MODALITIES OF RULES AND GENERALISING STRATEGIES OF YEAR 8 STUDENTS FOR A QUADRATIC PATTERN

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This paper reports on the performance of 167 eighth graders in Singapore making generalisations for a quadratic figural pattern presented in a non-successive format. Data were collected through administering a written test in which the students had to establish the functional rule underpinning the pattern. The findings revealed that the students constructed a variety of functional rules, expressed prevalently in symbols using a range of generalising strategies, some of which were novel in the literature.

BACKGROUND

Most generalising tasks used in pattern generalisation research involve linear rather than quadratic patterns. The quadratic patterns typically depict the widely-recognised square and triangle numbers (see Steele, 2008). Moreover, the patterns are all too often presented in the form of a successive sequence of numerical terms or configurations. The generalising strategies that students employ to formulate a rule for predicting any term of a linear pattern are well established. However, if the rule were to change from a linear to a quadratic relationship, would the strategies that students engaged in the former case change to suit the latter? What types of rules would the students then establish for the latter? To gain more insights, an empirical study was conducted on a group of Year 8 students in Singapore to examine how they construct the rule underpinning a quadratic pattern presented in figural form. Specifically, this paper addresses these research questions: What are the different forms of rules that the Singapore students formulate for a figural quadratic pattern? What is the modality of the rules that the Singapore students formulated? What are the generalising strategies employed by the Singapore students in formulating the quadratic rule?

THEORETICAL FRAMEWORK

Students are often asked to construct a rule to describe the pattern structure that they see in a generalising task. Their rules take on mainly two forms: *recursive* and *functional*. The *recursive* rule allows the computation of the next term of a sequence using the immediate term preceding it whereas the more powerful *functional* rule refers to the rule expressed as a function that computes the term directly using its position in the sequence. Consider the linear task comprising a square made of four matchsticks in Figure 1, a row of two squares made of seven matchsticks in Figure 2, and a row of three squares made of 10 matchsticks in Figure 3. A recursive rule for this matchstick task could be expressed as "*add three to get the next term*" and its functional rule in closed form is $3n + 1$.

2014. In Nicol, C., Liljedahl, P., Oesterle, S., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 2, pp. 305-312. Vancouver, Canada: PME.

The functional rules are often represented in three different modes: purely symbolic (S), purely in words (W), and in alphanumeric form (SW). These different modes of representation are referred to as the *modality* of the rules. The functional rule for the matchstick task above, $3n + 1$, is expressed entirely in symbols. This rule can be stated wholly in words as: *add one to three times the number of squares*. Written alphanumerically, it can take the form: $3 \times number\ of\ squares + 1$. Stacey and MacGregor (2001) reported that nearly half of their sample of 2000 Australian students in Years 7 to 10 described the functional rule underpinning a pattern in words. Mavrikis, Noss, Hoyles and Geraniou (2012) noted a student using the alphanumeric form in their study.

The wealth of research on students' generalising strategies suggests that students use a variety of strategies to derive the rule connecting the term and its position in a pattern. Bezuszka and Kenney (2008) identified three numerical strategies: (1) *comparison*, where the terms in a given number sequence are compared with corresponding terms of another sequence whose rule is already known, (2) *repeated substitution*, where each subsequent term in a number sequence is expressed in terms of the immediate term preceding it, and (3) *the method of differences*, which is an algorithm for finding an explicit formula when the pattern is derived from a polynomial.

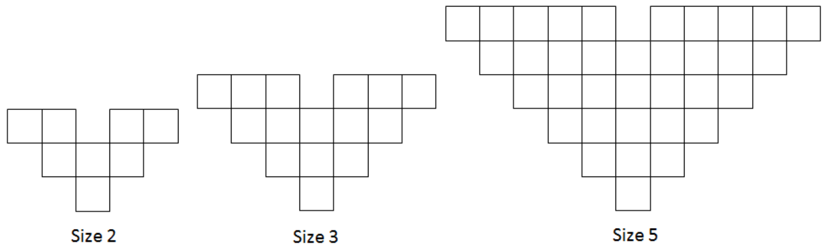
Different categories of *figural* strategy have also been identified. Rivera and Becker (2008) distinguished between (1) *constructive generalisation*, which occurs when the diagram given in a generalising task is viewed as a composite diagram made up of non-overlapping components and the rule is directly expressed as a sum of the various sub-components, and (2) *deconstructive generalisation*, which happens when the diagram is visualised as being made up of components that overlap, and the rule is expressed by separately counting each component of the diagram and then subtracting any overlapping parts. Chua and Hoyles (2010) introduced two further strategies into Rivera and Becker's (2008) classification scheme: *reconstructive*, which involves rearranging one or more components of the original diagram to form something more familiar, and *figure-ground reversal*, which entails augmenting the original configurations to become part of a larger composite configuration.

METHODS

167 Year 8 students (89 girls, 78 boys) of average learning abilities from three secondary schools participated in the study. The students had to complete two linear and two quadratic generalising tasks in 45 minutes and were asked to produce the functional rule in each task. Only one of the quadratic tasks, *Tulips*, in Figure 1 below is discussed here. Prior to participating in this study, the students had learnt the concept of variables and the topic of number patterns in the Singapore mathematics curriculum. They should also be far more familiar in dealing with linear patterns than with quadratic ones, which are less commonly featured in their mathematics textbooks.

Tony used identical square tiles to create flower designs of different sizes for his art project.

The diagrams below show two flower designs he made.



As the size number became larger, more square tiles were used.
Tony wanted to find the number of square tiles he had to use to make any size.
He used a rule to find this number.

Figure 1: Tulips

The *Tulips* task was deliberately designed to depict the pattern with three non-successive configurations starting with Size 2 and made less structured without any part questions that gradually led students to detect and construct the general rule. This was to allow the students a greater scope for exploring the pattern structure so that we could then see how they recognised and perceived the pattern without any scaffolding.

All the student responses for the *Tulips* task were analysed comprehensively to identify the types of rules produced and the generalizing strategies used. Several types of equivalent functional rules were observed and those with similar structure were collapsed into the same category after further examination, thereby developing the coding scheme for the types of rules. When two or more equivalent expressions of the functional rule were seen in a student response, the initial one, albeit simplified to another form subsequently, was coded. The rules were also coded for their modalities. The coding scheme for generalising strategies relied on *a priori* ideas drawn from different sources, including, mainly the research literature and our observations made during the analysis of the student responses. The generalizing strategy of every student was matched with the available codes and when it was not found to match any, a new code was created. Some student responses were subsequently selected and passed to a mathematics teacher for coding. After the inter-rater reliability was established, the frequencies of each type of rule and each type of generalising strategy were then determined.

RESULTS

93 students (56%) produced a correct functional rule for *Tulips*. Another nine students identified the first differences between consecutive terms correctly but only six of them articulated the recursive rule successfully.

Types of functional rules

Nine categories of different but equivalent expressions of quadratic functional rules were constructed, as shown in Table 1. The rules display variation in the mathematical operations used to join different terms together, involving both addition and subtraction. For instance, $n^2 + 2n$ illustrates the sum of two terms whereas $(n+1)^2 - 1$ exemplifies the difference of two terms.

	Rule Modality				Rule Modality		
Rule type	S	W	SW	Rule type	S	W	SW
$n(n+2)$	40	5	5	$n(2n+1) - n(n-1)$	1		1
$n^2 + 2n$	22	1	2	$(2n+1)(n+1) - n(n+1) - 1$	1		
$n + n(n+1)$, $n + (n^2 + n)$, $n + \frac{2n(n+1)}{2}$	8		1	$(n^2 - 1) + (n+1) + n$	1		
$3n + n(n-1)$	2	1		$n + 2[n + (n-1) + (n-2) + \cdots + 3 + 2 + 1]$			1
$(n+1)^2 - 1$	1						

Table 1: Rules and their modalities

The two most common functional rules are $n(n+2)$ and $n^2 + 2n$. Figure 2 below illustrates how Student A established $n(n+2)$ by means of producing the missing Size-4 configuration and rearranging it into a 4 by $(4+2)$ rectangle, followed by recognising the link between the dimensions and the size number.

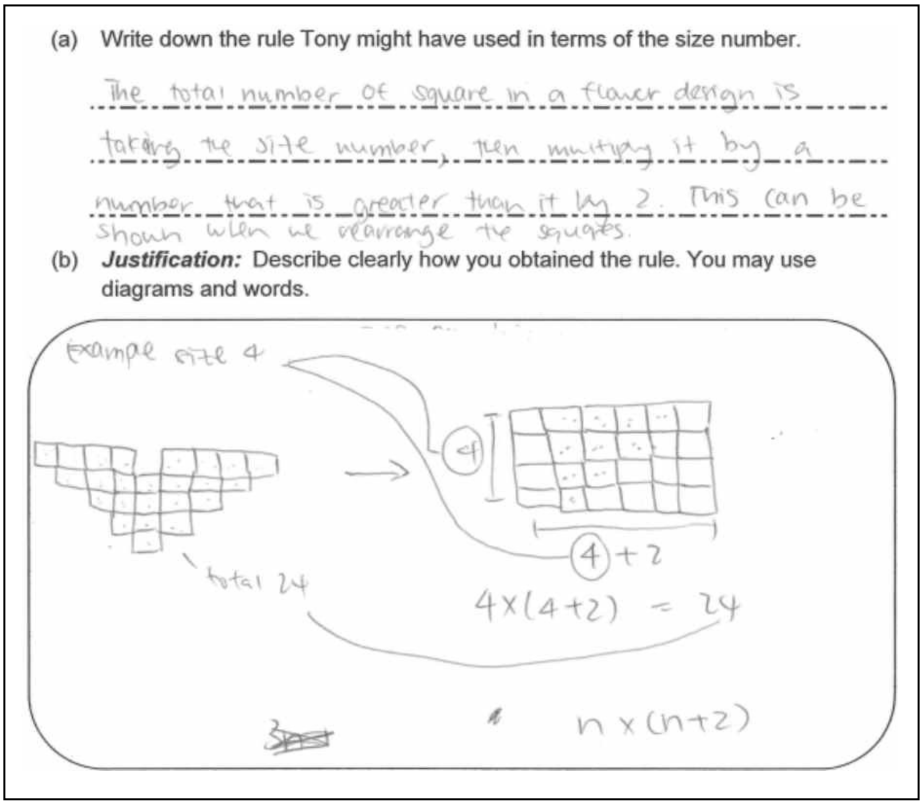


Figure 2: Functional rule $n(n+2)$

In Figure 3 below, Student B generated the rule, $n(2n+1) - n(n-1)$, by first shifting the bottom-most single tile to fill the gap in the top-most row, then imagining the resulting configuration as being formed by removing staircase-shaped tiles from each corner at the bottom left and bottom right of a "perfect" rectangle with dimensions $(2n+1)$ by n. The two sets of staircase-shaped tiles that are removed can be joined to form a rectangle of dimensions n by $(n-1)$, hence the rule.

The rule, $n + 2[n + (n-1) + (n-2) + \cdots + 3 + 2 + 1]$, is worth highlighting even though it occurred only once in this study. Although it describes the structure underpinning the pattern, it is not *algebraically useful* in Lee's (1996) language. This is because it does not allow the direct computation of the output when given an input.

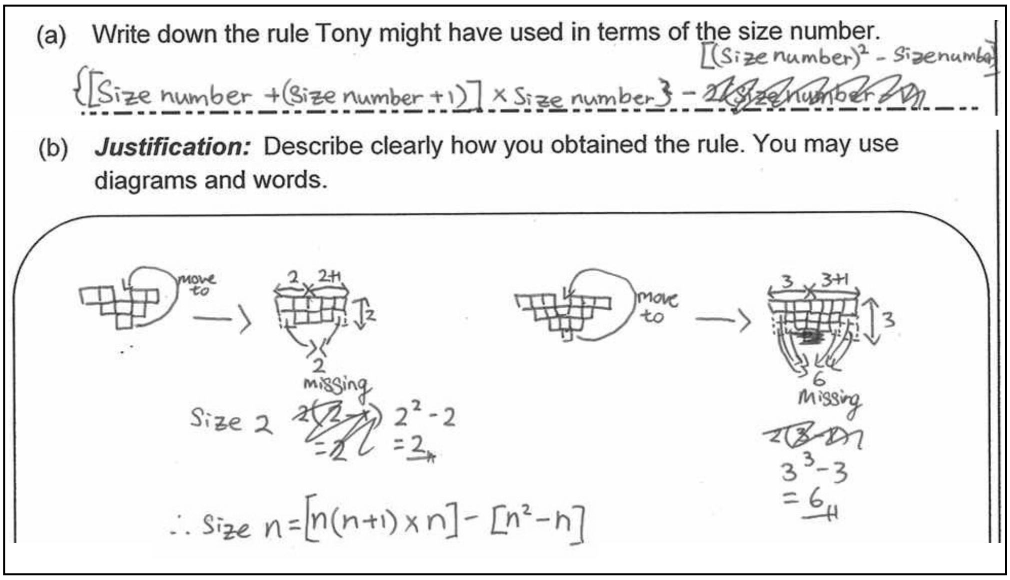


Figure 3: Functional rule $n(2n+1) - n(n-1)$

Modalities of rules

Three categories of modalities were identified, as indicated in Table 1. The functional rules were articulated predominantly in symbols, whilst the word and alphanumeric modes of representation were seldom used. Student A expressed the rule correctly in words and in symbols, thus the more sophisticated symbolic form was considered. Similarly, Student B also articulated the rule in two different forms: symbolic and alphanumeric, but the latter was considered because the former was incorrect (Note: $n(n+1) \times n$ should have been $[n + (n+1)] \times n$).

Generalising strategies

Eight different strategies were used, the most common being what we call a *combo* strategy involving both the *constructive* and the *comparison* strategies (see (d) below). Descriptions of the various strategies, excluding guess-and-check, now follow.

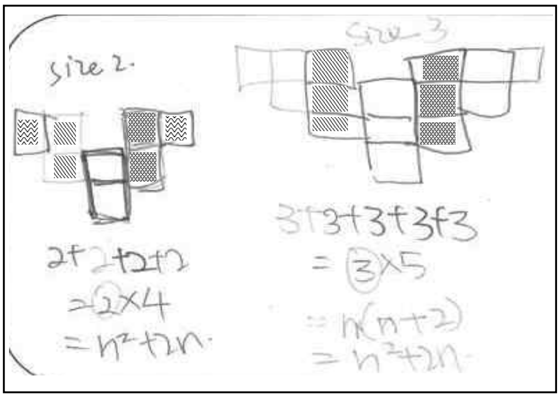
a. ***Grouping***. In Figure 4(a), the size number is used to generate the number of groups of tiles in each configuration: for instance, there were four groups of two tiles in Size 2, and five groups of three tiles in Size 3. Hence, there are $(n+2)$ groups of n tiles in Size n, or $n(n+2)$ tiles.

b. ***Reconstructive***. Figure 2 exemplifies this strategy where the original configuration is rearranged into a rectangle of dimensions $(n+2)$ by n.

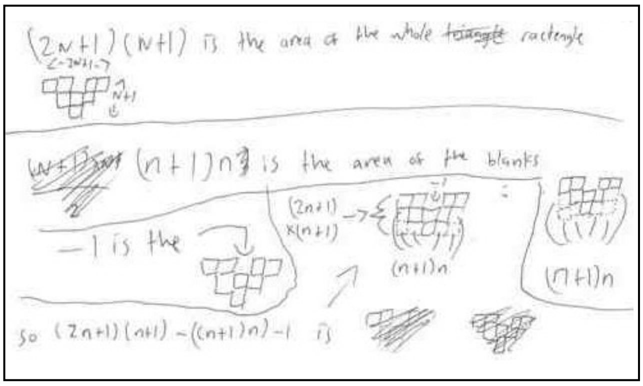
c. ***Figure-ground reversal***. The original configuration is visualized as being formed from a $(2n+1)$ by $(n+1)$ rectangle with two step-shaped components removed from its bottom-left and bottom-right corners alongside a tile in the top-most row. Given that the two step-shaped components can be repositioned to form a n by $(n+1)$ rectangle, the rule is thus $(2n+1)(n+1)-n(n+1)-1$ (see Figure 4(b)).

d. ***Constructive-comparison combo***. In Figure 4(c), each configuration is first viewed as comprising two non-overlapping parts: the top-most part made up of two rows, and the "step pyramid" (i.e., the *constructive* strategy first). The number of tiles in each "step pyramid" is then worked out and compared with the square numbers (i.e., the *comparison* strategy next).

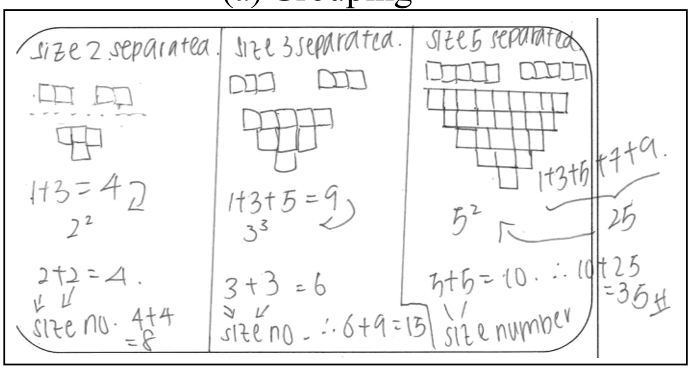
e. ***Constructive-reconstructive combo***. As Figure 4(d) shows, the discernment of the pattern begins with separating the original configuration into the "stalk" and "petals" (i.e., *constructive* first), then rearranging the "petals" into a rectangle before combining it with the "stalk" to form a larger rectangle (i.e., *reconstructive* next).



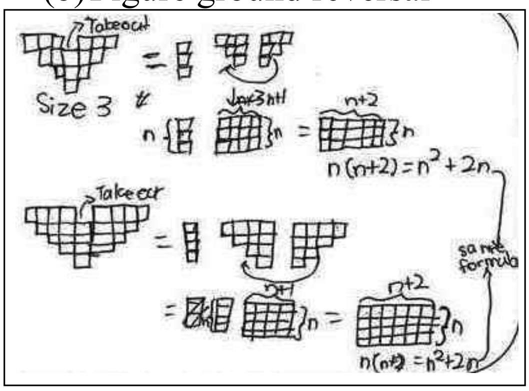
(a) Grouping



(b) Figure ground reversal



(c) Constructive -comparison



(d) Constructive-reconstructive

Figure 4: Generalising strategies

f. ***Reconstructive-constructive combo***. This strategy is similar to (e) except the order of applying the strategies is switched around.
g. ***Reconstructive-figure-ground reversal combo***. Figure 3 illustrates an example involving the repositioning of a tile (i.e., *reconstructive* first) followed by envisioning the resulting configuration being cut out from a larger rectangle (i.e., *figure-ground reversal* next).

DISCUSSION AND CONCLUSION

It is a fact that making generalisations for a quadratic pattern challenges secondary school students (see Jurdak & El Mouhayar, 2014; Steele, 2008). In Singapore, quadratic patterns are rarely used in mathematics textbooks. Moreover, with the *Tulips* pattern presented in a non-successive format, the task of finding a general rule might be even more testing. It is therefore surprising, yet encouraging, to see the students achieving moderate success in *Tulips*. A key to their success in detecting the inherent pattern structure lies in their recognising the need to use the size number as a generator of the term-to-position relationship.

The prevalence of functional rules expressed in symbols in *Tulips* stands in contrast to previous results by Stacey and MacGregor (2001). The fact that many Singapore students could develop the rule as an algebraic expression indicates that the concept of variables is generally well understood, a result of their prior experience with algebra where the teaching of number patterns follow the introduction of variables.

A marked observation to emerge from the analysis of the generalising strategies used in *Tulips* is the lack of *repeated substitution*, a common strategy for linear tasks. Using this strategy to generate the quadratic rule is not as straightforward as one might expect and students favouring it might have faltered and did not know how to employ it when the first differences of the pattern were not a constant, like in *Tulips*. Another remarkable finding is the use of certain strategies that are hardly described in the literature: *grouping* and the *combo* strategies such as the *constructive–reconstructive* and *constructive–figural-ground reversal* strategies.

To conclude, most studies on pattern generalisation have been undertaken in the west, offering a vast knowledge of students' generalising abilities and strategies. We hope this paper provides new insight into how Asian students visualise, think and reason about patterns, and opens the door for comparisons and future research.

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