VISUAL AND ANALYTICAL STRATEGIES IN SPATIAL VISUALIZATION: PERSPECTIVES FROM BILATERAL SYMMETRY AND REFLECTION

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This case study contrasts the strategies used by two students in solving bilateral symmetry and reflection tasks, based on the differential properties they attended to. The ninth grader focussed on congruence of sides as the main property of reflection whereas the eighth grader focussed on perpendicularity and equi-distance, as is the normative procedure. The inadequate criteria for reflection shaped the ninth grader's actions and equally served to validate her solutions, although in a flawed fashion. The visual strategy took over as a fallback measure. We attend to some of the well-known constraints that students encounter in dealing with symmetry, particular situations involving slanted line of symmetry. Importantly, we made an attempt to show how visual and analytical strategies interact in the production of a reflected image.

INTRODUCTION

Understanding the ways in which visual and analytical strategies interact in the solution of mathematical problems has been one of the challenging questions for mathematics educators. Some steps have been taken to explain such an interaction. For instance, Hoyles and Healy (1997) showed how students attempted to synthesize the visual anticipation of the solution and their analytic symbolic representations in a microworld environment which equally allowed dynamic actions. In their analysis of the role of visual reasoning, Hershkowitz, Arcavi, & Bruckheimer (2001) suggested that visualization can be an analytical process itself. In fact, symmetry has conceptual foundations that can be investigated through visual and analytic strategies.

Research conducted since the 1980s has consistently shown that the apparently simple concept of symmetry is problematic for many students. One of the first extensive studies conducted in this domain is by Küchemann (1981) who identified five essential variables that influence students' ability to perform bilateral symmetry, namely the slope of the line of symmetry, the slope of the object, the complexity of the object, the existence or absence of intersection between the object and the line of symmetry, and the presence or absence of a grid in the problem. The variables identified by Küchemann were analysed in further depth by Grenier (1985) in her dissertation study to investigate patterns of errors. Recent studies (e.g., Bulf, 2010; Ho & Logan, 2013) continue to highlight the influence of such variables in students' performance.

Although much is known about the type of variables that affect students' ability to perform bilateral symmetry and reflection, the source of the respective difficulties has not been thoroughly investigated. In this paper, we use constructs from the area of

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spatial visualization to analyse the processes behind students' strategies and errors (intuitive or learned) in performing bilateral symmetry and reflection tasks. We address the following research questions: (1) What are the sources of conceptual difficulties associated with slanted lines of symmetry? (2) How are visual strategies enacted in bilateral symmetry and reflection tasks? (3) How do visual and analytical strategies interact in the production of the image from the object?

CONCEPTUAL FRAMEWORK

We analysed the data using constructs from the domain of spatial visualisation which essentially refers to the ability to generate and manipulate images (Yakimanskaya, 1991). Yakimanskaya considers images as the basic operative units of spatial visualization. Additionally, according to Kosslyn (1990), imagery is used "when we reason about the appearance of an object when it is transformed, especially when we want to know about subtle spatial relations" (p. 75).

We now explain how we interpret spatial visualization in relation to the types of symmetry and reflection tasks that students are generally called upon to perform in school mathematics, as is the case in the present study. We distinguish between two types of mental actions where spatial images are involved in reflection tasks in terms of the following visual anticipatory action or imaginative construction:

(i) visual-mental reflection: The visual/mental action of anticipating the image of an object from a line of symmetry. This process occurs when a printed object on paper (either plain or grid paper) is to be reflected given a line(s) of symmetry.

(ii) visual-mental folding: The visual/mental action of imagining the shape of an object being folded to determine the one-to-one geometric or morphological correspondence between the parts of an object. This process occurs in finding the lines of symmetry of shapes or alphanumeric characters.

The two operations described above were defined on the basis of the observations that we made as the participants interacted with the tasks. We refer to a *visual strategy* when attention is given to the use of imagery as related to shape, location/position, orientation and global perception. Such a strategy may equally include kinaesthetic imagery, as will be shown in the data analysis. Reflection, as a transformation, constitutes an isometry as it preserves length, shape and angle. The two main properties that are useful to reflect an image on a line of symmetry are (i) perpendicularity between corresponding points on object and image and (ii) equidistance between object, line of symmetry and image. We use the term *analytical strategy* whenever explicit reference is made to the properties (in terms of following a rule) in performing a reflection, finding the line(s) of symmetry or in the construction of a symmetrical object.

We used the concept of local and global perception from the psychology literature (Enns & Kingstone, 1995) in understanding the strategies used or constraints encountered by participants. In fact, Kosslyn (1990) suggests that imagery and perception share common features. Psychologists use the term local perception to refer to the interpretation of an image when it is visually parsed into units. On the other hand, global perception refers to the overall structure of the image being processed. In analysing the videorecords, we could equally note how the participants were reorienting the diagrams by moving the worksheets or their posture, or used their fingers on the given diagrams in their imaginative actions. These observations indicated the importance of Presmeg's (1986) construct of kinaesthetic imagery.

METHOD

The two participants of the study are identified by the pseudonym Brittany (Grade 9, age 15 years) and Sara (Grade 8, age 14 years). Each participant was individually interviewed by twice. Accessing mental images and visualization processes are methodologically challenging. In our attempt to capture the moment-by-moment responses of the participants, two cameras were used to record the four interviews so as to focus on their inscriptions and the movements that they made to track their kinaesthetic actions. The students were allowed to complete each task before they were asked to describe their strategies so as not to distort their thinking processes as suggested by Gutiérrez (1996).

The students were presented with four sets of tasks. In the first set, they were required to find the line of symmetry of a polygon (square, rectangle, equilateral triangle, cross, parallelogram and rhombus) and alphanumeric characters (S, X and Z). In the second set, they were asked to find the image of a given line segment or polygon reflected on a line of symmetry on grid paper (see Figures 1(a), 1(d) and 2 for sample tasks). In the third set, they had to complete the object from the partial object and the given number of lines of symmetry (see Figure 3(a) for sample task). In the fourth set, they had to reflect objects on slanted lines of symmetry, without the support of a grid (see Figures 3(c) and (d) for sample tasks). Due to space constraints, we present only selected tasks and responses.

RESULTS AND DISCUSSION

The participants' prior knowledge of symmetry

Brittany described reflection in terms of "exactly opposite" and "folding". The analytical property of 'equal distance' was well-established for her. However, at no point she made any reference to perpendicularity. Her conception of reflection was based on the more intuitive congruence property (referred to as congruence criterion), specifically the congruence between corresponding lengths in the object and image. She gave explicit description of her visual strategy to find the number of lines of symmetry in the alphanumeric characters (S, X and Z). For example, with regard to the letter 'S', she explained that she visualized a solid object: "Like, the shape. It's the shape of the S. So the paper was like cut out in the shape of the S and you can fold it."

On the other hand, Sara had a well-articulated analytical conception of reflection in terms of equal distance between object and image and line of symmetry (referred to as equi-distance criterion). Particularly, her concept of perpendicularity (referred to as perpendicularity criterion) empowered her to unsparingly reflect objects in slanted line of symmetry. She would always focus on the ends of line segments or vertices of polygons in the application of the equi-distance and perpendicularity criteria. She clearly stated that points on a line of symmetry have no reflection. She described reflection in terms of a "mirror".

(1) WHAT ARE THE SOURCES OF CONCEPTUAL DIFFICULTIES ASSOCIATED WITH SLANTED LINES OF SYMMETRY?

The nature of the symmetry and reflection tasks dictates when perpendicularity is vital for successful problem solving. For vertical and horizontal lines of symmetry, perpendicularity is readily ensured. However, for slanted lines of symmetry such is not the case, although the visual appearance of the task in a grid may help. While the situations involving the horizontal and vertical lines of symmetry did not pose any constraint for Brittany, the slanted lines of symmetry revealed the inadequacy of her conception of symmetry. She focused on congruence of length as the main criterion of reflection and was not formally aware of the perpendicularity criterion. The second interview also confirmed that Brittany was not aware of the fact that the reflection of a point on a slanted line of symmetry is invariant under such a transformation.

We give a sample response to show the outcome of her focus on congruence of lengths for Task 1.8 (Figure 1(a)). As the given object (vertical line segment) crossed the line of symmetry, she interpreted the object as consisting of two parts. The motion of her pencil suggested that she was thinking about moving either to the right or left, at right angle to the object. She first decided to draw the image to the left of the object (see Figure 1(b)). She joined the two end points (labelled X and Y for explanation purposes) to find a means of getting equal distance between X and Y. Since the length on either side of the line segment XY was different, this led her to realize that this step is incorrect: "That's not really…". At a later point, she changed her solution by drawing the image on the right of the object (see Figure 1(c)) and mentioned: "Because well this is one square like pass it. So if we do this it's got one part on this side too, the same amount on the other side." Because the length of the object and image above and below the line of symmetry was the same, she felt confident that she performed the reflection correctly. Further evidence of her reliance on congruence could be observed by comparing the object and its corresponding image in Figures 1(d) and 1(e) respectively.

In contrast, Sara's consistent approach involving equi-distance and perpendicularity properties showed that she had a well-established scheme for lines of symmetry. For example, in Task 1.3 (see Figure 2(a)), she mentioned: "I used the boxes. I made a perpendicular line with my ruler. And then I saw how many boxes I needed."

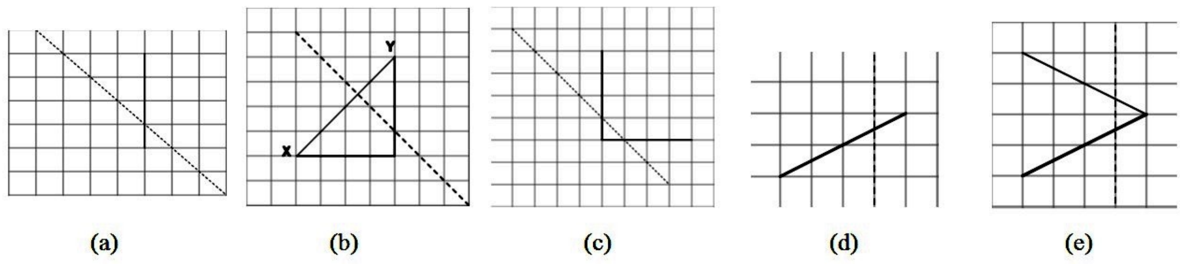


Figure 1: Brittany's responses to Set 2

Our findings led us to conjecture that students' responses to slanted line of symmetry tasks are also dependent on whether the objects are closed or open. A line segment (e.g. Fig. 2(a) and (c)) was more challenging to reflect as compared to when it was part of a figure (e.g. Fig. 2(d) where polygon L contains vertical and horizontal line segments). Furthermore, it appears that the orientation of the object relative to the slanted line of symmetry tends to suppress the global perception necessary to visually check the soundness of the image produced, an observation equally made by Küchemann (1981) and Grenier, (1985). In Task 1.7 (see Figure 2(c)), Brittany merely extended the object vertically down by 4 units, while in Task 1.6 (see Figure 2(b)), she produced the correct image.

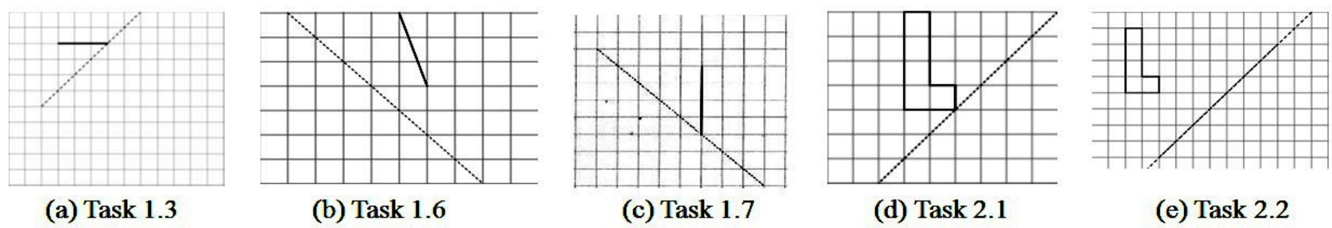


Figure 2: Sample tasks from Sets 1 and 2

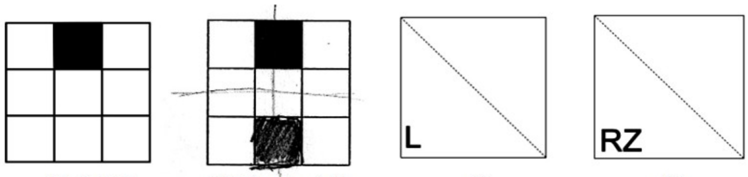
In summary, absence of formal awareness of the perpendicularity criterion and failure to recognize that points on line of symmetry are invariant under a reflection, accounted for the difficulties that Brittany experienced with slanted lines of symmetry.

(2) HOW ARE VISUAL STRATEGIES ENACTED IN BILATERAL SYMMETRY AND REFLECTION TASKS?

We observed four distinct ways in which visual strategies directed the participants' actions in performing the symmetry and reflection tasks.

Imagining lines of symmetry

We could access Brittany's visual strategy in Set 3 (see Figure 3(a) for sample task) by her actions of positioning an imaginary line of symmetry (as could be inferred by the motion of her pencil) generally in the sequence, vertical, horizontal and slanting to then mentally reflect the shaded cells given in the tasks. She focused not only on the shaded parts but equally looked at the continuous shape made by the unshaded parts. For example, in Task 3.1 (shade one more square so that the diagram has two lines of symmetry), she focused on the letter H formed on shading the required cell (see Figure 3(b)) to confirm that there were two lines of symmetry.



(a) Task 3.1 (b) Brittany's solution (c) (d)

Figure 3: Sample tasks from Sets 3 and 4

Reorientation of slanted line of symmetry to the vertical

To work with situations involving slanted line of symmetry, particularly when a grid was not available, Brittany and Sara would turn the line of symmetry in a vertical orientation. This reorientation was particularly apparent in Set 4 (Figures 3 (c) and (d)). The visual/perceptual facility afforded by the vertical reorientation of the line of symmetry was explicitly highlighted by Brittany: "If you tilt it (to the vertical) this way, I could just have to work out where it will be". Psychologists in the area of perception (Giannouli, 2013) made similar observations, claiming that the vertical orientation is favoured by human beings.

Reflection of part of object as a visual trigger

In some cases, the reflection of one part of the given object served to open the space for reflection of the whole object in the slanted line of symmetry. We could observe such a visual trigger in Figure 2(d). Brittany first reflected the horizontal segment (touching the slanted line of symmetry) in the letter L. Then she reflected the vertical segment (touching the slanted line of symmetry). These two initial constructions apparently served as a trigger for her to spontaneously identify the next part of the image and she quickly proceeded to construct the image, measuring the length of the different parts of the object by counting the number of cells. Küchemann(1981) described this strategy as semi-analytic. However, in Figure 2(e) involving the same object at a distance from the line of symmetry, she could not find the image of one part of the object to serve as a visual trigger for the whole image.

The visual strategy as a visual check

In a number of cases, we could observe how the students inspected their solution as a whole (global perception) to verify whether the image was correctly drawn. For instance, in Task 2.6 (See Figure 4(a)), where the line of symmetry was not inclined at 450, both of them could observe that their initial solution was incorrect. Another example is in Set 4 (Figure 3(c) and 3(d)) where Brittany compared the orientation of the alphanumeric characters and their image after reflection in the slanted line of symmetry. She justified her image by mentioning: "It looks correct". Although Sara used such a visual check, she tended to rely more on her local perception emanating from her primarily analytical approach. In other words, she implemented her equi-distance and perpendicularity criteria systematically right away and did not seem to find the necessity to rely on the visual approach, except as a check.

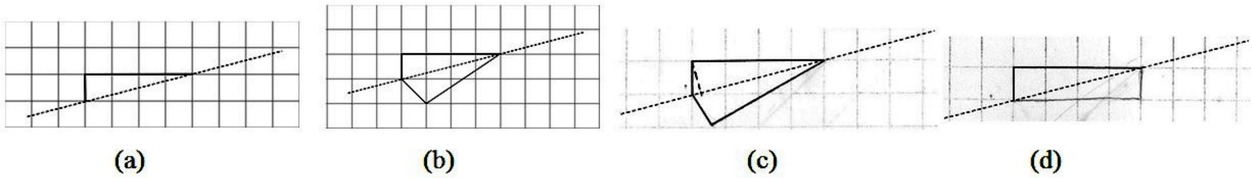


Figure 4: Task 2.6

(3) HOW DO VISUAL AND ANALYTICAL STRATEGIES INTERACT IN THE PRODUCTION OF THE IMAGE FROM THE OBJECT?

We present excerpts to show how the visual and analytical strategies were conjointly used to produce the image. In Task 2.6 (Figure 4(a)), Sara first drew Figure 4(b) and visually analysed the drawing to mention: “no, it can’t be good”. Then, she used her ruler to set the perpendicular distance (dotted line in Figure 4(c)) to help her draw the image. Brittany initial construction was similar to Figure 4(b) and she could observe that it was incorrect and mentioned: “trying to visualize the whole thing but…” She pursued further with her “congruence of length” criterion to draw the image (see Figure 4(d)). In other words, the incorrect appearance of the image prompted her to switch to the analytical strategy.

The visual strategy as a scaffold for te analytic strategy

In some of the tasks, the students asserted that they made a global picture of how the image would look like before actually applying the analytical properties of symmetry and reflection. In Set 4 (see Figure 3(c)), Sara first mentally folded the object before applying the equidistance and perpendicularity criterion. In cases where she experienced constraint, Brittany depended on the visual to scaffold her analytical strategy, “congruence of sides” as in Task 1.8 (see Figure 1(a)). Here, the visual strategy took over as a more intuitive fallback measure. The students also used kinaesthetic imagery in starting their solution. For instance, Brittany tended to pull the page up on the corner in imitating a folding action in Set 4 (See Figures 3(c) and 3(d)).

CONCLUSION

By focusing on spatial visualization, this study enhances our understanding of the subtle interaction between visual and analytical strategies in relation to symmetry and reflection tasks. It explains the constraints associated with the slanted line of symmetry identified by the seminal work conducted by Küchemann (1981) and Grenier (1985). Parallel to the work of Hoyles and Healy (1997), it explains how the meaning of symmetry is negotiated via visual and analytical strategies. More importantly, it attempts to make explicit the layers of complexity inherent in what is usually regarded as seemingly simple concepts, i.e., symmetry and reflection. It is acknowledged that this two-participant contrasting case study is bound to be limited in scope. However, the ways in which it portrays the explicit students’ actions with the symmetry and reflection tasks is informative for teachers. The constraints that Brittany encountered with slanted lines of symmetry are not uncommon among students and serve to highlight the necessity to give more attention to the perpendicularity criterion. Ignorance of this criterion may be carried over to adulthood. The data also prompts us to suggest the consideration of global perception as a visual check in instruction on reflection. This study equally brings forth the importance of local and global perception as influential elements in students' reasoning, an aspect that requires further exploration.

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