VALIDATING IN THE MATHEMATICS CLASSROOM

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The focus of this report is on the process of resolution of a task; specifically, on the validation of the mathematical model proposed by a group of students and the numerical result that is constructed within this model. Habermas' construct of rational behavior is used to describe validity conditions that emerge and are used by the students as means for validation. We take a classroom episode from a design experiment to examine how the emergence of these conditions points to a socially constituted mathematical epistemology in the secondary school mathematics classroom, to shared and tacit principles of the didactic contract concerning the knowledge there, and to non-mathematical references that are taken for granted.

CONTEXT AND RESEARCH QUESTION

According to Habermas (2003), accepting a validity claim is tantamount to accepting that its legitimacy may be adequately justified, that is, that the conditions for validity may be fulfilled. Following Boero, Douek, Morselli and Pedemonte (2010), we are interested in characterizing how, in terms of Habermas' rational behavior construct, mathematical activity is supported through the situated emergence and fulfillment of validity conditions. Specifically, in the context of the secondary mathematics classroom, we want to investigate what situated validity conditions (acceptance/ rejection) can be observed concerning the resolution process of a task. The current research, as well as previous studies by our team (e.g. Goizueta & Planas, 2013), is in line with the commitment to conceptualizing mathematical argumentation and learning in whole and small group discussions in the mathematics classroom.

We begin by showing how we complement Habermas' construct to better suit the complexity and specificity of the mathematics classroom. This perspective frames our understanding of classroom practices and our approach to data analysis through the integration of social and epistemological issues. We then present and analyze a classroom episode to discuss the emergence of validity conditions throughout the construction of a mathematical model in a problem-solving environment. From here, we briefly discuss the classroom mathematics culture that is being propagated.

AN INTEGRATED PERSPECTIVE

For the interpretation of what counts as validity conditions and how and why they emerge, we draw on epistemological and social issues. According to Habermas' construct of rational behavior and its adaptation by Boero et al. (2010), in the students' argumentative practices we can distinguish an epistemic dimension (inherent in the epistemologically constrained construction and control of propositions, justifications

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and validations), a teleological dimension (inherent in the strategic decision-making processes embedded in the goal-oriented classroom environment) and a communicative dimension (inherent in the selection of suitable registers and semiotic means to communicate within the given mathematical culture).

Steinbring (2005) suggests that a "specific social epistemology of mathematical knowledge is constituted in classroom interaction" (p. 35) along with a criterion of mathematical correctness. The socially constituted mathematics classroom epistemology and the mathematical activity are reciprocally dependent: the former shapes a frame in which the latter takes place and the latter develops the former to conform to the emergence of new legitimated mathematical discourses. In the context of a content-particular task, this relationship between classroom epistemology and mathematical activity must be considered in light of a specific, content-related, didactic contract (Brousseau, 1997). Furthermore, when considering students' interaction we also consider the references (statements, axioms, visual and experimental evidence, physical constraints, etc.) that may be associated with their argumentative activity. Some references are related to institutionalized corpora (e.g. school mathematical knowledge), but not all of them are. Douek (2007) introduces the notion of reference corpus, which is assumed to be unquestionable and shared; it is operatively used by the students to make sense of the task, semantically grounds their mathematical activity and backs their arguments. Thus, in a task-specific context, validity conditions are explicit or tacit constraints that allow students to control the coherence of the mathematical activity according to the socially constituted classroom epistemology, the reference corpus and the goals.

In what follows, we present the analysis of an episode to illustrate different emerging validity conditions behind the observed practices that students in the classroom enact to validate their arguments. We also account for possible explanations.

PARTICIPANTS, TASK AND DATA COLLECTION

The participants in our design experiment were thirty 14/15-year-old students and their teacher in two lessons in a regular classroom in Barcelona, Catalonia-Spain. It was a problem-solving session, with time for small group work and whole-class discussion. The following task for the two lessons was suggested by the researchers:

> Two players are flipping a coin in such a way that the first one wins a point with every head and the other wins a point with every tail. Each is betting €3 and they agree that the first to reach 8 points gets the €6. Unexpectedly, they are asked to interrupt the game when one of them has 7 points and the other 5. How should they split the bet? Justify your answer.

As we ascertained in a pilot experiment, this task can be approached and solved using arithmetical tools, without having been taught formal probability contents, which was the case of this group, although intuitive probabilistic thinking is fundamental to provide a mathematically sound answer. The novelty of the task was expected to lead students to develop models and negotiate meanings, while producing arguments to validate them and avoiding mechanical approaches based on well-established heuristics. For data collection, two small groups were videotaped and written protocols were collected. For each of the videotaped groups, students were collectively interviewed a week after the task; this set of data is not discussed here.

DATA ANALYSIS AND RESULTS

The excerpt below illustrates the attempts to cope with the task by the group made up of Anna, Josy, Vasi and Zoe. We point to the students' rational efforts to support the validity of their resolution process; we then account for different types of validity conditions and associate their emergence to different dimensions of the students' rational behavior.

Genesis of the initial validity conditions

112	Anna	This one only needs to get one point and this one three to get to six euros. But obviously, because it's random, the game, you know, one's got more chances because imagine that now, suddenly, if the game didn't stop, you could get three tails in a row and then this one would win. So A does have more chances of winning but B could win as well (…) From what we've got so far, player A would have to get more money… because he's got more points.

Anna interprets the need to split the money in relation to the advantage one player has over the other. By describing the situation as a random game and bringing chance to the fore, she tacitly proposes a frame with which to interpret the task, and within it, she draws on prior experiences with coin-flipping situations, shared notions about the characteristics and dynamics of the game and adequate words to talk about it. This cluster of references empirically and semantically grounds Anna's interpretation of the task and her reference corpus. The meanings that students may associate with this reference corpus act as constraints that any possible answer should meet; it gives the students an operational way to decide not only on the validity of any proposed answer but also on the validity of any mathematical model within which the answer is elaborated. Anna states what can be taken as a necessary validity condition for any possible answer: "player A would have to get more money… because he's got more points." Similarly, "B could win as well" might mean "B should get some part of the bet". This could be considered a validity condition for any forthcoming answer, but since she does not make it explicit, we cannot know the actual status of this statement at this point.

We observe the emergence of a first validity condition that any model of the situation should satisfy: player A gets more money than player B and, possibly, player B gets more than zero. In terms of Habermas' construct, it is the epistemic dimension of Anna's rational behavior that leads her to establish constraints that match her interpretation of the goal-oriented task according a specific reference corpus. This validity condition is epistemological in nature. Anna's rational activity supports two parallel processes: the abductive search for a plausible model to describe the situation and find a solution to the task, and the justification of its situated legitimacy.

First model: "one point, how much money"

116	Anna	So then, if six is the total...
117	Vasi	We've got to calculate, if we calculate how much a point is worth.
118	Anna	Wait, wait.
119	Zoe	What if we work out the percentage?
120	Anna	We've got to say one point, how much money.
121	Vasi	That's what I've just said!
123	Zoe	One of the eight...
124	Anna	Yeah, one of the eight, how much is it worth. You know? Six over eight? No, sorry. Yeah, six over eight?
125	Josy	Zero point seven five.
137	Zoe	… So, one point is zero point seven five.

When Vasi reminds the group of the need to resort to calculation, we recognize a constraint imposed by the didactic contract: any possible correct answer must be mathematics-related. Behind her suggestion of calculating "how much a point is worth" –marked by the use of 'have to'– and behind Zoe's suggestion of working out the percentage, we observe how the teleological dimension of the students' rationality guides their efforts to seek a suitable mathematical model, according to normative and goal-oriented constraints. The utterance "one of the eight … one point is zero point seven five" condenses the first model to describe and solve the situation. We may paraphrase it as, 'if by winning 8 points a player gets €6, for each point won a player should get €0.75'. We relate the emergence of this model to typical school problems about proportional costs, which tend to be solved by manipulating the numerical data appearing in the wording. The use of 'to be worth' and the proposed calculation support this interpretation. It is plausible that the focus of the students' speaking turns is on proportionality as an adequate mathematical content with which to engage in the task. Thus, the clause of the didactic contract stating that any possible correct answer must be mathematics-related acts as a necessary validity condition and forces the students to discard answers that might be somehow considered non-mathematical and seek mathematics-related ones. Of the rational efforts to solve the task, we can distinguish two different validity conditions: a first epistemic one about the need to account for the reference corpus-based interpretation of the task, and a second normative one about the need to conform to a basic premise of the didactic contract. Under these constraints, we observe the abductive emergence of a first proportional model providing an intermediate result: each player gets €0.75 for each point won.

Model falsification and new validity conditions

139	Josy	No, but then, they only get to six euros if they win eight points and here they've won twelve points.
140	Anna	That's true.

Josy checks the result obtained and realizes that accepting the proposed model necessarily leads to an incoherent interpretation of the situation, and in so doing, she is falsifying the proposed model. Her reasoning may be related to a well-established principle of the didactic contract about applying proportionality to this kind of problem: a correct result is confirmed by performing 'the opposite operation'. The rejection of a contradictory model accounts for the epistemic dimension of Josy's rational behavior. This falsifier will play a crucial role in deciding about the validity of the model by acting as a new necessary epistemic validity condition: any valid model must be immune to this falsifier. Due to the relation between falsifier and resolution, we call it heuristic. By acting as a validity constraint, this heuristic is at the root of the emergence of a second proportional model and its assessment.

Second model: "we've got to divide by twelve"

142 Anna	Then we've got to divide the seven, hold on, we've got to divide, seven plus five, twelve. So we've got to divide by twelve. How much is it divided by twelve? (…) Zero point five. So zero point five times seven? Calculate that a second.
149 Josy	Three point five.
151 Anna	And zero point five times five?
152 Zoe	Two point five.
153 Anna	And two point five plus three point five?
154 Josy	Exactly.
155 Anna	That's it!

In order to overcome Josy's falsification, Anna proposes a new model that corresponds to a distribution that is proportional to the points won by each player. Driven by the need for epistemic coherence, the students in the group assess this new model using an equivalent version of the heuristic falsifier developed (assuming that 12 points were won): the amounts of money that the players receive must add up to six. By producing a numeric solution for the task (€3.50 for player A and €2.50 for player B) the students prove the model's immunity to the heuristic falsifier and, on that basis, seem to validate the new model and the numerical result. The focus of the students' activity shifts therefore to showing that the new model cannot be falsified in the same way the previous model was, which for them appears to be a positive confirmation of adequacy.

Explaining the solution to the teacher

190 Anna	We thought that... well, player A has got seven points and B five points. We thought that if they won four points each, three euros for each one, and the distribution would be fair. Then we did six euros divided by eight, which is the total... by how many points... I mean, how much one point would be, eight points in total. You know? But then we said no, no, no. Because they got twelve points in total... and then we multiplied each point they won by the 0.5 that one point costs and we got it exactly [on the sheet, "player A: €3.50 and player B: €2.50, "2.5 + 3.5 = 6"].

Regarding the teleological dimension, we assume that Anna's intention is to convince the teacher about the validity of the answer and the model they constructed. Thus, her explicit discourse highlights what she considers relevant reasons for that purpose. She starts by introducing the situation of a tied game that can be considered a generic example, proposes a numerical solution and qualifies it as 'fair'. Anna introduces fairness as a taken-as-shared notion and as the criterion to describe the answer and determine its validity. She then accounts for the first model by focusing on the eight points (needed to win); then discards this model stating, as a reason, that the total points that must be considered is twelve (won points). The explicit, though unclear, mention of the falsifier suggests the relevance that the falsification process had for the students; however, it is difficult to grasp the epistemic roots of such falsifier and its role in the emergence of the second model.

Drawing on her written protocol, Anna then presents the second model, focusing on the role of 0.5 (money per point won) as the intermediate result they used in the group to get the answer. Anna makes their interpretation of the task evident (to split the money according to the points won) and tacitly proposes proportionality as a relevant and adequate mathematical model to solve it. Finally, she says "we got it exactly" while showing in her protocol that 3.5 and 2.5 add up to the original six euros that had to be split. This assessment of the result is significant in the light of the whole solution process, especially if we consider the role played by the heuristic falsifier in discarding the first model and supporting the emergence of the second one. The epistemic status of the result (necessary, plausible, possible...) is not made explicit, but the expression "we got it exactly" constitutes a positive assessment of both the model and the result's validity. This expression takes over the role that 'fair' played in the prior turn: while 'being fair' was the key feature of the proposed distribution that led its validity to be accepted, now 'getting it exactly' is a validation of the answer and becomes the guarantee of 'being immune to the heuristic falsifier.' Anna does not mention the developed validity conditions or their emergence as a rational process in the model's validation. Instead, what appears is the solving process' fit with the didactic contract. The discourse on the validity of the proportional model is evoked during the description of the process, but its recognition is left to the teacher's discernment.

CONCLUDING DISCUSSION

Using Habermas' construct of rational behavior, we have described students' mathematical activity as a twofold rational process: the abductive search for a model to describe the problem situation and solve the task, and the justification of its validity. Initially, the epistemic dimension of the students' rational activity is related to the semantic and empirical grounding of the task according to a reference corpus, leading them to develop an interpretation of the task and the initial necessary epistemic validity conditions. According to the teleological dimension, this suggests that establishing epistemic constraints is a relevant activity to support the construction of a suitable model and that this is done, in part, by creating specific epistemic validity conditions. Later, according to the teleological and communicational dimensions, the didactic contract-related need to provide a mathematics-related answer, acting as a necessary normative validity condition, is what drives the students' efforts towards the construction of a first mathematical model. It is on the epistemic side that the first model is falsified: its inadequacy to account for the available data leads to its falsification and rejection. A second model is then proposed to overcome what we have considered a heuristic falsifier. For the students, the immunity to the falsifier becomes not just a necessary epistemic validity condition but also a confirmation of the model's validity. However, despite the fact that they are key features in the process, the reference corpus-related epistemic constraints developed as well as the epistemic roots of the falsification are almost absent from the explanation to the teacher, which is instead centered on the link between the numerical solution and what is considered a suitable mathematical model. This indicates the relevance that the didactic contract has for the students when selecting what parts of their production to communicate.

Although limited to the analysis of brief excerpts, we have shown how Habermas' construct of rational behavior helps to investigate and account for the emergence of validity conditions as means to support the validity of mathematical activity. We argue that this is a relevant theoretical instrument to investigate the students' situated practices of validation while keeping track of the complex relationship between the epistemic and social dimensions of the mathematics classroom.

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