THE CASE FOR LEARNING TRAJECTORIES RESEARCH

Amy B. Ellis[1], Eric Weber[2], Elise Lockwood[2]

[1]University of Wisconsin-Madison, [2]Oregon State University

This paper addresses the role of learning progressions in informing many international standards documents, discussing the affordances and limitations of building standards and curricula from a learning progression model. An alternate model, the hypothetical learning trajectory, is introduced and contrasted with learning progressions. Using the example of exponential functions, learning progressions are compared to learning trajectories in terms of their theoretical origins and practical implications. Recommendations for further work building learning trajectories in secondary mathematics are discussed.

INTRODUCTION

Curriculum development increasingly relies on guidance from national content standards or benchmarks, with standards-based accountability growing as a movement internationally (e.g., Australian Ministerial Council on Education, Employment, Training and Youth Affairs, 2006; Ministry of Education of the People's Republic of China, 2003; National Governor's Association Center for Best Practices, 2010; UK Department of Education, 2009). Given the proliferation of content standards and their influence on curriculum development, the quality of such standards and their adherence to research on students' learning is a key concern. However, evidence suggests that mathematics content standards typically approach learning goals from the perspective of sophisticated mathematical expertise, failing to address students' conceptual development (Olive & Lobato, 2008). Lobato et al. (2012) conducted a survey of the mathematics content standards for seven countries focusing on quadratic functions and found that nearly all of the standards emphasized procedural knowledge and lacked specificity addressing conceptual knowledge.

This paper discusses a typical approach guiding the development of many standards documents, that of a learning progression, and considers some of the limitations of learning progressions for informing standards and curricula. Using the example of exponential functions, we contrast the theoretical underpinnings of learning progressions with an alternate construct, the learning trajectory, and argue for the merits of learning trajectory research for developing content standards.

BACKGROUND AND THEORETICAL FRAMEWORK: LEARNING PROGRESSIONS AND LEARNING TRAJECTORIES

A learning progression is a sequence of successively more complex ways of reasoning about a set of ideas (National Assessment Governing Board, 2008). This definition situates a learning progression as a tool for curriculum design; the progression is a

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construct for organizing mathematical content in order to provide a potential path through which students can traverse as they develop competence in the domain. Recent years have seen an increased focus on the development and elaboration of learning progressions, and in the use of learning progressions to inform standards documents. For instance, the American Institute for Research (2009) released a report calling for the development of standards based on learning progressions gleaned from analysing the content standards of three high-performing countries, Hong Kong, Korea, and Singapore. This study produced a set of composite standards guided by "learning progressions of specific competencies within each topic across grades" (p. 2). Similarly, Fuhrman, Resnick, and Shepard (2009) made the case for incorporating learning progressions into content standards documents by referencing high-performing countries such as Singapore, Japan, South Korea, and the Czech Republic, emphasizing the importance of building curricula "based on sequences, or progressions, of increasingly sophisticated concepts and knowledge applications" (p. 28). A learning progression characterizes movement from novice to expert through the acquisition of relevant facts, skills, and concepts (National Assessment Governing Board, 2008).

Learning progressions have at times been treated as interchangeable with learning trajectories, but the two constructs have significantly different theoretical origins (Empson, 2011). The notion of a hypothetical learning trajectory has different meanings among mathematics education researchers. Simon's (1995) original discussion offered a description of a hypothetical learning trajectory consisting of "the learning goal, the learning activities, and the thinking and learning in which students might engage" (p. 133). Clements and Sarama (2004) expand on this definition, describing a learning trajectory as an elaboration of children's thinking and learning in a specific mathematical domain, connected to a conjectured route through a set of tasks designed to support movement through a progression of levels of thinking. These definitions emphasize the construct as a teacher-researcher's model, a tool for hypothesizing what students might understand about a particular mathematical topic and how students' understanding may change over time in interaction with carefully-designed tasks and teaching actions.

A learning trajectory is an account of changes in a student's schemes and operations; as such, it is a tool that seeks to explain learning that occurs over time, specifying the particular schemes and operations in play and elaborating how accommodation occurs to build up knowledge. This view of a learning trajectory differs significantly from learning progression frameworks emphasizing strategies or skills.

Challenges with Basing Standards Documents on Learning Progressions

Learning progressions are based on the researcher's knowledge of the field of mathematics. Steffe and Olive (2010) describe this as first-order knowledge, "the models an individual constructs to organize, comprehend, and control his or her experience, i.e., their own mathematical knowledge" (p. 16). Much of the organization of international content standards is based on first-order knowledge. Critiques of the learning progression approach to standards documents emphasize, however, that the development of a progression cannot be based on an analysis of the discipline alone. In particular, content learning cannot be separated from activity and context; what students learn is intricately connected to the types of instructional tasks they encounter, the manner in which teachers foster students' thinking with those tasks, and the ways in which students interact with one another and with their teachers (Empson, 2011). Mathematical learning occurs in interaction, with teachers' actions profoundly influencing student thinking. One of the most difficult issues facing researchers constructing learning progressions, then, is the need to attend more explicitly to the role played by teaching interactions and to determine how instructional variation affects these progressions (Simon et al., 2010).

These concerns are borne out by the meticulous research base demonstrating the non-convergence of children's learning in some areas of mathematics, such as number, fractions, and ratio and proportion (e.g., Steffe & Olive, 2010). In addition, standards based on learning progressions may fail to account for how different students approach the same mathematical idea from different conceptual bases. A more efficacious approach may be one that attends to the variation in students' conceptual development, building trajectories of student understanding over time.

LEARNING TRAJECTORIES AS AN ALTERNATE MODEL

While learning progressions are typically based on first-order knowledge, learning trajectories are an elaboration of researchers' second-order mathematical knowledge, "the models observers may construct of the observed person's knowledge" (Steffe & Olive, 2010, p. 16). As such learning trajectories are concerned with identifying the mathematics of students, elaborating models of students' mathematical concepts and operations. Lobato et al. (2012) noted that an analysis of students' constructions can also inform the way researchers conceive of the mathematics itself; a construction of second-order models can inform our first-order knowledge of the domain.

A learning progression typically presents a target construct or skill, an associated learning goal, evidence for achievement of the learning goal, and tasks designed to foster that achievement. Table 1 contrasts the ways in which learning progressions and learning trajectories address each of these four categories in general. Using the specific example of exponential functions, we then compiled typical statements of mathematical constructs, learning goals, and evidence from international standards documents that included exponential functions, particularly Chinese Taipei (Ministry of Education of Taiwan, 2003), China (Ministry of Education of the People's Republic of China, 2003), and the United States (National Governor's Association Center for Best Practices, 2010) (see Table 2). Table 2 contrasts a learning progression approach with a learning trajectory approach for one sample construct/concept about exponential functions; due to space constraints, one example rather than an entire progression and trajectory is provided.

Learning Progression		Learning Trajectory	
Construct	• Based on 1st-order knowledge • Define levels in terms of subject-matter competencies • Constructs elaborated as formal mathematics	Concept	• Based on 2nd-order knowledge • Define stages of student thinking • Concepts elaborated in terms of students' mental activity
Learning Goals	• Description of skills and procedures • Specifies target performances	Character-ization	• Description of the nature of student's thinking • Identifies relevant schemes
Evidence	• Describes the necessary performance; focus on external performance • Identifies external strategies • Based on mathematical domain	Examples	• Describes conceptions based on strategies, language, activity • Identifies mental activity • Based on evidence of student activity
Tasks	• Developed from content analysis • Goal is to elicit target performances • Provided as stand-alone problems	Activities	• Developed from retrospective analysis of teaching experiments • Goal is to support emerging concept development • Provided with context and pedagogical connections

Table 1: Learning progressions contrasted with learning trajectories

Construct versus Concept

Constructs arise from adults' first-order knowledge of mathematics, and thus are developed according to the logic of the discipline. It is typical for constructs to describe formal mathematical ideas, strategies, or procedures. For instance, consider the case of exponential functions. A learning progression might describe a construct for exponential functions in terms of the desired subject matter competency without regard to the qualitative difference in thinking at different stages. The construct describes the mathematical idea, for instance, "Express a situation in which a quantity grows by a constant per-cent rate as $y = ab^x$." Rather than specifying a mental operation, the construct specifies a particular algebraic representation, as conceived by the researcher. This type of progression is concerned with identifying instructional goals framed in terms of target performances rather than target concepts.

We can contrast this approach with a learning trajectory approach, drawing on a learning trajectory describing middle-school students' initial understanding of exponential growth (see Ellis et al., 2013). A learning trajectory will define a concept in terms of student understanding, and would base concept definitions on existing knowledge of students' ways of operating. For instance, one conceptual stage students achieve when developing ideas of exponential growth is that they can coordinate multiplicative change in y with additive change in x. A concept at this stage would include the understanding "that the ratio of y_2 to y_1 for a corresponding change in x holds for any Δx value, even when Δx is < 1."

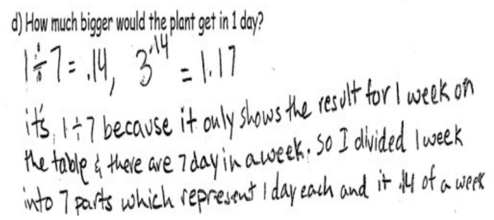
Learning Progression		Learning Trajectory	
Construct	Express situations in which a quantity grows by a constant per-cent rate per unit interval relative to another as $y = ab^x$ where b is a whole number and x is non-negative.	Concept	Coordinate change in y for any-value change in x: Understand that the ratio of y_2 to y_1 for a corresponding change in x holds for any Δx value, even when Δx is < 1.
Learning Goals	• Understand the meanings of the power in an exponential expression • Comprehend the calculations involving base numbers as whole numbers and exponents as non negatives • Interpret the parameters a and b in terms of a context	Character-ization	• One can coordinate the ratio of any two y-values for any-time gaps in corresponding x values. • Imagery is reliant on constant ratios, and is no longer grounded in images of repeated multiplication. • Understanding that the expression b^x can represent both a static height value and a measure of growth for two values x time units apart.
Evidence	• Use repeated multiplication to find missing table values • Write correct equations in the form $y = b^x$ and $y = ab^x$ • Perform correct calculations such as $3^2 \times 3^4 = 3^6$ • Recognize a non-zero a-value as the functions' initial value	Examples	
Tasks	• Missing-value tables and far-prediction problems • Cell growth, population growth, and compound interest modelling problems	Activities	• First provide tasks with only two data points with large-time gaps in which students must determine the growth factor. Large gaps will encourage shifts away from repeated multiplication. • Next, provide tasks in which students must determine amounts of growth for a half-unit or other fractional amount of time.

Table 2: Contrasting a progression with a trajectory for exponential functions

Imagine two students who are at two different stages in their developing understanding of exponential growth. The first student can coordinate the ratio of two y-values for corresponding x-values when $\Delta x \geq 1$, but his mental imagery is grounded in repeated multiplication. For instance, this student may compare the height of an exponentially-growing plant at two different time points: After 2 weeks, the plant is 4 inches tall, and after 5 weeks, the plant is 32 inches tall. This student can conceive of the plant at 5 weeks as 8 times as tall is it was at 2 weeks by taking the 4 inches at 2 weeks and doubling it three times: 8 inches, 16 inches, 32 inches. This student may even be able to express this idea as 2^3, but that expression is grounded in a mental operation of doubling the height three times. This student's ability to imagine a process of repeated multiplication has some limitations; because he must mentally go through the process of doubling in order to compare two values, he cannot extend that process for very large-week gaps, or make sense of gaps smaller than 1.

Imagine a second student whose imagery is no longer grounded in a process of repeated multiplication. This student has mentally truncated the process to the point at which she can think about multiplicatively comparing two heights for large-week gaps and does not have to go through the operation of doubling for each and every week between x_1 and x_2. This student can express the ratio R of two height values as $b^x = R$ for the growth factor b. This expression no longer represents a process of multiplying by the growth factor b x times, but instead is grounded in an image of a constant ratio change in y for any constant additive change in x. This student may use language and gestures to indicate a notion of continuous scaling or magnification, and her imagery enables her to make sense of growth even when Δx is not a whole number. In both cases, the students may write the same algebraic expression b^x, but the expression is a result of different ways of operating and means different things to the two students. A learning trajectory should account for these differences in students' thinking and aim to capture them in its description of conceptual stages.

Learning Goals versus Concept Characterization

In order to develop a learning progression one might engage in task analysis (Gagné, 1977) to identify the capabilities one must possess in order to perform a specific mathematical task. For exponential functions this may include using repeated multiplication to determine missing table values, writing correct equations and performing correct calculations with exponents, and identifying the parameter "a" as the initial value of a function when $x = 0$. Note that these learning goals are framed in terms of target performances.

In contrast, learning trajectories are built on empirical evidence from working with students. The exponential functions learning trajectory emerged from repeated cycles of retrospective analysis of two teaching experiments with groups of middle-school students (see Ellis et al., 2013). Each teaching experiment lasted approximately 15 1-hour sessions and was videotaped and transcribed. Rather than describing target performances, the learning trajectory characterizes the nature of students' thinking at a particular stage, for instance, by specifying that a students' imagery is grounded in constant ratios rather than repeated multiplication. One aim of these characterizations is to explain how students' ways of thinking, schemes, and operations provide an explanation for how they solve problems.

Evidence versus Examples

Learning progressions focus on elaborating the necessary strategies, performances, and other observable behaviour for determining whether a student has met the learning goals. Evidence of this nature does not address how students' conceptions will change as they progress from one level to the next. Rather than providing an account of learning that makes performance possible, the emphasis is on the performance itself, which is taken as evidence of learning. While much can be gained from a careful analysis of students' strategies, a focus on strategies to the exclusion of mental activity leaves much unknown about how learning progresses over time. In contrast, a learning trajectory builds evidence from students' actions in teaching-experiment settings. Ongoing and retrospective analysis informs the construction of models of students' thinking. Here the example evidence from Table 2 is from a task in which students had to predict, for a plant that tripled in height each week, how much larger it would grow in 1 day. One student wrote the expression "$3^{14} = 1.17$", explaining, "I divided 1 week into 7 parts, which represents 1 day each and it's .14 of a week." This is evidence that the student could make sense of a non-integer exponent and could conceive of the expression 3^{14} as a measure of growth, an important feature of coordinating the ratio of *y*-values for time gaps smaller than 1 week.

Tasks versus Activities

Tasks for learning progressions, like activities for learning trajectories, may come from empirical evidence with large or small groups of students. Such tasks may also be developed, however, from a curricular analysis or other investigations focusing more on the content domain than on students' thinking. One advantage of the learning trajectory approach is its empirical origins; descriptions of students' conceptions evolve in relationship to their interactions with activities. Thus a learning trajectory could provide a way to include instructional moves or other contextual suggestions along with related activities. In Table 2, the two sample problem types are briefly provided with explanations about their ordering and justification.

DISCUSSION

Building learning trajectories requires a great deal of work in identifying a precise set of schemes and operations to serve as a model for informing how a student might be operating at a particular stage. Some of this work has already been done, particularly in the work of early number, fractions, and measurement (e.g., Clements & Sarama, 2004; Steffe & Olive, 2010), but few models of this type exist for algebra and beyond. While there are promising steps in this direction, much work remains to develop tools to a) characterize qualitative distinctions in students' thinking at different stages of development, and b) identify mechanisms of learning driving students' transitions from one stage to the next (Simon et al., 2010). A stronger emphasis on learning trajectories research moving forward could support the development of standards and topic sequences that account for research-based findings on students' conceptual development over time, thus leading to more useful guides for teachers at all grade levels.

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