HOW THE KNOWLEDGE OF ALGEBRAIC OPERATION RELATES TO PROSPECTIVE TEACHERS' TEACHING COMPETENCY: AN EXAMPLE OF TEACHING THE TOPIC OF SQUARE ROOT

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This study is the part of a larger study on investigating Hong Kong (HK) prospective teachers' (PTs) Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). In this paper, five HK PTs' SMK and PCK on teaching one topic regarding square root were investigated. The results suggest that insufficient understanding on the concept of algebraic operation is the major obstacle to limit those student teachers from teaching students with mathematical understanding. The results further echo with the viewpoint that SMK and PCK are two interrelated constructs and rich SMK leads to high quality of PCK.

BACKGROUND INFORMATION

Results from international comparative studies such as TIMSS and PISA indicate that students from East Asian regions (including mainland China, Hong Kong, Taiwan, and Singapore) outperform their Western counterparts (e.g., Mullis, et al, 2008; OCED, 2013). Educational professionals believe that the "curriculum gap" is not the sole explanation for the performance discrepancies between West and East, and that the "preparation gap" of teachers, as confirmed by the results of IEA-study Teacher Education and Development Study in Mathematics (TEDS-M) (Tatto et al., 2012), is a fundamental concern. The results from TEDS-M study showed that potential mathematics teachers from two participating East Asian regions – Taiwan and Singapore – ranked the top in their achievement in both CK and PCK assessments among other participating countries. It is intuitively believed that the good performance in such international assessment exercises would be the consequence of well-equipped and competent teachers in the two regions. However, less is known about the reasons behind this relationship, and more explorations on other East Asian regions might help. In this study, we aim to contribute to the current knowledge by studying a group of HK PTs' teaching knowledge. HK had undergone substantial educational reform at the turn of the millennium, which requires a paradigm shift of teachers' teaching from teachers-centered to students-centered; therefore the extent to which HK PTs are ready to deliver such kind of effective mathematics teaching becomes a crucial issue.

In his most cited article, Shulman (1986) set out the multi-dimensional nature of teachers' professional knowledge. He identified, among other dimensions, three

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aspects of professional knowledge: pedagogical knowledge, subject knowledge, and PCK. In the subject of mathematics, this knowledge is further conceptualized by Ball and her colleagues and categorized into two domains: mathematical subject matter knowledge (SMK) and PCK. In their mathematics knowledge for teaching (MKT) model (Hill, et. al., 2008, p.377), PCK and SMK are treated as two separate components. PCK includes knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of curriculum, yet all three constructs under PCK connect with content knowledge in various ways. Indeed, the relationship between PCK and SMK is very vague. Despite that some studies separated the constructs of PCK and SMK empirically, a deep connection between the two constructs was found (e.g., Krauss, Baumert, & Blum, 2008). The impacts of SMK on PCK have been explored by scholars, in particular, which types of SMK can equip mathematics teachers for effective teaching are attractive. For example, Even (1993) studied the SMK of pre-service secondary mathematics teachers from the U.S. and its interrelations with PCK in the context of teaching the concept of functions. The study showed that insufficient SMK might lead pre-service teachers to adopt teaching strategies that emphasize procedural mastery rather than conceptual understanding. By comparing the US and Chinese mathematics teachers' teaching competency, Ma's (1999) found that Chinese teachers possessed profound understanding of fundamental mathematics (PUFM), that their American counterparts lack, facilitates them to conduct more effective teaching, Ball and her associates also included both common content knowledge (CCK) and specialized content knowledge (SCK) into their construct of SMK. In particular, they defined SCK, different from CCK, "that allows teachers to engage in particular teaching tasks, including how to accurately represent mathematical ideas, provide explanations of common rules and procedures, and examine and understand unusual solution methods to problem" (Ball, et al., 2005).

In a recent paper, Buchholtz et al. (2013) reemphasize and highlight Felix Klein's ideas "*elementary mathematics from an advanced standpoint*" (EMFAS) as another important category of teachers' professional knowledge. The results gained from their international comparative study indicate that the future mathematics teachers from top mathematics performing countries including HK still have the problems in linking school mathematics and university knowledge systematically. The construct of EMFAS looks different from SCK by definition. The former is stated as more mathematical, and the latter one is mathematics knowledge applied for teaching. However, we make the hypothesis that EMFAS and SCK should share some similarities in content, both of them lead to conceptual-understanding oriented mathematics teaching.

In this paper, we investigated HK PTs' PCK and SMK on one topic in lower secondary school algebra namely square root as one example. The fundamental to the learning of mathematics is the process of learning the abstraction (Mitchelmore and White, 2000). Therefore, when concerning the richness of their SMK and PCK in teaching this topic, we focus on if their SMK and PCK can facilitate them to teach students with algebraic abstraction. Specifically, the research questions we aim to answer in current paper are:

- What are the SMK and PCK that the HK PTs have for teaching the topic of square root?
- How PTs' algebraic thinking relates to the quality of PCK in teaching this topic?

METHODS

What is presented here is a portion of a larger project in which two groups of future secondary mathematics teachers in HK participated. They are either in the third or fourth year of their study towards a bachelor of Education (BEd) majoring in mathematics, or, during full or part time study in the program of postgraduate diploma in education (PGDE) in mathematics. The whole project comprises both quantitative and qualitative data collection, the former being a questionnaire tapping PTs' beliefs of the nature of mathematics and mathematics teaching and mathematics knowledge. Base on the results of this phase, five participants were selected to take part in the second phase which constitutes an interview aiming at capturing the PTs' PCK and SMK in teaching three topics. They were given the pseudonyms of Jack, Fanny, Mandy, Gary and Charles. In the second phase, video-based interview was employed. It was taken a TIMSS 1999 Hong Kong video which constitutes a 40 minutes lesson of Grade 8 class. During the interview, both researcher and interviewee sat next to each other. The researcher controlled the play of video and asked questions where appropriate. The interviewee watched video and sometimes wrote their responses on the whiteboard. The whole process was video-taped. The interview questions basing on Ball et al. (2008) MKT framework (with incorporation of EMFAS), are depicted in Table 1.

Figure 1: The context of video-based interview

		Interview questions	The context of video
PCK	KCT	What are your comments on this teacher's approach on how to introduce the topic of square root? If you were the teacher, how would you do?	The teacher in the TIMSS video told the students to find a number that, after multiplication of two identical numbers to give the resulting number. In general, the process of getting a square root as introduced by this teacher as a simple multiplication procedure.
	KCT	How to teach your students to find out the square roots of 9, i.e., $(a)^2$=9	The teacher in the TIMSS video put the focus on emphasizing the square roots of 9 could be either positive or negative, as for the value of square root is negative or positive, this can be judged by the sign in front of the

	KCS KCT	What's the thinking behind the student when student treat "a" as a positive number in writing the expression: $(-a)^2$=9 for figuring out a negative value for the square root of a? (KCS); If you were the teacher, how would you respond to this student? (KCT)	surd. The teacher in the TIMSS video wanted to enlighten student to find out negative square root of 9, so she wrote down a question: what is the negative square root of 9? One male student was invited to solve this problem on the blackboard. He immediately wrote down the expression: $(-a)^2$=9, thought for a while but could not find the answer. The teacher suggested him wipe out the negative sign in front of a.
SMK	CCK SCK (EMFAS)	What are your comments on this student's solution: $\sqrt{(-4)^2}$ = $(-4)^{2\times\frac{1}{2}}$= $(-4)^1$=-4? Is it correct or not? Please provide reasons to support your answere from the mathematical point of view?	In the textbook utilized in this video lesson, one exercise was to ask students think about if it is true that $\sqrt{(-4)^2} = -4$. To investigate the depth of student teacher's SMK, one hypothetical scenario was posted: One student demonstrated $\sqrt{(-4)^2}$= -4 is true, because $\sqrt{(-4)^2}$ = $(-4)^{2\times\frac{1}{2}}$= $(-4)^1$=-4

Table 1: Interview questions and how they correspond to PCK categories

FINDINGS

Some preliminary findings and analysis based on the HK PTs' responses to part of PCK and SMK illustrated in Table 1 will be presented.

KCT – Introduction the topic of square root

All informants tended to introduce the topic of square root by making a connection with the topic of square. There are two approaches of making this connection. The first approach is to start with introducing square numbers such as 4, 9, 16 and 25, for example, Mandy suggested to ask students,

> What is the square of 3? What is the square of 4? Thinking of 1^2, 2^2, 3^2… (Mandy)

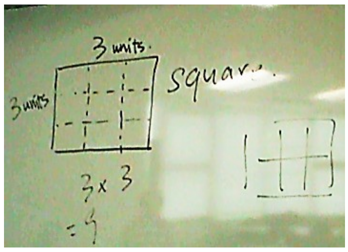

Figure 2: Fanny's picture to explain why 3^2 is not equal to 3×3

The second approach is to introduce the relationship between the area and the side of a square. Some student teachers tended to emphasize the notation of square and square root, and illustrated the concept of notations by a square image either explicitly or implicitly. For example, the picture presented below by Fanny demonstrates that 3 is the length of a side of the square with the area of 9. At the meanwhile, Fanny tries to help students to distinguish the concept of 3^2

from 3×2 (see Fig.2).

Not only Fanny, but other two student teachers Mandy and Jack mentioned explicitly that students might be confused the concept of the square of 3 with that 3 multiplies by 2. For example, Mandy justified the reason why she adopted the approach of introducing square numbers to students is to strengthen students' impression on the meaning of square, that refers to multiply by itself but not multiplying by 2.

KCT – Introduction of the notion "a^2= 9"

As for how to explain the equation: $a^2 = 9$, which is related to teach how to introduce the students about positive and negative square roots. The major approach that student teachers introduce this idea emphasizes the procedures. They highlighted the term "self -multiplication", and suggested to introduce students the concept that positive, positive turns out to be positive, and negative, negative turns out to be negative. Charles' approach is typical among other student teachers, since he wrote down 3×3=9 and (−3) × (−3) =9. However, not any visual representations were used by those student teachers to explain how to solve problems. Even through Gary was able to draw a picture to illustrate that "a" refers to the length of the side of a square whose area is 9, but he failed to use this similar image to explain why -3 is the square root of 9.

> Er… I might think of drawing a square. Three…three [is nine]…but I don't have ideas on how to draw the square with [the side] as negative 3? (Gary)

KCS and KCT – Reaction to a misconception

The student wrote down the negative sign in front of "a" in expression $(-a)^2$ =9 when he tried to solve a problem – what the negative value for square root of 9 is, what is the thinking behind him? The informants came up with two types of interpretations. One interpretation endorsed by three PTs – Charles, Fanny and Gary – is that the student is misled by the information "negative value". For instance,

> Em…he [the student] might not think that the value of this unknown number could be either positive or negative. He probably thought, taken it for granted, that a must be a positive number… because it is an unknown number, so the unknown number could be positive or negative? … but he did not think of the possibility that this number could be negative number. (Gary)

The second interpretation is endorsed by two PTs- Jack and Mandy and they attributed it as a piece of student's incorrectness.

> Well, in fact what he was thinking at that moment was what he had thought was totally wrong. I think he was empty in his mind. (Jack)

To respond this piece of student's thinking, the majority of informants just criticized this teacher's suggestion – ask the student to wipe off the negative sign. For example,

> The teacher should let him (the student) continue. In fact he (this student) was able to write this, why don't we let him finish it? That is … I think the teacher just wanted the student to write the equation…but I think that student have the whole plan in his mind,..., so we can

talk about what was in his mind and helped him to clarify the misconceptions in terms of format. (Jack)

Other methods were regarding how to stimulate this student to think of "a" could be either a positive or a negative number. For example, Mandy tried to provide the student some hints,

How about the number a is -3? How about the square of (-3)? (Mandy)

CK – Mathematical explanations for why $\sqrt{(-4)^2} = -4$ is not true

All PTs can make a correct judgement that $\sqrt{(-4)^2} = -4$ is not true. The knowledge they apply is CCK, i.e., the radicand is non-positive so $\sqrt{(-4)^2}$ is not equal to -4.They also commented that there must be wrong in some steps in the expression: $\sqrt{(-4)^2} = (-4)^{2\times\frac{1}{2}} = (-4)^1 = -4$, however, no PTs was able to point out the mathematical reasons for why this method did not work. Some thought that it should solve $(-4)^2$ first, and then deal with $16^{\frac{1}{2}}$because the order of calculating matters.

DISCUSSION

The analysis of the five HK PTs' PCK in teaching this current topic shows that they adopted a procedural and a purely calculating approach to teach students square root. As evidenced in their approaches of explaining to students that $a^2 = 9$, the most of them try to explain in the way that 3 times 3 equals 9, and negative 3 times negative 3 equals 9, however, this approach cannot help students with algebraic thinking, since teaching them to substitute numbers 3 and -3 is kind of trial and error, yet nothing related to the generalization of patterns. Similarly, in responding students' question- adding one negative sign in front of a, the only approach that those student teachers employed was to provide the hints that, "-3 is the square root of 9" in order to emphasize that "a" could be either positive or negative. Their response to the KCS question demonstrated that those student teachers tended to interpret students' confusions from literal understanding; for some student teachers, it is even worse, they attributed it as students' incorrectness or lack of mind. Some evidences show that those PTs embrace students' previous knowledge in learning the topic square, that is, how to interpret the operational meaning of superscript 2 in 3^2, yet this is nothing to do with the content of square root.

The results from an analysis of those PTs' SMK show that they could have sufficient CCK in making judgment, yet the failure to answer student's enquiry why $\sqrt{(-4)^2} \neq -4$ reflects the weakness in their SCK in teaching this topic. It relates to their insufficient understanding of $\sqrt{\ }$, weak knowledge in how to apply index law and composite function. In the case: $\sqrt{(-4)^2} = (-4)^{2\times\frac{1}{2}} = (-4)^1 = -4$, those PTs seemed to overlook the fact that the index law cannot apply in the case $\sqrt{(\text{negative number})^2} = \left(\sqrt{\text{negative number}}\right)^2$, because of the properties of composite functions.

Here, $f: x \mapsto x^2$, and $f^{-1}: x \mapsto \sqrt{x}$, ideally by the use the concept of composite function, we have $f \circ f^{-1}(x) = f[f^{-1}(x)] = x$, but making x, $f \circ f^{-1}(x)$ and $f[f^{-1}(x)]$ equal only if $f^{-1}(x)$ is well defined. However, in this case $f^{-1}(x) = \sqrt{x}$ is undefined when $x < 0$.

The lack of adequate SMK especially SCK could be the major reason to explain why their PCK in teaching this algebraic topic is procedural. Algebraic thinking involves the understanding of roles and properties of variables, and relevant operations among those variables. Learning algebra we often go from a less abstract state to a more abstract state. Mitchelmore and White (2000) identified the learning stages in term of the intensity of abstraction, namely *familiarization, similarity recognition, rectification and application*. In this current case, knowing computationally that the square of the number 3 or -3 is 9 learners only reach the *familiarization* and *similarity recognition* levels. While, the rectification level is only reached when learners identify that we can only take the positive sign when taking square root of a number. It is because we treat *squaring-taking square root* as a pair of function and its inverse. Knowing what constraint is in there when writing $f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$ will be in the level of *application* because the domain of the inverse function can only be applied to positive real numbers. However, when PTs' levels of abstraction cannot reach in rectification and application, how could they possess high quality of PCK that facilitates students to develop algebraic thinking?

CONCLUSION

In spite of the limitations, this study highlights the role of SMK especially SCK or EMFAS plays a significant role in those HK PTs' PCK in teaching the topic of square and square root. Consistent with the study conducted by Buchholtz et al. (2013), the results gained from current study show that those HK PTs could not connect relevant university mathematics with current topic. Lack of adequate knowledge of algebraic operation and functions leads those student teachers teach this algebraic topic in a procedural way. In addition, this study provided another perspective to evaluate the quality of PCK from the perspective of SCK and EMFAS. We hence rethink of the construct of PCK, which cannot be apart from CK especially SCK or EMFAS, which is more important than CCK in facilitating mathematics teachers to teach students with more conceptual understanding.

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