# TEACHING LINEAR ALGEBRA IN THE EMBODIED, SYMBOLIC AND FORMAL WORLDS OF MATHEMATICAL THINKING: IS THERE A PREFERRED ORDER?

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*This research project involved teaching linear algebra to second year undergraduates. Using Tall's three worlds of mathematical thinking as a theoretical framework, students were taught fundamental linear algebra concepts using each of the embodied, symbolic and formal dimensions. By varying the order in which these approaches were used in each topic we investigated students' perceptions of the combinations and their potential for understanding and learning. The results show that students seem to react positively to symbolic examples and embodied ideas but there is little effect overall of order on understanding.*

## BACKGROUND

Can we improve the way we introduce students to linear algebra concepts? Over the last decade Tall's (2004, 2008, 2010) theory of three worlds of embodied, symbolic and formal mathematical thinking, along with APOS theory (Dubinsky & McDonald, 2001), has been employed to construct a Framework of Advanced Mathematical Thinking (FAMT) (Stewart & Thomas, 2009) for investigating lecturing (Hannah, Stewart, & Thomas, 2013) and students' conceptual understanding of key linear algebra concepts (e.g., Thomas & Stewart, 2011). According to Tall's theory, in the embodied world we think using our mental perceptions of real-world objects and other forms of visuo-spatial imagery (Tall, 2004). In this world a vector might be thought of as directed line segment, or a quantity having magnitude and direction. In the symbolic world we calculate using symbols, arrays and equations. In this world a vector might be an *n*-tuple of real numbers. In the formal world objects are defined in terms of their properties, with new properties deduced by formal proof. In this world a vector is an object obeying the axioms for elements of a vector space. All three worlds are available to, and used by, individuals as they engage with mathematical thinking, and they interact so that "three interrelated sequences of development blend together to build a full range of thinking" (Tall, 2008, p. 3). When students first meet new concepts, such as *subspace* or *linear independence*, the question naturally arises as to whether there is a right, or best, order in which they should meet these three world views of the concepts. Tall observes that although school students usually meet embodiment first, followed by symbolism and then, finally, formalism, "when all three possibilities are available at university level, the framework says nothing about the sequence in which teaching should occur" (Tall, 2010, p. 22). So a fundamental research question for us is whether there is a preferred order of presentation of concepts in the FAMT framework.

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Traditionally, a typical first linear algebra course begins in the formal world, with an axiomatic presentation that took many years to achieve its present form (Harel & Tall, 1989). But the presentation could perhaps just as easily begin in the embodied or symbolic worlds. A number of recent studies have considered the relationship between formal thinking in linear algebra and students' other approaches and demonstrated that to develop teaching that promotes formal ideas a knowledge of student thinking prior to teaching is valuable. For example, Wawro, Sweeney, and Rabin (2011) considered students' concept images of the notion of subspace and found that they made use of geometric, algebraic and metaphoric ideas to make sense of the formal definition. In other work, Wawro, Zandieh, Sweeney, Larson, and Rasmussen (2011) found that students' intuitive ideas about span and linear independence could be employed to assist them to develop the formal definitions.

In this paper we present the results of a study where the lecturer (the second author) experimented with different orders of presentation for each of the main concepts in an introductory linear algebra course. A crucial feature of her lectures was the use of *contingent teaching* (Draper & Brown, 2004), which involves gaining responses from the whole class via clickers and immediately reacting to the data, so that the lecturer is constantly confronted with decision making. Research shows that although lecturers appreciate the feedback they receive through clickers (Abrahamson, 2006) the ability to react to these responses on the spot, according to students' needs, is challenging. In this setting "lecturers must develop their plans beyond the factory machine stage of executing a rigid, pre-planned sequence regardless of circumstances" (Draper & Brown, 2004, p. 91) and have relevant strategies on hand, depending on student responses.

## METHOD

This research project comprised a mixed methodology of action research, as a university lecturer examines and refines her teaching practice, along with a case study of student reactions to the teaching. The first phase of the project was conducted in the Fall of 2013 at a large research university in the USA. The researcher, who is the second named author, was teaching an introduction to linear algebra course to two classes of students (mainly from engineering and other science majors), C1 and C2.

To investigate whether the order in which the material is presented has an impact on students' learning and attitudes, each of the following possible combinations of teaching concepts was used: Embodied, Symbolic, Formal (ESF); Embodied, Formal, Symbolic (EFS); Symbolic, Embodied, Formal (SEF); Symbolic, Formal, Embodied (SFE); Formal, Symbolic, Embodied (FSE); Formal, Embodied, Symbolic (FES) and two most common ways of teaching with no embodied exposure at all: Formal, Symbolic (FS) and Symbolic, Formal (SF). The aim was to try to establish whether the order influences understanding of a particular concept. For example, concept A was taught in the morning to class C1 in the ESF order, whereas in the afternoon class C2 was taught in the FSE order. To expose students to as many orders as possible, concept

B was taught using SFE in the morning and SEF in the afternoon section, and so on. Hence, each concept was taught in all three worlds of embodied, symbolic and formal mathematical thinking to each class, but in different orders. To try to gain some measure of students' understanding the lecturer employed contingent teaching, incorporating clicker quiz questions into the presentation of the teaching material. The design of suitable quizzes, posed at the right moment, was a crucial part of the project. The students were also given clicker opinion questions throughout the lecture regarding their preference of the order of presentation, to gauge the reaction of the class and make sure everyone was following. These included questions such as: How would you like to be taught this particular concept? (a) by a definition, (b) an example, (c) a picture. Now that you have seen the examples, what would you prefer to see next? Students were also asked a number of True/False opinion questions regarding their understanding. (e.g. I fully understand this theorem. T/F). Data was collected from the student clicker quizzes to try to establish the effect of a particular order on student attitudes and learning. It was noticeable that this approach changed the class atmosphere and it appeared that students were more involved and engaged, started to respond better and embraced the lecture style.

Other forms of data gathering occurred through the lecturer's daily journals for each lecture, specific in-class activities, homework assignments, tests, final examination questions and student interviews, which are still under analysis. Of the 82 students in the classes 68 gave consent for their data and course material to be used. In addition, during the final two weeks of the course, 10 student volunteers from classes C1 and C2 were given semi-structured interviews by a colleague, using questions such as: Did you notice any difference in the way Dr. Stewart taught different concepts in her lessons this semester? If so, in what way were they different? If not, was her approach in teaching concepts always the same? If you prefer teaching to start with one particular approach, which one would it be? Can you explain why you prefer this approach? Do you think that step should always come first (second, third), or are there situations where you would prefer a different order? Which type of thinking do you prefer, or feel most comfortable with: embodied, symbolic or formal? Do you think any of these types of thinking is more important than the others in mathematics? If so, which one? What do you think about clicker questions (quizzes and opinion)?

The research questions for this part of the study are: Is there a preferred order of exposure to linear algebra concepts (based on Embodied, Symbolic and Formal)? Do different categories of students (eg geometric, symbolic and versatile thinkers) prefer different orders? Is there any influence of order of presentation on understanding?

## RESULTS AND DISCUSSION

We reiterate that concepts were taught to each class using all three worlds of embodied, symbolic and formal mathematical thinking, but in different orders. This section considers several examples of the effect of these different orders.

## Example 1: Linear Combination and Span, FSE versus ESF

The concept of linear combination was introduced in two different ways: one class, C1, met the concept first through its formal definition, then through symbolic examples and finally through embodied pictures (FSE), while the other class, C2, met the concept first through a pictorial embodied explanation, then symbolic examples and finally through the formal definition (ESF). Clicker responses were used to gather answers to questions. Of course, one problem with some of the categories used (see below) is that they are not necessarily mutually exclusive but still force a choice. For example, a student could think that their understanding is complete *and* that they didn't get much from the definition.

Following three geometric examples of linear combinations in $\mathbb{R}^2$ and $\mathbb{R}^3$, the ESF students in C1 were asked to use the clickers to select from: A) Pictures were fine but I need a definition to understand the concept and B) Examples are all I need to understand the concept. 36% chose A) and 64% B), indicating that even at this stage the embodied view was useful for a number of students, and most preferred to add examples than a definition. In contrast the FSE group of students, C2, was first show a formal, algebraic definition and asked to select from: A) Now that I have seen the definition my understanding is complete, B) I didn't get much from the definition, C) I need some examples and D) I need a picture. The percentages choosing each option were 33%, 14%, 42% and 8%, respectively, with 3% not selecting any. This suggests that a third of the students felt that the definition was sufficient for them to understand the concept, but a larger percentage still needed some examples.

The second step for the ESF group (C1) was the introduction of some symbolic, matrix-based examples and a link to consistent solutions of the equation $A\mathbf{x} = \mathbf{b}$. Once again they were asked to choose from the options A) I completely understand the concept, B) I need to study a bit more to understand this and C) I am ready to see a definition of the concept. 44% now claimed to understand the concept completely but 52% still needed more study to understand. Only 4% said they were ready for the definition. The FSE group (C2) had exactly the same symbolic, matrix-based examples and link, but were not asked about their understanding at that point.

The final phase for the ESF students was to be given the formal, algebraic definition of linear combination and following this they used the clickers to choose between A) now that I have seen the definition my understanding is complete and B) I didn't get much from the definition. In the event, 73% said they now understood completely and 27% did not get much from the definition. At this point the FSE group was given the same three embodied, geometrical examples that the ESF students started with. Finally they were given the choice between A) Pictures were great, I always learn better when I see a picture, B) I am not in favor of pictures, I learn mainly from examples and C) I first look for a definition. 36% were pleased to see the pictures (A), 50% said they learn mainly from examples and 14% went for the definitions option. They weren't asked about their understanding at this stage.

Following the lectures on linear combination the two groups were taught span, which, of course, is based on the concept of linear combination. The class C1 was taught this using the order FSE, the order C2 had received first, while the FSE students in C2 were presented with ESF ordered material, so both groups experienced each order of presentation. After the formal definition the C1 students responded to A) I completely understand the definition, B) This definition is very abstract – I don't understand it and C) I need more time to understand it, with just 13% saying they understood it, 45% saying they didn't and 39% wanting more time. After being shown a geometric picture 21% chose A) I am still not sure about the span, while 79% selected B) I am happy about the idea of span. Due to time constraints it was the start of the next week when the C1 students were asked to choose from the options: A) I can't remember much from last week; B) I remember the definition of Span; C) I remember some pics; D) I remember the story: *once upon a time there were two vectors, together they spanned the entire* $\mathbb{R}^2$ and E) I remember the examples. 30% claimed not to remember much, 9% said they remembered the definition and 9% some pics, while 39% could recall the story of the two vectors and 13% the examples.

For group C2, following the geometry 20% said A) I am not sure about the span. I don't really get it, but 80% chose B) So far what you are saying does make sense. Following some symbolic, matrix examples 63% were convinced that A) I completely understand it now, 8% said B) I need a concrete definition now and 26% went for C) I need more examples (and 3% were uncommitted). Interestingly, after they had been presented with the formal definition only 44% said A) I completely understand the definition, 14% thought B) This definition is very abstract – I don't understand it and 42% were in the category C) I need more time to understand it.

We see that the students preferred different routes to using the formal definition to gain understanding of linear combination and span, but this was an essential part of the picture for them, often cementing together their geometric and matrix ideas.

**Example 2: Subspaces, FSE versus ESF**

The concept of subspaces was introduced using the same orders FSE and ESF as linear combination. The contrast between the initial introductions is quite stark. After seeing the formal definition of a subspace and the theorem requiring to check closure for a non-empty subset to be a subspace, only 37% of the class felt they understood anything, the rest feeling lost (22%) or in need of more time to think about it (41%). On the other hand, after the pictorial embodied introduction 83% felt they had at least partial understanding (44% thought they had complete understanding). Asked what would help them understand better, about 80% of both groups of students wanted (symbolic) examples. Fortunately the plan for both classes was to supply that very need. After seeing some symbolic examples almost all students in both classes felt they had at least partial understanding (97% of the FSE class and 91% of the ESF class) but only 21% of the FSE class felt they had full understanding whereas 71% of the ESF class did. By the time each class had experienced all three worlds, however, their feelings were essentially identical, with 68% (FSE) or 69% (ESF) feeling they had full understanding, and another 29% or 23% (respectively) claiming partial understanding.

Students' actual understanding was sampled at the end of the same lecture, with two true-false questions, and again with another true-false question at the start of the following lecture. Students in the ESF class performed slightly better at the end of the first lecture (with 85% getting the first question correct and 95% the second, compared with 76% and 93% for the FSE class) but by the time of the following lecture there was hardly any difference between the two groups of students with the ESF class actually performing slightly worse this time (with 49% of the ESF class choosing the correct option and 54% of the FSE class).

**Student interviews**

In the interviews students displayed a wide variety of preferences while often cautioning that not all concepts would lend themselves to the same treatment and that not all students would have the same learning styles.

When asked to nominate which of the three worlds (embodied, symbolic or formal) they felt most comfortable, eight of the ten students chose the symbolic world. However, one of these (Ed) qualified his answer: "Symbolic is the easiest for me but I enjoy formal thinking the most." The other two students both saw themselves as visual learners, but Rod went on to say that "in the cases where the pictures won't work I guess symbolic would be the best" and Wade pointed out that in some cases "the picture either might throw you off or, if you don't know what it's talking about, it's not going to help you learn it." This is consistent with what we saw in Example 1, where about 80% of both classes asked for more (symbolic) examples when the concept had been introduced through either the formal or the embodied world in Example 1.

Students were also asked if they had a preferred order in which they would like to meet the embodied, symbolic or formal aspects of a new concept. Most, but not all, of the interviewed students felt that the formal aspect should come last. Typical of the non-visual people was Jenny: "I like to see the examples on how to work through it and then maybe go back and understand what we did from the definition side of it." On the other hand Rod identified himself as a visual learner: "The visual idea of something usually is enough to make it work out for me. So whenever I get examples and then definitions I can understand it better." On the other hand, two students could see reasons for looking at the formal aspect first. James, studying mechanical engineering, preferred to follow the habit of his engineering classes where:

> What we'll do is, first we'll prove or do a derivation of what we're about to do; then we'll do an example of what we just derived; and then our professor, most of the time, will show a visual representation of what we just did. So that's just how I think.

Andrew didn't "know if pictures would make sense coming before the theorem" but he saw a role for the formal aspect at the start and the end:

> but I do, like I said theorem I like to see before and after I think. I think it makes more sense because like seeing it before at least it introduces you to it even if you don't know what it means and then you see like examples and maybe a picture and then you see the theorem afterwards it makes, it kind of cements it a little bit more and then you can see how it relates to an actual example.

The majority view here perhaps reflects what we saw in Example 2, where presenting the formal aspect of subspaces first resulted in only 37% feeling they had understood anything, as opposed to 83% when the embodied aspect was put first.

Several students emphasized the importance of looking at all three aspects (embodied, symbolic or formal) even when they had definite preferences for the order in which they wanted meet these aspects. Sara preferred the order ESF but rejected the idea that one of these might be more important: "No, I think they all go together equally." But John knew that if he "had to say" which was more important, it would be the formal: "I'm trying to get better at formal. I'm a math major, I have to get better at that." But there was a feeling that this only applied to mathematics majors. James pointed out that most of the students in his class were engineering or meteorology majors and for them "application-wise, I think the symbolic world would probably be more important." Ed echoed this distinction: "if I'm going to enter mathematics as a profession, then I need to be very well grounded in formal mathematics, more so than in symbolic mathematics, like, symbolic is more the applied mathematics area."

## CONCLUSION

The initial analysis of the data above shows that students noticed the fact that the lecturer was tailoring the material to their needs, and they appreciated this. The students were always keen to see examples of the concepts and we found that student affect is much more positive when concepts are first met in the embodied or symbolic worlds, but that once students have met all three aspects of a concept there seems to be little difference in the levels of understanding gained. One of the aims of the lectures was for students to appreciate for themselves the power of formal world thinking, and that examples alone are often insufficient. Most students did value the formal definitions of concepts, whenever they were introduced, but often found them more challenging. This may be because it takes time to appreciate fully all the details of a formal definition and why they are important. By the end of the course student perspectives on formal aspects of mathematics, definitions, theorems and proofs, were much more positive than at the start. Integrating the power of the mathematical thinking in each of the three worlds is not a simple matter. We hope that the results of this study will contribute to the thinking and practice of the many university teachers who are seeking to do so.

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