AN ANALYTIC FRAMEWORK FOR DESCRIBING TEACHERS' MATHEMATICS DISCOURSE IN INSTRUCTION

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We illustrate an analytic framework for teachers' mathematics discourse in instruction (MDI). MDI is built on three interacting components of a mathematics lesson: a sequence of examples and related tasks; accompanying talk; patterns of interaction. Together these illuminate what is made available to learn. MDI is grounded empirically in mathematics teaching practices in South Africa, and theoretically in socio-cultural theoretical resources. The framework is responsive to the goals of a particular research and professional development project with potential for wider use.

INTRODUCTION

Recent reviews of research on mathematics teachers, teaching and teacher education evidence the growth of this work (e.g. Sullivan, 2008). In their review of such research in thirty years of PME, Ponte & Chapman (2006) conclude with a call for future research that attends to "…innovative research designs to deal with the complex relationships among various variables, situations and circumstances that define teachers' activities" (p. 488). The framework offered in this paper responds to this call. Our central concern is a framework that illuminates the complexity of teaching mathematics in ways that are productive in professional development research *and* practice; a framework that characterise teaching per se, across classroom contexts and practices, *and* captures shifts in practice.

The framework we present developed within the Wits Maths Connect Secondary Project (WMCS), a five-year research and professional development project aimed at improving the teaching and learning of mathematics in ten relatively disadvantaged secondary schools in one education district in South Africa, through ongoing engagement with what we have come to describe as teachers' MDI. MDI characterises the teaching of a mathematics lesson as a *sequence of examples/tasks* (which we distinguish below), and the *accompanying explanatory talk* - two commonplaces of mathematics teaching that occur within *particular patterns of interaction* in the classroom. In previous work in WMCS and a similar project in primary schools, we conceptualised MDI to examine coherence within a task, and so between the stated problem or task, its exemplification or representation, and the accompanying explanations; and more recently to examine coherence across a sequence of tasks/examples and accompanying explanatory discourse within a lesson, and in relation to the intended object of learning (e.g. Adler & Venkat, forthcoming). It was our empirical data that emphasized the need for coherence, and teaching that mediates

2014. In Nicol, C., Liljedahl, P., Oesterle, S., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 2, pp. 9-16. Vancouver, Canada: PME.

towards mathematics viewed as a network of scientific concepts (Vygotsky, 1978), and so towards generality (Watson & Mason, 2006), and objectification (Sfard, 2008).

There are clear commonalities with other frameworks, particularly aspects of the Mathematical Quality of Instruction (MQI) framework (Hill, 2010) and that of Borko et al (2005), both of which include attention to language/discourse (depending on their orientations to language), and to justification and/or explanation. In particular we share the concern of MQI to foreground the importance of generality in mathematics, and so what mathematically is made available to learn. Neither pay attention to examples, and so the specificity of example/task selection. This is a key element of the MDI framework, and we hope the elaboration that follows below illustrates its salience.

SOCIAL CONTEXT

It is common cause in South Africa today to hear that school mathematics is "in crisis". Learner performance in local, national and international comparative mathematics assessments are poor across levels, and while explanations increasingly acknowledge system wide failure, considerable 'blame' is placed on the knowledge of practice of mathematics teachers (Taylor, Van der Berg, & Mabogoane, 2013)

Of course, Teachers' MDI is only a part of a set of practices and conditions through which performance is produced, not least of which is social class and related material and symbolic resources in the school. That said, our concern from both a research and professional development perspective is to understand how teachers' MDI is implicated in what is made available to learn. In the majority of schools in South Africa (as is the case in schools serving disadvantaged learner populations in many parts of the world), schools provide the sole sites of access to formal learning. Within this, learners' access to mathematical learning resources is through the teacher's discourse. Understanding how teachers' MDI supports mathematical learning matters deeply. We want to be able to describe whether and how teachers' MDI shifts over time, in what ways, and how MDI is related to what is made available to learn in school.

SOME THEORETICAL ROOTS AND RESOURCES

MDI has its roots in analytic tools developed for describing the constitution of mathematics in mathematics teacher education practice (e.g. Adler & Davis, 2006). Based on Bernstein's insight that evaluation is "key to pedagogic practice" (2000 p.36), and following Davis' elaboration of this through the notion of evaluative judgment (Davis, 2005), we described three key features of mathematics pedagogy (school or teacher education). First, for something to be learned/taught, it has to be presented in some form. In mathematics, this is always a representation rather than the thing itself, one that as yet has to be invested with particular mathematical meanings. What then follows is reflection on this 'object' – semiotic mediation – so as to fill out its meaning. At some point reflection will need to end, and meaning fixed as to what can/does count as legitimate with respect to the 'object'. Description, while important, is not sufficient for linked research and development. In the first year of WMCS

(2010), we observed that teachers typically selected, sequenced and explained some examples for the announced focus of a session, often with poor levels of coherence between the example and its elaboration, and/or across a sequence of examples. Many lessons began and ended with teacher-directed whole class interaction. In some lessons there was opportunity for independent learner work on set problems. Across classroom contexts, opportunity for learner ideas to enter the discourse varied from none to substantive, with the former dominant.

The detail of our responsive professional development practice is not the focus here. Our position was that we needed to start where we all were – the teachers themselves, and their well-oiled practices; and the project team, with its goal of enhancing opportunities to learn mathematics. We constructed a simple framework foregrounding the intended object of learning: improved coherence, in our view, rested firstly on appreciation of that which was to be learned. We found further resonance with the work on examples (e.g. Watson & Mason, 2006) and variation theory (e.g. Runesson, 2006) as resources for bringing the object of learning into focus. This broad framing is operationalised into an analytic framework for describing teachers' lessons over time.

AN ANALYTIC FRAMEWORK FOR MDI

Table 1 below presents the framework. It is not possible here to elaborate it in full, nor illustrate it in detail. We briefly discuss each of the analytic resources, and how we have assigned levels in the example and explanation spaces constructed – increasing generality in examples; increasing complexity in tasks; towards objectified talk in naming; and towards generality and use of mathematics in legitimating/substantiating – and with respect to participation, towards increasing opportunity for learners talk mathematically, and teachers' increasing use of learners' ideas. We illustrate our use of this framework through a WMCS Grade 10 Algebra lesson.

Our unit of study is a lesson, and units of analysis within this, an event. The first analytic task is to divide a lesson into events, distinguished by a shift in content focus, and within an event then to record the sequences of examples presented. Each new example becomes a sub-event, as illustrated in Table 1 below. Our interest here is whether and how this presentation of examples within and across events brings the object of learning into focus, and for this we recruit constructs from variation theory (Marton & Pang, 2006). The underlying phenomenology here is that we can discern a feature of an object if it varies while other features are kept invariant, or vice versa, and different forms of variation visibilise the object in different ways. Variation through *separation* is when a feature to be discerned is varied (or kept invariant), while others are kept invariant (or made to vary); *contrast* is when there is opportunity to see what is not the object, e.g. when an example is contrasted with a non-example; *fusion* is experienced when there is simultaneous discernment of aspects of the object is possible; and generalisation is possible when there are a range of examples in different contexts so that learners can discern the invariants – an expanded form of separation. These four forms of variation can operate separately or together, with consequences for what is possible to discern – and so, in more general terms, what is made available to learn. In WMCS we are interested in analysing the teacher's selection and sequencing of examples within an event and then across events in a lesson, and then whether and how, over time, teachers expand the example space constructed in a lesson.

Object of learning – mediation towards scientific concepts				
Exemplification		**Explanation**		**Learner Participation**
Examples	**Tasks**	**Talk/Naming**	**Legitimating criteria**	
Examples provide opportunities within lesson for learners to experience **Level 1-** separation or contrast **Level 2-** any two of separation, contrast, and fusion **Level 3-** fusion and generalization	**Level 1** – *Carry out known operations and procedures* e.g. multiply, factorise, solve **Level 2** – Apply level 1 skills;& learners have to decide on (explain choice of) operation and /or procedure to use e.g. Compare/ classify/match representations; **Level 3** – *Multiple concepts and connections.* e.g. Solve problems in different ways; use multiple representations; pose problems; prove; reason.etc	**Level 1** – *Colloquial language including* ambiguous referents such as this, that, thing, to refer to objects **Level 2** – *Some math language* to name object, component or simply read string of symbols when explaining **Level 3-** *Uses appropriate names* of math objects and procedures	**Level 1**NM (Non- Math) *Visual:* Visual cues or mnemonics *Metaphor:* Relates to features or characteristics of real objects **Level 2M** (Math) (Local) Specific /single case (real-life application or purely mathematical) Established shortcuts; conventions **Level 3M** (General, partial) equivalent representations, definitions, previously established generalization but explanation unclear or incomplete, principles, structures, properties but unclear/partial **Level 4M** (General full)	**Level 1** –Learners answer *yes/no questions or offer single words* to teachers unfinished sentence **Level 2** –Learners answer (what/ how) questions in phrases/ sentences **Level 3-** Learners answer why questions; present ideas in discussion; teacher revoices / confirms/ asks questions

Table 1: Analytic framework for mathematical discourse in instruction.

Of course, examples do not speak for themselves. There is always a task associated with an example, and accompanying talk. With respect to tasks, we are interested in its cognitive demand in terms of the extent of connections between and among concepts and procedures. Hence, in column two we examine whether tasks within and across events require learners to carry out a known operation or procedure, and/or whether they are required to decide on steps to carry out, and/or whether the demand is for multiple connections and problem solving. These three levels bear some resemblance to Stein et al's (2000) distinctions between lower and higher demand tasks.

With respect to how explanation unfolds through talk, and again the levels and distinctions have been empirically derived through examination of video data, we distinguish firstly between naming and legitimating, between how the teachers refer to mathematical objects and processes on the one hand, and how they legitimate what counts as mathematics on the other. For the latter, we have drawn from and built on the earlier research discussed above, together with aspects of Sfard's (2008) word use and endorsements as key elements of mathematical discourse. Specifically, we are interested in whether the criteria teachers transmit as explanation for what counts is or is not mathematical, is particular or localised, or more general, and then if the explanation is grounded in rules, conventions, procedures, definitions, theorems, and their level of generality. With regard to naming, we have paid attention to teacher's discourse shifts between colloquial and mathematical word use.

Finally, all of the above mediational means (examples, tasks, word use, legitimating criteria) occur in a context of interaction between the teacher and learners, with learning a function of their participation. Thus, in addition to task demand, we are concerned with what learners are invited to say i.e. whether and how learners have opportunity to use mathematical language, and engage in mathematical reasoning, and the teacher's engagement with learner productions.

A LESSON

The illustrative lesson, as stated by the teacher, is a Grade 10 revision lesson on algebraic fractions leading to a focus on the operation of division of algebraic fractions. The lesson consists of five events, with a new event marked by a new key concept in focus. The first event focused on a review of the meaning of a *term* in an algebraic expression. The teacher presented six examples of expressions (sub-events) in increasing complexity, with each next example of an expression produced by her performing an operation on the present expression. The task for learners was to agree to the number of terms in the new expression. The second event reviewed a common factor using just one example of a binomial expression. Event 3 signals new work. The teacher presented a sequence of four examples (sub-events) of algebraic fractions. The task was simplifying (through factorization) the expressions in each of the numerator and denominator to produce a single term. Complexity increased in terms of the type of factorisation required in successive examples. The task in events 4 and 5 was division of algebraic fractions. The examples in event 4 were of positive algebraic fractions only and event 5 included examples with negative algebraic fractions. We illustrate the use of our framework through detailed analysis of Event 4, particularly sub-event 4.3. in the box on the following page

Our analysis of Event 4 shows the Teacher operating at Level 3 with respect to examples, Level 1 with respect to tasks (which remain at the level of learners carrying out known procedures), and interaction (learners answers yes, no questions, and provide words/phrases in response to teachers questions on what to do). With respect to explanatory discourse, the teacher's words while frequently including ambiguous referents, move on to rephrase using mathematical language to name objects and processes, and thus at level 2; criteria shift between emphasis on visual features of expression, conventions, with some reference to structure and generality and so across levels 1 - 3.

Event 4: Sub-events 4.1 – 4.4 Examples and tasks

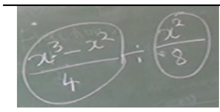
T writes example 4.1 on the board, asks questions mainly requiring yes/no answers, completion of sentences by learners in unison, leading to the solution. Occasionally learners respond with a phrase or sentence to a *what* or *how* question. Any *why* question she answers herself. Examples 4.2 and 4.3 follow the same form. The transcript extract below details the talk leading to the solution for 4.3. Example 4.4 is then given for learners to do independently.

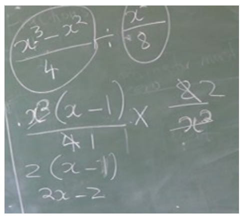
4.1 $\frac{2}{6} \div \frac{2}{3}$ 4.2 $\frac{2x}{6x} \div \frac{2x}{3x}$ 4.3 $\frac{x^3-x^2}{4} \div \frac{x^2}{8}$ 4.4 $\frac{x^2-x}{x^2+x-2} \div \frac{x^2+4x}{x^2-4} \times \frac{3x+12}{1}$

Examples: Level 3 - Variation is by separation, generalization and fusion. The structure of the division of one fraction by another is kept constant and terms varied (Separation). These range from simple to complex; from numerical to algebraic. Eg 4.4 extends to three fractions and a product (Generalization). Egs 4.3 and 4.4 require associating common factor with fraction division (Fusion).
Tasks: Level 1 - Perform the indicated operations to simplify expressions

Sub-Event 4.3 Talk and legitimating criteria
Analysis of explanatory talk highlighted as follows: *italics for colloquial* and underlining for formal language; and **bold type for criteria/legitimations;**

1.T: It's *one and the same thing.* They give you *something like this* (writes symbols on board),.... x cubed minus x squared *the whole thing over, over* four divided by x squared *over* eight...ok?
2. Ls: Yes
3. T: So it's, it's *one and the same* concept. *Over here* (points to number 4.1 ($\frac{2}{6} \div \frac{2}{3}$)) you just have two numbers, a fraction divided by a fraction, ok?
Ls: Yes





4. T: *Over here* (pointing back to 4.3) is *the same thing*. I've got, here's one fraction divided by one fraction (circles each fraction). So the examiner is just making your life difficult, ok?
5. T: So....what are we going to do *over here*? (points to first fraction)
6. Ls (some): we are going to divide
7. T: ...**remember the rule** that we learnt *over there*? (points to similar expression, Event 2,factors obtained to simplify fraction)
8. Ls: Yes.
9. T: For before we can go and divide, **what must I do**?
10.Ls: *Take out* the common factor.
11.T: *Take out* the common factor, ok?
12.Ls: Yes
13. T: So, the same thing applies here. It is everything that you, that you have learned, but they just put it into one thing to make it look a bit complicated. It's actually very simple...ok?
14. Ls: Yes
15. T: So*, over here* we need the common factor. Why? Because **we want to have one, one term *at the top* and one term *below***, ok?
16. Ls: Yes
17. T: So, what is common factor to the two terms?

[18-36] – not shown; includes reference to "**change the sign"** shift from division to multiplication

37. T: **So, you just apply the same principle, it's just that when it looks complicated just pause and say *what must I do* here?** Because I know terms *like this* (points to $\frac{x^3-x^2}{4}$), I cannot just...go and say *this* (pointing to $x^3 - x^2$) divided by *this* (points to 4) ...ok?

38. Ls: Yes 39. T: Make sure that you have got <u>one term</u> *at the top* and <u>one term</u> *below*. So from here I can, what must I do? … [T, together with Ls and with similar interactional pattern, produce the answer.]
Talk: Level 2 – Uses some math language (e.g. ln 3) to name individual components or simply read string of symbols when explaining **Legitimation: Level 1** Reference to visual features (e.g. ln 3, 4, 13) and **Level 2M (Local)** Established shortcuts; conventions (e.g. lns 7, 10, 11, 30) and **Level 3M (General)** Makes reference to structure/principle but not clear due to naming (e.g. ln 37)
Event 4: Interaction pattern
Interaction pattern: Dominantly Level 1 Ls answer yes/no questions or supply words to T's unfinished sentence; **Occasional Level 2** Ls answer what/how questions in phrases or sentences

DISCUSSION

Our MDI framework allows for an attenuated description of practice, prising apart parts of a lesson that in practice are inextricably interconnected, and how each of these contribute overall to what is made available to learn. It co-ordinates "various variables, situations and circumstances" of teacher activity (Ponte & Chapman, op cit) There is much room for the teacher to work on learner participation patterns, as well as task demand (and these are inevitably inter-related); at the same time her example space even in sub-event 4.3, evidences awareness of and skill in producing a sequence of examples that bring the operation of division with varying algebraic fractions into focus, hence the value of this specific aspect of MDI. While not within scope here, contrasting levels in earlier observation of this teacher indicates an expanded example space and more movement in her talk between colloquial and mathematical discourse. The MDI framework is thus helpful in directing work with the teacher (practice), and in illuminating take up of aspects of MDI within and across teachers (research); e.g. our analysis across teachers suggests that take-up with respect to developing generality of explanations is more difficult. We contend further that content illumination through examples is productive across pedagogies and so across varying contexts and practices.

The MDI framework provides for responsive and responsible description. It does not produce a description of the teacher uniformly as in deficit, as is the case in most literature that works with a reform ideology, so positioning the teacher in relation to researchers' desires (Ponte & Chapman, op cit). We have illustrated MDI on what many would refer to as a 'traditional' pedagogy. MDI works as well to describe lessons structured by more open tasks, indeed across ranging practices observed.

CONCLUSION

In this paper we have communicated the overall framework, and illustrated its potential through analysis of selected project data. What then of its wider potential? While we have suggested this in pointing to our use across a range of practice in our data, we recognize that MDI arises in a particular context. Its potential beyond the goals of the WMCS project needs to be argued. Analytic resources are necessarily selective, reflecting a privileged reading of mathematics pedagogy. We have made these visible and explicit, and hold that its generativity lies in their theoretical grounding.

Acknowledgements

This work is based on research of the Wits Mathematics Connect Project, University of the Witwatersrand, supported by Department of Science and Technology, and FirstRand Foundation, and administered by the National Research Foundation. Any opinion, findings, conclusions, recommendations are those of the authors. The authors also acknowledge the WMCS project team all of whom have contributed to MDI.

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