HOW STUDENTS COME TO UNDERSTAND THE DOMAIN AND RANGE FOR THE GRAPHS OF FUNCTIONS

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To understand the mathematical concept of function, students must understand certain subconcepts, such as domain and range. Many researchers have studied students' understanding of functions, but no study has focused on how students come to understand the domain and range for the graphs of functions. In this study, we identified four common strategies, two transitional conceptions, and two representational challenges evidenced by students. In general, determining the range was more difficult than determining the domain for the students.

HOW STUDENTS COME TO UNDERSTAND MATHEMATICS

Functions play a key role throughout the mathematics curriculum. The U.S. Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Office, 2010) states that high school students should be able to: a) create functions that model relationships between two quantities, b) analyze and employ functions using different representations, and c) interpret functions for applications in terms of the context of the situation. However, the concept of function is one of the most difficult for students to understand (Tall & DeMarois, 1996). To understand the concept of a function, students must understand numerous subconcepts, such as input, output, ordered pairs, and correspondence to name a few.

Additional complications can arise when students graph, use graphs to reason about, or try to understand graphs of functions. Prevalent evidence suggests that piecewise functions cause substantial difficulty for students (Norman, 1993). Graphs play a role in how students come to understand and work with functions. Functions and their graphs are of interest in an instructional sense because they tend to focus on relationships as well as entities. While many have studied students' conceptions of functions (Markovits, Eylon & Bruckheimer, 1983) and how they understand domain and range (Arnold, 2004), there has not been a specific focus on how students understand the graphical representation of a function's domain and range.

Mathematics ideas and relationships can be represented using a variety of multimodal resources (e.g., inscriptions, speech, gestures, and artifacts, for more see Moore-Russo and Viglietti (2012)). Representations "help to portray, clarify, or extend a mathematical idea by focusing on its essential features" (National Council of Teacher of Mathematics [NCTM], 2000, p. 206). For example, to express the domain and range of a function, students often use interval or inequality notation. The representations used often come to impact how students make meaning of the concept at hand. During

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the meaning making process, individuals often: a) rely on strategies to help develop their understanding and b) develop individual conceptions regarding the idea under consideration. Chiu, Kessel, Moschkovich, and Muñoz-Nuñez (2001) defined a *strategy* as "a sequence of actions used to achieve a goal, such as accomplishing a particular task or solving a particular problem" (p. 219). They defined a *conception* to be "an idea that is stable over time, the result of a constructive process, connected to other aspects of a student's knowledge system, robust when confronted with other conceptions, and widespread" (p. 219). Following Smith, diSessa, and Roschelle's (1993) recommendations, Moschkovich (1999) defined a *transitional conception* as "a conception that is the result of sense-making, sometimes productive, and has the potential to be refined" (p. 172). To study a particular individual's meaning-making process, it is imperative to consider the transitional conceptions that occur and the strategies employed when students are engaged in tasks. In this study, we explore students' transitional conceptions of the domain and range of a graphical representation of a function.

METHODS

Setting

For this research, a qualitative, multiple-case study was conducted. The research site was a community college adjacent to a large city in eastern region of the United States. The lead researcher administered a pre-test to all students enrolled in two precalculus classes to determine their performance on graphical tasks that involved the concepts of domain and range. Study participants for the study were selected from middle-achieving students whose test scores ranged from 51% to 79% on the initial instrument, since evidence suggested that these students were developing an understanding of domain and range, and were more likely to have transitional conceptions than those with very low or very high results on the initial instrument.

Data Collection

The data sources for the study were students' written answers to domain and range tasks that involved functions' graphs on four test sets as well as videos and transcripts of subsequent student interviews. Five participants were asked to solve short-answer items. Each participant completed items 1-20 first, either in the classroom or researcher's office. Then after completing the test without interruption, the lead researcher immediately interviewed the student asking about how the tasks were completed. After the first interview, the lead researcher administered a second set of tasks, items 21-40, to the participant after a short break. Upon completion a second interview was then conducted. In a later setting, the third and fourth interviews were administered in a similar format. All interviews were videotaped and transcribed within two weeks after the interviews were completed. The four test sets were designed with different purposes. The first set, items 1-20, was designed with basic questions and figures. The second set, items 21-40, contained more advanced problems whose graphs included more turning points, open points and horizontal sections. The third set, items 41-60, included more complicated piecewise function graphs. The fourth set, items 61-108, was designed to check the impact of the small graphical differences on the domain and range. For example, pairs of tasks might contain one item where all the turning points were closed and another related item where the same graph was given except all the turning points were open.

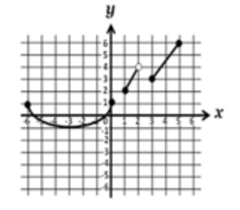
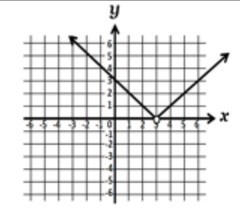
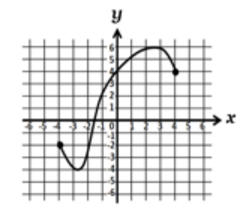
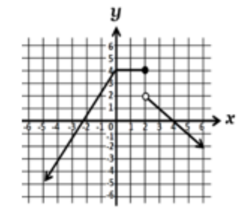
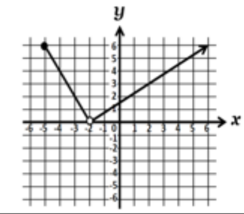
Data Analysis

The data were examined for emerging categories through a general inductive analysis. The research team used theoretical memoing (Glaser, 1998) to record and classify observations during the multiple passes through the data. This process was guided by the use of rich, thick descriptions of participants' activities and their responses. As the research team combed through the data, they began to cluster similar entries to form unifying categories. Upon studying the data, the research team determined it would be best to start with a loose structure of three broad classifications (strategies, conceptions, and representations) allowing more specific categories to emerge from the data under this structure. The research team then used these categories to make sense of observed activity (Thomas, 2006). During the constant comparison of data across participants as well as across interviews and written responses while considering what information might be of greatest benefit to instructors, there was a slight refinement of the overall structure to concentrate primarily on students' common strategies, transitional conceptions, and challenges with representations.

RESULT: STRATEGIES, CONCEPTIONS, AND REPRESENTATIONS

In this study, we considered two research questions: a) Which strategies and transitional conceptions are evident when students consider the domain and range of a graphical representation of a function? b) How do students' use of strategies and their understanding of concepts and representations impact their understanding of the domain and range of a graphical representation of a function? To address the first research question, the research team determined the most prevalent strategies and conceptions that students used when engaged in domain and range tasks for a given graph. To address the second research question, the research team analysed all data sources to see how students were using strategies, concepts, and representations related to the domain and range of a function's graph.

To determine and denote the domain or range of a graph, students need to be able to use appropriate **strategies** that fit the context and the problem at hand; they need to hold particular **conceptions** to understand and work with certain mathematical concepts; and they need to be able to represent their ideas and responses with an appropriate **representational notation**. Consequently, all three are needed to work with the domain and range of a function's graph. Next, we report our findings based on these three classifications: common strategies, transitional conceptions and representational challenges. We use representative examples, displayed in Figure 1, which are a subset of 10 items and 4 participants' responses to these items.

① Item 20 Mary	② Item 74 Mary	③ Item 22 Kara	④ Item 36 Mary	⑤ Item 10 Victor
				
20. Find the range	74. Find the range	22. Find the range	36. Find the range	10. Find the range
[-1, 1]∪[2,6]	**(0, ∞)**	**[-2, 4] → [-4, 6]**	**[4, 2) → [4, 2)∪(2, -∞)**	**(-2, ∞) → (0, ∞)**

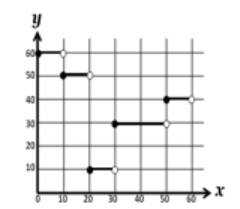
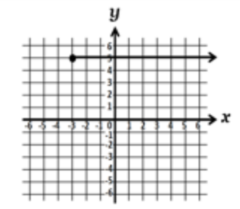
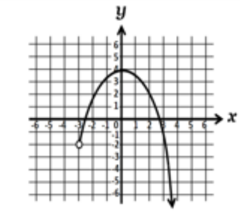
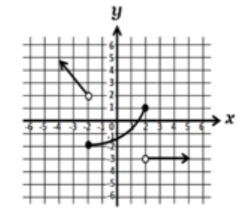
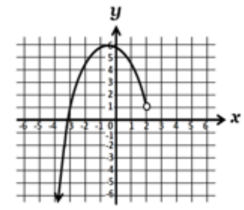
⑥ Item 16 Louis	⑦ Item 64 Louis	⑧ Item 12 Kara	⑨ Item 38 Victor	⑩ Item 24 Mary
				
16. Find the range	64. Find the range	12. Find the range	38. Find the range	24. Find the range
[10, 60]	**ø**	**(-∞, 4) → (-∞, -2) → (-∞, 4]**	**[-2, 1] (2, ∞) → y=-3∪[-2, 1] (2, ∞)**	**(6, -∞) → [6, 1)∪(1, -∞)**

Figure 1: Representative examples (For each item response, an arrow sign "→" represents a subsequent attempt during the interview process.)

Common Strategies

1. Projecting the graph onto the x-axis (or y-axis) to determine the domain (or range). Projecting the graph onto the x-axis strategy is a glancing-and-imagining method. With this strategy, students glanced at the graph and projected the graph onto the x-axis without any other body motion. They mentally projected the graph onto the x-axis and used the imagined horizontal segment or line to determine the domain. Projecting the graph onto the y-axis strategy is a strategy similar to projecting the graph onto the x-axis strategy. On item 20, see Figure 1-(1), Mary projected the piecewise function onto the y-axis. When determining the range she looked at the graph, staring the longest at the right side of the graph, where the two linear segments' ranges of [2, 4) and [3, 6] overlapped. She reported that she projected the graph to the y-axis with her eyes and merged the two intervals to determine the answer $[-1, 1] \cup [-2, 6]$.

2. Pushing the graph to the x-axis (or y-axis) to determine the domain (or range). Pushing the graph to the x-axis was an embodied strategy that involved student gesturing. To determine the domain of the graph, students used a motion with their hands or fingers as if pushing or pressing down the graph to imagine it as a horizontal segment (or line) on the x-axis. A slight variation of the pushing gesture, students used a clapping gesture to make a noise by actually clapping their hands as they imagined the graph physically being pressed to the x-axis. Pushing the graph onto the y-axis strategy is a strategy related to the strategy of pushing the graph onto the x-axis, with the only distinction being pushing to the *y*-axis rather than the *x*-axis. On item 74, see Figure 1-(2), Mary gestured by "pushing" the piecewise function to the *y*-axis using her both hands.

3. Focusing on the endpoints when tracing the graph from the starting point to the ending point. Tracing the graph is a graph-following strategy that involved eye or finger movement. When students traced a graph with their eyes or fingers, typically they focused on the endpoints then traced the graph from the starting to the ending point. While this strategy would yield a correct response for the domain, it would not necessarily do so for the range. In some instances, students traced from right to left, where the domain was $(-\infty, p)$ or $(-\infty, p]$ for some point p. For example, if a graph was bounded by a closed endpoint on the right yet unbounded to the left, often students reported they traced the graph from the right point to the left arrowhead because the arrowhead's direction caused their eyes to naturally follow the arrowhead's path. On Item 22, see Figure 1-(3), Kara answered [-2, 4] for the range of this graph. She traced the graph from the left boundary point to the right ending point. She initially used the *y*-coordinate values of the both end points, even though they were not the minimum and maximum values of the graph.

4. Not overlapping sections of a graph to determine its range. When a graph is not a one-to-one function, there is at least one portion of the graph were the *y*-coordinate values overlap. However, some students did not notice the overlapped portions. For example, if an open point exists in the overlapped portion, the open point's *y*-coordinate value should not be eliminated since the open point can be overlapped with another portion of the graph. On item 36, see Figure 1-(4), Mary determined the range [4, 2) on her first attempt focusing on the two end points. Upon noticing this in the interview she then responded $[4, 2) \cup (2, -\infty)$ on her second attempt, even though the desired answer was $(-\infty, 4]$. In her second response, she did not include the *y*-coordinate value of 2, even though the point (-1, 2) is on the left part of graph. In addition, she started at the top most part of the graph following the arrowheads down and then reported the interval in a nonstandard descending order.

5. Using the closest axis value; using x-coordinate values instead of y-coordinate values, and vice versa. When students determine the domain of a graph, they should focus on the *x*-coordinate values of the graph. However, if a graph intersected a *y*-axis or had a vertex on the *y*-axis, students often focused on the *y*-coordinate values of the point. This phenomenon seems to suggest that the students' eyes are attracted or drawn to the closest number on the *y*-axis. Similar situations occurred for range when a graph had a critical point on the *x*-axis. On item 10, see Figure 1-(5), Victor used the *x* value of -2 rather than the *y* value of 0 from the open point (-2, 0) when reporting the range of this graph.

6. Measuring the range from the lowest value to the highest value of a piecewise function. Some students used the lowest and highest values to determine the range even though the graph was vertically discontinuous. On item 16, see Figure 1-(6), Louis answered [10, 60] for this step function's range.

Transitional Conceptions

7. Belief that a horizontal line or segment of a line has no range. A horizontal line is a specific, special case since it has no change in its range. Some students believed that a horizontal line or segment had no range at all. Their belief stemmed from the conviction that the range should have some length or distance. On item 64, see Figure 1-(7), Louis' answer was an empty set "Ø" and his reasoning in the interview was "There is no range because it is a flat line."

8. Dealing with marked open or closed points as boundaries. Students felt it was especially difficult to determine the range of a horizontal segment when it included open points on the ends of its graph. They did not seem to recognize that a horizontal segment consists of infinitely many closed points.

Students preferred clearly designated points, either open or closed, when finding the range and when the graph was not horizontal. When a graph had an absolute maximum where the function was concave down, the point's y-coordinate value should be the range's maximum. However, some students hesitated to put the vertex's y-coordinate value as the range's greatest point. The reasoning was that when an absolute maximum is a part of curve, there is no clear point but a curve. Instead of using the absolute maximum (or minimum) on a curve, students preferred to use the open or closed point that was highlighted at the boundary points of intervals.

On a related note, some students would purposely use open parentheses in their responses when turning points where the absolute extrema. On item 12, see Figure 1-(8), Kara's original answer was (-∞, 4) for the range. She hesitated to use the maximum vertex, the point (0, 4), specifically mentioning because there was no closed point specifically marking a definite point.

Representational Challenges

9. Difficulty with the notation in representing the range of horizontal lines. The horizontal line is a specific case of a graph since it has only one point for its range. Even when students realized that a single point was the range, they did not know how this should be represented. In the special case of a single value, using conventional set notation, the degenerated interval is represented by braces { }. Students were at times unfamiliar and more often uncomfortable using this notation. On item 38, see Figure 1-(9), Victor did not include the horizontal part of the graph for the range originally. He thought that the horizontal ray's range did not exist because of an open point. His original answer was [-2, 1] (2, ∞). After he realized that there were many points that had the y-coordinate value -3, he added y = -3 to his response. His final answer was y= -3 ∪ [-2, 1] (2, ∞).

10. Representing an interval in descending order. By convention, the interval notation is written in ascending order with the smaller number located on the left side of the interval and the greater number located on the right side. However, many students used a descending order, especially when the graph decreased or when they traced an increasing graph that was bounded on the right. Often they traced from right to left following the direction of the arrowhead on the left hand. On item 24, see Figure 1-(10), Mary determined the range from the maximum point to the arrowhead. Rather than writing $(-\infty, 6]$, she wrote $(6, -\infty)$. She then rewrote the answer $[6, 1) \cup (1, -\infty)$, still using the descending order, which related to the transitional conception labelled #8 above.

DISCUSSION

Common Challenges

All five students had difficulty with the range of horizontal lines. This was related to the fact that they thought of graph as flat and without any vertical distance or length. They felt it should not have a range. Even when students began to recognize that a horizontal segment would have a range of a single point, they were often only considering the endpoints of the segment and felt that horizontal segments with open endpoints as boundaries would not have a range. Others recognized that the range would be a single point but often struggled with how to represent this. The fact that all five students had difficulty with this suggests that instructors should take this into account in their task selection.

When a graph extends infinitely and its representation includes arrowheads, students impulsively followed the arrowhead direction. When students used this strategy to determine the range (especially for the piecewise functions), they frequently created overlapping intervals. The instructional implication is that students often view parts of graphs that should be considered at the same time as separate entities. Of note was the fact that the students who used the "projecting" or "pushing" strategies rather than the tracing strategies seemed to treat the piecewise graphs as all belonging to a single whole and were less likely to give the overlapping interval responses.

One of the notable findings was when students used open parentheses when boundaries were not endpoints nor where they specifically designated (either open or closed) points. Three of five students did not want to use the y-coordinate value of the absolute maximum point since the turning point had no closed point. Instead, they favoured using only specifically represented closed points serving as boundaries for certain sections of the graph, since they were clearly specified. The underlying source of this transitional conception relates to the fact that either students did not realize that a line consisted of infinitely many closed points or they did not realize that points do not have to be represented by either open or closed circles (i.e., segments and other curves represent a continuous collection of closed points). This finding suggested that students' challenges related to domain and range may stem from not understanding the meaning of curved sections of graphs but may go back to understanding the fact that continuous curves represent an infinite set of closed points.

More challenges arose when students were determining the range, rather than the domain. Since the functional inputs are not repeated but are unique for the domain, there are no overlapping sections as can be the case for the range. This was a particular challenge for students for graphs of functions that were not one-to-one.

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