THE FREQUENCIES OF VARIOUS INTERPRETATIONS OF THE DEFINITE INTEGRAL IN A GENERAL STUDENT POPULATION

Steven R. Jones

Brigham Young University

Student understanding of integration has become a topic of recent interest in calculus research. Studies have shown that certain interpretations of the definite integral, such as the area under a curve or the values of an anti-derivative, are less productive in making sense of contextualized integrals, while on the other hand understanding the integral as a Riemann sum or as "adding up pieces" is highly productive for contextualized integrals. This report investigates the frequency of these three conceptualizations in a general calculus student population. Data from student responses show a high prevalence of area and anti-derivative ideas and a very low occurrence of summation ideas. This distribution held even for students whose calculus instructors focused on Riemann sums while introducing the definite integral.

INTRODUCTION

First-year calculus has received much attention in mathematics education in recent years due to its significance in science, technology, engineering, and mathematics (STEM) fields. In particular, the calculus concept of the definite integral has become a current topic of interest among mathematics education researchers (e.g. Black & Wittmann, 2007; Hall, 2010; Jones, 2013; Sealey & Oehrtman, 2007; Thompson & Silverman, 2008). The integral is an important topic to investigate because it is commonly used in subsequent mathematics courses (see Brown & Churchill, 2008; Fitzpatrick, 2006) and provides the foundation for many concepts in science and engineering coursework (see Hibbeler, 2012; Serway & Jewett, 2008).

However, several studies demonstrate that students are struggling to apply their knowledge of integration to subsequent courses (e.g. Beichner, 1994; Christensen & Thompson, 2010; Grundmeier, Hansen, & Sousa, 2006; Pollock, Thompson, & Mountcastle, 2007). This finding has led some researchers to begin to examine *why* students are having this difficulty. Sealey (2006) and Jones (2013) suggest that the "area under a curve" notion alone is not sufficient for understanding definite integrals. Thompson and Silverman (2008) promote the development of an "accumulation" conception of the integral in order to help students.

Jones (under review) subsequently conducted a more thorough analysis of the anti-derivative, area under a curve, and summation interpretations of the definite integral by students in both mathematics and science contexts. The results demonstrate that the "summation" conception proved highly productive for understanding definite integrals that are either situated in a larger context or that contain variables representing physical quantities. By contrast, the study confirms that the "area under a

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curve" and "values of an anti-derivative" conceptions are less productive in making sense of these types of contextualized definite integrals. While the findings do *not* imply that the area and anti-derivative ideas are not important (nor that they should not be learned) they *do* suggest that it is critical for students to have a robust and accessible summation conception of integration in their cognitive repertoire.

Based on these results, it is important to ask the question: Are calculus students generally constructing their knowledge of the integral in a way that promotes the beneficial summation conception? This paper seeks to answer this question by investigating (a) how common each of the three conceptualizations are when a large sample of calculus students are asked to think about integration, and (b) whether standard ways of introducing Riemann sums are sufficient for a general student population to internalize the summation conception.

THEORETICAL PERSPECTIVE

Symbolic forms

For this study, the manner in which students hold their knowledge of the integral is characterized through the lens of *symbolic forms* (Sherin, 2001). A symbolic form is a blend (Fauconnier & Turner, 2002) between a *symbol template* and a *conceptual schema*. The symbol template refers to the arrangement of the symbols in an equation or expression, such as $\int_{[]}^{[]}[]\,d[]$, where each "box" can be filled in with symbols. The conceptual schema is the meaning that underlies the symbols in the template. Jones (2013) documents students' symbolic forms of the definite integral that are associated with the notions of area under a curve, values of an anti-derivative, and summations. Note that the students in the study regularly ascribed "anti-derivative" meanings to both indefinite *and* definite integrals. Furthermore, Jones describes a "deviant" of the typical Riemann sum conception that was dominant in some students' thinking. These four symbolic forms aided the analysis of the student data by helping determine when students were drawing on each the area, anti-derivative, or summation conceptualizations of the integral. A brief description of each form is provided here.

Area and perimeter: This symbolic form interprets each "box" in the symbol template as being one part of the perimeter of a shape in the (*x*-*y*) plane. The differential, "*d*[]," represents the "bottom" of the shape by dictating the variable that resides on the horizontal axis. This symbolic form is associated with the "area under a curve" notion.

Function matching: This symbolic form interprets the integrand as having come from some "original function." The original function became the integrand through a derivative, and the differential "*d*[]" indicates the variable with respect to which the derivative was taken. This form is associated with the "anti-derivative" conception.

Adding up pieces: This form casts the differential as being a tiny piece of the domain, given by the limits of integration. Within each tiny piece, the quantities represented by the integrand and differential are multiplied to create a small amount of the resultant quantity. The integral symbol dictates an "infinite" summation that ranges over the domain. This form is related to the Riemann sum idea.

Adding up the integrand: This is a "deviant" of the *adding up pieces* form. The key difference is that within each tiny piece, only the quantity represented by the *integrand* is added up. This resulting "total" from the integrand is then multiplied by the entire domain (length, area, or volume) to get the resulting quantity. This form fails to adequately describe the Riemann sum process, but is rooted in ideas of summations.

Manifold view of knowledge

Symbolic forms can be considered a subset of "cognitive resources" (Hammer, 2000), which are any piece of cognition that can be drawn on and employed as a unit, whether large or small. The main idea from cognitive resources that is used for this paper is the push away from a "unitary view" of concepts to a "manifold view" of knowledge (Hammer, Elby, Scherr, & Redish, 2005). The theory of resources argues that a "concept" such as the integral is comprised of many small and large elements—like rectangles, graphs, functions, ideas about summation, areas, limits, anti-derivative rules, and so forth—that are too complex to be considered a single entity.

Thus, this study assumes that students can think about the integral by drawing on certain aspects of their "integral knowledge" while other aspects remain dormant. For example, a student may look at an integral and immediately think "area under a curve" without ever thinking about Riemann sums. This does not necessarily mean that the student does not have a Riemann sum conception in their cognition, but rather that the area conception is much more familiar and readily accessible to them. This has implications for learning integrals, since students need not only to "assimilate" a summation conception somewhere in their cognition, but that that conception needs to be created in a way that it is prevalent in their thinking to capitalize on its usefulness.

METHODS

Initial survey

In order to investigate the prevalence of the area, anti-derivative, and summation conceptions of the definite integral, 150 students at two major colleges in the Western United States, who had successfully completed first-semester calculus, were recruited to participate in a survey that asked them open-ended questions about definite integrals. A χ^2-test revealed no significant difference between the students at these two schools, in terms of the frequencies of the responses that were coded as belonging to the each conception of the integral used in this study (see below for more detail). This allows for the assumption that these students may be considered representative of the general calculus student population. The choice to use successful first-semester students is based on the fact that many key aspects of the integral are explored during the first semester: areas under curves, the Riemann integral definition, the Fundamental Theorem of Calculus (FTC), the Net Change Theorem, velocity/position applications, and anti-derivative techniques (including *u*-substitution).

To recruit students, several second-semester calculus courses were visited within the first two days of the semester in order to administer the survey. Students who had already taken second semester calculus (or the equivalent in high school) were asked not to complete the survey, to keep the focus of the study on students who had only successfully finished first-semester calculus. The students who participated were given fifteen minutes to complete the survey.

Two of the four items from the survey form the focus for this paper. The first item reads, "Explain in detail what $\int_a^b f(x)\,dx$ means. If you think of more than one way to describe it, please describe it in multiple ways. Please use words, or draw pictures, or write formulas, or anything else you want to explain what it means." The item clearly asks the students to express any and all ways that they conceive of the integral and this instruction was reiterated to the students when the surveys were administered. The second item reads, "Why does an integral need a '*dx*' on it? For example, why can't it just be $\int_0^1 x^3$ instead of $\int_0^1 x^3\,dx$? Explain in as much detail as you can." The main purpose of this question was to give the students a second context to discuss their ideas about integrals as well as to ask them to mentally break apart the integral symbol template, in order to discuss the integral in more detail. The way in which they explained the existence of "*dx*" was also compared to the symbolic forms of the integral described in the previous section, to see if the students were possibly invoking other conceptualization beyond what they used for their responses to item 1.

Responses to the items were coded into the "area," "anti-derivative," "summation," or "weak summation" categories. Many responses were coded into multiple categories if the students expressed more than one idea in their answer. Responses were coded based either on (a) an explicit statement regarding one of the three main conceptualizations, or (b) the inferences made by the correlation of a response to one of the symbolic forms of the integral. The "weak summation" category was included since several responses hinted at a summation notion, but were not articulated enough to be conclusive. Also, responses along the lines of the *adding up the integrand* symbolic form were placed in "weak summation." Confidence intervals (95%-level) were used to estimate the percentages of the overall calculus student population that might respond similarly to these survey items (see Triola, 2010).

Classroom observations and second survey

In order to investigate whether standard classroom instruction can adequately support the creation of a robust summation conception, two veteran instructors of first-semester calculus from one of the schools were recruited for observation. Both instructors taught large sections (200+ students) and the first five of their one-hour lessons on integration were observed and videotaped. Both instructors had taught calculus many times and used standard templates for their lesson schedule. A sample of students from their courses were surveyed (n = 55), using a "cluster sample" technique on the individual lab sections. This occurred during the same semester as the previous survey, ensuring that there was no overlap between the two samples. A χ^2-test was conducted for both survey items ($\alpha = .05$) to compare the frequencies of responses of these instructors' students to the responses of the general sample (see Triola, 2010).

RESULTS

General population sample

Table 1 illustrates the frequency of responses that fit under each conceptualization of the integral for items 1 and 2. Note that since many students provided elaborated answers that fit into more than one category, the frequencies add up to more than the sample size. Confidence intervals (95%-level) have been included as estimates for the percentage of the overall calculus student population that might respond similarly.

Conceptualization	Responses from item 1	Reponses from item 2
Area	131 (82.0% < p < 92.7%)	7 (1.3% < p < 8.0%)
Anti-derivative	60 (32.2% < p < 47.8%)	114 (69.2% < p < 82.8%)
Summation	10 (2.7% < p < 10.7%)	11 (3.2% < p < 11.5%)
Weak summation	7 (1.3% < p < 8.0%)	2 (*n/a, low frequency*)

Table 1: Frequencies of responses (n = 150), with confidence intervals

The data show a high prevalence of area and anti-derivative conceptions when students think about the integral. This in and of itself is not bad, since these two notions are helpful, useful ideas. However, what is surprising is the low frequency of students who invoked any type of summation conception. Even taking "summation" and "weak summation" together, only 17 out of 150 students (6.3% < p < 16.4%) made *any* kind of statement dealing with summations on item 1. Further, 117 out of 150 students (71.4% < p < 84.6%) made *no* mention of anything related to summations whatsoever on either item 1 or item 2. With confidence, I can assert that roughly three-fourths of *successful* first-semester calculus students leave their first-semester course without a familiar, accessible conception of the Riemann sum or any related "adding up pieces" idea. This, of course, is not to say that these students have *no* summation conception in their cognition; they may express something along these lines if pressed. Yet, given the important nature of the summation conception (Jones, under review), it is striking that so few students "choose" to activate that knowledge when asked to explain what a definite integral is or what it means. This has important ramifications for understanding integrals in further coursework where a Riemann sum conception is critical for making sense of a contextualized integral expression or equation.

Observed instructors and their students

Both observed instructors used Riemann sums to introduce integration, as a way to approximate the area underneath the graph of a function. They drew the familiar rectangles under the curve and walked through examples of calculating left-hand, right-hand, and midpoint approximations. Both instructors regularly discussed Riemann sums throughout the first two one-hour class sessions. During the third lesson both instructors moved to the Fundamental Theorem of Calculus and then, for the remainder of the observed lessons, the instructors focused on calculating anti-derivatives and discussing a variety of integral properties. In short, their instruction reflected how many textbooks present integration (e.g. Stewart, 2012; Thomas, Weir, & Hass, 2009). These observations show that Riemann sums were a significant portion of the early instruction regarding integration.

Based on these observations, one might hope that these students showed a stronger tendency to interpret integrals through a summation interpretation, in addition to the area and anti-derivative conceptualizations. Unfortunately, however, in essentially every way these students reflected the general population sample. Table 2 shows the breakdown of responses from these instructors' students (n = 55), including p-values from the χ^2-tests (α = .05) that were done on the frequencies of responses from these students versus the frequencies of responses from the general population sample.

Conceptualization	Responses from item 1	Responses from item 2
Area	47	1
Anti-derivative	14	38
Summation	8	9
Weak summation	4	3
(χ^2-test)	p = .13	p = .07

Table 2: Frequencies of responses (n = 55) and p-values from χ^2-tests

Neither p-value was below the threshold for statistical significance. The p-value for item 2 was close to the ".05" mark, but since neither test was significant, the results suggest that there is no important difference between these instructors' students and the overall population described previously. This outcome leaves us with the conclusion that the attention the instructors gave to the Riemann sum throughout their first two lessons did not make an impact in supporting the students' creation of a robust summation conception. Therefore, merely having Riemann sums present during instruction is *not* sufficient for accomplishing this goal. More, apparently, is needed.

DISCUSSION AND FUTURE DIRECTIONS

The results of this study suggest that simply giving attention to Riemann sums is *not* enough to help students construct a viable summation conception in regards to integration. By examining the manner in which these two instructors introduce integration, we see a common theme that may contribute to this issue.

Both instructors began their introductory lesson on integration by using the "area under the graph of a function" as the primary motivation for the study of integrals. Riemann sums were invoked only as a way to calculate the irregular shapes created by the graphs, since basic geometry could not be used. For example, one instructor began the lesson by saying, "We're going to draw a graph… And I want to find the area under this [graph], between the x-axis and this [graph]." The instructor then created rectangles underneath the graph to approximate the area. Similarly, the second instructor introduced integration by saying that areas under curves was the second "main idea" of calculus. "Second is integrals. Integrals can be thought of as area under a curve." He then drew a generic graph, with vertical lines at $x = a$ and $x = b$, and shaded in the area of the shape created by the graph and the vertical lines. "This area between the curve and the x-axis, that's represented by the idea of an integral." This instructor then also used Riemann sums to approximate these irregularly shaped areas.

It appears from these lessons that, even though Riemann sums are used, the central concept portrayed to the students is still that of "area under a curve." In fact, Riemann sums are used only as a "tool" for getting at these areas. By doing this, instructors may inadvertently be reducing Riemann sums to a procedure for calculating areas under curves—the more salient goal of the lesson. The summation conception does not have the chance to stand on its own as an important idea. Thus, when students encounter the FTC, they might decide that anti-derivatives are a better/easier "tool" for calculating areas, and the Riemann sum conception takes a cognitive "backseat" to the anti-derivative notion. This may, in part, explain the students' prevalent use of areas and anti-derivatives to explain integrals, while rarely appealing to summations.

In order to investigate this problem further, the author is currently involved in a design experiment that seeks to examine other ways of introducing the integral, in order to highlight the summation conception as the *basis* of the integral. This is done through several activities, based off those described in Jones (2013/14), that use Riemann sums *without* discussing areas under curves. For example, the accumulation of water spilled from a pipe can be estimated, using Riemann sums, from discrete data points. Or the mass of an object with non-uniform density can be estimated by selecting density data points and multiplying them to each small piece of volume. Preliminary results of the study are showing promising outcomes for conveying to students that the Riemann sum is the central, underlying conception (and definition) of definite integrals.

References

Beichner, R. J. (1994). Testing student interpretation of kinematics graphs. *American Journal of Physics, 62*(8), 750-762.

Black, K. E., & Wittmann, M. C. (2007). Epistemic games in integration: Modeling resource choice. *AIP Conference Proceedings, 951*, 53-56.

Brown, J., & Churchill, R. (2008). *Complex variables and applications* (8th ed.). New York: McGraw-Hill.

Christensen, W. M., & Thompson, J. R. (2010). Investigating student understanding of physics concepts and the underlying calculus concepts in thermodynamics. *Proc. 13th special interest group of the Mathematical Association of America on research in undergraduate mathematics education*, Raleigh, NC. Retrieved from http://sigmaa.maa.org/rume/crume2010/Archive/Thompson%20&%20Christensen.pdf

Fauconnier, G., & Turner, M. (2002). *The way we think: Conceptual blending and the mind's hidden complexities*. New York: Basic Books.

Fitzpatrick, P. M. (2006). *Advanced calculus* (2nd ed.). Providence, RI: AMS..

Grundmeier, T. A., Hansen, J., & Sousa, E. (2006). An exploration of definition and procedural fluency in integral calculus. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies*, *16*(2), 178-191.

Hall, W. L. (2010). Student misconceptions of the language of calculus: Definite and indefinite integrals. *Proc. 13th special interest group of the Mathematical Association of America on research in undergraduate mathematics education,* Raleigh, NC. Retrieved from http://sigmaa.maa.org/rume/crume2010/Archive/Hall.pdf

Hammer, D. (2000). Student resources for learning introductory physics. *American Journal of Physics, 68*(S1), S52-S59.

Hammer, D., Elby, A., Scherr, R. E., & Redish, E. F. (2005). Resources, framing, and transfer. In J. Mestre (Ed.), *Transfer of learning from a modern multidisciplinary perspective* (pp. 89-120). Greenwich, CT: Information Age Publishing.

Hibbeler, R. C. (2012). *Engineering mechanics: Statics* (13th ed.). Upper Saddle River, NJ: Pearson Prentice Hall.

Jones, S. R. (2013). Understanding the integral: Students' symbolic forms. *The Journal of Mathematical Behavior, 32*(2), 122-141.

Jones, S. R. (2013/14). Adding it all up: Reconceiving the introduction to the integral. *Mathematics Teacher, 107*(5), 372-377.

Pollock, E. B., Thompson, J. R., & Mountcastle, D. B. (2007). Student understanding of the physics and mathematics of process variables in PV diagrams. *AIP Conference Proceedings, 951*, 168-171.

Sealey, V. (2006). Definite integrals, Riemann sums, and area under a curve: What is necessary and sufficient? In S. Alatorre, J. L. Cortina, M. Sáiz, & A Méndez (Eds.), *Proc. 28th annual meeting of the North American chapter of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 46-53). Merida, Mexico: PMENA.

Sealey, V., & Oehrtman, M. (2007). Calculus students' assimilation of the Riemann integral into a previously established limit structure. In T. Lamberg & L. Wiest (Eds.), *Proc. 29th annual meeting of the North American chapter of the Int. Group for the Psychology of Mathematics Education* (pp. 78-84). Stateline, NV: PMENA.

Serway, R. A., & Jewett, J. W. (2008). *Physics for scientists and engineers* (7th ed.). Belmont, CA: Thomson Learning.

Sherin, B. L. (2001). How students understand physics equations. *Cognition and Instruction*, *19*(4), 479-541.

Stewart, J. (2012). *Calculus: Early transcendentals* (7th ed.). Belmont, CA: Brooks/Cole.

Thomas, G. B., Weir, M. D., & Hass, J. (2009). *Thomas' calculus* (12th ed.). Boston, MA: Pearson.

Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. In M. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 117-131). Washington, DC: MAA.

Triola, M. F. (2010). *Elementary statistics* (11th ed.). Boston, MA: Addison-Wesley.