BLACK AND WHITE MARBLES – OLDER PRIMARY STUDENTS' INTUITIVE CONCEPTIONS AND APPROACHES CONCERNING RATIOS

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We encounter ratios on a daily basis. They also play an important role as a basic construct of thinking in many areas of school mathematics. For example, a fraction can be interpreted as the ratio of a part to the respective whole. Many children appear to have difficulties with fractions and although the concept of ratios is crucial for this subject area, there has been hardly any scientific research on how the understanding of ratios is developed. In this article, we will highlight, using the "marbles problems", how children between 3rd and 6th grade handle ratios.

INTRODUCTION

In our everyday lives, we come across ratios in various situations. We find them in proportions, game and election results, probabilities, physical quantities such as velocity or density, mixing instructions in recipes, and in scales, to name just a few examples. In some of these cases, the information is expressed using fractions as special ratios; however, ratios are much more multifaceted. Fractions always show parts in relation to the respective whole (**part-whole ratio – PW**), while ratios also indicate the relationship between the parts of a whole (**part-part ratio – PP**), e.g. in game results or mixing proportions. A third way of interpreting the concept of ratio can be seen in the example of velocity, where quantities of different types are put into relation to each other. In the event of such **rate problems (RP)**, new quantities are often formed through reification, which can then be used as new objects of thinking. Furthermore, ratios can be used as *concrete ratios* to characterise a concrete situation or a certain object. According to Führer (1999), this is a formative description method ("*gestaltliches Beschreibungsmittel*"). Also, they can be applied as equivalent ratios in an *abstract* way. In colloquial (German) language use, the term "ratio" is used even more extensively. Führer (2004, p. 46) explains that the German word for "ratio" ("*Verhältnis*") is used very often, when at least two objects are related to each other in any way ("*[v]on einem Verhältnis spricht man oft schon, wenn mindestens zwei Objekte nur irgendwie in Beziehung gesetzt werden*"). For further details and the basic mathematical principles refer to Rink (2013).

It quickly becomes clear that PW ratios (fractions, percentages, probabilities etc.) and PP ratios (scales, similarities, intercept theorem etc.) play an extremely important role in school mathematics; however, in Germany this topic is not covered explicitly until 7th grade. Rate problems (RP), on the other hand, are only scarcely discussed beyond theoretical contexts in mathematics lessons.

Particularly in Anglophone countries, extensive research has already been done on the abilities in handling ratios (e.g. Hart, 1980; Karplus et al., 1983). However, the participating children were always at least 12 years old. With reference to Piaget (Piaget & Inhelder, 1973), it has been widely assumed that younger students are not able to handle ratios successfully. Some studies (e.g. Streefland, 1984) – though lacking a theoretical differentiation of the concept of ratio and respective variations of test items – and first studies conducted by the second author (Rink, 2013), however, show that primary school children absolutely have the potential to handle ratios successfully. Even if multiplicative thinking is considered to be the probably most important requirement for dealing with ratios successfully (Rink, 2013), children are actually able to discuss ratios on a qualitative level before being taught multiplication in school (Adhami, 2004; Streefland, 1984).

The importance of ratios in our everyday lives and in school mathematics on the one hand, and the apparently related high cognitive requirements on the other hand, seem to call for further systematic research on this matter.

RESEARCH QUESTIONS AND USED METHODS

The following pilot study shows only examples of the capabilities of older primary school children[1] in handling ratios, however studying a bigger group including pupils from four grades. The subjects of the study were primary students from 3rd to 6th grade. We were interested in their "natural" way of handling certain ratio problems before corresponding algorithms and concepts are systematically taught in school. The youngest participants of the study were pupils who had just started 3rd grade, which makes sense, because at this age children have a solid understanding of multiplication. This is considered to be crucial in the successful handling of ratios and, in Germany, is first taught in 2nd grade. Furthermore, this composition of participants gave us a chance to also look into possible effects of the systematic introduction of fractions in 5th grade. In doing so, our aim was not only to gather quantitative information of resolution rates, but also to particularly investigate the pupils' methods in a qualitative manner.

Regarding the high importance of the ratio types PW and PP in school mathematics, we decided to make them the main focus of this pilot study. Further, these ratio types allow for context-free problem situations, thus reducing the influence of previous experience and (mis-)conceptions from non-mathematical areas. In contrast, most previous studies on the subject of handling ratios placed the problems in various contexts. On the one hand, this ensured that the participants understood that they had to work with ratios, but, on the other hand, the results varied greatly and were barely comparable (e.g. Hart, 1980; Noelting, 1980; Streefland, 1984; Rink, 2013). Also, these studies usually did not take possible changes over several education levels into account.

Another aim of our research was to understand the pupils' ideas of the concept of "ratio" on a linguistic level as well as possible relations to their abilities of handling the

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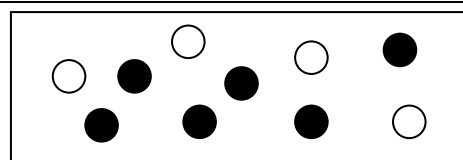[1] In the state of Berlin, primary school comprises grades 1 to 6.

respective problems. We are not aware of any previous studies on this subject, neither in Germany nor internationally.

The study was mainly carried out in Berlin primary schools during the first few weeks of the school year. In order to ensure as much heterogeneity within the group of participants as possible, eleven schools from different urban school catchment areas with and without mixed-level learning groups were involved. All 231 participating schoolchildren were asked to do the following exercise in writing:

In a box, there are 10 marbles –

black and white ones.
Take a look at the image.

1. How many white marbles would you have to remove from these 10 marbles, to leave half as many white as there are black marbles? Explain your solution.

2. How many black marbles would you have to add to these 10 marbles, so that three quarters of all marbles are black? Explain your solution.

3. What is the ratio between the white and the black marbles? Explain your solution.

Figure 1: Marbles problem

The first problem covers PP ratios, which play an important role in everyday life. Since "half as many" is a rather simple ratio and, moreover, the reference quantity is known, the question concerning the number of white marbles should be relatively easy to answer for many children. It is interesting to observe how, going from there, the pupils manage the transition to the second question, which is a PW problem. Not only does the ratio "three quarters" make it more challenging, but it is especially more difficult because the reference quantity is unknown. While the first two problems work without the term *ratio*, the third question requires the pupils to explain their intuitive understanding of the concept. This only happens in the last problem in order to keep the influence of possibly induced ideas, associations or affects on the first two problems to a minimum.

The data collection was carried out by student teachers, who were introduced in the study beforehand to guarantee a widely consistent organisational framework. Afterwards, the data was analysed in tandem by the two authors of this study. In order to conduct qualitative analyses of approaches, we developed descriptive categories based on the collected data (bottom-up), which also refer to elements of the theoretical analysis (top-down) of the ratio concept (cf. section 1).

RESULTS

The table below shows the resolution rates for the first two marble problems. These results confirm, on the one hand, our a priori estimation of the level of difficulty, which is also supported by the fact that only three of 231 participating pupils were able to

handle the second problem successfully without having answered the first question correctly.

grade	number of pupils	resolution rate of problem 1	resolution rate of problem 2
3	61	70.5%	19.7%
4	57	82.5%	14.0%
5	45	66.7%	15.6%
6	68	80.9%	23.5%
total	**231**	**75.8%**	**18.6%**[2]

Table 1: Resolution rates of problems 1 and 2

Even if this does not constitute an actual longitudinal study, the only slightly changing resolution rates and missing tendencies across the education levels suggest, on the other hand, that without systematic teaching of the ratio concept in mathematics lessons, there is no major capabilities increase in this field.
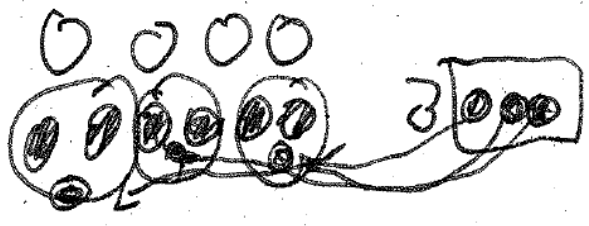
Problem solving and explaining approaches for the second marbles problem

Table 1 shows clearly that only a small part of the 3rd- to 6th-grade students participating in the study were able to solve the second marbles problem successfully. Additionally, only 68% of these children wrote an explanation for their answers. However, the results still show a broad spectrum of correct or improvable problem solving and explaining approaches, as presented below. On the one hand, this shows the existing capabilities and potentials of the group of participating schoolchildren and, on the other hand, it offers indications for the didactic organisation of teaching the ratio concept in mathematics lessons.

Anne (9 years old, 4th grade) solves the problem in the following way:	
Man müsste sechs schwarze hinzufügen weil 4·3=12 ist. Weil, wenn man die weisen 3x nimmt sind es 12 und 6+6=12 schwarz	Description: **Three times as many are black.** Even if the problem text description suggests a PW ratio, this pupil handles the exercise with a PP approach and triples the number of white marbles in order to determine the number of black marbles. Fractions or ratios, however, are not expressed.
Translation: *You would have to add six black marbles, because 4x3=12. Because if you multiply the white ones by 3, it's 12 and 6+6 (→black) = 12*	

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[2] Some pupils suggested the solution "three black marbles". This would be correct, if one were to take the result of problem 1 as a starting point. Under consideration of these suggestions, the resolution rate would be 24.2%.

Bea (11 years old, 6th grade) solves the problem in the following way:	
	Description: **¼ is 4, ¾ is 12.** From the quarter of a whole three quarters are calculated – similar to Anne's multiplication approach. The girl does not name the whole, which is why the approach can be interpreted as PP approach.
Translation: *- add 6* *- I counted the white and the black ones* *4=¼ x3=12=¾*	

Charles (9 years old, 3rd grade) uses a sketch.	
	Description: **PP approach with a sketch.** This boy draws 3 black marbles for each white marble. Because Charles forgets one marble, he receives an incorrect result.

Danny (11 years old, 6th grade) solves the problem in the following way:	
	Description: **3:4 → 4:12.** This pupil develops a PP ratio that goes with the PW ratio provided in the problem text description. Danny's exact approach cannot be reproduced.
Translation: *You would have to add six black marbles to get the ratio 4:12 (w.:b.) and 16:4 = 4 and 12:4=3.*	

Ethan (10 years old, 5th grade) solves the problem in the following way:	
	Description: **¼ of all marbles is white.** In this PW approach, the pupil uses the number of white marbles to determine the total number and, going from there, calculates the number of black marbles that need to be added.
Translation: *You have to add 6 black marbles, because 4 marbles are a quarter, because there are 4 white marbles and 4x4=16, so that's why it has to be plus 6.*	

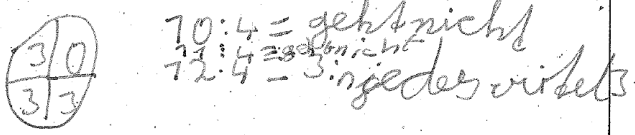| Frieda (8 years old, 4th grade) "tries" to reach a solution.	
	Description: **Trying (PW).** Frieda attempts to reach the required, modified PW ratio by trying different options.
Translation: *10:4 = doesn't work* *11:4 = doesn't work* *12:4 = 3 in each quarter*	

Figure 2: Problem solving and explaining approaches

We were able to identify the first approach in a particularly large part of problem solutions collected from the participating pupils. It seems remarkable that about three quarter of the exactly reconstructable approaches can be interpreted as PP approaches, even if the second marbles problem is actually a PW problem.

Pupils' conceptions on ratio

The participating schoolchildren's solutions for the third problem showed a very broad spectrum of understanding the term "ratio". We are seeking to illustrate this in a first approach by using the following answer categories, which have been developed based on the collected data:

- *ratio*: Answers of this category specify the ratio between the numbers of white and black marbles, e.g. in the forms "4:6" or "4 to 6", sometimes even naming the term "ratio". Pupils of higher education levels also used percentages, as in "40% to 60%", or "cancelled" ratios, such as "1:1.5" or "1:1½". None of the participants, however, used the basic ratio 2:3. All answers in which pupils specified a ratio were correct.
- *comparison*: A big part of the children compared the numbers of marbles according to their respective cardinality and said that there are "more black than white" marbles. Sometimes they commented on how many white marbles would have to be added in order to leave the same number of black and white marbles.
- *geometric*: Some of the children described the position or formation of the marbles, e.g. "*There are two rows of marbles*" or "*The marbles are facing each other*".
- *no answer*: A major part of the participating pupils did not write an answer.
- *others*

Table 2 illustrates that children of higher education levels showed, as expected, increasing capabilities in expressing scientifically correct ideas of the term "ratio". However, even among 6th-grade pupils, still less than half of the participants succeeded in putting an appropriate answer into writing. It was also rather surprising that answers of the category *geometric* were given more often by older pupils than younger ones.

	3rd grade	**4th grade**	**5th grade**	**6th grade**
ratio	23%	19%	37%	44%
comparison	9%	33%	19%	6%
geometric	0%	1%	8%	11%
others	23%	17%	13%	11%
no answer	45%	29%	22%	20%

Table 2: Pupils' answers to the third marbles problem

We did not detect any statistically significant links between the types of pupils' answers to the third marbles problem and the successful solving of the first two problems. This outcome matches the results of other studies which showed that children have the ability to work with ratios on a *qualitative* level at a very young age already, without needing any rather formal aspects (Adhami, 2004; Lorenz, 2011).

DISCUSSION

The pilot study presented in this article shows that most of the 231 participating 3rd- to 6th-grade pupils were able to solve an easy ratio problem successfully. However, the results of the second marbles problem also indicates that the required capabilities among the studied age group do not develop by themselves, but rather require systematic teaching in mathematics lessons. Furthermore, the problem solving and explaining strategies applied for the second problem suggest a certain flexibility and confidence of the pupils in working on PP problems. Therefore, PP problems and their respective solving approaches might possibly be used as starting points for suitable learning trajectories.

In an already planned follow-up study, the number of participating schools and thus the heterogeneity within the group of participants will be increased. This study will not only include systematically varying PP and PW problems, but also rate problems that are phrased with little context, e.g. using exchange situations.

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