INVENTION OF NEW STATEMENTS FOR COUNTEREXAMPLES

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From a fallibilist perspective, mathematics gradually develops with problems, conjectures, proofs, and refutations. To attain such authentic mathematical learning, it is important to intentionally treat refutation in mathematics classrooms, such as facing or proposing counterexamples and coping with them. In particular, analysing students' behaviour in response to counterexamples can lead to a design of teaching materials and instruction based on students' existing knowledge and strategies. In this paper, we construct a framework for capturing students' actions of inventing a new statement that holds for counterexamples to an original statement. We then illustrate a specific aspect of this framework with an episode that took place in an eighth grade classroom, and discuss two approaches to deductively generating a new statement.

INTRODUCTION

According to Lakatos (1976), mathematics progresses through the consideration of conjectures, proofs, and refutations, not just by monotonously increasing the number of indubitably established theorems. To introduce this authentic process in mathematics classrooms (Lampert, 1990), it is essential to deal with not only proving that a statement is true, but also refuting a conjecture by counterexamples, restricting the domain of the conjecture to exclude the counterexamples, and inventing a new statement to account for the counterexamples. In particular, it is fundamental to construct frameworks of analysis for students' behaviour in response to counterexamples, because these frameworks will enable mathematics teachers and educators to deepen their understanding of students' thought processes; such understanding may provide insights into a more effective design of teaching materials and instruction based on students' existing knowledge and strategies.

There are at least two research strands on students' behaviour related to counterexamples. The first centres on the production of counterexamples; researchers have investigated whether students and teachers can produce a proper counterexample to show that a statement is false, how they generate counterexamples, and what types of counterexamples they create (e.g. Hoyles & Küchemann, 2002; Peled & Zaslavsky, 1997; Weber, 2009). The second strand of research focuses on the recipients of counterexamples (Zazkis & Chernoff, 2008). In particular, some researchers utilise the mathematical actions shown in *Proofs and Refutations* (Lakatos, 1976) to analyse how students respond to counterexamples (Balacheff, 1991; Reid, 2002; Yim, Song & Kim, 2008). For instance, Larsen and Zandieh (2008) construct a framework that consists of "monster barring", "exception barring", and "proof analysis" (lemma incorporation), and they describe an undergraduate classroom episode to argue that this framework can serve as a description and explanation of students' mathematical activity. This paper intends to contribute to this second strand of research.

However, most researchers of the latter strand have focused on students' behaviour to exclude counterexamples, and they have not dealt with the invention of a new statement that holds for the counterexamples. In fact, monster barring, exception barring, and lemma incorporation were formulated as methods for excluding counterexamples in Lakatos (1976). Although Balacheff (1991) shows that some students created new conjectures to account for counterexamples to their initial conjectures, he summarises various student responses as "modification of conjectures" and does not examine in detail how the students modified the conjectures or what relationships the modified conjectures had with the original ones. It is valuable to focus on the invention of a new statement for counterexamples because this invention can be regarded as a brave attempt to explain the counterexamples rather than disregard them.

Consequently, this paper has two research purposes. First, we construct a framework for capturing students' action to invent a new statement that holds for the counterexamples to an original statement. Second, we illustrate a specific aspect of this framework by describing an episode that took place in an eighth grade classroom, and discuss two approaches to deductively generating a new statement.

THEORETICAL FRAMEWORK

Lakatos (1976) referred to the invention of new conjectures to account for the counterexamples to a primitive conjecture, though the description has not been sufficiently considered in mathematics education research. It was mentioned as "increasing content by deductive guessing", which means the deductive invention of a more general conjecture that holds even for the previous counterexamples (Lakatos, 1976, p. 76). Komatsu (2011) demonstrates that Lakatos's notion of increasing content by deductive guessing is useful for describing certain behaviour by ninth grade students.

However, it may not be appropriate to directly introduce this notion for describing students' behaviour in general because Lakatos's main interest lay in describing a process of growth in the discipline of mathematics, and there are differences between mathematicians' and students' behaviour. In addition, Lakatos seemed to think that his heuristic rules, which included increasing content by deductive guessing, were not universal or obligatory (Kiss, 2006). Therefore, in the following, we examine alternatives to increasing content by deductive guessing to construct a framework for capturing students' invention of a new statement that holds for previously given counterexamples.

There are two characteristics of increasing content by deductive guessing. The first is related to 'increasing content', that is, the product of invention. As mentioned earlier, increasing content by deductive guessing refers to inventing a general conjecture that holds even for the counterexamples to the previous conjecture. Therefore, the new generated conjecture is more general than the previous one in that it includes the counterexamples to the previous conjecture as its examples. However, there may be another case in which even if students can produce a statement for counterexamples to an original statement, the original statement and the produced statement do not always have such a particular-general relationship. In other words, the students may generate a statement separated from the original one, and these two statements may be regarded as just case analysis (see the following sections for an example).

The second characteristic is related to 'by deductive guessing', that is, the approach to creating a new conjecture. When Lakatos mentioned increasing content by deductive guessing, he seemed to consider the deductive invention of conjectures that were difficult to find through empirical or perceptual approaches (Lakatos, 1976, p. 82). However, there are types of mathematical reasoning other than deduction, such as induction and analogy. Therefore, it is expected that students may generate a new statement for previous counterexamples in non-deductive ways, such as through inductive, perceptual, analogical, and ad-hoc methods.

From the above, it is possible to construct a framework as shown in Table 1 for capturing students' actions to invent a new statement that holds for counterexamples to an original statement. Regarding the horizontal structure of this framework, a particular-general relationship is more desirable than a case-analysis relationship because the former can unify an original statement and its counterexamples under a new statement, without separating them (Nakajima, 1982). Although the vertical direction does not have this desirable structure, a deductive approach may be more difficult for students than a non-deductive approach. In addition, the vertical structure of this framework is relevant to the functions of proof (De Villiers, 1990). A deductive approach involves the discovery function of proof, especially if students use the proof of an original statement to generate a new statement. On the other hand, the verification and explanatory functions of proof are relevant to a non-deductive approach if students produce a statement in a non-deductive way and then prove it.

Invention approach	Relationship between original and new statements	
	Case analysis	Particular-general
Non-deductive	Type I	Type II
Deductive	Type III	Type IV

Table 1: A framework for invention of new statements to account for counterexamples

Lakatos's notion of increasing content by deductive guessing corresponds to Type IV in this framework, and this framework implies three possibilities of students' behaviour other than increasing content by deductive guessing. Nevertheless, this framework is derived from purely theoretical considerations, and it therefore needs empirical support, which we describe in the following sections.

METHODS

The classroom episode examined in this paper is taken from our larger study that aims to develop, through design experiments, a set of tasks and associated teachers' guidance to foster student engagement in proofs and refutations (Komatsu & Tsujiyama, 2013). We selected this episode because it is suitable for one of the purposes of this paper, that is, illustration of the framework in Table 1.

The second author, who has over 10 years of experience teaching in secondary schools, carried out a teaching experiment that consisted of two lessons (50 minutes per lesson) with 36 Japanese eighth graders (13–14 years old). He was not familiar with the above framework, but he took an active role in the lessons, encouraging the students to think of counterexamples and challenging the students' thinking. Both authors were involved in the lesson design, and the first author observed all the lessons.

On average, the students' mathematical abilities were above standard. They could prove geometric statements related to various properties of triangles and quadrilaterals, using conditions for congruent triangles, and had learnt counterexamples as well.

All the lessons were recorded and transcribed. The data for analysis included these transcripts, the students' worksheets, and field notes taken during the lessons. We analysed the data with a focus on the students' behaviour after proof construction, in particular, how they invented new statements to account for counterexamples to the original statement. We translated the problem sentences, the students' words and proofs from Japanese to English. All the students' names used here are pseudonyms.

RESULTS

Original statement and its proof

We used the problem shown in Figure 1 in our teaching experiment because it enables students to find counterexamples to the statement PQ = DQ – BP, as described later.

As shown in the right figure, we draw line l that passes point A of square ABCD, and perpendicular lines BP and DQ to line l from points B and D, respectively. Prove that PQ = DQ – BP.

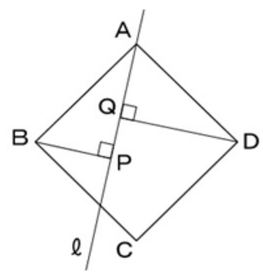

Figure 1: The problem in the lesson

The teacher presented this problem at the start of the first lesson. We describe only briefly how the students proved the statement, because the focus of this paper is on their processes after proof construction. After discussing a plan for solving the problem, the students worked individually. Next, the teacher had a student, Emi, write her proof on the blackboard. Her proof was examined in a classroom discussion, which revealed that the part which showed the congruence of angles ABP and DAQ was complicated for the other students. The teacher therefore had Mai give a complementary explanation with a different expression (Figure 2).

Proof by Emi and Mai:
In ΔABP and ΔDAQ,
From the supposition, $\angle$APB = $\angle$DQA = 90°
Since quadrilateral ABCD is a square, AB = DA
Let $\angle$BAP = a
Since the sum of the interior angles of triangle ABP is 180 degrees,
$\angle$ABP = 180° – $\angle$APB – $\angle$BAP = 90° – a
Since an interior angle of a square is 90 degrees, $\angle$DAQ = 90° – $\angle$BAP = 90° – a
Therefore, $\angle$ABP = $\angle$DAQ
Since the hypotenuses and a pair of corresponding angles in right triangles are equal, ΔABP $\equiv$ ΔDAQ
Since the corresponding sides of congruent figures are equal, AP = DQ and BP = AQ
Therefore, PQ = AP – AQ = DQ – BP

Figure 2: The proof constructed by the students

Counterexamples and new statements

After this proof, the teacher asked, "Now, we drew line *l* which passed point A like this [Figure 1], but when the place of this line *l* is different from here [Figure 1], is it possible to say that this [PQ = DQ – BP] is true?" A few students responded "maybe impossible". Then, the teacher told the students, "Draw various lines, *l*, which pass point A and investigate by drawing your own diagrams". The first lesson finished with the students individually drawing diagrams on their worksheets.

Analysing their worksheets after the lesson, we found that many students drew diagrams similar to those shown in Figure 3 (these figures are examples of the students' actual drawings). In the case of Figure 3-a, the students wrote, "Segment BP becomes longer than segment DQ" or "DQ – BP becomes negative". For Figure 3-b, they wrote, "Segment PQ is longer than segments DQ and BP" or "[DQ – BP] becomes negative as well". Their worksheets evidenced that they grasped these cases as counterexamples refuting the statement in the original problem, PQ = DQ – BP.

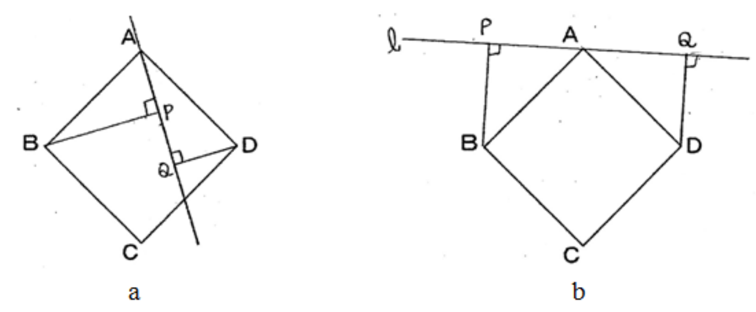

Figure 3: Counterexamples drawn by the students

In the second lesson, the students investigated what relationships among PQ, DQ, and BP held in the cases in Figure 3. At this point, the teacher told them they were allowed to utilise the previous proof by Emi and Mai (Figure 2).

After the students engaged in this investigation individually, the teacher had Manabu and Ken write their ideas on the blackboard. Regarding the case represented by Figure 3-b, Manabu wrote, "I prove the congruence of triangles ABP and DAQ as we did in the last lesson, and from PQ = AQ + PA, it should be true that PQ = BP + QD". Thus, Manabu deductively invented a new statement, PQ = BP + QD, for this case that had been a counterexample to the original statement, by utilising the congruence of triangles ABP and DAQ as a reason which, he thought, could be shown by the same proof as the previous one. Ken thought similarly to Manabu, writing his idea for Figure 3-a as follows: "[From the previous proof, I found DQ = AP and AQ = BP.] Since PQ = AQ – AP is true, the relationship among PQ, DQ, and BP is PQ = BP – DQ" (he wrote the square brackets on his worksheet, but not on the blackboard).

Next, the students examined whether the congruence of triangles ABP and DAQ could actually be shown by the same proof as Emi and Mai's one. For example, the teacher asked the students whether Emi and Mai's proof was directly applicable to the case shown in Figure 3-b. Daisuke answered that it was possible to apply this proof up to its part deducing AP = DQ and BP = AQ, and many students seemed to agree. Then, the teacher urged the students to inspect this applicability in more detail, and some students had doubts as to the part stating that since an interior angle of a square is 90 degrees, the degrees of angle DAQ are 90 – a (Figure 2). More concretely, Satoshi stated, "Because both angles DAQ and BAP are not inside it [angle BAD], I think it is not true". After that, other students added that it was enough to use the degrees of angle PAQ (180 degrees) to prove that the degrees of angle DAQ are 90 – a.

DISCUSSION

In this episode, the students proved the original statement (Figures 1 and 2) and then faced counterexamples that refuted it (Figure 3). In response, they produced new statements, that is, PQ = BP – DQ for the case as Figure 3-a, and PQ = BP + QD for the case as Figure 3-b. The original statement and these new statements written for the counterexamples do not have a particular-general relationship, and they are regarded as case analysis according to the position of line l. In theory, it is possible to generate a general statement that holds for all cases if we represent PQ as the absolute value of the sum of vectors BP and DQ (Shimizu, 1981). However, the students in this episode had not learnt vector, and it was impossible for them to consider such a generalisation.

The students invented the new statements for the counterexamples (Figure 3) in deductive ways, such as utilising a part of the previous proof as a reason for their thinking or constructing new deductive arguments. In addition to Manabu and Ken, Yuko wrote on her worksheet that "I had thought PQ = DQ – BP [in the case of Figure 3-a], similar [to the case shown in Figure 1], because the right and left were only reversed, but *I found [PQ = DQ – BP was] not true through copying [the previous] proof*". Toru also wrote that "In *the process of making proofs, I gradually understood* that I could represent the relationships between PQ, DQ, and BP by using + and –" (our emphases). In summary, these students' behaviour corresponded to Type III in the framework shown in Table 1.

This episode implies a possibility of dividing a deductive approach to invention of a new statement into at least two categories. The students in this episode thought that the congruence of triangles ABP and DAQ, which had been shown by Emi and Mai for the original statement, held for the counterexample shown in Figure 3-b as well, and they used this congruence as a reason to produce a new statement, PQ = BP + QD. At that point, they did not examine this congruence in detail, such as by considering whether the previous proof by Emi and Mai was directly applicable to the case as Figure 3-b. These students' behaviour can be regarded as *modularly deductive* in the sense that they thought a certain encapsulated part was true and invented the new statement by utilising this part as a reason for their thinking.

Considering an alternative to a modularly deductive approach, it is possible to think up a *sequentially deductive* approach that refers to confirming, from the beginning, that each detailed point is true and piling these points step by step to invent a new statement. This approach is only a research hypothesis because we could not directly capture the relevant students' behaviour in the episode reported in this paper. However, the relevant process can be seen in Lakatos (1976), which dealt with the Descartes-Euler conjecture on polyhedra, expressed as V – E + F = 2, where V, E, and F are the numbers of vertices, edges, and faces of polyhedra, respectively. In this literature, an imaginary teacher and students sequentially constructed polygons and polyhedra by marking points, connecting them, and pasting polyhedra whose values of V – E + F were already known. They then examined each increase and decrease in the numbers of vertices, edges, and faces to invent a more general conjecture than the above conjecture. In the future, it will be necessary to investigate whether a sequentially deductive approach can be observed in actual students' activity.

Another future task should explore the characteristics of a modularly or sequentially deductive approach. In the episode reported here, the students who took a modularly deductive approach first believed that part of the previous proof by Emi and Mai, up to deducing AP = DQ and BP = AQ, was directly applicable to the case shown in Figure 3-b. After that, when the teacher urged the students to inspect this applicability in more detail, they could realise the necessity of modifying the part showing that the degrees of angle DAQ are 90 – a. In addition to such a pitfall, it will be valuable to investigate the advantages of each approach.

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