CHARACTERISTICS OF UNIVERSITY MATHEMATICS TEACHING: USE OF GENERIC EXAMPLES IN TUTORING

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The aim of this paper is to report early findings from university mathematics teaching in the tutorial setting. The study addresses characteristics of the teaching of an experienced research mathematician and through interviews, her underlying considerations. An analysis of a teaching episode illuminates her use of generic examples to reveal aspects of a mathematical concept and links with the tutor's particular research practice, didactics and pedagogy emerge.

INTRODUCTION

Research on university teaching practice can inform mathematics education community's understanding of university mathematics teaching and produce resources that novice and experienced university teachers might access for professional development (Speer, Smith & Horvath, 2010). However, research regarding pedagogy and mathematics puts an emphasis on school mathematics teaching (Jaworski, 2003) and very little has been studied to date concerning the teaching practices and knowledge of university mathematics teachers (Speer & Wagner, 2009). Furthermore, the number of studies about teaching in the small group tutorial setting is still limited.

Petropoulou, Potari and Zachariades (2011, 2012) linked teaching practices in the format of lectures with research, teaching and studying experiences and argued that the process of thinking in mathematical research is used in university teaching (Petropoulou et al., 2011). The focus in this paper is on the characteristics emerging from one tutor's mathematics teaching and how they are linked with the sources of knowledge coming from her research practices and teaching (including epistemology, didactics and pedagogy). The aim of the wider study, on which this paper is based, is to produce understanding about mathematics teaching at university level; the setting of small group tutorials was selected since more opportunities of teacher-student interaction and dialogue emerge there.

THEORETICAL BACKGROUND

A systematic literature review from Speer et al. (2010) categorised published scholarship in university mathematics teaching and showed lack of research in actual university mathematics teaching practice. In particular, these authors report that most of the studies offer researchers' reflections on their own mathematics teaching and accounts of students' learning, and they insist that there is no systematic data collection and analysis focusing on teachers and teaching. They also make the distinction between teaching practice and instructional activities at university level, defining the

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former as teacher's judgements, thinking and decision making for the design, implementation and reflection on their teaching. Small group tutorials are seen as instructional activities along with lectures, whole class discussions, student's individual work on exercises and many other activity structures.

University mathematics education research is rapidly developing. A research focus has been the teaching of particular topics in undergraduate mathematics such as mathematical analysis (e.g. Petropoulou et al., 2011, 2012; Rowland, 2009) and linear algebra (e.g. Jaworski, Treffert-Thomas & Bartsch, 2009). The above studies are on the teaching of a large number of students in a lecture format; however, another focus of research is the teaching and learning of mathematics in alternative settings such as small group tutorials (e.g. Jaworski, 2003; Nardi, Jaworski & Hegedus, 2005; Nardi, 1996). In the context of small group tutorials, Jaworski (2003) investigated first year mathematics tutoring of six tutors. She distinguished tutors' exposition patterns as the main teaching aspect, with the most prevalent ones to accord with *tutor explanation*, *tutor as expert* and forms of *tutor questioning* and stressed that the teaching-learning interface is idiosyncratic to the tutor and to some degree to the particular students. Nardi et al. (2005) studied tutor's thinking processes collecting their interpretations of incidents from their teaching in small group tutorials concerning three strands: *tutor's conceptualizations of students' difficulties, tutor's descriptive accounts of pedagogical aims and practices with regard to these difficulties* and *tutor's self-reflective accounts with regard to these practices*. They produced a spectrum of tutor's pedagogical awareness with four dimensions namely *Naive and Dismissive, Intuitive and Questioning, Reflective and Analytic* and *Confident and Articulate* and indicated a development in tutor's readiness to respond in self-reflection questions over time. Nardi (1996) conducted her PhD research in undergraduate tutorials and explored the learning difficulties that first year students experienced in mathematics. Subsequently, Nardi (2008) investigated mathematicians' perceptions of their students' learning and reflections on their teaching practices based on data rooted in their small group tutorials and students' work. In the above studies, the tutor's teaching practice is examined in accordance with his or her students' learning outcomes and difficulties. In our study, we are interested in the tutor's teaching practice as well as the sources of knowledge that frame it. In this paper, we focus on the tutor's teaching on features of mathematical concepts by using generic examples.

Petropoulou et al. (2011) introduced the idea of an example used to illustrate critical characteristics of concepts as one of the lecturer's strategies to construct mathematical meaning in lectures. Our interpretation is that the use of an example to illustrate critical characteristics of concepts is what other researchers call a generic example. A generic example is an example that is presented so as to carry the genericity ("the carrier of the general") inherently (Mason & Pimm, 1984, p. 287). In other words, the general (argument) is embedded in the generic example "endeavoring to facilitate the identification and transfer of paradigm-yet-arbitrary values and structural invariants within it" (Rowland, 2002, p. 176). An example-of-a-generic-example that Rowland

(2002) routinely chooses for the introduction of the notion “generic example” is the calculation of the sum from 1 to 100 with Gauss’ method. Gauss added 1 to 100, 2 to 99 and, so on, and computed fifty 101s. The genericity of his method is that it can be generalised to find the sum of the first *2k* positive integers, which is *k*(*2k+1*). The sum from 1 to 100 is a generic example of Gauss’ method and as such it is “a characteristic representative of the class” (Balacheff, 1988, p.219, cited in Rowland 2002) of the sum of the first *2k* positive integers. Nardi et al. (2005) reported that the use of generic examples was amongst the most discussed strategies that tutors used to enhance their students’ concept image in tutorials.

METHODOLOGY

The context of the study

The study is being conducted in small group tutorials for first year mathematics students at an English University. Tutorials are 50 minute weekly sessions and a group include 5 to 8 students. Tutors are lecturers in modules offered by the mathematics department and conduct research in mathematics or mathematics education. The modules that are usually tutored are analysis and linear algebra, but tutors are sometimes flexible to provide assistance in other modules, as well. This study is part of a PhD project, which draws on data of tutorials of 26 tutors and data that systematically follow three out of the 26 tutors for more than one semester. Zenobia is one of the three tutors. She is an experienced lecturer, holds a doctorate in mathematics and does not prepare a design for her tutorial. Her tutees decide what questions or topics they all struggle with and usually select with her to deal with one or two relevant exercises from the problem sheets that follow each chapter in lectures. During the tutorial time and through the exercises, Zenobia put emphasis on concepts and mathematical thinking rather than computations.

Data collection and analysis

The first author observed, audio recorded and transcribed Zenobia’s small group tutorials. Observation notes were also kept and a discussion with Zenobia about each tutorial was audio recorded and transcribed. The discussions concerned characteristics related to Zenobia’s teaching practice and from her reflections we gained insight into her underlying considerations. The characteristics emerged through a grounded analytical approach of a small number of tutorials and were subsequently traced throughout the data. The following episode has been selected from a vast amount of data as a paradigmatic case that characterises the use of generic examples allowing us to reveal key issues in practice.

RESULTS

In Zenobia’s tutorials, characteristics were identified through the process of coding and categorisation. These involved the use of examples to practice algorithms before tackling proofs in a more abstract setting; graphs to provide a visual intuition for formal representations; repertoires of strategies and techniques for the work on mathematics; minimal information for elegant proofs; questioning in students' valid or invalid definitions and claims for conceptual understanding; counterexamples to refute invalid arguments; and generic examples to reveal features of mathematical concepts.

In this paper, we analyse a teaching episode from a small group tutorial, which was about calculus revision for exam preparation purposes. One of the exam questions was to show that a function is bijective and for this exercise, they chose to work first of all on injectivity. This episode concerns the use of generic examples to reveal that "a strictly monotonic function (in other terminology monotonically increasing or monotonically decreasing function) on an interval of its domain is injective on this interval", a property that can be used as an alternative to the definition to prove injectivity on an interval. Before showing injectivity for the exam question, Zenobia used these examples and the following discussion occurred:

Zen: Are there any kinds of functions that you know are going to be injective, for instance? Is there anything about a function that you… Ok. So, let's draw some functions on the board, shall we? So, here's an example of a function. [The graph of $f(x)=x^2$]. And here's another example of a function. [The graph of $f(x)=sin(x)$.] And here's an example of a function [the graph of $f(x)=ln(x)$], and here's an example of a function [the graph of $f(x)=x$]. So, if you wanted to determine some domains on which all of these are injective, how would you do it? How would you do it for this one? [Zen. points to the graph of $f(x)=x^2$]. How would you find your domain of injectivity? Is it injective on anything?

S1: From 0 to ∞. [Zen. draws a red line from 0 to ∞ to show the domain on which $f(x)=x^2$ is injective.]

Zen: Right. This is definitely not injective on the whole thing, right? Because if I go off in opposite directions, I'm going to the same thing. Ok. But if I go from here on, that's injective, right? Ok. And what about down here? [Zen. shows the graph of $f(x)=sin(x)$.] Do you want to have a go at that? You're very close. I know you can do it. Just draw a little red line on the domain axis.

S2: I hope I'm right. I think. [S2 draws a red line from 0 to ∞.]

Zen: You think? Ok. So, what does "injective" mean? It means that there shouldn't be any two points that are at the same height. No, that's definitely not right.

S2: Can you have two parts to the domain? [S2 draws a red line from -π to π.]

Zen: I guess you could, sure. You just do it. I mean, it's conventional to choose a connected interval, but you don't have to.

S2: It must be from here to here. [S2 draws a red line from –π/2 to π/2.]

Zen: Excellent. Good, good. Right. So, what did you notice? You noticed that you can't have it go up and down, basically.

Zen: So, what can I say about… Ok, what about this function? [Zen. shows the graph of $f(x)=ln(x)$.] Is this injective? Is this an injective function?

S3: Yeah. It is injective.

Zen: It is injective. What about this one? [Zen. shows the graph of $f(x)=x$.]

S3: Yeah.

Zen: Ok. So, what can you say about this part of this function, this part of this function, this function and this function? [Zen. shows the previous functions restricted on the domain of injectivity.] What do they all have in common?

S4: They're monotonically increasing.

Zen: They're monotonically increasing, right. So, a function that's either monotonically increasing or… I could easily have chosen this, instead. [Zen. plots the graph of $f(x)=log_a(x)$ where $0<\alpha<1$.] I could have chosen this part instead. [Zen. shows the graph of $f(x)=sin(x)$ restricted on [π/2, 3π/2].] So, either monotonically increasing or monotonically decreasing is automatically going to be injective.

Using a range of examples, which included very simple and more complicated ones, Zenobia attempted to build up students' awareness of the feature of monotonicity on an interval: "a strictly monotonic function is injective". Through these examples, she introduced layers of generality of monotonicity on an interval so that her students could connect monotonicity on intervals with injectivity. In my discussion with her, she referred to the four functions as "standard" ones meaning that they "have many applications" and "are very special classes of functions; polynomial, trigonometric and logarithmic functions". On lines 4-6, we see the first three examples she devised (the graphs of $f(x)=x^2$, $f(x)=sin(x)$ and $f(x)=ln(x)$) each of which is a generic example of the feature of monotonicity on an interval. All four functions [lines 4-7] have the property that makes them monotonic on an interval; however, they should have a high level of generality about them in order to be generic examples. The linear nature of the graph of $f(x)=x$ indicates that it is not a generic example, since not all strictly monotonic functions are linear [line 7]. However, this function along with the parabola fit in the class of polynomial functions and also, the linear function is odd and the parabola is even. The logarithmic function carries more generality of monotonicity on the domain than the linear function, because it has not null curvature and its domain is not the whole ℝ. The periodic trigonometric function can be divided into intervals where it is either monotonically increasing or monotonically decreasing. As Zenobia argues:

> Everything you see in polynomials [regarding monotonicity on intervals] is already seen in these two functions so adding any additional polynomial you don't get anything new, whereas you never see periodicity or natural domain less than a whole axis in polynomials.

In the interview, she also informed us about her didactical and pedagogical intentions and links between the particular epistemology, didactics and pedagogy emerged:

The particular epistemology

In Zenobia's discussion with the first author, reflecting on her teaching approach with generic examples, she related it to the research mathematicians' practice of "decoding and encoding". She drew on her research practice experiences and explained:

> The first step [in doing research] is the decoding where you are given a problem and you have to understand what the problem is, what everything mean [e.g. by experimenting with images against definition], why it is a problem; the second step is with this picture that you have got from the decoding process, you get some intuition, you play around with things in your head a little bit and then you get this sort of 'aha I figured it out, I have got this idea now of why that works' and then you have got the encoding process [i.e. the third step] where you write it down [formally]. […] [In this teaching episode,] through examples I tried to extract from the complicated language that core intuition [of step 2]. I tried to teach them to decode the problem to something where they can sort of see 'oh of course that's how it works' and then figure out how to write it in their proof back into a formal language.

Zenobia also related her inductive thinking approach to a way of producing mathematical definitions in research informed by the history of mathematics.

> It's a situation where –from the set of examples that we have– we've come up with an ideal idea, and then we can actually rigorously then check that something is in that or not. […] We come up with new definitions any time we recognise that there are some sets of structures that have some relevance. But it really does emerge out of the examples. *And if you look at the history of mathematics, it's not that people have had the idea of a function. It's that they've had lots of examples of functions and they've tried to distil what the critical characteristics of a function are.* So, I think it's a very natural way to think about the relationship between examples and theories – it's that we don't define definitions just off the tops of our heads. We define them because they capture a behaviour we see in examples that have interesting kinds of properties.

The didactics

The teaching episode provides an example where through the use of the four functions Zenobia applies in her teaching the process of decoding of what it means to be strictly monotonic and what it means to be injective. The three out of four functions are carefully selected so as to be generic examples of monotonicity on intervals. Then, the observation of the commonality among the graphs of the functions leads to the "core intuition" (of step 2 in research) that 'a strictly monotonic function is injective'. The encoding process is the writing of the proof that 'a strictly monotonic function is injective'; however, this step is not included in this episode.

An interesting detail is that the last excerpt, concerning the history of mathematics, fits into Zenobia's discussion with her students in the tutorial studied, after the solution of the exam question. In the italic text of this extract, she explained to students the historical roots of the use of generic examples to reveal features of concepts. From a didactical point of view, through this discussion Zenobia enculturated students into this mathematical practice of the community of mathematicians and gave them historical evidence of its significance. Discussing with her, she reflects:

> I wanted to explain to them what it is to be a mathematician. […] It is important for students to learn this [encoding and decoding] process because that's a lot of the process of doing mathematics. And I think that's a lot of what mathematicians do on a daily basis.

The pedagogy

The teaching episode is also an example of her teaching in practice towards her aims (e.g. students' enculturation into being mathematical). The decoding process and the "core intuition" are collective among the students since different students showed the intervals of injectivity [lines 11, 26, 31, 33] and noticed the common property [line 38]. Zenobia focused all students' attention on the key features of monotonicity on intervals by explanations to each student concerning his or her claim [lines 13-15, 27-28, 39-44]; rhetorical questions [lines 20, 27] and the question about the commonality of functions restricted on intervals where they are monotonically increasing [lines 34-37]. Focusing on her explanations to students during the decoding process [lines 13-15, 27-28], we extract her use of everyday language. In the interview, she stresses:

> At this [intuitive view] point I am not trying to make them phrase things in a mathematical language. I do that quite a bit when I am trying to get into the intuition first and I really don't want to burden it with technical vocabulary. I bring up the vocabulary later and by the end I really make them put things in a very strict mathematical formulation.

CONCLUSIONS

In this paper, we reported characteristics of tutorial teaching, which occurred in one tutor's small group tutorials. The teaching episode discussed was "a particular moment in the zoom of a lens" (Lerman, 2001, cited in Jaworski 2003) and illuminated one of these characteristics. Zooming in we provided an example of the characteristic "generic examples to reveal features of mathematical concepts" and the suggestion of a path of informing: from the history of the development of mathematics to the research practices of the mathematician to his or her didactics to his or her pedagogy. This path contributes to mathematics education community's understanding of university mathematics teaching and offers a way to lecturers to consider their teaching, with regard to their epistemology, didactics and pedagogy, in order to enculturate students into the core of mathematics and being mathematical. An example of how a tutor implements her teaching towards this aim is revealed through the teaching episode. Zooming out and from analyses as a whole, we suggest that teaching practices are informed by research practices. This influence accords with findings in the format of lectures (Petropoulou et al., 2011, 2012) and has implications to the pedagogy of the tutor. In future studies, we will analyse data from the other tutors and search for common or different characteristics in teaching and underlying considerations.

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