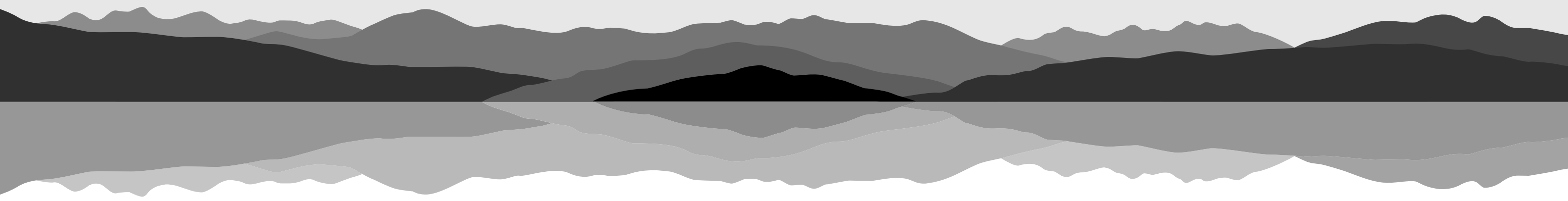
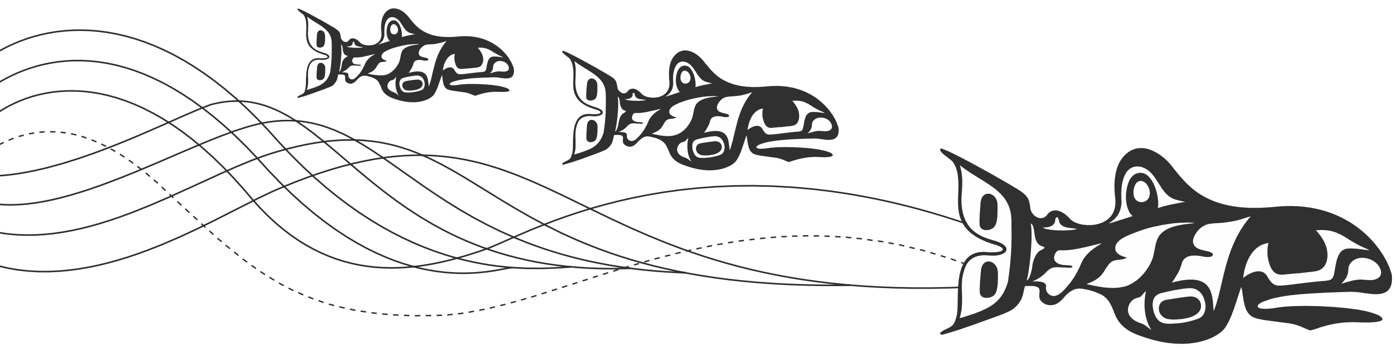


PME 38 / PME-NA 36

Proceedings

Vancouver, Canada | July 15-20, 2014

Of the 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education



Volume 1

Editors | Peter Liljedahl, Cynthia Nicol, Susan Oesterle, Darien Allan

PME 38 / PME-NA 36

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PREFACE

We are pleased to welcome you to PME 2014. This year's meeting is a joint conference of the 38th meeting of the International Group for the Psychology of Mathematics Education (PME 38) and the 36th meeting of the North American Chapter of the Psychology of Mathematics Education (PME-NA 36). PME is one of the most important international conferences in mathematics education and draws educators, researchers, and mathematicians from all over the world.

PME 2014 convenes in Vancouver, Canada, located on the traditional and unceded territory of the Musqueam, Squamish, and Tsleil-Waututh First Nations. On behalf of the conference local organizing committee we are very excited to welcome you to Vancouver. This is a conference of firsts. This is the first time PME has returned to Canada since 1987 PME 11 in Montreal. This is also the largest PME conference to date. For the first time in PME history more than 800 people will attend the conference, representing almost 50 countries around the world. This is the first time that a session devoted to supporting new researchers in mathematics education will be included in the conference meeting and, for some of you, it will be your first time in Vancouver. We are honoured to host you at this prestigious conference.

Mathematics Education at the Edge has been chosen as the theme of the conference. Academically, the theme provides opportunities to highlight and examine mathematics education research that is: 1) breaking new ground or on the cutting edge of innovative research and research methodologies; and 2) exploring issues with groups that are often positioned at the edge or periphery of educational research, such as social justice, peace education, equity, and Indigenous education. Geographically, the theme *Mathematics Education at the Edge* describes the very place of the conference setting, Vancouver, a city situated at the edge of Canada on the Pacific Ocean and Coast Mountain Range.

The papers in the six volumes of these proceedings are organized according to the type of presentation. Volume 1 contains the presentations of our plenary speakers, Research Forum activities, Discussion Group activities, Working Session activities and the National Presentation of mathematics education in Canada. Volumes 2 – 5 contain the Research Reports of the conference, while Volume 6 consists of the Short Oral and Poster Presentations.

The organization of PME 2014 is a collaborative effort involving teams of colleagues at the University of British Columbia and Simon Fraser University. We are grateful for the support received from each university. The conference is also organized with the support of three committees: the International Program Committee for PME 2014, the International Committee of PME, and the Local Organizing Committee, along with the PME Administrative Manager. We acknowledge the tremendous support and effort of these various committees. PME 2014 is possible only with the time and energy of the many volunteers who have dedicated hundreds of hours before, during and following the conference. Thank you to each of you. Finally, we thank each PME participant for

making your journey to PME 2014 in Vancouver and for your contributions to this conference.

Place has a powerful impact on what and how we learn. We hope Vancouver, as the place of PME 2014, will evoke and inspire exciting, critical, and difficult discussions that are on the *edge* of mathematics education research.

Cynthia Nicol and Peter Liljedahl
PME 2014 Conference co-Chairs

SPONSORS

We are grateful for the generous support received for this conference from the following sponsors:

Faculty of Education, Simon Fraser University

Vice President Academic, Simon Fraser University

Faculty of Education, University of British Columbia

Department of Curriculum and Pedagogy, University of British Columbia

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THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION (PME) & THE NORTH AMERICAN CHAPTER (PME-NA)

HISTORY AND AIMS OF PME

The International Group for the Psychology of Mathematics Education (PME) is an autonomous body, governed as provided for in the constitution. It is an official subgroup of the International Commission for Mathematical Instruction (ICMI) and came into existence at the Third International Congress on Mathematics Education (ICME 3) held in Karlsruhe, Germany in 1976.

Its former presidents have been:

Efraim Fischbein, Israel	Carolyn Kieran, Canada
Richard R. Skemp, UK	Stephen Lerman, UK
Gerard Vergnaud, France	Gilah Leder, Australia
Kevin F. Collis, Australia	Rina Hershkowitz, Israel
Pearla Nesher, Israel	Chris Breen, South Africa
Nicolas Balacheff, France	Fou-Lai Lin, Taiwan
Kathleen Hart, UK	João Filipe Matos, Portugal

The current president is Barbara Jaworski, United Kingdom.

PME-NA

The North American Chapter of the PME (PME-NA) is affiliated with PME and shares the same major goals as PME.

THE CONSTITUTIONS OF PME AND PMENA

The constitution of PME was adopted at the Annual General Meeting of August 17, 1980 and changed at the Annual General Meetings of July 24, 1987; August 10, 1992; August 2, 1994; July 18, 1997; July 14, 2005 and July 21, 2012.

The constitution of PME-NA was adopted at the PME-NA Annual General Meeting of October 24, 1982 and changed at the PME-NA Annual General Meeting of October 28, 2006. The major goals of the group are the same as PME:

The major goals of both groups are:

- to promote international contact and exchange of scientific information in the field of mathematical education;

- to promote and stimulate interdisciplinary research in the aforesaid area; and
- to further a deeper and more correct understanding of the psychological and other aspects of teaching and learning mathematics and the implications thereof.

All information concerning PME and its constitution can be found at the PME Website: www.igpme.org

All information concerning PME-NA and its constitution can be found at the PME-NA Website: www.pmena.org

PME MEMBERSHIP AND OTHER INFORMATION

Membership is open to people involved in active research consistent with aims of PME, or professionally interested in the results of such research. Membership is on an annual basis and depends on payment of the membership fees. PME has between 700 and 800 members from about 60 countries all over the world.

The main activity of PME is its yearly conference of about 5 days, during which members have the opportunity to communicate personally with each other about their working groups, poster sessions and many other activities. Every year the conference is held in a different country.

There is limited financial assistance for attending conferences available through the Richard Skemp Memorial Support Fund.

A PME Newsletter is issued three times a year, and can be found on the IGPME website. Occasionally PME issues a scientific publication, for example the result of research done in group activities.

WEBSITES

All information concerning PME, its constitution and past conferences can be found at the PME Website: www.igpme.org

All information concerning PME-NA, its constitution and past conferences can be found at the PME Website: www.pmena.org

HONORARY MEMBERS OF PME

Efraim Fischbein (Deceased)
Hans Freudenthal (Deceased)
Joop Van Dormolen (Retired)

PME ADMINISTRATIVE MANAGER

The administration of PME is coordinated by the Administrative Manager

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Germany

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INTERNATIONAL COMMITTEE OF PME

Members of the International Committee (IC) are elected for four years. Every year, four members retire and four new members are elected. The IC is responsible for decisions concerning organizational and scientific aspects of PME. Decisions about topics of major importance are made at the Annual General Meeting (AGM) during the conference.

The IC work is led by the PME president who is elected by PME members for three years.

President:	Barbara Jaworski	United Kingdom
Vice President:	Stefan Ufer	Germany
Secretary:	Stephen Hegedus	USA
Treasurer:	Bettina Dahl Søndergaard	Denmark
Members:	Olive Chapman	Canada
	Marta Civil	USA
	Marj Horne	Australia
	Keith Jones	United Kingdom
	Guri A. Nortvedt	Norway
	Anke Lindmeier	Germany
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	Masakazu Okazaki	Japan
	Núria Planas	Spain
	Leonor Santos	Portugal
	Michal Tabach	Israel
	Tai-Yih Tso	Taiwan
	Wim van Dooren	Belgium

STEERING COMMITTEE OF PME-NA

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Treasurer:	Ji-Eun Lee	Oakland University (USA)
Graduate Rep:	Julie Nurnberger-Haag	Michigan State University (USA)
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Past Conference:	Mara V. Martinez	University of Illinois at Chicago (USA)
Current Conference:	Peter Liljedahl	Simon Fraser University (Canada)
Current Conference:	Susan Oesterle	Douglas College (Canada)
Future Conference:	Kristen Bieda	Michigan State University (USA)
Future Conference:	Tonya Bartell	Michigan State University (USA)
Members:	Marcy Wood	University of Arizona (USA)
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	Jennifer Eli	University of Arizona (USA)
	Donna Kotsopoulos	Wilfred Laurier University (Canada)
	Wendy Sanchez	Kennesaw State University (USA)
	Amy Hackenberg	Indiana University (USA)
	Erik Tillema	Indiana University – Purdue University Indianapolis (USA)

THE PME 2014 CONFERENCE

The 2014 joint meeting is the 38th meeting of the International Group for the Psychology of Mathematics Education (PME 38) and the 36th meeting of the North American Chapter of the Psychology of Mathematics Education (PME-NA 36). Given the common roots and the uncommon numbering we have chosen to refer to this joint meeting as PME 2014.

Two committees are responsible for the organization of the PME 2014 Conference: the International Program Committee (IPC) and the Local Organizing Committee (LOC). Because PME 2014 is a joint conference the IPC is constituted by an intersection of members representing the PME International Committee (PME), the PME-NA Steering Committee (PME-NA), and the Local Organizing Committee (LOC). These representatives are working together to ensure that the joint conference delivers a first rate scientific program. The joint conference will operate under PME policies and practices.

THE INTERNATIONAL PROGRAM COMMITTEE (IPC)

Barbara Jaworski	Loughborough University (United Kingdom) *PME President*
Peter Liljedahl	Simon Fraser University (Canada) *PME 2014 Co-Chair (LOC), PME-NA representative*
Cynthia Nicol	University of British Columbia (Canada) *PME 2014 Co-Chair(LOC)*
Kim Beswick	University of Tasmania (Australia) *PME representative*
Olive Chapman	University of Calgary (Canada) *PME representative*
Susan Oesterle	Douglas College (Canada) *PME-NA representative, LOC representative*
Stefan Ufer	Ludwig Maximilians University of Munich (Germany) *PME representative*
Deborah Moore-Russo	University of Buffalo (USA) *PME-NA representative*

THE LOCAL ORGANIZING COMMITTEE (LOC)

The local organizing committee for PME 2014 in Vancouver is comprised of volunteers from Simon Fraser University and the University of British Columbia, as well as a number of other institutions in British Columbia:

Peter Liljedahl, *Simon Fraser University*, and Cynthia Nicol, *University of British Columbia* (co-chairs).

Darien Allan, *Simon Fraser University*, Ann Anderson, *University of British Columbia*, Alayne Armstrong, *University of British Columbia*, Leicha Bragg, *University of British Columbia/Deakin University*, Sean Chorney, *Simon Fraser University*, Natasha Davidson, *Douglas College*, Gaya Jayakody, *University of the Fraser Valley*, Harpreet Kaur, *Simon Fraser University*, Judy Larson, *Simon Fraser University*, Minnie Liu, *Simon Fraser University*, Rob McDonald, *University of British Columbia*, Susan Oesterle, *Douglas College*, David Pimm, *Simon Fraser University*, Diana Royea, *University of British Columbia*, Rina Zazkis, Simon Fraser University.

We also acknowledge and are grateful for the volunteer support provided by many others during the conference including graduate students at the University of British Columbia and Simon Fraser University and high school students from Vancouver and Burnaby schools.

HOSTING INSTITUTES OF PME 2014

PME 2014 is co-hosted by Simon Fraser University (SFU) and the University of British Columbia (UBC) on the beautiful UBC campus.

SFU SIMON FRASER UNIVERSITY ENGAGING THE WORLD

UBC a place of mind

PROCEEDINGS OF PREVIOUS PME CONFERENCES

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No.	**Year**	Location	ERIC number, ISBN/ISSN and/or website address
1	**1977**	Utrecht, The Netherlands	Not available in ERIC
2	**1978**	Osnabrück, Germany	ED226945, ISBN 3-922211-00-3
3	**1979**	Warwick, United Kingdom	ED226956
4	**1980**	Berkeley, USA	ED250186
5	**1981**	Grenoble, France	ED225809
6	**1982**	Antwerp, Belgium	ED226943, ISBN 2-87092-000-8
7	**1983**	Shoresh, Israel	ED241295, ISBN 965-281-000-2
8	**1984**	Sydney, Australia	ED306127
9	**1985**	Noordwijkerhout, The Netherlands	ED411130 (vol. 1) ED411131 (vol. 2)
10	**1986**	London, United Kingdom	ED287715
11	**1987**	Montréal, Canada	ED383532 ISSN: 0771-100X
12	**1988**	Veszprém, Hungary	ED411128 (vol. 1) ED411129 (vol. 2)
13	**1989**	Paris, France	ED411140 (vol. 1) ED411141 (vol. 2) ED411142 (vol. 3)
14	**1990**	Oaxtepex, Mexico	ED411137 (vol. 1) ED411138 (vol. 2) ED411139 (vol. 3)
15	**1991**	Assisi, Italy	ED413162 (vol. 1) ED413163 (vol. 2) ED413164 (vol. 3)
16	**1992**	Durham, USA	ED383538
17	**1993**	Tsukuba, Japan	ED383536
18	**1994**	Lisbon, Portugal	ED383537

19	**1995**	Recife, Brazil	ED411134 (vol. 1) ED411135 (vol. 2) ED411136 (vol. 3)
20	**1996**	Valencia, Spain	ED453070 (vol. 1) ED453071 (vol. 2) ED453072 (vol. 3) ED453073 (vol. 4) ED453074 (addendum)
21	**1997**	Lahti, Finland	ED416082 (vol. 1) ED416083 (vol. 2) ED416084 (vol. 3) ED416085 (vol. 4)
22	**1998**	Stellenbosch, South Africa	ED427969 (vol. 1) ED427970 (vol. 2) ED427971 (vol. 3) ED427972 (vol. 4) ISSN: 0771-100X
23	**1999**	Haifa, Israel	ED436403 ISSN: 0771-100X
24	**2000**	Hiroshima, Japan	ED452301 (vol. 1) ED452302 (vol. 2) ED452303 (vol. 3) ED452304 (vol. 4) ISSN: 0771-100X
25	**2001**	Utrecht, The Netherlands	ED466950, ISBN 90-74684-16-5
26	**2002**	Norwich, United Kingdom	ED476065, ISBN 0-9539983-6-3
27	**2003**	Honolulu, Hawai'i, USA	ED500857 (vol.1) ED500859 (vol.2) ED500858 (vol.3) ED500860 (vol.4) ISSN: 0771-100X http://www.hawaii.edu/pme27
28	**2004**	Bergen, Norway	ED489178 (vol.1) ED489632 (vol.2) ED489538 (vol.3) ED489597 (vol.4) ISSN: 0771-100X www.emis.de/proceedings/PME28

29	**2005**	Melbourne, Australia	ED496845 (vol. 1) ED496859 (vol. 2) ED496848 (vol. 3) ED496851 (vol. 4) ISSN: 0771-100X
30	**2006**	Prague, Czech Republic	ED496931 (vol. 1) ED496932 (vol. 2) ED496933 (vol. 3) ED496934 (vol. 4) ED496939 (vol. 5) ISSN: 0771-100X http://class.pedf.cuni.cz/pme30
31	**2007**	Seoul, Korea	ED499419 (vol. 1) ED499417 (vol. 2) ED499416 (vol. 3) ED499418 (vol. 4) ISSN: 0771-100X
32	**2008**	Morelia, Mexico	ISBN: 978-968-9020-06-6 ISSN: 0771-100X http://www.pme32-na30.org.mx
33	**2009**	Thessaloniki, Greece	ISBN: 978-960-243-652-3 ISSN: 0771-100X
34	**2010**	Belo Horizonte, Brazil	ISSN: 0771-100X http://pme34.lcc.ufmg.br
35	**2011**	Ankara, Turkey	978-975-429-262-6 ISSN: 0771-100X http://www.arber.com.tr/pme35.org
36	**2012**	Taipei, Taiwan	http://tame.tw/pme36 ISSN: 0771-100X
37	**2013**	Kiel, Germany	ISBN: 978-3-89088-287-1 ISSN: 0771-100X http:// http://www.pme2013.de/

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REVIEW PROCESS OF PME 2014

RESEARCH FORUMS (RF)

The goal of a Research Forum is to create dialogue and discussion by offering attendees more elaborate presentations, reactions, and discussions on topics on which substantial research has been undertaken in the last 5-10 years and which continue to hold the active interest of a large subgroup of PME and PME-NA. A Research Forum is not supposed to be a collection of presentations but instead is meant to convey an overview of an area of research and its main current questions, thus highlighting contemporary debates and perspectives in the field. Seven Research Forums proposals were submitted to PME 2014 and five were accepted by the International Program Committee (IPC):

- *The challenges of teaching mathematics with digital technologies—the evolving role of the teacher* with Alison Clark-Wilson, Gilles Aldon, Analisa Cusi, Merrilyn Goos, Mariam Haspekian, Ornella Robutti, and Mike Thomas
- *Mathematical tasks and the student* with David Clarke, Heidi Strømskag, Heather Lynn Johnson, Angelika Bikner-Ahsbahs, and Kimberly Gardner
- *Mathematical modeling in school education: Mathematical, curricular, cognitive, instructional, and teacher education perspectives* with Marcelo Borba, Rita Borromeo, Jinfa Cai, Michelle Cirillo, Gabriele Kaiser, John A. Pelesko, and Gloria Stillman
- *Spatial reasoning for young learners* with Nathalie Sinclair and Cathy Bruce
- *Habermas' construct of rational behavior in mathematics education: New advances and research questions* with Paolo Boero and Núria Planas

DISCUSSION GROUPS (DG)

The objective of a Discussion Group is to provide attendees with the opportunity to discuss a specific research topic of shared interest. The idea of a Discussion Group may be the result of an Ad hoc Meeting or an intensive discussion of a Research Report during the previous conference. Discussion Groups may begin with short synopses of research work, or a set of pressing questions. A Discussion Group is exploratory in character and is especially suitable for topics which are not appropriate for collaborative work in a Working Session because they are not yet elaborate enough or because a coherent research strategy has not been identified. A successful Discussion Group may result in an application for a Working Session one year later. 20 proposals were submitted for PME 2014 and 11 Discussion Groups were accepted by the IPC:

- *Exploring horizons of knowledge for teaching* with Nicholas H. Wasserman, Ami Mamolo, C. Miguel Ribeiro, and Arne Jakobsen

- *Mathematical discourse that breaks barriers and creates spaces for marginalised students* with Roberta Hunter, Marta Civil, Beth Herbel-Eisenmann, and David Wagner
- *Negative numbers: bridging contexts and symbols* with Laura Bofferding, Nicole Wessman-Enzinger, Aurora Gallardo, Irit Peled, and Graciela Salinas
- *Numeracy across the curriculum* with Merrilyn Goos, Helen Forgasz, and Vince Geiger
- *Observing teachers observing mathematics teaching: Researching the unobservable* with David A. Reid, Richard Barwell, Lisa Lunney Borden, Dominic Manuel, Elaine Simmt, and Christine Suurtamm
- *Preparing and supporting mathematics teacher educators: Opportunities and challenges* with Rachael Mae Welder, Amanda Jansen, and Andrea McCloskey
- *Researching 'Thinking Classrooms'* with Gaye Williams and Peter Liljedahl
- *School mathematics curriculum in centralized and decentralized educational systems* with Zahra Gooya and Soheila Gholamazad
- *The affordances and constraints of multimodal technologies in mathematics education* with Stephen Hegedus, Yenny Otalora, Lulu Healy, and Nathalie Sinclair
- *Visualization in teacher education: Toward a pedagogy* with Barbara Mary Kinach and Andrew Coulson
- *What is quality teaching-research?* with Bronislaw Czarnocha

WORKING SESSIONS (WS)

The aim of Working Sessions is that participants collaborate in joint activities on a research topic. For this research topic, there must be a clear research framework or research strategy and precise goals so that a coherent collaborative activity is ensured. Ideas for a Working Session can result from Discussion Group sessions of previous conferences where a topic was elaborated upon and a research framework or strategy was developed. Each Working Session should be complementary to the aims of PME and PME-NA and ensure maximum involvement of each participant. 9 proposals were submitted for PME 2014 and 7 were accepted by the IPC:

- *Teacher noticing: a hidden skill of teaching* with Molly H Fisher, Edna O Schack, Jennifer Wilhelm, Jonathan Thomas, and Rebecca McNall-Krall
- *Developing preservice elementary teachers' mathematical knowledge for teaching* with Lynn Cecilia Hart and Susan Oesterle

- *Key issues regarding teacher–student interactions and roles in assessment for learning* with Guri A. Nortvedt and Leonor Santos
- *Mathematics teacher educators' knowledge* with Kim Beswick, Merrilyn Goos, and Olive Chapman
- *Special education and math working group* with Helen Thouless, Ron Tzur, Susan Courey, Marie Fisher, Jessica Hunt, Katherine Lewis, Robyn Ruttenberg, and Yan Ping Xin
- *The use of eye-tracking technology in mathematics education research* with Patrick William Barmby, Chiara Andrà, David M. Gómez, Andreas Obersteiner, and Anna Shvarts
- *A discussion on virtual manipulatives* with Patricia Moyer-Packenham and Jennifer Suh

RESEARCH REPORTS (RR)

Research Reports are intended to deal with topics related to the major goals of PME and PME-NA. Reports should state what is new in the research, how it builds on past research, and/or how it has developed new directions and pathways. Some level of critique must exist in all papers.

The IPC received 486 RR proposals. Each paper was blind-reviewed by at least two peer reviewers. As the review capacity offered by the eligible reviewers was insufficient, reviewers were asked to accept a larger number of reviews. The majority of the members contacted responded to the request and thanks to their efforts this crucial task was successfully completed.

All papers with two positive reviews were accepted directly. For the proposals with one positive and one negative review, a third review was assigned. In cases of more than one negative review, the IPC considered the reasons for the rejection and made a final decision whether to reject or invite a resubmission as a Short Oral Communication (SO) or a Poster Presentation (PP). In the end, 222 proposals were accepted, 141 were recommended as SOs, 68 as PPs and the remaining RR submissions were rejected.

SHORT ORAL COMMUNICATIONS (SO)

Short Oral Communications are intended for research that is best communicated by means of a short oral communication instead of a full research report. 331 proposals were submitted. Of these, the IPC accepted 183, recommended 55 as PPs, and the remaining submissions were rejected. In the end, considering resubmissions of Research Reports as Short Orals, 281 Short Orals were accepted.

POSTER PRESENTATIONS (PP)

Poster Presentations are intended for information/research that is best communicated in a visual form rather than as a formal paper presentation. 96 proposals were submitted. The IPC accepted 63 proposals and rejected the remainder. In the end, considering resubmissions of Research Reports and Short Oral proposals as Poster Presentations, 141 posters were accepted for presentation.

The reviewing process was completed during the 2nd Meeting of the International Program Committee at the beginning of April 2014. Notifications of decisions of the International Program Committee to accept or reject the proposals were available by mid April 2014.

LIST OF PME 2014 REVIEWERS

The International Program Committee of PME 2014 thanks the following people for their help in the review process:

A

Abu-Bakare, Veda *(Canada)*
Adler, Jill (South Africa)
Aizikovitsh-Udi, Einav *(Israel)*
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Alsalim, Lyla Ali *(Canada)*
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B

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C

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D

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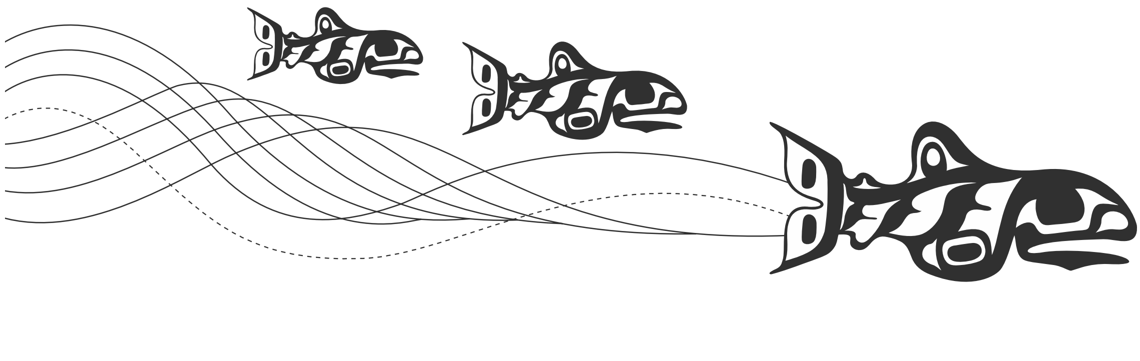
PLENARIES



ON TEACHERS *AND* STUDENTS: AN ETHICAL CULTURAL-HISTORICAL PERSPECTIVE

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My purpose in this paper is to try to understand the ideas that underpin our conceptions of mathematics teachers and students. The path that I follow rests on the thesis that our conceptions of teachers and students are related to, and derive from, conceptions of classroom forms of mathematical knowledge production and forms of human cooperation. In the first part of the paper I focus on two influential educational paradigms of Western modernity and late modernity—the transmissive program and the "progressive" educational program. In the second part of the article, I discuss a historical and cultural conception of teachers and students that is based on a non-utilitarian ethical dialectical materialist logic of production of knowledge and subjectivities.

INTRODUCTION

In *Politics*, Aristotle remarks that when it comes to determining what should be the aim of education "The existing practice is perplexing; no one knows on what principle we should proceed." Should we teach children what is "useful in life," or "higher knowledge," or what promotes "excellence" (that is, what contributes to virtue)? (Aristotle, 1984, p. 2121; 1337a33-1337b22)

Were Aristotle able to come back from his grave, he would certainly be disappointed in the utilitarian outlook of our 21st century school mathematics. He would be surprised to see school board officials, principals, and teachers meticulously absorbed in the implementation of abstract consumerist skills and competencies through business-like management techniques. And he would definitely be confused by our tremendous fixation on subjecting students to regional, provincial, national, and international evaluations.

We may guess that, as a good Athenian, Aristotle would regret the disappearance of curriculum principles concerning the common good and truly social communitarian life. He would be appalled by the instrumental conception of teachers and students. And maybe he would be very curious about how we ended up where we are today.

My purpose here is not to suggest how we should tell Aristotle our history. My purpose is rather to try to understand the ideas that underpin our conceptions of teachers and students. For instance, what is it that has moved us to conceive of the students in an instrumental manner? That is, how did we end up conceiving of students as rational- and instrumental-oriented cognitive problem solvers denuded, as it were, of all subjectivity?

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There are, of course, different ways in which to carry out this investigation. The path that I shall follow rests on the thesis that our conceptions of students and teachers are related to, and derive from, conceptions of classroom forms of knowledge production and forms of human cooperation. I will be arguing that the classroom forms of knowledge production and human cooperation are not created *in situ*, on the spot. On the contrary, they are related to culturally and historically constituted political and economical forms of production of human existence. Within this context, to conduct my investigation, I need to pay attention to the cultural forms of production of material, social, and spiritual life. Hence I will be trying to scrutinize what we can term *cultural logics of production* out of which knowledge and subjectivities are co-produced in the classroom. I am interested here in discussing the conceptions of teachers and students conveyed by two influential educational paradigms of Western modernity and late modernity. These are the transmissive educational program of the 20th century, and the "progressive" educational program that sought to replace the previous one during the last third of the 20th century and that continues to serve to a large extent as a general model for mathematics education today. In the second part of the article, I focus on conceptualizations of mathematics teachers and students from an ethic cultural-historical perspective.

ON TEACHERS AND STUDENTS

A few months ago, our research team was videotaping in one of the schools we regularly work with. At lunchtime we went to the staff room and joined a discussion about the forthcoming provincial assessments. Quickly, the discussion moved to questions about best teaching practices and, unavoidably, to the expectations we set about our students. These expectations do not merely refer to the reasonableness of having the students learn this or that mathematical content in a certain span of time. More than anything else, they refer to the very idea we adopt about what it means to be a student. Let us pause for a second and think what we take a student to be beyond, of course, its legal definition—that is, an individual subjected to a learning institution.

The idea of the student that we articulate in education in general and in mathematics education research in particular derives from conceptions about how knowledge is produced and the role of students and teachers therein. Conceptions of knowledge production are in turn embedded in cultural conceptions of the production of existence more generally and in their ensuing conceptions of the individual. This is why our conceptions of students are not insulated from general conceptions of the individual. The latter serves, indeed, as the basis to create expectations about the students —e.g., how students should behave, what they should or should not do, what they should learn and how. What are these conceptions?

The transmissive educational program

In modern times, one predominant conception was developed during the first half of the 20th century. It was underpinned by a bureaucratic idea of agents as implementers of solutions that were required in the business production context. Individuals were expected to acquire skills and use them in order to cope with well-defined problems. This conception of the individual nourished the idea of teachers and students conveyed in Paulo Freire's "banking concept" of education. In *The pedagogy of the oppressed*, Freire (2005) pointed out that students were receiving knowledge in a passive manner by a knowledgeable teacher who treated the students as deficient beings—empty containers or depositories that, as education progressed, were gradually filled with static, monotonous and irrelevant knowledge. Education, Freire argued,

> becomes an act of depositing, in which the students are the depositories and the teacher is the depositor. Instead of communicating, the teacher issues communiqués and makes deposits which the students patiently receive, memorize, and repeat. This is the "banking" concept of education, in which the scope of action allowed to the students extends only as far as receiving, filing, and storing the deposits. (p. 72)

What is the *logic of production* on which the transmissive educational program rests? The *logic of production* provides an educational program not only with a general framework to operate (e.g., by indicating how knowledge is produced and reproduced), but also with the parameters within which teachers and students are conceptualized.

The transmissive educational program, which was supported by behaviourist psychology, resorts to a *logic of production* that conceives of individuals as "adaptable, manageable things" (Freire, 2005, p. 73) and of knowledge as possession. Knowledge is treated as a commodity or merchandise that can be moved from one place to another and that can be passed from one individual (the teacher) to another (the student). Within this logic of production, the teacher *owns* knowledge and gives it to the student. In turn, the student comes to *own* knowledge through unreflective drill and repetition. Knowledge delivery is sanctioned and ensured by an institutional process where commodified knowledge is endowed with capital value.

What is the ultimate characteristic of this educational model? As we can see, the ultimate characteristic of this model is to consider knowledge as a commodity and to conceive of teachers and students as related to each other through processes of transmission of goods. Knowledge is considered a commodity that teachers *possess* and students *acquire*. Within this educational model, teachers and students are conceived of as *private owners*. The logic of production that underpins this conception is the *private owner logic of production*.

Although this conception of teachers and students has not completely vanished from education—at least not in practice—there are other conceptions nowadays. There is in particular a much more elaborated and sophisticated conception of the student that, roughly speaking, emerged in the second half of the second part of the 20th century and that Canadian psychologist Jack Martin (2004) describes in detail in a famous article—"The educational inadequacy of conceptions of self in educational psychology."

The "progressive" educational program

In the aforementioned article, Martin refers to a concept of the individual that has influenced contemporary educational and psychological research. The concept is a self-regulated adaptive

> individual labouring in relative solitude, constituted of componential mechanisms, processes, parts, and strategies . . . an individual actor capable of simultaneous action and reflection on this action, much like a stereotypic scientist in close scrutiny and judgment of experimental phenomena of interest . . . [An individual] whose most vital resources are apparently available within its detached internality . . . a self that already knows its business, one that requires only a facilitative grooming to become more fully socialized and intellectually engaged. (pp. 193-194, 197)

Drawing on this cultural conception of the modern individual, the modern conception of the student is largely based on the ideas of rational self-regulation, autonomy, and self-sufficiency. It assumes that the origin of meaning, knowledge, and intentionality is located *within*, and must come *from*, the individual.

This rational conception of the modern individual and its concomitant idea of the student do not come out of the blue. Both are a historical invention. Morris (1972) locates the first steps of this invention in the late Middle Ages. Traces of this historical invention are also found in the Renaissance—when some merchants and bankers emancipated from traditional societal structures and started conceiving of themselves as owners and crafters of their own destiny. However, it was only in the 17th and 18th centuries that Descartes, Kant and other philosophers articulated, in its clearest form, the modern idea of the individual as a sovereign, rational autonomous subject. During the nineteenth and twentieth centuries, the idea was gradually translated to the educational context, leading to the educational movement that has been called "progressivism" and its chief idea that "knowledge is … [a] personal acquisition, obtained by learning from experience" (Darling & Nordenbo, 2002, p. 298). Although "progressivism" evolved differently in German, England, and other countries, stressing with various nuances the learner's autonomy, the role of investigation and play (see, e.g., Neill, 1960/1992), bit by bit, from the aforementioned idea of knowledge as personal acquisition emerged the idea of the student as someone who is not there to be taught but rather someone expected to think and learn through his/her own deeds. For instance, drawing on Kant's ideas, Piaget (1973) asserted towards the end of his life that "The goal of intellectual education is in learning to master the truth *by oneself*" (p. 106; emphasis added). Piaget and educators of that time (e.g. Dearden, 1972) were instantiating the general view of the modern student, already advocated by Jean Jacques Rousseau in his *Émile*, written in 1762, that resulted in a Piagetian inspired child-centred "progressive" educational reform in the last third of the 20th century. The reform was based on the idea that knowledge is something that each student has to construct *by him/herself*—as opposed to something that can be passed on or learned from others. Within this context, leading the students towards an idea that did not come from them was often understood as constraining the students' freedom and autonomy: it was seen as coercing the students' own solutions and imposing the teachers' meanings upon them (Lerman, 1996; Radford, 2012).

What is the logic of production on which the Piagetian inspired "progressive" educational reform rests? As we can see, it rests on a logic of knowledge production that equates *doing* and *belonging*: *what belongs to me is what I do by myself. What I do not do by myself, does not belong to me*.

From the previous discussion we clearly see that, as in the case of the traditional educational program, the student of the educational "progressive" reform of modernity and late modernity is still conceived of as a *private owner*. It is hence not surprising that we treat them as private owners and talk about them as if what they do and should do is to produce their own wealth (in our case, mathematical knowledge). In their interaction with others we come to see them as *negotiating* meanings regulated by didactic contracts—as entrepreneurs negotiate goods in their own transactions bound by commercial agreements. We organize the curriculum around the idea of credits. And as a speaker did during the 2013 Ontario Educational Research Symposium, we talk about the students as *assets* of society. Within this context, as can be expected teachers are put at a difficult juncture. If students are to build their own knowledge through their own initiative and autonomy, what is then left to the teachers? There is no much left. What teachers can do is to give students their freedom (Darling & Nordenbo, 2002). An unending series of terms have been used to try to come up with the teacher's job description: coach, aide, helper, stimulator, consultant, guide, and so on. Whatever the term, in the end, within the logic of private ownership, they work indirectly, as *financial advisors*, helping the students secure and increase the knowledge they are supposed to create and grow by themselves.

Naturally, the private owner conception of the student is problematic on several counts. For one thing, it makes it difficult to understand the role of culture, history and society in the formation of the student. "There certainly is little here," Martin (2004) argues, "that might speak to the possible socio-culture, political, and moral constitution of personhood" (pp. 193-94). In the last few years—as a result of a flux of human migration and immigration, and the generalization of national capitalisms to a global capitalist economy and in tune with the condescending attitude of neoliberalism—cultural elements such as sensitivity to multiculturalism, as well as tolerance towards cultural and social differences have been added to the educational picture. However, in the end, social, cultural, historical and political factors remain considered as peripheral or partially understood in our constitution as individuals. Within the given logic of cultural production, social justice amounts to a mere redistribution of, and access to, the material.

Let us recapitulate. The transmissive educational program is based on the private owner production logic. It can only offer an alienating and oppressing structure where the teacher acts as a "bank-teller" that pays out wealth to the student. The student acts as someone who acquires wealth from the teacher through drill and repetition. The Piagetian inspired "progressive" educational program is based on the exact same logic, but reverses the roles of the agents. In the transmissive educational program the teacher assumes power while the students are relegated to a passive role. In the Piagetian "progressive" educational program the students assume power while teachers are relegated to an ancillary role. There is a nuance, though. In the first case, students receive knowledge. In the second case, the students no longer receive knowledge; they are expected to produce (and hence *own*) it. If a piece of knowledge has not been produced by the student, then it is not his/hers, and learning has not occurred. In the first case, the teacher's agency is emphasized to the detriment of the student's agency. In the second case, we have the exact opposite situation. Regardless of the nuance, in one case as in the other, education as a social practice is left structurally *the same*: the forms of knowledge production are those of private ownership. Although there is a displacement in the distribution of power and agency, both practices are equally alienating, since in both cases the student and the teacher remain alienated from each other, and from the broad historical and cultural context, without a truly possible connection. The teacher and the student are like screws placed in different parts of a machine, connected as it were by a formal link–a piece of metal that holds them together.

Now, since the structures of educational praxis remain the same, oppression is not removed: the autonomous student of the "progressive" educational program remains as oppressed as the student of the transmissive program. Freire (2005) saw this problem coming: "The truth is . . . that the . . . solution is not to 'integrate' them [the oppressed] into the structure of oppression, but to *transform* that structure so that they can become 'beings for themselves'" [that is, authentic critical beings- LR] (p. 74; emphasis added)

Late modernity expands and generalizes in more sophisticated and global terms the forms of production of modernity. But the forms of production of human existence remain basically the same. They are not *transformed*. It is hence not surprising that the corresponding educational projects—i.e., the transmissive and Piagetian "progressive" programs— remain unable to overcome alienation as well as their most striking contradiction, namely their conception of the individual as an acultural and ahistorical subject. What is at stake in this contradiction is the understanding of the relationship between the individual and society, and the student and knowledge. And the chances are that we will remain in the impasse we are in today, if we are not able to conceive of students other than as private owners. Indeed, to move beyond this predicament we seem to be in need of new conceptions of classroom knowledge production providing non-alienating roles for students and teachers. What could these roles be? Freire (2005) gives us a hint: "Education," he notes, "must begin with the solution of the teacher-student contradiction, by reconciling the poles of the contradiction so that both are simultaneously teachers *and* students (p. 76; emphasis in the original).

In the rest of the paper I sketch a different logic of production that provides room for exploring the aforementioned "reconciling contradiction" of teachers and students on the basis of a different conception of knowledge and its production.

A CULTURAL-HISTORICAL UNDERSTANDING OF TEACHERS AND STUDENTS

The cultural logic of production that I outline here draws on a Hegelian-Marxist dialectical materialist conception of knowledge. Within this logic of production, knowledge is not something that individuals, possess, acquire or construct. As a result, the relationship between students and teachers is not predicated in terms of individuals who possess knowledge and individuals who lack it.

Now, if knowledge is not possession, what is it? In dialectical materialism, knowledge is not a psychological or mental entity. The dialectical materialist idea of knowledge rests on the distinction between the *Potential* (something that may happen, i.e., possibility) and the *Actual* (its happening). Knowledge and its individual kinds, that is, *concepts*, are social-historical-cultural entities: a "complete totality of possible interpretations—those already known, and those yet to be invented" (Ilyenkov, 2012, p. 150). Knowledge includes possibilities of making calculations, or thinking and classifying spatial forms in certain "geometric" manners; possibilities of taking courses of action or imagining new ways of doing things, etc. This is what school knowledge is when the student crosses for the first time the school door—*pure open possibility.*

Let me note that knowledge as possibility is not something eternal, static, or independent of all human experience (as in Kant's concept of things-in-themselves or as in Plato's forms). In fact knowledge results from, and is produced through, human social labour. Knowledge is a *cultural synthesis* of people's doings. More precisely, knowledge is a dynamic and evolving implicit or explicit culturally codified way of doing, thinking, and relating to others and the world.

Knowledge as possibility means that knowledge is indeterminate, general (Hegel, 2009). Knowledge is not representable. In order for it to become an object of thought and consciousness, knowledge has to be set into motion. That is to say, it has to acquire cultural determinations. And the only manner in which knowledge can acquire cultural determinations is through specific *activities*. Let us take the example of algebraic knowledge—an example that I will develop with more detail below. Algebraic knowledge is not the sequence of signs we see on a paper. Algebraic knowledge is pure possibility—possibilities of thinking about indeterminate and known numbers in manners that are opened up by certain historically constituted analytical ways of thinking. Algebraic knowledge can only become an object of thought and interpretation by being put into motion and being made into an object of senses and consciousness through sensuous and sign-mediated specific problem-solving and problem-posing activities.

In more general terms, through activity knowledge moves from an indeterminate form of possibilities to a determinate singularized form filled with concrete determinations (e.g., the singularized knowledge-form that results from dealing with some *specific* equations). In this context the general/singular (or abstract/concrete) are not two opposed, disjoint kinds. They are two entangled ontological categories—two moments in the becoming of knowledge. This is why, in its becoming through activity, knowledge and concepts are simultaneously abstract and concrete.

The characterization of knowledge as movement from indeterminate possibilities to their determinate actualization or concretion through the mediation of activity offers room to envision in new ways the relationship between teachers and students. In engaging in activity, knowledge is something that teachers and students *produce*. That is, knowledge is something that they "bring forward." This is what 'to produce' means etymologically: to bring something forward (in this case, possibilities of mathematical action and reflection). In the transmissive and the "progressive" programs, the production of knowledge appears impoverished as a result of the structure of the knowledge-mediating-activity. In the transmissive program, it is the teacher who is essentially in charge of the activity. In the "progressive" program, it is the student who is essentially in charge of the activity. Within the dialectical materialist logic of production that I am outlining here, teachers and students carry out the knowledge-mediating-activity *together*. Knowledge is produced *collectively*. The collective nature of knowledge production means that students and teachers *work together* in order to bring forward possible mathematical interpretations and courses of action. Knowledge production refers to emergent classroom collective and dynamic ways of thinking and doing arising against the backdrop of culture and history. They include modes of mathematical inquiry, conceptions of truth, evidence, mathematical argumentation, symbol use, and meaning making.

Forms of human collaboration

There is something missing in the previous account. In engaging in classroom activities teachers and students do not merely produce knowledge. They co-produce themselves too. They co-produce themselves in accordance not only to the forms of knowledge production but also in accordance to the activity's *forms of human collaboration*. In the "progressive" educational program, forms of human collaboration are usually reduced to utilitarian tools. Thus, interaction is considered a form of reciprocity where agents mutually trade services driven by instrumental self-interest (see, e.g., Piaget's (1967) reckonable reciprocal interactionism; for a critique, see Radford & Roth, 2011). Within this context, students end up considered as purely rational- and instrumental-oriented cognitive problem solvers (Valero, 2004) denuded, as it were, of all subjectivity and unconcerned by questions of social existence.

As Mészáros (2010) put it, within

> the contemporary liberal orientation, we see that in society and in our schools, the legitimately feasible objectives of human activity must be conceptualized in terms of material advancement . . . remaining blind to the social dimension of human existence in other than essentially functional/ operative and manipulative terms. (p. 29; italics in the original)

Human collaboration, Mészáros concluded, is reduced to a "material factor of production" (p. 29).

The dialectical materialist cultural logic of production of knowledge and subjectivities that I am outlining here—what we have termed the 'theory of objectification' (Radford, 2008)—puts forth a *communitarian ethic* that promotes forms of human interaction driven by *solidarity*, *commitment*, *responsibility*, and *caring*. This communitarian ethic is based on a non-essentialist conception of the individual. It is a conception of the individual as constituted in a relation to *alterity*, that is to the *Other*—that which is not *I*.

Individuals, indeed, are seen as subjects ontologically constituted with others (Nancy, 2000): individuals who co-produce themselves as they engage in historically, economically, and politically configured social activities to produce and reproduce their material, interpersonal, and spiritual life. The individual that we portray is an individual whose "interiority," to borrow an Augustinian metaphor, is not *inside* the individual, but *outside*, in the exterior. More precisely, the individual's "interiority" is located in the ensemble of culturally, historically constituted social relations that the individual finds in front of him/her—an ensemble of social relations that reflexively shape each one of us through the activities in which we participate and through which we come to think, feel, hope and grow as cultural beings. Hegel remarked once that, to find their foundational support, plants use their roots, which reach outside themselves. Humans are not different: our constitutive foundational ontological support lies in our exterior, in material and spiritual culture, and in its individuals.

Although there seems to be a pre-reflective sensibility or pre-conceptual proclivity to attend to, and to attune with, others (Roth, 2013; Tomasello, 2009), more sophisticated ethical forms of relations between subjects are of a cultural and historical nature. They are embedded in culturally evolved forms of *being*. Like knowledge, being appears as *pure possibility*. But instead of presenting possibilities for knowing they are pure possibilities for *becoming*.

The ethical forms of human collaboration that we emphasize drive a general attitude towards the world and serve to configure the teachers' and students' participation in the classroom. The classroom appears as a public space of debates in which the students are encouraged to show openness towards others, solidarity, and critical awareness. The classroom indeed appears as a space of encounters where teachers and students become what Freire called "presences in the world" (Freire, 2004, p. 98). That is to say, the classroom appears as a space of encounters where teachers and students become individuals who are *more* than *in* the world, individuals with a vested interest in one another and the joint enterprise; individuals who intervene, transform, dream, apprehend, and hope.

To recapitulate, the cultural-historical perspective previously outlined rests on a dialectical materialist logic of production that offers an alternative by which to conceptualize teachers and students. Teachers and students are not conceptualized as private owners. They are conceptualized as individuals that engage in joint activity in the production of knowledge and subjectivities. Such a production rests on the idea that knowledge and being are pure possibilities—possibilities of knowing and becoming in collective emergent and unending processes configured by history and culture.

Within this cultural-historical perspective, the forms of knowledge production and human collaboration that mediate classroom activity are based on collaborative modes of action and interaction that seek: (1) to foster deep and varied mathematical understandings and interpretations, and (2) to create space for critical and reflexive subjective and inter-subjective growth to occur.

In the rest of the paper I discuss a classroom example that will help, I hope, to illustrate the previous ideas.

TEACHING-LEARNING ALGEBRA

The example that I present here comes from a 6-year longitudinal program where our research team followed a class of students from Grade 2 on. The research program revolved around the teaching and learning of algebraic pattern generalization and equations. I focus here on the latter.

As mentioned previously, within our cultural-historical perspective, algebraic knowledge (in this case knowledge about equations) is not considered a psychological or mental entity. It is a culturally codified and historically evolved way of thinking and doing that features an *analytical* manner of calculating with determinate and indeterminate quantities (Radford, 2013b). For our Grade 2 students, algebraic knowledge was a pure possibility—possibility to bring forward forms of action, understanding, and interpretation.

The culturally codified and historically evolved ways of thinking are *general* or *abstract.* As such, they cannot be sensed (perceived, touched, heard, etc.). They are pure possibility. To become objects of consciousness and thought, they have to be set into motion through activity. In common parlance, however, the term activity is often used with different and sometimes contradictory meanings. Sometimes activity refers to a mere set of actions carried out by an individual, sometimes by various individuals. Activity appears hence as a mere background or as an empirical phenomenon. This is not the dialectical materialist meaning. In dialectical materialism it is *in* and *through* cultural-historical activity that individuals produce knowledge and co-produce themselves. To emphasize this dialectical materialist meaning of activity, I shall use rather the original Hegelian term *labour*. I can hence rephrase my previous sentence by saying that, to become objects of consciousness and thought, algebraic knowledge has to be set into motion through classroom *joint labour*.

In joint labour teaching and learning are *fused* into a single process: the process of teaching-learning—one for which Vygotsky used the Russian word *obuchenie*. In this sense, teachers and students “are simultaneously teachers *and* students” (Freire, 2005, p. 76). They are simultaneously teachers *and* students, but not because both are learning (Roth & Radford, 2011). They are, of course. However, the real reason is because teachers and students are *labouring together* to *produce* knowledge.

Through students-and-teachers' joint labour algebraic knowledge is hence going to be *produced*, that is, knowledge is going to be *brought forward*. The bringing forward of knowledge, however, is always partial, incomplete, unfinished (Radford, 2013a). It is an actualization of possibilities made up of concrete and singular or individual interpretations, which depend on the characteristics of joint labour. These characteristics depend, in turn, on the didactic quality of the mathematical questions, the richness, deepness, and variety of forms of mathematical inquiry. To ensure a didactic quality we conduct an *a priori* analysis that allows us to gauge the epistemological density (e.g., sedimented meanings) of knowledge (in this case, algebraic knowledge). The epistemological analysis also provides us with the ground to select and conceptually organize the questions that will serve as the starting point for joint labour to occur. The forms of production of algebraic knowledge are hence related to the collective and emergent forms of thinking that will arise in the classroom out of conceptually charged and epistemologically informed questions that the students tackle with the teacher.

One important point to notice is the *division of labour* that arises as a result of the manner in which teachers and students are aware of the *object* (in Leont'ev's (1978) sense) of their joint labour. The teacher and the students have a different grasp of this *object*. The object, which has a didactic intention, is not necessarily clear for the students from the outset. The students' and teachers' difference vis-à-vis the object of labour creates tensions and (dialectical) contradictions (see Williams & Ryan, in press). In our example, the object of joint labour is to foster deep and varied mathematical interpretations and understandings about equations. These include those algebraic interpretations that have been built historically and culturally. It is in the course of joint labour that those historical interpretations may be brought forward and become objects of consciousness and thought. The culturally and historically built mathematical interpretations have been the object of successive refinements, organized theoretically in complex forms of thinking—in our example, historically constituted ways of algebraic thinking cast in more and more complex semiotic systems. Because these historical forms of thinking are not natural but cultural, they are not necessarily clear for the students. The teacher, hence, has a particular role to play in joint labour. But regardless of how much the teacher knows about algebra, she cannot set algebraic knowledge in motion by herself. She *needs* the students—very much like the conductor of an orchestra, who may know Shostakovich's 10th Symphony from the first note to the last, needs the orchestra: it is only out of joint labour that Shostakovich's 10th can be produced or brought forward and made an object of consciousness and aesthetic experience.

Now, the 'need' to which I am referring here is not merely rational. The 'need' is thoroughly *emotional*. It requires a deep emotional connection between participants. Thus in the best musical performances, the conductor and the musicians work truly collectively, attuning and responding to each other. Maybe one of the best examples of joint labour is the amazing tuning of Venezuelan conductor Gustavo Dudamel and the Simón Bolivar Youth Orchestra (see http://www.youtube.com/watch?v=XKXQzs6Y 5BY#aid=P9NiWZi3QJQ particularly from 27:50 to 35:00). This musical example intimates that forms of (musical) knowledge production are deeply entangled with forms of human interaction and cooperation. The same is true of classroom mathematical knowledge. The mathematics teacher and the students *need* to labour together to bring forward various mathematical interpretations and make them the object of an intellectual, reflective, and aesthetic experience. This joint labour is, simultaneously, intellectual and emotional; they cannot be separated. They are two sides of the same coin. We may conclude, then, by noticing that, although there is a division of labour that is induced by the manner in which teachers and students engage in their joint labour—division of labour that has to do with the teacher's awareness of the didactical intentions, etc.—the teacher and the students *need* each other to bring knowledge forward.

The communitarian ethic mentioned in the previous section finds its full expression in the theoretical articulation and practical elaboration of this mutual 'need' of teachers and students. We are interested in an ethic that fosters modes of collaboration of a non-utilitarian and non self-centred nature—modes of human collaboration and interaction that rather promote solidarity, critical stance, and responsibility.

Indeed, since the communal ethic we target is not something that will necessarily emerge in the classroom naturally, we have to create the conditions for it to appear. To promote forms of human collaboration aligned with our ethical perspective, since the beginning of research program, the class that we worked with was divided in small groups of 2 or 3 students since the beginning of the research program. The students were encouraged to discuss the emerging ideas, to listen to and try to understand the other students' perspectives, to compare them critically to what they produced and to engage in dialogue to improve the ideas generated in the classroom.

At the very beginning of the program the Grade 2 students were confronted with equations having the unknown on one side of the equation only. Then they tackled equations having the unknown on both sides. Here is an example of the latter. The equation was introduced under the form of a story that the teacher (T) read to the students:

> Sylvain and Chantal have some hockey cards. Chantal has 3 cards and Sylvain has 2 cards. Her mother puts some cards in three envelopes making sure to put the same number of hockey cards in each envelope. She gives 1 envelope to Chantal and 2 to Sylvain. Now, both children have the same amount of hockey cards. How many hockey cards are in an envelope?

The equation was illustrated in the blackboard (see Figure1).

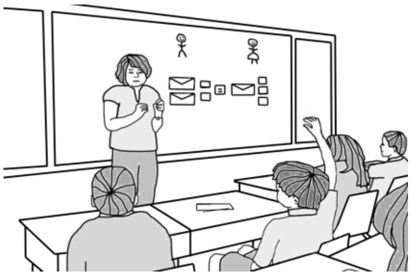

Figure 1: The Sylvain and Chantal equation

In the beginning the students were resorting to interpretations of an arithmetic nature: they were resorting to trial and error methods. Thus, Willy (W) suggests the following:

1 W: Um, I think there is… there is 1, um is, 1 hockey card in each card (*meaning envelope*), because… euh, there are 3 cards just there (*Chantal's cards*) and if there is just 1 in the cards (*meaning envelopes*), that means there are 4, and there are 2 cards just there (*meaning Sylvain's cards*) and 2, and … there are 2 in the 2 envelopes.

2 T: Uh, huh (*trying to make sense of W's strategy*). So, if I understand well Willy, you used the trial and error strategy?

3 W: Uh? Huh…

4 T: That is, you said: Ah! I will pretend that there is 1 card here, 1 card here, 1 card here (*referring to the envelopes*). Is that what you did?

The teacher asks for other ideas. Aided by Sue (S), Joe (J) suggests removing one envelope from each side of the equation:

5 J: Um, I think there is 1 [card] in each [envelope], because I would like to remove Chantal's envelope there…

6 T: OK.

7 S: And Sylvain's envelope and…

8 T Why do you remove an envelope here, and an envelope here?

9 J: Um, because if, because Chantal has 3 [cards], and Sylvain has 2 [cards], and if, and if there is a card in this [envelope], (*pointing to the remaining envelope in Sylvain's side of the equation*), it will make equal to, it's equal.

10 T: […] Ok, so you found the solution like that? You, you isolated a little bit, but you didn't isolate completely, eh? That was your solution, you removed the envelopes, eh?

11 J: Yeah…

Joe's strategy seems to draw on a discussion that the class had the previous day about removing envelopes to *simplify* the equation. In turns 5 and 7 he suggests removing one envelope from each side of the equation. Since this is a crucial idea in solving an equation algebraically (Filloy, Rojano, & Puig 2008), the teacher invites Joe to articulate the idea in an explicit way (Turn 8). Yet, removing the envelopes from each side of the equation is made to *simplify* the equation, but not to *deduce* the value of the unknown: in Turn 9 Joe indeed *assumes* that the number of hockey cards in an envelope is 1 and concludes that both sides of the simplified equations are equal.

The design of the activity provided the class with the opportunity to produce or bring forward mathematical interpretations of a varied nature and to become aware of the limitations of trial and error methods. Naturally the students did not know that they were resorting to trial and error methods (see Willy's surprise in Turn 3). Working together with the students, the teacher brings forward terminology and conceptualizations that allow the students to make sense and better understand the methods they imagine. In labouring with the students, the teacher also provides room for the classroom to collectively become conscious of the subtleties of a historically and culturally constituted way of thinking about equations. Thus, in Turn 10, the teacher distinguishes between isolating "a little" and "completely" the unknown. In doing so, the teacher calls attention to the fact that the equation was simplified but not totally algebraically solved—although of course, she does not use those terms. Rather, she calls attention to a subtle aspect of a way of thinking where equations are successively simplified until the value of the unknown is *deduced* (as opposed to *guessed*).

The previous short excerpts illustrate the forms of knowledge production that we foster: they are not based on the private owner logic of the transmissive and the "progressive" educational programs. Knowledge is not something that an individual possesses: knowledge is potentiality that is actualized through joint labour. By being actualized, knowledge (in this case algebraic knowledge) becomes the object of consciousness and thought for the students. In labouring together, the teacher and the students have accomplished important things. In Turn 4 the teacher articulates for the students a strategy that so far remained ostensibly shown. She uses the term "pretend," which allows the students to better seize the conceptual nature of the trial and error method they imagined. As pointed out above, in Turn 10 a delicate distinction is made apparent to differentiate between isolating the unknown "a little" and "completely." Although knowledge is being bestowed with mathematical determinations and, in doing so, is progressively becoming an object of consciousness for the students, the teacher and the students still have a long way to go.

Naturally, the teacher is not interested in the particular equation under discussion. Actually, she is not interested in *any* particular equation. The interest is rather put on *ways of thinking* about solving linear equations. But the understanding of such ways of thinking (the historically and culturally constituted way of thinking that we term algebraic included) can only appear through the solving of particular equations. The teacher and the students talk hence about particular equations (like the equation in the dialogue), but what is becoming object of consciousness is not how to solve *that* particular equation but the manner in which to think about equations like *that*. More precisely, what is becoming an object of consciousness for the students is a system of ideas: something general (i.e., algebraic knowledge). We have termed this lengthy social and sign-mediated process of becoming conscious of systems of ideas, in this case algebraic ideas, *objectification* (Radford, 2008). Now, the fact that the teacher is not interested in particular equations *per se*, does not mean that the discussed equations do not bear any relevance. The depth and scope of the awareness that is produced in the course of a process of objectification depends on the quality of the didactic choice of equations, their conceptual organization, etc. It would have been, for instance, of limited interest not to tackle equations with the unknown on both sides of the equation, given the object (i.e., the didactic purpose) of the joint labour. But the depth and scope of the awareness that is produced in the course of a process of objectification depends *also* on the characteristic and complexity of the classroom discussions, which in turn depends on forms of classroom interaction and human collaboration.

To foster sophisticated forms of classroom interaction and collaboration, we encourage the students to produce ideas in small groups and to discuss them with other groups in a deep manner. To provide a concrete example, I turn now to an example that comes from the following year—when the students were in Grade 3 (the students were 8–9-years old). In the beginning, the teacher plays a salient role in suggesting what is to be discussed and in the organization of the forms of interaction. The teacher's organizing role decreases as the collective gains cohesion and a common understanding of their joint labour. Thus, in the Grade 3 example, organization of interaction was divided in four steps. In the first step, the students worked in small groups to produce a text that included: a story of their invention, the translation of the story into an algebraic equation, and the solution of the equation (see Figure 2, pictures 1 and 2) (for more details, see Radford, 2012). Each group had a "corresponding" group with which an exchange in consecutive phases will occur.. In the second step, one text goes to the corresponding group, and vice-versa. Each group proceeds to read and evaluate the other group's production (see Figure 2, pic. 3). We ask the students to assess the corresponding group text on the basis of several elements, such as:

> (1) Is the text clear? (2) Do they find the answer to be right? (3) Do they find the solution convincing? (4) Do they find the solution beautiful?

Once they have finished critically studying the other group's text, the two groups get together (see Figure 2, pic. 4). The groups take turns presenting their results, emphasizing what they liked about the text and what they think should be improved and how. The teams also react to the critique and the teacher may also be part of the discussion. After having discussed the groups' texts, as a last step, they work together in trying to come up with a text that would be an improvement of what was initially submitted. They are also encouraged to share the final text with other groups.

The question, of course, is not merely how to come up with a better mathematical solution. Although this is important, equally important is the fact that in going through this process, the students have an opportunity to understand others and in understanding others to better understand themselves.

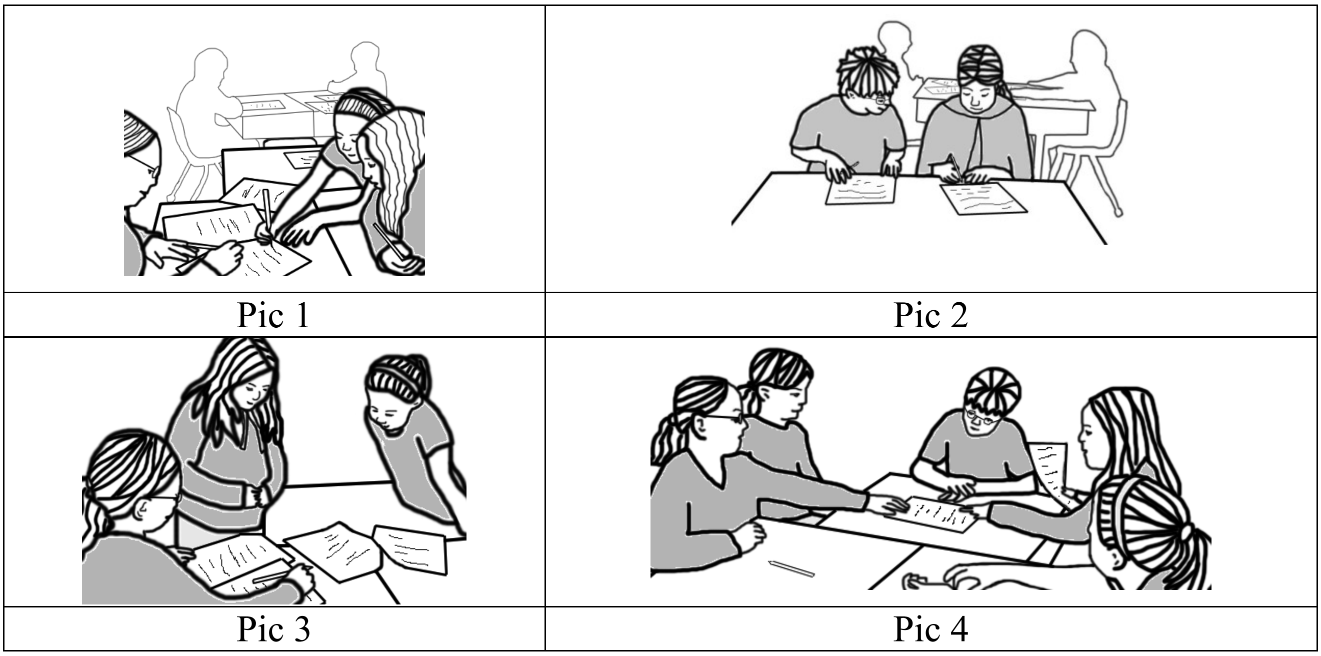

Figure 2: In Pics 1 and 2 Teams 1 and 4 work independently towards the production of a mathematics text. In Pic 3, Team 1 critically examines Team 4's text. In Pic 4, the members of Teams 1 and 4 meet to discuss their texts.

Team 1 (made up of Christina (Ch), Elisa (E), and Sara (S)) produced the following story: "Martine has a collection of 10 stamps. For her birthday she receives an envelope with stamps. Cassidy has 6 stamps in her collection. And [she receives] 2 envelopes with stamps for Christmas. How many stamps are there in each envelope?"

Team 2 (made up of Carl (C) and Sandra (Sd)) produced the following story: "For Christmas Calin received three boxes with *Webkinz* and Samantha received one box. He [Calin] has already 4 *Webkinz*. And Samantha has already 28 *Webkinz*. Now both have the same amount of *Webkinz*."

The translation and solution of the equation appear in Figure 3.

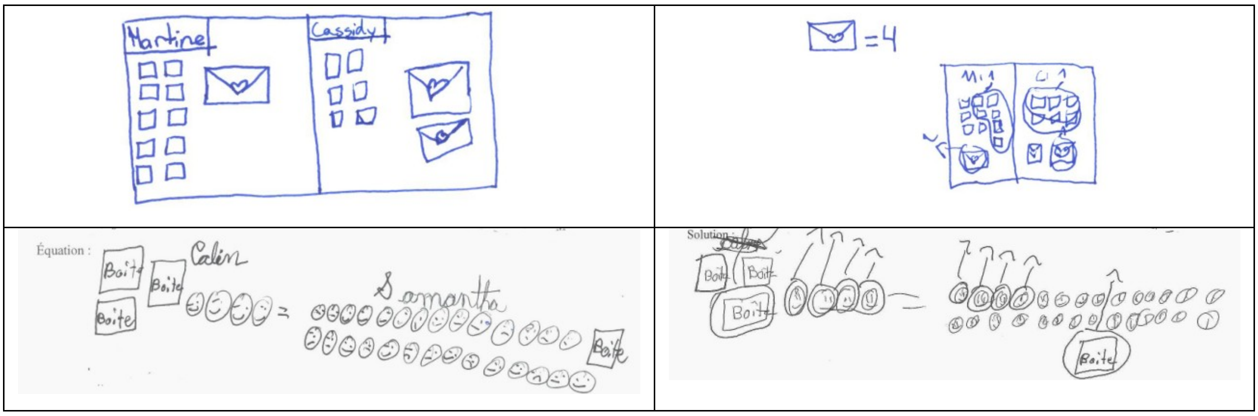

Figure 3: Top row, Team 1's equation and solution.
Bottom row, Team 4's equation and solution.

During the discussion, three elements were discussed. (1) Team 4 pointed out that Team 1's equation does not have an equal sign and that they did not specify the nature of the answer (they wrote that an envelope is equal to 4, but that such an answer is ambiguous). (2) The teams discussed the validity of removing 6 items at once from the equation (as Team 1 did), as opposed to removing items one at the time (as Team 4 did). (3) Team 1 complained that Team 1 did not ask a question.

The teacher (let's call her T', as she was different from the Grade 2 teacher) was attending to other groups when Team 1 (T1) and Team 4 (T4) started the conversation:

1	C:	(*Addressing theme 1*) Um, what we liked about your story is that it was clear, it was nice, there were no mistakes, we could read it well. That's about your story.
2	Sd:	(*Talking about the solution and addressing theme 2*) Here, here what we liked is that you put "envelope = 4."

Then, they pointed out what they did not like:

3	C:	What we did not like… You did not put the equal sign in the equation.
4	Sd:	And you have to put it.
5	C:	(*Addressing theme 3*) You did not [remove the equation terms] one at the time.

T1 agreed with T4's remarks. When it was T4's turn, T4 argued that T1's story did not include a question in the story and that without a question one cannot know what one is looking for. When the teacher (T') arrived, she found the students in a vivid and unsettled discussion. They summarize their discussion for her.

6	T':	So (*talking to Team 1*), is there a question missing?
7	Ch:	There is no question! (*Answering the teacher's question*) Yes!
8	E:	Yes, the question is missing!
9	T':	Ah! but why do you think that…
10	C:	(*Interrupting*) Yeah, but…
11	T':	(*Talking to Carl*) We'll ask the question here (*meaning T1*)… That's OK, you will be able to defend yourself. (*Talking to T1*:) Why do you think that it is important to ask a question?
12	Ch:	Because if you don't, what are you going to do?
13	Sd:	You don't need to ask a question!

The discussion continued without agreement for a while before the teacher decided to ask:

14	T':	For someone who is reading the story … do you think that it is important to ask the question?
15	C:	I would say no…
16	T':	I do think that in a story like this, it is important to have a question if …

In the end, the question remained unsettled. And the goal was not to settle it. The goal was to create the conditions of possibility for the students and the teacher to engage in classroom debates underpinned by non-utilitarian forms of human interaction and collaboration out of which the teacher and the students speak out and position themselves in the public space.

SYNTHESIS AND CONCLUDING REMARKS

My goal in this paper was to inquire into current conceptions of mathematics teachers and students. My inquiry was based on the idea that conceptions of what individuals are within a definite historical period are related to the manner in which human existence is culturally produced. I scrutinized hence the conceptions of teachers and students through the lenses of what I termed the cultural logics of production, which included *forms of production* and *forms of human collaboration*. The examination of the traditional educational model (which Freire called the "banking concept" of education) through this methodological approach reveals that teachers and students are produced by an alienating educational structure that turns them into private owners. When we apply the same inquiry to the Piagetian "progressive" educational reform we cannot fail to notice that the educational structure is basically kept intact. The illusions of change appear only in terms of re-distribution of power and agency, but in the end students are subjected to the same utilitarian production logic—namely the private ownership of production logic—with the same alienating results. It turns out that, upon closer scrutiny, the progressive educational reform is not the antithesis of traditional teaching but its dual model. In the second part of the paper, I discussed a historical-cultural conception of teachers and students that is based in a non-utilitarian logic of production of knowledge and subjectivities. The forms of production of knowledge in the classroom rest on a dialectical materialist conception of knowledge as historically synthesized labour. Mathematical knowledge is not something possessible. It is not yours or mine. Mathematical knowledge appears as pure potentiality—virtual possibilities for mathematical understandings, meanings, and course of action.

To be materialized, knowledge has to be set into motion through teachers' and students' labour. It is through classroom labour that knowledge moves from an indeterminate form of possibilities to a determinate singularized form filled with concrete determinations (in our examples, in Grade 2, the concrete determinations revolved around guessing the answer through trial-and-error methods and issues about the algebraic simplification of equations; in Grade 3 the concrete determinations revolved around symbol-use, the translation of story-problems into algebraic non-formal symbolism, etc.).

The classroom labour that mediates knowledge and makes it an object of thought and consciousness may be alienating or fulfilling. All will depend on the classroom forms of knowledge production and forms of human collaboration. The dialectical materialist approach to the production of knowledge and subjectivities outlined in this paper puts forward a communitarian ethic that is based on a non-essentialist conception of the individual that allows us to understand teachers and students as engaged intellectually, emotionally and ethically in joint labour. Teachers *and* students are in the same boat, *producing knowledge and learning together*. In their joint labour, they sweat, suffer, and find gratification and fulfillment with each other.

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THINKING WITH AND THROUGH EXAMPLES

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In my talk I will discuss the roles that examples play (or could play) in mathematical thinking, learning, and teaching. I draw mainly on research I've been doing for over a decade that addresses this broad topic from several perspectives.

In terms of learning – I look at mathematical concepts (e.g., periodic function) and meta-concepts (e.g., definition) and examine the way interacting with examples may enhance understanding of these concepts. In terms of mathematical thinking, I look at mathematical proof and proving as a site for developing mathematical thinking (here I draw on my experience in designing and implementing an undergraduate course on Mathematical Proof and Proving (MPP), on my current work with Eric Knuth and Amy Ellis on the roles of examples in learning to prove, and my previous work with Uri Leron on generic proving, and with Orly Buchbinder on the roles of examples in determining the validity of mathematical statements). In terms of teaching, I try to unpack pedagogical considerations that teachers encounter when constructing or selecting instructional examples (this work I have done mainly with Iris Zodik), and try to characterize this kind of knowledge for teaching mathematics that appears to be crafted through experience.

I propose examining the field through three teaching/research settings that elicit example-use and example-based reasoning: Spontaneous example-use, evoked example-production, and example-provisioning (by the teacher or researcher).

WHY DEAL WITH EXAMPLES?

Over the past decade there has been an increasing interest among the community of mathematics education researchers in studying the roles, use, and affordances of examples in learning and teaching mathematics, at all levels. For example, at the 30th conference of PME in Prague, a Research Forum was devoted to exemplification in mathematics education (Bills, Dreyfus, Mason, Tsamir, Watson, & Zaslavsky, 2006). A special issue of *Educational Studies in Mathematics* (Vol. 69, No. 2) came out of that Research Forum and was published in 2008, followed by another special issue of *ZDM – Zentralblatt fuer Didaktik der Mathematik* on examples in mathematical thinking and learning (Antonini, Presemeg, Mariotti, & Zaslavsky, 2011).

The increasing attention to examples stems from the central role that examples play in learning and teaching, in general, and in mathematics and mathematical thinking, in particular. Examples constitute a fundamental part of a good explanation - a building block for good teaching (Leinhardt, 2001). According to Leinhardt (2001, p. 347), "For learning to occur, several examples are needed, not just one; the examples need to encapsulate a range of critical features; and examples need to be unpacked, with the features that make them an example clearly identified." Current work on mathematical exemplification indicates that using examples for explaining to oneself or to another person is a non-trivial challenge.

In this paper, the term *example* refers to a mathematical object for which one can answer the question: "What is this an example of?" In other words, the person who generates or selects it is able to articulate what property, principle, concept, or idea the specific example is a case of. Note that any example carries some attributes that are intended to be exemplified and others that are irrelevant. Skemp (1987) refers to the irrelevant features of an example as its 'noise', while Rissland (1991) suggests that "one can view an example as a set of facts or features viewed through a certain lens" (p. 190). When dealing with geometric concepts the notion of example has a unique nature (Zodik and Zaslavsky, 2007a). For example, there is no way to give an example of a 'general' triangle (except by a verbal description), since whatever triangle we sketch, it will always have salient features that are not general, as it cannot be both acute-angled and obtuse-angled. The above suggests that 'examplehood' is in the eyes of the beholder.

Examples are an integral part of mathematics and a critical element of expert knowledge (Rissland, 1978). In particular, examples are essential for concept formation, generalization, abstraction, analogical reasoning, and proof (e.g., Buchbinder and Zaslavsky, 2009; Dahlberg and Housman, 1997; Ellis, Lockwood, Dogan, Williams, and Knuth, 2013; Ellis, Lockwood, Williams, Dogan, and Knuth, 2012; Hazzan and Zazkis, 1999; Hershkowitz, 1990; Mason, 2011; Sandefur, Mason, Stylianides, and Watson, 2013), though there could be drawbacks in terms of use of examples, e.g., for proving (Iannone, Inglis, Mejía-Ramos, and Weber, 2011; Zaslavsky and Peled, 1997).

One way to examine how an individual understands a concept is by identifying elements of his or her concept image or example space. The collection of examples to which an individual has access at any moment, and the richness of interconnection between those examples, constitute his or her accessible example space (Bills et al., 2006). Example spaces are not just lists, but have internal structure in terms of how the elements in the space are interrelated. In my work, I consider an example space as the collection of examples one associates with a particular concept at a particular time or context. According to Mason and Goldenberg (2008), what determines the use of a concept is the example space one associates with it. This notion is closely related to Vinner and Tall's idea of concept image (1981, 1983). Vinner and Tall use the term concept image to describe the total cognitive structure that is associated with a particular concept, which includes all the mental pictures and associated properties and processes. "It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures… Different stimuli can activate different parts of the concept image, developing them in a way which need not make a coherent whole." (Tall & Vinner, 1981, p. 152). Example spaces are also dynamic and evolving. Thus, in orchestrating learning or conducting research it is important to identify (limited) concept images and prototypical views of certain concepts, which the learners hold, and facilitate the expansion beyond "more of the same" examples.

Some parts of an example space may be more accessible at a given time than others (Mason and Goldenberg, 2008). The less accessible parts await an appropriate trigger to be used. Watson and Mason (2005) regard the notion of a personal example space as a tool for helping learners and teachers become more aware of the potential and limitations of experience with examples. In a group activity or discussion, an example suggested by one member may trigger access to a further class of examples for other members. When learners compare their examples, they often extend and enrich their example space. Moreover, once a connection is made it is strengthened and more likely to come to mind in the future (Mason and Goldenberg, 2008). Learners' example spaces play a major role in what sense they can make of the tasks and activities in which they engage. Zaslavsky & Peled (1996), who coined the term *example space*, point to the possible effects of limited example spaces that (practicing and prospective) teachers hold with respect to a binary operation on their ability to generate examples of binary operations that are commutative but not associative or vice versa.

Research on exemplification in mathematics education has become very broad and includes different approaches to studying it, as well as competing and even seemingly contradicting findings. For example, Dahlberg and Housman (1997) indicate that students who generated examples of an unfamiliar concept more widely and visually gained a better understanding of the concept and were able to solve subsequent related problems better. On the other hand, based on their findings, Ianonne and her colleagues (2011) warn that simply asking students to generate examples is not necessarily productive for proving. In both studies the students were explicitly asked to use examples. Sandefur et al. (2013) argue that in order to understand the value of exemplification one needs to examine cases of spontaneous use of examples. They build on studies that suggest that example generation is beneficial for problem solving and proving when the participants generate and use examples spontaneously (Alcock and Inglis, 2008; Watson and Chick, 2011). In contrast to the two settings that foster learners' generation of examples – either spontaneously or deliberately, there are also studies that examine instructional examples that are provided by a teacher (or researcher) with a certain goal in mind (e.g., Hershkowitz, 1987, 1990; Zodik and Zaslavsky, 2008). The intentions of such studies are twofold: to unpack design principles, and to examine the affordances that such examples or sequence of examples create, in terms of learning. These settings can be actual teaching sessions or research sessions (e.g., individual or group interviews, teaching experiments).

Whatever the learning environment (or research setting) involves – spontaneous generation of examples, deliberately evoked production of examples, and provisioning of examples, there is still a web of other factors that play a role, such as the mathematical topic and complexity of the focal concept or problem, prior experiences and knowledge (Alcock & Inglis, 2008), the kind of interactions that are facilitated (between the teacher/interviewer and the learner, and /or between learners in a small or large group). For this matter, learners can be students at all levels as well as teachers who are engaged in these types of activities.

In my presentation I will use examples from my own work and work with colleagues of mine to illustrate what example generation and use can look like in these three different settings, and what possible affordances and limitations are entailed in each setting, acknowledging that each example has its specifics (in Skemp's terms – "noise"). In this paper, I elaborate on one example of a process of an evoked example -production, and mention briefly other cases (on which I will elaborate further in my presentation). These examples are related mostly to concept formation and proving. They serve as 'existence proofs' of exemplification in the service of learning mathematics and learning to teach mathematics. Included in learning to teach mathematics is using example generation as a diagnostic tool to identify students' strengths and weaknesses (Zaslavsky & Zodik, in press; Zazkis & Leikin, 2007).

I turn to the three settings mentioned above.

SPONTANEOUS EXAMPLE-USE

Spontaneous-use of examples does not occur automatically. There are conditions that foster this type of behavior. For this to occur, special attention needs to be given to the nature of the task in which students engage. Typically, tasks that create a sense of uncertainty (Zaslavsky, 2005; Zaslavsky, Nickerson, Stylianides, Kidron, and Winicki-Landman, 2012) have the potential of raising the need to try out examples spontaneously. The uncertainty can be about a conjecture (i.e., whether it is true or false), or about a problem-solving situation for which the solver has no readily solution strategy that works (problems that require proving are included in this type). The examples can be generated randomly, just to get a sense if the conjecture holds for all cases, with the goal of building an intuition for whether or how to prove or disprove the conjecture. The examples may also be carefully selected and represented in a way that provides structure and potentially shed light on the main ideas of a proof, such as in generic proving (Leron and Zaslavsky, 2013; Malek and Movshovitz-Hadar, 2011; Rowland, 2001).

This type of setting allows studying what may come naturally to learners and experts, and what productive (or problematic) uses of examples can be anticipated. Moreover, the intention in studying spontaneous example-use is often aimed at building on this in other contexts or with other learners, for example, by explicitly introducing such forms of example use.

In a study on the roles of examples in learning to prove[1], we identified several manifestations of students' productive use of examples, without prior direct instruction on uses of examples for proving. In particular, in a task-based individual interview

[1] Examples Project (NSF grant DRL-1220623, Eric Knuth, Amy Ellis, & Orit Zaslavsky, principal investigators).

some students used examples generically, and were able to build on a generic example to form a deductive argument that supports their assertion. One task was about the following conjecture: “When you add any consecutive numbers together, the sum will always be a multiple of however many numbers you added up”.

One student chose an example of 5 consecutive numbers and wrote: 5+6+7+8+9. Then he represented the sum in the following way: (7-2)+(7-1)+7+(7+1)+(7+2). The latter representation allowed him to see this example as a general case for any 5 consecutive numbers, and even for any odd number of consecutive numbers (since there will always be a middle term).

Zaslavsky and Shir (2005) studied students’ conceptions of a meta-concept: a mathematical definition. They did it trough task based group sessions, in which the researcher did not interfere at all in the students’ discussions. The tasks required examining several possible definitions of a mathematical concept and determining whether each one is acceptable as a definition of the given concept. Interestingly, this task elicited rich example-based reasoning that led to shifts in students’ conception of a definition.

Note that although the settings described here elicit spontaneous (and often productive and desirable) example-use, it does not occur incidentally. It relies on careful choice and design of appropriate tasks.

EVOKED EXAMPLE-PRODUCTION

There are numerous manifestations of the value of explicitly asking students to generate examples of a concept or use examples for problem solving or proving (Watson & Mason, 2002, 2005; Hazzan & Zazkis, 1999; Zaslavsky & Zodik, in press). In terms of concept formation, this requirement may serve to get an idea of a person’s concept image (or example space) and motivate towards its expansion.

I turn to a group activity at a professional development workshop for in-service secondary school teachers, that explicitly required example-generation of a particular mathematical concept – a periodic function, followed by example-verification (for more detail see Zaslavsky and Zodik, in press). We (ibid.) used the generic task of: ‘Give an example of…, and another one…, and now another one, different from the previous ones…’. This type of task has been discussed in the literature (e.g., Hazzan & Zazkis, 1999; Watson & Shipman, 2008; Mason & Goldenberg, 2008; Zaslavsky, 1995). This activity calls for generation and verification of examples. The assumption is that the learners know the definition of the concept.

We chose the concept of a periodic function, in order to examine the current concept images teachers held, and motivate them to expand their example spaces associated with periodicity (Shama, 1998). Van Dormolen and Zaslavsky (2003) discuss the meta-concept of a mathematical definition, and illustrate it with the notion of a periodic function. They suggest the following pseudo-definition (p. 92, ibid) that is useful pedagogically, as it conveys the essence of the notion of a periodic function:

A pseudo-definition of a periodic function

A periodic function is a function that can be constructed in the following way: Divide the x-axis into equal-length segments, such as for example…, [−39,−26], [−26,−13], [−13, 0], [0, 13], [13, 26], [26, 39], … Take any of these segments, no matter which one, and define a function on it, no matter how (e.g., as in Fig. 1).

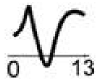

Fig. 1

Then define another function on the whole x-axis, such that on each segment it behaves in the same way as the first function (as in Fig. 2).

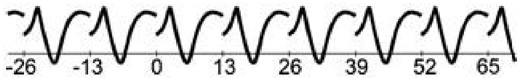

Fig. 2

Then the new function is a periodic function. Its values are repeated regularly.

Van Dormolen & Zaslavsky, 2003, p. 92

Inspired by this work, it appeared that the notion of a periodic function would lnd itself well to generating and verifying examples with the goal of expanding the participants' example spaces, or concept images of a periodic function; it was anticipated that the teachers would mainly think of trigonometric functions as examples of periodic functions, thus, there would be many learning opportunities to expand their example spaces. In particular, we anticipated that the idea of constructing an example of a periodic function without knowing, or even being able to know, its analytic representation, basically similar to the above 'copying' approach of van Dormolen and Zaslavsky (2003), would be a new way of thinking of a periodic function.

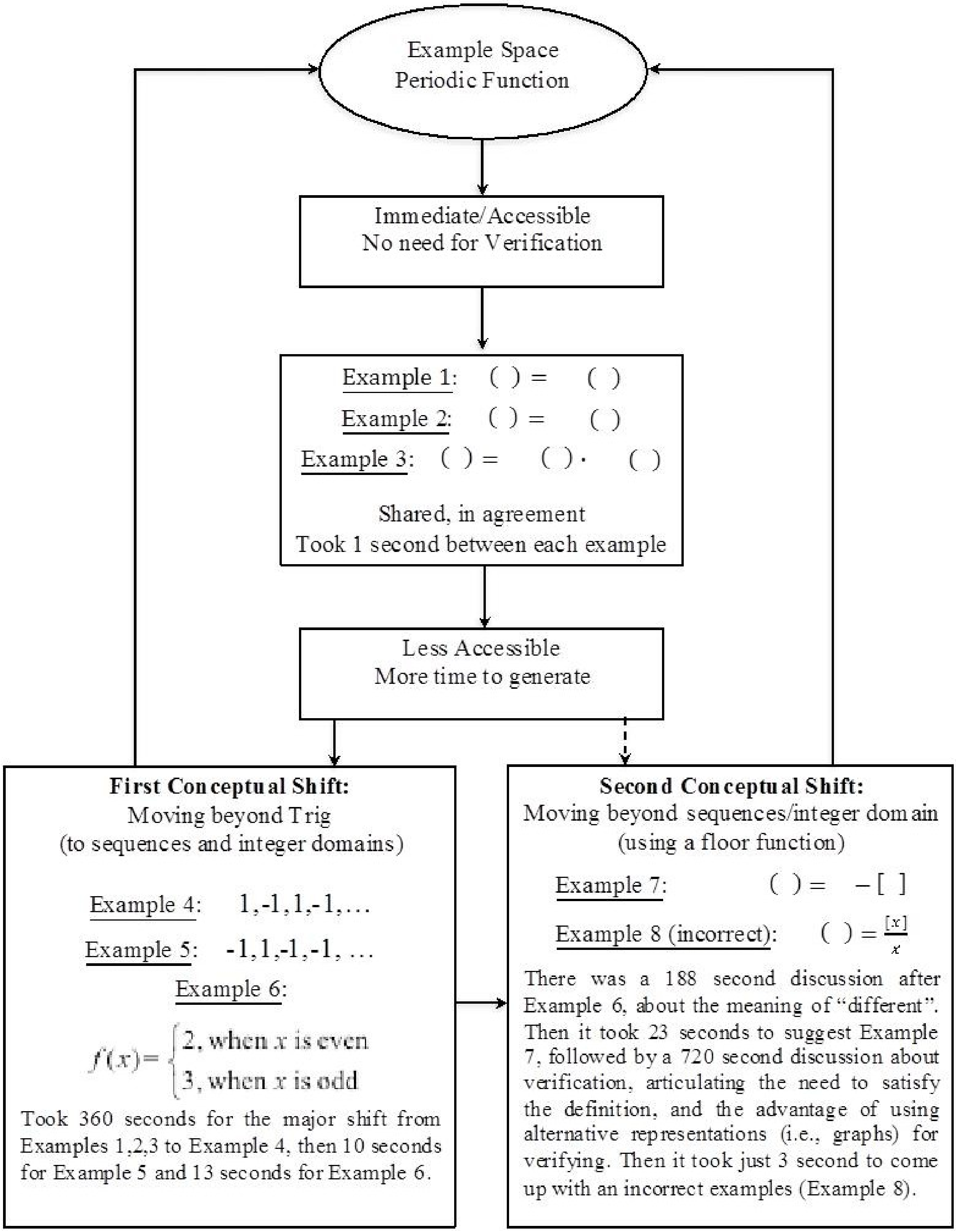

Figure 1: Collaborative expansion of an example space of a Periodic Function by evoked example-production

Figure 1 conveys the turning points in how the participants viewed a periodic function. After giving 3 familiar, rather prototypical and highly accessible examples (examples 1, 2, & 3), all drawn from the domain of trigonometric functions, they ran out of examples. This led to a discussion, in which one of the participants, Reli, suggested moving from the domain of trigonometric functions to special kinds of sequences. Based on Reli's idea, the group members helped her construct a specific example. This learning event triggered the first shift from regarding periodic functions as mostly (or even solely) combinations of trigonometric functions, to non-trigonometric functions. Example 4 can be seen as a conceptual shift in participants' views of a periodic function. It opened their eyes to similar cases and led to the construction of examples 5 and 6, by Hassan and Mary, respectively. These examples triggered the next discussion, as several participants, including Hanna, questioned the extent to which examples 4, 5 and 6 are essentially different. As a result, a more sophisticated example emerged (example 7), involving a floor function (by Hassan). This example led to a long discussion including group work surrounding ways to verify that example 7 is indeed a periodic function. Some verified this based on the symbolic representation of the function and some used its graphical representation. Interestingly, while periodicity lends itself naturally to visual representations, this did not come up spontaneously. However, at a certain point, participants suggested also approaching examples of a periodic function graphically (in the spirit of van Dormolen and Zaslavsky, 2003).

For examples 7 and 8, it was not easy to check whether they satisfy the definition of a periodic function. For that, several participants moved to a graphical representation, such as the one in Figure 2.

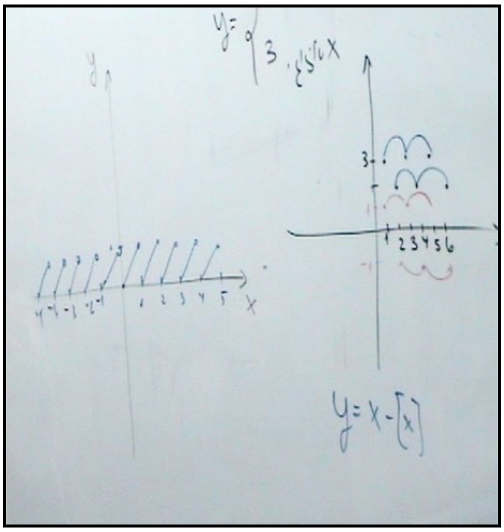

Figure 2: A graph of Example 7 that a participant drew on the board

For example 8 the graphical representation was difficult to draw, so it remained indecisive whether it was a periodic function or not. Only at a later stage they were able to find its graph (see Figure 3) and realize that it was not a periodic function.

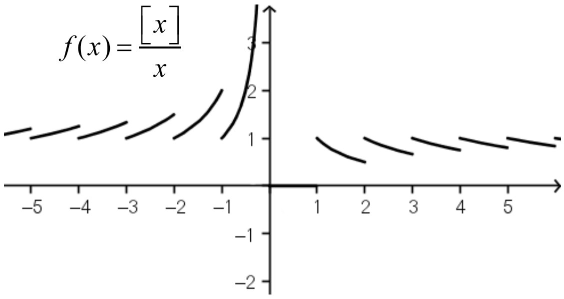

Figure 3: The graph of Example 8

The case of the periodic function represents an example-generation and verification eliciting learning environment that is characterized by an open-ended generic task of "give an example of…". This learner-centered environment is characterized by the ongoing activity of generating examples of a given concept followed by the naturally evolving need to verify that the proposed examples satisfy the definition of the given concept or other sufficient conditions of the concept. The teacher's or researcher's main roles are choosing the focal mathematics concept, and orchestrating the discussions; it is critical that the teacher persists and pushes the learners to continue generating more and more examples that are different than the previous ones. As we see in Figure 1, the learning occurs once we go beyond the familiar and the accessible.

PROVISIONING OF EXAMPLES

Studies on how people learn from worked-out examples point to the contribution of multiple examples, with varying formats (Atkinson, Derry, Renkl, &Wortham, 2000). Such examples support the appreciation of deep structures instead of excessive attention to surface features. Other studies dealing with concept formation highlight the role of carefully selected and sequenced examples and non-examples in supporting the distinction between critical and non-critical features and the construction of rich concept images and example spaces (e.g., Hershkowitz & Vinner, 1983; Vinner, 1983; Petty & Jansson 1987; Watson & Mason 2005; Zaslavsky & Peled, 1996). In these studies, it is the role of the teacher or researcher to select and provide the specific examples that the learners will encounter. The choice of examples then is related to the learning or research goals.

The choice of an example for teaching is often a trade off between one limitation and another. Choosing examples for teaching mathematics entails many complex and even competing considerations, some of which can be made in advance, and others only come up during the actual teaching (Zaslavsky, 2010; Zodik & Zaslavsky, 2007b, 2008, 2009). The specific choice and treatment of examples are critical as they may shape students' understandings by facilitating or hindering learning (Zaslavsky & Zodik, 2007; Rowland, Thwaites, & Huckstep, 2003).

The knowledge teachers need for meeting the challenge of judiciously constructing and selecting mathematical examples is a special kind of knowledge. It can be seen as core knowledge needed for teaching mathematics. In Ball, Thames, and Phelps' (2008) terms, it encompasses knowledge of content and students and knowledge of content and teaching, as well as "pure" content knowledge unique to the work of teaching. Teachers' treatment of examples may reflect their knowledge base (Zaslavsky, Harel, & Manaster, 2006). Moreover, engaging teachers in generating or choosing instructional examples can be a driving force for enhancing these elements of their knowledge (Zodik & Zaslavsky, 2009). Teachers' use of examples often leads to learning opportunities for themselves through which they gain pedagogical and/or mathematical insights (Zodik & Zaslavsky, 2009).

In examining the quality of instructional examples there are two main attributes that appear to make an example pedagogically useful, according to Bills et al. (2006). First, an example should be 'transparent' to the learner, that is, it should make it relatively easy to direct the learner's attention to the features that make it exemplary. It should also foster generalization, that is, it should highlight the critical features of an example of the illustrated case, and at the same time point to its arbitrary and changeable features.

This notion of transparency is consistent with Mason and Pimm's (1984) notion of generic examples that are transparent to the general case, allowing one to see the general through the particular, and with Peled and Zaslavsky's (1997) discussion of the explanatory nature of examples. Examples with some or all of these qualities have the potential to serve as a reference or model example (Rissland, 1978), with which one can reason in other related situations, and can be helpful in clarifying and resolving mathematical subtleties.

Provisioning of examples calls for special attention to how the learner interprets the example and what the learner notices (or fails to notice). The following example, taken from Zodik and Zaslavsky (2007b), illustrates this point. In a geometry lesson introducing the concept of a median of a triangle, the teacher used the following example (Figure 4) to illustrate a median:

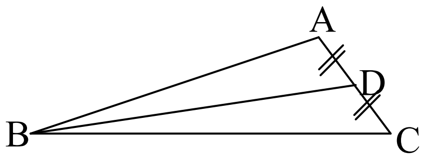

Figure 4: An initial example of a Median (BD)

Based on this example, a student suggested that any median is also an angle bisector. Following the student's remark, the teacher modified the original example and presented the following case (Figure 5).

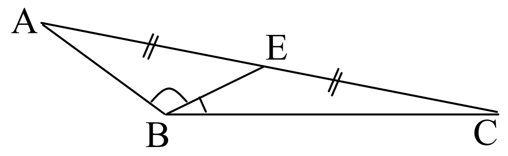

Figure 5: A revised example of a Median (BE)

Apparently, there are special entailments in visual/geometric examples. Basically, there is much ambiguity with respect to what visual information one is allowed to attend to and infer from a drawing and what not. Yet, the possible "mismatch" between a teacher's intention and what students attend to is not restricted to visual examples.

CONCLUDING REMARKS

The three settings – spontaneous example-use, evoked example-production, and provisioning of examples are interrelated. Often in research we first want to examine what students do spontaneously and only at a later stage direct them to example use. There are times when we evoke students to generate their own examples, and then present them with additional examples that were either "missing" with respect to a goal we have in mind, or that can help shed light on their thinking.

In my talk I will elaborate on the affordances and limitations of each setting, draw connections and point to similarities across these settings.

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PROFESSIONAL KNOWLEDGE OF (PROSPECTIVE) MATHEMATICS TEACHERS – ITS STRUCTURE AND DEVELOPMENT

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Recent research on the professional knowledge of mathematics teachers, which has been carried out in the last decade, is in the focus of this paper. Building on the international IEA Teacher Education and Development Study – Learning to Teach Mathematics (TEDS-M), this paper describes a more situated way of evaluating the professional knowledge of teachers. The theoretical framework of the follow-up study of TEDS-M takes up the novice-expert framework and analyses via video-based assessment instruments the structure and development of the professional knowledge of mathematics teachers. More recent concepts on noticing and interpreting classroom situations and students' activities are also incorporated into the analysis. Connecting the results of the study TEDS-FU with the study TEDS-M gives insight into the development of the professional knowledge of mathematics teachers.

INTRODUCTION

Studies on the knowledge of mathematics teachers have gained significant relevance in the last decade (for an overview see for example Blömeke & Delaney, 2012). Criticism about the inefficiency of teacher education has long been voiced in many Western countries. Teacher education in general has been described as a weak intervention compared to one's own school experience and later professional socialisation (Richardson, 1996). More particularly referring to mathematics teacher education, Klein (1932, German original 1908) criticised already at the beginning of the last century in his famous metaphor of a "double discontinuity" the lack of impact of university education on teaching practice in school.

In the light of the growing importance of international comparative studies on students' achievements in mathematics like TIMSS or PISA the professional knowledge of mathematics teachers and its influence on the development of the knowledge of students at school has become of special interest. The effectiveness of mathematics teacher education, i.e. the question how far universities succeed in the development of the professional knowledge of future mathematics teachers during their study, is a core question within this debate.

In the last decades a substantial number of national and international studies on mathematics teacher education have been carried out. As Krainer and Llinares (2010) pointed out in their comprehensive survey on the state-of-the-art on mathematics teacher education (MTE), three trends can be identified in the literature on mathematical learning of the three groups of prospective teachers, teachers, and teacher educators, namely "(1) teacher educators' and researchers' increasing attention to the social dimension and (2) attention to teachers' reflections" (p. 702).

The first trend including the social dimension of mathematics teacher education incorporates a shift from the perspective of the training of individual future teachers and teachers to practice and research emphasising the social dimension in teacher education has led to a strong change in the discussion on teacher education. For example, Krainer and Llinares (2010) point out that the concepts of collaborative learning, teacher-inquiry groups, communities of practice have played an important role in the recent discussion on mathematics teacher education, which is reflected in a strong shift towards the inclusion of sociological and sociocultural theories in research papers in the conference proceedings of PME.

The second trend, with a focus on teachers' reflective practice, is partially connected with the social shift described above and refers to the growth of teachers as professionals. For example, the research developed in the last decade on teachers' noticing when they observe their classes, how they interpret the observations made and how these interpretations change their practice, belongs to this developing aspect of research. The third trend described by Krainer and Llinares as

> increasing attention to the general conditions of teacher education (e.g., time, structure, institutional settings, and human resources), is newer and can be seen as an influence of work done on the practice and research in MTE in other fields, for example, organizational development. (p. 702)

Krainer and Llinares (2010) make a strong plea for

> taking these three trends seriously and regarding them as the challenges for the future" (p. 704). They comment that a further challenge is the fact that many studies on mathematics teacher education use qualitative research methods and argue that "more external and quantitative research are needed, in particular, looking at the outcomes of different types of teacher education or at longitudinal studies of mathematics teachers' learning and career. In all these cases, large populations are necessary to test relevant hypotheses. (p. 705)

They describe the creation of competence models for prospective teachers as challenge for the future in order to analyse different kinds of knowledge of teachers and prospective teachers. Referring to the work by Adler et al. (2005) they state: "Overall, there is a future challenge to combine qualitative and quantitative research methods and to integrate systematic reflections of teachers into research projects" (p. 705).

This research-oriented view on mathematics teacher education and student achievement is complemented by discussions in the light of international comparative studies. Such studies yield constantly strong differences in mathematics achievement between East Asian and Western students. Based on the results of large-scale studies like TIMSS or PISA, Leung and Park (2002) ask the question, whether the "competence of the East Asian students can be attributed at least partly to the competence of their teachers" (p. 128). This assertion leads to the question whether in teacher education the same achievement differences between Eastern and Western students prevailing over the last two decades are valid for prospective teachers as well and if yes, how far different systems of teacher education lead to these achievement differences.

The questions of how effective different educational systems on mathematics teachers are, and to what extent do country-specific differences exist, has lead the International Association for the Evaluation of Educational Achievement (IEA) to implement an international study on the effectiveness of teacher education at primary and lower secondary level, the so-called "Teacher Education and Development Study – Learning to Teach Mathematics (TEDS-M)" (see Tatto et al., 2008) in the last decade. In the following sections, an overview on the discussion of the professional knowledge of (future) mathematics teachers will be presented including the TEDS-M study on teacher education and a follow-up study on the professional knowledge of practising teachers, the so-called TEDS-FU study in which the transition of mathematics teachers from teacher education into the profession is examined.

SURVEY ON THE PROFESSIONAL KNOWLEDGE OF (PROSPECTIVE) MATHEMATICS TEACHERS

In their comprehensive survey on the state of research on the assessment of teacher knowledge across countries, Blömeke and Delaney (2012) point out that warning signs exist about the low proficiency levels of mathematics teachers in Western countries. However, prior to TEDS-M there appeared to be no systematic evidence on the state of these proficiencies. Since the late 1990s several small-scale comparative studies on mathematics teacher education and its efficiency have been carried out (cf. Ma, 1999). The survey, presented at ICME-10 in Copenhagen (Adler et al., 2005), the 15th ICMI Study (Even & Ball, 2008) and published in the International Handbook of Mathematics Teacher Education (Wood, 2008), provided a huge step forward and had the potential to fill many gaps in research concerning the efficiency of mathematics teacher education. Concerning the knowledge domain, the scope of these studies was limited, as many of these studies were either case studies or based on self-reports. Other studies did not include the knowledge domain and focused instead on beliefs or other concepts. To summarise the state of research prior to TEDS-M, we refer in the following to the extensive survey by Blömeke and Delaney (2012) on the professional knowledge of (prospective) mathematics teachers and restrict ourselves to a few selected results (for details see Blömeke & Delaney, 2012).

In the area of the professional knowledge of prospective mathematics teachers earlier work characterised pre-service teacher education as teacher learning, understanding teacher education as a kind of an apprenticeship. The 1990s have then seen a growing number of empirical studies on mathematics teacher education. However, many of these studies were conducted within their own education institution (cf. Chick et al.,), which implied several limitations as Adler et al. (2005) point out. Further research on teacher education turned more strongly to the knowledge base of teachers' classroom practice and developed theoretical conceptualisations in close relation to teaching practice (cf. the studies contained in the book edited by Rowland & Ruthven, 2010).

More recent studies are on the one hand similar to the studies described above, but are on the other hand characterised by a more analytical approach of defining and distinguishing between different knowledge facets functional for teaching and stressing the importance of mathematics content knowledge. These studies depart from a notion of competency related to competency-oriented approaches in international comparative studies on students' achievements such as PISA. Modelling the resources for proficiency in teaching mathematics in a multi-dimensional way is one important source for the theoretical framework as it has been described by Schoenfeld and Kilpatrick (2008) and further developed by Schoenfeld (2011), who sees teaching as a knowledge-intensive domain with different knowledge and affective-motivational facets.

Several large-scale studies on mathematics teacher knowledge share this common theoretical orientation, the already mentioned TEDS-M study, which will be described in detail in the next chapter, the study Mathematical Knowledge for Teaching (MKT), developed by the Learning Mathematics for Teaching Group of researchers from the University of Michigan (Ball & Bass, 2000) and the Cognitive Activation in the Classroom Project (COACTIV) developed by German researchers (Kunter et al., 2013). While TEDS-M and COACTIV are linked to the seminal classification of the different facets of professional knowledge of teachers developed by Shulman (1986), the MKT framework was inspired by Shulman's idea of pedagogical content knowledge and categorises the domains of knowledge needed to teach (see Ball et al., 2008). The COACTIV as well as the MKT study connect the professional knowledge of teachers with the growth of students' mathematical achievements, which is not the case with TEDS-M. The focus of TEDS-M is on an international comparison of the professional knowledge of *prospective* teachers for primary and secondary level, thus examining how their knowledge can be fostered during teacher education in contrast to the other two studies.

Apart from these differences it can be summarised that research on the professional knowledge of prospective teachers has increased dramatically with many small-scale and a few large-scale studies. These studies develop different descriptions of the structure of the professional knowledge of prospective teachers as they distinguish different facets of the knowledge base, including affective aspects such as the belief systems of the teachers. The common core of most studies can be aligned with the description of pedagogical content knowledge (PCK) of teachers following Shulman's (1986) seminal work in which PCK is defined as "that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special from of professional understanding" (p. 8). In their extensive survey on the current discussion around PCK, Depaepe et al. (2013) point out the special importance of this concept used by many studies.

However, despite the general agreement on PCK as connection between content and pedagogy and its dependence on the particular subject matter, no general consensus exists in empirical research on the facets of this important concept. Further, Depaepe et al. (2013) argue that there is an important group of empirical studies that do not define any component of PCK, although PCK was the central topic of this group of studies. Their study revealed consequences of the ongoing debate on the two principally different views on the conceptualisation of PCK, namely “whether mathematical knowledge in teaching is located ‘in the head’ of the individual teacher or is somehow a social asset, meaningful only in the context of its applications” (Rowland & Ruthven, 2011, p. 3).

Adherents of the cognitive perspective define according to Depaepe et al. (2013)

> – in line with Shulman – a limited number of components to be part of PCK and distinguish PCK from other categories of teachers’ knowledge base, such as content knowledge and general pedagogical knowledge. By contrast, proponents of a situated perspective on PCK as knowing-to-act within a particular classroom context, typically acknowledge that the act of teaching is multi-dimensional in nature and that teachers’ choices simultaneously reflect mathematical and pedagogical deliberations. (p. 22)

These paradigmatic differences in the conceptualisations of PCK have, according to Depaepe et al. (2013), an impact on the way in which PCK is empirically investigated.

> Advocates of a cognitive perspective on PCK believe it can be measured independently from the classroom context in which it is used, most often through a test. They typically focus on gaps in individual teachers’ PCK, on how PCK is related to and distinguished from other categories of teachers’ knowledge base …. Adherents of a situated perspective on PCK, on the contrary, typically assume that investigating PCK only makes sense within the context in which it is enacted. Therefore, they often rely on classroom observations (in some cases supplemented with other data sources such as interviews, lesson plans, logbooks) …. (p. 22)

The analyses by Depaepe et al. (2013) characterise the paradigmatic disagreement among scholars on the way how to conceptualise and evaluate teachers’ professional knowledge, including PCK, within different perspectives. Depaepe et al. (2013) conclude by calling for the integration of the cognitive perspective and the situated perspective, because both perspectives have their pitfalls, for example, neglecting the socio-cultural background of teaching or ignoring of the interactions of different knowledge categories within the cognitive perspective. Both perspectives provide powerful insights into teacher professional knowledge and so should be harnessed in a way that furthers understanding of how this aspect of teacher education influences teaching and learning.

In the following we will describe the results of the TEDS-M study and its continuation in TEDS-FU in order to show, how both kinds of research can be integrated.

DESIGN AND STRUCTURE OF TEDS-M

The comparative "Teacher Education and Development Study: Learning to Teach Mathematics (TEDS-M)", carried out under the auspices of the International Association for the Evaluation of Educational Achievement (IEA), evaluated the effectiveness of teacher education in terms of teacher knowledge and teacher beliefs both across countries and subject-specifically for the first time (for an overview see Blömeke et al., 2014; Tatto et al., 2008). TEDS-M was the first large-scale assessment of higher education that included direct testing covering graduates from 16 countries from East and West. The study includes a primary study and a lower-secondary study. The focus of TEDS-M were prospective teachers in their final year of teacher education who would receive a licence to teach mathematics in one of the grade 1 through 4 (primary study) or in grade 8 (lower-secondary study). The two studies were based on nationally representative samples and had to follow the rigorous IEA quality control mechanisms of sampling, data collection, coding, and data analysis. About 23,000 prospective teachers participated in the two studies, which took place from 2007-2009, the results were released in 2010.

The main questions of TEDS-M were multi-layered, namely as follows:

1. What are the professional competencies of future mathematics teachers?
2. How distinctive are the institutional conditions of mathematics teacher education?
3. What are the national conditions of mathematics teacher education?

We will limit ourselves in the following on the first question. Because teaching is the core task of teachers, and thus the development of teaching abilities internationally constitutes the main function of teacher education, teaching abilities – called 'professional competencies'– are the starting point of the theoretical framework of TEDS-M. According to Weinert (2001), professional competencies can be divided up into cognitive facets (in our context, teachers' professional knowledge) and affective-motivational facets (in our context, e.g., professional beliefs). The professional knowledge of teachers can again be divided into several facets. Referring to Shulman (1986), the following facets were distinguished in TEDS-M: mathematics content knowledge (MCK), mathematics pedagogical content knowledge (MPCK), including curricular knowledge, and general pedagogical knowledge (GPK).

TEDS-M examined also the professional beliefs held by the future teachers, due to the fact that beliefs are crucial for the perception of classroom situations and for decisions how to act, as Schoenfeld (2011) pointed out. Based on Richardson (1996), beliefs can be defined as stable, psychologically held propositions of the world around us, which are accepted to be true. In TEDS-M, several belief facets were distinguished, in particular epistemological beliefs about the nature of mathematics and beliefs about the teaching and learning of mathematics (Thompson, 1992). In addition, beliefs and affective traits such as motivation, and also metacognitive abilities such as self-regulation, are indispensable parts of the professional competencies of teachers (as displayed in Figure 1).

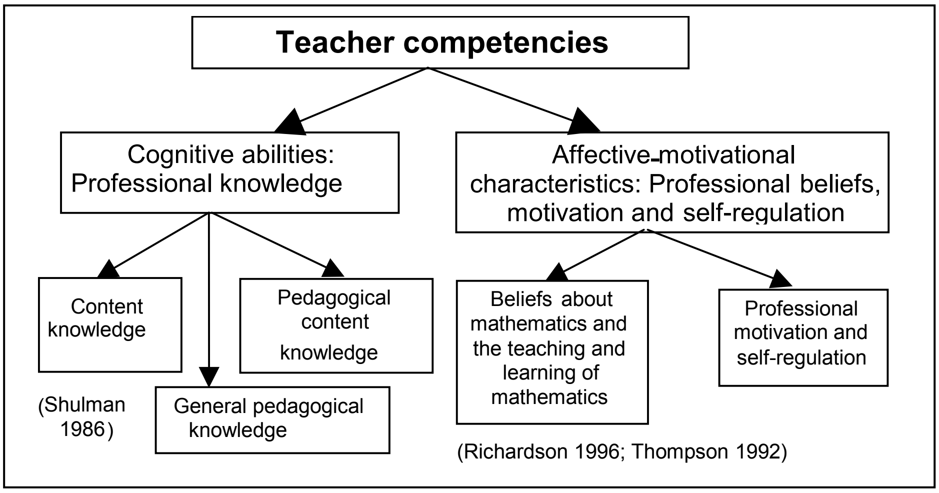

Figure 1: Conceptual model of teachers' professional competencies

These facets of professional knowledge are further differentiated: mathematical content knowledge covers the main mathematical areas relevant for future teachers, mathematics pedagogical content knowledge covers curricular knowledge, knowledge of lesson planning and interactive knowledge applied to teaching situations (see Figure 2).

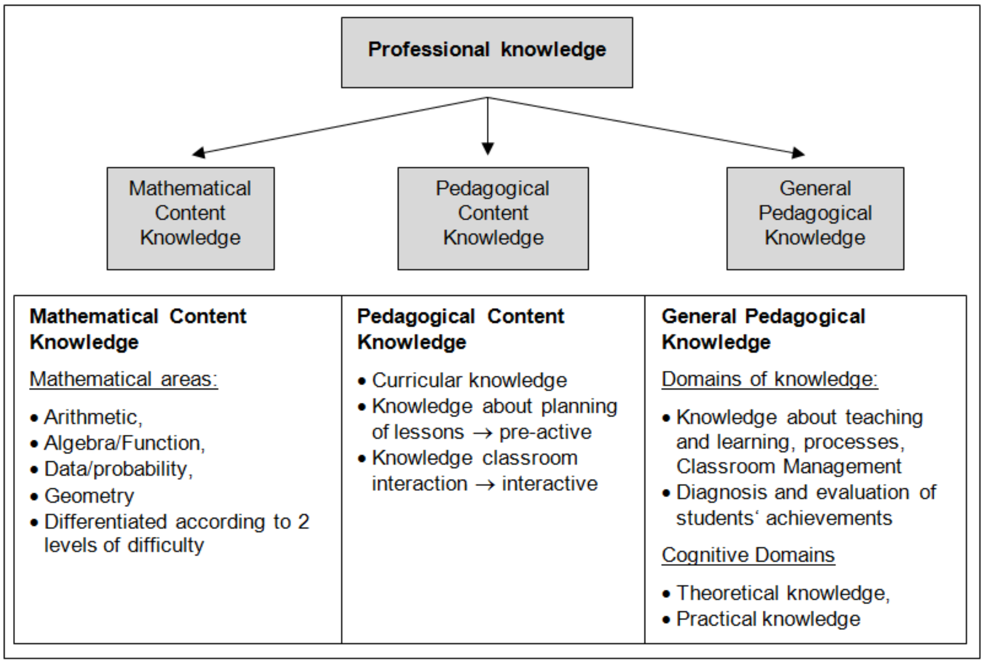

Figure 2: TEDS-M model of professional knowledge (Tatto et al., 2008)

TEDS-M examined the effectiveness of mathematics teacher education using the instruments of a future teacher survey, teacher educator survey, expert survey, document analysis of a sample of course offerings. The cognitive and affective-motivational facets of the future teachers' competencies were measured as criteria for effective teacher education. The future teachers' MCK and PCK were assessed in every participating country of TEDS-M, as well as their subject-related beliefs and professional motivations. Germany, Chinese Taipei and the USA assessed the GPK in a supplementary study using an instrument developed by König et al. (2011). Metacognitive abilities, however, were not part of the TEDS-M surveys.

Due to space limitations we cannot describe item examples, but refer to the extensive descriptions in Blömeke et al. (2014) and *ZDM – The International Journal on Mathematics Education*, issue 3 in 2012.

PROFESSIONAL KNOWLEDGE OF PROSPECTIVE MATHEMATICS TEACHERS – RESULTS OF TEDS-M

The results of TEDS-M on the prospective teachers' achievement revealed huge differences between the participating countries, both concerning MCK and MPCK. In the primary study the participants from Chinese Taipei and Singapore showed the highest performance in MCK, significantly distinct to the performance of the other participating countries. The results of prospective teachers from USA and Germany were marginally above the international mean, the difference to the achievement of future teachers from Chinese Taipei and Singapore added to approximately one standard deviation. The achievement of future teachers from USA and Germany was not only lower than those of the future East Asian teachers, they were also significantly lower than the future teachers from Switzerland. Concerning MPCK, the performance pattern was quite similar: The future primary teachers from Singapore and Chinese Taipei achieved much higher test results than the future teachers from the other countries. German students' attainments were around the international mean, the difference from the students' achievements of Singapore and Chinese Taipei was again about one standard deviation. In addition, the MPCK results from the German students were significantly lower than the attainments from the students from Switzerland, the USA and Norway.

In the secondary study, participants from Chinese Taipei outperformed all other participants, in relation to MCK as well as MPCK. Participants from Russia, Singapore, Poland and Switzerland followed the Chinese Taipei prospective teachers with their achievements in MCK, German and US American prospective teachers achieved slightly above the average, whereas in relation to MPCK, prospective teachers from Russia, Singapore, Switzerland, Germany and Poland achieved the highest results after the Chinese Taipei participants, with prospective teachers from the USA close to the international mean. These results point to interesting differences between prospective teachers for primary level and secondary level and confirm the superior performance of Eastern prospective teachers compared to their Western counterparts in most areas. This is consistent with the achievement differences at student level in respective countries (for details see amongst others the comprehensive overview on the TEDS-M results in Blömeke et al., 2014 and Tatto et al., 2012).

A comparison of the relative strengths and weaknesses in MPCK and MCK (using ipsative values) reveal interesting results. Comparing the achievements of the prospective primary teachers country-wise in the area of MCK and MPCK allow to develop country specific achievement profiles:

- Relatively strong achievement in MCK compared to international mean differences between MCK and MPCK – from Asia, the prospective teachers from Chinese Taipei and Thailand belong to this group, from East and Middle Europe the future teachers from Russia, Poland, Germany and Switzerland can be assigned to this group.
- Relatively strong achievement in MPCK compared to international mean differences between MCK and MPCK – several Eastern and Western countries contribute to this cohort, namely the future teachers from Norway, the USA, Spain, Chile, Malaysia, and the Philippines.
- Knowledge relatively levelled and close to international mean differences between MCK and MPCK – one East Asian country, namely Singapore, and one country from the former Soviet Union, namely Georgia, belong to this group as well as Botswana.

The absolute level of achievement does not influence this pattern, apparently neither a particularly strong emphasis on MCK nor on MPCK supports the overall achievement of the prospective teachers of a country. It is remarkable that the two East Asian countries belong to different groups, although cultural traditions seem to have influenced this diverse pattern. The tradition of Confucianism in East Asian countries, labelled as Confucian Heritage Culture (CHC), sees the teacher as an expert, who possesses the content knowledge students need to acquire. This tradition leads to a high importance of content knowledge in teacher education in many East Asian countries. In Continental Europe, content-related approaches also place traditionally high emphasis on knowledge strongly connected to content-related reflections but this within PCK (being one strand within the European didactics traditions), which explains the high importance of content knowledge in Germany and Switzerland. Eastern European countries have historical roots linked to the Continental European educational systems including teacher education, content knowledge and content-related didactics, which is reflected in the high importance of MCK in Russia and Poland. These very different traditions may have led to the relatively high level of MCK compared to MPCK of the future teachers from East Asian and East European countries.

In contrast, in Scandinavian countries, North and South America, and in countries shaped by US-American influence such as the Philippines or Singapore a so-called “progressive education” with child-centred approaches characterises school and teacher education are employed. These traditions may have led to the high level of MPCK compared to MCK of the future teachers from Scandinavian and American countries (for details see Kaiser & Blömeke, 2013). The situation is even more varied for prospective teachers for secondary level, which shows the strong, but not exclusive dominating influence of culture on education.

In further analyses going beyond country means, country-specific strengths and weaknesses in the knowledge of prospective teachers were detected by using differential item functioning (DIF). The item-by-item analyses reveals that due to differences in the cultural context, teachers from different countries responded differently to subgroups of test items with certain characteristics such as those stemming from certain particular domains, requiring similar cognitive demands or using the same item format. The analyses show that prospective teachers from Chinese Taipei and Singapore were particularly strong on mathematics content and constructed-response items. Prospective teachers from Russia and Poland were particularly strong on items requiring nonstandard mathematical operations. The USA and Norway achieved strongly on mathematics pedagogical content and data items. These results point once more to the influences of the cultural context on mathematics teacher knowledge.

Cultural influences on the results of TEDS-M cannot only be seen at the achievement level, but also in the area of the future teachers' beliefs. TEDS-M has evaluated in detail epistemological beliefs on the nature of mathematics and on the genesis of mathematical knowledge, i.e. the nature of mathematics teaching and learning. The studies explore amongst others the extent to which a country's culture can be characterised by an individualistic versus a collectivistic orientation using the cultural-sociological theory of Hofstede (1986). The collectivism-individualism antagonism describes the extent to which the individuals of a society are perceived as autonomous, the role and the responsibility of the individual for knowledge acquisition plays an important role.

The analyses (based on ipsative values) show that prospective teachers from more collectivistic-oriented countries such as Malaysia, Russia, Thailand, and the Philippines agree much more strongly to static aspects of mathematics (seeing mathematics as theory and a set of rules) in relation to dynamic aspects (describing mathematics as process to develop new mathematics insight) than it happened on average across the participating countries. In contrast, prospective teachers from highly individualistic countries such as Norway, Switzerland, and Germany much more strongly emphasised the dynamic nature of mathematics. Prospective teachers from countries that cannot be characterised as individualistic or collectivistic, namely Spain, Chinese Taipei, and Singapore, emphasised both aspects of mathematics in line with the international average (for details see Blömeke et al., 2014).

Currently, the question of the effectiveness of mathematics teacher education is of great interest. Disappointing first results demonstrate the limited influence of MPCK courses on the development of teacher professional knowledge (Blömeke et al., 2011) although this could be mitigated by a more differentiated and more extensive analyses. Internationally it was possible to identify two teacher profiles at the end of pre-service courses: teachers with a cognitively demanding and dynamic-constructivist accented competence profile and teachers with a lower achieving competence profile with more static and transmission-oriented beliefs. As explanatory features of the assignment to the profile the aspects gender, MCK and MPCK opportunities to learn as well as the coherence of the education could be identified.

The results lead to direct consequences for possible reform processes in teacher education. Furthermore, the high explanatory power of opportunities to learn in MPCK is of high relevance. These results lead for the first time to different conclusions regarding the importance of the different opportunities to learn: former analyses emphasized mathematics as predictive instance for the different educational attainment results. Looking at teacher competence as a multidimensional construct, the influential effect of MPCK courses come into the foreground (Blömeke et al., 2012). More important results of TEDS-M can be found in relevant journals or in Blömeke et al. (2014).

DESIGN AND STRUCTURE OF TEDS-FU

In the follow-up study of TEDS-M, TEDS-FU, the question of how mathematics teachers' professional knowledge develops after the end of teacher education in the first years of their school career based on the framework and the instruments of TEDS-M is explored. In addition, it is examined how professional knowledge can be analysed in a more performance-oriented way and how teacher expertise develops. Building on work from expertise research (for a review, see Li & Kaiser, 2011), professional competence of teachers is characterized by a high degree of integration of knowledge with multiple links, a modified categorical perception of teaching situations and by increasing integration of the different dimensions of professional knowledge. From the perspective of MPCK, this means an increase in conceptual understanding, the differentiation of a repertoire of heuristic strategies and metacognitive control strategies, an increasing competence through teaching and an increase in knowledge of school mathematics in depth and width (Schoenfeld & Kilpatrick, 2008).

In addition to MCK, MPCK and GPK as central cognitive facets of the professional competence of teachers the following practice-oriented, situated indicators of teacher expertise were considered: the precise perception of different mathematical classroom situations, described as perception accuracy or "noticing" (Van Es & Sherin, 2002) under the perspective of "selective attention" (Sherin, 2007) and their adequate analysis and interpretation as well as the flexible reaction on it, described as "knowledge-based reasoning" (Sherin, 2007). Due to the high importance of speed within the teaching profession we identify as further indicator for teacher expertise the fast recognition of mathematical student errors. Research on expertise points out that fast and adequate identification of errors is indeed a measure for differences in expertise level.

In the study TEDS-FU, carried out from 2010 to 2013, participants from the TEDS-M primary and secondary study were tested on a voluntary basis. The tests were web-based and the professional knowledge of the teachers was evaluated using video vignettes with short teaching sequences dealing amongst others with effective classroom management, heterogeneity, individualisation, teaching strategies, continuation of the teaching sequences with possible teaching options. This approach using classroom situations was intended to evaluate the professional knowledge of teachers in a performance-oriented way as requested by Blömeke, Gustafsson and Shavelson (2014) summarising the discussion around competence assessments. Furthermore, the knowledge of students' error and its speedy recognition was tested with a time-limited test. In order to allow sound descriptions on the development of the professional knowledge partly shortened versions of the original TEDS-M tests on mathematics, mathematics pedagogy and general pedagogy were carried out transferred into a web-based design. 171 teachers from the secondary cohort and 130 teachers from the primary cohort participated once more in the study.

PROFESSIONAL KNOWLEDGE OF MATHEMATICS TEACHERS – RESULTS OF TEDS-FU

First results of the study FU on the development of early career teachers' professional knowledge reveal interesting insight into the structure and development of lower secondary teachers' professional knowledge.

The average level of MCK of these young teachers has decreased significantly between the first testing in 2008 at the end of their teacher education within TEDS-M and 2012 within TEDS-FU. By contrast, the average level of MPCK remained stable. The first result has been expected but the latter is more surprising as a decrease would have been plausible due to the nature of the paper-and-pencil test assessing partly declarative knowledge and measurement issues, i.e. the regression to the middle within repeated measurements and a positively selected sample. This result indicates the relevance of practical experience as learning opportunity for the development of MPCK, which is stated by the research on expertise for other professions already for a long time.

An analysis of the rank order of the participants regarding their achievements in MCK and MPCK in 2008 and 2012 yields interesting differences between MPCK and MCK: in MCK the rank order remains nearly unchanged, i.e. the knowledge level of the prospective teachers at the end of their education predicted very strongly the achievement level after four years of teaching practice. The situation concerning MPCK is varied: the level of MPCK at the end of teacher education predicts significantly the level of MPCK after four years of teaching, but the rank order of the mathematics teachers is less stable in this knowledge facet than in MCK. Referring to the research on expertise we can tentatively conclude that the MPCK of young teachers at the beginning of their career may be more flexible here. Teaching experience may be a strong opportunity to learn, influencing both knowledge facets. However, this influence may be much stronger concerning MPCK than towards MCK, which might be explained by differences in the nature of MPCK and MCK (see Buchholtz et al., 2014). In addition different ways in dealing with the experiences made in school practice might be relevant, a so-called "deliberate practice" can be important for the early career teachers' development but may vary inter-individually and by context.

Based on the TEDS-FU results, the relation between the knowledge facets and the young teachers' performance-oriented skills to perceive and interpret mathematics classroom situations analysed via path models cannot be described with a simple competence model, but require complex description. MCK and MPCK at the end of teacher education both predict significantly how well mathematics teachers can recognise time pressured student errors and how adequately they can notice the relevant activities in the classroom, interpret them and anticipate adequate options for further actions. However, the path model fits much better and explains more variance in the teachers' skills if the MCK and MPCK development between TEDS-M and TEDS-FU is taken into account (Blömeke et al., in press).

The ability to notice classroom situations adequately and reason appropriately is influenced strongly by both knowledge facets, whereas the ability to recognise student errors depends more strongly on MCK than on MPCK. These results reveal once more the differences in the nature of MCK and MPCK (see Buchholtz et al., 2014).

Further evaluation of the TEDS-FU data on the nature of teacher expertise – describing the relation between knowledge, noticing and reasoning in classroom situations, and the speed of student error recognition – reveal unexpected results. If one distinguishes the facets of noticing and reasoning in classroom situations under an applied perspective, i.e. either content-related or pedagogical-oriented, the study points out that teacher expertise can neither be adequately described via models claiming either homogeneity of these indicators for expertise or by distinctions of facets according to domains or assessment methods. Based on our data, expertise can best be described with a two-dimensional model distinguishing between content-related knowledge (MCK, MPCK and speed in mathematics error recognition) and performance-related competencies (GPK, noticing and reasoning) (see Blömeke et al., re-submitted after revisions).

Analyses (based on IRT scaling and exploratory factor analysis) on GPK point out that the abilities to noticing and reasoning knowledge-based are in fact two loosely connected but different dimensions. The level of GPK at the end of teacher education does not predict these two abilities, which suggests that teachers' cognitions are reorganized during the transition into teaching. However, there exist relations between the current level of GPK and the ability to reasoning knowledge-based in contrast to noticing (for details see König et al., 2014).

Until now, it remains an open question as to whether teachers from primary levels have a similar structure of expertise, and if professional knowledge develops in the same manner or differently because of their different teaching practice. To summarise, the results of the studies described above show the differentiated nature of the expertise of mathematics teachers, the complicated interplay between the different facets of the professional knowledge of teachers and the high relevance of teaching practice for the development and the organisation of the professional knowledge of teachers in order to become true experts in their field.

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PLENARY PANEL

THE CALCULUS OF SOCIAL CHANGE – MATHEMATICS AT THE CUTTING EDGE

Mamokgethi Setati Phakeng (Chair)

University of South Africa

The aim of this plenary panel is to explore the idea of a Mathematics Education that is at the cutting edge and on the cusp/crest of making a difference, hence the title, "The Calculus of Social Change – Mathematics at the Cutting Edge". This theme will be addressed by four panellists. In this introduction I give the rationale for the plenary panel and describe how it has been organised.

INTRODUCTION

My participation in mathematics education conferences began in 1994 in South Africa and in 1998 I attended my first international conference, PME-22, which was held in Stellenbosch, South Africa. Since then I have attended many international mathematics education conferences, notably PME, MES and ICME. I have continued to attend these congresses because not only have they created an opportunity to share my work but also allowed me to reflect on 'why' mathematics, what I am doing in mathematics, who I am in mathematics, why I do the research that I do and what the point of my research is in terms of shaping what happens in my country, my continent and perhaps in the world. These questions continue to trouble me especially when I attend international conferences, which sometimes feel like a contest somewhere on a glorious stage away from real life: A place where we, who are most privileged, come together to debate the merits of our epistemological positions and argue over semiotics. At the same time many children all over the world, especially in developing countries, are being failed by mathematics itself and thus declared failures in life. These are the very children who are also dying of hunger, poverty and disease while others loll in obscene opulence.

There is no doubt that my life has changed for the better simply because I studied and succeeded in mathematics at a time in my country's history when it was unusual for people like me (black, woman, poor and relatively young) to do so. So I understand very well the argument that mathematics education plays a role in keeping the powerless in their place and shoring up the strong in positions of power. I can see it in my context – mathematics education remains the key discipline in the politics of education. Mathematics qualifications remain an accepted gatekeeper to employment and a better life, so it is not surprising that managing success in mathematics has become a way of controlling the job market. So the question is: What is our role as mathematics education researchers who understand all this? Do we continue to do business as usual? Is it possible to use our understanding of mathematics and the

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mathematics of change as well as the politics of mathematics education to bring about some social change?

The need for social change is something that all mathematics education researchers would agree to and support. However, the fight for social change, regardless of the form the battle takes, is deeply imbued with emotion: Often we want to hear about the life changed, the child educated, the mathematics transformed and the poverty eradicated. We also want to be sure that we contribute to a cause that seeks to better the world, or that volunteering to serve a cause, giving one's time, money, sweat and tears will achieve a positive impact, however small.

This plenary panel was organised in a way that cuts through the emotions and the mere talk and gets to answer a simple question: How can we create a school mathematics education context that is built on democratic principles? As researchers we critique easily, but this panel discussion is an attempt to go beyond critique and theorising to think deeply and engage with the challenges of effecting social change on the ground . I understand very well the limitations of dealing only with the practical as much as I understand those of only theorising, hence my attempt to ensure that the two interact to enable the PME community to re-imagine critical mathematics education in practice.

In his paper entitled, 'Critical mathematics education for the future', Skovsmose (2004) argues that while mathematics education can empower, it can also suppress, and while it can mean inclusion, it can also mean exclusion and discrimination. Mathematics education, Skovsmose explains, does not contain any strong 'spine', because it can collapse into forms of dictatorship and support the most problematic features of any social development, or it can contribute to the creation of a critical citizenship and support democratic ideals. The socio-political roles of mathematics are neither fixed nor determined. Both roles, and a range in between, of being a hero or a scoundrel, are available to be enacted through mathematics education. This possibility of creating a critical citizenship and supporting democratic ideals is what the plenary panel is attempting to attend to. It is a possibility of going beyond talking about the virtues of critical mathematics education and working on what it may look like in practice. I am aware that there are no straightforward procedures for 'determining' the functions of mathematics education, as they might depend on many different contexts in which the curriculum operates. However, the possibility explored by this plenary panel is inspired by the need to challenge mathematics education research to examine what 'could be' rather than focusing on critiquing, analysing, describing and exploring 'what is'.

WHAT IS THE PROBLEM?

In preparing for this plenary panel, panellists were asked to participate in a simulation. They have been sent individual letters by a fictional retired mathematics education researcher, Dr Thuli Dlamini, who is familiar with their work and was active in PME before retirement. While Dr Dlamini is well travelled and has lived in different parts of Africa, she is currently based in South Africa and has made some money since she retired from mathematics education ten years ago. She now wishes to use this windfall to set up a school. The four panellists were sent individual letters of invitation to serve as advisors to Dr Dlamini who needs specific advice on how best to run the school. Below is a letter that was sent to one of the panellists. Since the letters were individualised to speak directly to the research of each panellist, I provide only one example here. Despite their different backgrounds and areas of specialisation, all panellists were given the same scenario in their letters and they were required to respond to the same request from Dr Dlamini as indicated below:

> Dear Prof Wagner
>
> It is great to be in contact with you again after so many years since I left academia. The last time we met was during the PME conference in Bergen, Norway where you presented a research report entitled, Critical awareness of voice in mathematics classroom discourse: learning the steps in the 'dance of agency'. Despite the fact that I am no longer in academia, I remain interested in Mathematics education, and have kept up to date of your work. The published conversation that you did with David Stocker entitled, 'Talking about teaching mathematics for social justice' (2007) caught my attention and reminded me once again of the excellent work that you do not only as a mathematics education researcher but also as an advocate and activist for equity and social justice in general and mathematics education. I was thus prompted to write to you with a request that you join a team of international advisors for a school that I am setting up in South Africa.
>
> The school will have capacity for 100 Grade 8 learners in the first year and we aim to grow it by a 100 learners each year until we get to Grade 12 with a total of 500 learners in the school.
>
> The school will cater for children from low socio-economic backgrounds in an area that is neglected in terms of educational resources, other services and infrastructure. Low levels of literacy prevail and the community is multilingual because of migrant labour and the legacies of colonial social engineering. Numerical and digital literacies are very uneven. Parents have had a poor school education themselves, some to only primary level, and rely heavily on teachers to educate their children.
>
> The average monthly income of households in the area is US$300, cost of living is high because of the distance to the city, and access to high quality education is difficult and very few children achieve success in mathematics. And those who do matriculate at the end of their high-school careers generally have weak grades.
>
> The location has been chosen because it is a frontier between rural and urban communities, with some prospects for employment in secondary industries and some mining enterprises. Transport is expensive and households have unreliable power supplies. Electricity is also costly. Households therefore rely on alternative sources of fuel at certain times of the month. Communication is by mobile devices, but with low specifications. Internet connectivity through private service providers is also expensive. Many households have television, but are confined to a small range of national channels.
>
> Community education is offered by some NGOs, churches and clubs, but few local inhabitants have a post-school education. If they do, they tend to leave the area to seek better employment in the neighbouring town or head for the nearest city. High

unemployment levels are prevalent and the fickle informal economy is often the only means of livelihood.

I have specifically chosen this area as an appropriate location for the school to provide access to high quality mathematics education for the learners in a deprived region of the country which is potentially a reservoir of labour in a developing province. While the problem of out-of-school, primary-age children still exists in the area, the numbers fell substantially and what remains a serious problem is access to and success at high school level. This is the reason why I have decided that the school should start at Grade 8 level, which is first year of high school in this country.

Your role as a member of the international advisory board is to advise me on the following two issues I am struggling with:

1. How can I ensure that this school is built on democratic principles?
   - Given the context in which the school will be located, how can we deal with issues of learner selection? Whom do we select, how and why?
   - What should be included in the mathematics curriculum to ensure that they do not only have access to higher education in science, engineering and technology but that they are also socially aware?
2. What should the projected identity of the school be and how should it be constructed?
   - How can we engage with issues about what the learners are becoming as a result of being in the school?
   - How can the school deal with the challenge of constructing an identity that is not elitist in a context where success in mathematics is regarded as elitist?

Please send your advisory notes of not more than five pages to Prof Mamokgethi Phakeng by 20th March 2014, and be ready to present them to an international audience at the upcoming conference, 'The Calculus of Social Change – Mathematics at the Cutting Edge' in Vancouver during week of 14th July 2014.

Regards

Dr Thuli Dlamini

WHOSE PROBLEM IS IT?

As expected, the responses submitted by the panellists are very diverse – they range from critique to applause for Dr Dlamini's endeavour. However, all of them raise complex political issues that Dr Dlamini has to confront. The responses highlight the difficulties of having to engage outside the comfort of research or academia, as if the problem of changing mathematics education at the grassroots level is not the problem of research or of academia.

Wagner and Valero begin their responses by telling Dr Dlamini what does not qualify to give advice which can or should be relied on.

> I am not more than a researcher and my expertise is in researching, not in building schools. People who, in your country and many other countries in the world, have built schools in

> areas of “disadvantage” have for sure expertise and extremely valuable understandings of this situation and the challenges you will face. I highly suggest that you listen to them carefully and do not get tricked by the legitimacy that researchers’ voice have in this society. (Valero)

Valero’s caution to Dr Dlamini is well meant, but Dr Dlamini’s request for advice was not so much about the building (bricks and mortar) of a school as about the social dynamics of the school and the politics of mathematics education in the school. Unfortunately, those who are in the business of building schools do not worry much about the mathematics being taught in the school they are building or about its projected identity. It is Valero and other excellent researchers whose work focuses on mathematics education and society who spend their time critiquing ‘what is’ and hence they are the relevant people to engage with when thinking about ‘what could be’, as Dr Dlamini is trying to do. The fact that Valero feels unqualified to do so is perhaps worrying for the future of mathematics education.

Wagner, in his response, struggles with whether he should accept or reject the invitation to participate. He says,

> For me, it is important to decide whether I am willing to be part of a project that makes change in another culture. I am a creature of privilege: white, relatively wealthy, well-educated, male, and a citizen of a relatively safe and prosperous country. (Wagner)

If indeed ‘who Wagner is’, disqualifies him from participating as an advisor to Dr Dlamini then we have to ask what qualifies him to do the kind of research that he does, which in many ways gives hope to those who teach and learn mathematics in contexts of poverty and inequality. Wagner continues to explain his hesitation to participate as follows:

> I worry about reproducing or resembling colonialist relationships. In particular, I would not want to be associated with a project that values the knowledge and experience of one group more than others. I ask myself how outside consultants can privilege local knowledge. (Wagner)

While Wagner’s concerns above are valid, he is ignoring the fact that Dr Dlamini selected him to be his advisor precisely because of his awareness of all these issues, and with the hope that in his advice he will ensure that what we know about colonial relationships is not reproduced. Sometimes the anxiety about repeating the colonial turn becomes an alibi for inertia, which in some cases ironically has the same effect.

Walshaw’s concerns are even more troubling because they are not about ‘who she is’ and ‘what she knows’, but about ‘what can serve as a guarantee of the production of an inequitable mathematics experience’.

> Privately funded initiatives and policy incursions might heighten social awareness and seek solutions to the educational problem by introducing new initiatives, yet they cannot shore up the guarantee of the production of an equitable mathematical experience. (Walshaw)

So the question remains, whose problem is it? And why do we do what we do as mathematics education researchers?

It is curious that of all the panellists, only Halai does not raise concerns about the Dr Dlamini's initiative. Is it because of 'who Halai is'? Or perhaps where she comes from and where she is currently practising? Is it because of her epistemological assumptions or the theoretical framework? Halai's stance may be influenced by the sense of urgency which is prompted by being in the situation every day and seeing how mathematics serves as the gatekeeper to participation in the decision-making processes of her society. Living in both Pakistan and Tanzania, I have no doubt that Halai deals with many instances of how access to participation in mathematics also influences 'who will move ahead' and 'who will stay behind'.

ORGANISATION OF THE PRESENTATION

The plenary panel will begin with an introduction by the co-ordinator. The co-ordinator will introduce the problem that the panel discussion focuses on, the rationale and the manner in which the panel discussion will be conducted. The co-ordinator will then introduce Dr Thuli Dlamini who is the project owner and sponsor to whom the panellists will offer advice.

The panellists will then be invited to present their advisory notes to Dr Dlamini in front of the audience. In addition to presenting their responses to Dr Dlamini, the panel members will be required to respond to each other's paper as follows

- Wagner presents his advisory note
    - Halai responds to Wagner
- Halai presents her advisory note
    - Walshaw responds to Halai
- Valero presents her advisory note
    - Wagner responds to Valero
- Walshaw presents her advisory note
    - Valero responds to Walshaw

It is important to note that the manner in which I have scheduled these presentations is informed by the content of the papers. The person requested to act as Dr Dlamini will be given the opportunity to ask questions throughout the panel discussion – these questions will serve to evoke a discussion among and spur the panellists to think beyond their papers and consider the challenge at the heart of the scenario. Members of the audience will also be given opportunity chance to ask questions and interact with the panellists in an effort to assist with Dr Dlamini's challenge.

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PRIVILEGING LOCAL CULTURES AND DEMOGRAPHICS IN THE MATHEMATICS CLASSROOM

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I respond to a proposal for a new school in rural South Africa, built on democratic principles. First, I reflect on my prospective role as an outsider giving advice in this context foreign to me. Second, I consider the nature of democratic mathematics teaching and learning. I describe my participation in an analogous context in Canada, a First Nations community. Third, I ask how selecting learners can be democratic.

I feel honoured to be invited to consult on this significant project. It connects with two experiences I have had. First, I taught grades 7-12 mathematics both in Canada and in Swaziland before starting graduate studies. The juxtaposition of those two teaching contexts helped me understand that both mathematics and pedagogy practices are culturally situated. This insight drove me to my research interests. Second, I have been privileged to participate in conversations about mathematics teaching and learning in First Nations communities in Eastern Canada. Such cross-cultural experiences have enriched my understanding of education. However, cross-cultural pedagogical relationships are inherently problematic.

DECIDING WHETHER TO PARTICIPATE IN A MULTI-CULTURAL ENGAGEMENT

Many mathematics educators from privileged contexts have taken on positions of guidance and leadership in contexts that are foreign to them—within their own countries and beyond. For me, it is important to decide whether I am willing to be part of a project that makes change in another culture. I am a creature of privilege: white, relatively wealthy, well-educated, male, and a citizen of a relatively safe and prosperous country. The project Dr. Dlamini proposes takes place in a community that has been marginalized in various ways. I worry about reproducing or resembling colonialist relationships. In particular, I would not want to be associated with a project that values the knowledge and experience of one group more than others. I ask myself how outside consultants can privilege local knowledge.

My conversations in First Nations communities have sharpened my critical reflection on the relationships at play in such cross-cultural interactions. Lisa Lunney Borden and I (Wagner & Lunney Borden, 2012) acknowledged the inevitability of power relationships that could be connected to colonialist histories but we agreed that there are greater dangers in avoiding cross-cultural relationships. Without cross-cultural interactions, which help us develop understanding of others and ourselves, we are doomed to stagnate in our present worldviews and positionings.

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It is tempting to assume that I do not need to worry about colonialist relationships because someone from within the community has invited my interaction. Though the agenda comes from within the community, I know that any community comprises people with a variety of agendas. The history of colonialism in Canada reminds me that we ought not to assume that an individual speaks for his/her community. European settlers made agreements with First Nations individuals as if they could speak for the entire community, often to the detriment of the community.

DEMOCRATIC PRINCIPLES

The fact that the new school aims for democratic values helps me be confident that we can have good collaboration. When I think about democracy, I am inclined to think about the discourse I would like to see in the context. I ask: What kinds of conversation would I like to see in classrooms and elsewhere?

We are gathering our advice for the new school under the title *The Calculus of Social Change*. Calculus is described as the study of change so it is an apt metaphor for analysing change. With calculus we analyse change in the smallest possible increments to garner insight into the larger trend. Indeed, any large trend comprises infinitely many small changes. In the same way, I am interested in zooming into discourse practices in mathematics classrooms.

Even though we educators and planners are interested in larger democratic interactions, we need to pay attention to the smallest interactions in classrooms. I envision a new school that directs attention to democratic discourse at all levels of interaction. As students and teachers make connections to power dynamics in local micro-interactions within the classroom and school, larger societal interactions, and the many levels of interaction in between, students will discover what democracy looks like and develop skills for building and critiquing it. In this way, a democratic school might be a force for positive democratic social change. I do not know of any empirical evidence for this claim. I would like to see more research that makes explicit connections among discourses in small classroom interactions, classroom cultures, academic curriculum, school systems, regional political and social networks, and very large-scale discourses including gender and race.

CURRICULUM THAT SUPPORTS THE DEVELOPMENT OF DEMOCRATIC PRINCIPLES

This leads me to Dr. Dlamini's question about curriculum in relation to democratic principles. What kinds of interactions would I want to see in the school's mathematics classrooms? I encourage any school to guide its children to address explicitly the many levels of discourse I noted above and to make connections among them. In order to be taken seriously in the complex discourses that they will seek to change, the children will need to develop an understanding of the discourses as they currently exist. Thus I would promote curriculum that develops learners' skills and knowledge that align with the curricula in other South African schools while also looking critically at this knowledge, its structure, the cultures it favours, and the discourses that surround it. Thus there would be elements of traditional mathematics classrooms that ignore local culture, elements of radical mathematics teaching (e.g., Gutstein, 2008), and even critique of the whole structure of mathematics curriculum and its privilege (Pais & Valero, 2012). Derrida's (1976) concept of erasure may help us envision the coexistence of these conflicting curricula.

The elders and other leaders in First Nations communities in which I have had conversations desired this approach to curriculum. The children of their communities need to understand and connect with the dominant discourses in the country and also understand how these discourses create and sustain conditions in their communities. With these two kinds of understanding, children can be equipped to stand up to and change unjust social structures in their communities and beyond.

Bakhtin (1975/81) used the metaphor of complementary centripetal and centrifugal forces to describe the way all utterances simultaneously draw in and lead away. When I address someone I have to appeal to shared meaning in order for my interlocutors to make sense of what I say (this is the pull toward the centre of conventional discourse) but I also refer to the conventional in order to make change and to push at its boundaries. He described the two forces in the metaphor as heteroglossic and unitary language. Other scholars have described the distinction using the terms *open* or *closed* dialogue. As Bakhtin noted, there would be evidence of both forces in any instance of interaction, but we should be attentive to which one is favoured. An important principle of democratic dialogue is that diverse views are voiced. In other words, open dialogue is necessary for democracy.

I characterize the field of mathematics education as one that promotes open dialogue in solving problems and developing procedures. Children should understand that multiple approaches are possible and that a new approach can produce new insights. For example, students should be given mathematical tasks that invite multiple approaches. Furthermore, research frameworks and pedagogical frameworks have been developed to draw attention to strategies that develop student autonomy. I consider it unfortunate that mathematics classroom practices tend toward closed dialogue in which children are not invited to see the possibility of multiple approaches and possibilities. Teachers too frequently fail to raise the possibility of students' autonomy. The quantitative analyses of a large body of classroom transcripts that I have done with Beth Herbel-Eisenmann (Herbel-Eisenmann & Wagner, 2010) show that classroom dialogue is still markedly closed, at least in the context of the analysis. On the basis of my conversations with mathematics educators around the world, it seems that the nature of classroom dialogue is similar elsewhere.

This will be a challenge in the new school. Given that attempts to change classroom discourse practices have been difficult in other contexts—even with the overwhelming commitment to open dialogue among mathematics education scholars—how might the new school succeed in this endeavour?

To address this question, I want to tell you about my experience working in the First Nations communities mentioned above, because Lunney Borden and I noticed significant changes to discourse patterns as a result of some conversations in which we took part. The key to the change in the patterns of discourse seemed to lay in the fact that teachers gave students the responsibility to investigate their community. Lunney Borden and I had had conversations with elders and other honoured community members to identify mathematics in their traditional and current practices. The elders were very receptive to this ethnomathematical research, although they thought it was funny to think of their common sense as mathematical. Lunney Borden and I noticed that we would be positioning ourselves as mediators of knowledge between the elders and the community children if we passed the results of our ethnomathematical work to the children. We raised this concern with the elders and some teachers. Together we invented an event called Show Me Your Math, in which we invited children to identify mathematics in their community. Children were invited to talk with elders and investigate local artefacts to identify mathematics done in the community. They were also invited to identify the mathematics that the government uses to cheat their community. Lunney Borden and I saw three emergent qualities that aligned with community values (Lunney Borden & Wagner, 2011). First, cross-generational relationships were developed as children talked with community members about their practices. Second, the synthesis of community practices and academic mathematics adjudicated aspects of the colonizing cultures in terms of community values. Third, with this synthesis, the event supported holism for children as they were invited to bring together different aspects of their lives.

Some of the strongest critiques of ethnomathematics come from South Africa (e.g., Vithal & Skovsmose, 1997). Although I find these critiques warranted, I am reluctant to give up on the study of mathematics at work in cultures. I note that identifying political conflicts could benefit from mathematical analysis as well as from identifying ways in which actors in the conflict use mathematics to argue their cases. Identifying the mathematics in cultures of dominance is a form of ethnomathematics.

SELECTING LEARNERS FOR PRIVILEGE

In my view, the most challenging aspect of the proposed new school relates to 'learner selection.' I resonate with the term 'learner' because I am uncomfortable with the idea of calling children students, which would foreground one aspect of their experience and background other important experiences (the opposite of the holism I mentioned above). Teachers, administrators, and students are all learners, each with their own rich experiences beyond and within the context of the school. Careful selection of the participants will shape the nature of the discourse within the school.

In my research I have not focused on learner selection, probably because in my country (Canada) all children have the right to schooling and most schools are provided with good resources (OECD, 2013). The funding of schools in First Nations communities is a notable exception historically and presently (FNEC, 2009). However, the question of learner selection reminds me of a girl whose father died in her last year of secondary school in Swaziland. Without his advocacy, her family refused to pay her school fees for her final months because they felt education was wasted on girls. Her experience suggests that the new school may need to seek out its students instead of relying on passive selection from applicants.

Choosing students for the privilege of schooling seems antithetical to democracy. I suggest that the school aim for a learner body that is representative of the people significant to the community in terms of demographics. In this way it may privilege individual participants but could help the school showcase possibilities for active engagement of the local demographic. I would hope to see both the student and teacher body include approximately equal numbers of females and males, representatives of the various national and linguistic backgrounds in the community, including a few with privileged backgrounds. I do not have experience with engineering such classrooms, but I suggest that conversations about community challenges (local and national) and their connections to a history of abuse by White people and to migrations of people would be enriched with the inclusion of White children, immigrant children, and a majority of community children with local roots. To the extent that the school showcases possibilities for democratic engagement with local issues including people representing local demographics, local concerns and local people and their values will be privileged.

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SOCIAL JUSTICE THROUGH MATHEMATICS EDUCATION: SKILLING YOUTH FOR SOCIETAL PARTICIPATION

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This paper responds to issues in offering a relevant, skills focused secondary school mathematics education in a multi-cultural, remote, poverty-ridden community. It is proposed that the mathematics education in the school could be structured within Fraser's social justice framework of re-distribution, recognition and participation (Fraser, 2000). A mathematics education from a social justice perspective would reach out to marginalized learners and enable them to realize their goal of mathematics learning by skilling them for an active participation in the society.

SETTING THE SCENE

The case study under consideration of the panel raises several issues for a secondary school education that realizes the promise and potential of mathematics to leverage a community out of poverty. These issues include: low socio-economic background of the learners in the community; limited access to quality secondary education and uneven numerical and digital literacy among the learners; remoteness of the community from opportunities of employment; little exposure to media and internet; and the multi lingual and multicultural nature of the student intake.

In the context of the issues above, two main challenges are posed to the panel: How can the democratic principles be incorporated in building the school, and what should be the projected identity of the school. To propose recommendations that are principled and pragmatic the case is located within a framework of social justice because social justice is a significant pillar of a democratic education concerned with inclusion in education of the marginalized sections of the community.

A FRAMEWORK FOR SOCIAL JUSTICE IN EDUCATION

The recent Global Monitoring Report shows that in spite of huge strides in providing access to basic education, access has not translated into positive learning outcomes, and education has not realized its potential in leveraging individuals and communities out of poverty, or addressing persistent social inequities on the basis of poverty, gender or culture (UNESCO, 2014). For example, consider the case of Tanzania in East Africa, where access to primary education has increased to 98% and secondary education is at about 49% (Tanzania Education Sector Analysis, 2011). However, the quality of education is in crises. A regional study of student achievement in literacy and numeracy found that in Tanzania "Only 3 in 10 standard 3 (primary school) pupils can add subtract and multiply [--] Only 1 in 10 Standard 3 pupils can read a basic story (Uwezo, 2011, p. 7)". Furthermore, the high stakes Form Four secondary school national examination results for 2012, especially in mathematics evidenced huge failure rate of 65% leading to urgent national education reform (MoEVT, 2014).

It is certainly not the case that the young Tanzanian learners do not know any mathematics. Indeed the work for example of Paulus Gerdes (2010), shows that highly complex and sophisticated mathematics is embedded in the basket weaving and other traditional cultural practices embedded in the East African culture, bearing testimony to the mathematical history and potential of the learners in this community. However analysis of mathematics syllabus and practice in the public schools in the country shows that socio-culturally embedded mathematics knowledge is not a part of the syllabus, and the transition from Kiswahili as a medium of instruction in primary schools to English in the secondary schools poses conceptual and syntax related hurdles in learning (Kajoro, forthcoming). A high rate of failure in mathematics in secondary schools suggests among other factors alienation with the curriculum so that learners are not able to see the relevance and purpose of the mathematics they learn with their lives and prospects (Valero & Pais, 2011).

To conclude from the foregoing, access to education has to be seen in conjunction with the relevance of education *for* and *with* the community being served. This would entail a critical examination of the assumptions underpinning the purpose and relevance of education within a framework of social justice issues in education access.

Fraser, (2007) proposes a useful framework to make sense of the social justice issues in education with three key dimensions of social justice i.e. "redistribution, recognition and participation" (p. 17). This framework is usually employed with the country as a unit of change, to redistribute access to education across the socio-economic divide. However, the framework could be employed at the level of schools and classrooms where social justice issues are experienced locally (Atweh, 2009). For example, in the classroom, the teacher has the authority to ensure that the cultural capital is distributed to all learners for them to be able to learn effectively and succeed in school examination. Here, cultural capital is seen from Bourdeau's perspective including forms of mathematics knowledge, skills and attributes that could potentially give the learners an advantage to succeed in mathematics (Bourdeau, 1977). Recognition of diverse needs of learners from various social and cultural contexts would require that the teacher acknowledges these diverse needs in the classroom, and creates opportunities for their optimal participation in learning.

However, participation is contingent upon recognition which is inherently political in nature because recognition demands that the larger social and cultural forces that are played out in the classroom dynamics are challenged to allow for the participation of the marginalized learners. What follows is a discussion of the three dimensions of social justice with reference to the specific issues in the case.

Redistribution and Recognition

Selection criteria and entry into the secondary school would need to be inclusive so that redistribution of the cultural capital is among a wider population and not just those few who might have performed well in primary school. Beyond selection and entry, effort to re-distribute cultural capital would need to ensure that curricular processes recognise the diversity in the classroom. Recognition of learners who are marginalized due to socio-economic status, gender, language or other factors would mean questioning deep seated assumptions that underpin the organising structure and process of classrooms, in this case mathematics classrooms. For example, in patriarchal societies with roles defined on the basis of gender, teachers often subscribe to the dominant social and cultural views that boys are inherently better in mathematics thereby marginalizing girls in terms of participation in mathematics (Halai, 2011). In such situations "affirmative remedies" could reinforce the prevalent views and not questioning those deeply held cultural views which inhibit participation of both, boys and girls.

Redistribution of cultural capital in mathematics would take into account the requisite 21st century skills for learners such as, numerical, digital, problem solving and critical thinking skills. For skills development, process of teaching and learning in the mathematics classrooms would move away from routine memorization of procedures and algorithmic knowledge towards participatory learning involving application of mathematics knowledge to problems. Mathematics knowledge embedded in the history and culture of the learners would be a significant element of the cultural capital being re-distributed. This would socio-culturally embed mathematics learning and reduce alienation of learners with school mathematics (Gerdes, 2010).

Participation

Participation in mathematics learning from the perspective of social justice means that learners have a voice, and intellectual and social space to take part in the process of learning and achieve their learning goals (Atweh, 2009; Fraser, 2007). Learners' identity is not that of passive recipients of knowledge dispensed by the teacher, they identify themselves positively as becoming mathematically proficient. The dynamics of the power relations between the teacher and the learners would need to change to position teachers and learners as co-participants in the teaching learning process.

RECOMMENDATIONS

Several recommendations are made for an education aimed at preparing learners skilled mathematically and digitally, for active participation in society, and leveraging their community out of poverty.

Selection of learners

In a context with limited access to secondary schooling, issues of learner selection are complex. An attempt to select on the basis of performance in primary school examination could exclude students who know mathematics embedded in their culture, experience and language because the standard school system is usually set up on the basis of an academic mathematics, encoded in a national/international language of instruction. Hence, selection of students should aim to re-distribute access through a critical interrogation of any prevailing admission policy in favour of a multi-faceted admission policy that includes other criteria besides performance in primary school leaving examination. Other criteria could include demonstrated skill in application of knowledge of mathematics rooted in learners' culture and experience. Beyond entry, affirmative action in the form of "bridging classes" in the afternoon could support learners requiring upgrading of numeracy and digital skills.

Relevance of mathematics content and process

Relevance of mathematics content and process is an important element of recognition of diversity in learners' needs. Curriculum content in this school would need to go beyond the usual emphasis on Euro-western mathematics to recognize the mathematics rooted in the historical, cultural background of the learners. Pedagogic process would espouse social justice principles i.e. inclusive in terms of high expectations from all learners irrespective of their socio-economic background, culture or gender, and creation of space in the classroom dynamics for learners to participate in construction of mathematics knowledge (Valero & Pais, 2011).

Strategies such as *internship placement* could be employed for learners in mathematics to work on real mathematics problems in work-place situations in the local community. Recognizing the work place as a legitimate site and source of knowledge would also challenge the traditionally powerful position of the school as the sole site of knowledge.

Teacher recruitment and professional development

Appropriately qualified teachers would most likely not come from the local community that has a low level of education. Adequate measures for recruitments and retention would include liaison with key education stakeholder, and a system of rewards and incentives on the basis of a nuanced understanding of education quality that goes beyond the traditional focus on achievement in examination.

Teachers knowledgeable in mathematics, skilled digitally, and professionally competent would be crucial to re-distribute cultural capital in mathematics to the marginalized learners. Teachers would require regular mentoring to enable them to sustain their practice in highly challenging classroom contexts (Halai, 2006; 1998). School budget and policy would need to provide for regular professional development for mathematics teachers to keep them updated with new developments in the field. Given the remote location of the school it would be beneficial for mathematics teachers to become members of a professional association network.

School community links

School community links play a significant role in monitoring the quality and outcomes of education (UNESCO, 2014). A strategically formed board of advisors with members from the local community and the professional community including mathematicians and mathematics educators could support quality assurance.

To conclude, a secondary school in the midst of a highly disadvantaged community could strongly support the youth by skilling them to participate in the society and leverage their education to come out of poverty. However, it would have to be an education framed within the social justice framework as noted above.

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CUTTING THE CALCULATIONS OF SOCIAL CHANGE WITH SCHOOL MATHEMATICS

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Dear Dr Thuli Dlamini

Thank you for your invitation to be part of this panel of expert researchers to advice you forming your project of a new school. I appreciate the information provided regarding the context of the school that you would like to build and some of your visions for such a school. In my answer to your request, I would highlight different aspects of what we —myself and the group of researchers[1] with whom I work— have to say about your initiative.

THEORY IS ACTION!

I am honoured that you had considered asking for advice from a researcher. I am not more than a researcher and my expertise is in researching, not in building schools. People who, in your country and many other countries in the world, have built schools in areas of "disadvantage" have for sure expertise and extremely valuable understandings of this situation and the challenges you will face. I highly suggest that you listen to them carefully and do not get tricked by the legitimacy that researchers' voice have in this society. As a humble researcher, I am in no position to tell you what would work in practice. It would be extremely arrogant on my side to say that on the grounds of our research, my team and me could give you the key for success. In other words, I cannot offer you an "implementation" in practice of my research results, because I do not believe that such "implementation" is a straightforward matter that could effectively lead to a good school.

However, that does not mean that researchers have nothing to offer. The only thing I can offer is theory and analysis, and not more than that. And it is indeed the most powerful tool you could ever have in thinking your school. Politicians, administrators and researchers alike tend to think that theory is just a bunch of words that people like me, accused of never leaving the wall of the Ivory Tower of their universities, write and publish in high impact factor journals. Such a way of thinking may be connected with the historical division between mind (and theory) and body (and doing and action) at the heart of the rationalism of Western philosophy. In this way of thinking, thoughts and theories are enunciated but are not performed, and therefore are believed not to be action. Against this view I would argue, thinking with Foucault and Deleuze, that there is no more powerful practice than the practice of thinking and theorizing (Foucault &

[1] Melissa Andrade, Gloria García, Gelsa Knijnik, Alexander Montecino and Aldo Parra helped me commenting and discussing a response to this invitation.

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Deleuze, 1977). Theorizing is not contemplation. Theorizing is the very same political act of imagining what is not yet possible to imagine because of existing epistemological framings for practice. Theorizing as a critical practice means articulating the cracks of the truths of knowledge within which we innocently act. It is only in the cracks of the truth narratives that govern our action that we may find a new way, a possibility to be invented, a new language to be articulated and, with it, a choice to act for breaking existing forms of government and power. Therefore theorizing is action. It is a political commitment of the researcher—would rather say the *intellectual*, even if the word intellectual seems to have vanished in favour of the understanding of the educational researcher as an engineer. Without thinking seriously there is no change!

Allow me to interpret your invitation not as an advisory task, but rather as an encounter between the humble perspective of an intellectual and the hard work of you and the people who will set up your school. My intention will be that of raising questions on the ways it is possible to think mathematics education as part of the larger cultural politics of schooling in a country like South Africa, and how the practices that will take place in the school could fabricate the subjectivities of children through the functioning of the mathematics curriculum. My invitation is to a critical dialogue about the possible effects of the many good intentions in the foundation of your school.

MATHEMATICS EDUCATION AND POWER EFFECTS

Let me start by spelling out an assumption I depart from. Although frequently seen as the practice that transmits valuable knowledge, mathematics education is part of a power dispositif that governs the conduct of all its participants in desired directions. Seen in this way it is possible to connect the micro politics of educational practices with its different technologies within the larger politics of government in the state. In that sense, mathematics education is no other than political, and effects power through its organization of the population, the schools, the classrooms and the teachers and pupils as well. Such organization though, does not happen in a vacuum but is part of historically constituted forms of thinking about the self and the other. Elements of the theoretical position that I deploy in thinking your case have been developed in our recent work (e.g., Valero, García, Camelo, Mancera, & Romero, 2012).

Such theoretical position invites me to consider in a critical way some of the assumptions made explicit in your invitation to advice your initiative. In particular it calls my attention the idea of a school that explicitly empowers students with mathematics, and the issue of who the students will become.

MATHEMATICS EDUCATION AND POVERTY

With respect to "developing countries" such as South Africa, Brazil, Colombia or Chile—the countries where my collaborators and I have carried out research—high mathematical achievement is seen as an indication of national and personal progress. Such thinking makes desirable to "empower" children and the nation with providing "democratic access to powerful mathematical ideas". In the particular case of South Africa, you express your desire of empowering children in your school with mathematics. We need to place such desire in a broader rationality to understand why in the first place such intention is desirable, and to think which may be "side effects" that you may not have considered. Several researchers have argued that in South Africa mathematics education cannot be understood in isolation from the operation of the mechanisms of racial segregation installed by the apartheid regime, even nowadays, 20 years after the transition to democracy (e.g., Setati, 2005). The interesting thing is that the blunt exclusion of black population from educational opportunities, even in mathematics, is an extreme case of what seems to be the case in other developing —and even developed—countries. The documented low achievers in mathematics are children belonging to the groups positioned in the low ranks of a particular society. If this is the case, then we can say that low mathematical achievement evidences that a low socioeconomic positioning is strongly connected to losing in the game of getting credit and value through education. That national and international statistics evidencing such connection are not only the numbers that represent a social fact; those very same statistics and their production are an important element in the construction of the strong connection between low school (mathematics) achievement and poverty as a simple, unquestionable truth.

One of the effects of this truth and its associated ways of reasoning is the motivation of mathematics educators to put their good intentions and efforts in promising an empowerment of people with mathematics. This empowerment sometimes results in some stories of success. At least in Colombia, very few students attending a very poor school, even if they excel in mathematics, would make it out of poverty. And of course any story of success is a gaining because it means the realization of the promise of a brighter future, of social and economic mobility, for at least one individual and one family. However, the success of one or of few is only the success of one or few among thousands. It is not the success of the many. Why? All good intentions and promises of redemption with and through mathematics are subordinated to the ordering of power and differentiation in society. Education is a very powerful institution that classifies, selects, and grants credit to some and, at the same time, has to inscribe failure in others as the very same pre-condition for its functioning. In thinking this, I take the analysis that Alexandre Pais (2012) has been proposing when connecting mathematics education with the ideological functioning of capitalist societies and their power mechanisms. The narrative of salvation and empowerment with mathematics are part of the technologies of government that detract our attention from the fact that educational failure in mathematics is the very same condition of the ordering of power. In other words, the failure of many in mathematics is the precondition for the success of the very few. It is failure of the many what grants value to the few who succeed. If all, in reality all, could have success in school and in school mathematics, education would not be a central field of government and power.

MATHEMATICS EDUCATION AND THE GOVERNMENT OF SUBJECTIVITIES

You raised the question of how to avoid that mathematics be perceived by people in the as an elitist activity and how to harmonize it with democratic goals. My first answer is that it is impossible to detach school mathematics practices from a perception of elitism for various reasons. First of all, from the time of the Ancient Greeks, access to mathematics was the key element in the education of those with "gold in the soul" who in fact were the ruling elite of the polis (Radford, 2008a). Second, despite the expansion of arithmetic and further mathematics in massive educational systems during the 20th century, mathematics remains still the type of knowledge and the school subject that operates the selection of the "smart" and "intelligent" from those who are not. The association of mathematical ability with intelligence is part of the discourses circulating in popular culture, expressed in, for example, the views on who mathematicians and what is connected to their success (Moreau, Mendick, & Epstein, 2010). Third, achieving success in mathematics demands acquiring competence and fluency in a vertical, hierarchical code, which differs from the language codes of everyday life. Thus, those who can "speak the codes of mathematics" have gained a "rarefied", specialized language that clearly breaks with the structure and relative "simplicity" of the everyday linguistic forms of communities (Jablonka & Gellert, 2011). Fourth, the adoption of hierarchical codes is not simply a new linguistic habit or a change in cognition. If objectifying as knowing is inseparable from being, as Luis Radford has postulated (Radford, 2008b), succeeding in mathematics is also a process of becoming subject. As we have argued before, the subject that school mathematics seeks to fabricate is the Modern, rational, cosmopolitan child (Valero et al., 2012). Becoming that child has a double effect in the governing of the self in relation to the power. On the one hand the technologies of mathematics education inscribe in children desired forms of thinking which make the child the desired subject for the political and economic organization of society; and at the same time effects a rupture between the child and the forms of reasoning of the communities they live in. The "(mathematically) educated child" becomes a subject breaks apart from his/her community and its forms of knowing and being. This rupture is inevitable. It is the very same condition of education.

DECENTERING THE CURRICULUM

As you can already notice, my analysis is not as optimistic as your intentions. Not being optimistic does not mean that such an initiative should not be realized. I simply want to make a strong point that no matter what we do in education, there are always power effects. The rethinking of (mathematics) education is not an easy task if a strong political concern is taken into consideration. The research that we have carried out in Colombia (García et al., 2009), in situations similar situations to the ones you describe, has taught us two important lessons. First of all, as part of the cultural politics of education, school mathematics practices govern children, effect classifications, and inscribe in them forms of reasoning about themselves. Such systems of reason both include some who become the desired child for the dominant political and economic organization, and exclude others whose forms of life and being continue to be marginalized. Mathematics education effects power through the fabrication of children's self, that is, their subjectivity. Secondly, as a consequence of the former, one possible alternative for the development of curricula is to articulate it around the construction of children's subjectivity. While curricula have been traditionally organised around central mathematical ideas or competencies, an organization around children's subjectivities displaces the core of mathematical concepts as the centre of the curriculum and opens the space for subjectivity to become the articulating axis around which mathematical forms of reasoning and acting could be organised. It becomes then possible to include other forms of being while also expanding the meanings of mathematics beyond the realms of a disciplinary core. In such a project being and objectifying become entangled in new ways that may be worth exploring.

We look forward to hearing about the advances in your initiative!

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RAISING POLITICAL, PSYCHOANALYTIC, AND CULTURAL QUESTIONS OF A PROPOSED EDUCATIONAL INTERVENTION

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This paper provides a response to a proposal that seeks to make a difference to students in one location in South Africa. It raises troubling questions about how to change education in one setting in a way that is just as well as educationally and culturally meaningful. Comprehending the proposal through theory draws attention to political and psychoanalytic questions and the imperative to consider the future alongside the cultural present and past.

INTRODUCTION

The immediate response to the intention to establish a school in South Africa is that it feels like a welcome breath of fresh air. The proposal is both an instructive response and an emancipatory gesture to one of the most complex and pressing issues facing mathematics education in South Africa today. Aspirations that advocate for the rights of students to education carry considerable persuasion, particularly to outsiders from communities which experience universal free education. Viewed from the perspective of transition, development and progression, the proposed initiative represents a compelling emancipatory and utopian vision, promising to create a better balance amongst global communities. Interventions predicated on an interest in establishing new norms of educational relations, can at the individual level, it is believed, open up possibilities that enrich the student's present and future, enabling the student to move beyond repressive practices and fulfil unrealised dreams, while, simultaneously, advancing the local economy.

A more considered response reflects on the serious challenges that South Africa is currently experiencing—challenges in relation to a repressed economic environment, dire poverty and the realities of multi-cultures, formed through space, place, and race as well as history and language. Moreover, these realities are nested within the tensions between the national and global economies, as well as between colonial and postcolonial educational cultures. In a context like this, the achievement of sustainable educational futures, like the initiative proposed, becomes exceedingly tenuous. The plan for innovation and reform in relation to the establishment of a school located at a frontier between rural and urban communities in South Africa weighs heavily in the balance.

Across longitudes and latitudes, from specific coordinates where every child *does* have access to mathematics education, I wish to underline the significance of the proposal on the basis that is forward-thinking and is organised around a belief that students on the wrong side of the social-capital divide should be given opportunities. But in a shift in registers I wish to convey what Spinoza has named the unceasing challenge of the 'not yet'. What precisely does lie ahead? Proposals are gifts but, as Derrida has cautioned, in their realisation, they produce events, new forms of action, new practices, and new forms of organisation. All these signal that the task ahead will not in any way be straightforward. Privately funded initiatives and policy incursions might heighten social awareness and seek solutions to the educational problem by introducing new initiatives, yet they cannot shore up the guarantee of the production of an equitable mathematical experience.

My response does not offer the magic bullets or guarantees that you might be seeking. Rather, it is more tentative, raising troubling questions about how to change the texture of the world in one setting in South Africa in a way that is just as well as educationally and culturally meaningful. The imperative is to consider the future, along with the present and the past. Comprehending these through theory might provide a sobering counterpoint to the promise of your proposal.

THE POLITICAL QUESTION

Without a doubt, the proposal demonstrates a large reservoir of hope. However, viewed from the perspective of post-critical theorists (e.g., Ellsworth, 1992; Lather, 1992) a project based on liberal democratic principles is fundamentally problematic precisely because a collective enterprise is always already operative upon and within the individual. Students, like their teachers, participate in a social web of power. What this means is that a new proposal for mathematical access and opportunity will govern, regulate, and discipline students as well as teachers. In other words, the identities that students construct of themselves will be made in and through the proposal's pronouncements, its interests, and its investments in others. Power will do its work through the material, discursive, pedagogic and technological forms, as well as through the proposal's discourses that relate to categories of gender, ethnicity, and a range of other social determinations. In short, a vision of change that is conceived of as emancipatory will always, at the same time, be regulatory.

We are beginning to get a sense of the political nature of educational development. To put politics into the immediate context, let's consider the proposal that the school will enrol 100 Grade 8 learners in its first year of operation, and thereafter growing each year by 100 learners at the same level until a total of 500 learners from Grades 8-12 are enrolled. The fundamental Foucauldian knowledge/power postulate maintains that these students will be produced as subjects under the specific discursive conditions made available to them by the terms of the proposal.

It follows, then, that interventions like the one proposed, will produce new subjects and practices and will regulate both. Interventions are part of a wider range of technologies that are involved in the production of the modern subject. As a regulatory apparatus, they impose certain meanings, subjecting individual students and teachers into particular understandings of themselves as students and teachers. In that sense, the potential dangers of all discourse, including a proposal whose stated objective is to liberate, come to the fore. The utopianism of interventionary action must be considered in light of its constraining and inhibiting practices. The Foucauldian question to consider concerns how we might address the point that there is no emancipatory space ‘outside’ normalising discourses.

THE PSYCHOANALYTIC QUESTION

There is another issue, too, that deserves attention. The students who never enter into the process of being produced as a subject within the discursive conditions of the intervention are, in a sense, ‘de-produced’. That is not to suggest in any way that they are not constituted within that production. It means, rather, that their exclusion from the horizon of recognisable subject is fundamental to the very production of the recognisable subject. The fact is that some students will be produced as learners within your discursive construction of a secondary school student at this school, and others will not.

We need to think through the price, not in economic but psychic terms, that needs to be paid for the inclusion of some students and not others. This is your question of learner selection: Whom do we select, how and why? The question can be framed as a psychoanalytic one, relating to subjectivity: “At what price does one become a mathematics student in this school?” This is an important question but it tends to be ignored by the mainstream discourse on educational development. For those who are included, the price will depend on the selection criteria and its presumptions, and what needs to be surrendered to satisfy the criteria. The obliteration might involve generational and parental knowledge. It could involve ethnicity. It could be friendships with ‘de-produced’ students, and so forth. For those who are excluded, the price will accompany the realisation of being ruled-out of education in this school, and will likely be lived out through forms of suffering that tend to go hand-in-hand with practices of exclusion.

THE CULTURAL QUESTION

What might be included in the mathematics curriculum? There has been much transportation of educational theories about curriculum developed within major nation states, just as there has been a practice of exporting ideas about educational reform, child development, teaching, learning, and assessment. We might think of the practice as situated within a developmental paradigm of one-way border crossing, in which ideas, practices, expertise and materials promoted by aid and development programmes, UNESCO, and so on, are considered universally true for every school system, every community and every student cohort. In New Zealand it took us a long time to appreciate that educational ideas, tests, and textbooks, imported from other parts of the world, were not necessarily generalisable for our specific context.

A curriculum transported globally overlooks important local knowledge about language, culture, family systems and values, citizenship and community. In other parts of the world, a number of researchers (e.g., Civil, 2002; D’Ambrosio, 1985) have privileged local knowledge and attempted to design their curricula around it. They have engaged with the local culture, its history, its demographics, and identified points of difference within that culture, not at a superficial level, but at the level that generates understanding of the day to day practices, the belief systems, and the power struggles. Engagement with these realities is a crucial starting point. The engagement will likely not only unravel details of life 'lived at the edges' but also important modes of operating within the social order. As one instance, you note that transport is expensive. Rather than centralising the teaching and learning opportunities in one site, you may choose to create village campuses—hubs of teaching and learning, not reliant on technology, in a number of localities.

Critical scrutiny might reveal that pedagogy is teacher-centred, that students have a high respect for their teachers, and that basic skills are lacking. A low-definition curriculum might be called for. An analysis of current classroom and everyday life might suggest a need to go against the grain of mainstream theories of teaching and learning. A rigorous assessment will pave the way for informed curriculum policy decisions—decisions that are not simply based on inherited ideas from other cultures in order to play the game of developmental 'catch-up'. A curriculum developed with careful thought will give expression to the key point that curriculum policy development sits at the nexus of culture, history and place.

CONCLUSION

Some years ago, in relation to the debate centred on progressive education, Valerie Walkerdine (1992) asked: "An idealist dream, an impossible fiction, or something to hope and struggle for?" Posing that same question for the proposed new school in a deprived setting within South Africa, my answer returns to the point made in the first paragraph: The proposal is both an instructive response and an emancipatory gesture to one of the most complex and pressing issues facing mathematics education in South Africa today. While it is indeed ambitious, its realisation is truly something we would want to hope and struggle for. But to make the dream a reality we need to move away from understanding the project as one which perceives the Other as the problem, and the 'liberator' as the solution to that problem.

My advisory notes have outlined the fundamental points for the calculus of the proposed social change. But there is one further point to be made: an ethical response to emancipatory efforts recognises that people have different histories and different 'presents' and attempts to preserve the difference of the Other. An ethical approach turns upon itself to examine the emancipatory discourse itself, constantly interrogating its pronouncements and the new norms of social and educational relations it engenders. In leveraging the potential of the Other, what eventuates is a "truly educational experience…connected to past and future educational experiences and to other on-going life experiences" (Noddings, 2012, p. 776). Mathematics at the edge within one setting in South Africa might then be reformulated as mathematics at the cutting edge. Mathematics not above, not below, but beside.

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RESEARCH FORUMS

THE CHALLENGES OF TEACHING MATHEMATICS WITH DIGITAL TECHNOLOGIES – THE EVOLVING ROLE OF THE TEACHER

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This Research Forum highlights the most recent research on the development of the role of the teacher of mathematics within mathematics classrooms that involve the use of technological tools, with an emphasis on teachers' experiences within both formal and informal professional development programmes. We foreground the theoretical ideas and methodological approaches that focus on the development of classroom practices at the levels of both individual teachers and communities of teachers, charting their respective development over time. The RF makes reference to a previous forum at PME37 on the theme of Meta-Didactical Transposition (Aldon et al., 2013a), a theoretical framework that has evolved from research in this area.

INTRODUCTION

The earlier research concerning digital technologies directed their lenses on the processes and outcomes of pupils' mathematical learning. However, it is now widely acknowledged that the earlier visions for how pupils' learning might be transformed by the inclusion of technology have not translated into widespread changes in classroom practices. This is partly due to an underdeveloped knowledge of how teachers' practices are impacted by the use new of technologies, and subsequently how teachers embed them within their professional lives, for the purpose of improving pupils' mathematical learning. More recent research has focused on the development of teachers' knowledge and practices within technology enhanced classroom environments. For example, the instrumental approach used in didactics of mathematics (Artigue, 2002; Trouche, 2005), initially used to analyse students' interactions with technology in mathematics learning, has been applied to the study of teachers' professional development through its central notion of "instrumental genesis", using the concept of orchestration and its extension (Drijvers et al., 2010; Trouche, 2005). During PME37 the development of teacher's practices with technology has also been discussed extensively at a Research Forum on *Meta-Didactical Transposition* (*MDT*) (Aldon et al., 2013a; Arzarello et al., 2014). Other ways to describe the use and knowledge of technologies by teachers is given by theories such as *Pedagogic Technological Knowledge* (*PTK*) (Hong & Thomas, 2006; Thomas & Hong, 2005), *Technological, Pedagogical and Content Knowledge*

(*TPACK*) (Koehler & Mishra, 2009; Mishra & Koehler, 2006), and the *Structuring Features of Classroom Practice* framework (Ruthven, 2009). A comprehensive discussion comparing *TPACK*, the *Structuring Features of Classroom Practice Framework* and the *Instrumental Orchestration Approach* can be found in (Ruthven, 2014). Further to this, research on teacher identities has also contributed insights into how and why teachers develop their practice (or not) as users of digital technologies. From a sociocultural perspective, teachers' learning is conceptualised as the evolution of their participation in practices that develop their pedagogical identities, which Wenger describes as "a way of talking about how learning changes who we are" (1998).

As this Research Forum is focused on making visible the dynamic processes of teachers' development of their classroom practices with and through technology over time, the theoretical frameworks have been chosen as they enable this temporal element to be seen. However, our choices are not exhaustive!

The adoption of a holistic view of teachers, their practices and their professional learning concerning the teaching of mathematics with digital technology raises many questions (about practices, about training etc.). Newer constructs have been developed to articulate the teachers' learning processes with and about mathematical digital technologies, such as *critical incidents* (Aldon, 2011), *hiccups* (Clark-Wilson, 2010) and the notions of *instrumental distance* (Haspekian, 2005) and *double instrumental genesis* (Haspekian, 2006, 2011).

Many studies evidence the importance of the role of the teacher *from different perspectives*: the teacher in the classroom, the teacher as a learner of mathematics, the teacher as member of a community of professionals (Sfard, 2005). For instance, Wenger (1998) argued that teachers have to reconcile multiple identities that result from their participation in various communities of practice into a single core identity that holds across contexts. Theorising teacher learning as identity development in multiple contexts provides a dynamic perspective on the evolution of teachers' knowledge and practices. This approach is useful for investigating how teachers engage with any kind of educational innovation, whether this involves the introduction of digital technologies, other teaching resources, or changes to curriculum or assessment.

As Wenger's theory suggests, any research into teaching practices must confront the issue that teaching practices embody several different dimensions (social, institutional, cognitive…). Consequently researchers have to make choices about the 'grain size' of their focus of analysis, to different levels of detail, whilst also respecting the dynamicity and interconnectivity of the related processes. The analysis appears to be even more complex when digital technologies are introduced, both as tools for teaching and as tools within teacher education. When the research lens is trained on the mathematics teacher in his/her interaction with the technology in the class and during professional development activities, it is a challenge to maintain a deep focus on multiple aspects.

This Research Forum aims to respond directly to this challenge. It is focused on the role of the mathematics teacher within both the classroom and during teacher education activities, where the mathematical, pedagogical and wider communication tools include increasingly ubiquitous digital technologies. The Forum aims to advance research on teaching practices in general by drawing from the substantial research of the last 5-10 years on teachers' uses of digital technologies in school mathematics in order to explore and propose stronger connections with the wider body of research on teachers' practices with technology and learning from cognitive, psychological, and social perspectives. The main objective is to contribute to a critical debate on the wider implications of the selected set of research themes on initial and continuing teacher education.

MAKING SENSE OF THE EVOLVING ROLE OF THE TEACHER

We start by postulating that the process through which teachers develop their professional identity and associated practices over time is experienced as 'professional development', which encompasses the full range of individual and collaborative activities in which a teacher might engage, within and outside of their school setting, to include: traditional courses; within-school initiatives; participation in research projects; and professional networks.

The term 'professional development' is being conceived as both a product (i.e. a tangible set of professional activities with structure, content, a timeline, etc.) and as a process, which involves a range of participatory actions. This is analogous to the idea of a mathematical proof, where the final product can be conceived as the outcome whereas the process of proving may well have involved exploration, argumentation, justification, communication etc.

In order to analyse the process of mathematics teacher professional development holistically and from different theoretical perspectives, we have identified some key questions that address three axes of research concerning teachers' practices: the professional development of the individual teacher; the role of digital technological tools; and the role of institutions.

- How can we observe and describe change, evolution of practices and innovation within mathematics teachers' professional development concerning digital technologies?
- How does the use of digital technological tools impact upon the role of the teacher and their associated professional development?
- What roles do the institutions play (e.g. national curriculum, national/international assessment, school inspection regimes, etc.) in supporting changes within mathematics teachers' professional development at large scale?

Different theories that try to describe the activity of teaching involve different dimensions. In order to address our key-questions we have identified among these dimensions five themes that include a consideration of the process of professional development concerning digital technologies:

- The institutional context and its impact upon teachers' roles.
- The design of selected mathematics teachers' professional development programmes (from the perspective of the designers).
- The professional development activities of teachers with technologies, within and outside of formal professional development programmes.
- Teachers' implementation of technologies in their classes.
- Meta-level reflections by teachers and researchers on the processes of professional development facilitated by the use of digital technologies.

These themes coexist, intersect and interact, possibly – but not necessarily – in sequence with each other. Moreover, this list does not aim at being exhaustive in that other dimensions could be considered (the affective dimension, the intercultural dimension etc.), but they are beyond the scope of our analysis.

This RF seeks to compare, combine and connect the most pertinent theoretical perspectives, in tune with an idea connecting theories (Prediger et al., 2008, see Figure 1) to describe and explain the whole process of mathematics teacher professional development with technologies.

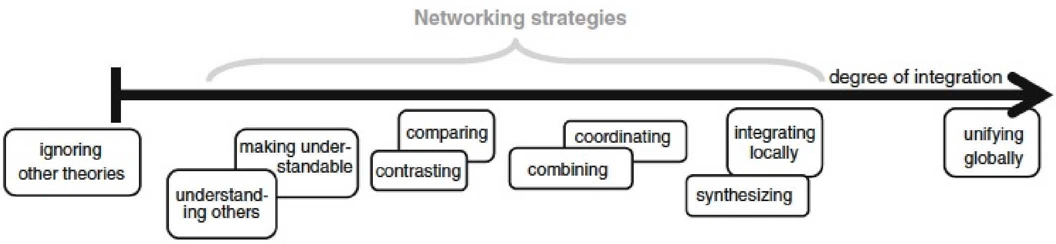

Figure 1: Networking strategies to connect theoretical approaches (Prediger et al., 2008, p. 170).

The idea is to explore what each theory can and cannot illuminate and to try to explain how they can work together. Thus, this contribution, the result of this co-working, develops around the afore-mentioned five themes. For each theme, we present examples from a variety of relevant studies from different contexts (country, professional development setting, type of technology, school phase, mathematical focus, pre- and in-service teachers, etc.), analysed according to different elements from the identified frameworks. This analysis is conducted with reference to specific sub-questions associated with each theme and makes it possible to highlight how the different theoretical ideas support the development of new understandings. The emphasis is on the usefulness of theories that enable both the temporal and personal aspects of teachers' trajectories to be described, with the teachers' voices as a central and essential element.

However, as the model of *Meta-Didactical Transposition* (*MDT*) may prove to be a useful tool for the analysis of different aspects involved in the whole process of teacher professional development, prior to focusing on each dimension and the corresponding examples taken from our studies, we briefly present an overview of *MDT* and highlight its main characteristics.

THE META-DIDACTICAL TRANSPOSITION AS A TRANSVERSAL LENS

The *MDT* model has been conceived to take into account the complexity arising from the intertwining of the processes involved during a teacher education program. It considers some main variables in the teacher education processes (community of teachers, community of researchers, role of the institutions) and accounts for the evolution of their mutual relationships. It includes a consideration of teachers' practices (both during professional development and in their activities in the classroom) and provides tools to analyse if and how teachers' knowledge and practice evolve during these processes. This evolution is observed as changes to and integrations of new teaching practices, mathematical technologies and research issues, both in the mathematics classrooms and within mathematics teachers' professional activities (programming didactical plans, designing tasks, planning assessment, etc.). Moreover, this evolution takes account of the teachers' relationships with institutions on the one hand and with the researchers' community on the other hand. Beginning with the assumption that institutions (i.e. national curricula, national assessment tools, the constraints of teachers' time and space, etc.) play an important role in the school context, the theoretical background for the *MDT* model is derived from Chevallard's *Anthropological Theory of Didactics* (Chevallard, 1985, 1992). In particular the model refers to the notions of didactical transposition and praxeology. Chevallard defines didactical transposition as the transition from knowledge regarded as a tool to be put to use, to knowledge as something to be taught and learnt (Chevallard, 1989). The notion of praxeology, which is the core of this theory, refers to the tasks that are to be performed and can be conceived as a quartet, constituted by two main blocks: (a) the technical-practical block, a task and a technique, that is the "know how" (which includes a family of similar problems to be studied, as well as the techniques available to solve them); and (b) the technological-theoretical block, constituted by the technology/technologies and the theory/theories that represent the argument that justifies or frames the technique for that task, that is the "knowledge" (García et al., 2006).

Since the aim of the *MDT* model is to frame and reflect on teacher education programs, the term "didactical" has been substituted with "meta-didactical" to stress that the processes under scrutiny are, in this case, the practices and the theoretical reflections developed within teacher education activities. In other words, in the case of teacher education programmes, fundamental issues related to the didactical transposition of knowledge are faced at a meta-level. Through the *MDT* model teacher education processes are analysed from a dynamic point of view, highlighting the interactions between the community of teachers involved in a professional development and the community of researchers who design and coach the activities. Initially, the two communities of researchers and teachers have their own praxeologies, associated to specific tasks. During the process of the *MDT*, as a result of the dialectical interactions between the communities, both the praxeologies of the community of researchers and the teachers' community change and sometimes evolve in a shared praxeology, which constitute the core element of the whole process.

Many factors have enabled us to identify the *MDT* model as a possible useful transversal lens that could act as a "binding agent" for the analysis of the examples we have chosen to discuss the five dimensions: (1) the stress on the role played by institutions and the constraints they impose; (2) the dynamic interplay and the interactions it allows to describe at different levels; (3) the focus on the different actors involved in these processes and on their mutual interactions; (4) the possibility it gives to highlight the evolution of teachers' and researchers' praxeologies over time through the notion of shared praxeology. Other aspects of the *MDT* model will be recalled and discussed in the analysis of specific examples, to include: the change of the status of some components of teachers' and researchers praxeologies from external to internal and vice versa; the brokering role played by teachers and researchers within the different communities; and the notion of double dialectic as a fundamental aspect typical of the processes aimed at fostering teachers and researchers' reflections and comparisons.

THE INSTITUTIONAL CONTEXT

There are two sub questions relating to this theme: To what extent can teachers develop individual agency in the face of institutional constraints, and what role can researchers play in this process? and How can researchers impact on the institutions in the planning of large scale professional development programs?

As Chevallard (1987) testifies, the relationships between the institutions connected with the teaching system and society are most relevant,

> The teaching system is not a thing in one piece. It does not consist only of teachers and students, textbooks, homework assignments, and so forth. Like any social institution, it has to attend to the maintenance of its relations with society as a whole. Accordingly, a part of it will specialise in the overseeing of the relationship between the teaching system proper and its societal environment. This is a quite general requirement of social life, which no institution can elude. (p. 2, our synthesised translation)

The *Anthropological Theory of Didactics* (*ATD*) focuses on the institutional dimension of mathematical knowledge and puts the activity of learning mathematics within the bulk of the human activities and of the social institutions (Chevallard, 1999). Some examples of the relevant institutional variables are: the national curriculum; the ministry of education; national education programmes; national assessments; the textbooks; the schools and classes in which the implementation occurs; the communities of teachers of the same subject; and the communities of teachers involved in the same projects. These institutional components will vary in accordance with the national context in which they are situated.

The institutional context in teacher education activities is important in that it influences the choices made by stakeholders, researchers and trainers when a new programme of professional development is designed. By taking the institutional variables into account, it is possible to contextualise educational initiatives for teachers into the school setting, and indirectly, as a product of the professional development, the teachers' changed practices can have a positive impact on students and their mathematics competences. For example, many large-scale initiatives of this kind have happened in Italy over the few last years, and their impact is tangible from different points of view: the use of technologies by teachers and students; the increasing scores in international and national assessment of students; the diffusion of the new national curriculum; and so on (PISA, 2012). In addition, the European Union is promoting lifelong education as strategic element for the development of countries, and in this context the institutional dimension is related not only to the educational one, but also to the political and social one. According to *ATD*, a mathematical object in school exists "since a person, or an institution acknowledges that *it* exists" (Chevallard, 1992, p. 9). Consequently, we can also claim that a didactical object exists in a teacher education context since a researcher, or an institution, acknowledges that it exists. For example, an education programme based on teaching geometry through open problems with the support of a dynamic environment such as GeoGebra can be planned in a specific country taking into account: the national curriculum, the time teachers can spend in lifelong learning, the time they have available in school to introduce such types of tasks, the availability of classes, suites of computers, interactive whiteboards, and so on, namely all the variables that comprise the institutional dimension. In designing the activities for teachers, the researchers have to make these variables clear and make choices in relation to the specific objectives of the activities such that it is possible to impact upon teachers' learning. Alternatively, the teachers may not choose to take part in the training activity, or participate without consideration of the usefulness of professional development programme for their purposes at school. From the point of view of researchers who are involved in the design and implementation of the educational innovation by institutions as the Ministry of Education, or international organisation, or local institutions, it is very important to have not only the possibility to train teachers, but also to take the opportunity to disseminate key ideas from research within schools (contextualised through the curriculum, traditional methodologies, textbooks etc.). In this way, these are the mediating ideas between the institutional dimension related to teachers and that related to researchers (and in some cases the external institutional dimensions of private companies, the European Union, or others).

With reference to the model of *MDT*, we can say that the criteria on which the choice of the variables in the institutional dimension is based are part of the researchers' praxeologies. An example is a national project in Italy, *Piano Lauree Scientifiche* (*PLS – Scientific Degree Plan*), for which one sub-project of teacher education is "Problem solving with GeoGebra" (Robutti, 2013). The organization of the programme began with an analysis of the new Italian national curriculum *Indicazioni Nazionali* (Ministero dell'istruzione dell'università e della ricerca, 2010) in order to select the curriculum statements that could constitute the starting point for teacher professional development activities. The sections of the curriculum chosen for the design of tasks for teachers focus on both general aspects, related to the purposes of mathematical activities, and specific aspects, such as the role played by geometry, modelling, open problems, and the use of the technology in these domains.

According to the model of *MDT*, the research community has the *task* of selecting (with some *techniques*) the variables (geometrical concepts, use of software, and modelling) to focus on the educational programme within the institutional dimension represented by the national curriculum. This selection is carried out with reference to the aims of the project, which are part of the technological-theoretical part of the researchers' praxeologies. Other variables, coherent with the researchers' theoretical background, may be taken into account in the design of teachers' activity (e.g. mathematics laboratory, open problems, mathematics discussion). This is an example of what we have previously called a change in status of some components of teachers' and researchers' praxeologies from external to internal. Initially, these variables may be external to "ordinary" teachers' praxeologies. However, through the professional development programme, they become progressively internal, a result of the meeting of teachers' and researchers' praxeologies, as evidenced by several cases from within the *PLS* project. The problems connected to a change in a curriculum in countries where the schools have to follow national recommendation for its implementation are those related to the change: teachers have difficulties in change something in their praxeologies, and researchers can help them in doing it. *MDT* is a model to describe the process of development, giving insights into the elements that change in this process.

Chevallard poses the question of integration in anthropological terms that are the *viability of technological tools in the class* and stresses the importance of the teachers and the institutional contexts in which they act. Indeed, this viability is conditional upon by many aspects that have been considered within research on the integration of technical objects: the epistemological effects of the tools, the mathematical renewals which can result from them, etc. But this integration can be only partial a weakly viable if we forget the teachers' role. Chevallard explains the origin of the weak integration thus: one tends to retain only the knowledge ("*le savoir*") and the student's "*rapport au savoir*", forgetting that those cannot exist alone, in a didactic vacuum, without a functionally integrating didactic "intent", which is left, in practice, under the teachers' responsibility, however seconds are these aspects implicitly judged (Chevallard, 1992, p. 6, our synthesised translation). Chevallard's *ATD*, by focusing on the institutional dimension of mathematical knowledge, obliges researchers who want to study teachers' practices in mathematics to situate this activity within social institutions.

Several examples of research applying this point of view can be found in Clark-Wilson, Robutti and Sinclair (2014). For instance, in Haspekian (2014), teachers' difficulties in spreadsheet integration have been explained by the changes that the spreadsheets introduce within mathematical objects, techniques and representations. The research was concerning the domain of algebra, where the use of spreadsheet was planned by the observed teacher in order to help 12 years students to enter in algebra. But the changes introduced by the spreadsheets in the algebraic domain impacted upon the praxeologies that are usually viable in this domain in the French education institution for this grade. The table below gives a quick insight of the distance between the whole algebraic culture in the French secondary education and the algebraic world carried out by spreadsheets.

"Values" of algebra	**In paper pencil environment**	**In spreadsheet**
Objects	unknowns, equations	variable, formulae
Pragmatic potential	tool of resolution of problems (sometimes tool of proof)	tool of generalization
Process of resolution	"algorithmic" process, application of algebraic rules	arithmetical process of trial and refinement
Nature of solutions	exact solutions	exact or approached solutions

Table 1: Algebraic worlds

This instrumental distance introduced by the tool goes beyond Balacheff's *computer transposition* (1994) as it concerns all of the mathematical and didactical organisations that are usually viable in the classroom for the institution concerned.

Thus, the new praxeologies did not immediately, nor easily, fit with the institutional constraints that weigh on teachers' shoulders: national curriculum, inspection regimes, education programmes, national assessments, textbooks… that are all institutionally situated. By considering the whole institutional context, one understands better the difficulties of integrating spreadsheets for teaching and learning algebra.

THE DESIGN OF MATHEMATICS TEACHERS' PROFESSIONAL DEVELOPMENT

We reiterate our perspective that 'professional development' encompasses a wide range of individual and collaborative activities across a broad range of structured and informal opportunities, which are constrained by country-specific and cultural boundaries and expectations. Central to all of these activities lies the development of a teacher's mathematical, pedagogical and technological knowledge and practice. Consequently, the notion of an explicit 'design' implies that there has been some fore-thought. Whilst there have been some research studies that have sought to articulate the processes and outcomes of more informal professional development activities (see Clark-Wilson et al. (2014) for examples), here we will focus on professional development that has been constructed for the purpose of developing teachers classroom uses of technology.

The importance of design in the planning of both teacher professional development projects and specific related tasks for teachers (and their students) is pervasively recognized (de Geest et al., 2009; Even & Loewenberg Ball, 2009). The research community has a prominent role in designing activities for teacher education, and this design of meta-didactical trajectories is the task of researchers involved in education programmes, while the teachers involved learn to design didactical trajectories for their classes. In this way, design can be described at two levels: of the teachers' activities and of the students' activities.

Design of teachers' activities may include also the design of students' activities and a team of researchers involved in this design may work with various methodologies, according to the cultural tradition of the country. In Italy, for example, the team is usually constituted by academic researchers and teacher-researchers who plan activities for teachers' programmes that include students' activities. In most of these programmes (i.e. *M@t.abel* or *Piano Lauree Scientifiche-PLS*) teachers are asked to experiment the proposed activities in their classes, during or after the training, in order to observe processes and discuss them in a final meeting of the research team and the teachers.

Design includes not only different types of tasks (i.e. open/closed problems, tutorial activity with technologies, etc.) and different types of lessons (lecture, workshop, working in groups). It also includes the design of initial questionnaires, interviews, materials used for the lessons, references to institutional aspects and logbooks to observe and record processes in the class. According to the paradigm of *MTD* the information acquired in the initial questionnaires/interviews supports researchers to identify the teachers' usual praxeologies when teaching mathematics with technologies.

In the following example, we present some data related to an initial questionnaire proposed to teachers involved in an educational programme in Italy in the national project *PLS*. This data may help the participant of the Research Forum to discuss and respond to the questions, "What is the role of the use of digital resources as a component of teachers' professional knowledge?", and, "How we can describe possible changes in their use by teachers, when they meet researchers in educational programmes?".

For example, in the programme design the team can prepare questions such as: Do you use different technologies in your class? What software do you choose? What kind of problems do you propose to the students in order to use technology to solve them? In which ways do you think technology can be useful for the learning process? These kind of questions make it possible not only to make inferences about the technology used by the teacher, but also on the teaching practices adopted and the teacher's ideas about the role of technology in learning processes, that is, the teacher's praxeology. For example, an older, experienced teacher of secondary school responded to the previous questions with these words: "Then, usually, for example in this class we have an IWB, so usually I do not prepare some special kind of things, but surely it is like having a projector and computer there, so for example it is quite normal that we use GeoGebra to explain, depending on the subject, but ... this definitely."

We can infer that the praxeology of this teacher implies her sole use GeoGebra by herself, to demonstrate something at the whiteboard, to pose a task and to solve it, to justify a procedure, without including students in the work on problem solving. By observing this teacher during the professional development programme, and then in the classroom with her students, researchers may collect data on her praxeologies and identify if and when there is some change in them, related to the use of technology. For example, researchers may highlight if a certain teaching practice far from her traditional way of working, at a certain point, become a consolidated praxis in her activity with students. The *MTD* model helps researchers in describing this passage as a modification in one or more components, which from external become internal, and mark a change in the evolution of teacher's praxeology, as a result of the meeting with the research team.

A second example, in this case, is the work done in the Comenius Project *EdUmatics* (Aldon et al., 2013b) – aimed at developing resources for mathematics teacher education in the field of integrating technology into mathematics teaching. The resources for professional development, are directed towards teachers, and include a range of tasks for school students, aimed at: giving an introduction to the use of technologies, using representations in static and dynamic way, making use of videos for teacher training, obtaining functions as models of phenomena and mathematical configurations. These themes offer a choice of different uses of technologies in teaching mathematics, depending on the motivation and the preliminary knowledge and skills. During the design of these resources, the research team met in order to share not only the tasks to prepare, but also the teaching practices to extent the tasks in the classes, and a develop materials related to didactical suggestions. The general aim was not only to give teachers didactical resources, but moreover to give ideas about some of the important research themes that have underpinned the design of the resources. This activity of the EdUmatics team can be described in *MDT* with the praxeologies of the research team, shared by the various countries groups involved in the project. Using the terminology of the *MDT*, these praxeologies are made of task-techniques (the design of activities and teaching practices, considering the institutional dimension of secondary school and the teaching practices and technologies to be enhanced); technologies-theories (all the reasons to implement such tasks, teaching practices and technologies, such as the theoretical references adopted by the research teams – in this case, for example, the multi-representation of mathematical objects in technological tools and multimodality as two sides of the same coin, the documentational approach and didactical incidents, the use of CAS in classes from a theoretical point of view, and instrumental orchestration). The EdUmatics project gave the research team the opportunity to work together and to learn each others, sharing praxeologies of research and of resources design. The collaboration during the EdUmatics project is an example of a co-production in which researchers and teachers brought in the design of resources their expertise and competencies, as co-producers (Kieran et al., 2013).

PROFESSIONAL DEVELOPMENT ACTIVITIES WITH TECHNOLOGIES

Digital technology encompasses technology as both a tool to experiment with representations of mathematical objects and a medium through which to find and communicate information (Hegedus & Moreno-Armella, 2009). The context for the following example is a local professional development offer in which teacher trainers used a web-based platform. In the preceding year, eleven teams of teacher trainers volunteered to modify and augment their usual training sessions through a combination of on-line and face-to-face instruction. The main aim of the training session was to allow trainers to change from their usual in-service format to a blended learning system, as explained in Aldon et al. (2013a). Researchers who participated in this training session evaluated its outcomes by investigating how some teachers implemented the ideas that had been presented. In this particular example, the subject of the training session was "the use of algorithms and programming to do mathematics". This subject is part of the French national curriculum for students in high schools (from 16 to 18 years old), especially for students following a scientific stream.

It is often interesting to analyse the failure of a program to achieve its objectives as a means of showing the importance of theoretical aspects. As it happens, the institutional context and the lack of shared praxeologies brought about difficulties in the professional development of the teachers. The training session was organized into four different phases. The first involved presentation (of trainees, of trainers, of the aim of the training session, of the programming languages). The second phase was a face-to-face session during which fundamental algorithms were presented and implemented on computers. At the same time, trainees started the design of lessons for the classroom. The third phase was conducted in "distance" mode with the aim being for teachers to implement lessons in their classes, to share observations and analysis, and to present further development of algorithms. The fifth and last phase was a face-to-face phase of discussion about the classroom implementation.

Teachers following the training session were volunteers, but chose to participate more because of the subject matter than for the hybrid modes of presentation. What is highlighted in this example is the difficulty of bringing these teachers to really take advantage of distant and asynchronous exchanges. It was clearly apparent that the modification of the institutional contract (responding to the didactical contract) was too great for teachers to alter their training habits. During the first phase, all trainees logged on to the web platform and participated in the presentation activity. During the face-to-face session, the trainees began the elaboration of mathematical courses including use of algorithms and programming for their own class context. However, and despite the efforts of the trainers to animate the forums, send relaunches, and offer new contents and challenging problems, the trainees did not concur with the organisation of the training session and did not keep up their participation in the course. An important aspect of the training session was for teachers to develop reflexive thinking on their professional behaviour relative to the use of computers in their mathematics courses.

Algorithms used in the teacher training session can be considered as praxeologies and the interesting thing is to compare the trainers' praxeologies and the trainees' praxeologies in order to understand why the training session did not lead to a shared praxeology. Let us take the example of the work on Graham's algorithm, which is a method of computing the convex hull of a finite set of points in the plane with time complexity O(n log n). The trainers' praxeologies included the justification for the study of this particular algorithm by trainees – as a link between mathematical knowledge and algorithmic knowledge at the level of mathematics teachers' knowledge. The implementation of such an algorithm in the classroom was not planned but the transposition to the classroom of the idea of linking mathematical problems and algorithmic solutions was seen by trainers as a consequence of this task. At the same time, trainees considered this task as an application of sorting algorithms without possible applications in the classroom. The lack of discussion in the third phase of the session meant that the two praxeologies remained separate without ever becoming a shared praxeology. The consequence of the misunderstanding of the institutional contract was a rupture in the dynamic of the *MDT*, despite the mediation by the researchers.

A second example illustrates the situation where a teacher sought out informal professional development opportunities through professional networks and participation in research projects, rather than through working with teacher trainers. Again, however, the role of institutional contexts is evident. This teacher came to integrate digital technologies into his practice as a way of helping his students (secondary school age) to access the curriculum and succeed in learning mathematics. This summary of his development draws on data from his participation in several research projects between 2001 and 2010. Until the 1990s he would have described himself as a traditional teacher who tried to explain mathematical concepts to students as clearly as possible. He expected students to copy what he did and to demonstrate their recall on tests. However, he was confronted with the reality that most students did not understand what he was teaching because, only a few weeks after passing the test, they seemed to have forgotten everything they had learned. Rather than blaming the students for their apparent inability to learn, he returned to the university where he had completed his initial teacher education to look for new ideas in mathematics teaching through discussion with academic researchers and reading current literature. In subsequent years he volunteered to participate in research projects investigating the role of digital technologies in mathematics teaching and learning. In this way he created his own pathway of development in response to the pedagogical problems he wanted to solve.

As a result of his professional reading, this teacher became influenced by the work of Paul Ernest on constructivism, and he began to reform the curriculum and teaching approaches in his school along constructivist lines. He had previously attended professional development workshops on the use of graphics calculators (which were, at the time, a new form of technology being introduced into secondary school mathematics). Initially he saw graphics calculators and other technology as being "interesting but not essential." However, when his teaching philosophy changed he realised that technology was a way of helping students to access concepts that would otherwise be beyond their understanding. In addition, at this time the use of graphics calculators and computers had been made mandatory in senior secondary mathematics curricula. As Head of the school's Mathematics Department he developed a new junior secondary curriculum incorporating manipulatives and digital technologies, with a blend of student-centred small group work followed by whole class teacher-led discussion. Other teachers were initially resistant to this new approach because it demanded more pedagogical flexibility than they were accustomed to using. However, they quickly became convinced of the benefits when they saw that students whom they thought incapable of learning could succeed when given appropriate tasks – many of which were technology-enriched – in contexts that encouraged dialogue and experimentation.

In this school the teacher began to develop a new identity by participating in new professional practices – those centred on both his own learning and his students' learning (Wenger, 1998). The institutional context was an important influence on his developmental trajectory, offering potential enablers and hindrances. For example, the university offered access to academic experts and research literature that "seeded" the teacher's thinking about constructivist pedagogies. Without this, the graphics calculator workshops in which he had participated would not have been seen as useful. His school context could have hindered his development due to lack of resources and teacher resistance, if not for the support he received from the Principal in initiating change. The development of new senior secondary mathematics curricula that mandated technology use gave the teacher another argument for introducing graphics calculators and computer applications in the junior secondary years. This example shows that institutions can have many, sometimes conflicting, and influences on teacher professional development. These influences are not static; instead, they interact with a teacher's search for professional development opportunities that align with his/her goals and problem solving needs. Conceiving of teacher development as identity formation makes it possible to trace out the dynamic, temporal dimension of professional learning.

This section has presented two brief examples of the evolving role of teachers in terms of their professional development activities with technologies. The examples have illustrated partial successes and failures, using different theoretical lenses. Together, however, they allow us to observe teacher change (or not) (i.e., the first key question posed by this Research Forum) and the role of institutions such as school curricula, professional development regimes, and societal expectations in supporting or hindering change (the second question for this RF). A related question that could be posed is whether the extent to which a teacher has mastered a mathematical digital tool supports them to transform the tool into a didactical professional instrument (see below). The relevance of this question is less evident in the second example than in the first, where teachers' mastery of virtual communication technologies came into play. Nevertheless, as both examples suggest, mastery of the digital tool is but one of many factors that may influence teachers' use of technologies in their classes.

As we saw in the earlier section, which considered institutional contexts, some research has stressed the importance of taking into account of the instrumental distance generated by the tool between the different praxeologies that are viable in the different environments, the new technological environment and the usual paper-pencil one. The necessary work to relate these praxeologies is part of the teacher's professional instrumental genesis. Using the frame of the Instrumental Approach, Haspekian (2011) uses Rabardel's notion of instrumental genesis and distinguishes personal from professional genesis by distinguishing two different instruments for the teacher. From a given artefact, the personal instrumental genesis leads to the construction and appropriation of an instrument for mathematical activity. From, or along with, the previous instrument, the professional instrumental genesis leads to the construction and the appropriation of a didactical instrument for mathematics teaching activity. Indeed, the teacher has to turn the digital tool into a didactical tool in order to serve her learning objectives. This task is non-trivial, even if teachers have been made aware of the digital tool's didactic potentialities, and even if didactic work in terms of situations has been already done. The situation becomes more complex when the digital tool is a non-educational one, encompassing a personal genesis and a professional genesis on the teacher's part. The relationship between personal and professional geneses is accentuated in the case of technologies that are not initially made for mathematics education and are, such as spreadsheets, imported into classrooms to teach mathematics. The case studied in (Haspekian, 2014), shows that teachers' personal and professional instrumental genesis cannot be independent and that this double instrumental genesis of the teacher can also interfere with the students' development.

TEACHERS' IMPLEMENTATION OF TECHNOLOGIES IN THEIR CLASSES

There is a substantial body of research on how particular teachers in particular settings have integrated particular technologies within their classroom settings (Hoyles & Lagrange, 2009). As has been previously stated, although the earlier studies used this context to research the mathematical outcomes from the students' perspectives, more recent studies have focused on the process of the teachers' development of the knowledge and classroom practices over time. This has led to a number of global and local theories that served both to explain particular classroom outcomes and to inform the development of professional development programmes and ongoing support through professional learning communities. This section focuses on three evolving approaches – the Technological Pedagogical Content Knowledge (*TPACK*), the Instrumental Approach (*IA*) and *MDT*. The question of technology is therefore tackled using concepts emanating either from an ergonomic approach (instrumental genesis as in *TPACK*), or from an anthropological approach (as didactic transposition in *MDT*), or from both (as in the *IA*). The section is exemplified by particular examples of individual teachers' trajectories. These individual stories provide insight into how the particular features and functionalities of the different digital mathematical tools impact upon teachers' motivation and confidence to integrate them into classroom teaching and how they respond to the challenges of task design involving mathematical digital technologies. They illustrate the use of different theories

TPACK (Mishra & Koehler, 2006; Koehler & Mishra, 2009), similarly, but complementary to Pedagogical Technology Knowledge (*PTK*) (Thomas & Hong, 2005; Hong & Thomas, 2006) has demonstrated merit in analysing factors related to the challenges that teachers face in using digital technology, and provides an indication of teacher readiness for implementation of technology use. A critical review of the *TPACK* frame, including an analysis of its affordances and constraints can be found in Graham (2011). While *TPACK* takes a more generic approach, *PTK* is mathematics focused, recognising that mathematics has its own important nuances of content knowledge, as exemplified in Ball and Bass's framework of mathematical knowledge for teaching (*MKT* – Hill & Ball, 2004). In turn, it places an emphasis on the epistemic value of the technology, how it can be used to produce knowledge of the (mathematical) object under study (Artigue, 2002; Heid et al., 2013). Both frameworks build on Shulman's pedagogical content knowledge by adding aspects of (digital) technology knowledge (*PCK* – Shulman, 1986). Here, *TPACK* articulates each of *PCK*, technological pedagogical knowledge (*TPK*) and technological content knowledge (*TCK*) and the relationships between them. In the framework, *TCK* involves an understanding of the manner in which technology and content influence and constrain each another, while *TPK* is an understanding of how teaching and learning can change when specific technologies are used in particular ways (Koehler & Mishra, 2009). The definition of technology knowledge (*TK*) used in *TPACK* to form the constructs of *TCK* and *TPK* is close to that of Fluency of Information Technology (FITness), as proposed by the Committee of Information Technology Literacy of the National Research Council (Koehler & Mishra, 2009, p. 64). Also, *PTK* includes the crucial element of the personal orientations of the teacher who is using the technology and their role in influencing goal setting and decision-making. Hence, it suggests teachers need to understand information technology broadly enough to apply it productively at work and in their everyday lives, to recognise when information technology can assist or impede the achievement of a goal, and continually to adapt to changes in information technology. In contrast, *PTK* highlights the principles, conventions, and techniques required to teach mathematics through the technology. This includes the need to be a proficient user of the technology, but more importantly, to understand the principles and techniques required to build and manage didactical situations incorporating it and enable mathematical learning through the technology.

Thus, *PTK* employs the theoretical base of instrumental genesis, with its explanation of how tools are converted into didactic instruments, while *TPACK* relates to "knowledge of the existence, components and capabilities of various technologies as they are used in teaching and learning settings, and conversely, knowing how teaching might change as a result of using particular technologies" (Mishra & Koehler, 2006, p. 1028). However, while there are differences in the frameworks it is clear that both provide useful conceptual lenses for analysing classroom practice, and should be viewed as complementary rather than competitive.

The *PTK* and *TPACK* frameworks suggest that the ability of a teacher to employ digital technology to construct and use tasks with epistemic value requires sound technological, pedagogical and content knowledge, along with positive orientations towards learning and teaching with technology, good *MKT* (Hill & Ball, 2004) and sound instrumental genesis. Thus, a teacher's perspective on the technology, their familiarity with it as a teaching tool, and their understanding of the mathematics and how to teach it are all crucial factors. A teacher with strong technological, pedagogical and content knowledge can understand the principles and techniques required to build didactical situations incorporating digital technology, comprising tasks that enable mathematical learning to emerge, mediated by the technology. We exemplify such knowledge here in the case of two secondary mathematics teachers.

The first case, reported fully in Thomas and Hong (2013), describes a teacher who had moved forward in the use of digital technology. In spite of over six years' experience of using graphic calculators (GC) in her teaching she admitted "Sometimes it's hard to see how to use it effectively so I don't use it as continuously as I should." Her confidence was, however, at a level where she had "… done some exploratory graphs lessons where students get more freedom to input functions and observe the plots." Thus, she was happy to loosen control of the students and let them explore the GC and help one another: "Students learn a lot by their own exploration…In past lessons I have never had a student get lost while using a graphics calculator. Sometimes friends around will assist someone."

She expressed a desire for her students to appreciate the challenge of the depth of mathematics: "The success for me as a teacher is when they want to learn more and students show a joy either in what they are doing or in challenging themselves and their teacher with more deeper or self-posed mathematical problems." She was convinced that the technology could be used to challenge and motivate students in this way "The calculator puts a radiant light in the class… With a graphics calculator lesson no one notices the time and no one packed up." An orientation, a belief, crucial to her pedagogical technology knowledge was related to the complementary roles of by hand and technology approaches. This was revealed through her comment that "Today we find a lot of maths does not need underlying understanding…I feel as teachers what we need to really be aware of is what the basics are that students must know manually… when we sit down to work with graphics calculators we need to consider carefully what still should be understood manually."

One of her lessons, with a class of 17 year-old students, considered families of functions with the aim of exploring exponential and hyperbolic graphs and noting some of their features, "we're going to utilise the calculator to show that main graph and then we're going to go through families of $y = 2^x$". Her pedagogical technology knowledge enabled her to direct them to link a second representation, "Another feature of the calculator I want you to be aware of..[pause] you've got also a list of x and y values already done for you in a table." Her instrumental genesis was such that she had moved away from giving explicit key press instructions, instead declaring "I want you to put these functions in and graph them and see what's going on." and "You can change the window if you want to see more detail, and if you want to see where it cuts the x-axis, you can use the "trace" function." A copy of her whiteboard working can be seen in Figure 2.

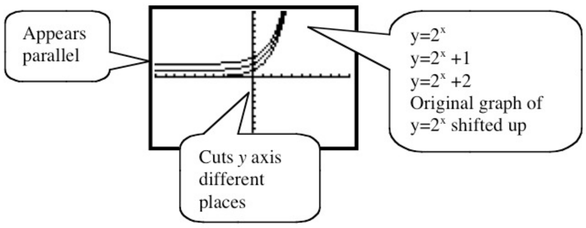

Figure 2: The teacher's whiteboard working (Thomas & Hong, 2013)

She was also moving towards an investigative mode of teaching "if you're not sure where the intercepts are, you can use the "trace" key, remember, and I want you to observe what is happening", encouraging students to use the GC in a predictive manner, to investigate a different family.

> We want to do some predictions⋯ Looking at the screen try to predict where 3 × 2*x* will go then press "*y* =⋯" and see if it went where you expected it to go. You may get a shock ⋯ Can you predict where "*y* = 4 × 2*x*" will be? Now you learned from that, so can you predict where it'll lie. The gap between them gets smaller. If you're interested put in "*y* = 100 × 2*x*". Does it go where you expect?

The epistemic value of the teaching was noticeable since mathematical concepts were a focus of attention. For example, she linked 2×2^x with 2^{x+1} and during an examination of the family of equations $y = 2^x$, $y = 2^{x+1}$, $y = 2^{x+2}$, said of $y=2^{x+1}$. "We expect this to shift 1 unit to the left [compared with 2^x]. Did it?" In this way she encouraged versatile thinking by linking with previous knowledge of translations of graphs parallel to the x-axis, and reinforced this with the comment that "With this family, when you look at the graph can you see that the distance between them stays the same because it's sliding along 1 unit at a time. The whole graph shifts along 1 unit at a time." This relationship between the functions is not so easily seen by students from the graphs and hence she linked to the algebraic expression and the foundation of a previously learned mathematical concept. In addition, there was a discussion of the relationship between the graphs in the family of $y =2^x + k$, and the relative sizes of 2^x and k.

> …as the exponential value gets larger, because we're adding a constant term that is quite small, it lands up becoming almost negligible. So, when…all they're differing by is the constant part, you'll find that they appear to come together. Do they actually equal the same values ever? Do they ever meet at a point? No, because of the difference by a constant, but because of the scaling we have, they appear to merge.

In summary, she had demonstrated good technological, pedagogical and content knowledge. Her technological knowledge had reached the point where she showed strong instrumentation and instrumentalisation of the technological tool. Thus, she was able to use the affordances of the technology (within acceptable constraints) to provide an epistemic focus on mathematical constructs. This included the idea of testing concepts against definitions, a strong emphasis on the crucial process of generalisation (Mason et al., 2005), and the use of student investigation to form and test conjectures. In addition, she had a high level of confidence in using the technology to teach mathematics and positive orientations, including a strong belief in the value of technology as a tool to learn mathematics.

Teachers' implementation of technology in their classes can also be studied at the local level of instrumental geneses using the Instrumental approach (*IA*). An example of interference of the teacher's double instrumental genesis and students' ones is given in the case of spreadsheet already mentioned in the sections before. In this study of the teacher integrating spreadsheet for algebraic learning, these relationships are constrained by:

- The mathematical knowledge aimed at (statistics, algebra, etc.)
- Pupils' instrumentation (that is how to make pupils work mathematics *through* spreadsheet, encompassing instrumental and mathematical knowledge, for example: frequency, dependence through the change of the value in the cell)
- Pupils' instrumentalisation (that is which functionalities, schemes of use are aimed at? For example: relative references, recopy, incrementation with the copy, but not absolute references, $ sign and its different behaviour in the copy)

Managing all these constraints at once is not easy as a spreadsheet is not automatically a didactical instrument, the case studied here shows that such an instrument is only progressively built along a complex professional-oriented genesis and that the professional and the personal geneses interfered one on the other.

For example, in preparing the task for pupils, the teacher modified her spreadsheet file 3 times! (See Figure 2.) In its 1st version, the formula calculating the frequency (in B7) was: =**B6/50*100**. This formula, if copied along line 7 is convenient for Q a) but not anymore for Q b) (The formula refers to the value 50 for the total. If one changes the value of any cell, then the total will change and the form becomes wrong)

The day before the lesson, the teacher realised the mistake and changed the formula into: =**B6/F6*100**. She confided she did not feel yet totally comfortable with spreadsheet. If her own instrumental genesis with spreadsheet-as-a *mathematical* instrument probably plays a role here, we also see that the key point of the problem comes from the spreadsheet-as-a *didactic-oriented* instrument. It is the didactical aim (showing the mathematical dependency between the numbers and the frequencies) that led the teacher to add Qb) and make pupils change the number in C6, which turned wrong the formula. She did not realise it when she built first her formula. At that moment, the personal instrument stands at the front of the scene, and covers up the professional and its didactical aims (the Qb.).

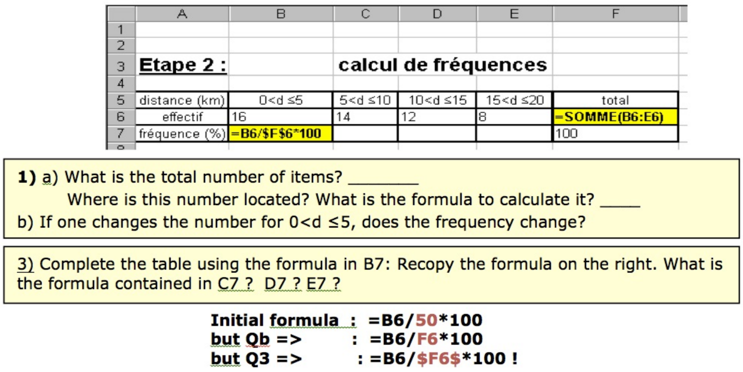

Figure 3: The teacher's spreadsheet and accompanying worksheet (Haspekian 2011)

In this example, the teacher's spreadsheet session has been disturbed because the teacher wanted to avoid mentioning the $ sign to the pupils, but it came out during the session! Facing pupils' questions, she was compelled to explain but she just said that it is not important to write it in paper-pencil. This link with the paper-pencil work is a strong preoccupation for teachers and is precisely linked to the instrumental distance generated by the tool evoked within Section 1.

The final example of implementation of technologies in the classroom is analysed in term of *MDT* in the context of the European project EdUmatics (Aldon et al., 2013b). In France, a high school teacher (called Jean in the following) worked in collaboration with the French Institute of Education (ENS de Lyon) and the Italian team in Turin (made of researchers and teachers). The purpose of the project was to develop professional development activities for teachers of mathematics in Europe. It was therefore necessary, to transform classroom situations into training situations. Jean's role was to adapt and analyse a mathematical task for students that had been created by the colleagues from Turin. Jean said, "All of this work led me to reflect on professional actions from a training perspective. This reflection is of course beneficial for my own training! […] The preparation work was often meticulous, observing the influence of gestures or seemingly innocuous words in the course of a session, which helped me to improve my classroom management. This experience has allowed me to build some of my pedagogical beliefs, including the conclusion that the exchange and mutual building of knowledge with students may be preferable to a lecture" (EducTice-Info 2, 2012). In this case, and with reference to the *MDT* framework, the implementation of technologies within the classroom is the result of the evolution of praxeologies taking into account both the point of view of the research and the point of view of teacher professional development. The task, designed in another institutional context, find its justification in the French institutional context because of the *a priori* analysis leading to a shared praxeology by researchers and teachers.

This section has presented detailed examples of analyses with three different lenses of teachers' implementation of technologies in their classes. These different lenses provide tools to analyse on one part the evolution of practices of mathematics teachers with technologies; on the other part the impact of this use of digital tools upon teacher's professional development, which were two of our key questions in this Research Forum.

The lenses are different but complementary For instance the evolutions of the two teachers in the first example (*TPACK*) can be complementary tackled with the tools of the Instrumental approach. In the case of the first teacher, it has been said that her instrumental genesis moved from giving explicit key press instructions to a more exploratory mode ("put functions in and graph them and see what's going on"). In fact, the genesis at stake here is that of the *professional instrumental genesis* because it is the GC as a *didactic* tool that is being progressively built here, not the GC as a personal tool for the teacher (calculating or plotting). This evolution implies constitution of schemes of instrumented action as the one described in this example 'moving from key press instructions to open questions; moving "towards an investigative mode of teaching", "encouraging students to use the GC in a predictive manner"…).

Last but not least, the implementation of technology in classroom also poses the question of its link with educational programs for teachers. Thus, related questions to this section could be: To what extent can teachers develop individual agency in the face of institutional constraints, and what role can researchers play in this process? and How can researchers impact on the institutions in the planning of large scale professional development programs?

META-LEVEL REFLECTIONS BY TEACHERS AND RESEARCHERS IN THE PROCESS OF PROFESSIONAL DEVELOPMENT FACILITATED BY THE USE OF DIGITAL TECHNOLOGIES

Teachers involved in the different activities which characterise the process of professional development, according to the different roles they could play (as teacher-researchers, or trainers, or ordinary teachers), may reflect on their activity and evolve in their praxeologies over time, if motivated to the importance of that, and if helped by researchers.

The meta-level reflections that teachers and researchers can carry out are part of the professional development as a whole process. Moreover, there are recent studies that actually highlight that involving teachers in reflective practices where classroom dynamics are object of a careful scrutiny, enables the teachers' deep beliefs emerge, so that the reconstruction of a new identity for the teacher becomes possible (see for instance Goos, 2013; Jaworski, 2012).

The notion of *double dialectic*, a component of the *MDT* model (Aldon et al., 2013a; Arzarello et al., 2014), could enable to introduce and analyze this dimension. This construct has been conceived to highlight a typical feature which characterizes those teacher education programs that are based on the study of the teachers' practice: the engendering of dynamics which enable the teachers develop an awareness about their role during classroom activities and also possible gaps between their knowledge and beliefs and their classroom actions.

The double dialectic encapsulates two interrelated processes: (1) a *first dialectic*, which is at the *didactic level* in the classroom, between the personal meanings that students attach to a didactic situation to which they are exposed and its scientific, shared sense; (2) a *second dialectic*, which is at the *meta-didactic* level, between the interpretation that the teachers give to the first dialectic according to their praxeologies and the meaning that the first dialectic has according to the community of researchers, which results from researcher praxeologies. It is through this double dialectic that, thanks to the constitution of a shared praxeology, a significant evolution of teacher professional competences could be fostered. The use of digital technologies as tools to promote teachers' reflections on the educational processes in which they are involved (Hegedus & Moreno-Armella, 2009) further facilitate the engendering of this double dialectic.

The first example is therefore related to the use of technologies as tools for teachers to communicate and interact with researchers and mentors. It proposes possible activities that could activate this double-level process: those connected to the *Multi-commented transcripts* methodology, developed within the *ArAl Project*. The *ArAl Project* (ArAl is an acronym for "Arithmetic and Algebra") is aimed at proposing a linguistic and constructive approach to early algebra starting from primary school or even kindergarten and is also meant to constitute an integrated teacher education program (Malara & Navarra, 2003; Cusi et al., 2010). The *Multi-commented transcripts* are the results of a complex activity of critical analysis of the transcripts of audio-recordings of classroom processes and associated reflections developed by groups of teachers and researchers involved in the same teaching experiment within the *ArAl Project*. The teachers who experiment the project activities in their classes send the transcripts, together with their own comments and reflections, to mentors-researchers, who make their own comments and send them back to the authors, to other teachers involved in similar activities, and sometimes to other researchers. Often, both teachers and researchers make further interventions in this cycle, commenting on comments or inserting new ones. This process, which is carried out through email exchanges, is characterized by a sort of choral web participation because of the intensive exchanges via e-mail, which contribute to the fruitfulness of the reflections emerging from the different comments. These activities have been conceived, in a perspective of lifelong learning, starting from the hypothesis that involving teachers in the critical-reflective study of teaching-learning processes, to be developed within communities of inquiry (Jaworski, 2003), could enable their development of awareness about the "subtle sensitivities" (Mason, 1998, 2008) that could guide their future choices and determine their effective action in the classroom. Through the Multi-commented transcripts teachers have the possibility to become aware of: (1) the contrast/interaction between the personal sense their students attribute to class activities and the institutional meaning of both the same activities and the mathematical concepts involved (*first-level dialectic*); (2) the possible different interpretations, given by teachers and researchers, of the dynamics activated during class activities (*second-level dialectic*). The tension developed as a result of this double-level dialectic fosters the development of new teachers' praxeologies, related both to the roles they should activate in their classrooms and to the ways of pursuing their professional development.

Digital technologies have enabled an evolution of the Multi-commented transcripts. The initial activation of the ArAl Project official website (www.aralweb.unimore.it) and the recent activation of a work-in-progress blog (http://progettoaral.wordpress.com) have, indeed, created "virtual places" where teachers can find clarification and further materials on mathematical, linguistic, psychological, socio-pedagogical, and methodological-didactical issues and also prototypes of didactical sequences aimed at giving them a stimulus for their own elaboration of teaching processes. The blog, in particular, is a source of information for all the teachers who are interested in classroom innovation and, therefore, a "place" where the dialogical comparison typical of the ArAl Project can further develop.

The evolution of the Multi-commented transcripts, the *Web Multi-commented transcripts*, are interactive PDF-files conceived as learning tools to enable the reader to develop an in-depth analysis of the presented activities, through web-links to both the website and the blog that highlight: (a) specific theoretical terms used in the teachers and researchers' comments; (b) contents related to theoretical, methodological and disciplinary aspects; (c) some FAQ, possible answers aimed at clarifying important aspects often highlighted by many teachers involved in the project through their comments.

The methodology of Multi-commented transcripts have therefore evolved from professional development tools for the teacher's own reflections to tools to be shared within the whole community of teachers and researchers, specifically conceived to be used as formative web objects to mediate theoretical aspects and classroom practice. The Web Multi-commented transcripts are therefore examples of how technologies as communication infrastructures can impact and strengthen specific tools conceived for teacher education, enabling also to highlight the role played by teachers as protagonists of their own professional development, through the interaction between their "voices" and researchers' voices in educational programmes.

A second example comes from the *Cornerstone Maths* Project (Hoyles et al., 2013; Clark-Wilson et al., in press), which has developed a *curriculum activity system* (Vahey et al., 2013), comprising curriculum teaching units with integrated dynamic software and accompanying professional development and community support for selected mathematical topics in lower secondary mathematics education. This national project is researching the design and impact of the introduction of dynamic mathematical technologies at scale, with over 230 teachers and 6000 students currently involved. In this case, the participating teachers are introduced to the online community during face-to-face professional development and encouraged to continue to use the fora to discuss ongoing aspects of their developing classroom practices, share their lesson adaptations and reflect upon their students' learning outcomes. Just over two thirds of the teachers who have completed their teaching of the first Cornerstone Maths unit of work on linear functions (n=78) reported that they had made use of the Forum beyond the initial face-to-face PD. They cited the following uses for the community: to keep up to date with the project news (n=28); to read questions and comments by the community (n=45); to post questions or comments to the community (n=11); to access electronic copies of the pupil workbook and teacher guide (n=17); upload resources they had created to share with the community (n=3) and to download resources created by others (n=3). Although this is early data from the project, and further analysis of the qualitative data contained within the community's written exchanges will reveal the nature of its role within teachers' professional learning trajectories, the results do justify the creation of such online communities in both establishing a professional community and in enabling ongoing professional discourse for the project's participants.

CONCLUSIONS AND PERSPECTIVES ON THE FUTURE RESEARCH

Teaching mathematics with digital technologies is a challenge that teachers in different countries have faced to differing degrees both individually and as practitioner communities. In this paper we have described how research can approach this theme and describe it from different perspectives, which are more or less integrated. Our approach has been to consider the different aspects of teaching within both professional development – to include the design of activities by researchers and/or teachers that are mindful of the institutional considerations – and within the classroom. These aspects are contextualised within the process, which often pass through face-to-face or distance learning phases, through to the post-implementation reflections on those activities. In this way, the term professional development is intended in a wide sense, and is seen as a process involving many actors (researchers, trainers, teacher-researchers, teachers as learners and as teachers in their classes, mentors). In this process, all the actors may change their ideas and approaches to the use of technologies, and so their praxeologies may evolve, thanks to their dialogical interaction with the other actors.

In this paper we have highlighted the elements that signal this evolution, both from the point of view of research and of teaching practices. The selected frameworks were used in a co-working approach to describe the variables within the professional development process concerning teaching mathematics with technologies. The various frameworks highlighted the description of the professional development in different ways, taking into account the activity of teachers, the institutional aspects and the relationships within professional development settings. For example, *MTD* accounted for the dynamic aspects of PD and allowed us to consider both the perspectives of both teachers and of researchers (or teacher trainers) in a joint action. By contrast, *PTK* revealed a picture to help us understand teachers' classroom practices and the relationships between teachers orientations and the possible use of technology. Instrumental genesis, combined with an analysis of pedagogical and technological knowledge, provide tools that give a clear description of mathematical constructs with technology and enable us to tackle the complexities of the dynamic process of the related instrumental geneses (personal/professional or students'/ teachers' ones).

Such frameworks give tools to highlight the important evolution of teachers' professional learning about using technology in mathematics lessons and enable us to capture the importance of the didactical and pedagogical aspects, which are linked to the constraints and potential of technology, They also point the importance of the role of institutions. For example, the *MTD* model and the Instrumental Approach, are both connect to Chevallard's ATD through the notions of institution, didactic transposition, or praxeologies. Thus, the variety of theories mentioned here share the common point of having this "sensitivity" to contextual and individual factors that may account for the evolving role of the teacher in technology-enriched mathematics teaching. Moreover, teachers' personal beliefs about what represents a good teaching are situated in specific institutional cultures (Goos, 2014).

Examples that have been developed in the text also show, if needed, the importance of the phase of designing a lesson when using technology as well as the accompanying role of research. Researchers and teachers, when working together, provide examples of possible evolutions of teaching practices with technology that takes on a share of professional development advancement, particularly by developing meta-levels of reflections on both the educational processes and the results of these processes.

Evolution of practices and innovation have been observed and described through theoretical frameworks that allow understanding the use of digital technological tools for teachers and the associated professional development. A large scale mathematics teachers' professional development and its institutional implications need to be developed in future researches, leaning on the first results described in this paper. Particularly, evolution of practices and innovation within mathematics teaching may be accompanied by strong researches giving evidence that would be usable in the designing of PD sessions.

Finally, we acknowledge that it was not possible to be exhaustive in our coverage of research perspectives and approaches that exist for analysing technology mediated teaching. For example, the *Documentational Approach* (Gueudet & Trouche, 2009) and *Semiotic Mediation* (Bartolini Bussi & Mariotti, 2008). However we expect the Research Forum to provide the opportunity to discuss and debate other related theories.

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MATHEMATICAL TASKS AND THE STUDENT

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Mathematics Education has at its core a conception of the mathematical performances that represent the aspirations of the mathematics classroom and curriculum. These performances are constituted through teacher and student participation in the activities stimulated by mathematical tasks selected by the teacher for the realization of an instructional purpose. In this nexus of activity, intention, interpretation and consequence, the mathematical task occupies a central place. This Research Forum provides an opportunity to explore and reflect upon the role that mathematical tasks play in the achievement of the goals of the international mathematics education community. Further, consistent with current curricular and theoretical priorities, the agency, attributes and activities of the student are foregrounded in the discussion of the instructional use of mathematical tasks. The contributors to this Research Forum represent a wide variety of theoretical perspectives and report research undertaken in different school systems and different cultures. These different perspectives offer a useful exploration of the theme: Mathematical Tasks and the Student.

RATIONALE

Attempts to model the complexity of the mathematics classroom have generated increased interest in theories capable of accommodating consideration of artifacts[1] as well as individuals. Theories such as Activity Theory (Engeström, 1987) and Distributed Cognition (Hutchins, 1995) foreground the mediational role of artifacts in facilitating learning, and locate tasks among those mediating artifacts.

> Mediating artifacts might be mathematics textbooks, digital technologies, as well as tasks and problems, [and] language. (Rezat & Strässer, 2012)

Rezat and Strässer (2012) identify the students' mathematics-related activity as an example of the Vygotskian conception of an instrumental act, where the student's interaction with mathematics is mediated by artifacts, such as mathematical tasks. Most importantly, recognizing the function of mathematical tasks as tools for the facilitation of student learning leads us to the further recognition that (à la Vygotsky) the use of a tool (i.e. a task) fundamentally affects the nature of the facilitated activity

[1] Either artifact or artefact are acceptable spellings to denote "arte factum" (Latin) as something made through the use of skill. We have employed Rezat and Strässer's (2012) spelling in this proposal, which also corresponds to North American usage.

(i.e. student learning). Rezat and Strässer (2012) have re-conceptualized the familiar didactical triangle (teacher-student-mathematics) as a socio-didactical tetrahedron, where the vertices are teacher, student, mathematics and mediating artifacts. This reconception of didactical relationships recognizes that the connections represented by the sides of the original didactical triangle require mediation. The vehicles of this mediation are artifacts, which include everything from textbooks and IT tools to tasks and language. Use of the socio-didactical tetrahedron provides us with an important tool by which to give recognition to the mediational role of tasks in the teaching and learning of mathematics.

One virtue of the socio-didactical tetrahedron is that it facilitates the separate consideration of the triangles forming each face of the tetrahedron and the vertices of each of those triangles. In this Research Forum, we focus attention on the task as mediating artifact and address the question of how the resultant socio-didactical tetrahedron (Fig. 1) might structure our consideration of research into the function of tasks in facilitating student learning and into the dynamic between student and task.

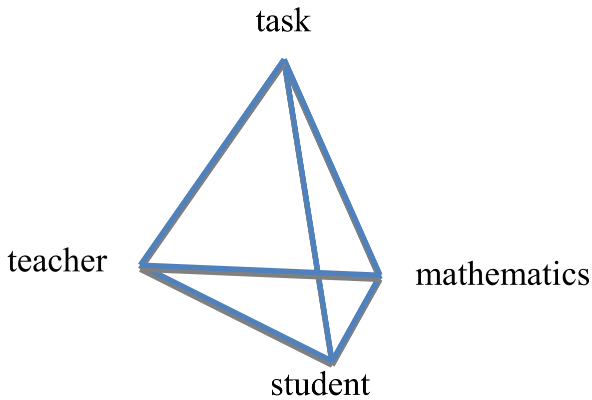

Figure 1: The socio-didactical tetrahedron (Rezat & Strässer, 2012)

To paraphrase Rezat and Strässer (2012, p. 645): Each of the triangular faces of the tetrahedron stands for a particular perspective on the role of tasks within mathematics education: the didactical role of the teacher is best described as an orchestrator of student mathematical activity as represented by the triangle *teacher-task-student* (Face A); the triangle *student-task-mathematics* represents the student's task-mediated activity of learning mathematics (Face B); the triangle *teacher-task-mathematics* depicts the teacher's task-mediated activity of representing mathematics in an instructional setting (Face C); the original didactical triangle constitutes the base of the model (i.e. *student-teacher-mathematics*) (Face D). The tetrahedral structure offers an important representation of the complexity of classroom teaching/learning that affords a level of detailed reflection on the didactical role of tasks. In utilizing this more complex conception of the instructional use of mathematical tasks, significant agency is accorded to *each* component (student, teacher, mathematics and task) in the determination of the actions and outcomes that find their nexus in the social situation for which the task provides the pretext.

Research into the design and use of mathematical tasks in instructional settings must accommodate student intentions, actions and interpretations to at least the same extent as those of the teacher. Research in this area is important, but fragmented. This Research Forum brings a variety of research studies together into a discussion intended to yield a more coherent picture and has been designed to assist in structuring the field of task-related research and to equip researchers to better situate the student within research on instructional task design.

Goals framing the Research Forum:

- (i) To present research into the instructional use of mathematical tasks, with a specific focus on the associated student activity and the implications for task design, classroom practice and the mathematics curriculum internationally;
- (ii) To focus attention specifically on the agency of the student during the completion of mathematical tasks in educational settings and examine the performative expression of this agency in different settings and in response to different task types;
- (iii) To highlight, through the reporting of selected research studies, particular issues associated with the instructional use of mathematical tasks, including: teacher intentionality, student interpretation, implicit and actual task contexts, considerations of task sequence, and the distinction between the stated task and its realization as a social activity involving teacher and students;
- (iv) To bring together researchers from a variety of countries, who share an interest in both the instructional use of mathematical tasks and the intended and resultant student activity;
- (v) To draw to the attention of PME members some of the issues associated with the instructional use of mathematical tasks, particularly those arising from the assumptions implicit in different instructional theories, which may conceive the instructional purposes of mathematical tasks and optimal student activity very differently.

The Research Forum has been structured around the following issues:

- (i) Differences in the instructional deployment and function of mathematical tasks and the nature of student task participation in different instructional settings;
- (ii) Utilizing mathematical tasks to promote higher order thinking skills;
- (iii) Differences in the theoretical frameworks by which the instructional use of mathematical tasks might be better understood (particularly from the perspective of the student) and thereby optimized;
- (iv) The accommodation of student agency within the instructional use of mathematical tasks.

Each issue can be usefully addressed in the form of a question.

Focus Question 1. What are the possible functions of a mathematical task in different instructional settings and how do these functions prescribe the nature of student task participation?

Focus Question 2. What contingencies affect the effectiveness of a mathematical task as a tool for promoting student higher order thinking skills?

Focus Question 3. How might we best theorize and research the learning processes and outcomes arising from the instructional use of any mathematical task or sequence of tasks from the perspective of the student?

Focus Question 4. What differences exist in the degree of agency accorded to students in the completion of different mathematical tasks and with what consequences?

The sequencing of the forum contributions constitutes a research narrative aligned with the issues listed above and structured by the socio-didactical tetrahedron already discussed. It is the construction of structure within substantial research diversity that provides a key motivation for this Research Forum.

ISSUE ONE: DIFFERENCES IN THE FUNCTION OF MATHEMATICAL TASKS AND THE NATURE OF STUDENT TASK PARTICIPATION IN DIFFERENT INSTRUCTIONAL SETTINGS

1: MAKING DISTINCTIONS IN TASK DESIGN AND STUDENT ACTIVITY

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The design principles below have developed during the time of our collaboration, over a period of fifteen years (e.g., Brown & Coles, 1997). The principles are drawn both from the enactivist theory of cognition and learning (Varela, Thompson, & Rosch, 1991) and the pedagogic ideas of Gattegno (1987). We developed these principles within a community centred around one school (School S) in the Bristol area of the UK. Laurinda made this school her main research site and visited, where possible, weekly. We focus on one particular community in the spirit of ‘particularization’ (Krainer, 2011, p. 52), to draw out general principles from an in-depth study of one case. Our data comes from transcripts of video recordings of lessons as well as the scheme of work of School S.

We believe task design that centres around activities that provoke differences in student response can allow the opportunity for students to make mathematical distinctions and for teachers to introduce new skills. Our task design principles are:

- starting with a closed activity (which may involve teaching a new skill).
- considering at least two contrasting examples (where possible, images) and collecting responses on a ‘common board’.
- asking students to comment on what is the same or different about contrasting examples and/or to pose questions.

- having an open-ended challenge prepared in case no questions are forthcoming.
- introducing language and notation arising from student distinctions.
- opportunities for students to spot patterns, make conjectures and work on proving them (hence involving generalising and algebra).
- opportunities for the teacher to teach further new skills and for students to practice skills in different contexts.

Our data analysis indicates these design principles operate to inform: (1) teacher planning, (2) teaching actions in the classroom and (3) students' mathematical activity. Firstly, the principles inform teacher planning. For example, the offer of contrasting examples (principle 2) can be used to focus students on mathematical distinctions, from which questions and challenges can be generated that provoke further work with that distinction. Secondly, we have evidence from video recordings that, over time, our design principles inform teacher actions in the classroom. In particular, the principles seemed to support teachers in School S adapting tasks in the light of student responses. Thirdly, there is evidence from transcripts that the principles can inform (implicitly) student actions in the mathematics classroom; through making distinctions, students notice and extend patterns, they ask questions and generalize (principle 6).

There is a significant problem, identified in the literature, around the student experience of tasks compared to the intentions of the designer or teacher (Watson & Mason, 2007). Mason, Graham and Johnston-Wilder (2005, p. 131) raise the issue of how an expert's awarenesses get translated into instructions for the learner that do not lead to those same awarenesses.

Our results indicate that the making of distinctions within mathematics can become a habit and a normal way of engaging in tasks for students. Creating opportunities for students to make distinctions within mathematics can also become a habit for teachers and a normal way of both planning activity and informing decisions in the classroom. When this happens, there is a convergence of planned and actual activity. With a focus on distinctions, there is a potential route out of the problems highlighted by Mason et al. (2005) around the divergence of teacher intention and student activity. With a focus on distinctions, the expert (teacher) can plan, initially via the choice of examples, to support students in making the same distinctions as a mathematician, leading to the same awarenesses.

2: ORDER OF TASKS IN SEQUENCES OF EARLY ALGEBRA[2]

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[1]Barcelona University, [2]Institute of Education, University of Minho

Rationale

It is assumed that epistemic and cognitive aspects are fundamental to build sequences of tasks. We investigated different aspects that appear when we analyse the process as a teaching experiment and examined how teacher intentions evolved according to interactional and ecological suitability.

Our research focused on student-related aspects influencing the ordering of tasks and how student responses are accommodated, using the case of early algebra. It is well known that structured investigative activities provide opportunities for meaningful learning of mathematical concepts. We consider task design as a crucial element of the learning environment, and describe a teaching experiment in which class discussion introduces unexpected new perspectives to an initial a priori instructional scheme. Our perspective relates to Realistic Mathematics Education, where the designer conducts anticipatory thought experiments by envisioning both how proposed instructional activities might be realized in the classroom, and what students might learn as they engage in them.

Framework and Methodology

It is important for our design process, a task analysis, to identify difficulty factors providing frameworks for hypothetical designs inspired initially by developmental cognition according to levels of abstraction. We decided to choose an early algebra task as the basis for a situated study supporting the perspective in which algebraic reasoning could be strongly promoted as a tool intertwined with arithmetic building through their interconnection in order to promote success by developing both arithmetic and algebra together, one implicated in the development of the other (Smith, 2011). The study supporting this paper has been done with two classes of 8-9 years old students. The basis for building our sequence of tasks and test analysis was to promote algebraic thinking by overcoming relational apprehension and the use of patterns in connection with a search for order or structure. Therefore regularity, repetition and symmetry are frequently present because of their relevance to the development of abstraction, generalization and the establishment of relations. Next step concerns the experimental task design process based upon a refined sequence of tasks. The principles for our task design are the following: (1) ensure the possibility of using arithmetic number sense related to algebraic reasoning; (2) apply suitability criteria for analysing mathematical activities; (3) use mathematical examples, using relations and diversity of representations but not letters for the unknowns; (4) prioritise the voice of the students for analyzing and promoting mathematisation and retention. The tasks

[2] Work partially funded by Ministry of Economy & Competitivity of Spain. EDU2012-32644.

were meant to be diverse, some leading to an exploratory and investigative open activity to improve meaningful construction. In our study, we considered one class solving 6 sequential tasks and then six structural tasks and another class solving six structural tasks and then six sequential tasks (Palhares, Giménez, & Vieira, 2013). A typical sequential task would ask the student to "Observe carefully the sequence of numbers: 6, 10, 14, 18, 22, . . . What will be the 20th term of the sequence? . . . Explain how you found the 20th term of the sequence. Will the number 63 be part of this sequence of numbers? Justify your answer." A typical structural task would ask students to "Observe carefully the four 'number machines' (shown below). Replace the question mark with a number that follows the rule of the other three machines."

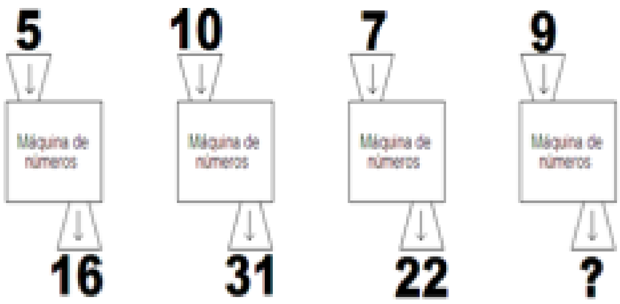

Figure 1

The research design focused directly on the consequences of task sequence.

Results and Final Comments

Statistical results show that there are significant differences starting with sequential or with structural tasks. Sequential tasks are better for starters and apparently provide a solid foundation for the work with structural tasks. The study is a first step for reconsidering the tasks for the next redesign stage in which a new cycle of testing could lead to small or big changes in task sequence. It is clear that students who started with the sequential tasks seemed to be capable of establishing broad generalizations, when the other group could not. These findings argue for redesigning in terms of stability and improving connectivity in self-regulation processes as synthesis activities. Also, the group that started with sequential tasks appeared to retain their performance more robustly as stable across time. The experiment did not consider any modelling situations from the real world. We assume that this would improve and enrich not only structural, but sequential examples in providing students with new learning experiences.

3: TASKS TO PROMOTE HOLISTIC FLEXIBLE REASONING ABOUT SIMPLE ADDITIVE STRUCTURES

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Our team is conducting a 3-year research project funded by the Quebec Ministry of Education on additive problem solving in early grades of elementary school. The goals of the project are: 1) to develop a pedagogical approach that would promote holistic and flexible reasoning about simple additive structures; 2) to design and test a set of tasks and didactical scenarios that implements the new approach; 3) to propose a related teacher professional development program. Our research team consists of two researchers (Savard and Freiman), a designer (Polotskaia), and a school board consultant responsible for the teachers' professional development (Gervais). We want to support teachers to guide their students on solving additive structures problems.

There are two paradigms in which additive problem solving can be seen. The *Operational Paradigm* puts the focus on addition and subtraction as arithmetic operations. From this position, additive word problems can be seen as exercises where the knowledge about arithmetic operations can be applied or further developed. Contemporary research (Thevenot, 2010) shows that some problems are particularly difficult because they require a flexible and holistic analysis of their mathematical structure while easy problems do not require such analysis.

The *Relational Paradigm*, appears in the work of Davydov (1982) and more recent studies (Iannece, Mellone, & Tortora, 2009). According to Davydov (1982), the concept of additive relationship is, "the law of composition by which the relation between two elements determines a unique third element as a function" (p. 229). Davydov (1982) advanced the premise that an adequate understanding of the additive relationship is the basis for the learning of addition and subtraction and should be taught prior to calculation. The analysis of the additive relationships present in the situation yields the following task design principles:

1. The task should be based on a situation involving a simple additive relationship between three quantities.
2. The task should involve students in the mathematical analysis of the described relationship as a whole. It should help students to discover different properties of the relationship, and to see how different arithmetic operations can be used in the described situation for different purposes.
3. The task should use a socio-cultural context in which students can identify themselves as active agents.
4. The task should not contain any explicit and immediate questions that could be answered by finding one particular number. This criterion is to prevent students from immediately calculating the answer. However, the task should include an

intriguing element, which would support students' natural interest and commitment.
5. The goal of the task, which is learning to analyze the situation, should be explicitly communicated to students.
6. The text of the task should be very short and should contain simple words and expressions that the students are familiar with.
7. The mathematical discussion of the situation should integrate appropriate graphical representations as a method of analysis.

We provide here one example of the task that we named 360° *situation* to highlight the main goal – holistic analysis of the mathematical structure of the situation. This is an example of a text proposed to students.

> Peter, Gabriel and Daniel are playing marbles. Peter says, "I have 5 marbles." Gabriel says, "I have 8 marbles." Daniel says, "Peter has 4 marbles less than Gabriel".

We introduce this text as a strange situation or as a situation where one of the persons made a mistake. Students are invited to explain why the text is unrealistic and how it can be corrected considering different quantities involved. The objective of the first is to make explicit the fact that all three quantities are related to each other and that the choice of two values implies one (and only one) third value. At the next step, we invite students to construct a graphical representation, which can support discovering of the appropriate arithmetic operations. Each quantity should be evaluated to figure out a correct numeric value in the condition where the other two quantities are fixed. At this step, the formal use of arithmetic operations can be discussed. Finally, the numbers in the text can be replaced with different ones to further generalise the initially discussed quantitative relations. This will complete the 360° tour around the situation.

The teachers we worked with had a tendency to return to the traditional teaching behaviours as soon as they start to work with traditional problems. For example, once the numerical answer was found for the problem, the discussion of the problem often ended abruptly. Thus, the focus of the activity was often shifted towards the use of the correct representation or the calculation of the numerical answer. A one year follow-up provided for each teacher-participant was needed for a sustainable change in teaching habits.

ISSUE TWO: UTILIZING MATHEMATICAL TASKS TO PROMOTE STUDENTS' HIGHER ORDER THINKING SKILLS

4: HYBRID TASKS: PROMOTING STUDENT STATISTICAL THINKING AND CRITICAL THINKING THROUGH THE SAME MATHEMATICAL ACTIVITIES

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[1]Beit Berl College, Israel, [2]Ludwigsburg University of Education, Germany, [3]University of Melbourne, Australia

In a well-known definition of Statistical Literacy by Gal (2004), a "critical stance" is included among the key attitudes for successful statistical thinking (ST) – hence, Gal includes such attitudes in his definition of statistical literacy. However, being critical in statistical contexts is not only an attitude. It is possible to describe specific abilities that have to be used in order to critically evaluate statistical data. Two key concepts or overarching ideas in statistical thinking relevant for a critical evaluation of data are *manipulation of data by reduction and dealing with statistical variation*.

Critical thinking (CT) skills rely on self-regulation of the thinking processes, construction of meaning, and detection of patterns in supposedly disorganized structures (Ennis, 1989). Critical thinking tends to be complex and requires the use of multiple, sometimes mutually contradictory criteria, and frequently concludes with uncertainty. This description of CT already suggests links with ST, such as dealing with uncertainty, contradictions and a critical evaluation of given claims. Dealing critically with information – a crucial aspect for both domains – demands critical/evaluative thinking based on rational thinking processes and decisions (Aizikovitsh-Udi, 2012).

In order to explore thinking processes related to tasks in the domains of both Statistical Thinking and Critical Thinking, individual semi-structured interviews were conducted with mathematics teachers. By using mathematics teachers as subjects, basic content competence can be assumed and it becomes possible to examine their content-related higher order thinking skills, both in terms of statistical thinking and critical thinking. The interviews focused on thinking-aloud when solving tasks and each lasted about 40–50 minutes. Figure 1 shows a sample task.

Looking at both CT and ST, the interviews appeared to highlight how elements of CT can contribute to ST, for example when evaluating data, its presentation and analysis, planning data collection, etc. Conversely, aspects of ST like dealing with statistical variation and uncertainty were shown to contribute to CT, especially when it comes to decisions in non-determinist situations, where full data is unavailable. This study has shown that both ST and CT skills can be evoked by the same task. We suggest that this models authentic and useful thinking practice more effectively than a more closed task that stimulated only statistical thinking and the application of taught procedures.

Connections clearly exist between Statistical Thinking and Critical Thinking at the level of individual reasoning practices. We suggest that an instructional program of hybrid tasks could provide the opportunity to employ Statistical Thinking, while simultaneously introducing students to the practices and structure of Critical Thinking.

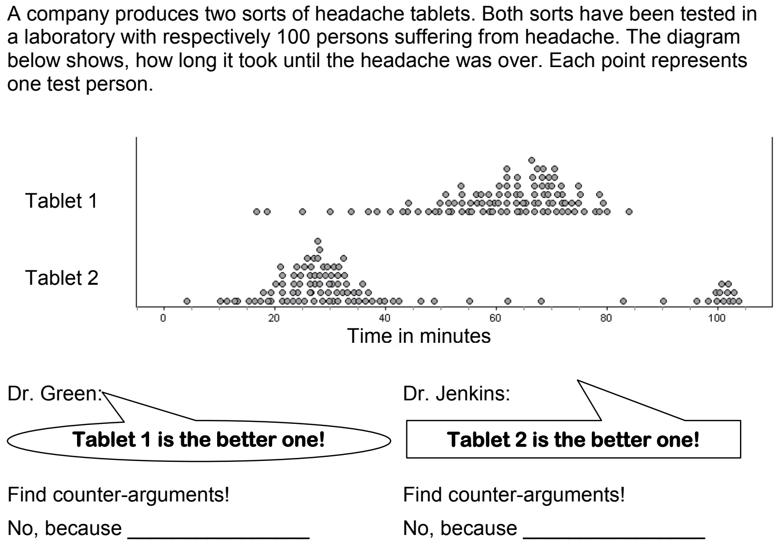

Figure 1: Task "tablets" (Kuntze, Lindmeier, & Reiss, 2008)

5: DESIGNING COVARIATION TASKS TO SUPPORT STUDENTS' REASONING ABOUT QUANTITIES INVOLVED IN RATE OF CHANGE

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Researchers using mathematical tasks involving dynamic representations of covarying quantities have supported secondary students' forming and interpreting relationships between changing quantities (e.g., Johnson, 2012; Saldanha & Thompson, 1998). Taking into account students' emergent conceptions of rates of change, the design of this covariation task sequence provided opportunities for students to use non-numerical quantitative reasoning in situations involving constant and varying rates of change. By covariation tasks, I mean tasks that involve forming and interpreting relationships between changing quantities.

Adapting the bottle problem to design covariation tasks

I designed covaration tasks by adapting Thompson, Byerly, and Hatfield's (2013) version of the well-known bottle problem (see Figure 1).

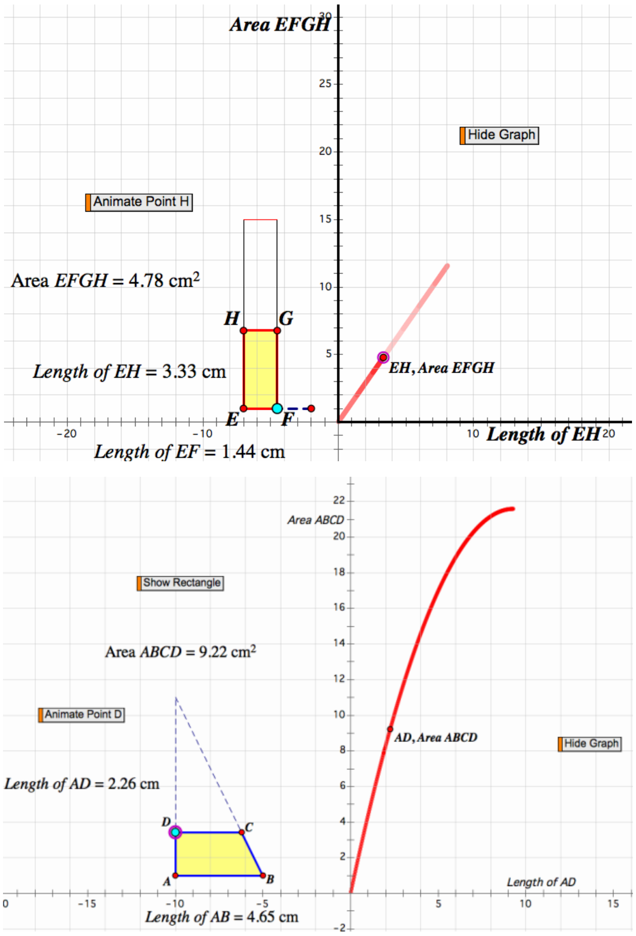

Figure 1: Filling Rectangle and Filling Triangle Sketches

The adaptation for middle school students resulted in a sequence of tasks. To accompany each task, I developed dynamic sketches linking a rectangle or right triangle "filling" with area to a graph representing shaded ("filled") area as a function of height (Figure 1). Students could vary the height of the rectangle or triangle by animating or dragging points H (Figure 1, top) or D (Figure 1, bottom), respectively, then predict and create a corresponding graph representing shaded area as a function of height. Additionally, students could drag point F (Figure1, top) to vary the width of the rectangle. Anticipating that students might interpret linked graphs iconically (Leinhardt, Zaslavsky, & Stein, 1990), in particular that graphs would represent pictures of filling rectangles or triangles, I chose to represent the height of the shaded region on the horizontal rather than the vertical axis. By affording students' manipulation of dynamically linked representations, the dynamic sketches provided opportunities for students to form and interpret relationships between quantities.

Task design principles

In designing the task sequence, I provided students with opportunities to demonstrate that they conceived of rate of change as some attribute of a situation that could be measured. In the case of the filling rectangle and triangle situations, such a conception of rate of change could entail a student being able to envision the filling area as increasing in relationship to another changing quantity.

To investigate how students might conceive of rate of change in the context of a filling rectangle or triangle situation, I began by asking students what changed and what stayed the same. This prompt provided students the opportunity to identify different attributes of the situation that could be measured. Once students demonstrated evidence of attending to a rate of change as something that could be measured in the context of the situation, I provided students with representations of constituent quantities (e.g., a graph representing area as a function of height) that could be used to quantify the measurable attribute students had just described.

Task implementation results

Students reasoning about area as a result of a numerical calculation interpreted variable increase as if it were constant. These students made sense of unfamiliar graphs by connecting shapes of objects to shapes of graphs such that rectangles elicit one type of graph and triangles elicit another type. Students' work suggests that iconic interpretations of graphs extend to dynamic graphs such that dynamic graphs are pictures in motion. Students reasoning about area as a measurable attribute of a rectangle or triangle attended to variable increase in area when interpreting and/or predicting features of a graph relating area and side length. These students attended to variation in amounts of change in area, identified sections with different kinds of increases in area, and described variation in how area could increase as side length continually changed. Students attending to variable increase in area also interpreted dynamic sketches and graphs as relationships between quantities.

Concluding remarks

Using non-numerical quantitative reasoning, students can make predictions and create representations indicating how quantities might change together. Although representations included in the tasks explicitly indicate quantities of area and height, students may interpret the graphs shown in Fig. 1 as representing a relationship between area and elapsing time rather than area and height. The possibility for such interpretation highlights the complexity of designing tasks to provide students with opportunities to engage in rate-related reasoning. Future iterations of implementation and analysis could provide further explanation as to how students' non-numerical reasoning develops when constructing relationships between quantities.

ISSUE THREE: THEORETICAL FRAMEWORKS BY WHICH STUDENT PARTICIPATION IN MATHEMATICAL TASKS MIGHT BE BETTER UNDERSTOOD AND OPTIMIZED

6: APPLYING THE PHENOMENOGRAPHIC APPROACH TO STUDENTS' CONCEPTIONS OF TASKS

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Tasks serve a communicative purpose between teacher and student, by conveying the teacher's intent for learning and the student's conception of that intent. Often, responses or work produced by students from a task reveal a disconnect between the teacher's learning expectation and the true depth of knowledge attained by the student. By applying the descriptions of an outcomes space from a phenomenographic inquiry to student work samples, I will discuss how this approach informs a framework for connecting a student's conception of learning to the quality of the individual's task engagement.

Phenomenography is a research methodology with its own theoretical framework that accounts for the qualitatively different ways people experience learning. From this theoretical stance, the impact a task has on learning may be analysed using the outcome space of student conceptions about the learning. By analysing a student's conception of, and approach to learning, the relationship between focal awareness and task performance is further documented. The analysis is guided by the question: "What do students focus on when assigned a task, and in what way does the work produce communicate to the teacher the student's personal epistemology of the content to be learned?"

Learning is defined as perceiving, conceptualizing, or understanding something in a new way by discerning it from and relating it to a context. Furthermore, learning involves two aspects: i) what is to be learned, and ii) how one goes about learning (Marton & Booth, 1997). The learner's perspective of what is to be learned is derived from the student's definition of the direct object of learning. How the learner assigns meaning to the learning object is determined by the learning strategies the student utilizes to meet personal learning goals.

To maintain consistency with the phenomenographic definition of learning, a task is characterized by its relationship to the structural and referential aspects of the learning experience. A task is a situation requiring the learner to experience the object of learning in such a way that the learner must discern components of the situation and how they are related (structural aspect), then assign a meaning to the situation (referential aspect). The task analysed in the study assessed student understanding of descriptive statistics and data analysis.

Since the student's conception is the unit of analysis, an explanation of what a student is attentive to when engaged in completing a task is warranted. The basic components of awareness are appresentation, discernment, and simultaneity (Marton & Booth, 1997). *Appresentation* refers to being conscious of a perceptual or sensual experience in the presence of concrete or abstract entities; *discernment* involves recognizing a foreground-background structure of a situation; *simultaneity* means knowing how the discerned parts are related to the whole structure. The structure of a student's focal awareness directly informs the way the student understands content, which leads the student to perceive that something has been learned.

Collectively, the various levels of student performance in the class fell into the first three conceptions of the learning of statistics outcome space. The majority of the students met the level of knowledge attainment deemed acceptable to teacher. This finding supports the proposition that the meaning and purpose a student assigns to a task seem to be aligned with the student's meaning of learning, approaches to learning, and capabilities sought as a result of learning.

7: THE MILIEU AND THE MATHEMATICAL KNOWLEDGE AIMED AT IN A TASK

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Sør-Trøndelag University College

Context and theoretical background

The research question addressed in the paper is: How does the *milieu* devolved to the students for algebraic generalisation of shape patterns influence their mathematical activity? A gap between the teacher's intention with a task and the students' mathematical activity is explained in terms of a lacking coordination between the knowledge aimed at (an equivalence statement) and the milieu (Brousseau, 1997) devolved to the students.

Participants in the reported research are two groups of three student teachers enrolled at a teacher education programme for primary and lower secondary school in Norway, and a teacher educator who teaches mathematics to these students. The data are a mathematical task and transcripts from video-recorded small-group sessions where the students engage with the task. The theory of didactical situations in mathematics (Brousseau, 1997) has been used to analyse the empirical material.

A shape pattern in elementary algebra is usually instantiated by some consecutive geometric configurations in an alignment imagined as continuing until infinity. According to Måsøval (2011), there are two types of shape patterns: *arbitrary patterns* (Figure 1), *and conjectural patterns* (Figure 2).

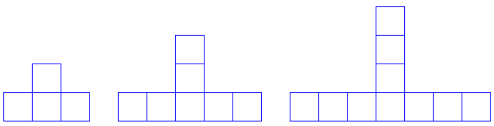

Figure 1: An arbitrary pattern

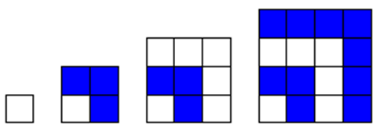

Figure 2: A conjectural pattern

These patterns correspond respectively to two different mathematical objects aimed at in the process of generalising (Måsøval, 2011): *formula* (for the general member of the sequence mapped from the shape pattern; e.g., $a_n = 3n+1$ in Figure 1), and *theorem* (a general numerical statement; e.g., $1+3+5+\cdots \quad 1 = n^2$ in Figure 2).

A priori analysis: the milieu

The pattern in Task 3 (with which the students engaged) is intended to be a conjectural pattern, aiming at the formulation of a theorem. It is made of a first milieu (Shape pattern 1, in Figure 3) that evolves (Shape pattern 2 with white squares, in Figure 4).

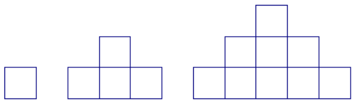

Figure 3: Shape pattern 1

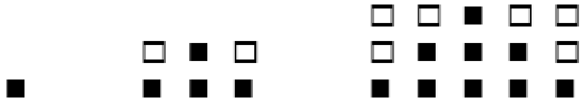

Figure 4: Shape pattern 2

For the teacher, the role of Shape pattern 1 is to provide students with the elements to formulate the theorem "the sum of the first *n* odd numbers is equal to the square of *n*", first in words and then algebraically: $1+3+5+\cdots \quad 1 = n^2$. It is important to notice that the solution of the problem (proof of the theorem) can be reached without the algebraic formulation by direct manipulation the elements of the pattern. A generic example of this manipulation (made by me) is shown in Figure 5.

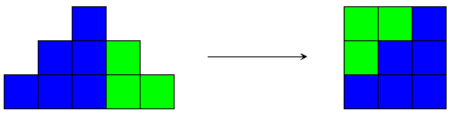

Figure 5: The third element manipulated into a 3x3 square

An alternative shape pattern that would illustrate that the *n*-th square number is equivalent to the sum of the first *n* odd numbers is the pattern shown in Figure 2 above (where the relationship is visualised directly). The pattern would then play the role of a "real milieu" in the sense of Brousseau (1997).

Because of that, the algebraic formulation $1+3+5+\cdots \quad 1) = n^2$ does not appear as a necessary tool to construct the proof of the theorem; it is just a way to formulate a mathematical statement with symbols. In this respect, the pattern is a real milieu when it is considered as a geometrical representation of an arithmetical sequence, in that the elements of the pattern can be represented arithmetically ($1 = 1^2$, $1+3 = 2^2$, $1+3+5 = 3^2$, etc.) and serve as a "model" that can guide a process of algebraic thinking that aims at the equivalence statement $1+3+5+\cdots+ \quad -1 = n^2$. Here, the elements of the pattern serve as referents for first arithmetic and then algebraic symbols, the algebraic formulation being here only a tool to state the equivalence.

Results from the analysis of the transcript data show that: 1) The students produce adequate solutions to subtasks, but this does not constitute a milieu for the formulation of the mathematical statement aimed at. This is consistent with the *a priori* analysis presented above. 2) There is a weakness in the milieu caused by missing clarification of the concept of mathematical statement.

Task 3 is focused on calculations (how many), but the intended knowledge is *theoretical*. Hence the focus should be on *why* the sum of the first *n* odd numbers is equal to the square of *n*. This question has potential to create the need to use algebra.

ISSUE FOUR: ACCOMMODATING STUDENT RESPONSES AND STUDENT AGENCY WITHIN THE INSTRUCTIONAL USE OF MATHEMATICAL TASKS

8: WRITING THE STUDENT INTO THE TASK: AGENCY AND VOICE

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The classroom performance of a task is ultimately a unique synthesis of task, teacher, students and situation. Task selection by teachers initiates an instructional process that includes task enactment (collaboratively by teacher and student) and the interpretation of the consequences of this enactment (again, by teacher and student). In undertaking this study, we examined the function of mathematical tasks in classrooms in five countries. A three-camera method of video data generation (see Clarke, 2006), was supplemented by post-lesson video-stimulated reconstructive interviews with teacher and students, and by teacher questionnaires and copies of student work. Our analysis characterized the tasks employed in each classroom with respect to intention, action and interpretation and related the instructional purpose that guided the teacher's task selection and use to student interpretation and action, and, ultimately, to the learning that post-lesson interviews encouraged us to associate with each task.

The eighth-grade mathematics classrooms that provided the sites for our analysis were drawn from the data set generated by the Learner's Perspective Study (LPS) (Clarke, 2006). Our initial goal in the analysis of mathematical tasks undertaken in these classrooms was the selection of tasks that could legitimately be described as distinctive because of the character of the mathematical activity or because of the teachers' didactical moves in utilising the tasks to facilitate student learning.

The tasks were selected for their disparity across the key attributes: mathematics invoked (both content category and level of sophistication); figurative context (real-world or decontextualised); resources utilised in task completion (diagrams and other representations); and the nature of the role of the task participants. Two examples are noteworthy:

Japan School 3 – Lesson 1 (the Long Task)

In this task, the seemingly simple pair of simultaneous equations $5x + 2y = 9$ and $-5x + 3y = 1$ engaged the class for a fifty-minute lesson (and indeed was the discussion point for the first fifteen minutes of the following lesson). A feature of the performance of this task was the extent to which student suggestions, responses and the articulation of their thinking were regarded as instruments for developing understanding.

Shanghai School 3 – Lesson 7 (the Train Task)

In relation to mathematical tasks, Clarke and Helme (1998) distinguished the social context in which the task is undertaken from any 'figurative context' that might be an element of the way the task is posed. In this sense, the task:

> Siu Ming's family intends to travel to Beijing by train during the national holiday, so they have booked three adult tickets and one student ticket, totalling \$560. After hearing this, Siu Ming's classmate Siu Wong would like to go to Beijing with them. As a result they buy three adult tickets and two student tickets for a total of \$640. Can you calculate the cost of each adult and student ticket?

has a figurative context that integrates elements such as the family's need to travel by train and the familiar difference in cost between an adult and a student ticket. The social context, however, could take a wide variety of forms, including: an exploratory instructional activity undertaken in small collaborative groups; the focus of a whole class discussion, orchestrated by the teacher to draw out existing student understandings; or, an assessment task to be undertaken individually. In each case, the manner in which the task will be performed is likely to be quite different, even though we can conceive of the same student as participant in each setting.

Students were given a significant "voice" in the completion of each task, but the nature of their participation reflected differences in the extent and character of the distribution of responsibility for knowledge constructed in the course of task completion. This distribution of responsibility (or enhanced agency) is a consequence of each teacher's strategic decision, moment by moment, of how best to orchestrate student work on the task. We see task performance as the iterative culmination in the joint construction, not only of the task solution, but of the mathematical principles of which the task is model and purveyor.

Concluding Remarks

Of particular interest in our analysis were differences in the function of mathematically similar tasks, dealing with similar mathematical content (those relating to systems of linear equations), when employed by different teachers, in different classrooms, for different instructional purposes, with different students. The "entry point" for our analysis was a tabulation of the details related to the social performance of the task. Using these tables, our analysis drew on the video-stimulated, post-lesson interview data to identify intention and interpretation and relate both to social performance of the task.

The significance of differences between social, cultural and curricular settings, together with differences between participating classroom communities, challenges any reductionist attempts to characterize instructional tasks independent of these considerations. The attention given by competent teachers to student voice and student agency, and the mathematical tasks that they employ to catalyse that voice and agency, support our belief that the maximization of student agency and voice in the performative enactment of a mathematical task should be recognized as a key principle of task design and delivery.

9: EMERGENT TASKS: SPONTANEOUS DESIGN SUPPORTING IN-DEPTH LEARNING

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According to Bruder (2000), a task can be regarded as a triplet of an *initial state*, a *final state* and a *transformation* that transforms the initial state into a final one. Even adaptive mathematical tasks such as self-differentiating tasks designed before the lesson can only support optimal learning if the teacher also is able to spontaneously transform the situation into a fruitful epistemic process (Prediger & Scherres, 2012). How can such transformations be achieved? This question is addressed by the concept of *emergent tasks*. Emergent tasks are ad-hoc tasks created by the teacher when the teacher conceives the mathematical potential of a learning opportunity and translates it into a task, so that

- the students' interest present in the situation is taken up and
- acute mathematical problems and questions are addressed adaptively.

Our investigation of emergent tasks aims at elucidating how the gap between the students' epistemic needs and the affordances of a task can be bridged.

In order to identify emergent tasks in empirical situations, four types of tasks are distinguished (see Vogt, 2012, p. 35):

Task type	Students express interest	The teacher formulates an adaptive task for a situation
prepared task	-	-
spontaneous task	-	yes
missed emergent task	yes	-
emergent task	yes	yes

Table 1: Types of tasks

A *prepared task* is constructed before the lesson, it may or may not be adaptive or meet the students' interests. A *spontaneous task* is acutely created by the teacher in order to support a specific learning situation, it is not a requirement that it meets the students' interest. However, if a student shows interest in a problem but the teacher does not take this opportunity up to transform the situation into a suitable learning opportunity the teacher has missed setting an emergent task, in such a case we observe a *missed emergent task*.

Emergent tasks often appear when the initial and/or the final state of a problem are not clear to the students. If a student expresses epistemic interest for clarification, the teacher may translate this task into a more adaptive one, thus creating an emergent task. The students' may also *explicitly* express a different epistemic need, in this case the teacher has the chance to set an adaptive, hence, emergent task. If the students' epistemic need is *implicit*, the teacher may act in a sensitive way for instance by prompts ("please tell us what you mean") to make the student's problem visible and then formulate an emergent task. In addition, we found *emergent tasks that unveiled an epistemic gap* that initially remained unnoticed by the students.

Our investigations of emergent tasks has yielded two results: (1) An emergent task has the tendency to initiate further emergent tasks leading to a sequence of fruitful learning opportunities that sometimes shape more than one lesson; (2) based on an initial emergent task we gained five design principles for building a task sequence on learning a procedure: *emergent task* (the teacher is reacting to a student's interest), *presenting and questioning* (the students' solutions of the task are presented and questioned), *using and checking* (an interesting student solution is used and checked by the other students), *expanded use and application* (the potential in use is evaluated by an expanded task), and *institutionalization* ((individual) textualization of the procedure).

On the part of the teacher our studies point to the following conditions that enable the teacher to perform appropriate translations of learning situations into emergent tasks: "The teacher must

- have mathematical knowledge that extends the content of the lesson,
- show interest in the students' learning processes,
- and be open for unusual ways on the part of the students. She or he must be willing to abstain from the planned course" (cf. Bikner-Ahsbahs & Janßen, 2013, p. 162).

MATHEMATICAL TASKS AND THE STUDENT – MOVING FORWARD

The didactical relationship between the student as learner of mathematics and the mathematical task as facilitating that learning

The research reports present complementary perspectives on the student-task relationship and demonstrate just how diverse are the considerations affecting the instructional deployment of tasks and their role in facilitating student participation in particular types of mathematical activity Furthermore, considerable diversity is evident in the descriptions of the positioning of students within that mathematical activity, particularly with respect to the agency afforded to students to determine the nature of their participation. The socio-didactical tetrahedron provides a reflective structure within which to discuss the various research reports.

Teacher-student-task (*Face A*): In the mathematics classroom, the teacher, the student and the tasks provide the key structural elements through which the classroom's social activity is constituted. There has long been a tacit assumption that the completion of mathematical tasks chosen or designed by the teacher will result in the student learning the intended mathematics. This view is persistent despite research that suggests this is not a direct relationship (Margolinas, 2004, 2005).

Student-task-mathematics (*Face B*): For some time, theories of learning have viewed cognitive activity as not simply occurring in a social context, but as being constituted in and by social interaction (e.g., Hutchins, 1995). From this perspective, the activity that arises as a consequence of a student's completion of a task is itself a constituent element of the learning process and the artifacts (both conceptual and physical) employed in the completion of the task serve simultaneous purposes as scaffolds for cognition, repositories of distributed cognition and as cognitive products.

Teacher-task-mathematics (*Face C*): Task development, selection and sequencing by teachers represents the initiation of an instructional process that includes task performance (collaboratively by teacher and student) and the interpretation of the consequences of this enactment (again, by teacher and student).

Teacher-mathematics-student (*Face D – base*): The original didactical triangle has the virtue of connecting the classroom participants with the knowledge domain that provides the pretext for their interaction. As noted, however, the connections represented by the sides of the original didactical triangle require mediation by artifacts; in this case, tasks. The theory of didactical situations (Brousseau, 1997) provides a conceptualisation of the didactic relationship between the teacher, the mathematics and the student. Here, the mathematical task is part of the *milieu*, which models the elements of the material and intellectual reality on which the students act.

One of the dangers for both research and instructional design lies in the disconnection of the elements of the socio-didactical tetrahedron for separate, typically pairwise, study. For example, analysis of student response to a particular task independent of the instructional/learning context in which the task is encountered could understate the complexity of the activity under investigation by backgrounding considerations central to task completion, such as teacher intention, student interpretation, and curricular and organisational context. During the process of task completion, the effectiveness of the task in promoting learning will also be contingent on student intention (with respect to the task) and teacher interpretation (with respect to the students' activity). These socio-mathematical considerations are central to any attempt to understand (and thereby optimize) the function of tasks in catalyzing student mathematical activity and consequent learning in institutionalized settings such as mathematics classrooms.

Some of these considerations can be summarised in the form of questions:

- What problem does the student think she is solving?
- What student-related factors determine the optimal selection and sequencing of tasks for instructional purposes?
- What are the student-related considerations affecting the use of mathematical tasks to promote students' higher order thinking skills?
- What contribution does the student make to the performative shaping of the task and how is this contribution accommodated within available theoretical frameworks?
- What degree of agency can the student realistically be afforded in the framing and performance of a mathematical task, if the teacher's instructional agenda is to be achieved?

These questions have been addressed to varying degrees in the papers that comprise this Research Forum. It is useful to review some of the key points made by each contribution.

A recurrent theme in the framing of this Research Forum was the tension between the teacher's instructional intentions and consequent student activity. Coles and Brown suggest that an emphasis on making distinctions foregrounds the targeted mathematical awarenesses that are otherwise only indirectly prompted by instruction based on different principles. This reduces the possibility of divergence of teacher intention and student activity by actively stimulating those student capabilities directly. Giménez, Palhares and Vieira investigated the role of task order in promoting algebraic thinking, by making comparison between instruction that commenced with sequential or structural tasks. This sensitivity to task sequence rather than simply to the quality or effectiveness of the individual tasks per se, introduces an additional consideration to the question of how best to utilise tasks to promote student learning. Savard, Polotskaia, Freiman and Gervais examined the contemporary premise that some problems (or tasks) are particularly difficult because they require a flexible and holistic analysis of their mathematical structure while easy problems do not require such analysis. The emphasis on the capacity of tasks to facilitate student consideration of mathematical relationships rather than simply mathematical operations introduces additional considerations in the design of instructional tasks.

In combination, these three studies usefully demonstrate the diversity of considerations invoked by the different aspirations pertaining to specific organisational and curricular settings. The interplay of these considerations can be seen in the significance of the students' response to a task and the sensitivity of that response to task characteristics, including task order. This interplay is most evident in the implicit compromise between prescription and devolution, undertaken in order to provide opportunities for the expression of student agency, while still holding out some hope that student activity and learning might resemble the teacher's instructional intentions.

The papers by Aizikovitsh-Udi et al. and Johnson identify some of the challenges faced by task designers hoping to elicit something more sophisticated than the replication of a taught procedure. The dynamic between promoting the development of mathematics-specific skills and modes of thought and meeting the more encompassing aims of contemporary curricula is presented as potentially a productive symbiosis by Aizikovitsh-Udi and her co-authors. Johnson's investigation of mathematical tasks involving dynamic representations of covarying quantities necessarily also documents student hypothesis formulation and associated mathematical reasoning. The capacity of her tasks to frame, shape and facilitate sophisticated student reasoning mirrors the capacity of the hybrid tasks of Aizikovitsh-Udi et al. to simultaneously stimulate statistical and critical thinking. Given the aspirations of contemporary curricula towards promoting higher order thinking skills, these two papers provide cause for optimism.

Our use of the socio-didactical tetrahedron to frame this Research Forum has already placed a Vygotskian slant on our conception of the process of mathematics learning and the role of instructional tasks in facilitating that learning process. Without wishing to be theoretically exclusive, we would argue that recognizing the function of mathematical tasks as tools for the facilitation of student learning leads us to the further useful recognition that the use of a tool (i.e. a task) fundamentally affects the nature of the facilitated activity (i.e. student learning). This does not preclude the use of other theoretical perspectives in the analysis and optimisation of task use in instruction. Phenomenographic approaches, as illustrated by Gardner, precisely capture the reflexive connection between the teacher's use of tasks and the students' conceptions of those tasks. The prioritisation of student perception of the object of learning aligns Gardner's perspective with aspects of the paper by Coles and Brown. However, Gardner adds a layer of sophistication in her consideration of the student's perception of and response to a given task as the social enactment of the student's conception of learning. This perspective accords a level of significance to student intellectual agency that both complicates and enhances our consideration of the student-task axis and its significance within the socio-didactical tetrahedron. The paper by Strømskag draws together several considerations: the tension between intention and activity, and the role of the task in creating a mileu (Brousseau, 1997) conducive to the promotion and use of the targeted mathematical knowledge. The conditions governing the teacher's capacity to orchestrate the creation of a milieu suitable for the development of the targeted mathematical knowledge are a direct consequence of the choice of instructional task.

The research narrative concludes by directing attention to student agency. Examination by Mesiti and Clarke of task functionality through the lens of international comparison highlights differences in instructional purpose and curricular context, which shape the particular activity arising from the instructional use of a task in differently situated classrooms. In the paper by Bikner-Ahsbahs, tasks encompass initial and final states [of knowing] and their connecting transformation. Emergent tasks appear, fractal-like, where the learning situation requires the revision, refinement, or elaboration of the intended task, including the insertion into the lesson of an entirely unintended task, called upon in response to the demands of the particular didactical situation. In an interesting way, emergent tasks embody the teacher's pedagogical agency through their incarnation of the teacher's response to an instructional situation not anticipated in the lesson's original planning. The implication is that teacher agency is best expressed in reflexive relation to student agency, but also in the provision of opportunities for the expression of that student agency. This recognition returns us to the assertion by Mesiti and Clarke that "the classroom performance of a task is ultimately a unique synthesis of task, teacher, students and situation" and reinvokes the socio-didactical tetrahedron.

As a final recapitulation: There is a tension between the teacher's instructional intentions (and associated actions) and the students' consequent activity (and ultimate learning). This tension is probably inevitable and even productive. The existence of this tension should reassure us that student agency has not been precluded entirely from our classrooms.

Equally, the tension is not one of opposition, but rather the recognition of the need for continual mutual adjustment. Both teacher and students are complicit in the construction of classroom practice; if the teacher appears to exert the greater control through task selection, the students can, by their responses, significantly determine the nature of consequent classroom activity. Within this process of incremental and iterative adjustment, the task serves as the frame for activity, while the activity constitutes the performance of the task.

In the preceding discussion and the research narrative constituted through the various research reports, we have attempted to examine the instructional use of mathematical tasks, the roles played by students in the performance of those tasks, and the anticipation of those roles by teachers and task designers. The results of several of these analyses have been interpreted as indicating principles for instructional (task) design. Tasks and their social performance provide both a window into the practices of mathematics classrooms internationally and the means to realise our curricular ambitions.

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MATHEMATICAL MODELING IN SCHOOL EDUCATION: MATHEMATICAL, COGNITIVE, CURRICULAR, INSTRUCTIONAL AND TEACHER EDUCATION PERSPECTIVES

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ABOUT THE RESEARCH FORUM

The purpose of this Research Forum is to present and discuss five perspectives on research and practice in the teaching and learning of mathematical modeling in K-12 school mathematics classrooms and to engage participants in advancing our understanding of the teaching and learning of mathematical modeling.

In today's dynamic, digital society, mathematics is an integral and essential component of investigation in disciplines such as biology, medicine, the social sciences, business, advanced design, climate, finance, advanced materials, and many more (National Research Council, 2013). In each of these areas, this work demands an understanding of and facility with mathematical modeling to make sense of related phenomena. Mathematics education is beginning to reflect the increased emphasis of mathematical modeling. In fact, mathematical modeling has been explicitly included in national curriculum standards in various countries. For example, in the United States, real-world applications and modeling are recurring features throughout the Common Core State Standards for Mathematics (CCSSM; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010).

In the past several decades, the mathematics education research community has made great efforts to study the issues related to the teaching and learning of mathematical modeling (Blum & Niss, 1991, Galbraith et al., 2007; Houston, 2009). Recent interest in mathematical modeling has been stimulated by OECD's PISA study, which assessed students' mathematical literacy, as well as the publication of the CCSSM in the United States. However, despite the increased interest in mathematical modeling, a large number of questions remain unanswered (see, e.g., Lesh & Fennewald, 2013). Blum (1994) pointed out "a substantial gap between the forefront of research and development in mathematics education, on the one hand, and the mainstream of mathematics instruction, on the other." (p. 7). Twenty years later, this gap still exists (Kaiser, 2013). The main goal of this forum is to help narrow this gap with respect to the important area of mathematical modeling. In particular, this Research Forum

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provides a venue for researchers around the world to present findings and discuss issues surrounding the teaching and learning of mathematical modeling from the following five perspectives: Mathematical, Cognitive, Curricular, Instructional, and Teacher Education Perspectives. In each perspective, we list a set of research questions to be discussed.

MULTIPLE PERSPECTIVES FOR RESEARCH ON MATHEMATICAL MODELING: RESEARCH QUESTIONS

In this section, we first identify a few research questions in each perspective. In the next section, we provide some initial thoughts on some of the research questions.

Mathematical Perspective

The world of mathematics and the world of mathematics education interact, but do not completely overlap when they communicate with each other about mathematical modeling (Burkhardt, 2006; Pollak, 2003). Taking a parallel example, research on mathematical proof has shown that students and teachers hold different conceptions from those held by research mathematicians (e.g., Weber, 2008). Similarly, the notion of mathematical modeling in school mathematics is different from the way it is understood by practicing mathematical modelers. In fact, Lesh and Fennewald (2013) pointed out that one of the major challenges in the teaching and learning of mathematical modeling is the "conceptual fuzziness" about what counts as a modeling activity. Even those researchers who have long been conducting research on mathematical modeling have not come to an agreement on the processes of modeling and how to conceptualize mathematical modeling (Zawojewski, 2013). In this Research Forum, we specifically invite mathematicians and mathematics educators to directly interact and discuss these research questions about mathematical modeling. (1) *If we view mathematical modeling as a bidirectional process of translating between the real-worldand mathematics, what are its essential features?* (2) *Which of those essential features differentiate mathematical modeling from problem solving in school mathematics?* (3) *From the viewpoint of a practitioner of mathematical modeling, what are the essential competencies and habits of mind that must be developed in students to allow them to become competent mathematical modelers?*

Cognitive Perspective

In order to improve students' learning, it is necessary to understand the developmental status of their thinking and reasoning. Teachers' knowledge of students' thinking has a substantial impact on their classroom instruction, and hence, upon students' learning (e.g., Hill et al., 2007). Although we know a great deal about the cognitive processes of students' mathematical problem solving (see. e.g., Schoenfeld, 1992), we know less about how students approach modeling problems (Borromeo Ferri, 2006). Some researchers have theorized that students hold mental models that connect mathematics and the real-world (Borromeo Ferri, 2006). Even though there is little agreement about the fundamental cognitive features of mathematical modeling, there is some consensus that the *process* of getting from a problem outside of mathematics to its mathematical formulation in mathematical modeling begins with the formulation of research questions (Pollak, 2003). Prior research has demonstrated that students are quite capable of posing mathematical problems from given situations (Cai et al., in press; Silver, 1994), but it less clear how students formulate mathematical problems based on true real-worldsituations. It is important to note that the situations that have been used in problem-posing research are typically much less complex than the situations that occur in mathematical modeling. Hence, there is still much to learn from the cognitive perspective on mathematical modeling. (4) *What are factors that have an impact on students' formulation of researchable questions in modeling situations?* (5) *If we view mathematical modeling as ill-structured problem solving, how does one convert an ill-structured problem into a well-structured problem with specified research questions?* (6) *What are cognitive differences between expert modelers and novice modelers?*

Curricular Perspective

Historically, worldwide, changing the curriculum has been viewed and used as an effective way to change classroom practice and to influence student learning to meet the needs of an ever-changing world (Cai & Howson, 2013). In fact, curriculum has been called a change agent for educational reform (Ball & Cohen, 1996) and the school mathematics curriculum remains a central issue in our efforts to improve students' learning. Although some ideas fundamental to mathematical modeling have permeated school mathematics textbooks for some time (e.g., *Realistic Mathematics* in the Netherlands and *Standards-based mathematics curricula* in the United States), mathematical modeling is usually not a separate course, nor do there exist separate textbooks for mathematical modeling. Thus it will be useful to understand international perspectives on research questions from the curricular perspective. (7) *Looking within existing mathematics textbooks, are there activities specifically geared toward mathematical modeling?* (8) *Is it possible or even desirable to identify a core curriculum in mathematical modeling within the general mathematical curriculum?* (9) *In CCSSM in the United States, mathematical modeling is not a separate conceptual category. Instead, it is a theme that cuts across all conceptual categories. Given this orientation, how might mathematical modeling be integrated into textbooks throughout the curriculum?*

Instructional Perspective

Although curricula can provide students with opportunities to learn mathematical modeling, classroom instruction is arguably the most important influence on what students actually learn about modeling. Thus, the success of efforts for students to learn mathematical modeling rests largely on the quality of instruction that might foster such learning. Researchers have documented a number of cases of teaching mathematical modeling in classrooms (e.g., Lesh & Fennewald, 2013). In this Research Forum, we synthesize and discuss these findings to explore the following research questions: (10) *What does classroom instruction look like when students are engaged in mathematical modeling activities?* (11) *What mathematical-modeling tasks have been used in classrooms, and what are the factors that have an impact on the implementation of those tasks in classrooms?* In addition to devoting an appropriate amount of time to mathematical modeling tasks, teachers must also decide what aspects of a task to highlight, how to organize and orchestrate the work of the students, what questions to ask to challenge those with varied levels of expertise, and how to support students without taking over the process of thinking for them, and thus eliminating the challenge (NCTM, 2000). Subsequently, there is a need to consider how productive discussions around modeling activities can be facilitated. (12) *What is the nature of classroom discourse that supports students in becoming successful mathematical modelers?*

Teacher Education Perspective

There is no doubt that teachers play an important role in fostering students' learning of mathematical modeling and students' learning of mathematics through engagement in mathematical modeling. However, it is well documented that modeling is quite difficult for teachers because real-worldknowledge about the context for modeling is needed, and because teaching becomes more open and less predictable when students engage in more open-ended modeling situations (e.g., Freudenthal, 1973). In general, teachers' initial and in-service training as well as the curricular contexts of schooling have not readily provided opportunities to make mathematical modeling an integral part of daily lessons (Zbiek & Conner, 2006). A number of researchers in different countries (e.g., Kaiser & Schwarz, 2006) have started to develop mathematical modeling courses for in-service teachers. Likewise, a number of teacher education programs around the globe have included mathematical modeling as part of their initial teacher education program requirements (Galbraith et al., 2007). In this Research Forum, we discuss the various course offerings for teachers around the globe and address key research questions. (13) *Are there programs worldwide which successfully support pre-service and in-service teachers to teach mathematical modeling, and what are the features of these successful programs?* (14) *What level of familiarity with disciplines other than mathematics is it necessary for pre-service and in-service teachers to have in order to successfully teach mathematical modeling?*

MULTIPLE PERSPECTIVES FOR RESEARCH ON MATHEMATICAL MODELING: SOME INITIAL THOUGHTS

The first sub-section was written by John A. Pelesko, an applied mathematician. It presents a first-person perspective that represents a direct form of communication of ideas about mathematical modeling between an applied mathematician and mathematics educators.

1: INITIAL THOUGHTS ON THE MATHEMATICAL PERSPECTIVE

John A. Pelesko
University of Delaware

Having spent the better part of the last twenty-five years engaged in teaching and doing mathematical modeling as an applied mathematician (see, e.g., Pelesko & Bernstein, 2003; Pelesko, Cai, & Rossi, 2013), it is hard to overstate the joy I felt upon realizing that the Common Core State Standards for Mathematics (NGA & CCSSO, 2010), the new standards adopted widely across the United States, , placed a special emphasis on mathematical modeling. This ascension can be credited, in part, to the long term efforts of researchers such as Pollak (2003, 2012), Lesh and Doerr (2003), and others who have argued that it is not just applications of mathematics that should be incorporated into the mathematics curriculum at all levels of education, but that the practice of mathematical modeling itself is an essential skill that all students should learn in order to be able to think mathematically in their daily lives, as citizens, and in the workplace (see, e.g., Pollak, 2003). Now that the importance of mathematical modeling is being recognized by the mathematics education community at large, appearing as both a conceptual category and a Standard for Mathematical Practice in CCSSM, it is critical that those who do mathematical modeling engage deeply with the K-12 mathematics education community around the issues of teaching and learning the practice. It is important to note that mathematical modeling is practiced far and wide – across the natural sciences, engineering, business, economics, the social sciences, and in almost every area of study in one form or another. Hence, the set of stakeholders in this conversation is large, and we should be careful not to substitute any one practitioner's perspective for the whole. Nevertheless, in an attempt to contribute to this conversation, here I provide one practitioner's perspective.

What is Mathematical Modeling?

Given the lack of attention that has been paid to mathematical modeling in the US educational system, especially in mathematics teacher education programs (see Newton et al., 2014), it is not hard to imagine that many mathematics educators, upon reading the CCSSM, found themselves asking this question. The brief description of mathematical modeling found in the standards document (pp. 72-73), and the fact that this description appears only within the high school standards, likely adds to this confusion. Further confusion is likely to occur as educators digest the US Next Generation Science Standards (NGSS Lead States, 2013), which make use of the term "model" both in and out of the context of "mathematical model."

To address the question "What is mathematical modeling?" it is then perhaps useful to first consider the question "What is modeling?" My answer? Modeling is the art or the process of constructing models of a system that exists as part of reality. By "model," I mean a representation of the thing that is not the thing in and of itself. The model captures, simulates, or represents selected features or behaviors of the thing without being the thing. By "mathematical model" I mean a model or a representation that is constructed purely from mathematical objects. So, mathematical modeling is the art or process of constructing a mathematical model. *That is, mathematical modeling is the art or process of constructing a mathematical representation of reality that captures, simulates, or represents selected features or behaviors of that aspect of reality being modeled.*

Now, we should note that mathematical models have a special place in the hierarchy of models in that they have both *predictive* and *epistemological* value. The epistemological value is a consequence of the idea that *mathematical modeling is a way of knowing*. The predictive value of a mathematical model gives mathematical models a special place in "science," loosely and broadly defined, in that a mathematical model can take the place of direct ways of knowing, in other words, experiment. A good mathematical model is both an instrument, like a microscope or a telescope, allowing us to see things previously hidden, *and* a predictive tool allowing us to understand what we will see next.

Note that an especially "good" mathematical model, that is, one with a high level of predictive success, often ceases to be thought of as "just a model." Rather, it attains a different status in the scientific community. We don't say "Newton's mathematical model of mechanics;" rather we say "Newton's Laws." We don't say "Schrodinger's model of the subatomic world;" rather we say "Quantum Mechanics" or the "Schrödinger Equation." Yet, each of these examples is, in fact, a mathematical model of the thing, and not the thing in and of itself. These examples have attained the highest possible level of epistemological value. They have become *the* way of knowing, understanding, describing, and talking about their subjects.

Now, we have diverged into abstract territory, and we do not want to leave the reader with the impression that mathematical modeling is *hard*, something to be left to the Newtons and Schrödingers of the world. Rather, we hope the reader is left with the impression that mathematical modeling is *exceedingly useful* and that by helping our students master this practice, we will be adding a tool to their mental toolkit that will serve them well, no matter what their future plans.

Thought Tools for Modeling

The question then becomes: How exactly does someone become a proficient mathematical modeler? In the United States, as evidenced by textbook after textbook on mathematical modeling (see, e.g., Pelesko & Bernstein, 2003), the answer has been "Modeling can't be taught, it can only be caught." Now, I take a different perspective and argue that it is useful to think of the mathematical modeler as having discrete "thought tools," each of which can be discovered and taught. As a consequence, we see that many "modeling cycles" unintentionally hide much of the real work of mathematical modeling.

We borrow the term "thought tools" and this framework for meta-thinking from the philosopher and cognitive scientist, Daniel Dennett. In Dennett (2013) he quoted his students as having made the observation that "*Just as you cannot do much carpentry with your bare hands, there is not much thinking you can do with your bare brain*." Dennett then proceeded by analogy with saws, hammers, and screwdrivers, to introduce thought tools of informal logic such as reductio ad absurdum, Occam's razor, and Sturgeon's Law. Applying this notion of thought tools to the mathematical modeler, we argue that they must possess a set of thought tools that lie in three different categories: Mathematical Thought Tools, Observational Thought Tools, and Translational Thought Tools.

Mathematical Thought Tools are those tools we attempt to add to our students' toolkits when we teach mathematics. These include notions such as algebraic thinking, the principle of induction, the pigeonhole principle, and any tool that lets students think about and do mathematics. Note that these thought tools are directed at mathematics and their utility is generally tied to thinking in the mathematical domain.

Observational Thought Tools are those tools we typically think of as being used by "scientists." These include the ability to think in terms of cause and effect, to observe spatial and temporal patterns in the real world, and to look deeply at reality. Note that these thought tools are directed at the real-worldand their utility is generally tied to thinking in the domain of the real world.

Translational Thought Tools are those tools that allow the mathematical modeler to take questions formed in the observational domain, translate them into the mathematical domain, and translate answers and new questions uncovered in the mathematical domain back again to the observational domain. These include knowledge of conservation laws, physical laws, and the assumptions that must be made about reality in order to formulate a mathematical model. Note that these thought tools are directed both toward reality and toward mathematics. Their utility lies in their usefulness in translating between these two domains.

In a typical "modeling cycle," such as appears in the CCSSM (see Figure 1), one moves from the "real world" or the "problem" to the "formulation" via a single small arrow. Buried in this small arrow is the use of Observational and Translational Thought Tools. The remainder of the cycle, up to the point of comparing results with reality, generally relies purely upon Mathematical Thought Tools. While we can argue over whether or not we are properly equipping our students with the proper Mathematical Thought Tools they will need in their journeys around the modeling cycle, I would argue that generally we pay little attention to the Observational and Translational Thought Tools they will need to even begin their journey. Identifying, unpacking, and learning how to equip our students with these sets of tools is an essential step in learning how to teach mathematical modeling.

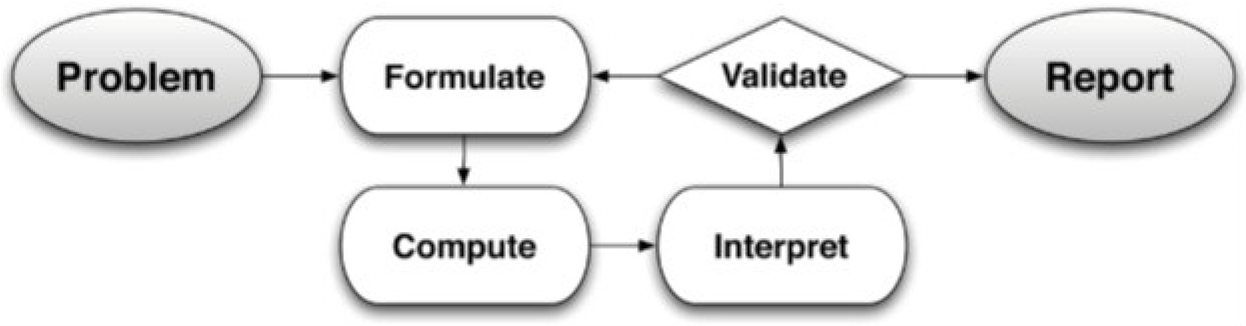

Figure 1: The mathematical modeling cycle from CCSSM (2010, p. 72)

As an example of how the mathematical modeler wields these tools, I ask the reader to imagine drops of morning dew on a spider web. Scientists, using their observational tools, notice these droplets and wonder why they are all roughly the same size. The mathematical modeler recalls that nature acts economically and often in a way that minimizes some quantity. They cast forth a hypothesis that here, nature is acting to minimize surface area, and that this leads the dew to break into droplets of nearly uniform size. They recast this observation and hypothesis into mathematical terms, already anticipating the mathematics from the presence of the notion of "minimizes" and wields their Mathematical Thought Tools to predict the size of the droplets given the presence of the dew. Comparing the predicted size with the size of actual droplets, the modeler refines and perfects the model, and acquires an understanding of *any* droplets on *any* spider web at *any* point in time.

In summary, mathematical modeling is a practice worth sharing and teaching. It is a powerful way of knowing the world, and it can be taught rather than simply caught. In the United States, we have much work to do in order to bring this new toolkit to our students. It will take the efforts not only of mathematics educators and applied mathematicians, but of mathematical modelers of every stripe in order to do so. Here, I have sketched out one avenue of approach that in many ways parallels recent work in unpacking the thought processes behind mathematical proof (see Cirillo, 2014). A similar effort to identify and unpack the thought tools of the mathematical modeler holds the promise of helping us train a wide range of students in the art of mathematical modeling.

2: INITIAL THOUGHTS ON COGNITIVE PERSPECTIVE

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Past and Present

Cognitive perspectives on students' learning from modeling have long been debated within the international community. Nearly thirty years ago, Treilibs (1979) and

Treilibs, Burkhardt and Low (1980) from the Shell Centre in Nottingham analysed, at a micro level, the videotaped modeling processes of groups of university students. They mainly focussed on determining how learners build a model and hence concentrated on the so-called "formulation phase". They visualized this construction process of a model with "flowcharts" through which several modeling steps of individuals were represented graphically. One central result of their study was that building a model is a very complex activity for individuals and, at the same time, not easy to communicate for university professors during lectures. Because they only investigated university students, there was no empirical evidence about cognitive processes of primary, middle-school, or high school students. Unfortunately this group from Shell-Centre did not work on further studies.

Matos' and Carreira's (1995, 1997) research 15 years later placed a special emphasis on 10th-grade learners' cognitive processes and representations while solving realistic tasks. They analyzed the creation of conceptual models (interpretations) of a given situation and the transfer of this real situation into mathematics. The results of their study show the numerous and diverse interpretations learners use while modeling and that the modeling process is not linear. Similar to the studies of Treilibs, Burkhardt and Low (1980), the research of Matos and Carreira did not emphasize the analysis of the complete modeling process.

Galbraith and Stillman (2006) also stressed cognitive aspects. They tried to identify the "blockages" that fourteen- and fifteen-year-old students experience while modeling, and pointed out that the overall modeling process is cyclic rather than linear. On the basis of their in-depth analysis, Galbraith and Stillman were able to identify in which parts of the modeling cycle individuals have blockages that hinder solutions. Their more recent research (e.g., Stillman, 2011) shows the important role of meta-cognitive activities while modeling, as does the research of Mousoulides and English (2008), which we address later.

Other significant research on cognitive perspectives includes the extensive work of Richard Lesh and his colleagues (cf. amongst others, Lesh & Doerr, 2003). They adopted a theoretical approach drawing upon upon the ideas of Piaget (1978) and Vygotsky (1934).

Also worthy of notice is the project DISUM (Blum & Leiss, 2007), which focused on the investigation of modeling processes of middle school students within a seven-step modeling cycle and on teacher interventions during these modeling activities. The results showed several micro-processes of students' work and how the situation model was built. The COM²-project (Borromeo Ferri, 2010) had a far stronger cognitive view than the project DISUM, with a focus on cognitive theory behind the analysis (Mathematical Thinking Styles). The central result of COM² was evidence of the reconstruction of "individual modeling routes" of pupils while undertaking modeling activities in the classroom. It became clear that mathematical thinking styles have a strong influence on the modeling behavior of students and teachers concerning their focus on "reality" and "mathematics" (Borromeo Ferri, 2011.

Summarizing some of the central research studies in this field, it becomes evident that cognitive views on modeling were highlighted in the international arena 30 years ago, but were then neglected for a long time and, in general, and were overtaken by other perspectives such as modeling competencies. However, the cognitive research increased especially after the ICMI-Study 14 on mathematical modeling, where the Discussion Document (Blum et al., 2002) argued that the cognitive psychological aspects of individuals during their modeling processes should be strongly emphasized in further studies.

The Cognitive Perspective – "An Additional Perspective"(?)

Kaiser and Sriraman (2006) offered a classification of five central perspectives on modeling, with a main focus on the goals intended for teaching modeling: realistic or applied modeling, contextual modeling (recently described as the "MEA-approach", Borromeo Ferri, 2013), educational modeling, socio-critical modeling, and epistemological modeling. These theoretical perspectives are understood as research perspectives. This classification was mainly a result of extensive discussions of international researchers during several European Conferences (ERME) within the group "Mathematical Modeling and Applications." As an additional perspective "cognitive modeling" or the cognitive perspective on modeling was formulated. Kaiser and Sriraman (2006) described "cognitive modeling" also as a "meta-perspective", because it is focusing on specific research aims and not on goals for teaching modeling, in contrast to the other approaches. When developing this classification, the general consensus was that this cognitive perspective can be combined with the other approaches depending on the research aims one likes to have in a study. Furthermore, Kaiser and Sriraman pointed out that the research aims of cognitive modeling are to describe and understand students' cognitive processes during modeling activities (Kaiser & Sriraman 2006).

Following the call from the ICME-14 Discussion Document, further research was done in the field of cognitive modeling. Results of empirical studies offered more knowledge about cognitive processes during modeling activities, especially concerning potential barriers or so-called red-flag situations (Stillman & Brown, 2011). When looking at the different modeling cycles (Borromeo Ferri, 2006), mostly a seven-step-modeling cycle (Blum & Leiss, 2007;Borromeo Ferri, 2006) is used as a basis or an instrument for analysing cognitive processes along several steps. Within the current discussion the seven-step-cycle is described as the "diagnostic modeling cycle" because this cycle includes the step, "construction of a situation model." Building a situation model or a mental representation of the situation is a very individual process, because one has to understand the problem and visualize the given situation (Blum & Leiß, 2010; Borromeo Ferri, 2010).

On the one hand there are a lot of studies that have a focus on theory-building, but on the other hand, we now have a lot of implications, core concepts, and empirical evidence that this cognitive view is no longer exclusively a research perspective or an "additional perspective" as described in the initial classification of Kaiser and Sriraman (2006). These researchers argued that the cognitive view on modeling is mostly integrated in empirical studies, because it is a crucial part of modeling activities. But we believe that this "additional perspective" is far more than a "meta-perspective" and should have an equal position to the other named perspectives.

Cognitive Modeling in School

Within the cognitive perspectives on modeling we give an additional characterisation of such perspectives on the basis of several studies done by Borromeo Ferri (e.g., 2007, p. 265): "If modeling is considered under a cognitive perspective the focus lies on the individual thinking processes which are expressed mainly through certain verbal and non-verbal actions in combination with written solutions during modeling activities of individuals (including teachers)."

A further example of this cognitive perspective can be found in the research of Mousoulides and English (2008). They reported on the mathematical developments of two classes of ten-year-old students in Cyprus and Australia as they worked on a complex modeling problem involving interpreting and dealing with multiple sets of data. The MEA problem ("The Aussie Lawnmower Problem") required students to analyse a real-worldbased situation, pose and test conjectures, and construct models that are generalizable and re-usable. Their findings revealed that students in both countries, with different cultural and educational backgrounds and inexperienced in modeling, were able to engage effectively with the problem and, furthermore, adopted similar approaches to model creation. The students progressed through a number of modeling cycles.

In the first cycle, the students focused only on some of the problem data and information. This resulted in a number of initial, interesting approaches to model development, but these approaches were inadequate because the students did not take into account the whole problem data. The students quickly moved to a second cycle when they realized that their initial approaches were not successful, since a number of contradictions arose in their results. Consequently, almost all groups in both countries moved to mathematizing their procedures by totalling the amounts in each given table of data and, for the Australian students, by finding the averages. This was a significant shift in the students' thinking. In the third cycle, the students in both countries identified trends and relationships to help them find a solution to the problem.

Also of significance in Mousoulides and English's (2008) study is students' engagement in self evaluation: groups in both countries were constantly questioning the validity of their solutions, and wondering about the representativeness of their models. This helped them progress from focusing on partial data to addressing all data in identifying trends and relationships in creating better models. Although the students did not progress to more advanced notions such as rate (which was beyond the curriculum level in both countries), they nevertheless displayed surprising sophistication in their mathematical thinking. The students' developments took place in the absence of any formal instruction and without any direct input from the classroom teachers during the working of the problem.

3: INITIAL THOUGHTS ON THE CURRICULAR PERSPECTIVE

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As Cai and Howson (2013) pointed out in their discussion of "What is a Curriculum?" there is no agreement over a definition of the term. Taking curriculum to refer to intentions, it can be considered as formal documentation that sets out what is to be taught and learned and as such, it encapsulates an epistemology with historical precedence. However, as Travers and Westbury (1989) highlighted, it is possible to broaden consideration of the curriculum by not only focusing on what is *intended* but also what is *implemented* and what is *attained*. This removes mathematics from the pages of official documents and brings it to life in the schools and classrooms where it is taught (*implemented*) and learned (*attained*). It is in such classroom ecologies, or in the classroom milieu as Brousseau (1989) called it, that mathematics is lived and defined for students. Ultimately mathematics becomes something uniquely defined for each individual through the mathematical activity in which they take part, both socially and alone, although there are certainly common and strong trends that emerge in classrooms, schools and indeed nationally (Givvin et al., 2005). Taking a socio-cultural view, mathematics, its teaching and its learning, can be considered as in mutually recursive relationship with the classroom community in which teachers and students live and learn. It is in this coupling of human activity with mathematics as a discipline that modeling as a mathematical practice seeks to find a place.

Historically, worldwide, changing the intended curriculum through carefully designed (re-)specification has been viewed and used as an effective way to change classroom practice and to influence student learning to meet the needs of an ever-changing world (Cai & Howson, 2013). In fact, curriculum (intended and specified) has been called a change agent for educational reform (Ball & Cohen, 1996) and the school mathematics curriculum as such remains a central issue in our efforts to improve students' learning. Further, in terms of bridging from strategic and tactical design (Burkhardt, 2009) to classroom practice, through the technical design of classroom materials, we find little support. Although some ideas fundamental to mathematical modeling have permeated school mathematics textbooks for some time (e.g., *Realistic Mathematics* in the Netherlands and *Standards-based mathematics curricula* in the US), mathematical modeling is usually not a separate course, nor do there usually exist separate textbooks for mathematical modeling. Thus it will be useful to understand international perspectives based on research questions from the perspective of curriculum.

Discussion is further complicated if we consider different understandings of what modeling as a mathematical practice means and the different aspects of it. For instance, if one considers modeling in the classroom from the perspective of connecting mathematics to real-worldproblems or problems from everyday life, it is possible to think of changes in textbooks that can include these aspects. But if one considers modeling from a perspective in which the emphasis is on the choice of the problem by the students the situation may change. Borba and Villarreal (2005) see modeling as "a pedagogical approach that emphasizes students' choice of a problem to be investigated in the classroom. Students, therefore, play an active role in curriculum development instead of being just the recipients of tasks designed by others" (p. 29).

In such an approach the curriculum is not pre-defined and specified, it is negotiated between teachers and students, and consequently the students' interests are a priority. The authors suggested that such an approach would approximate the practice of applied mathematicians, who deal with new issues, and in which one of the main tasks is "building the problem", defining the variables and then trying to solve the resulting mathematical model, usually under time pressure. João Frederico Meyer, an applied mathematician, in a book written with two mathematics educators, reinforces the idea that there is time pressure and that finding the problem is a big part of applied mathematics (Meyer, Caldeira and Malheiros, 2011). If this is the case, new questions may arise; for example, "Do we need to have a list of topics to be taught?" Taking such a view requires us to consider new directions in discussions about curriculum as it is intended, implemented and attained.

Authors such as Skovsmose (1994) also propose, and have done so for a long time, that modeling may (or should) be closely linked to social and political issues. He identifies critical mathematics education as being closely connected to modeling. In such a perspective, it is not so relevant that the choice of problems is made by the students, but it is important that the theme discussed in the classroom is closely connected to issues such as social equality and justice. We should perhaps also add other issues such as those relating to gender differences and the environment to reflect emerging concerns of citizens throughout the world. From such a perspective one can ask: is it possible to enroll students in political discussion, with capital P, if we have a problem that was chosen by the teacher or is from a textbook?

A long time ago, Borba (1990) asked similar questions when he connected ethnomathematics with modeling in informal education settings in one of the slums of Brazil. If ethnomathematics – with its concern with cultural background of students is brought into curriculum debate – is combined with modeling, then different issues and questions may arise, such as: (1) Can a common textbook be used with students from different backgrounds in different parts of a country? and (2) How do we deal with multicultural classrooms?

Borba (2009) and Meyer, Caldeira and Malheiros (2011) have debated the synergy between modeling and digital technologies. Authors such as these have discussed how modeling can be transformed with technology as students can be released from calculations and focus on problems that could not be handled if digital technologies were not available. Soares and Borba (2014) have shown how an inversion of topics can be made in an introductory Calculus course for Biology majors if software such as Modellus is available. It was found that such students could start, from day one, dealing with a modeling activity related to malaria, using a model that was important in the second half of the 20th century. This model included a system of differential equations with students computing graphical solutions and graphically displaying these using Modellus. Such a model was used with these students to introduce several concepts in precalculus and calculus, including the notion of differential equations by the end of the course. This approach shows a clear possibility of how inversion of the order of topics taught in the curriculum is possible due to the use of technology-based modeling tools. This leads to further potential research questions such as, “To what degree do students need to learn the formal mathematical techniques of differentiation and integration, for instance, when students are able to model with access to digital technologies?”

A further perspective we might explore focuses on modeling by workers in settings out of school. Most recently Wake (2014) has suggested how mathematics in general education might learn from activity in workplaces. In summarizing findings from some dozen case studies of the mathematical activity of workers he reported: “Workplace activity with mathematics as central often relies on relatively simple mathematics embedded in complex situations (Steen, 1990). Making sense of this also provokes breakdowns, problem solving and modeling” (Wake, 2014).

The complexity of the situations that workers deal with is considerable, but of course it is an integral part of their daily life, and consequently, in their work, they often do not recognize that what they are doing involves mathematics at all. It certainly seems to bear little resemblance to the mathematics they met in school. This raises the important question: How can we better provide experiences of modeling in school that ensures good preparation for activity of this type in out of school settings such as workplaces?

Wake went on to suggest one way that we might reframe mathematics curricula by suggesting a model that could support the didactical transpositions that Chevallard (2002) identified as necessary in adapting mathematical knowledge for use in the day-to-day interactions of mathematics classrooms. This recognized how we must attend to the design that is essential if we are to bring into reality our aims and values in relation to modeling. As we highlight here, there are many different perspectives that might inform approaches to developing appropriate mathematics curricula, and these raise many different potential research questions. It is clear that a comparative approach to such research would be beneficial by providing additional insight as we have increased opportunities to test our hypotheses in a range of different cultural settings. A starting point is to focus on curriculum intentions, but the real richness of such work will be revealed as we explore modeling activity in classrooms throughout our international community.

4: INITIAL THOUGHTS ON THE INSTRUCTIONAL PERSPECTIVE

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[1]Catholic University, [2]University of Hamburg

While instruction in mathematical modeling shares many of the characteristics of quality teaching and learning in mathematics. at the same time, it is inclusive of a range of practices that are not a part of the traditional mathematics classroom (Niss, Blum & Galbraith, 2007). Approaches to teaching modeling can involve traditional methods or be based on innovative teaching practices such as inquiry methods, collaborative group based learning, and use of digital technologies. The nature of instruction in mathematical modeling varies according to many factors including: level of education, national context, curriculum intention and expectation, type of modeling tasks, and availability of teaching resources. Modeling tasks on which instruction is based can be drawn from a range of real-life situations including industry and the workplace, social and political issues, or daily life. Different contexts have implications for the design of modeling tasks and the selection of associated pedagogies.

This paper provides a brief synthesis of selected aspects of instruction in mathematical modeling. In doing so, we consider types of modeling activities and tasks and approaches to mathematical modeling teaching practice.

Modeling Cycles, Activities, and Tasks

The process of mathematical modeling remains a source of debate within the mathematical modeling community. The dominant perspective depicts mathematical modeling as a cyclic process in which mathematics is brought to bear on real-worldproblems through a series of steps or phases. While various forms of the modeling cycle are described in the literature (e.g., Blum, 1995; Kaiser, 1996; Pollak, 1968), these typically coalesce around a number of core activities: central influencing factors are identified; the real problem is simplified in order to build a manageable model of the situation; assumptions based on known factors are made to accommodate missing information; the real situation is translated into an idealised mathematical model; an initial solution is generated from the mathematical model; proposed solutions are tested against the initial real-worldsituation; a decision is made about the validity of a solution; and the process is revisited until an acceptable solution is established. These phases can take place in a linear fashion or frequent switching between the different steps of the modeling cycles may occur in generating a final solution (Borromeo Ferri, 2011). The modeling of real-worldproblems is challenging and so students will typically experience blockages to their progress (e.g., Stillman and

Galbraith, 2006). These blockages can be related to limitations in their content knowledge, cognitive impasses, and obstacles associated with beliefs or attitudes.

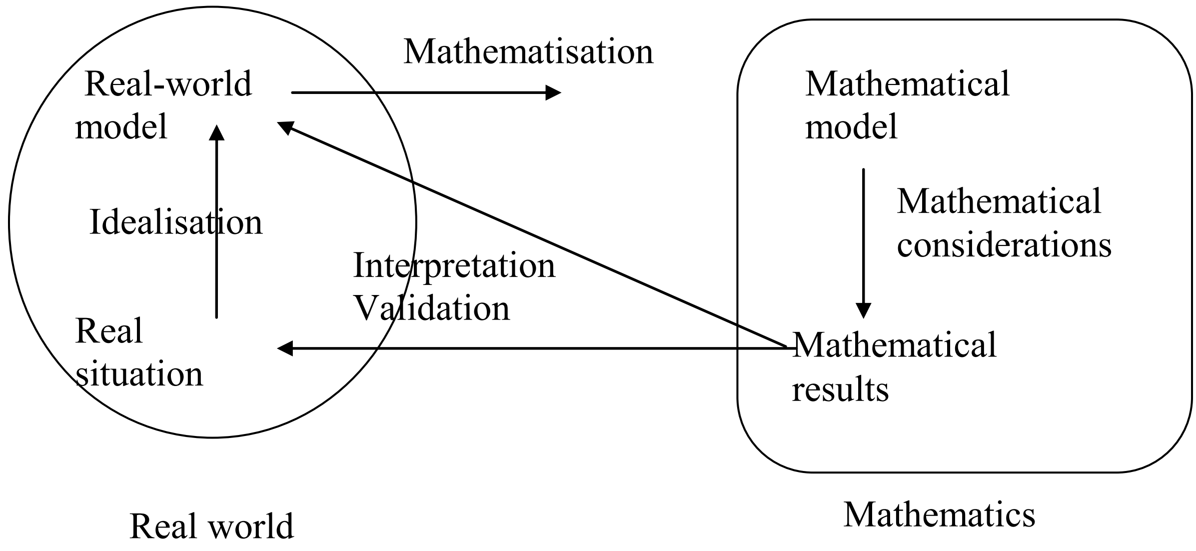

Figure 2: Modeling process from Kaiser-Meßmer (1986) and Blum (1996)

Other modeling approaches place cognitive analyses in the foreground and so include an additional stage within the modeling process, the understanding of the situation by the students. In this approach students develop a situated model, which is then translated into the real model (Blum, 2011). This approach is represented in Figure 3.

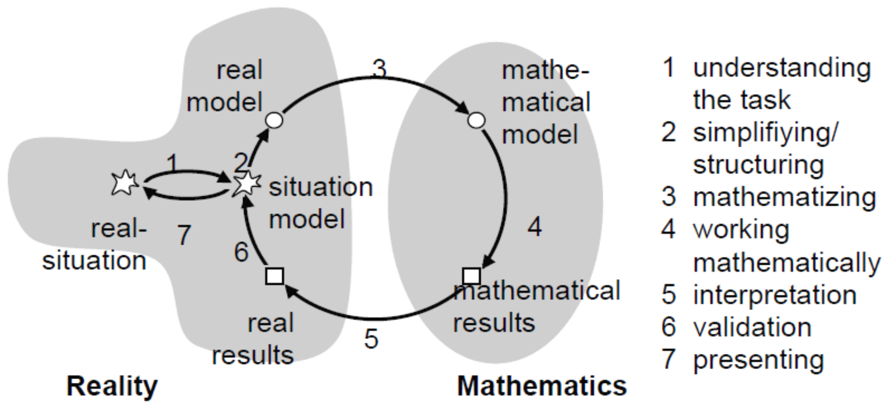

Figure. 3: Modeling process by Blum (2011)

While this cyclic process is consistent with the way many real-worldproblems are modeled, others argue for a broader definition for modeling that accommodates a wider range of context aligned mathematical activity. Modeling is considered by Doerr and English (2003), for example, as ‘‘systems of elements, operations, relationships, and rules that can be used to describe, explain, or predict the behaviour of some other familiar system’’ (p. 112). From this perspective, modeling makes use of mathematical thinking within realistic situations to accomplish some purpose or goal but may or may not involve a cyclic process. Alternatively, Niss (2013) distinguished between descriptive and prescriptive types of modeling. In descriptive modeling a real-world problem is specified and idealized, assumptions are made, relevant questions are posed, leading to the mathematization of the problem. Answers are then derived and justified and de-mathematized and finally validated. Thus, the processes associated with descriptive modeling are consistent with the cyclic view of mathematical modeling. By contrast, the purpose of prescriptive modeling is not to explain or make predictions about real-worldphenomena but to organize or structure a situation, for example – where should a new power plant be located? As the nature of prescriptive modeling cannot involve the validation of an initial solution, the process is not cyclic. Thus, Niss' insight into the nature of mathematical modeling suggests that the real-world phenomenon being investigated influences the way it is modeled, which in turn has implications for how instruction is organized to support students to work on a problem.

Approaches to Modeling Practice

The purpose of modeling from an instructional perspective can be considered as an objective in itself or as a method to achieve the goal of mathematics knowledge construction (Ikeda, 2013). The first purpose is based on the premise that the capacity to model and to find solutions to life related situations is a competence that can serve the individual in daily life and in the workplace. The second purpose is achieved when an individual constructs new knowledge or re-constructs knowledge they have already acquired (Van Den Heuvel-Panhuizen, 2003) when engaging with the process of modeling. As modeling requires the use of previously acquired mathematical knowledge in different ways it promotes a flexible and adaptable mindset in relation to the utilization of mathematical competencies. Challenging modeling problems, however, demand the appropriation of new mathematical facts, skills and processes, thus requiring the construction of new knowledge.

Niss and Blum (1991) distinguished six different approaches to instruction related to mathematical modeling and applications:

- Separation – in which mathematics and modeling are separated in different courses;
- Two-compartment – with pure and applied elements within the same course;
- Islands – where small islands of applied mathematics can be found within the pure course;
- Mixing – in which newly developed mathematical concepts and methods are activated towards applications and modeling, although the necessary mathematics is identified from the outset;
- Mathematics curriculum integrated – here real-life problems are identified and the mathematics required to deal with them is accessed and developed subsequently;

- Interdisciplinary integrated – operates with a full integration between mathematics and extra-mathematical activities where mathematics is not organized as separate subject.

While these approaches to instruction in mathematical modeling are distinct, they should not be seen as mutually exclusive, but rather as a choice to be made by teachers that reflects their intention when planning for instruction. This choice will impact the way they design modeling tasks (e.g., Geiger & Redmond, 2013). The design of tasks is also framed by the affordances and constraints of educational systems and school based circumstances. Tasks can be extended complex modeling problems in co-operative, self-directed learning environments (e.g., Blomhøj & Hoff Kjeldsen, 2006) through to more constrained versions of modeling tasks embedded taught within a traditional curriculum (e.g., Chen, 2013).

The nature of modeling task design, however, becomes increasingly complex once digital technologies are introduced into the range of resources available to students and teachers. Research into the role of digital technologies in supporting mathematical modeling indicates that more complex modeling problems become accessible to students (Geiger, Faragher, & Goos, 2010), but the successful implementation of technology “active” modeling tasks is largely dependent on the expertise and confidence of teachers as well as their beliefs about the nature of mathematics learning.

5: INITIAL THOUGHTS ON TEACHER EDUCATION PERSPECTIVE

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Earlier in this document, it was pointed out that the teaching profession faces difficulties in teaching mathematical modeling as mathematical content in its own right and using mathematical modeling as a teaching strategy to engage students in the learning of mathematics. Further, this becomes problematic for many teachers because of the different practices teachers must employ or adopt associated with allowing students more freedom to drive their own learning and the amount of specific domain knowledge that might be required. García and Ruiz-Higueras (2011) suggested that this problematic issue can be viewed from the perspective of renewal of the profession as a whole thus taking a top-down approach in researching issues associated with it, or alternatively, as a problem of the teacher in the classroom in renewing their models of teaching leading to research that focuses on more of a bottom-up approach. Both of these approaches are evident in the research literature associated with research into teacher education related to teaching modeling, whether it be researching in-service or pre-service teachers. In this section we examine the extent to which such research has taken as its focus (a) programs that support pre-service and in-service teachers in teaching mathematical modeling, and (b) interdisciplinary or extra-mathematical knowledge requirements for successfully teaching mathematical modeling. We also suggest where there are current gaps and the implications for future research.

Nature of Research into Teacher Education in Modeling

Many of the reports of studies into teacher education with respect to modeling are small-scale qualitative research studies involving the reporting of rich data from a few teachers usually from case studies (e.g., Villareal, Esteley, & Mina, 2010). This can be seen as either a sign that the research field is emerging or of the complexity of the phenomenon being studied (Adler et al., 2005). Both are clearly true. A third possibility is the way research is predominately reported in the field. Much research in this area is reported in short conference papers (e.g., Ng et al., 2013; Widjaja, 2010) or short book chapters (e.g., Stillman & Brown, 2011) in edited research books, and authors might not see these as ideal contexts for reporting larger studies. The focus of this research is teachers in teacher preparation and in-service courses. We have not found any studies where the reported focus is the teacher educators themselves and their expertise in supporting the teaching profession to address modeling so this is an area for future research.

Researching Programs Supporting Pre-service and In-service Teachers in Teaching Mathematical Modeling

Several programs for supporting pre-service teachers to teach mathematical modeling have begun to be developed and described around the world (e.g., Biembengut, 2013; Hana et al., 2013; Kaiser & Schwarz, 2006; Kaiser et al., 2013). A common approach is to involve pre-service teachers in modeling activities in order to develop a connected knowledge base in mathematics of both skills and concepts that can be applied to a variety of phenomena. There has, however, been limited research of the effectiveness of such programs. Often, the research is more of an exploratory nature investigating how modeling experiences can be infused into existing programs (e.g., Widjaja 2010, 2013). Table 1 shows a small selection of studies with pre-service teachers (PSTs) as the focus and selected claims or findings from these. In-depth evaluation studies identifying the ingredients of successful programs that can be scaled up for large course offerings should be the focus of future research.

In contrast, professional development (PD) programs or courses for in-service teachers have received much more research attention (e.g., de Oliveira & Barbosa, 2013) as these usually have been part of a funded project (e.g., LEMA see Table 2) of fixed duration with a research and evaluation study attached to it contingent on its successful completion in an expected time frame. Many results are localised to the context in which the programs were conducted but others clearly transcend contexts. Table 2 shows a small selection of studies with in-service teachers as the focus and selected claims or findings from these.

Program	Studies	Selected Findings/Claims
Brazillian PSTs	Biembengut (2013): study of course offerings across Brazil	Too little emphasis on MM although present in courses in all states; potential usefulness of MM developed in PSTs through such courses
Indonesian PSTs	Widjaja (2010); Widjaja (2013): study of MM activities	Must encourage PSTs to state assumptions & real-worldconsiderations of model in order to validate its appropriateness & utility
US elementary PSTs	Thomas & Hart (2010): models & modeling approach with Model Eliciting Activities (MEAs)	PSTs struggle with ambiguity of modeling activities; need to develop PSTs' ability to engage collaboratively with MEAs
Singaporean secondary mathematics PSTs	Tan & Ang (2013) using MM activities	PSTs need to experience MM for themselves developing meta- knowledge about modeling through such experiences
South African PSTs	Winter & Venkat (2013) using realistic word problems	PSTs abilities to reason within problem context critical; must develop deep, connected understanding of elementary mathematical content for successful modeling through such experiences

Table 1: Exemplar studies with pre-service teachers as focus

PD Program/Course	Reports	Findings/Claims
LEMA (Learning and Education in and through Modeling and Applications) 2006-9	Schmidt (2012): Pre, post & follow-up questionnaire for participants in training course and a control group; supplemented by interviews	Motivations to include MM in teaching which increased after the training course: Increases students ability to calculate & think more creatively, work independently & see relevance of mathematics to everyday life; modeling tasks have long term positive effects in mathematics lessons & beyond these and lesson teacher's workload
Making Mathematics More Meaningful M4	Berry (2010): design based research study	Refined group observation & teacher self-coaching tools designed & tested for teacher facilitation of optimizing student functioning in group work on MEAs

Experience 2004 with 3 secondary mathematics teachers & a university teacher	Villareal et al. (2010): main focus of report is student & task	MM offers space to construct new meaning for use of Information and Communication Technologies (ICTs) & ICTs are the media to think with and produce MM processes; Teacher, students & ICTs constituted a powerful thinking collective of Humans-with-Media
Training Program for non-certified teachers in Brazil	de Oliveira & Barbosa (2013)	Tensions in discourses can contribute to teacher PD through actions & strategies to deal with them; discussion of these tensions should be part of PST education
German in-service secondary mathematics teachers in academic-track schools	Kuntze (2011): quantitative comparative study of views	In-service teachers compared with PSTs saw a higher learning potential for tasks with higher modeling requirements; were less fearful of the inexactness of MM tasks; did not report good meta-knowledge about modeling.

Table 2: Exemplar studies with in-service teachers as focus

Researching Interdisciplinary or Extra-mathematical Knowledge Requirements for Successfully Teaching Mathematical Modeling

Within the studies of teacher education examined, there were few studies that addressed interdisciplinary or extra-mathematical knowledge requirements for successfully teaching mathematical modeling directly although some explained their findings (e.g., Tan & Ang, 2013; Winter & Venkat, 2013) by suggesting pre-service teachers isolated their modeling from the real-worldsituation in focus (e.g., car stopping distances), activated real-worldknowledge and attempted to incorporate such considerations into their modeling (Widjaja, 2013) or used contextual knowledge to interpret final mathematical answers (Winter & Venkat, 2013) within the problem context. Many classroom studies were found that alluded to the necessity for teachers, even in elementary settings, to have the knowledge background to make this knowledge visible to students. Mousoulides and English (2011), for example, when investigating the classroom activities of 12-year-old students exploring natural gas worldwide reserves and consumption, asked:

How we might assist students in better understanding how their mathematics and science learning in school relates to the solving of real problems outside the classroom and how we might broaden students' problem-solving experiences to promote creative and flexible use of mathematical ideas in interdisciplinary contexts?

They highlighted the issue of how the nature of engineering and engineering practice that relates to such problems can be made visible to these students. Studies which directly address interdisciplinary or extra-mathematical knowledge requirements for successfully teaching mathematical modeling are an area for future research.

CONCLUSION

This Research Forum starts to address a set of research questions in each perspective. Through the presentations and discussion, we hope to present a state of the art about the research on mathematical modelling from each perspective. After the conference, the organizers plan to develop a journal special issue and a book on the teaching and learning of mathematical modeling based on this Research Forum. We welcome all participants to contribute their ideas and papers.

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SPATIAL REASONING FOR YOUNG LEARNERS

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This Research Forum proposal arises from a recent focus on spatial reasoning that began with a multi-year collaborative project amongst a diverse group of researchers (mathematicians, psychologists, mathematics educators) from Canada and the US, and continues to expand with the goal of: mapping out the terrain of established research on spatial reasoning; consolidating that research within a nuanced discussion of the actualities and possibilities of spatial reasoning in contemporary school mathematics; offering a critical analysis of the theories and practices that define contemporary curriculum and pedagogy of school mathematics in a range of countries and contexts, and, offer examples of classroom emphases and speculations on research needs that might help to bring a stronger spatial reasoning emphasis into school mathematics.

BACKGROUND

Currently little time is spent in early years classrooms focusing on geometry and spatial thinking (Uttal et al., 2012). In fact geometry and spatial reasoning receive the least attention of the mathematics strands in North America (Bruce, Moss, & Ross, 2012; Clements & Sarama, 2011). However, there are several reasons to believe that this situation can and will change. The first is the extensive body of research over the past twenty years that has consistently shown the strong link between spatial abilities and success in math and science (Newcombe, 2010). Converging evidence from psychology research has revealed that people who perform well on measures of spatial ability also tend to perform well on measures of mathematics (Gathercole & Pickering, 2000) and are more likely to enter and succeed in STEM (science, technology, engineering and math) disciplines (Wai, Lubinski, & Benbow, 2009).

Second, there is a growing amount of evidence both from psychology and mathematics education showing that children come to school with a great deal of informal spatial reasoning (see Bryant, 2008), which is often not formally supported until much later in the curriculum, when numerical and algebraic ways of thinking have already become dominant. As early as four years old for example, children come to school with informal awareness of parallel relations, and although such relations are highly relevant to their work in two-dimensional shape identification and description, they are not formally studied until middle school. While such a concept might strike some educators as overly 'abstract' or 'formal', researchers have shown that given the appropriate and engaging learning environments, K-2 children can, indeed, develop very robust understandings of parallel lines (see Sinclair, de Freitas, & Ferrara, 2013).

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This leads to the third reason for change, which concerns the increase of digital technologies for young learners. In contrast to older software, such as Logo-based programming, which require numerical and/or symbolic input, or older mouse- and keyboard-drive hardware input, which can present motor dexterity challenges, newer touchscreen and multi-touch environments can greatly facilitate mathematical expression (Bruce, McPherson, Sabbeti, & Flynn, 2011). Research has already shown how new digital technologies that promote visual and kinetic interactions can help support the teaching and learning of spatial reasoning (Clements & Sarama, 2011; Highfield & Mulligan, 2007; Sinclair & Moss, 2012). These new technologies are challenging assumptions about what can be learned at the K-5 level; they are also showing that long-assumed learning trajectories might change drastically if spatial reasoning becomes a more central and explicit component of the curriculum.

Each of the sessions includes a plenary opening presentation that provides background on the historical, epistemological, psychological, mathematical and curricular contexts of spatial reasoning. Following each plenary, a coherent grouping of poster presentations illustrating classroom-based empirical studies will be used to incite small group discussions. Each round of posters will be followed by a facilitated whole group discussion (30 minutes) that highlights common themes and possible contradictions from the small group discussions and demonstrations. The first session critically analyses the role of spatial reasoning in the mathematics curriculum. The second session focuses on the role of different technologies in 'spatialising' the mathematics curriculum.

SESSION 1. THE ROLE OF SPATIAL REASONING IN MATHEMATICS LEARNING

This section contains two papers, followed by a five poster presentation descriptions. These will be used as a basis to motivate discussion around the following key questions:

1. What other factors may be contributing to the low emphasis on spatial reasoning in the curriculum? How might spatial reasoning fit into the numeracy strategies that many countries are pursuing?
2. What other strategies can be used to "spatialise" the curriculum and what impact might these have on assessment and professional development?

1: WHERE MATHEMATICS CURRICULUM COMES FROM

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The history of school mathematics is so tangled that attempts to offer a fulsome accounts necessarily drift toward fiction. The intention here is thus more to hint at its complexity.

With all the resources devoted to curriculum development, one would be justified in thinking that school mathematics undergoes constant reinvention – but curricular similarities across nations and centuries suggest otherwise. In other words, most curriculum developers don't actually develop curriculum. Their roles are more toward engineering programs of study.

How, then, did school mathematics come to be a trek from arithmetic through algebra to calculus, with modest diversions into geometry, statistics, and other amusements? To answer, I focus on three moments in its history, using a handful of personalities as metonyms for clusters of developments. To re-emphasize, the aim is neither completeness nor accuracy. For those I defer to others (Bishop et al., 1996; Howson, 1973; Menghini et al., 2008; Schubring, n.d.; Stanic & Kilpatrick, 2003).

Moment 1– up to the mid-1600s

The Pythagoreans (c. 500 BCE) are often credited with many innovations to schooling, both structural and curricular, but their major contributions are conceptual and philosophical. They helped to collect a rather disjoint set of facts and insights into a coherent, powerful system of knowledge that would later be at the heart of Plato's liberal arts – arts that are freeing.

However, "mathematics" was not part of those liberal arts, simply because it didn't exist as a coherent disciplinary domain. The devices were not yet invented to unite Logic, Arithmetic, Geometry, and other domains, but Euclid (c. 300 BCE) took a major step in that direction as he employed logic to prove, connect, and extend geometric truths. This massive intellectual leap set the stage for a unified discipline … but that unification had to wait almost two millennia until the 1600s when René Descartes brought together arithmetic, analysis, geometry, and logic through the masterstroke of using a coordinate system to link number and shape. That extraordinary contribution marked the emergence of the system of knowledge that we know as mathematics, affording a means to gather not just notions of number, shape, and argument, but also a host of other foci now recognized as properly mathematical.

However, Descartes did very little to shape school mathematics. Major contributions in that regard actually predate him – and that fact is telling. The content and foci of what was to become school mathematics were largely defined before mathematics cohered as a domain of inquiry, as exemplified in Robert Recorde's work. His 1540 textbook emphasized notations, procedures, and applications.

Moment 2 – mid-1600s to mid-1990s

There have actually been two distinct school mathematics through most of modern history, only merging in the mid-1900s within a broader education-for-all movement.

In the elementary schools of the newly industrialized world, curriculum was more aligned with the work of Recorde than that of Descartes. It was about ensuring that the workforce would have a functional numeracy.

Secondary school mathematics was quite a different beast. Intended more for elites, there mathematics more reflected what Descartes had brought forth. However, as the two projects converged, its contents were increasingly selected and framed through the utilitarian mindset of elementary schools.

Moment 3 – from the mid-1900s

Even so, the merging of elementary and secondary school mathematics was never made seamless. Teachers have experienced strategies aimed at unification as pendulum swings between rote competence and deep understanding; students have experienced them as discontinuities, reflected in the too-frequent confession of being "good at math until Grade 6."

Even so, revisions since the mid-1900s have proceeded as though school math were unified. Triggered by a sequence technological and economic rivalries with the USSR, Japan, and China, the progression of New Math, Reform Math, and the New New Math represented efforts to force coherence onto topics that were never chosen for that purpose. Consequently, while pedagogical emphases changed, the substantive content remained stable.

What's holding current curriculum in place?

With that stability, we force children to master competencies that are increasingly (if not completely) irrelevant as we ignore topics of growing necessity. Ironically, mathematics curriculum appears to find its stability in the conflicting interests of stakeholders. Facets of these include a culture of examination, a profit-driven textbook industry, inflexible university mathematics departments, and a self-perpetuating cycle of teachers teaching as they were taught – coupled to the fact that curriculum is a result, not an input. It cannot serve as a mechanism to effect change; it can only co-evolve with shifts in belief and expectation.

2: HOW IS SPATIAL REASONING RELATED TO MATHEMATICAL THINKING AND HOW IMPORTANT IS EARLY EXPOSURE TO SPATIAL ACTIVITIES?

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There is an urgent need to foster students' spatial reasoning skills. This need is recognized in the recent NCTM Focal Points that places its emphasis on geometry as one of the three key curriculum focal points (NCTM, 2006). At the same time, empirical evidence is emerging that suggests that those with a strong sense of space tend to be successful in mathematics and more broadly in the STEM disciplines (e.g., Newcombe, 2010; Wai, Lubinski, & Benbow, 2009). One goal of this presentation is to synthesize the recent findings that support the link between spatial reasoning and performance in mathematics and other STEM disciplines. Another goal is to discuss studies addressing the link between early spatial experience and later spatial sense. Below I will briefly present the studies conducted in my lab.

How is spatial reasoning related to mathematical thinking?

In one study, we asked 114 high school students (56 females) to fill out various measures of spatial thinking as well as to take a geometry test that consisted of age appropriate items taken from publicly available sources such as National Assessment of Educational Progress (NAEP, 2006). The goal was to examine whether we find differences in geometry performance between those who scored high on spatial measures and those who scored low on such measures. Because spatial cognition is a multi-faceted construct, we used three existing measures that assess different aspects of spatial thinking: Mental Rotation Task (MRT; Vanderberg & Kuse, 1978); Paper Folding (Ekstrom, French, Hartman, 1976); and Snowy Pictures (Ekstrom, French, & Harman (1976).

Using the composite scores, we identified 23 participants to be strong in spatial reasoning (high spatial) and 25 participants to weak in such reasoning (low spatial). There was a large, significant difference between the two groups on the test of geometry. In addition, high spatial participants had higher grades in geometry. They also earned higher grades in algebra but this difference did not reach statistical significance. When the geometry test items were separated into either 2- or 3-dimensional items, the high spatial group did significantly better on the 2-dimensional items than the other group and the difference on the 3-dimensional items approached significance in favor of the high spatial group.

These findings demonstrate a strong link between spatial sense and mathematical performance (at least in geometry). Lack of statistical differences in algebra grades and 3-dimensional items are not necessarily discouraging. After all, these high school students were taught mathematics that did not emphasize geometry or spatial solutions to algebraic problems. Had geometry and spatial thinking been emphasized, spatially oriented students would have excelled not only in 2-dimensional geometry but also in ways to approach complex situations spatially.

How do first graders understand geometric shapes?

We are currently conducting a study to understand how first-grade children reasoning about plane and solid shapes. We are particularly interested in how children interpret various geometric shapes when asked to compose or decompose them.

Our initial effort includes 15 first graders. We devised four composition and four decomposition items. For each item, there was a stimulus item and four option shapes. One option shape was exactly the same as the stimulus shape; two others matched but included distracting features; and one item was non-match. For each of the composition items, children saw a stimulus item that is a 2D diagram of a plane geometric figure. They were then shown four other geometric figures that are solid or plane 3D figures. Their task was to choose among the four options the figures in which the stimulus shape is contained. That is, the stimulus shape is a composite of the selected figure. For each of the decomposition items, children saw a stimulus item that is a 2D diagram of a solid geometric figure. Similar to the composition items, they were then shown four solid or plane 3D figures. Their task was to choose among the four options whose figures consist of all or part of the stimulus shape. That is, the stimulus shape that is solid must be decomposed to 2D components that make that solid shape.

Our preliminary findings indicate that first graders were able to find option shapes that were exact matches to the stimulus shapes. But when some distracting features were included, they had difficulty recognizing matched features in either composition or decomposition items. Interestingly, children found triangular vertices ("pointiness") as a significant feature to accept or reject option figures. Overall, we found inconsistency in children's criteria for choosing features of geometric shapes in deciding a match or non-match.

How important is early exposure to spatial activities?

Our interest in this line of work is to examine types of activities in which highly spatial individuals have participated throughout their lives. We are particularly interested in activities in which they participated early and those in which they continue(d) to participate for a prolonged time period. We are currently collecting data from junior high, high school and college students. Here, we describe our first effort that investigated gifted 7th and 8th graders.

We identified 14 students (9 females) as highly gifted in quantitative and/or verbal reasoning. They all filled out a survey that listed 70 spatially oriented activities in five different areas: computers, toys, sports, music and art. The group as a whole reported that their favorite activities included video games, blocks, board games, soccer, piano, and drawing. Because our participants included students who were identified as gifted in quantitative and/or verbal reasoning, we decided to give two types of spatial measures to further identify those who are strong in mental rotation abilities (MRT) and those who use verbal cues to process spatial information (verbal-spatial task (VST); adapted from Hermelin and O'Connor, 1986). This resulted in five students who scored high on MRT but low on VST and five students who were high on VST but low on MRT. We found that high MRT/low VST students favoured activities similar to the rest of the students in the study but for a longer period of time. These students consistently favored these activities from early childhood to the present time. On the other hand, there were no consistent patterns that emerged in the favored activities by the high VST/low MRT students.

These findings show that gifted students in general and those strong in mental rotation in particular tend to favor activities that include highly spatial elements and the latter group in particular engaged in such activities for a long period of time. This study suggests there is a link between spatial activities and spatial thinking (mental rotation in this study). As we collect more data from a wider age range of students, both gifted and non-gifted, as well as use measures that assess different aspects of spatial reasoning, we hope to more clearly explain how early spatial experiences contribute to later spatial and geometric reasoning skills.

POSTERS: (RE) 'SPATIALISING' THE CURRICULUM

3: CHANGING PERCEPTIONS OF YOUNG CHILDREN'S GEOMETRY AND SPATIAL REASONING COMPETENCIES: LESSONS FROM THE "MATH FOR YOUNG CHILDREN" (M4YC) PROJECT

Joan Moss[1], Zachary Hawes[1], Sarah Naqvi[1], Beverly Caswell[1],
Catherine D. Bruce[2], Tara Flynn[2]
[1]OISE, [2]Trent University

We are witnessing an unprecedented political and academic focus on mathematics in early years classrooms which has included a call for a greater emphasis on geometry and spatial reasoning. Spatial reasoning in early years is foundational, not only to later success in mathematics (Mix & Cheng, 2012), but also to success in the STEM disciplines (Wai, Lubinski, & Benbow, 2009; Newcombe, 2010). Unfortunately, not all children have equal access and exposure to spatial reasoning (e.g., Casey, et al., 2008): indeed, recent research has shown striking SES-related differences in spatial reasoning in children as early as 3-years of age (Verdine, et al 2013).

The Math for Young Children Project

For the last 3 years, we have been working on a professional development research project to promote and enhance the teaching and learning of geometry and spatial reasoning in early years classrooms. Using a Japanese Lesson Study approach, the Math for Young Children project has, to date, collaborated with more than 15 teacher-researcher teams which have included more than 100 early years teachers (pre-school to second grade) and their students. Demographically, we have been working in underserved populations, typically in schools with low provincial test scores.

Our work with our teacher-researcher teams has been the co-design and implementation of lessons, activities, assessment tools and trajectories that build on the development of children's geometrical and spatial reasoning. An important mission of this project has been to gather data to demonstrate that young children – regardless of SES background – are capable of exceeding current expectations in geometry and spatial reasoning given carefully crafted learning experiences.

In this poster we focus on the work of one team of 8 teachers and their Kindergarten and Grade 1 students from a large urban low-SES school. We present the design, implementation and results of two "Research Lessons", both of which involved knowledge and geometric reasoning well beyond curriculum expectations.

The first lesson, conducted with 3-5 year olds, centered on the "pentomino challenge" involved students' in discovering the twelve unique shapes, composed with 5 squares (see Figures 1 and 2).

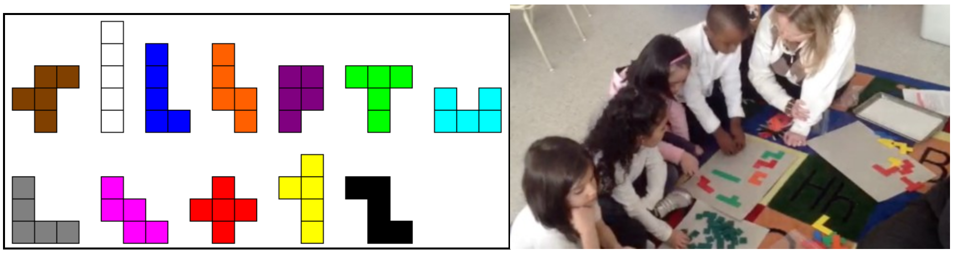

Figure 1: Set of 12 pentominoes

In the second lesson, "The Upside Down World, Grade 1 students were challenged to recreate 'buildings' composed of multilink cubes in their upright orientation and use spatial language to describe the composition of the 'buildings for other class members to build accordingly (see Figure 2).

Figure 2: Grade 1 students working with teacher on Upside-down World lesson

Results

Overall the results revealed that the majority of the 4-, 5- and 6-year-old students performed well above expected levels on activities involving aspects of geometry typically reserved for older students in later grades. Specifically, among our findings were that the kindergarten students were able to recognize, rotations and reflections in 2-dimensional figures, demonstrate an understanding of congruence and, collectively, find all twelve pentomino configurations (see Figure 2). The grade 1 students showed the ability to: 1) copy and assemble 3-dimensional shapes composed of multi-link cubes; 2) discriminate between congruent and non-congruent 3-dimensional figures; and, 3) describe to classmates the necessary steps required to rebuild a shape that was flipped upside down from its intended orientation. Moreover, students as young as kindergarten age demonstrated a great interest in the activities and were able to sustained their interest and engagement in the tasks and lessons.

4: THE APPLICATION OF AMBIGUOUS FIGURES TO MATHEMATICS: IN SEARCH OF THE SPATIAL COMPONENTS OF NUMBER

Michelle Drefs, Lissa D'Amour
University of Calgary

Ambiguous figures allow for a single image to be correctly perceived and interpreted in more than one way. A classic example is the "My Wife and My Mother-in-law" illustration, in which the image can be viewed as either the side profile of a young woman or as an old woman from the front (Boring, 1930). Which image first emerges depends on what information the viewer attends to and privileges within the display (e.g., de Gardelle, Sackur, & Kouider, 2009). The same principles can be seen to apply when examining the tasks and activities used to develop students' understanding of number and number systems. Specifically, many commonly used number tasks are viewed as predominately, if not completely, numerical in nature. However, evidence is accumulating in support of mathematical knowledge as having a strong spatial component (e.g., Bishop, 2008). In fact, both domain-specific (e.g., Approximate Number System; Butterworth, 1999) and domain-general (e.g., visuo-spatial processing) spatial mechanisms are believed to undergird numerical competencies and mathematical thinking (see Bonny & Lourenco, 2012; Sasanguie, Göbel, Moll, Smets, & Reynvoet, 2013). In alignment with this, several researchers have identified spatial aspects as involved in a number of commonly utilized numeracy tasks and activities. For example, LeFevre and collaborators (2010) identified that spatial attention plays an important role in preschool and kindergarten children's ability to successfully complete number naming, numeration, and symbolic magnitude comparison tasks as well as to areas of mathematic achievement two years later. Ansari et al. (2003) similarly found visuo-spatial abilities to account for typically-developing children's success on standard "How many?" and "Give a number" cardinality tasks. Aside from directly impacting student's number knowledge, spatial cognition may additionally contribute indirectly with, for example, spatial visualization supporting word problem-solving performance (e.g., van Garderen, 2013). The importance of spatial cognition to number knowledge is also highlighted in the work of Mowat and Davis (2010) in which they discuss the importance of utilizing sensori-motor experiences to deeply understanding a particular math concept. They argue that inherent to an understanding of number as a "position in space" is the spatial component of movement along a continuous measurement scale.

To identify aspects of "spatial cognition" of relevance and importance to mathematical tasks and activities requires, as with ambiguous figures, a shift in what information is being attended to and privileged. The purpose of this poster session is thus to explore this "ambiguous" interplay between number and spatial cognition as it relates to number as a focal point in the curriculum. Specifically, it is intended that participants will be provided an opportunity is to explore the possible intersects between number and spatial cognition. Emphasis will be given to: (1) outlining several cognitive mechanisms that provide support for a spatial component of number, and (2) providing common number tasks and activities that attendees can examine and discuss with respect to the contributions of both spatial and number knowledge.

5: YOUNG CHILDREN'S THINKING ABOUT DIFFERENT TYPES OF DYNAMIC TRIANGLES

Harpreet Kaur
Simon Fraser University

The aim of this study is to show how young children (age 7-8, grade 2/3) can exploit the potential of dynamic geometry environments to identify, classify and define different classes of triangles (scalene, isosceles, equilateral). Our motivation was to explore what can be further done on the topic of triangles in the lower primary school while remaining within the topic of the curriculum but extending the geometric applicability and sophistication.

We developed the triangle Shape Makers sketches (see some examples in Figure 1 a, b, c, d) for different types of triangles (scalene, isosceles, equilateral triangles, right triangle) to extend the work of Battista (2008). Each triangle type had a different colour (pink for scalene, red for equilateral, blue for isosceles and green for right). In the sketch shown in Figure (1a) only the middle triangle is constructed to be equilateral. Sketches in Figure 1 (b, c, d) were used as a way of focusing attention on the inclusive relations.

(a) Drag each of these triangles around. What do you notice about the kind of shapes they can each make?

(b) & (c): Which coloured triangles can fit into given triangle outlines?

(d) Whether a scalene triangle can fit into the given equilateral triangle outline (top) and vice versa (bottom)?

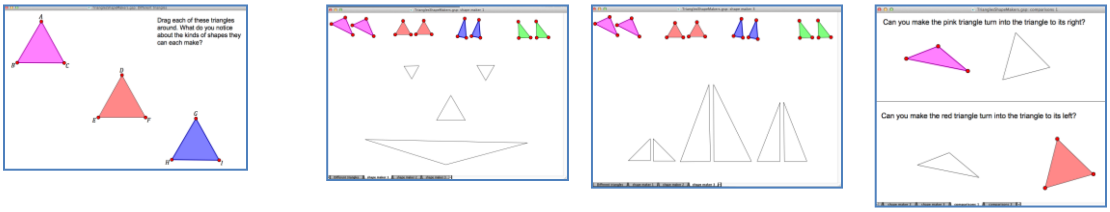

Figure 1: Different triangle sketches

The research was undertaken in the context of a classroom-based intervention. Three lessons on triangles were conducted with the 24 children seated on a carpet in front of an interactive whiteboard. Previous lessons with *Sketchpad* involved the concepts of symmetry and angles, but the children had never received formal instruction about classification of triangles before. We used Sfard's (2008) communicational approach for looking at children's discourse (thinking) about dynamic triangles along with their gestures, use of diagrams during their interactions with dynamic sketches.

(a) (b) (c)

Figure 2: Snapshots of various gestures by children

Over the course of their interaction with the dynamic triangles, children's discourse started with noticing the informal properties based on dragging behaviour and eventually moved to formal properties (e.g. angles are staying same in the equilateral triangle). The children and the teacher used "if…then" statements extensively to talk about the behaviour of the different triangles in the dynamic environment. The dynamic behaviour of the triangles prompted the children to make connections of the restricted or free movements of the dynamic triangles with real life experiences of the restrictive mobility of humans (e.g. children used word 'paralysed' for isosceles triangles). Children made ample use of gestures (see Figure 2 a, b, c) during the intervention (e.g. stretching both arms to show moving behaviour of sides of isosceles triangle, gesturing a triangle with the fingers of both hands, tilting head sideways to recognise non-prototypical isosceles triangle,). The children's communication shifted from specific to more generalized statements as intervention progressed, which included inclusive descriptions of classes of triangles (in other words, they thought about equilateral triangles as special types of isosceles triangles). This kind of reasoning emerged as a result of the children's attempts to overlap or fit different triangles on each other in Sketchpad. This study also provides the initial evidence that the teaching of concepts like symmetry and angles in early years can lead to whole set of new possibilities of geometric reasoning about shape and space for young children.

6: KINDERGARTENERS' ABILITIES IN PERSPECTIVE TAKING

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Contemporary early childhood curricula and educational programs emphasize the need to start with 3D geometry at an early age (NCTM, 2008; Van den Heuvel-Panhuizen & Buys, 2008). This approach is in line with Freudenthal's (1973) view regarding early geometry learning: "Geometry is grasping space […] that space in which the child lives, breathes and moves. The space that the child must learn to know, explore, conquer, in order to live, breathe and move better in it" (p. 403)

Thus, spatial ability is important for young children to learn and therefore, it is worthwhile to gain more insight into how they develop this ability. A major component of spatial ability is the competence of imagining objects from different perspectives of the viewer (e.g., Hegarty & Waller, 2004). In the present study, we focused on this specific spatial competence of kindergartners, namely, imaginary perspective taking competence (IPT), which means that children are able to mentally take a particular point of view and that they can reason from this imagined perspective.

Flavell, Everett, Croft, and Flavell (1981) proposed and validated a distinction into two abilities of perspective taking. The Level 1 competence concerns the *visibility of objects*, that is, the ability to infer which objects are and are not visible from a particular viewpoint. The Level 2 competence is related to the *appearance of objects, that is, the ability* to make judgments about how an object looks from a particular viewpoint.

The aim of this study was to gain more insight into kindergartners' IPT and specifically IPT type 1 (visibility) and IPT type 2 (appearance). Also, we intended to identify cross-cultural patterns in this competence and therefore we included children from two countries in our study. In particular, we investigated how able kindergartners are in IPT type 1 and type 2, how these competences are related, and whether the IPT competence is related to children's kindergarten year, mathematics ability and gender. Furthermore, we examined whether there are cultural similarities and differences in these IPT competence issues.

The sample consisted of 4- and 5-year-old kindergartners in the Netherlands (N=334) and in Cyprus (N=304). Children's IPT competence was assessed by a paper-and-pencil test of various perspective-taking pictorial items which require either IPT type 1 or IPT type 2. In Figure 1 two test items are given. The *Duck* item (instruction: “The duck has fallen into the hole. He looks up. What does he see?”) is meant for measuring IPT type 1, while the *Soccer* item (instruction: “Two children are playing soccer. How do you see it if you look from above like a bird?”) is meant for measuring IPT type 2.

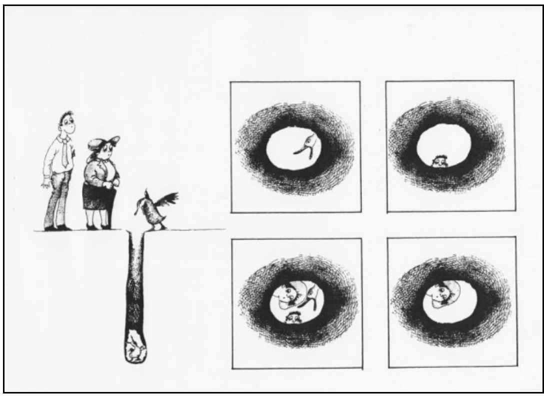


Figure 1a: *Duck* item

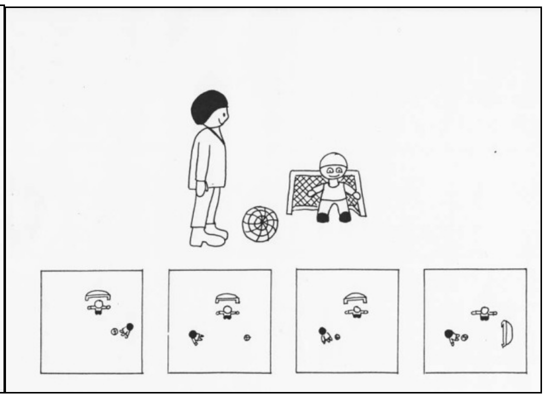


Figure 1b: *Soccer* item

The results revealed interesting common patterns for the two IPT types in both countries. Specifically, IPT 2 items were significantly more difficult than IPT 1 items and children’s success on the former items implies success on the latter items. Also in both countries, IPT 1 appeared to develop during the kindergarten years. For IPT 2 this was the case only in the Netherlands. In the two countries, there were no significant gender differences for kindergartners’ IPT competence. However, the relationship between children’s IPT competence and mathematics ability was not so clear, as in the Netherlands and in Cyprus significant interaction effects were found.

7: A SPATIAL-VISUAL APPROACH TO OPTIMIZATION AND RATES OF CHANGE

Robyn Ruttenberg[1], Ami Mamolo[2], Walter Whiteley[1]
[1]York University, [2]UOIT

Optimization and rates of change are familiar terms to calculus students, but what do these concepts look like when lifted from that context and interpreted through a spatial-visual lens? We present research on elementary students’ spatial-visual with a well-known optimization problem: the “popcorn box problem.” We will illustrate how different tools and representations, and their affordances– in our case 3-D models – can provide enriched learning experiences for pupils by allowing them entry and access to sophisticated mathematics without the need for computation or calculation.

The popcorn box problem

> Given a square sheet of material, cut equal squares from the corners and fold up the sides to make an open-top box. How large should the square cut-outs be to make the box contain maximum volume?

To investigate this problem, we introduced a network of activities which culminated in the physical exploration of pairs of clear-plastic boxes with coloured foam inserts that represent volume lost and gained when comparing two boxes (see Figure 1, below). Our poster highlights key features of the activities, as well as our design considerations for scaffolding the exploration. Via engagement with the activities and models, several mathematical ideas and observations emerged in a manner accessible to learners of various ages and mathematical sophistication. We identify a few key ones accessible to young pupils here:

Figure 1: Pairs of clear plastic 'popcorn boxes'

(i) The volume of the boxes can change, and boxes with different shapes could have the same volume.
(ii) There is a largest volume.
(iii) The maximum volume is *not*: at either extreme, in the cube shape, or the 'middle' between extremes.
(iv) Volume and surface area can be physically represented, and physically compared in multiple consistent ways.
(v) Change in volume between pairs of boxes (Figure 1) involves both volume lost (on the sides of the outside box) and volume gained (on the top of the inside box), as the cut size is increased.
(vi) Given the uniform thickness of these gains and losses (the size of the increase in the cut), these changes in volume (loss and gain) between pairs of boxes can be compared with clarity by naïve overlay strategies (see Figure 2).
(vii) This reasoning can be extended to other pairs of boxes by adapting approximate loss-gain representations.
(viii) That the side ratios of the optimal shape were invariant under scaling (proportional reasoning).

We present some of the challenges and coping strategies that emerged as pupils engaged with the activities. Challenges such as how to accurately compare two boxes "close" in volume (but possibly very different in shape), and how to compensate for the physical constraints of imperfections of the tools, led to the negotiation of new strategies for comparing boxes as well as discussions of how to refine comparisons. Results suggest that big ideas about volume, and changes in volume, that students at the Grade 9 level had not noticed with regular ‘volume related’ calculations in their curriculum, are accessible to young pupils through our approach.

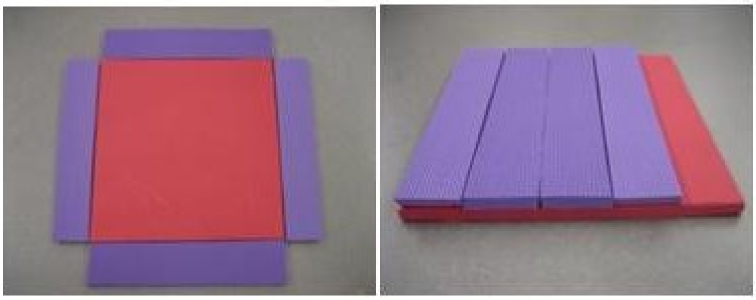


Figure 2: Foam inserts removed and compared via overlay

SESSION 2. SPATIALIZING THE CURRICULUM: CLASSROOM TOOLS AND TECHNOLOGY IMPLICATIONS

The second part contains two papers and five descriptions of poster presentations. These will motivate discussion on the following key questions:

1. What role do different tools (both digital and non-digital) play in promoting spatial reasoning in learning mathematics?
2. How might spatial reasoning be linked to the kinds of kinaesthetic and haptic forms of interaction that are available in new multi-touch platforms?

8: THE MALLEABILITY OF SPATIAL REASONING AND ITS RELATIONSHIP TO GROWTH IN COMPETENCE

Diane Tepylo, Joan Moss, Zachary Hawes
OISE

In what ways can we support the development of young children’s mathematical thinking; ways that are equitable to all young children? In this presentation we highlight the potential role of spatial reasoning and argue for the importance of a “spatial education” for young children.

Over a century of research confirms the close connection between spatial thinking and mathematical performance (Mix & Cheng, 2012). Recently, researchers have found that spatial thinking is not only correlated with mathematics but also predictive of later mathematics performance. To investigate the influence of early spatial learning skills on later mathematics, Verdine and colleagues (2013) carried out a longitudinal study with 3- to 5-year olds, in which they assessed children’s performance on standard vocabulary and mathematics measures, as well as on measures of spatial reasoning in the form spatial assembly tasks. A surprising finding was that children’s spatial reasoning skills at age three were the strongest predictor of mathematical skills at age five—even more than the three-year olds' math skills (Farmer et al., 2013). Another surprising finding was the significant SES-related difference in the spatial skills of three-year-olds.

In another longitudinal study, researchers investigated how the quality of kindergarten students' block play related to their performance in mathematics up to ten years later. Remarkably, kindergarten students' block building skills predicted their mathematics success in high school, even after controlling for IQ (Wolfgang et al., 2001). Furthermore, brain imaging studies corroborated the link between spatial and mathematical processing, revealing that overlapping brain regions are active during the performance of both spatial and of mathematical tasks. Given the proven relationship between spatial thinking and mathematics, an important question emerges: Is it possible to improve spatial reasoning skills?

Once believed to be a fixed trait (e.g. Newcombe, 2010), there is emerging evidence that spatial thinking is malleable. A recent meta-analysis of 217 training studies surveying more than 20 years of research, concluded that spatial thinking can be improved in people of all ages through diverse sets of training activities. The implications are far reaching, especially with respect to early interventions which have been repeatedly shown to be most effective in bringing about long-term change (Heckman, 2006). Many spatial training efforts carried out with young children demonstrate that it is possible to improve the children's spatial skills, reduce or eliminate early gender differences and, critically, reduce differences attributable to SES.

These findings lead to the next question: can training in spatial reasoning support mathematics performance? This is a long-standing issue in the cognitive science field: making a causal link between spatial training and math performance has proved to be difficult. Recently, however, two studies clearly link spatial training and improved math ability: the first study, with at risk children in a long term afterschool program Grissmer et al., (2013), and the second study with typically developing children, in a single 20 minute training program (Cheng and Mix, 2013).

In a controlled random assignment afterschool intervention study, Grissmer and colleagues (2013) invited kindergarten and Grade 1 children to construct and copy designs made from a variety of materials including Legos®, Wikki Stix®, and pattern blocks. A control group was given a non-spatial curriculum. After 7 months, 4 days a week, the children in the experimental group made substantial gains in their mathematics and spatial reasoning, moving from the 30th to the 47th percentile on a nationwide test of numeracy and applied problems. Most striking, there was no "mathematics" taught as part of the intervention, nor did the instructors ever specify any connections between the construction activities and mathematics. There were no gains in mathematics performance in the control group.

In the second intervention, Cheng and Mix (2013) randomly assigned 6 and 7 year old students to either a single-session mental rotation-training group or a crossword puzzle group. Children in the spatial training group, but not the crossword group, demonstrated significant improvements on their abilities to solve problems involving place value and addition and subtraction. Surprisingly, the greatest improvement was on the difficult missing terms problems such as 5+ __ = 11. The researchers posit that the spatial intervention may have encouraged children to mentally rotate the question into the more common equation: 11 – 5 = __.

There is growing awareness of the foundational role spatial reasoning has in mathematics and many scientific disciplines, but, notably, spatial reasoning is rarely taught. The NRC expresses concern in their 2006 spatial reasoning review: "spatial reasoning is not only under supported, under-appreciated, and under-valued, but it is underinstructed" (p. 5). We join them in their commitment to the development of spatial thinking across the curriculum.

9: THE ROLE OF TOOLS AND TECHNOLOGIES IN INCREASING THE TYPES AND NATURE OF SPATIAL REASONING TASKS IN THE CLASSROOM

Cathy D. Bruce[1], Nathalie Sinclair[2]
[1]Trent University, [2]Simon Fraser University

Since the inception of Kindergarten, mathematics learning tools have been envisaged as a feature of most mathematics classrooms for young children. For example, Friedrich Froebel, the progenitor of Kindergarten, and Maria Montessori, were powerful influences in the first policies related to Kindergarten, proposing that young children could engage in serious intellectual mathematical work; and both developed specific mathematical tools for students to use in order to explore these mathematics concepts. In fact Froebel's first ten sets of thinking tools, known as 'gifts' for the students, were 3-D figures that encouraged children to explore spatial reasoning and geometry (http://www.froebelgifts.com/). More recently, digital learning tools have been introduced to learners, the first of which relied on mediating objects such as a mouse, joystick or keyboard to operate the technology.

Over the past decade, several researchers have argued for the appropriateness and benefit of using "virtual manipulatives" (VMs) in the early grades, which build on the familiarity of physical ones, but which may also provide a range of added affordances. These researchers have questioned the assumption that "concrete" tools are more appropriate for young children and have argued that physical manipulatives are limited in their ability to promote both mathematical actions and reflections on these actions (Sarama & Clements, 2009). These authors point specifically to a VMs potential for supporting the development of *integrated-concrete* knowledge, which interconnects knowledge of physical objects, actions on these objects and symbolic representations of these objects and actions. Beyond VMs, there are also digital technologies that have little or no relation to physical manipulatives, but that also have unique potential for younger learners—especially dynamic mathematical technologies that help focus attention on mathematical relations and invariance as well as providing rich, visual examples spaces for mathematical objects such as triangles and numbers (Battista, 2007; Bruce et al. 2011; Highfield & Mulligan 2007; Sinclair & Crespo, 2006; Sinclair, de Freitas, & Ferrara 2013; Sinclair & Moss 2012).

With the advent of interactive whiteboards, touchscreen technologies were introduced to classrooms, and the role of technology in learning instantly expanded well beyond the initial observed roles. In an early publication on touchscreen technologies, Pratt and Davison (2003) describe the "visual and kinesthetic affordances" of the interactive whiteboard. Visual affordances relate to "the size, clarity and colourful impact of the computer graphics, writ large on the whiteboard" (p. 31). Kinesthetic affordances relate to "the potential impact of dynamically manipulating the screen in such a way that the teacher's (or child's) agency in the process is far more impressive than merely following a small mouse arrow" (p. 31). With further advances of touchscreen technologies, including tablets and handheld devices, direct-touch response has once again sparked new directions in educational research. Researchers are observing that in comparison to handling physical tools, engaging with virtual tools on touchscreens provides students with *different* kinesthetic experiences. The Technological-pedagogical interactivity (see Figure 1) which is engendered in a technology-mediated learning environment (Bruce & Flynn, 2012) is particularly salient in relation to spatial reasoning.

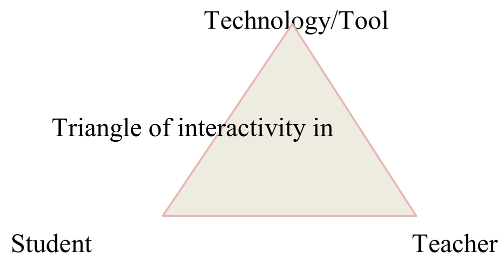


Figure 1: One conception of the technological-pedagogical interactivity triangle in technology mediated learning environments (Bruce & Flynn, 2012)

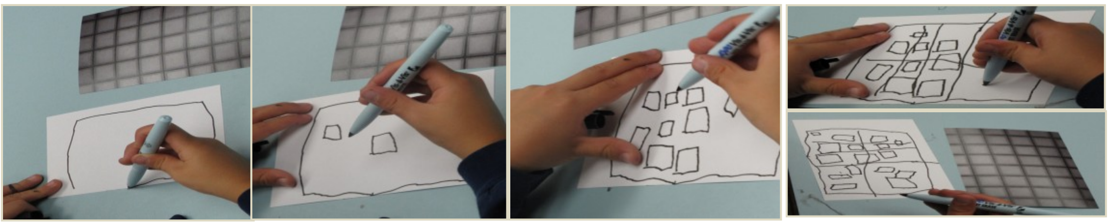
In an alternative approach to tool-use, de Freitas and Sinclair (2014) view the student, teacher, tool, mathematical concept as an assemblage that intra-acts because the distinctions between each "body" is not necessarily pre-determined. The focus thus shifts from the student or the tool to the interactive student-tool relation. In terms of spatial reasoning, they show how particular finger movements on the screen enable new ways of seeing numbers and operations. Gestures are one of the significant forms of finger movements used on touchscreen technologies and recent research that draws on the role of gestures in thinking and learning mathematics more broadly has pointed to their potentially unique role within the interactive environment of touchscreen technologies (Bruce et al., 2011; Sinclair, de Freitas, & Ferrara 2013) and number (Sinclair & Heyd-Metzuyanim, 2014; Sinclair & Pimm, 2014).

Recent technologies and tools are also putting into question the typical estimations of what children can do and understand spatially. Consider for example, how iPad applications such as *TouchCounts* (Sinclair & Jackiw, 2011) and *Spot the Dots* (Bruce) encourage young children to use spatial reasoning and gesture to cement foundational number concepts of quantity, magnitude, ordinality, cardinality and composition of number. In a unique tool design study, Hawes et al. (in press) have developed physical materials that are proving how children as young as 4.5 years old can imagine complex rotations of 3-D figures – a skill previously demonstrated to be too difficult for this age.

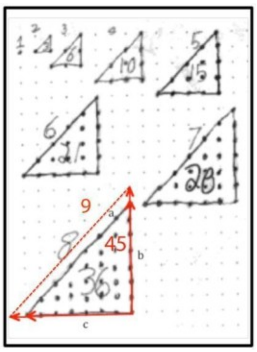
POSTERS: THE ROLES OF TOOLS IN SPATIAL REASONING

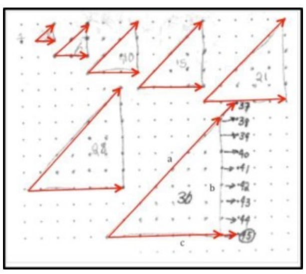
10: CHILDREN'S DRAWINGS: A BODYING-FORTH OF SPATIAL REASONINGS

Jennifer S. Thom[1], Lynn M. McGarvey[2]
[1]University of Victoria, [2]University of Alberta



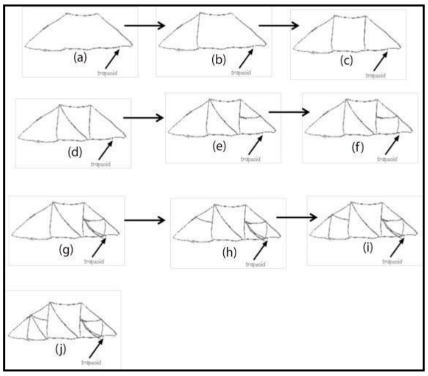
There is a growing interest in mathematics education to better understand children's spatial reasoning, not only in a general sense (e.g., its development) but also in terms of the particular manners in which children's spatial reasonings occur. In response to this call, we examine the role that drawing plays in the growth of children's conceptual understandings concerning properties and spatial relationships within and amongst two-dimensional shapes as well as the three-dimensional world.





Currently, prominent perspectives assume children's drawings to be external representations of their inner mental functioning. As such, establishing a child's level of understanding is achieved by comparing drawings against a hierarchical set of developmental and cognitive criteria (e.g., Piaget & Inhelder, 1956; Mulligan &

Mitchelmore, 2009). Our research, in contrast, takes an alternative approach. We conceive children's drawings to be ever-emergent artefacts and multi-sensory motor processes of *thinking forth of a world or worlds* (Woodward, 2012). In this way, drawing is "a matter of learning as much as it [is] a matter of thinking" (Cain, 2010, p. 32). Thus, any deep insights we might gain about children's spatial reasonings necessitate inquiry into the moment to momentness of their acts of drawings and the conceptualizations that occur within these moments.



Our research reveals instances in which drawing plays diverse and compelling roles in children's spatial reasonings. Through examplars taken from case studies across kindergarten through the second grade, we illustrate: how the children come to draw as a mode of thinking; the different ways that they draw and use drawings to attend to important mathematical ideas (Depraz, Varela & Vermersch, 2003); and the conceptions that arise with and in drawing that contribute substantially to the growth of their geometric and spatial reasonings.

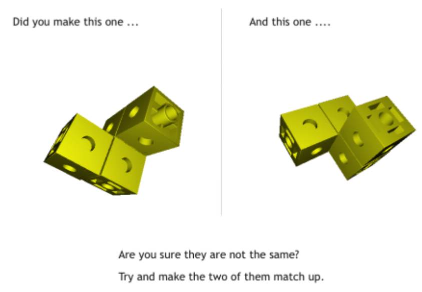
11: USE OF THE iPAD AS A MEDIATOR FOR THE DEVELOPMENT OF SPATIAL REASONING

Catherine D. Bruce
Trent University

Given the recent explosion of handheld technologies and tablets in education, researchers Bruce, Davis, Moss and Sinclair of the IOSTEM spatial reasoning working group have been investigating the role of iPad technology as a mediator of spatial reasoning for young children. There are three aspects to these explorations: i) Identifying the visual-spatial affordances of the iPad; ii) Capitalizing on the use of gesture to increase spatial reasoning; and, iii) 'Spatializing' mathematics, within but also beyond geometry to mathematics domains such as number sense and numeration. Design Research provides a useful research methodology and framework for iPad product development because it emphasizes development-testing-refinement-testing, and embraces complex contexts such as mathematics classroom and home learning environments.

Visual-spatial affordances of the iPad

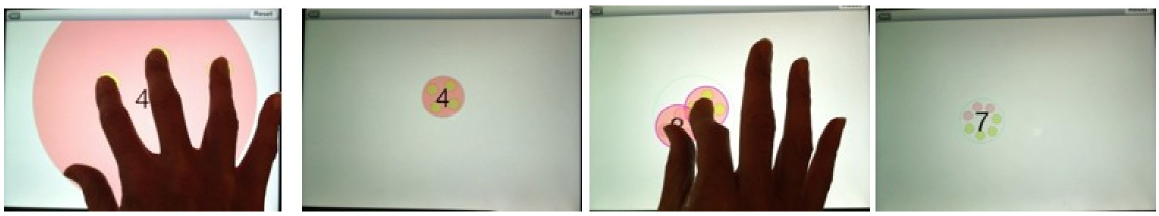
In the four-cube challenge iBook generated by Bruce, children are encouraged to use interlocking cubes to make all possible combinations of four cubes. When checking to see if they have found the comprehensive set, they can look at pre-generated figures on the iPad screen and use a directional tracing gesture in the direction they wish to spin the figure, which makes the figure rotate on the screen.



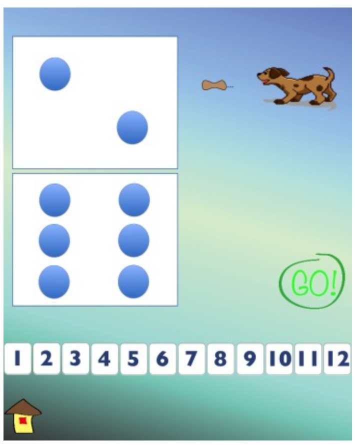
This enables rotation for visual comparing of interlocking cube figures to screen figures by aligning orientation of real and pictorial images, but it also enables comparison of multiple figures on the iPad for congruence. The power of the visual images, combined with their dynamic properties pushes children to consider the similarities and differences of mirror figures.

Capitalizing on the use of gesture when using the iPad

Compared to handling physical tools, engaging with virtual tools on an iPad provides students with *different* kinesthetic experiences. Given that much of the input on an iPad directly employs one's fingers, the potentially unique role of gestures within the interactive environment of tablet technologies is worth exploration. Sinclair and Jackiw's (2011) *TouchCounts* application, takes advantage of gesture by having users perform actions with their fingers that mirror the mathematics they are engaging in. For example a two finger pinching gesture brings circles of quantities together in order to add them together – If the child has 3 dots in one circle and 4 dots in a second circle, and then pinches these together (frame 3), the sets are joined to make one circle of 7 dots (frame 4).



Spatializing Number through Apps



Spatial arrangements that are non-verbal and illustrate quantity in organized and familiar structures, help students build fluency with quantity and addition.

In the iPad game Spot the Dots, developed by Bruce, children may reason via spatial magnitude information, that the quantity in the lower square (see screen capture) is greater than the top square. The arrangement of dots in the bottom square consists of two rows of three (6). The child may count-on from 6, to arrive at 8. Alternately, the child may see two columns of 4 dots, treating the squares as one figure.

12: ROBOTICS AND SPATIAL REASONING

Krista Francis
University of Calgary

In anticipation of upcoming changes to include spatial reasoning in elementary mathematics standards and curriculum, identifying and describing spatial reasoning in educational contexts is crucial for making informed decisions. Most of what is known about spatial reasoning comes from psychology (see Casey, Dearing, Vasilyeva, Ganley, & Tine, 2011; Kayhan, 2005; Levine, Huttenlocher, Taylor, & Langrock, 1999), where tasks are diagnostic measures. Educational tasks differ from psychological tasks because they not only require opportunities for teachers to assess ability (diagnostic), they also permit opportunities for students to learn and improve (develop). A study was designed to identify and observe spatial reasoning in an educational context in order to develop observational protocols and to begin imagining spatial reasoning curriculum outcomes.

This poster will present findings from this study. The robotics task aligned perfectly with the mathematical processes described in the front matter of the Alberta Program of Studies (Alberta Education, 2007): communication, connections, problems solving, reasoning, technology, and visualization. However, no specific outcomes aligned in any of the general outcomes: number, pattern, space and shape, or statistics and probability. Thus prompting the question: How might spatial reasoning outcomes be developed?

Study participants included 21 children and 5 teachers during a 4-day long Lego Mindstorms^{TM} robotics camp. Data collected included videos of children building and programming their robots. Video data permitted opportunities for detailed observations and allowed the researchers to repeated view the robot building at both slower and faster speeds. Analysis of the video was based initially on Bruce et al.'s (2013) list of skills associated with spatial reasoning. Descriptions of observed spatial reasoning skills were compiled. The observations and descriptions formed an imagined spatial reasoning curriculum.

In this poster presentation, we will present our imagined curricula and show a short two-minute video clip of one boy completing Steps 8 and 9 in the Lego instruction booklet for building a robot.

The video exemplifies the imagined spatial reasoning outcomes and the how the boy engaged in multiple spatial reasoning skills almost concurrently. Rather than isolated, sequential and fragmented, the boy repeatedly cycled through many spatial reasoning skills. The observation of the cyclical engagement of spatial reasoning skills highlights a caution for simply adding specific outcomes to the existing Program of Studies. At risk is a fragmentation of integrated process of spatial reasoning skills. The video also illustrates how the educational task permitted assessment of the boy's capabilities with rotation (a spatial skill) and his ability to learn. Poster observers will have opportunities to try to complete building the same Steps 8 and 9 of the robot.

Figure 1: Video of boy building a robot at http://www.ucalgary.ca/IOSTEM/files/IOSTEM/video1-spatial_reasoning-480.mov.

13: A MISNOMER NO MORE: USING *TANGIBLE* CUBE-FIGURES TO MEASURE 3D MENTAL ROTATION IN YOUNG CHILDREN

Zachary Hawes[1], Catherine D. Bruce[2]
[1]University of Toronto, [2]Trent University

In studies with adolescents and adults, three-dimensional (3D) mental rotation skills have proven to be a powerful predictor of mathematics achievement (Casey, Nuttall, Pezaris, & Benbow, 1995; Tolar, Lederberg, & Fletcher, 2009; see Figure 1). However, little is known about the development of 3D mental rotation in young children and even less is known about how the skill might relate to later mathematics learning (Mix & Cheng, 2012). This lack of knowledge is likely a result of developmentally inappropriate testing paradigms (Hoyek, Collet, Fargier, & Guillot, 2012). The primary objective of this study was to design a measure of 3D mental rotation appropriate for children aged 4-8. Our task differs from traditional measures because it uses *tangible* block figures, is not time-limited, and provides a coloured ‘anchor’ cube to reduce executive function demands. A second objective was to examine the onset and development of 3D mental rotation in young children.

Methods

165 children (94 boys) between the ages of 4 and 8 participated (M = 6.0 years, SD = .9, range = 4.3 to 8.0 years). The 3D Mental Rotation Block Task (3D-MRBT) consisted of one practice item and 16 test items (Figure 2). For each item, participants were presented with a target figure and three response figures, one of which was a perfect replica of the target figure but positioned in a different orientation. Participants were asked to indicate the figure that could be rotated to match the target (Figure3).

Results

Performance on the 3D-MRBT task was significantly correlated with age $r(161) = .44$, $p < .001$ (see Figure 4). One-sample t-tests were carried out for each age group to assess performance above chance, defined as 5.33 (i.e., 16 items divided by 3 answer choices). With the exception of the youngest age group, $t(6) = .30$, $p = n.s.$, all other age groups performed above chance, $p < .05$. These data indicate the ability to mentally rotate 3D figures emerged between 4 ½ to 5 years of age and performance improved linearly as a function of age.

Discussion

To our knowledge, this is the first test of 3D mental rotation that (1) uses tangible figures to assess the skill, and (2) demonstrates that children as young as 4 and ½ are capable of 3D mental rotation. The early onset of 3D mental rotation reported here contrasts the late onset reported by other researchers using traditional measures of 3D mental rotation (e.g., see Hoyek et al., 2012).

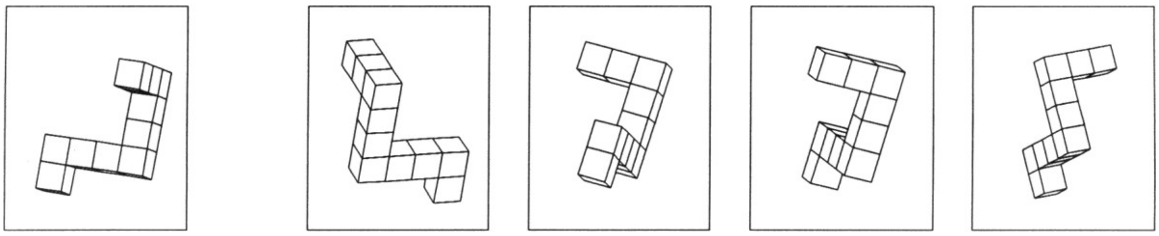


Figure 1: An example item from the Vandenberg and Kuse (1978) 3D mental rotation task. Participants are presented with a target item (far left) and four response items.

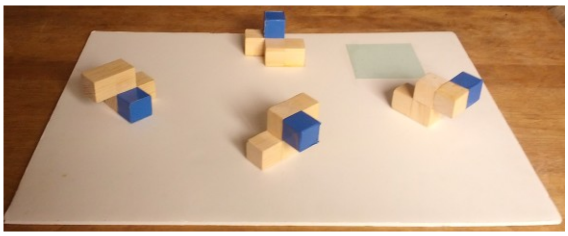


Figure 2: Example of an item from the 3D mental rotation block task (3D-MRBT)

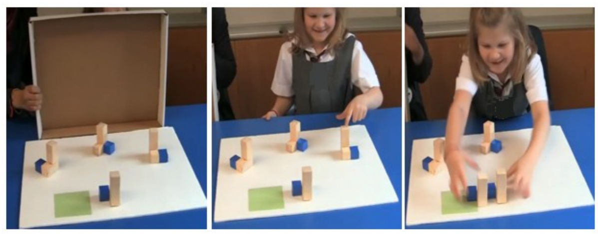


Figure 3: 3D-MRBT: To begin, each test item is shielded from view. Participants are then asked to look carefully at the three options and point to the item that matches the target. Once an item has been selected, participants are asked to place the item next to the target and show how it can be rotated to match.

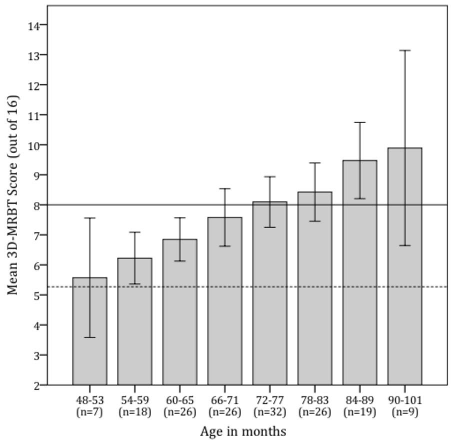


Figure 4: Performance by age; the dotted line denotes chance (33% correct) and the solid black line denotes a conservative estimate of 3D mental rotation ability based on performance at or above 50% correct. Bars represent 95% confidence intervals around each mean.

14: "THEY ARE GETTING MARRIED!" TOWARDS A DYNAMIC, FUNCTIONAL UNDERSTANDING OF SYMMETRY IN PRIMARY SCHOOL

Oi-Lam Ng
Simon Fraser University

Our goal in the current study is to discuss the potential for children to develop new forms of thinking about symmetry with teaching interventions, drawing on the dynamic nature of DGEs. In response to Bryant's call for more intervention studies to be done in the area, we are interested in examining how children's understanding of symmetry evolves within a teaching experiment. In addition, our work with DGEs has prompted us to try to better understand the effects that such a digital technology might have on children's thinking of symmetry, particularly in relation to its dynamic nature. What does teaching children with a dynamic approach of symmetry look like? What effect does a DGE have on children's thinking about symmetry?

We use Sfard's communicational framework to study children's discourse (thinking) while they engage in a sequence of lessons on symmetry. In addition, we focus on children's word use, gestures, and use of diagrams during the lessons involving interaction of dynamic visual mediators. Within this theoretical perspective, our aim is to study how the dynamic environment changes the way the children think of symmetry and to identify the particular tools that serve as instruments for semiotic mediation in their learning.

This teaching experiment involves three lessons, each occurring two weeks after the previous one, taught in an elementary school in Western Canada. Each lesson was taught to two different groups of children (a grade 1/2 split and a grade 2/3 split—each having about 22 students) and lasted approximately one hour. Each lesson included both computer-based activities as well as pencil-and-paper activities. Lessons 1 and 2 involved interactions with the discrete symmetry machines sketches shown in Figure 1.

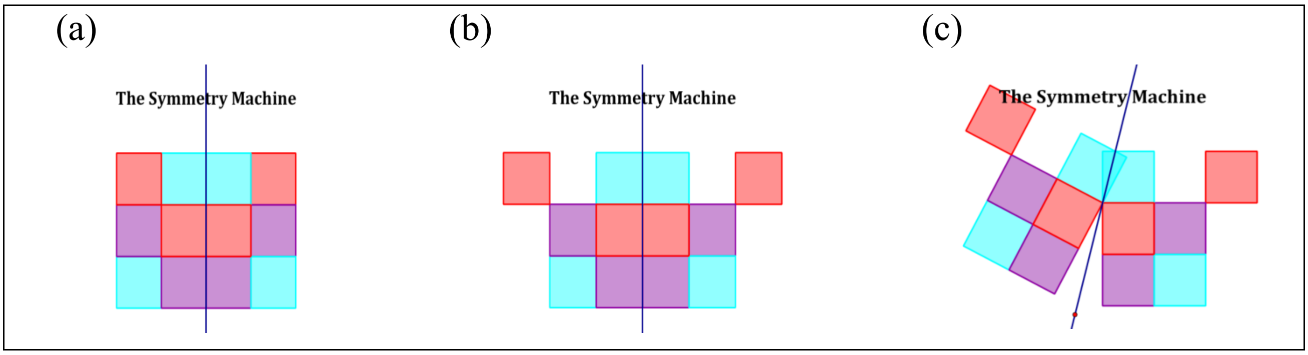


Figure 1: (a) The discrete symmetry machine; (b) After dragging one block away from the line; (c) After rotating the line of symmetry

Lesson 3 involved interaction with the continuous symmetry machine sketch shown in Figure 2. This is a blackbox sketch in which dragging a point causes the other point to move—in this case, in such a way that it remains symmetric with respect to a hidden vertical line of symmetry.

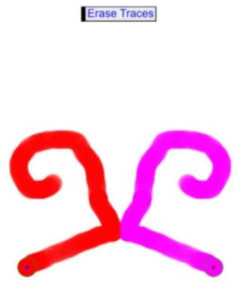
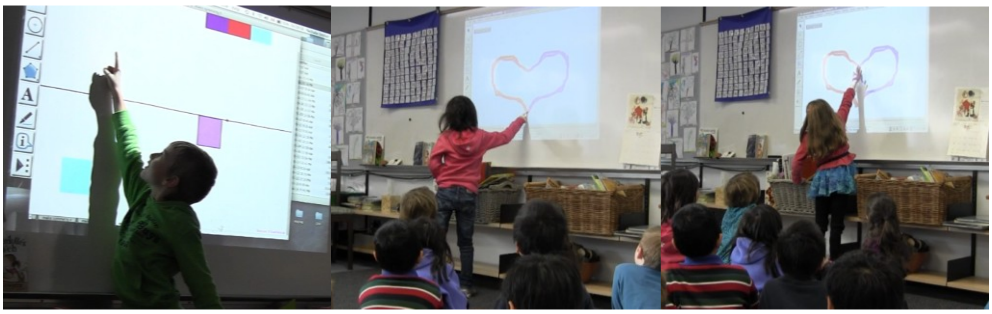


Figure 2

Over the course of the three lessons, which included a large component of whole class discussion and interaction with the projected images in *Sketchpad*, as well as opportunities for the children to create drawings based on both the discrete and continuous symmetry machine, the children changed their thinking about symmetry. They began with a static discourse on symmetry that was focused on the intrafigural qualities of shapes and that featured a small example space of shapes with vertical reflectional symmetry. The children began to talk about interfigural qualities, focusing on the functional relationships of a pre-image and its image. This shift was occasioned by the processes of semiotic mediation in which the dragging tool, as well as the language and gestures of the teacher, became signs that enabled communication about central features of reflectional symmetry including: the way in which one side of a symmetric design is the same as the other; the way in which one component of a symmetric design is the same distance away from the line of symmetry as its corresponding image; the way in which a pre-imagine component and its image have to be on the same line relative to the line of symmetry; and, the way in which a pre-image and an image gives rise to parity.



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HABERMAS' CONSTRUCT OF RATIONAL BEHAVIOR IN MATHEMATICS EDUCATION: NEW ADVANCES AND RESEARCH QUESTIONS

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Habermas' construct of rational behavior deals with the complexity of discursive practices according to three interrelated elements[1]: knowledge at play (epistemic rationality); action and its goals (teleological rationality); communication and related choices (communicative rationality). Thus, it seems suitable for being applied to mathematical activities like proving and modeling that move along between epistemic validity, strategic choices and communicative requirements. The following aspects of Habermas' elaboration (1998, pp. 310-316) are relevant for us.

CONCERNING EPISTEMIC RATIONALITY

> In order to know something in an explicit sense, it is not, of course, sufficient merely to be familiar with facts that could be represented in true judgments. We *know* facts and have knowledge of them only when simultaneously know why the corresponding judgments are true. (...) Whoever believes that he has knowledge at his disposal assumes the possibility of a discursive vindication of corresponding true claims. (...) This does not mean, of course, that rational beliefs or convictions always consist of true judgments. (...) Someone is irrational if she puts forward her beliefs dogmatically, clinging to them although she sees that she cannot justify them. In order to qualify a belief as rational, it is sufficient that it can be held to be true on the basis of good reasons in the relevant context of justification – that is, that it can be accepted rationally. (...) The rationality of a judgment does not imply its truth but merely its justified acceptability in a given context.

These remarks concern the intentional character of rational behavior on the epistemic side and align with a view of development of knowledge (the qualifying element being the tension towards knowing "why the corresponding judgments are true"). Then a connection is established with speech and action (i.e. teleological rationality)—the latter being related to the evolutionary character of knowledge:

> Of course, the reflexive character of true judgments would not be possible if we could not represent our knowledge, that is, if we could not express it in sentences, and if we could not correct it and expand it; and this means: if we were not able also to learn from our practical dealings with a reality that resists us. To this extent, epistemic rationality is entwined with action and the use of language.

[1] Habermas (1998, p. 310) makes a distinction between behaving "rationally" (for a person who "is oriented performatively towards validity claims") and to be "rational" (for a person who "can give account for his orientation towards validity claims". Thus, criteria for the three "roots of rationality" (p. 310) establish a horizon for "rational behavior".

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CONCERNING TELEOLOGICAL RATIONALITY

> Once again, the rationality of an action is proportionate not to whether the state actually occurring in the world as a result of the action coincides with the intended state and satisfies the corresponding conditions of success, but rather to whether the actor has achieved this result on the basis of the deliberately selected and implemented means (or, in accurately perceived circumstances, could normally have done so).

Regarding problem solving in its widest meaning (including conjecturing, proving, modeling, finding counter-examples, generalizing, and so on), this sentence brings forth the quality of the process, which may be qualified as rational (on the teleological side) even if the original goal is not reached. The relevant feature of teleological rationality consists of the action intentionality (including the choice and use of the means to achieve the goal) and the reflective attitude towards it:

> A successful actor has acted rationally only if he (i) knows why he was successful (or why he could have realized the set goal in normal circumstances) and if (ii) this knowledge motivates the actor (at least in part) in such a way that he carries out his action for reasons that can at the same time explain its possible success.

The second condition represents a projection from the past to the future—namely, a conscious enrichment of strategies, in the case of problem solving situations.

CONCERNING COMMUNICATIVE RATIONALITY

> (...) communicative rationality is expressed in the unifying force of speech oriented toward reaching understanding, which secures for the participating speakers an intersubjectively shared lifeworld, thereby securing at the same time the horizon within which everyone can refer to one and the same objective world.

From this ideal practice of communicative rationality, which creates an "intersubjectively shared lifeworld"—thus the possibility of referring to the same "objective world", Habermas moves to an evaluation of actual individual rational behavior on the communicative side:

> (...) The rationality of the use of language oriented toward reaching understanding then depends on whether the speech acts are sufficiently comprehensible and acceptable for the speaker to achieve illocutionary success with them (or for him to be able to do so in normal circumstances).

Here again the intentional, reflective character is pointed out (for the specific case of communicative rationality).

Habermas' elaboration offers a model to deal with important aspects of mathematical activity like those above, without demand to capture *all* the aspects (see below). It has been initially used as a tool to analyse students' rational behavior in proving activities according to the researchers' (and teachers') expectations (see Boero, 2006; Boero & Morselli, 2009). But its application to analyses that also use other constructs gradually developed it as a toolkit with various applications:

- To plan and analyse students' argumentative approach to the culture of theorems, in geometry and in elementary theory of numbers (e.g., Boero et al., 2010; Douek & Morselli, 2012; Morselli & Boero, 2011).
- Integrated with Toulmin and a semiotic lens, to identify different levels of awareness and control, which relate to rationality and are needed to manage arguments in advanced mathematical thinking (Arzarello & Sabena, 2011).
- To identify potential (or real) students' rationalities in elaborating arguments and face the issue of their transition to the levels and kinds of rationality aimed at by the teacher (e.g., Durand-Guerrier et al., 2012).
- To identify "ideal" rational behaviors in different mathematical fields as a means to develop teachers' awareness about the different ways of performing the same activity (e.g. proving) according to epistemic, teleological and communicative criteria (Boero et al., 2013).

Coming to the content of this RF, we may observe that the application of Habermas' construct to the analysis of mathematical activities may capture aspects that are mainly related to discursive practices, in particular those under intentional control by the subject (being she a mathematician, a student or a teacher). Other delicate issues need further elaboration: (i) the relativity of truth and the acceptability of judgments (cf. Douek and Ferrara & De Simone); (ii) the co-ordination between the creativity involved in problem solving processes and their intentional and reflective aspects (cf. Douek, 2007); (iii) the nature of cognitive processes that develop, connected with epistemic and teleological aspects of rational behavior (cf. Martignone & Sabena).

Also, Habermas considers social interaction just in relation to communicative rationality; especially, the negotiation of validity claims and the social construction of strategies are not focal points in his work. Nevertheless, his thoughts about epistemic rationality presuppose a social context for "acceptance"—at least, for a subjective presumption of "acceptance" (see above under *epistemic rationality*). We can generally recognize that Habermas does not deal with the educational problems related to rational behaviors in the classroom. Crucial aspects need further elaboration, for example: the agency of the teacher in the development of students' rationality (cf. Ferrara & De Simone); the forms of students' participation in the interaction (cf. Goizueta); teacher education to enable her to use the Habermas' construct as a tool for didactical choices (cf. Morselli et al.).

Focus on the teacher and on social interaction challenges in many ways the ideals of communication and rationality, as well as the progress of mathematics teaching and learning. The productivity of the students, in terms of effective individual learning, and that of the teacher, concerning effective creation of collective learning contexts, has to do with social aspects that intervene in broadening the reach of participation (cf. Ferrara & De Simone, and Goizueta).

The lack of interest in social interaction in Habermas is intentional and due to his effort of establishing a foundation for discursive rationality. In this Research Forum we try to put the construct of rational behavior close to that of social interaction in the planning and analysis of classroom situations with students involved in the resolution of mathematical tasks. Integrating, in a pragmatic way, an interactionist perspective with Habermas' communicative view is complex, since it requires the use of notions that were created with different purposes and within different frameworks. But we argue that it is complementary and convenient (cf. Goizueta).

The teacher should promote the development of students' rational behavior (taking care of the limits of the pertinence of her decision-making: cf. Douek). The communicative actions could be prompted by the task to be faced (e.g. request of the teacher to «explain» and «justify»; cf. Douek and Martignone & Sabena (2007)), or by the emotional engagement and expectations of the teacher (cf. Ferrara & De Simone), as well as carried on by students in interaction with peers (cf. Martignone & Sabena).

The Research Forum in its whole aims to share and discuss with participants: recent research developments and critical considerations that regard the use of Habermas for the analysis of mathematical teaching and learning (with special focus on discursive activities); and further extensions of the toolkits that have been or can be developed.

We are also interested in sharing with RF participants problems beyond what we have faced till now. We are in fact committed to identifying, revising and exploiting the potential of using Habermas in mathematics education research, even what is missing or under-theorized. In particular, once attention is drawn to dialectical relationships among participants in the activities, mutual understanding in communication is much more than just a need for guaranteeing communication. It is the result of a temporary achieved value and of a mutual acceptance. *The conditions for the creation of such value and mutual acceptance are not easy to identify and to frame whether we use Habermas' construct*. One of the tasks of the researcher, as well as of the teacher in the classroom, is to realise the value assigned to a student through the observation of how others refer to her expected productions. The ideals of communication and rationality are then linked to the capacity of altering and keeping expectations. But rationalization as an extreme and exclusive position may lead in practice to the loss of the views (and lifeworlds) of those who are not directly involved (although physically present) in the processes of reaching rational consensus. The fact that some students can be communicatively absent or unable to participate in reaching rational consensus affects the productivity of all the subjects, both teachers and students. It also fosters inequality structures. In this regard, a far goal of mathematics education research that want to take Habermas into account *is to deal with cases in which, for different reasons* (cf. Douek), *there are students who are not discursively given opportunity to participate in the construction of shared understanding*.

1: PRAGMATIC POTENTIAL AND CRITICAL ISSUES

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I considered the Habermas construct as sharing the modelling character of scientific constructions: it offers strategies to conceptualize classroom interaction and find ways to handle its complexity. As such, limitations may emerge at pragmatic or theoretical levels, as it is utilised.

Educational aims and tools in evolution

I developed my reflection on the use of Habermas' construct under three assumptions:

- A Vygotskian didactical perspective, conceiving teaching-learning as a dialectical construction of "scientific concepts" in relation to "everyday concepts". Among every day concepts I include spontaneous individual practices, whatever their socio-cultural roots are. Scientific conceptualization is characterized by conscious management of concepts, their properties and related practices on a general level. This perspective implies the gradual construction of class references—backing scientific conceptualization—through cycles of individual production, discussed then synthetized collectively under teacher's guidance. For scientific conceptualization argumentation is a means and an aim, as it is involved in proving and conjecturing.
- An epistemological perspective considering mathematical theories and activities as built on axioms and also on socio-cultural practices, complemented with an ethnomathematical perspective for the purpose of determining and analysing the objects and the cultural context of mathematics education.
- A conception of genuine problem solving as combining various modes of reasoning and references, not all being mathematical constructions. We can schematically identify two complementary directions: a structuring one organizing arguments and strategies, and an exploring one, when trials, metaphors, and transformational reasoning prevail. They rely on different rules of validity, but evolve dialectically.

On these bases, problem solving—in its widest meaning—is approached and enhanced through grounding activity in culturally meaningful situations for students; addressing knowledge like theorems, procedures and technical practices, and also modes of reasoning; and stimulating both creativity in exploration, and argumentation based on mathematical established references. A current didactical tool to approach these aims is based on Bartolini Bussi's (1996) construct of mathematical discussion. I interpret it as a canvas to develop students' mastery of their activity and knowledge in problem solving through two main questions: *how did you do it*? *Why is this true*? to share, criticize and develop procedures, and to identify, develop, or produce mathematical knowledge for backing and questioning certainty or data through argumentation. I consider that Toulmin's model of argumentation (1974) helps the teacher to identify the arguments' components, and to orchestrate such discussion. Toulmin's "argumentation domain" could be interpreted as the web of mathematical constructions the students have to rely upon. But, as I search a space for students' everyday conceptualization and a way to relate it to scientific conceptualization of school mathematics, I characterize "argumentation domains" by discursive practices shaped within communities sharing a cultural background (forming a cultural area of coherency).

Developing the didactical toolkit by using Habermas' construct

The idea that "*the rationality of a judgment does not imply its truth but merely its justified acceptability in a given context*" (Habermas, 1998, p. 312) fits the quest to define a wide argumentation domain to back a diversified epistemic component. Moreover, this construct allows discussing teleological reasons and communicational choices. Hence the idea of developing the mathematical discussion along the three components of rationality. On a more general level, this construct allows to identify fine grain components of potential argumentation lines enriching problem solving, and to identify different levels of rationality within the curriculum development from one school level to another, and different kinds of rationality according to different mathematical domains (cf. Morselli et al.). Thus it enriches the potential of mathematical discussion and allows improving its planning and management. My interpretation of the Habermas construct affects the articulation of the above mentioned educational choices. To create a classroom context suitable to promote the Vygotskian dialectics and develop scientific conceptualization and argumentation, supposes to: 1) introduce students to cultural interest for finding reasons that have a theoretical relevance and/or can be shared as valid references; 2) make them aware that reasons can be various, not all based on the class references, and understand the relations between those and the specific references related to everyday concepts; 3) develop attention and concern for interaction and ability to adequately express one's views in a given socio-cultural context; 4) develop consciousness of one owns positioning and a critical attention to it; 5) establish mathematical references collectively—theoretical knowledge and practices—under teacher's guidance, built upon various sources, including students' contributions and cultural experience.

To reach such aims, the teacher needs to stimulate and offer a model of rational behaviour and discourse (according to the school level and the specific subject) and of acceptance of a variety of justifications related to a variety of backings—as long as they are made explicit. The corresponding development of the didactical toolkit can be presented as a canvas of ***rational questioning*** to organize the mathematical discussion according to the three components of rationality as a way to introduce the students, and lead them, to behaviours shaped by rationality requirements: *why do you think that it is true? Why do you need to do that for...? Did you make yourself understandable to...?*

Did you (the others) understand why she…, how he…, etc. And in order to allow relativization of justifications to an argumentation domain (or level): *is this reason different from…? From his point of view could you say the same thing?* In this way, students' voices are received and deepened, bringing as much as possible their roots to consciousness and making some links emerge, in particular with class references. The teacher can support students' "rationalization" of discourse (i.e. afterward fitting rationality requirements) and the relativization of reasons according to references collectively put to light, dialectically forming argumentation domains.

Critical considerations

The requirement of intentionality and awareness from the learners' part

To sustain students' scientific conceptualization means to draw them to be able to refer to the classroom (or some outer acknowledged) constructions as theoretical references, mobilize them consciously, and understand their potential generality. This attitude in solving a problem is difficult to handle in the beginning of the process, while exploring the situation. Nor is it easy to establish a strategy while trying to find one's way. This is the case, whether the students try to rely upon classroom constructions or on specific local references. Genuine problem solving generally needs stimulation from others. Rational questioning should favour student's maturation (from the ability to act and develop a discourse about acting towards the ability to organise strategies and express them a priori, when the situation is mastered enough) by supporting—in between—going back and forth from "action" to its rationalization, accounting for validity of statements and strategies, and produce autonomously a conclusive rational discourse. Such a sequence should include explicit relating, contrasting and combining organization aspects of activity and exploration (which may involve everyday conceptualization) in order to develop problem solving abilities, on a long term perspective.

The "race to 20" example presented by Martignone and Sabena shows a didactical situation where the gradual transition from exploration to a rational attitude to organize explanation takes place (the teacher relies upon a didactical contract inducing students' efforts of explanation). Explanations cannot occur at the first trials. Students move gradually from playing the game, to describing and making claims, then to justifying them. When the teacher encourages them to organize and generalize the justification, they attain both a justification of why the solution works (based on arithmetic knowledge), and an organized description of the activity bearing a general character: they produce a rhythmic exposition of the game moves, with voice and gesture, that point to the various "variables" affecting the moves, and to hierarchy on their treatment. They gain ground along the three components of rational behaviour once they had explored and then gradually move to accounting for strategies and related reasons of validity.

Can we always recognize the rationality of another? The necessity of doubt

Excavating reasons and elaborating convenient communication modes may induce a real logical muddle. The reasons a person produces—after or before action, as well as communication efforts, depend on her perception of the context and the related argumentation domain(s), and on her perception of the listeners' expectations—that may not coincide with hers. As I understand Habermas' construct, those speaker's perceptions are essential elements for her intentional choices (intentionality is a criterion of rationality). Can the listener/observer be sure the speaker searched for reasons and convenient backing, and correctly perceived the listeners' expectations? Are they able to grasp them? Who can legitimately control validity criteria?

In an educational context, can we be sure we are able to welcome students' efforts of rationalization? Don't we risk inducing them to adopt non significant references and to impose links that hinders coherency between cultural roots—that they may not master consciously—and school's constructions? The issue is difficult to deal with. Teacher and student's positions are not symmetrical, and agreement can result from an authoritarian process instead of cooperative rationalization. But education is an insertion into a cultural community! The difficulty is striking when educators and students do not share a common culture. Moreover, there can be a gap between possibilities of activity and possibilities to develop a related discourse, especially in exploration: the student may lack discursive ability; or not grasp all the reasons behind one's behaviour because conceptualization is not yet sufficiently developed (e.g. elements affecting reasoning are not all conscious); or have deep difficulty to express oneself through a logically organised discourse (e.g. transformational reasoning is not easy to justify, and even to describe completely).

On the didactical level, all these difficulties do not imply to renounce to rational questioning within class discussions, but to seek a way to give space to doubt and to suspend conclusions, as a collective agreement that needs to be shared sometimes.

Recognizing students' rationality is almost always a challenge. In a teaching experiment Habermas' construct was used to analyse secondary school students trying to elaborate a proof (see Morselli & Boero, 2011; Douek & Morselli, 2012). A game was used to introduce algebra as a tool for proving: they were asked to choose a number then transform it through prescribed simple calculations. The transformation eliminated the chosen number from the final result. They had to explain why the result is constant whatever number is chosen. Two productions were exploited. Ric's explanation was close to an algebraic expression. It was analysed as based on adequate epistemic reasons, produced according to efficient teleological choices and well communicated (though many schoolfellows did not grasp it). Tor's reasons were better understood by his schoolfellows; his procedure to prove was well organized, but his calculations were incorrectly formulated (according to standard syntactic requirements). The first analysis pointed to a lack of epistemic rationality. But deeper reflection showed that his organization of calculations resembled computer programs (systematically naming X the result of each calculation step). Somehow, the process of analysis moved from an implicit and unconscious relativization to the targeted algebra field towards a conscious relativizing of the analysis to a new specific argument domain. Within the wider epistemic component including computer science, Tor's expression was almost correct, and revealed "good" epistemic basis. This critical reflection permitted to legitimate a potential epistemic rationality the student could not uncover, and to express its potential "scientific coherence". But it was not possible to share with the class this structuration of his justification—as coherently related to a system. In general, to identify students' potential rationality, welcome it, and relate it with classroom construction is a difficult challenge. Here, the teacher brought most students to understand both argumentations, but put to light only the pertinence and validity of the "algebraic" production according to the requirements of the mathematical community, on the epistemic and teleological sides.

Problem solving creativity and communicative rationality

Problem solving intertwines creative exploration and rationalization. Exploration needs to be freed from stereotyped modes of reasoning, and to evolve through doubt, using uncertain metaphors, approximate semiotic representation, transformational reasoning, interpretation and links. This relates to communicating with oneself. The rational questioning, be it the student's reflective activity or be it resulting from interaction should be a method in organization phases, on the structuration side. On the exploration side, it should allow to bring to consciousness creative behaviour; and to elaborate a critical view on innovative ideas, and on the necessity, difficulties and benefits of combining structuration and exploration. But it may hinder aspects of exploration and disturb interpretation and communicating with oneself, in transformational reasoning or in producing metaphors for example. It may happen that the need for validating statements results in the uncritical use of established knowledge; and that aiming at correct solutions results in the search for established methods. These well known phenomena depend on the didactical contract. They may be related to the lack of acknowledged space for doubt, to teacher's premature or misleading request, and/or request at inopportune phases. Thus, creative exploratory phases need a flexible exploitation of rational questioning, allowing didactical treatments ranging from releasing from justification and accepting doubt, to efforts to remove doubt accompanied by teacher's care for genuine students' accounting—at the convenient moment—for validity of statements and strategies.

2: THE EMERGENCE OF VALIDITY CONDITIONS IN THE SECONDARY MATHEMATICS CLASSROOM: LINKING SOCIAL AND EPISTEMIC PERSPECTIVES

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Introduction

It is widely accepted that argumentative competencies should be developed within the mathematical activity in the classroom, both as a product of this activity and as a means to support it. We share Boero's (2011) idea that a 'culture of argumentation' is to be developed in the classroom and that it should include practices on the production of conjectures, meta-mathematical knowledge about the acceptability of references advanced for the validation (acceptance/rejection) of claims and knowledge about the role of counter-examples and generality. It should include elements for evaluation of mathematical productions and general ideas about the use of all this knowledge within argumentative practices, along with the needed awareness to allow deliberate and autonomous control of the process.

In our study we address some specific aspects of the broad challenge of fostering such culture in the classroom. Our main interest is to investigate the epistemological basis of argumentative practices in the mathematics classroom and, particularly, how is validity interactively negotiated and constructed, as a rational enterprise, in a rich problem-solving mathematical activity. In the following, we will show how Habermas' construct of Rational Behavior is used for that purpose within our study and how we complement it with other theoretical constructs in order to better suit our needs, accounting for the social and epistemic complexity and specificity of the mathematics classroom. This theoretical integration frames our understanding of classroom argumentative practices and gives us a tool for investigating epistemic features of these practices in order to analyse data coming from students' interaction.

Theoretical framework

Following Steinbring (2005), we do not understand mathematical knowledge as a pre-given, finished product but, instead, as the situated outcome of the epistemological conditions of its dynamic, interactive development. We assume that a "specific social epistemology of mathematical knowledge is constituted in classroom interaction and this assumption influences the possibilities and the manner of how to analyse and interpret mathematical communication" (p. 35). Within this socially constituted mathematics classroom epistemology a criterion of mathematical validity is interactively negotiated between the participants. Mathematical activity and mathematics classroom epistemology are reciprocally dependant: the later shapes a frame in which the former takes place and the former develops the later to conform to the emergence of new legitimated mathematical (and meta-mathematical) discourses. A central consequence of these assumptions is the basic necessity for interpretative research to reconstruct the situated conditions in which (and from which) mathematical knowledge is interactively developed. Although 'conditions' might be considered in a broader sense, we are particularly interested in the epistemological assumptions at stake, the references (mathematical and not) that might be considered as relevant and the social environment in which the process is embedded.

According to Habermas' tridimensional description of rational behavior (Habermas, 1998), a person acts rationally when she is able to explain in a reflective attitude (and thus is aware of) how is her action guided by claims to validity, accounting for what she believes, does and says in accordance to the intersubjectively shared culture. If we think of the mathematics classroom as a social environment in which knowledge is interactively constructed according to evolving epistemic specificities and in which a particular culture of argumentation is to be intentionally constructed, Habermas dissection of rational behavior constitutes a very appealing descriptive tool. It allows focusing on specific features and issues at stake, and to plan and exercise adequate control to foster culturally accepted practices, promote noetic transparency and allow students to gain awareness about the intended culture.

Adapting Habermas construct to the classroom

In the situated context of the mathematics classroom, the general relation between classroom epistemology, mathematical activity and social environment must be considered under the light of a specific, content-related didactical contract (Brousseau, 1997). The didactical contract corresponds to the reciprocal expectations and obligations perceived within the didactical situation by the teacher and the students with respect to the knowledge in question. Mathematical acceptability of students' explanations is linked to these expectations and to the mathematical contents at stake (or perceived as being at stake). Thus, when faced with a problem, students might bring up mathematical knowledge and references they consider relevant for the proposed didactical situation in order to cope with the task; a common clause of the didactical contract may indicate them to do so. Nevertheless, not all the emerging references are linkable to well established and intersubjectively shared mathematical knowledge. We might also need to consider other references (statements, visual and experimental evidence, physical constraints, etc.) that are not part of institutionalized corpora, such as scholar mathematics, and are used de facto as taken-as-shared, unquestionable knowledge (Douek, 2007). A contextual corpus of references is necessary to support everyday argumentation but also mathematical argumentation at any level; it might be tacitly and operatively used by the students to make sense of the task, semantically ground their mathematical activity and back their arguments. Accounting for the reference corpus at stake might be particularly relevant when considering problem-solving settings in which empirical references are to be considered as part of the proposed milieu.

By complementing Habermas' notion of rational behavior with this situated view of the students' activity and the classroom social and epistemological environment, we try to better understand the argumentative practices of the classroom and to inductively identify underlying epistemic constraints that might be behind observed practices students enact to validate their arguments. Because these practices are multifarious and often implicit in the interaction, specific analytic tools are needed to observe the conversational level with the needed focus and detail.

Adapting Habermas construct to analyze the conversational level of interaction

Toulmin (1958) structurally describes an argument as consisting of some data, a warrant and its backing and a modalized claim. The backing supports the warrant, which allows inferring the claim from the advanced data. Conditions under which the warrant does not support the inference (rebuttal conditions) might be also considered. Toulmin's model allows one to focus on arguments as the basic analysis unit and to structurally dissect them, while Habermas' construct allows one to focus on the main motive of the ongoing discursive activity and apprehend those structural parts on their epistemic, teleological and communicative dimensions. We use Toulmin model strictly in order to structurally describe argumentation and account for validity evaluation in accordance to our theoretical perspective. According to Habermas (1998), accepting a validity claim is tantamount to accepting that its legitimacy can be adequately justified, that is, that conditions for validity may be fulfilled. Under our theoretical perspective, validity conditions are explicit or tacit constraints that allow students to control the coherence of the mathematical activity according to the socially constituted classroom epistemology, didactical contract, reference corpus and shared goals. Our main interest is to reconstruct the emergence of validity conditions and the process of fulfilment of this conditions, be it successful or not, as they are brought up by students as means for validation.

Because this analysis mainly occurs at the conversational level, we incorporate yet another analytic tool in order to focus and apprehend particular features of the conversation. According to Sfard and Kieran (2001) an exchange is considered effective if “all the parties involved view their expectations as fulfilled by the interlocutors” (p. 51). They propose to assess effectiveness through the analysis of the 'discursive focus': a tripartite construct consisting of a 'pronounced focus', what is actually said (and is thus public), an 'attended focus', what attention is directed to (including the attending procedure) and an 'intended focus', “a cluster of experiences evoked by the other focal components plus all the statements a person would be able make on the entity in question” (p. 53). Through this 'focal analysis', referred to our theoretical perspective, we bring to the fore particular aspects of the conversation that might be indicators and descriptors of the mainly tacit emergence and fulfilment of validity conditions. Through the inductive-interpretative analysis of discursive foci we identify the illocutionary intention of establishing and fulfilling validity conditions in order to perform the illocutionary speech act of validating claims.

The design experiment

We conducted a design experiment were thirty 14/15-year-old students and their teacher worked in two lessons in a regular classroom in Barcelona, Catalonia-Spain. It was a problem-solving setting, with time for small group work and whole-class discussion. The following problem was suggested by the researchers and intended as an introductory task to probability theory:

> Two players are flipping a coin in such a way that the first one wins a point with every head and the other wins a point with every tail. Each is betting €3 and they agree that the first to reach 8 points gets the €6. Unexpectedly, they are asked to interrupt the game when one of them has 7 points and the other 5. How should they split the bet? Justify your answer.

The novelty of the task was expected to lead students to develop models and negotiate meanings, while producing arguments to validate them avoiding mechanical approaches based on well-established heuristics. The teacher was asked to avoid hint-based guidance and favor reflection by proposing different numerical cases. Crucial cases were planned to problematize expected wrong proportional answers. For data collection, two small groups were videotaped and written protocols were collected. Some groups were collectively interviewed a week after the task.

Overview of the presentation within the research forum

We will present the case of a group of four students working on this problem. Drawing on a proportional model (corresponding to the points won), they come up with the solution: €3.5 for the winning player and €2.5 for the other. This model emerges as the result of the fulfilment of certain validity conditions. The teacher asks them to “check different situations to see if that reasoning holds” and proposes the case '2 points to 0'. This case was expected to problematize their model by producing a counter-intuitive result, driving the students to seek for a better fitting new model.

During the interview students were asked to watch on video this episode and then were asked about the relevance of testing the model in different numerical situations.

1	R:	So, the teacher comes, you explain to her what's going on and then she asks you to test it in other situations. Why do you think she asks for this?
2	Zoe:	To check it. If it happens the same.
3	Josy:	To check it.
4	Vasi:	If it always holds up. I mean, if it is not only in this case.
5	R:	And why would it be important that it 'holds up' in other cases?
6	Anna:	Because that way you verify that your method (…) is correct.
7	R:	And what does it mean for a method to be correct?
8	Anna:	Well, in this case, that the distribution is fair for both of them, and that it works not only in this case but also in others.

By directing the attention to the teacher's request, the intended focus of the interviewer in 1 is on the relation between particular case exploration, generality and explanatory power of the model when considering empirical situations. The students do not point, as was expected, to the necessity of the model to accord to shared empirical references about the game. Instead, our analysis shows that the deictic used by Zoe in 2, which corresponds to her intended focus, refers to the validity conditions that originated the model and their fulfilment. These conditions are intrinsic to the model and thus their fulfilment is guaranteed. In 6, Anna considers the fulfilment of the developed validity conditions as sufficient to positively assess the model as “correct” (although the epistemic status of the claim is not clear) and, in 8, makes 'correction' equivalent to 'fairness' in this case. By operationalizing the notion of fairness in terms of the developed validity conditions the requested testing of the model becomes a self-fulfilling process that inductively reinforces the model. This explains why these students support a counter-intuitive result when working with the example '2 points to 0', instead of problematizing the model.

In our presentation we will develop this example and use our analytical approach to show how the epistemic and social conditions of emergence of the model end up constituting an obstacle that prevents students from challenging it.

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3: ANALYSIS OF ARGUMENTATION PROCESSES IN STRATEGIC INTERACTION PROBLEMS

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In this study we use Habermas construct to analyse argumentation processes related to strategic interaction problems. These problems provide suitable environments to develop and analyse students’ planning and control processes, of a paramount importance in mathematical problem-solving. Some theoretical tools to study planning and control processes will be integrated with Habermas construct through the analysis of excerpts from a classroom discussion in grade 4. This integrated analysis will highlight specific features of the argumentative discourses, brought to the fore in strategic games.

Strategic interaction problems

In strategic interaction problems, two or more decision makers can control one or more variables that affect the problem results. The decisions of each player influence the final result of the game. Game Theory offers different mathematical models for the winning strategies, based on specific assumptions about how ideal, hypercalculating, emotionless players would behave (Von Neumann & Morgenstern, 1947). However, analysing strategic interaction games as problem-solving activities, Simon (1955) argues that the limited capabilities of the human mind (memory system and the development of calculus, attention span, etc.) combined with the complexity of the external context, make often impossible the elaboration of the strategic choices predicted by Game Theory. As concerns limits on iterated thinking, the data collected by Camerer (2003) show that, during the resolution of strategic interaction problems faced for the first time, only few subjects are able to develop many thinking ahead steps (*limited strategic thinking*).

Planning and control in mathematical problem solving

Our research is based on the assumption that strategic interaction problems constitute suitable environments to develop and analyse important features of genuine problem-solving, such as *planning* and *control processes*. Planning processes are related to the possible actions to be performed across time: for this reason, the studies about mind times can be useful to interpret the specific related cognitive processes. Considering problem-solving activities, Guala and Boero (1999) identify some examples of mind times (i.e. time of past experience, contemporaneity times, exploration time, synchronous connection time) involved in the imagination of possible actions over time. In particular, they analyse the *exploration time* as the projections that can be developed from the present onward (e.g. “which strategies can I develop to find the solution?”, “How can I manipulate the data to solve the problem?”) or from the future back to the past (e.g. “I think up a solution and explore it in order to find the operations to be performed, depending on available resources”). Planning processes can be analysed deeper by taking into account the cognitive studies about the human ability to remember (Tulving, 2002) and imagine facts and situations in the course of time (Martignone, 2007). “Remembering” and “projecting” need the ability to conceive *the self* in the past and future, which goes beyond simple “knowing” about past events and future facts. In particular, considering the imagination of possible future events, we can distinguish between the knowledge that we possess about an event (*semantic future thinking*), versus thought that involves projecting the self into the future (*episodic future thinking*) to “experience” an event (Atance & O’Neill, 2001). Knowledge supports and structures imagination processes through frames or scripts that influence the expectations on stereotyped situations.

Note that in episodic future thinking the imagination is not given free reign, but rather, the projection is constrained. For instance, envisaging my forthcoming vacation might require me to consider such factors as how much spending money I will have, how much work I will have completed before I go, and so on. (*ibid*., p.533)

Besides planning processes, also control processes play a fundamental role in problem-solving activities. As introduced by Schoenfeld (1985), control deals with “global decisions regarding the selection and implementation of resources and strategies” (p. 15). It entails actions such as: planning, monitoring, assessment, decision-making, and conscious meta-cognitive acts. In the context of argumentation and proof activities, Arzarello and Sabena (2011) show how students’ processes are managed and guided according to intertwined modalities of control, namely semiotic and theoretic control. *Semiotic control* relates to knowledge and decisions concerning mainly the selection and implementation of semiotic resources. For instance, semiotic control is necessary to choose a suitable semiotic representation for a problem (e.g. an algebraic formula vs a Cartesian graph). *Theoretic control* requires the explicit reference to the theoretical aspects of the mathematical activity: it intervenes when a subject use consciously a certain property or theorem for supporting an argument.

Our study integrates the identification of planning and control processes in strategic actions development, with the study of the communicative actions as rational discourses according to Habermas construct (cf. Boero & Planas).

Methodology

On the base of the theoretical discussion above, teaching-experiments are planned and analysed with the collaboration of classroom teachers. Activities develop around classical strategy games, such as NIM, Chomp, Prisoner Dilemma, etc., and alternate game phases with reflection phases (collective discussions, written reports). Video-recordings of the discussions and students' written reports are collected and analysed on a qualitative and interpretative base.

In the following we consider a case study in grade 4, based on a strategic interaction game called "Race to 20", which was used by Brousseau to illustrate the Theory of Didactical Situation (Brousseau, 1997). The rules of the game are the following. There are two players: they know the possible alternative choices and the relative outcomes, they do not cooperate and do not know in advance the adversary strategies. The first player must say a number between 1 and 2. The second player must add 1 or 2 to the previous number, and says the result. Now the first player adds 1 or 2, and so on… The player who says 20 wins the game.

Analysis of the collective discussion

We analyse some excerpts of the collective discussion organized by the teacher after the children have played the game several times, at first individually, and then in teams. In the discussion, students are asked i) to describe possible winning strategies and ii) to justify them. Numbers 14 and 17 are soon identified as "winning numbers". Justifications are based on the possible moves of the two players. We report Elena's contribution:

> Elena: it's necessary to arrive before at 14 and then at 17. Because if you do from 14 you do plus 1 and arrive at 15 and then you do plus 2 and arrive at 17, which then…you do plus 1 and arrive at 18 and the other does plus 2 and arrives at 20. Rather, if you do plus 2 from 14, you arrive at 16 and the other does 1 and arrives at 17, the other if he does plus 2 arrives at 19, you do plus 1 and arrive at 20. Hence anyway from 14 to 17 you arrive anyway at 20.

Elena carries out her argumentation by describing the possible moves of the players (*episodic future thinking*) who start from two particular positions (14 and 17). The *teleological aspec*t is clear: she wants to describe the winning choices. Because the steps of thinking are limited (*limited strategic thinking*), she manages to plan ahead only close to the winning end, i.e. number 20. When the teacher asks the students if there are other numbers like 14 and 17, different scenarios are explored by students.

The number line helps them to *control* the winning positions and strategies, relying on the semiotic representation at the blackboard (Figure 1).

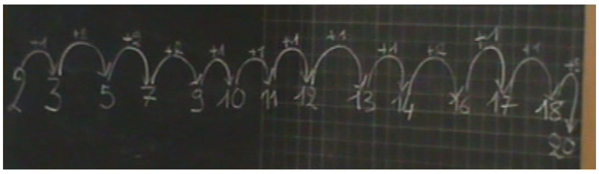

Figure 1: The written representation used to play the game.

The students' attention is focused both on backwards movements, in search of previous winning position, and on the forward movements, to check the efficacy of the elaborated strategies.

Diego: 11 maybe is an important number, because maybe my team adds 2 and it is 13, the other team adds 1 and arrives at 14, I add 1, 15, they add 2 and it is 17

Elisa: but if they are stupid they make plus 1 and arrive at 18, we make plus 2 and arrive at 20; but they are not so stupid to do plus 1, eh!

Also Diego and Elisa rely on the *episodic future thinking* to imagine the possible moves of the players and justify their hypothesis about number 11. The steps of thinking ahead are always "close" to the new winning numbers (*limited strategic thinking*). In the rational discourse of Elisa, the *teleological component* is linked to the goal of the game—getting to number 20—and it is guided by her knowledge (stressed in the discourse) that the other players have the same information and capacity. After that many students have expressed similar arguments, Giulio proposes a general rule, which can drive all strategies:

Giulio: I think that for the winning numbers you always remove 3: from 20 you remove 3 and you arrive at 17; from 17 you remove 3 and you arrive at 14, I think that another winning number could be 11, could be…8, could be…5, could be…2

Teacher: Explain well this idea

Giulio: Because…that is I don't know, if I arrive at 2…I don't know, I begin, I make 1, no I make 2, he arrives and makes 1 (gesture in Fig. 2a), I put 2 and I arrived at 5 (Fig. 2b), which I think is a winning number… yes, arrived at 5…it is a winning number, I think. Then…he adds 2, say (Fig. 2c), I add 1 and I arrived at 8, which is another winning number. He adds 1, I add 2 and I arrive at…12, which is another winning number. He adds 2, I add 1, and I arrived at 14, which is another winning number, he adds 1 I add 2, we arrive at 17 which is a winning number, he adds 1 or 2, I add 1 or 2 and I win

Figure 2 a,b,c: Giulio's gestures in his argument.

In his first sentence, Giulio expresses the rule in a general way, as an a-temporalized relationship between numbers ("you always remove 3" from 20). Giulio's argument is based on the backward induction (similar to what described in Game Theory): he starts from the winning result and moving backward he identifies the best strategy to win the game. In the argumentative discourse on the winning strategy, *the teleological and epistemic components* of rationality are on the foreground. The epistemic plane relies on the relationships between numbers, and in particular on the (both *semiotic* and *theoretical*) *control* over the number line model, recalled in the written schema introduced by the teacher to play the game (Figure 1). This representation has now become a thinking tool for Giulio. Asked to better explain his ideas (*communicative component*), the boy imagines a match, and describes the moves in a temporalized way (*episodic future thinking*). The subtraction turns into an onward movement starting from the very first move (number 2). This movement is produced by means of a rhythmical repetition of the same linguistic structure: "he adds…I add…and I arrive at…, which is a winning number". Linguistic repetition is co-timed with gesture repetition during the entire argument: gestures are synchronous with the added and obtained numbers in the imagined game. Gestures and words together constitute a schema through which the generality of the argument is conveyed. Gesture in Figure 2a (open hand as holding something) is co-timed with the words "makes 1": while saying "1", Giulio is indeed meaning "let's say 1" or "any move of the player", something similar to what Balacheff in proving processes called "generic example" (Balacheff, 1987). This interpretation is confirmed, besides by the voice intonation, by the gesture-speech combination of Figure 2c: a similar gesture is performed within a similar speech schema, but now the generic nature of the example is made explicit by the word "say".

Concluding remarks

In this paper we analysed some excerpts of a discussion about the winning strategies in a particular strategic interaction problem: the Race to 20. It provided a suitable context to study the students' argumentations, intended as rational discourses in the Habermas construct. The analysis integrated the cognitive studies about mind times, limited strategic thinking and semiotic aspects of control processes with the study of communicative actions. As a result of the use of these different interpretative tools, an important distinction in the teleological component of the argumentations emerged. In fact, we can identify two teleological planes: a *pragmatic* plane, related to the goal of the game ("Which strategy can I choose or develop in order to win the game?"), and a *theoretical* plane, related to justifying the chosen strategy ("How can I justify that my strategy is the best one?"). Furthermore, as we could see in the reported excerpts, the two planes are deeply intertwined: theoretical considerations can fruitful ground on pragmatic ones, and—more important—can also be justified on a pragmatic base (see Giulio's argument and the generic example therein). This feature is inherent to the specific didactic engineering based on strategic interaction problems: besides the game phases, in fact, it is the *request of sharing and justifying their strategies* what makes these activities a suitable context to develop communicative actions, as rational discourses, from early school grades. As a matter of fact, on the *theoretical* plane, the *teleological* dimension strongly intertwines with the *communicative* one, when students are asked by the teacher to explain and justify their strategies to their mates (for the analysis of possible teacher's intentions see Morselli *et al*., in this RF).

4: USING HABERMAS IN THE STUDY OF MATHEMATICS TEACHING: THE NEED FOR A WIDER PERSPECTIVE

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The teacher as a rational being

In the last decade, Habermas' notion of rationality has received lots of attention from mathematics education research whose main focus is on the learners' argumentative activities. In one chapter of the book *On the pragmatics of communication*, Habermas (1998) offers an analysis of the complementary relationship between the discursive activity of a rational being and the reflection on it. Drawing on this, we argue that we need to grapple with the mathematics teacher *as* a rational being, who participates in discursive activities and whose orientation towards validity claims has to do with her decision-making. Our aim is to examine the '*rational being*' of the teacher in the classroom rather than the thinking processes of individual learners.

Following Habermas in recognizing the complementarity above, we propose that, for talking about the teacher's rationality in her decision-making, we need to consider reflection on personal activity, what involves values and beliefs of the acting subject. This entails a true complexity for our study, since the beliefs and the background of the teacher contribute to her choices. The frame is even more complex whether we accept that beliefs are *not* agent-neutral, which means that the affective sphere of the teacher also affects her beliefs.

So, *what happens* in the mathematics classroom is only a part of the story. We need a wider perspective, which leaves room for *the feeling of what happens* that the teacher brings to activity. Briefly speaking, if we remain clung just to what happens we might lose the reasons for which that specific 'what' happens in the way it does. There is here an implicit assumption: what happens in the classroom is entangled with feelings of what happens that can be ascribed to individual teachers. Our perspective to study rationality in mathematics teaching expands to include also the affective domain, especially the 'emotion side'.

Rationality and emotion

The recent research on affect in mathematics education has increasingly recognised that affect and cognition are strictly related and unlikely separable (e.g. Zan et al., 2006; Hannula, 2012). Other than emphasizing the richness of the field through a variety of perspectives, many studies have marked the delicate role of *emotion* in the landscape of affect. Zan et al. (2006) have pointed out "how repeated experience of emotion may be seen as the basis for more 'stable' attitudes and beliefs" (p. 116). Hannula (2012) affirms: "people can have very stable patterns for emotional arousal across similar situations, which is the foundation of the whole concept of attitude" (p. 141). Mainly in the same years, some criticism of Habermas' work has highlighted the demoted role given to affect-related aspects:

> However, as soon as not just purely theoretical questions but practical ones are concerned—questions that bring values, attitudes and emotions into play—agreement will not be reached exclusively through arguments, as Habermas demands of all agreement reached communicatively—but rather […] through all sorts of non-argumentative means of influence, such as the way arguments are presented, affection or dislike for the one presenting the argument, unconscious group dynamics, etc. There is not, in point of fact, any agreement in practical questions where such factors do not play a role. (Steinhoff, 2009, p. 205)

By their very nature, emotions are closely connected with both social systems and the biological human body. Research in neuroscience has supported evidence of the interplay among cognition, metacognition and affect. In particular, the studies of Damasio re-evaluated the role of emotion and feelings in decision-making: "certain levels of emotion processing probably point us to the sector of the decision-making space where our reason can operate most efficiently." (Damasio, 1999, p. 42). More recently, Immordino-Yang and Damasio (2007) have considered the relevance of affective and social neuroscience to education, proposing to consider emotion as a "basic form of decision-making, a repertoire of know-how and actions that allows people to respond appropriately in different situations" (p. 7).

As the perspective is widened to count emotion with respect to the teacher's beliefs and decision-making, the rationale of our study becomes clear. But we have sort of face questions like: How can we talk about the discursive activity of the mathematics teacher taking into account her emotional engagement in it? How can we talk about the *entanglement* of rationality and emotion in mathematics teaching? Looking at mathematics education research again, we have found a possible answer drawing on Brown and Reid (2006)'s study, which offers the notion of *emotional orientation* as a theoretical construct to analyse teachers' decision-making.

Emotional orientation(s)

Brown and Reid's study is relevant for our research for its interests in that particular emotional aspect of human behaviour, which the authors feel as neglected and see as "related to the decision-making that happens before conscious awareness of the decision to be made occurs." (p. 179). Following Maturana (1988), Brown and Reid refer the idea of emotional orientation to the criteria for acceptance of an explanation by members of a community and emotions to the foundation of such criteria. The criteria for accepting an explanation (*x*s) *cannot* be the same as the criteria for accepting the criteria ('meta-criteria' *y*s). We can draw on this distinction to interpret emotions as being at the subtlest degree, that is, as *moving* those *y*s for accepting the *x*s, figuring out emotional orientation as set of meta-criteria. We stop here in order to avoid an infinite regress. So, the teacher's rationality will allow us to talk about what happens—in terms of *x*s, while her emotional orientation will inform us of the feeling of what happens—in terms of *y*s. The two aspects are purely *intertwined* and joined in a unique frame that intends to speak directly to mathematics teaching in contextual situations. But we need to identify the emotional orientation of the teacher. Recalling Damasio, Brown and Reid introduce somatic markers as those structures that inform our action and decision-making, pushing us to decide something since "it feels right"—in terms of its acceptance in a community. In decision-making, they say, "many possibilities are rejected because they are associated with negative somatic markers" (p. 180), while *positive* somatic markers entail possible behaviours that reveal the teacher's decisions in the activity. They see emotional orientation as set of somatic markers, to which emotions related to *being right* are attached.

Following Brown and Reid in seeing 'the being right' as crucial, we characterize the emotional orientation as follows. We focus on the teacher's beliefs concerning the context, the content, the subject matter and her experiential background—beliefs that she declares in an a-priori interview. We identify her *expectations* concerning the activity—expectations that are attached to the beliefs and that we recover from videos of her actual activity in the classroom. The word 'expectation' is used for its positive meaning of wait and anticipation, which we can refer back to emotions of being right. Briefly speaking, the set of expectations shapes the teacher's emotional orientation, which entails belief-related actions that reveal the rationality of her decision-making. In the next section, we illustrate the example of a teacher.

The emotional rational being of Carlotta

Methodology

For space constraints, we only sketch the context and methodology of the study. The study is part of a wider research, whose focus is on the rationality of the teaching of linear equations at secondary school, and involves 3 teachers and their grade 9 classrooms, in Northern Italy. Each teacher was first interviewed and asked about her beliefs on linear equations and algebra in general. The interviews were twenty minutes long and were videotaped with the camera facing the interviewer and the subject. The teachers' activity in the classroom was also videotaped. The videos were transcribed for data analysis.

Classroom culture and knowledge stability

During the interview, Carlotta said:

> For me (*sighing*), the greatest problem I'm trying to solve—I realised, in these last years it's becoming tragic—is the problem of the stability of knowledge; I feel that, in many classrooms, (*speeding up*) apart from the good ones, students don't remember what we did and for me this is serious. For example, in grade 10, I'd like to refer to something that I did in grade 9, on which I've even insisted, without having to repeat it entirely… the big problem to solve, in which I persist a lot, is to be able to find a way to construct a core (*miming a base*), a base of knowledge (*miming a list*), of abilities that stay. For me, aside from time economy—'cause, maybe, it's annoying having always to recall—it's really a matter that has to do with cognitive science, I don't know, I wouldn't know how to face it well, but it's becoming a generalised problem, then… we should look for, the problem is looking for meaningful activities that allow for… fixing things.

Carlotta feels that knowledge stability is a "serious problem" in her teaching and she sees classroom culture as a possible solution for having a "core of knowledge". From this belief, she *expects to construct new knowledge from what has been already done in the classroom*, as it is shown by many classroom moments. An example is given when Carlotta decides to present the "properties of linear equations" starting from the laws for equalities that were introduced at the beginning of the year (and that regard the substitution properties: adding/subtracting a number to—multiplying/dividing by it—both sides of an equality does not change the equality):

Carlotta:	How do the laws for equalities translate into properties for equations? (*forward-facing, with a hand on the desk, raising her eyebrows*) What can you say? … That, if you have an equation, right? What do you do?
S3:	If we multiply or divide both sides of an equation by the same value, we will get an equivalent equation
S8:	Even subtracting we get…
Carlotta:	Let's say: For the first law, given an equation, if we add the same number to both sides or we subtract […] it means to sum the opposite, right? We can speak of sum. Then, if we sum both sides of the equation (*miming them with both hands*) we get (*nodding, waiting for the students to speak*)
S6:	An equivalent equation
Carlotta:	An equation equivalent (*nodding*) to the given one. Instead, for the second law… (*nodding, biting her close lips, gesturing a fist in the air*; Figure 1)
S3:	If we multiply or divide (*Carlotta nods, keeps her lips close and the fist in the air*; Figure 1)
S5:	By a number different from zero
S7:	Both sides

S3: We get an equation that is equivalent to the given one

Figure 1: Carlotta's expression and gesture

A fabric of emotion and rationality

In this context, we talk about the entanglement of emotion and rationality by looking at how the expectation about classroom culture narrows, marking Carlotta's *positive* emotion in constructing the properties of linear equations from the known laws for equalities: "for the first law, given an equation". Emotions are revealed by her *somatic* engagement—gesture, gaze, head movement, facial expression, tone of voice—in interaction with the classroom. Carlotta creates an *intersubjectively shared* space where to encounter the students and make them to disclose the same connection through what they know ("What can you say?"), which represents their horizon of reference. Her *epistemic rationality* refers to the nature of the laws for equalities—like, in the case of the first law, the possibility of speaking always of sum, instead of distinguishing between adding and subtracting—and brings at play the relationship with the properties of linear equations. This generates actions to accomplish the goal: having students to discover the connection and apply it for gaining an "equation equivalent to the given one" (*teleological rationality*). Carlotta uses many ways to communicate: gestures, gazes, facial expression and tone of voice changes, pauses and demands. The repeated use of the pronoun "we" is also part of her *communicative rationality*, and reveals the attempt to actively engage students, as well as her participation to learning construction—often expressed with nodding.

Concluding remarks

In the previous section, we have shown a brief example of how we can talk about the discursive activity of a teacher considering her emotional engagement in it. We have also proposed an example of how rationality and emotion are entangled in teaching, pointing out how to integrate the analysis of what happens in the classroom with taking into consideration the fact that in decision-making actions are belief-related. The emotional orientation of the teacher allows us to talk about her beliefs when they are generating actions that are oriented towards reaching goals and understanding. The rationality and emotional orientation of the teacher are inseparable as the weave and warp of a fabric. The weave and the warp together shape the fabric in the same way as rationality and emotion together characterize the teacher. The warp is related to the emotional orientation as well as the weave is related to the actions. If we look at the backwards of the fabric, we find all prior experience of the teacher, without which she would not be the teacher she is now, with her beliefs and background. The interview is methodologically relevant with this respect: it is a means to investigate prior experience, looking for expectations that can constitute emotional orientations.

Our study also points to the entanglement of emotion and rationality as a possible way to rethink *intentionality*. Habermas argues that all action is intentional, yet he is not interested in treating emotion when dealing with the intentional and reflective character of rational behaviour. We suggest instead that for grasping the teacher's intentionality we need to consider her *emotional being* in decision-making, by virtue of the association of the latter with reflection on personal activity.

5: PERSPECTIVES ON THE USE OF HABERMAS' CONSTRUCT IN TEACHER EDUCATION: TASK DESIGN FOR THE CULTURAL ANALYSIS OF THE CONTENT TO BE TAUGHT

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This contribution presents some recent advances in our use of Habermas' construct of rationality in pre-service teacher education. The construct of rationality is adapted and made suitable to describe different forms of rationality in different mathematical domains; such a description is shown to be efficient to guide the planning and implementation of teacher education tasks. Furthermore, the construct of rationality may be progressively acquired by prospective teachers as a theoretical tool for their a priori analysis of classroom tasks and for the analysis of students' productions. As such, the construct of rationality may work as a tool both for teacher educators (guiding their design of a teacher education sequence) and for teachers (when they perform the Cultural Analysis of the Content to be taught—CAC).

The contribution may be situated in the stream of research on task design for teacher education (Watson & Mason, 2007); it may be linked also to the crucial issue of theories as tools for teachers, see Tsamir (2008).

In the first part of this contribution, the idea of CAC (Boero & Guala, 2008), as an important competence to be developed in teacher education, is briefly presented, together with the need of choosing suitable mathematical domains and tasks for it. In the second part, some examples of a-priori analyses of the same tasks tackled in different mathematical domains, performed according to the three dimensions of rationality, show how we elaborated the design of teacher education activities that may generate occasions for CAC. More specifically, we will deal with the design of teacher education activities that on the long term should enable teachers themselves to identify different forms of rationality, typical of different mathematical domains. Focus will be on the a-priori analyses of the first two tasks, then we will outline the whole teacher education sequence.

The cultural analysis of the content to be taught (CAC)

Boero & Guala (2008) present and discuss the Cultural Analysis of the Content to be taught (CAC) as one of the most important goals of teacher education. In the authors' words, CAC goes beyond Mathematics teacher professional knowledge, as described by Shulman (1986), "by including the understanding of how mathematics can be arranged in different ways according to different needs and historical or social circumstances, and how it enters human culture in interaction with other cultural domains" (p. 223). Moreover, the authors point out that CAC can help the teacher to reveal the nature of some difficulties met by students "as related to didactical obstacles inherent in the ways of presenting a given content in school, or to epistemological obstacles inherent in its very nature"(p. 226).

A crucial issue is how to promote CAC during teacher education programs. Boero and Guala point out that it is important to select suitable mathematical subjects and involve teachers in suitable activities. Nevertheless, "exemplary CAC activities on well-chosen topics can have an effect on other topics, if teacher education challenges teachers to reflect on those experiences and their cultural meaning beyond the specific content" (p. 229). In this way, well-chosen topics may serve a broader aim, overcoming the limitation of the specific subject at stake. The ideal routine of a teacher education program in a CAC perspective encompasses: individual problem solving, guided discussion of individual solutions (selected by the teacher educator), individual analysis of the given task, collective discussion in which the teacher educator acts as a mediator and offers some elements of CAC. Additional activities are the creation of tasks for students (individual creation and collective discussion).

CAC and rationality

In a former contribution to PME (Boero, Guala & Morselli, 2013) the issue of different rationalities in different mathematical domains was presented. The working hypothesis of the present contribution is that performing activities across different mathematical domains, and promoting a reflection on different rationalities at issue in those domains, may be a major component of teacher education in a CAC perspective. In particular we will consider the case of the differences between synthetic geometry and analytic geometry rationalities: they can be traced back to the history of mathematics, thus promoting a view of mathematics as dynamic and cultural product. Furthermore, forcing teachers to solve a problem in different ways according to those different rationalities may challenge their beliefs about "closed" mathematical domains and put into evidence the possibility of having multiple solutions for the same problem by crossing the borders between different domains.

The teacher education activity

The contribution refers to a prospective secondary teacher education course carried out in 2013 at the University of Genoa. 12 prospective teachers were proposed the following task to be solved individually:

The parabola task

a. To characterize analytically the set ***P*** of (non degenerated) parabolas with symmetry axis parallel to the ordinate axis, and tangent to the straight line $y=x+1$ in the point (1,2).
b. To establish for which points of the plane does it exist one and only one parabola belonging to the set ***P***.
c. To find straight lines that are parallel to the ordinate axis and are not symmetry axes of parabolas belonging to the set ***P***.

In terms of a-priori analysis, we may say that part (a) is formulated in a way that addresses students towards a solution with an analytic geometry method, or—eventually—a calculus method (but also a third method, referring to synthetic geometry considerations, might be possible).

Let us consider the set of parabolas through the point (1,2): $y=ax^2+bx+2-a-b$.

To get the expression for those parabolas that are tangent to $y=x+1$ in (1,2):

analytic geometry method: you need to find relationships between a and b, such that the intersection points between $y=x+1$ and the parabola (which depends on a and b) are coincident in (1,2); after the system between the equation of the generic parabola through (1,2) and the equation of the straight line $y=x+1$, you substitute $y=x+1$ in the equation of the parabola, then you move to consider the condition of coincidence of solutions: discriminant $\Delta=0$, and you get $b=1-2a$.

calculus method: it is based on the meaning of the derivative $f'(c)$ as the slope of the tangent line in the point $(c, f(c))$; thus you get $f'(1) = 2a+b$ and you write $2a+b= 1$ (slope of the straight line $y=x+1$) . Finally you get $b=1-2a$.

The different methods encompass different forms of rationality, as already outlined in (Boero, Guala & Morselli, 2013). Here we consider a brief account of the rationality inherent in the solution of the part a) of the task with the analytic geometry method:

Based on the assumption that the solution of the problem is represented "within" the system, the solution may be made explicit by deriving an equation from the system and then getting the relationship between a and b through the analysis of that equation (Teleological Rationality, TR). Controls need to be performed on the different steps of the process (steps of algebraic transformations, algebraic substitutions, steps of the treatment of the equation $\Delta=0$, and so on) (Epistemic Rationality, ER). Effective communication requires the use of both algebraic and geometric language (Communicative Rationality, CR). We may observe how, apart from communication, verbal language plays a planning (TR) and control (ER) role, while algebraic language plays a prevailing executive role (TR).

Part (b) will be an object of work during the Research Forum session.

As regards part (c) (*To find straight lines that are parallel to the ordinate axis and are not symmetry axes of parabolas belonging to the set **P***), we may note that the problem may be solved easily by means of a synthetic geometry method:

Among all the straight vertical lines, the only line that cannot be symmetry axis of a parabola of the set ***P*** is the line passing through (1,2) because, if it were symmetry axis, the vertex of the parabola (the point (1,2), would be on that line and the tangent line in that point would be horizontal, against the fact that the tangent line has slope 1.

Here again we may see how an analysis according to the components of rationality may be performed: Through a visual exploration of the problem situation (TR) the conjecture of the exclusion of the vertical straight line through (1,2) is produced; the validation of the conjecture (ER) is performed by combining (TR) visual evidence related to the shape of the parabola with the hypothesis about the slope of the given straight line (that is not horizontal) and getting a contradiction (ER). Communication (CR) is based on the verbal and iconic language of geometry, with a narrative role for verbal language, together with some easy algebraic expressions to communicate technical details. Verbal language plays also a treatment role.

On the contrary, solving part (c) of the problem by means of analytic geometry looks much more difficult.

$x=1-(1/2a)$ is the equation of the symmetry axis of a parabola of the set ***P***. The exclusion of the line $x=1$ derives from the fact that $1=1-(1/2a)$ would imply the infinity of a. This solution is difficult to attain because it is necessary to explore the relationships between x and a in the equation $x=1-(1/2a)$.

In this case, the solving strategy (TR) is based on the algebraic modelling of the situation, and on the interpretation of an algebraic equation; ER consists in the control of the different steps of the modelling process (algebraic formalization; and interpretation of the relationships between x and a); CR needs a narration of the process and a good technical verbal presentation of the discussion of the algebraic equation. Verbal language and algebraic language play a double role of communication (CR), and of treatment (TR). Control role (ER) is mainly played by verbal language.

The exemplified a priori analyses suggested that the task was a promising task for pre-service teacher education in a CAC perspective, since prospective teachers may appreciate the fact that different solutions are possible for the same task and that one can move from a problem formulated in a domain, to a solving process in another domain (with different rationalities).

Prospective teachers met big difficulties to solve parts (b) and (c) of the task. In the part (b), analytic geometry methods brought to several mistakes in performing rather complex algebraic transformations, with deadlocks depending on results "*without any meaning*", as a prospective teacher said. No prospective teacher tried to solve (b) through an easier synthetic geometry method. Interestingly, most teachers were unable to check their solutions through synthetic geometry considerations. A few teachers solved (c) by a synthetic geometry method, while most of those who tried an analytic geometry solution were unable to get the correct solution (and did not move to another method).

After working individually on the task, in a subsequent session prospective teachers analysed their own solving processes with the teacher educator. A special care was devoted to the analysis of the difficulties they met during the solving process, and to the difficulties met by some prospective teachers in the previous year. Indeed qualitatively similar data had been collected when the same task had been solved in an entrance examination for graduate students in Mathematics, willing to become mathematics teachers (see Boero, Guala & Morselli, 2013).

As a second step, prospective teachers were proposed a modified task that "forced" them to solve the same problem in a given way:

> In question b) [referring to the previous task], how is it possible to exclude the right line $y=x+1$ and the right line $x=1$, within synthetic geometry (i.e. with your knowledge on the shape and other geometric features of the parabola)?
>
> In question c) [referring to the previous task], how is it possible to exclude the right line $x=1$, within synthetic geometry?

After the individual solution, the difficulties met by participants were discussed and exploited to promote prospective teachers' need for performing the cultural analysis of the content to be taught. In doing so, the adaptation of Habermas' construct to mathematics education purposes (Boero & Morselli, 2009) was explicitly proposed to prospective teachers as a tool to identify specific, different features of typical activities in different mathematical domains; then the construct was used by them to perform gradually more autonomous a-priori analyses of tasks and of the difficulties students may meet to cross the borders between different mathematical domains. In this way, the construct of rationality was not only a tool for the teacher educator: it became a tool for prospective teachers.

Results

The contribution to this Research Forum refers to the use of Habermas' construct as a tool for teacher education aimed at promoting CAC; a consequent result is the idea of inserting the construct of rationality into the professional knowledge of teachers, as a tool for them to perform the CAC and design suitable tasks for students.

Habermas' construct of rationality was refined to describe the different forms of rationality in different mathematical domains. Such construct was used for the design of a sequence of tasks conceived in the CAC perspective. The construct was also proposed to teachers as a tool to identify and compare specific features of activities in synthetic geometry and in analytic geometry, and of synthetic and analytic methods.

During the Research Forum session, examples of teachers' solutions and their trials to analyse them according to rationality criteria will be provided and discussed.

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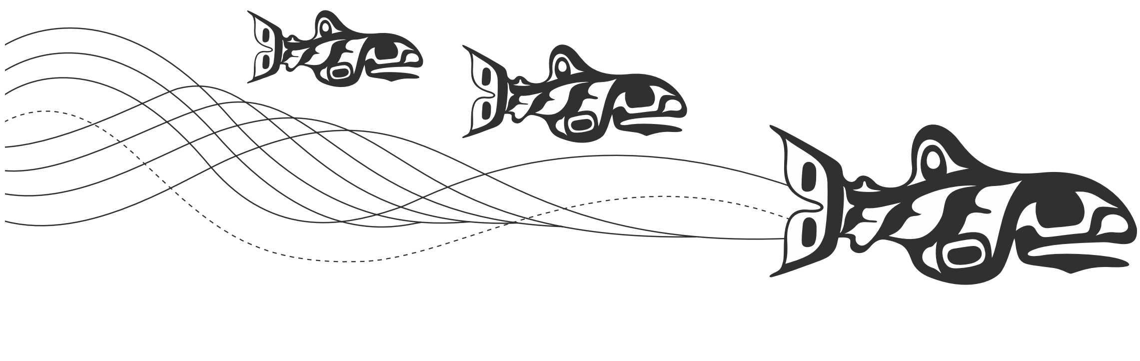
DISCUSSION GROUPS



WHAT IS QUALITY MATHEMATICS TEACHING-RESEARCH?

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Mathematics Teaching-Research (MTR) is investigating the relationship between classroom teaching and learning. It is a process of identifying classroom problems, what leads to the formulation of teaching-research questions, design of the intervention to be implemented in the classroom together with the collection of the data, followed by its analysis to answer the teaching-research questions, which can be generalized or deepened in the next cycle of studies. Some TR schools of thought (Japanese Lesson Study; TR/NYCity, Chinese Keli approach) emphasize the necessity of two consecutive cycles to be conducted by the teacher-researcher in order to design and implement the teaching improvements suggested by the previous cycle; the double TR cycle enriches TR investigations by facilitating teacher's creativity in the design of improved instruction and helps to develop skills of adaptive instruction. This teaching-research framework rests on the Action Research formulation by (Lewin, 1946); teaching-research in service of the curriculum research formulated by (Stenhouse, 1975) and the teaching-experiment methodology formulated by the Vygotsky school (Kantowski, 1978). *The research question of the Discussion Group is What is High Quality Teaching-Research?* The research strategy: high quality TR reports from several TR schools of thought will be chosen from available literature[*] by the organizers. The first session will start with short, intuitive responses of participants, followed by the collaborative analysis of the chosen reports with the aim to identify the quality components of each report. The second session will synthesize these components into the theoretical model of quality MTR followed by the collaborative design of possible classroom teaching experiments to be conducted in the intervening year. The reports from those classroom experiments will become the basis for the 2nd Handbook of MTR.

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NEGATIVE NUMBERS: BRIDGING CONTEXTS AND SYMBOLS

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At the latest PME-NA (35), a working group met on the topic of negative numbers. Groups presented on their research within one or more categories related to negative integers: role of contexts, models, historical development, algebra, student understanding, and teacher knowledge (Lamb et al., 2013). We began a productive discussion on issues related to each of these categories and would like to broaden the discussion with the inclusion of members from the international community. Further we aim to develop research collaborations around mutual areas of interest.

SESSION 1

During session 1, after introductions, Dr. Laura Bofferding and Nicole Wessman-Enzinger will present a summary of negative number research from past PME and PME-NA proceedings, highlighting common themes, any areas of discrepancy, and theoretical frameworks that underlie the different research paradigms. Participants will discuss additional frameworks that support their research around negative numbers to add to a research categorization document that was started during an initial working group at PME-NA 35. Participants will explore the intersection of the categories, in particular how research on contexts versus symbolic-only problems can inform each other, and identify gaps in the research.

SESSION 2

During session 2, we will present a summary of the discussion from the previous day. We will have two presentations focused on the different uses and conceptions of negatives from Dr. Irit Peled and Dr. Aurora Gallardo, followed by discussion. Then we will break up into groups (depending on the interests of the group) to discuss future research directions and begin preliminary plans for collaboration. Potential foci could be on exploring the use of the difference meaning of subtraction, symbolic understanding, RME models, and contexts to support negative number understanding. Groups will share out their areas of interest and initial research ideas to allow others in the group to have the opportunity to join the collaborations.

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NUMERACY ACROSS THE CURRICULUM

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This Discussion Group will explore international perspectives on embedding numeracy across the school curriculum. *Numeracy* is a term used in many countries, such as Canada, Australia, South Africa, the UK, and New Zealand, whereas elsewhere it is more common to speak of *quantitative literacy* or *mathematical literacy*. Being numerate involves more than mastering basic mathematics because numeracy connects the mathematics appropriated within formal learning situations with out-of-school contexts that additionally require problem solving, critical judgment, and making sense of non-mathematical contexts.

Steen (2001) maintains that, for numeracy to be useful to students, it must be learned in all school subjects, not just mathematics. Although not new, this is still a challenging notion that is now being taken up in some countries. For example, in Australia our research team has been working with teachers to embed numeracy across the school curriculum (Geiger et al., 2013). To do so, we introduced teachers to a theoretical model of numeracy that addresses real-life contexts, application of mathematical knowledge, use of representational, physical, and digital tools, and positive dispositions towards mathematics. These elements are grounded in a critical orientation towards mathematics. Our goal in this Discussion Group is to engage researchers in other countries with these ideas with a view to future collaborations.

We will begin the first session with a synopsis of our theoretical model and current research. Small groups of participants will then discuss the following questions:

1. What theoretical perspectives underpin different conceptualisations of numeracy?
2. How can we work with teachers to embed numeracy across the curriculum?

Groups will report their responses at the end of the first session. In the second session new groups will be created, each centred on either a theoretical perspective or approach to researching with teachers, to formulate questions to guide future research into numeracy across the curriculum.

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SCHOOL MATHEMATICS CURRICULUM IN CENTRALIZED AND DECENTRALIZED EDUCATIONAL SYSTEMS

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The debate on the kind and the degree of centralization in school system and its effect on mathematics curriculum and assessment as one of the central school subjects, has been going on for years. The level of centralization includes national, state or province, district, or even as local as school level. Last 30 years, the world has been witnessing the efforts of decentralized systems to move towards some kind of centralization in the forms of "national curriculum", "common Core Standards" and else. On the other hand, many of the centralized systems, are seeking new means to lessen the degree of their centralization by again, calling for one "national curriculum" and approved textbooks (Gooya & Ghadaksaz Khosroshahi, 2007, Gooya, 2010). Thus, this discussion group aims to engage participants in facing the challenges that these two directions have created for all aspects of mathematics education including curriculum, textbooks, teacher education and assessments around the world.

SESSION 1

The curriculum issue is relatively new in PME and deserves more attention. In the first 20 minutes, we provide participants with the brief introduction in this issue. We then devote 30 minutes to 3 presenters from different education systems to share their experiences with the group. In next 30 minutes, the participants will make small groups and discuss the issue in more details. Each group is asked to have a coordinator among themselves to report the major points to the whole group. In last 10 minutes, we sum up the session 1.

SESSION 2

The second session starts with reports from different groups. This takes 30 minutes. We then, have 40 minutes whole group discussion to better understand the points and challenges that are brought up by small groups. In last 20 minutes, we formulate some questions to be followed in the next meeting of this DG.

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THE AFFORDANCES AND CONSTRAINTS OF MULTIMODAL TECHNOLOGIES IN MATHEMATICS EDUCATION

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Multimodal technologies have evolved in various disciplines and applications including computer visualization and simulations, medicine, and architecture. Their use is also increasing in education, particularly in early learning and special needs education. Although less widely explored, we believe it is timely to examine specific uses and limitations in mathematics education contexts. Learners can now virtually touch, manipulate, hear or experience reactive force feedback through a wide variety of technological advances. Some of the most ubiquitous uses today are the combination of sight with touch through multiple inputs, for example, Smartphones, tablets/iPads. In a learning environment, these can be used by one user with multiple fingers or with several users interacting in a variety of ways, potentially reconfiguring the interactive and investigative space around a single device.

The design and implementation of mathematical activities using such technologies have trajectories from theoretical frameworks in mathematics education focused on dynamic, interactive software and semiotic mediation (Falcade, Laborde, & Mariotti, 2007) and more broadly in special education (Thompson Avant & Heller, 2011).

The goals of this discussion group would be to explore the new possibilities of such advances in a critical way. In the first session, the organizers will present 4-5 examples in the form of 3 minute nano-talks of how such technologies are presently being used in formal and informal settings to stimulate an active, critical dialogue by participants on their perceived benefits and limitations. The second session will move to a collective discussion based on the critical dialogues of session one, focused on (but not limited to) the following areas and questions:

1. Future activity spaces – which areas of mathematics education can such technologies be used or not used?
2. What are the pedagogical challenges for continued use in the future?

Expected outcomes will be a small brief that would frame future work at PME in the form of a working group or a critique for dissemination.

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MATHEMATICAL DISCOURSE THAT BREAKS BARRIERS AND CREATES SPACES FOR MARGINALISED STUDENTS

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THEORETICAL BACKGROUND AND AIMS

Classroom discourse has been accorded considerable attention in research and policy documents in the past two decades. Common to the body of literature is recognition of how opportunities students have to access mathematical content and discourse practices impacts on their identity as knowers and users of mathematics (Hunter & Anthony, 2011). Access to these discourse practices is closely related to who gets to participate in the mathematics classroom (Civil & Planas, 2004). This discussion group will consider ways in which marginalised students are provided with space to equitably access the mathematical discourse and practices. The work will be grounded on two approaches to analysing participation in mathematical discourse, the communication and participation framework by Hunter and Anthony (2011) and the four categories of obligation and choice in Herbel-Eisenmann and Wagner (2010).

SESSION STRUCTURE

This discussion group will invite participants to share experiences and research related to how barriers to the discourse have been identified and removed for different groups of marginalised students. In the first session the co-leaders will present the two different frameworks and examples of their own work. Participants will analyse transcripts and video clips (e.g., English language learners in the U.S., Pasifika students in NZ) using and extending the two frameworks. The second session will be used to discuss and create a new framework as a tool to be used to both scaffold and analyse marginalised students' access to the mathematical discourse and practices. An overall aim is that the two sessions will facilitate opportunities to discuss and develop a research agenda that focuses on evidence-led practices which support marginalised students' access to the classroom discourse.

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VISUALIZATION AS LEARNING TOOL: WHAT SHOULD PROSPECTIVE TEACHERS KNOW AND TEACHER EDUCATORS TEACH?

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Researchers in mathematics education increasingly recognize the role visualization plays in mathematics learning. Indeed, ZDM recently devoted a special issue to visualization as an epistemological learning tool (Rivera, Steinbring, & Arcavi, 2013).

While the ZDM studies yield important convergent knowledge about visualization in mathematics thinking and learning, Presmeg (2013) argues for deeper analyses of the role visualization plays in the development of mathematical knowledge by posing questions for future research cited here: 1) How can teachers help learners to make connections between visual and symbolic inscriptions of the same mathematical notions? 2) How may visualization be harnessed to promote mathematical abstraction and generalization? 3) How do visual aspects of computer technology change the dynamics of learning mathematics? 4) For prospective teachers at all levels to become aware of the affordances and challenges of using visualization as a learning tool in mathematics education, what aspects of teacher education programs are effective? This DG provides a forum for discussion and future research on these questions as they relate to mathematics teacher preparation.

During the first session, after a brief introduction to visualization research, the authors will survey participants on their use of visualization in mathematics teacher preparation. What tasks? Tools? For what purpose? In preparation for session two, a range of visualization tools for learning mathematics will be discussed and two types contrasted: virtual manipulative applets (VMA, e.g., National Library of Virtual Manipulatives) and interactive virtual game sequences (VGS, e.g. Spatial Temporal Mathematics Program Video Games). We will invite participants to explore the tools and discuss their didactical use and the challenges of professional development.

Session two centers on visualization task-design in mathematics teacher preparation. Results from an exploratory study of the affordances of visualization tasks in the author's mathematics methods course will launch discussion. Small groups will design visualization tasks using the visual tools above. Discussion will focus on how the tasks respond to Presmeg's pedagogical challenge. Participants are invited to bring methods syllabi and laptop to facilitate sharing of tasks using google docs.

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OBSERVING TEACHERS OBSERVING MATHEMATICS TEACHING: RESEARCHING THE UNOBSERVABLE

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Many researchers face the challenge of researching the unobservable (e.g., reasoning, beliefs, etc.). The Observing Teachers study makes use of an anthropological methodology (Tobin et al. 1989) to address this challenge, in particular to research middle school mathematics "pedagogy" across regions of Canada. "Pedagogy" is used to refer to the implicit cultural practices of teachers that guide teaching practice. We make use of the enactivist insight that "everything said is said by an observer" (Maturana, 1987), but this raises other issues, including the roles of both researchers and teachers as observers and observed. Approaches to the challenge of observing the unobservable will be discussed and advantages and disadvantages of our methodology will be explored.

The discussion group will address the methodological question: How can we research the unobservable? We will use examples from Observing Teachers study to provoke discussion of this key question.

ACTIVITIES PLANNED AND ATTENDEE PARTICIPATION

Day 1:	Introduction to the questions	10 min.
•	Discussion: Propose examples of specific research studies where the key questions matter and how they are/were addressed.	20 min.
•	Introduction to the research programme	10 min.
•	Example: Analysis of Auto-ethnography including theoretical frames and key results	20 min.
•	Group activity: Comparative analysis of two short pieces of rich data	30 min.
Day 2:	Group reports and Debrief of group activity	30 min.
•	Insider/Outsider perspectives	10 min.
•	Example: Ethno-ethnography	10 min.
•	Discussion & closing: Revisiting questions with insights from examples	40 min.

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EXPLORING HORIZONS OF KNOWLEDGE FOR TEACHING

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Scholars have begun to shape and frame the construct of horizon content knowledge (HCK), a part of mathematical knowledge for teaching proposed by Ball, Thames, and Phelps (2008). Different approaches have inspired dialogue among researchers (e.g., Jakobsen, Thames, & Ribeiro, 2013; Wasserman & Stockton, 2013; Zazkis & Mamolo, 2011), and we propose a discussion group session to foster exchange and interaction.

The purpose of this DG is to examine current research related to HCK in order to identify a coherent research strategy and promote further individual and collaborative efforts. In the first session, three 15-minute presentations will provide examples of different approaches to HCK, each followed by 15 minutes of discussion regarding the underlying framing of the problem of content knowledge, the relationship of content knowledge to teaching, the value of identifying specific HCK-curricular content, and the relevance for teachers' identities, learning, and use. During the second session, participants of the DG will engage in discussion around five topics: (i) examples of classroom episodes in which HCK is involved; (ii) HCK as related to and distinct from other MKT sub-domains; (iii) potential impact of HCK on the work of teaching (e.g., on teaching practices and instructional interactions); (iv) methodological approaches and challenges in researching HCK; and (v) mathematical terrain of HCK, including primary dimensions, distinctive features, and most useful representations. Following the session, we will draft and circulate a memo proposing a coherent overall research strategy and next steps. Other invited participants include: C. Charalambous, S. Delaney, L. Figueiras, M. Thames, and R. Zazkis.

The aim of the DG is to actively discuss and further develop this important domain of MKT. Clarifying and developing the role of HCK will help promote scholarly exchange and study about the role that knowledge of mathematics plays in teaching.

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PREPARING AND SUPPORTING MATHEMATICS TEACHER EDUCATORS: OPPORTUNITIES AND CHALLENGES

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Participants in this Discussion Group will explore opportunities and challenges for supporting mathematics teacher educators in their work with prospective elementary teachers in the context of doctoral programs and employing institutions.

Mathematics teacher educators (MTEs) are responsible for ensuring that prospective elementary teachers (PTs) are provided opportunities to develop deep Mathematical Knowledge for Teaching (Ball, Thames, & Phelps, 2008). Yet, research shows that most instructors of mathematics courses for PTs in the U.S. do not themselves have elementary teaching experience (Masingila, Olanoff, & Kwaka, 2012). Included in this majority are the organizers of this group, MTEs who teach content and methods courses for PTs and prepare future MTEs through their work with doctoral students.

The goal of this Discussion Group is to explore ways in which MTEs can be supported in their work with PTs. This group will focus on the professional learning opportunities that may contribute to the work of MTEs, in lieu of teaching experience, which connects with the work of PME 37 DG 3: Mathematics Teacher Educators' Knowledge. This inquiry will be approached from the contexts of both doctoral programs and MTE-employing institutions. The sharing of initial results of the organizers' research (e.g., at PME-NA 2013) has already inspired deep conversations among educators interested in this topic from multiple perspectives.

The first session will include a brief review of relevant research including descriptive statistics from a survey of 69 early-career MTEs. Participants will share a) challenges faced in their work preparing PTs and MTEs and b) experiences contributing to their successes. The organizers will share emergent themes and recommendations gleaned from interviews with a subset of the surveyed MTEs who teach courses for PTs but have no elementary teaching or research experience (n=8). During the second session, participants will discuss the knowledge and dispositions of effective MTEs and formulate ideas for creating opportunities for MTEs to develop such skills. One organizer will share her doctoral program's model for preparing future MTEs for this work. Lastly, participants will discuss potential paths for future research in this area.

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RESEARCHING 'THINKING CLASSROOMS'

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Our 2013 Discussion Group (DG) demonstrated the PME community's interest in the nature of 'thinking classrooms' and the community's wealth of knowledge about them. This 2014 DG extends those discussions to Researching Thinking Classrooms. By generating and exploring research questions, we will illuminate features of thinking classrooms. Questions generated in 2013 include "How can we identify a 'thinking classroom'?" "Does thinking continue beyond a thinking classroom?" "How do students perceive thinking classrooms. Does this affect their learning opportunities?" We will brainstorm possible research designs for generating rich data to answer such questions. Research designs previously employed to study data employing various theoretical perspectives will inform this process (e.g., cognitive, social, affective, emotional, and psychological aspects of student learning, and teachers' practices and beliefs). These designs include: self-reports of affective experiences (Liljedahl, 2013), 'talk-out-loud' during problem-solving (Krutetskii, 1976), video of own classroom practice (Lampert, 2001), video-stimulated student interviews (Williams, 2006), and emoticons to study student emotions (Ainley, 2010).

Session 1: The session commences with a brief overview of 2013 findings. Participants then work in small groups to identify research topics with the potential to progress our understanding of thinking classrooms. These are shared before participants 'opt in' to a topic of choice. Groups then develop one, or several, creative research questions with the potential to illuminate that research topic.

Session 2: Questions and preliminary ideas on research designs will be shared and discussed. Groups will then add detail to their research designs. Ideas will be shared and discussed. This session ends with brainstorming of ways to form and sustain international research collaborations on these topics. We will then ascertain interest in a Working Session on Thinking Classrooms in 2015.

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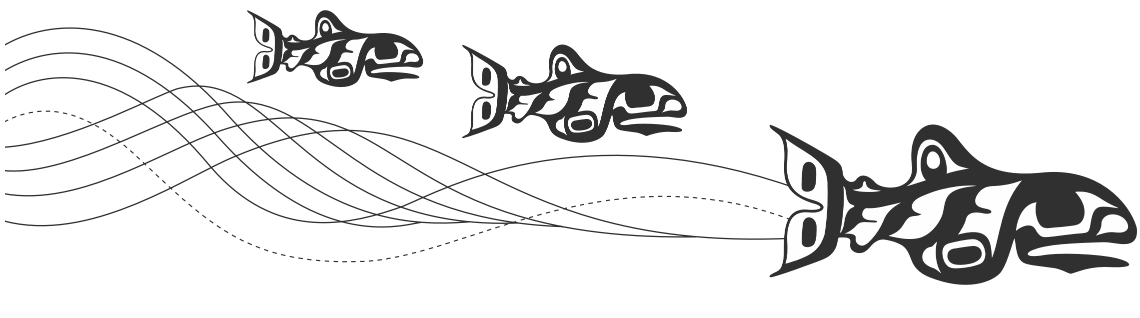
WORKING SESSIONS



THE USE OF EYE-TRACKING TECHNOLOGY IN MATHEMATICS EDUCATION RESEARCH

Patrick Barmby[1], Chiara Andrà[2], David Gomez[3], Andreas Obersteiner[4], Anna Shvarts[5]

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With the increased availability of advanced hardware, eye-tracking technology has become an important tool for gathering information about an observer's visual attention on particular stimuli. It allows researchers to make inferences about what the observer views as important whilst observing the said stimuli. Eye-tracking studies with a mathematics education focus include investigations of students' approaches to arithmetic (Suppes, 1990), comprehension of word problems (Hegarty, Mayer, & Monk, 1995), geometry (Epelboim & Suppes, 2001), and the role of representations in mathematical learning (Andrà *et al*., 2014). However, the use of eye-tracking in mathematics education research is still limited.

Therefore, in line with the conference theme for PME 38, the aim of this group is to discuss the potential of this innovative approach to mathematics education research, including ways in which the approach can be superior to other methodological approaches. In the first session, there will be two short presentations on how eye-tracking technology can be incorporated into quantitative and qualitative approaches. This will be followed by a group discussion of areas of research interest to see how eye-tracking could be incorporated into these areas. In the second session, following a bringing together of the possible areas of research from the first session, there will be one short presentation on possible analytical techniques. This will be followed by further group work to develop research ideas based around specific problems/tasks, with discussion on the advantages and limitations of eye-tracking methodology in possible studies.

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MATHEMATICS TEACHER EDUCATORS' KNOWLEDGE

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This Working Session (WS) builds on the Discussion Groups (DGs) on Mathematics Teacher Educators' Knowledge at PME37 (Beswick & Chapman, 2013) and ICME-12 (Beswick, Chapman, Goos & Zaslavsky, 2012) conferences. The DGs aimed to bring together researchers in this emerging area to explore the topic and to set directions for future work in the field. It was apparent from the DGs that there is increasing interest in the area and that relevant research was planned or being conducted in different countries. Three themes that emerged as central to these studies are: the nature of mathematics teacher educators' knowledge; the acquistion and/or development of mathematics teacher educators' knowledge; and issues in researching mathematics teacher educators' knowledge. Participants at PME37 expressed interest in a follow up session at PME38 conference. This WS is intended to provide a space for participants to discuss their research projects and developing manuscripts for contributions to a special issue of the *Journal of Mathematics Teacher Education*. For example, it will allow prospective authors of papers for the special issue to receive feedback on their draft manuscripts and to continue to refine their work. The WS will also provide an opportunity for researchers with a developing interest in the field to gain insight into relevant conceptual frameworks and the nature of research being conducted in the area.

In the first session, after the coordinators will provide a brief overview of the journey to and goal of the WS, participants with draft manuscripts based on research of one of the three themes noted above will briefly present their work. Presentations will focus on their theoretical frameworks, methodologies, results and conclusions. This will be followed by open discussion and feedback by all in attendance at the WS.

Outcomes of the first session will inform the precise structure of the second session but it is anticipated that participants will work in small groups with prospective authors, to discuss a draft manuscript of particular interest in greater detail. The coordinators will provide assistance to prospective authors and facilitate discussions as needed.

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TEACHER NOTICING: A HIDDEN SKILL OF TEACHING

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This working session continues the work of PME-NA 2013 teacher noticing working group. Participants will seek an operational definition of teacher noticing and continue to plan for: a teacher noticing conference, extending teacher noticing to the science arena, the creation of a teacher noticing website and a monograph that incorporates both mathematics and science teacher noticing.

PROFESSIONAL NOTICING FRAMEWORK

Mathematics teacher noticing involves the nearly invisible thought processes of teachers as they observe and instruct students. Research in mathematical noticing has grown considerably in recent years and while there is consensus of the value of teacher noticing to student learning, the definition of teacher noticing is not always consistent. Much of the research on noticing focuses on observation, or attending, although Jacobs, Lamb, and Philipp (2010) proposed this as only one of three interrelated components, the remaining two being, interpreting and deciding. This working session will seek to operationalize the definition of teacher noticing, extend the research to the discipline of science, and establish avenues for presenting new and original teacher noticing research to the mathematics and science education communities via a teacher noticing conference, website, and monograph.

WORKING SESSION TASKS AND ACTIVITIES

Session 1 will engage participants with examples of teacher noticing in mathematics and science to familiarize the audience with the existing state of the field (45 minutes). Additionally, the opening activities will serve to further operationalize the definition of teacher noticing. Three focus groups will be formed, each taking the lead on: 1) Proposal writing for conducting a teacher noticing conference, 2) Research discussions of extending mathematical teacher noticing to science leading to a monograph, and 3) Designing a teacher noticing website to communicate our operationalized definition, conference plans, and current research on noticing in mathematics and science. Session 1 will end with focus groups developing organizational strategies (45 minutes) to be reported at the opening of Session 2 (30 minutes). Focus groups will continue planning in Session 2 striving to complete a timeline of activities to reach each group's respective goal (30 minutes). The session will culminate with focus group reports to large group and discussion of next steps.

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DEVELOPING PRESERVICE ELEMENTARY TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING

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The working group on Developing Preservice Elementary Teachers' Mathematical Knowledge for Teaching aims to examine significant cognitive and non-cognitive factors influencing the preparation of preservice teachers from multiple, diverse perspectives, including those of mathematicians and mathematics educators.

Identifying factors that impact the development of appropriate and adequate mathematical content knowledge in preservice elementary teachers is a complex issue in mathematics education research. The PME meeting in 2014 is the third meeting of the working group focused on this issue. The group is organized into special interest subgroups, as described below.

Mathematical Tasks: This subgroup is grounded in a task development cycle of *design, enact, reflect,* and *modify/re-design* phases. In addition to reviewing and synthesizing the literature on task design, this group will share their research on task development using tasks originally designed for children.

Children's Thinking: Based upon earlier research from Cognitively Guided Instruction and more current studies focused on interpreting children's mathematical thinking, this subgroup will share how artefacts of children's thinking can promote mathematical understanding with preservice elementary teachers.

Mathematical Habits of Mind: This subgroup unpacks the notion of mathematical habits of mind. Building from the literature, they offer example tasks for developing mathematical habits of mind and discuss how these tasks can be used to both raise awareness of and foster these ways of thinking in preservice elementary teachers.

Affect: Another influential factor in the preparation of preservice teachers is the affect (e.g., attitudes, beliefs, emotions) they bring to and acquire during university mathematics content courses. This group reviews and summarizes the state of research in this area to reveal implications for preservice elementary teachers' learning of mathematics content.

Three International Perspectives: This subgroup examines common principles that should be included in post-secondary mathematics courses for elementary teachers, taking a broader look at concepts, models, and approaches within various contexts.

In the first session, each subgroup will share their work. Following brief presentations, newcomers and prior working group members are invited to respond, first in plenary and then in themed subgroups. This will continue in the second session, with the last hour dedicated to discussing compilation of the final papers into an edited book to support mathematics instructors of preservice elementary teachers.

 2014. In Liljedahl, P., Nicol, C., Oesterle, S., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36, Vol. 1, p. 256. Vancouver, Canada: PME.

A WORKING SESSION ON VIRTUAL MANIPULATIVES

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THEORETIC BACKGROUND AND QUESTIONS TO GUIDE THE WORK

In 2013, Moyer-Packenham and Westenskow published a meta-analysis on virtual manipulatives (VMs). This major review reflects a shared interest by researchers worldwide in the study of these objects for mathematics teaching and learning. Researchers use different terminology to describe these objects, including online math objects, digital objects, math cognitive tools, online math applets and virtual manipulatives (VMs) defined by Moyer, Bolyard and Spikell (2002) as "an interactive, Web-based, visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (p. 373). Researchers in this working session will explore theoretical rationales and concepts to better understand VMs as resources/tools for learning and development by examining questions such as: What terminology and frameworks help researchers communicate about this work? What big ideas can we glean from the research being conducted worldwide? In what ways does research on concrete manipulatives and other visualizations inform the work on VMs? What methods are researchers using and what research questions are important to our collective field studying the impact of VMs on achievement and affect? How do we advance research on VMs as these dynamic objects move from web-based environments to hand-held, touch-screen and augmented platforms?

GROUP GOALS, ACTIVITIES AND PARTICIPATION

International researchers from Australia, Canada, Germany, Sweden, Turkey, and the USA have committed to participate in the session. During the 2-day working session, different researchers will provide brief 5-minute overviews of key topics on VMs, followed immediately by discussion on the topics. Multiple key topics will be initiated and discussed consecutively during the two days. Examples include: Larkin (iPad apps); Highfield (interactive technologies); Martin, Ladel & Kortenkamp (VMs); Lindström & Holgersson (Fingu iPad game); Durmus, English & Osana (manipulative use); Namukasa (VMs frameworks); Ozel (design of VMs); Tucker (app affordances); Jamalian (manipulatives interacting with apps). The group's goal is to explore the topics above resulting in a book proposal with chapters on virtual manipulatives.

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KEY ISSUES REGARDING TEACHER–STUDENT INTERACTIONS AND ROLES IN ASSESSMENT FOR LEARNING

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This working session (WS) builds on the outcomes of a discussion group (DG) on assessment for learning and diagnostic teaching at PME 37 in Kiel. While Bell (1993) outlined the relationship between tasks, students and opportunities for learning in his framework for diagnostic teaching, this relationship is often overlooked in much of the literature on assessment for learning, where the focus, to a large extent, has been on the teacher (see, for instance, Black & Wiliam, 2012). Fostering assessment for learning by allowing students to take part in self- and peer-assessments might necessitate a major shift in student and teacher roles compared to those found in more traditional classrooms (Hayward, 2012; Hopfenbeck, 2011). Considering communication as an essential part of the assessment for learning process, the overall aim of the working sessions is to identify and reflect on key shifts in teacher-student interactions, roles and opportunities for learning mathematics. An expected outcome of the WS is an agenda for an edited book on the topic of AfL in mathematics classrooms.

The first working session will focus on situations in which teachers strive to provide feedback to students about their mathematical competence. The focus will be on the quality of tasks and criteria for analysing the impact of teacher feedback (see, for instance, Wiliam, 2007). Sample tasks and teacher evaluations will be analysed.

The topic of the second working session will be classroom interaction and changes the student role. Excerpts of student-teacher interactions will be analysed, focusing on student responsibilities in assessment situations. In both sessions, participants will work in small groups to analyse the provided material using predefined criteria from the assessment literature. This will be followed by a general discussion of identified key issues and the applicability of the criteria.

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SPECIAL EDUCATION AND MATH WORKING GROUP

Helen Thouless[1], Ron Tzur[2]
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This working group has been focused on developing a research agenda to explore pedagogical approaches for fostering conceptual knowledge of mathematics in students with special needs. The work is rooted in a twofold premise: (1) students with mathematics difficulties are capable of and need to develop conceptual understanding and mathematical reasoning skills, and (2) special education instruction and assessment needs to transition toward this focus.

THEORETICAL BACKGROUND

Procedural instruction is a major feature of instruction and instructional research for students with learning disabilities (LD) (Gersten et al., 2009). This type of instruction presents mathematics as a series of isolated procedures to be learned, and seems to make little attempt to help students construct mathematical ways of thinking and connections among mathematical ideas. To help students with LD develop and abstract concepts in mathematics, research is needed that focuses on students solving “problems that are within [their] reach [while] grappling with key mathematical ideas that are comprehendible but not yet well formed” (Hiebert & Grouws, 2007, pp. 387).

PLAN FOR WORKING GROUP

During PME-NA 2013 this working group decided on a research agenda to study the development of thinking/hypothetical learning trajectories in multiplicative reasoning for students with learning disabilities. Subsequently members of the group have familiarized themselves with a multiplicative reasoning instrument in readiness to design a collaborative study to explore this research agenda.

Session 1	Session 2
a) Briefly share past discussions	d) Decide on research methodology
b) Review use of instrument	e) Identify sources of funding
c) Decide on specific research questions	f) Start co-writing a grant proposal

Table 1: Goals and activities for working group

References

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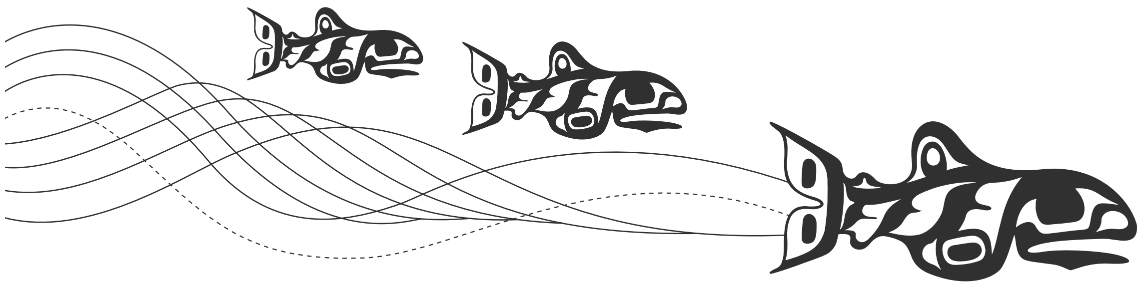
NATIONAL PRESENTATION



MATHEMATICS EDUCATION IN CANADA
PME 2014 NATIONAL PRESENTATION

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INTRODUCTION (David A. Reid)

As we began to prepare for this national presentation, the question arose whether it makes sense in general to think of national approaches to mathematics education research, and especially whether it makes sense to speak of *Canadian* research in mathematics education. “Research is either international or the activity so called is not research at all,” was one comment made in our discussion.

This uncertainty about our national identity is one aspect of a Canadian perspective on mathematics education research. In some places, such as France, Germany and the Netherlands, there are much stronger national perspectives on mathematics education than in Canada. These include shared theoretical frameworks, research foci, and assumptions about teaching and teacher education. In Canada circumstances make the emergence of a national perspective difficult.

One response to this is an enthusiasm for international contacts, which can be seen in Canadian involvement in PME, PME-NA, ICMI and other international organizations. Canadians have been involved in PME since the beginning, both as participants and as members of the IC, and PME has met twice in Canada, in 1987 and 2014. Canadians were strongly involved in the founding of PME-NA, which met in Canada in 1983 and 2004 in addition to the two joint conferences with PME. Canada hosted ICME-7 in 1992, and Canadians have served on the ICMI executive committee, first in 1952, and continuously from 1991 to 2009.

All this international activity may be related to the multicultural nature of Canadian society. Most Canadians are descended from immigrants from Europe, Asia and Africa, and continue to self identify according to their origins. Of course this is true for most countries in the Americas, but Canadian identity seems more ‘hyphenated’ than identity in the rest of the Americas.

Another aspect of Canadian multicultural identity that must be mentioned is the important role of Aboriginal Canadians in Canadian society. Aboriginal peoples shaped Canada’s history, and its present day culture in many ways. Some of the ways

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Aboriginal cultures interact with mathematics education will be described later in this report.

David Wheeler observed, on the occasion of the ICME-7 congress in Québec City, that "Canada's size, location, and federal structure pose special problems for any organization aiming at nationwide status" (1992, p. 5). However, there are two national institutions that have an influence on Canadian mathematics education, the Canadian Mathematics Education Study Group and the Social Sciences and Humanities Research Council of Canada.

The Canadian Mathematics Education Study Group

The Canadian Mathematics Education Study Group is a unique organization that brings together mathematicians and mathematics educators from across Canada once a year to discuss in depth topics in mathematics education. The group grew out of a mathematics education conference held at Queen's University on the general theme: of "Educating teachers of mathematics: the universities' responsibility." The participants felt that further opportunities to meet would be useful and a follow-up conference was organized in June 1978, at which the decision was taken to establish a continuing group, the Canadian Mathematics Education Study Group / Groupe Canadien d'Étude en Didactique des Mathématiques (CMESG/GCEDM). The format of the annual conference is unusual. From the first conference in 1977 working groups have had a central role, and the 1978 meeting set the pattern for all subsequent meetings with three full days, beginning with working group sessions, sandwiched between arrival and departure half-days including plenary lectures and panels. The working groups meet for nine hours in total and the themes chosen are designed to be as inclusive as possible, which is essential for a conference that is attended by mathematics educators ranging from primary school teachers to university mathematicians with no direct connection to schools. A full list of the working group themes over the years can be found on the group's website at http://cmesg.ca.

In 1992 David Wheeler wrote:

> Attendance at CMESG/GCEDM meetings has varied between 30 and 70, with most in the 50-60 range. This is a good size for the kind of meetings the Group organises: small enough to give a feeling of community while large enough to ensure a mix of interest and experience. Two-thirds of this number are usually regulars who attend most of the meetings. Membership is predominantly but not exclusively Canadian. The Group benefits a lot from the presence of a few non-Canadians, though it is watchful that the proportion does not grow too large. (p. 6)

Ironically, as soon as the 50-60 participant range had been described as "a good size" the group began to grow, with most meetings in the following decade being attended by about 70 people. Since 2005 the median attendance has been over 100. Participants continue to be "predominantly but not exclusively Canadian." The proportion of "regulars" has decreased as the conference has grown, but the absolute number of regulars has remained the same; a core of about 40 people attends almost every meeting. While the composition of the group of “regulars” has changed gradually over the years, its existence ensures a valuable continuity.

The Social Sciences and Humanities Research Council of Canada

The Social Sciences and Humanities Research Council of Canada funds educational research, among other things. Most countries have national research funding agencies, but SSHRC is interesting in that its grants are adjudicated by committees that are interdisciplinary and whose membership changes significantly from year to year. That means that the research that is funded by SSHRC is less likely to fit a single model than if there were a stable committee with a narrow focus. When researchers do not get funded they like to complain about this structure, but it does allow more unusual research to get funded in Canada than would otherwise.

National circumstances

It is difficult to present a national portrait of mathematics education in Canada, due to a number of circumstances, including Canada’s size, population density, climate, and linguistic diversity. After describing these circumstances, this report will present four portraits of mathematics education in regions of Canada.

Canada is the second largest country on earth, but much less populated than similarly sized areas like Europe, China and the United States. The low population density of Canada is both caused and mitigated by Canada’s climate, which is cold. Most Canadians live in the southern, warmer regions, which spreads most of the population out over what is effectively a long thin country 200 km wide and 5000 km long. And not all of that is populated; there are stretches of the trans-Canada highway where the only sign of human presence is the highway itself. In addition to the challenges this poses for national unity, Canada’s low population density also makes it difficult to provide educational opportunities everywhere.

A significant marker of Canada’s multiculturalism is the diversity of languages spoken in Canada. The main languages are English and French, but about 20% of the population speaks another language at home. Most provinces are either English speaking, French speaking, or both, but Aboriginal languages also have official status in the Northwest Territories and in Nunavut. Canadians love talking about language politics, but the circumstance most relevant for the development of mathematics education in Canada is that fact that the only officially French province, Québec, and the only effectively bilingual province, New Brunswick, lie between the mostly English speaking provinces on the Atlantic coast, and the rest of Canada.

Canada’s physical, climactic and linguistic circumstances divide Canada into four regions: The mostly English speaking provinces in the east, separated from the rest of English speaking Canada by French speaking Québec, then Canada’s most populous province, Ontario, which is separated from western Canada by the sparsely populated area north of lake Superior.

These regions have no official status, but they are reflected in mathematics education policy. Under Canada's federal system, each province sets its own standards for education, including teacher preparation and curriculum. Until the 1990s every province had its own mathematics curriculum. Then two regional groupings formed, in the west and in the east, in which provinces developed and implemented mathematics curricula collectively. At this point the curricular map of Canada reflected the regions mentioned above exactly. There has since been further curricular consolidation, as the Atlantic provinces decided to adopt the curriculum framework used in the West, but the remainder of this report will present regional portraits from the West, Ontario, Québec and the East.

THE WESTERN REGION (Ann Anderson & Jennifer Thom)

The Western Region of Canada spans the four provinces (Manitoba, Saskatchewan, Alberta and British Columbia) west of Ontario. The geography extends from the Pacific coast, across the Rocky Mountains into the prairies, with about 2382 km of highway joining Victoria, British Columbia to Winnipeg, Manitoba. Within the Western Region, there are 31 universities of varied sizes and foci, including nine large to mid-size, research-intensive universities.

Curriculum

Throughout the Western Region as in other jurisdictions, curriculum is governed by provincial Ministries of Education (MoE), although each province's mathematics curriculum is informed by the Western & Northern Canadian Protocol (WNCP), which provides a coherent vision for mathematics education in schools in the western provinces, as well as the Yukon Territory, the Northwest Territories and Nunavut. The WNCP Common Curriculum Framework (CCF) for mathematics (K-9) was first released in 1995 in English & French and revised 2006; the CCF for mathematics (Grades 10-12) was first released in 1996 and revised in 2008. These documents were developed by representatives of MoE, teachers, administrators, post-secondary educators and other stakeholders from the seven jurisdictions, and reflect the beliefs, student outcomes and assessment indicators for mathematics upon which they agreed. There is an evident influence of the NCTM, with an emphasis on conceptual learning, a problem-solving approach and seven interrelated mathematical processes. The provincial mathematics curriculum documents in all four provinces, for the most part, align with the WNCP; however, as each province produces and revises its own mathematics curriculum, there also exist differences.

Teacher Education

In Western Canada, mathematics teacher education programs are offered in 19 colleges & universities. Current research in mathematics education informs the development of these programs both generally and directly through instruction provided by active mathematics education researchers. Most B.Ed. (mathematics) programs are offered for one or two-years, following an initial four year B.Sc. mathematics degree, although options exist to do dual degrees or conjoint degrees over a 4-5 year period in many Western universities. Two colleges offer their programs in French and most universities in the Western Region include First Nations teacher education programs (e.g., NITEP @ UBC), which support Aboriginal students' academic success in culturally responsive ways. The prominence of mathematics education courses in each of the teacher education programs varies, with as little as one mathematics education course for elementary teachers (24 contact hours).

Research

In an attempt to represent the current state of research in the Western Region of Canada, it is important that we recognize our Elders whose scholarly contributions inspired many of our journeys. These include David Robitaille's continuing work in TIMSS and international assessments of mathematics; Tom Kieren's considered inquiries into learning and enactivism; Werner Liedtke's as well as Jack Hope's research into young children's mathematics learning; Walter Szetela's studies in problem solving; David Pimm's work in language and mathematics; and Susan Pirie and Tom Kieren's studies of mathematical understanding, to name just a few.

Currently, research throughout the Western provinces, conducted by more than 35 mathematics educators, is diverse, wide ranging and international in scope. Many of these researchers draw from ecology, cultural perspectives, complexity science, enactivism and theories of embodiment to investigate children's and teachers' individual as well as collective mathematical understanding. Much of this work is longitudinal, contextual, and collaborative, engaging teachers and students of mathematics in numerous settings. In addition to the studies carried out within our rural and urban Canadian communities, many of our colleagues are involved in scholarly and curriculum work with mathematics teachers internationally (e.g., Dadaab refugee camps; bilingual teachers in Oaxaca, Mexico; elementary teachers in Tanzania). Similarly, throughout all four provinces researchers are dedicated to understanding Aboriginal/Indigenous issues as they relate to mathematics education, collaborating with and learning from these communities (e.g., Haida Gwaii). Research on mathematics in the early years has recently come to the fore in BC and Alberta. Here, scholars explore areas such as parent-child mathematical engagement prior to school and young children's spatial understanding, while a collective of researchers involved in an early mathematics initiative conduct a concept study with teachers in the early years. Across the provinces there is considerable emphasis on 'mathematics for teaching' and elementary and secondary teachers' understanding of mathematics. Another important research area focuses on innovative practices (e.g., lesson plays, video cases, use of metaphors) and reforms toward improving mathematics teaching. Like our colleagues in other parts of the country, mathematics educators in Western Canada are researching and developing various technologies for mathematics teaching and learning (e.g., dynamic geometry, Lego Mindstorms, apps for mobile devices). Finally, studies involving aesthetics and creativity, gender relations, teachers' assessment practices, gesture & genre, teacher curiosity, collaborative action research, inquiry, changes in students' approaches to learning, and language and how it relates to mathematics, further characterize the diversity and depth of research in Western Canada.

ONTARIO (Chris Suurtamm & Ami Mamolo)

Ontario is home to roughly 40% of Canada's population, with nearly half of that figure living in the Greater Toronto Area. The multicultural character of Canada is most strikingly apparent in Toronto, where over 40% of the population was born outside of Canada. Ontario is also home to approximately 20% of Canada's First Nations and Métis peoples. The diversity in Ontario's demographics, and a population density of about 14 per square kilometre, offer both challenges and opportunities for educators, many of which are reflected in the scope and aims of the researchers across our provincial mathematics education communities.

Ontario universities

Ontario has a very collaborative community engaged in mathematics education research, policy, and practice. Several associations work together to support teachers of mathematics at all levels, and to support and work collectively with mathematics education researchers. There are 23 publicly funded universities across Ontario, 11 of which host mathematics education researchers, programmes, and centres. The oldest university, the University of Toronto, was established in 1827, and the newest, Algoma University, was established in 2008.

Ontario research in mathematics education

Each of the 11 universities involved in mathematics education hosts several researchers engaged in a variety of areas of research. Research is done at all levels of mathematics education: early childhood, elementary, secondary, and tertiary as well as in teacher preparation and professional development. At the elementary and early childhood education levels, research foci include topics such as spatial visual reasoning, early algebraic thinking, mental mathematics, and productive mathematical interventions on the part of educators and parents. At the secondary and tertiary level, there is interest in mathematical modelling, the development of abstract thinking, transitions between secondary and tertiary education, and a variety of modes of engaging students in mathematical thinking. The majority of research carried out cuts across the elementary, secondary, and tertiary divisions, with researchers specializing in areas of study that have importance for learners of all ages and stages. Studies in cognition and mathematics learning, equity and social justice, multi-lingual learning, semiotics and ambiguity, technology enhanced learning, multi-modal reasoning, teacher efficacy, assessment, and epistemology underpin both teacher education reform and developing classroom practice and resources. Issues in mathematics teacher classroom practice and professional learning are also prominent focal points for researchers in this province. Much of this research has provided crucial information for the implementation of Ontario's new teacher education programs, which are set to commence in Fall 2015.

Much of the research in mathematics education across the province is done collaboratively with researchers from various universities working together to develop and implement agendas and projects. This is facilitated by several mathematics education hubs such as the Nipissing University Mathematics Education, Research, and Information Centre (NUMERIC), the University of Ottawa's Mathematics Education Research Unit (MERU) and Pi Lab, Trent University's Math Education Research Collaborative (TMERC), the Numerical Cognition Laboratory at Western University, and the Dr. Jackman Institute of Child Study (ICS) at the University of Toronto. These research collectives host teams and outreach programs that tend to be cross-disciplinary, with connections to psychology, cognitive science, computing science, and literacy research and include research initiatives in mathematics teacher professional learning. In particular, the ICS offers prospective teachers a research-based program with international perspectives, as well as a laboratory school that includes children from age three to 12. Ontario's second largest university, York University, complements this focus on early childhood learning with a Master's programme in mathematics for teaching geared toward instructors of learners from age 13 to adulthood. A diversity of perspectives, opportunities, initiatives, and agendas, and integration and negotiation within this disparity, best characterizes Ontario and the research done there.

Collaborative work with other stakeholders

Collaborative work also includes partnerships between mathematics education researchers and others interested in mathematics education. Mathematics education leadership is strong in Ontario, through such organizations as the Ontario Association for Mathematics Education (OAME), the Ontario Math Coordinators Association (OMCA), and the Ontario College Mathematics Association (OCMA). These organizations host annual conferences, publish quarterly professional journals, and sponsor research initiatives that support collaborations across universities, public schools and school boards, and special interest groups. Further to this, the Ontario Ministry of Education calls upon mathematics education researchers to act in advisory capacities or to conduct research in particular areas of mathematics education curriculum development and implementation, as well as professional learning in mathematics teaching and learning. The Fields Institute for Mathematical Sciences has been instrumental in facilitating many of these collaborations through its Mathematics Education Forum. This Forum meets on a monthly basis throughout the academic year and brings together members from all of the above-mentioned groups.

QUÉBEC (Carolyn Kieran & Jerôme Proulx)

There are 18 universities in the largely French-speaking Canadian province of Québec. Three are anglophone: Concordia, McGill, and Bishop's; the rest are francophone. But only about half of the universities are active centres of research in mathematics education. The oldest university is Université Laval in Québec City, established in 1663. The newest is Concordia University in Montreal, which dates from 1974.

History of relations with PME, PME-NA, and ICME: A few glimpses

Québecers have been active in PME ever since its beginnings. In 1976 at the Karlsruhe ICME, it was a Québecer, Nicolas Herscovics, from Concordia University, who proposed forming a subgroup of ICMI that would permit mathematics education researchers to meet every year and which led to the formation of PME. (Of the 100 or so who showed up at the ad hoc meeting during the Karlsruhe congress to discuss the possible formation of a new group, six were Canadian and they were all from Québec: Claude Dubé, Nicolas Herscovics, Joel Hillel, Claude Janvier, Dieter Lunkenbein, and David Wheeler.) The first PME international conference hosted in Canada was in Montreal in 1987, co-organized by Jacques Bergeron, Nicolas Herscovics, and Carolyn Kieran.

As well, Québecers were equally active in PME-NA, right from its beginning in 1979. Nicolas Herscovics was the only Canadian member of the first PME-NA executive. In 1981, he was joined by Jacques Bergeron. Together, and with the collaboration of Claude Janvier and Dieter Lunkenbein, they organized the Fifth Annual Meeting of PME-NA, held in Montreal in 1983. To bring out the bilingual character of Canada, the texts of the two plenary papers and the two reaction papers appeared in both English and French within the proceedings; in addition, all English papers offered a French abstract and vice versa.

International collaboration among mathematics educators was encouraged by the reconstitution of ICMI in 1952, and the start of the ICME congresses in 1969 (Kilpatrick, 1992). ICME-1 in Lyon attracted 23 Canadians: 16 from Québec, 6 from Ontario, and 1 from Manitoba. At ICME-2 in Exeter, 52 Canadians were present, including 14 from Québec, for example, Claude Gaulin, Claude Janvier, and Richard Pallascio. That number has increased with each successive ICME, with Québecers and fellow Canadians organizing the 1992 ICME in Québec City – Bernard Hodgson and Claude Gaulin being the congress co-organizers and David Wheeler being the chair of the Program Committee. (By the way, Claude Gaulin is the only one of two people in the world to have attended all ICME congresses since 1969; Jerry Becker is the other.)

This internationalism is a great strength of Québec mathematics education research. The research community that has developed in Québec over the past 50 years has not developed in isolation. The influences of, and interactions with, mathematics education researchers in other countries, and in other regions of Canada, have all served to shape the Québec community. In addition, this same internationalism has underpinned the development of a strong local research community in Québec.

Facets of the Québec research community *of didactique des mathématiques*

Claude Janvier once remarked: "Québecers were not researchers in the late 1960s and early 1970s; they were university teachers looking for the best curriculum and the best approaches for teaching that curriculum" (Kieran, 2003, pp. 1727-8). However, the growth of universities and the concomitant spurt in research activity in the late 1960s and 70s soon changed that. In 1970, the national funding agency, the Social Sciences and Humanities Research Council, awarded its first grant in mathematics education research to Zoltan Dienes of the Université de Sherbrooke. As well, in 1970, the province of Québec set up a parallel funding agency. The formation of this second granting agency for Québec researchers was instrumental to the rapid growth of the Québec mathematics education research community during the 1970s and 80s, as many projects were funded and led to the development of various research programs for numerous researchers active at the international level.

In the same year, the Groupe de didactique des mathématiques du Québec (GDM) was set up (see https://sites.google.com/site/gdidmath/). This association of researchers and other persons interested in *didactique des mathématiques* has continued to meet annually to discuss questions related to current issues in mathematics education *research* (and not necessarily to address questions related to teaching practice, as other teaching associations filled this role). As might be expected, the early actors in this association were some of the same individuals who were present on the international scene.

Another force in the developing strength of the Québecois community was the establishment of the CIRADE research centre at the Université du Québec à Montréal in 1980. CIRADE had a significant impact on the emergence of the growing research community with its invited international scholars such as Bauersfeld, Wenger, von Glasersfeld, Brousseau, and Chevallard, not to mention the role played by the organization of international seminars on emergent topics in mathematics education research, ranging from representations (e.g., Janvier, 1987), constructivism (e.g., Bednarz & Garnier, 1989) and mathematical understanding, to social interactionism.

During the 1980s, the *didactique* section of the mathematics department of the Université du Québec à Montréal was perhaps the largest of any university group of mathematics education researchers in the world, with 18 full-time professors. Many second- and third-generation Québec researchers/university professors trace the roots of their own professional development to these UQAM pioneers. A central focus of the mathematics education group at UQAM has always been teacher training. With a strong mathematical orientation and inspired by cutting-edge research in the teaching and learning of mathematics, the UQAM group developed a unique approach to the training of future mathematics teachers at the secondary level: a four-year programme that requires students to take several courses in *didactique*, one for each of the main mathematical areas in the curriculum – a programme that could be said to be the envy of mathematics teacher training programs across Canada. Québec's *didacticiens* have also played a role in advising the government as to the content of school mathematics programs, a content that focuses on both basic skills and the development of problem-solving ability. The noteworthy performance of Québec students in international evaluation studies in mathematics is a testimonial to the particular *didactique* underpinnings of the curriculum, as well as to the nature of the training in *didactique des mathématiques* experienced by Québec teachers. For this, the Québec mathematics education research community can take a small measure of credit.

EASTERN CANADA (Lisa Lunney Borden & Mary Stordy)

What we are calling the Eastern region is usually referred to as the Atlantic region of Canada, and this reflects the importance of the Atlantic ocean to the region's geography, history and culture. The Eastern region is comprised of the four provinces of New Brunswick, Nova Scotia, Prince Edward Island and Newfoundland and Labrador and is the smallest region, in terms of area (539 064 km^2, a little over 5% of Canada's area) and population (about 2.3 million, 7% of Canada's population). However, the region's small land area does not mean that travel is easy within the region, due to the intervening bodies of water. About 50% of the population lives in the region's cities, with the rest being spread out in rural and sometimes remote communities.

There are eleven universities engaged in mathematics teacher education in the region. Most are small universities, even by Canadian standards. Only two, the University of New Brunswick and Memorial University of Newfoundland enrol more that 10 000 students in all academic programmes (11 000 at UNB, 18 000 at MUN). The remaining universities enrol about 5000 students. Two of these smaller universities are francophone.

The Atlantic region produces very few doctorates in mathematics education (one in recent years). This means that most faculty members come from or have studied elsewhere, primarily outside of Canada or the province of Alberta. This creates international ties as well as connections to colleagues in the West. The decision of all four of the Atlantic provinces to base their mathematics curricula on the WNCP Common Curriculum Framework may reinforce this connection to the West. The Atlantic region also has a long tradition of educating teachers who end up teaching in the more prosperous West, in Alberta in particular, and it is anecdotally argued that there are as many Atlantic Canadian teachers in some parts of Alberta as there are in Atlantic Canada.

Due to their size, most universities in the Eastern region have only one or two tenured faculty involved in mathematics education, which makes it difficult to form strong research communities at a single institution. In spite of this, there is a high level of involvement in research and curriculum development among mathematics educators in the East. This includes SSHRC funded research on positioning and authority in mathematics classrooms, mathematics pedagogy in regions of Canada, social perspectives on disparity in mathematics performance, and reasoning in dynamic geometry environments. Other research and development foci in the region include mathematics education and Aboriginal peoples, mathematics teacher beliefs, ontology of elementary mathematics teachers, problem solving, language number concept development, and intersections between mathematics and art.

The provinces in the Eastern region consistently perform below the Canadian average, but most are at or above the international average, on national and international assessments like PISA. There has been considerable debate, but very little research, related to this situation. The relatively high performance of Alberta prior to the most recent PISA may have been a factor in the adoption of the WNCP Common Curriculum Framework in the Atlantic provinces. Other than the curriculum, disempowerment of teachers and poor funding of the schools have also been suggested as factors contributing to low performance in the Eastern region. Yet, Atlantic Canadian provinces rank first, second, third, and fifth with respect to rates of high school graduation in the country. Only Ontario has a similar rate of over 80%. Also, the Mi'kmaw Kina'matnewey schools, a collective of community controlled schools, in Nova Scotia has achieved an 88.8% graduation rate, which is nearly double the national average for Aboriginal students.

CONCLUSION

Mathematics education research in Canada is multifaceted, outward looking, and vibrant. An international focus is a great strength of Canadian mathematics education research, but it also makes it difficult to distinguish a definitively Canadian approach to mathematics education. Nonetheless, the four regional approaches to mathematics education research within the Canadian context that have been described in this report demonstrate the existence of strong research communities, focussed on research questions of regional and international importance.

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