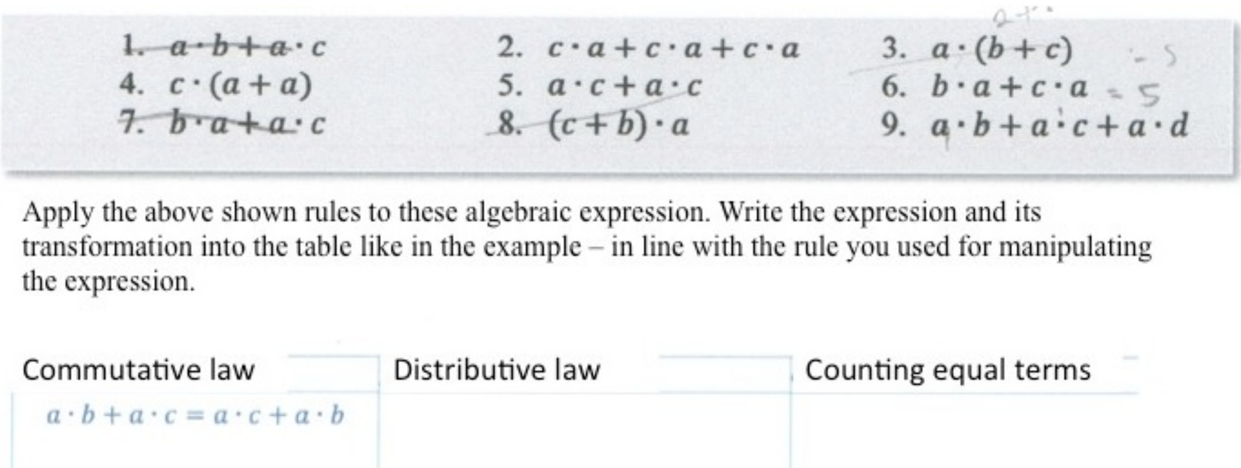
The model of cognitive activities was used for a design research study on students’ repertoire of basic structures and their abilities to give relevance. Design research intends to investigate learning processes of a given learning content by iteratively conducting design experiments. Each iteration of a design research experiment builds upon the empirical insights from the previous iteration. In order to do that, design research experiments start with a conjectured learning trajectory, which is based on findings about learning processes to the given content (Prediger & Zwetschler, 2013).

Design of the learning arrangement and the focus task

The learning arrangement in the case study presented here aims to enable students to see patterns in rule-based manipulations of algebraic expressions. To that end, it focuses on simple algebraic expressions with no more than four subexpressions and which can be manipulated by three previously given rules, namely the distributive law, the commutative law and the rule $ab+ab+ab=3ab$ (called “counting equal terms”). The case study presented here is part of the first design iteration.

This paper focuses on students’ cognitive processes while working on one task of the learning arrangement, which was especially designed to support the cognitive processes in the model. In this focus task, students are asked to write down their manipulated expressions together with the original expression as an equation into the respective column. Each column stands for one of the above mentioned three algebraic rules (see Figure 3). The more algebraic expressions students manipulate, the more diverse become the equations in the columns, while, at the same time, these equations have the same underlying structure. This way, the task supports the cognitive processes and manipulation activities in the model of cognitive activities by:

- Helping students to gain awareness of rules by asking them to make rules explicit through writing the expression into the respective column.
- Supporting students to gain a more elaborate notion of basic structures underlying each rule. The more equations are written into a column, the more material students have to acquire more elaborate basic structures.
- Supporting students to give relevance, by enabling them to come back to previous manipulations in the column. This might help students to give relevance to those parts of an expression that constitute its underlying structure.



1. $a \cdot b + a \cdot c$ 2. $c \cdot a + c \cdot a + c \cdot a$ 3. $a \cdot (b + c)$
4. $c \cdot (a + a)$ 5. $a \cdot c + a \cdot c$ 6. $b \cdot a + c \cdot a$
7. $b \cdot a + a \cdot c$ 8. $(c + b) \cdot a$ 9. $a \cdot b + a \cdot c + a \cdot d$

Apply the above shown rules to these algebraic expression. Write the expression and its transformation into the table like in the example – in line with the rule you used for manipulating the expression.

Commutative law	Distributive law	Counting equal terms
$a \cdot b + a \cdot c = a \cdot c + a \cdot b$		

Figure 3: Task with table (with Bianca's notes in it), translation A.M.

Data gathering in design experiments and data analysis

The learning arrangement represents the first iteration of a design experiment, that is going to have three design circles. The learning arrangement in each iteration encompasses three school lessons (3 * 45min). Previous to each design experiment, the teacher uses specifically designed teaching materials in the classroom, which supports the description of geometric shapes with algebraic expressions.

The design experiment was conducted in a laboratory setting. Three 7th-grade students, Bianca, Daniela and Andrew worked – separated from the class – under the supervision of the researcher (author) on the tasks of the design research experiment. The students were chosen by the teacher according to their active participation in the mathematics classes. The sessions were videotaped. The resulting video was transcribed and the relevant sequences were translated to English by the author.

The data is based on the students work on the above described focus task (Figure 3). It stands exemplarily for the wider dataset of the design experiment. The method of analysis is a category-driven sequential discourse analysis. The three main categories of the analysis are based upon the two cognitive processes of the model of cognitive activities and the actual manipulation activities. The model is not an analytical tool, but was adapted in order to arrive at categories: The sequences, in which students manipulate algebraic expressions, are analysed for the nature of the underlying cognitive processes and of the actual manipulation, that is conducted or discussed. The aim of the analysis is to characterize the cognitive activities, which guide the students' manipulations of simple algebraic expressions.

RESULTS

At some point in the task, the students are confronted with the expression $c \cdot a + c \cdot a + c \cdot a$. They already have seen the rule $ab + ab + ab = 3ab$ in the previous task. The students are now asked to manipulate this expression by applying one of the three given rules. Daniela, when confronted with the expression, immediately says:

319 D: There you could count equal terms, you could do 3 times c times a then.

It is apparent, that Daniela has no problems applying an already known rule to the slightly different expression. For her, the expression in itself seems to be a familiar structure, it is a basic structure in itself. Thus, the cognitive processes of giving relevance seems to be secondary, because relevance is given only in the sense that the subexpressions are recognized as being equal – and this is the precondition of the rule "counting equal terms". Thus, this cognitive activity, where an expression as a whole is associated with a known rule, is called "associating with a known rule".

At a later point, the students are confronted with the expression $ab + ac + ad$. The students already know the rule $ab + ac = a(b + c)$. The following exchange occurs:

334 B: Three times a times b times c…[…]

336 D: One could somehow everywhere, one could again exchange *[colloquiual for applying the commutative law, A.M.]*

337a A: One could count equal terms

337b B: One could exchange these *[said at the same time as 337a]*

338 D: Yes, but these are no equal terms.

When confronted with this expression, the students could not decide easily, which rule might be applicable. It seems that recognizing a basic structure in the expression is not guided by the expression itself, but by what most likely seem to be guessing processes: In turn 334, relevance seems to be given to the fact that there a three subexpressions and variables in alphabetical order, which might lead to the wrong rule $ab + ac + ad = 3abc$. The next utterances (turn 337a and 337b) might hint at a process similar to the above example, where the expression at hand is wrongly associated with known rules.

In turn 338, Daniela counters Andrews proposal. She gives relevance to the different features of the subexpressions. This suggests that Daniela is aware of the declarative content of the rule "counting equal terms", which might suggest that her basic structure behind this rule is elaborated: she interprets this rule as a proposition about the relations of equal parts in an algebraic expression. Thus, the underlying structure of the rule "counting equal terms" in her basic structure includes the declarative content of the rule, which allows her to give relevance to the features of the subexpressions. The cognitive activity behind this manipulation is called "interpreting the declarative content of a rule".

The above described situation continues, and now Andrew takes the lead:

351	A:	A times left bracket b plus c plus d.
352	D:	What?
	A:	Bam!
353	B:	One could just b times a times…
354	D:	Yes, right, that is b plus c times d, that probably works.
355	A:	It is the same as a times left bracket b plus c, only with a number more, you know. This also works.

In turn 351 Andrew suggests a new transformation of the algebraic expression $ab+ac+ad$, namely $a\cdot(b+c+d)$. After a short interjection, Daniela approves this transformation (turn 354). Andrew also justifies his suggested expression in turn 355 using an analogy to a previously applied (and negotiated as a correct rule in various conversions of algebraic expressions) rule $a\cdot(b+c)$. He adds "with a number more, you know". In Andrew's view, the variables $b+c$seem to represent numbers – accordingly, he can see d as an additional number. Thus, he gives relevance to the individual variables in the subexpression $(b+c)$. At the same time, through the lens of this subexpression, he brings the variable d in relation to his basic structure underlying $ab+ac=a(b+c)$. This bridges the gap between the expressions $ab+ac$ and $ab+ac+ad$.

The cognitive process, which guides the manipulation of the expression, might be characterized with two features. Firstly, Andrew is giving relevance to one part of his basic structure of the rule $ab+ac=a(b+c)$, namely the expression $(b+c)$ and the variables ("numbers") in it. This allows Andrew, secondly, to build analogies between $ab+ac+ad$ and the rule $ab+ac=a(b+c)$, perhaps through the lens of the subexpression $(b+c)$ and relating d to this subexpression. This expands the basic structures that are available to him. Andrew's cognitive activity is thus called "building analogies by focusing on a subexpression".

CONCLUSION AND DISCUSSION

In this paper, a model for analysing the cognitive activities involved in the rule based manipulation of simple algebraic expressions is suggested and used to analyse a case study. Data from the first design iteration suggests different cognitive processes, which can be characterised by their underlying processes of recognising basic structures and giving relevance (Table 1).

The data from this first experiment does not cover, whether the found cognitive activities are generalizable; that is, if these cognitive activities are also employed in cases where students are confronted with more complex algebraic expressions. In further iterations of the here presented design experiments, the above shown model is going to be applied to design supportive means for manipulating more complex alge¬braic expressions. The here presented model of cognitive activities allows improving on future design experiments: the below illustrated cognitive activities (Table 1) can now be specifically initiated by supporting their underlying cognitive processes.

Basic structure	Giving relevance	Cognitive activity
Declarative content of rule is embedded	Giving relevance to subexpressions and their features (“not equal”)	Interpreting declarative content
a(b+c+d) reconstructed through *a(b+c)=ab+ac*	Giving relevance to one subexpression and its composition (“numbers”)	Building analogies by focusing on a subexpression

Table 1: Example of cognitive activities in the manipulation of algebraic expressions.

More generally, the here suggested framework has proven successful in gaining insight into the nature of students’ cognitive activities of manipulating algebraic expressions. In the here discussed focus task, the students employ three cognitive activities to manipulate algebraic expressions. In spite of the simple algebraic expression used in this study and their ‘simple’ structure, rather complex cognitive processes were identifiable. This might suggest, that existing models of structure sense might not have allowed grasping the students’ activities of manipulating expressions in such detail, because of their sole focus on the structure of expressions.

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