Collaborative Gesture as a Case of Extended Mathematical Cognition

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Abstract

Gestures have been shown to play a key role in mathematical reasoning and to be an indicator that mathematical understanding is *embodied* – inherently linked to action, perception, and the physical body. As learners collaborate and engage in mathematical discussions, they use discourse practices like explaining, refuting, and building on each other's reasoning. Here we examine how gestural embodied actions become distributed over multiple learners confronting mathematical tasks. We define *collaborative gestures* as gestural exchanges that take place as learners discuss and explore mathematical ideas, using their bodies in concert to accomplish a shared goal. We identify several ways in which learners' gestures can be used collaboratively and explore patterns in how collaborative gestures arise while proving geometric conjectures. Learners use collaborative gestures to extend mathematical ideas over multiple bodies as they explore, refine, and extend each other's mathematical reasoning. With this work, we seek to add to notions of important *talk moves* in mathematical discussions to also include a consideration of important *gesture moves*.

Key words: gesture; geometry; proof; distributed cognition; extended cognition; classroom discussion

Collaborative Gesture as a Case of Extended Mathematical Cognition

1. Introduction

Gestures – movements of the hands that often accompany speech – have been found to be a powerful component of reasoning in a variety of domains (Alibali, Spencer, Knox, & Kita, 2011; Beilock & Goldin-Meadow, 2010; Glenberg, Gutierrez, Levin, Japuntich, & Kaschak, 2004), including mathematics. The gestures learners formulate can reveal information not expressed in speech (Church & Goldin-Meadow, 1986), can show an association with conceptual performance (Goldin-Meadow, 2005; Cook & Goldin-Meadow, 2006), can be manipulated to give students new actionable ideas (Goldin-Meadow, Cook, & Mitchell, 2009; Nathan et al., 2014; Novack, Congdon, Hemani-Lopez, & Goldin-Meadow, 2014), and when prevented, can impair reasoning (Hostetter, Alibali, & Kita, 2007). In addition, teachers use gestures to communicate ideas to students in a multimodal manner (Alibali & Nathan, 2012; Valenzeno, Alibali, & Klatzky, 2003), which may be particularly important in mathematics classrooms where gestures can make spatial and relational aspects of mathematical concepts come alive (Alibali & Nathan, 2012). Enyedy (2005) observes that "gestures are often found to be the glue that binds together other resources into a coherent whole" (p. 432).

Theories of embodied cognition (e.g., Wilson, 2002) posit that learners process and understand ideas through their bodies and their senses, and that the body has a central role in shaping cognition. Embodied theories often include the claim that cognition is *extended* (Clark & Chalmers, 1998) - the idea that a student's cognitive system extends beyond their own minds and bodies, into their environment and those around them. Wilson (2002) describes how "The forces that drive cognitive activity do not reside solely inside the head of the individual, but instead are distributed across the individual and the situation as they interact. Therefore, to understand cognition we must study the situation and the situated cognizer together as a single, unified system" (p. 631). From this viewpoint, external body-based representations like gestures constitute part of the cognitive system and help learners exchange ideas around shared goals. Together, these theories allow for a consideration of how gestures can be central to collaborative activities that take place between students in mathematics classrooms.

While considerable research has been conducted on how teachers and learners use gestures while engaging in mathematical reasoning (see Alibali & Nathan, 2012 for a review), less work has systematically examined gesturing as a collaborative activity that is part of mathematical discussion and argumentation between different learners. Indeed, the Common Core Standards (2010) for Mathematical Practice call for students to "construct viable arguments and critique the reasoning of others," but little research has detailed how the effective use of the body – particularly gesture - might facilitate students meeting this standard. Examining mathematical discourse practices is important because these moves can both provide access to mathematical ideas and contribute to the formation of a community that promotes mathematics learning. Collaborative discourse can surface gaps in students' understanding and improve students' ability to reason logically and reflect on their own thinking processes (Chapin, O'Connor, O'Connor, & Anderson, 2009), and gesture may be a key element of effective discourse.

Here we define *collaborative gestures* as gestures that are physically and gesturally taken up by multiple learners. In this way, collaborative gestures hold a meaning that is explicitly dependent upon the gestures that have been physically enacted by other interactional partners. These gestures represent an important case of extended cognition in that they demonstrate how body actions can influence and extend over multiple learners' mathematical reasoning in a way that is fundamentally different from traditional modes of exchanging mathematical information (i.e., speech and written work). The modality of gestures thus can serve to distribute collective problem solving.

An example of a collaborative gesture would be one learner showing a triangle growing and shrinking with their thumbs and forefingers to show mathematical similarity, and then another learner taking up and using this same gesture to explain mathematical similarity in their own words. This gesture exists as part of a joint, collaborative space where students' cognition has been extended over multiple physical bodies. Gestures can also be used collaboratively to exchange ideas or build on each other's' ideas, rather than simply echoing them – one learner might use a gesture of a rotating triangle to show why he or she disagrees with a mathematical argument made via gesture involving why three side lengths determine a unique triangle. An alternate definition of collaborative gesture might include the initial gestures made by the first learner as also being collaborative in nature – particularly if the gestures were purposefully designed to be meaningful to their recipients (Koschmann & LeBarron, 2002). However, here we use a narrower definition of collaborative gesture where a requirement is that *multiple* learners have *physically engaged* with the gestural activity.

We propose that learners can use their bodies, particularly their hands, in collaborative ways to reinforce, extend, and redirect the mathematical ideas of others, which were also expressed through physical movement. We explore learners' use of collaborative gestures to show how cognitive processes become extended over the embodied experiences of multiple learners. Collaborative gestures provide a window into understanding how the body can be leveraged to help students reason about mathematical ideas as they work jointly on challenging tasks. In the present paper, we examine collaborative gestural activity that groups engage in while proving geometry conjectures. In addition to examining mathematical talk moves – like revoicing or refuting the contributions of their peers - we also examine gestural moves that these learners make to exchange and discuss mathematical ideas. We identify several categories of collaborative gesture and explore how learners' use of collaborative gesture varies by experience, type of gesture, and their position in the social structure of their group. Our goal is to explore how gesture and the body can be brought into research and practice relating to collaborative mathematical discussions among learners, which is a vitally important area of study (Michaels, O'Connor, & Resnick, 2007). By identifying the centrality of gesture and the body to the exchange of mathematical ideas during collaborative discussions, we seek to shed light upon ways in which effective modes of student-student interaction can be facilitated.

2. Theoretical Framework

2.1 Gesture as Simulation Action

The theory of Gesture as Simulated Action (GSA; Hostetter & Alibali, 2008) provides an empirical, embodied cognition account of how the multimodal production of gestures comes about through individuals' cognitive processes. Gestures in the GSA framework arise during speaking when pre-motor activation, formed in response to internal mental simulations of motor or perceptual imagery, is activated beyond a speaker's current gesture threshold. The threshold is the level of motor activation needed for a simulation to be expressed in overt gesture; this threshold can vary depending on factors such as the current task demands (e.g., strength of motor activation when processing spatial imagery), individual differences (e.g., level of spatial skills), and situational considerations (e.g., social contexts). Gestures are often co-speech, but can also occur in the absence of speech (i.e., co-thought gestures; Chu & Kita, 2011). An important point in GSA is that gestures for the same idea can take many different forms, depending on the currently-salient elements of a learner's mental simulations. Gestures can vary in factors like speed, form, content, segmentation, and viewpoint, and the differences in mental simulations and in the communicative situation that give rise to these differences is under-specified in current theories of gesture (Hostetter & Alibali, submitted). One interesting way to study this might be to examine how people gesture in *collaborative* contexts – how people gesture differently about the same idea as they work together and share information. In a recent review of research on gesture (Hostetter & Alibali, submitted), significant empirical evidence was found to back each claim of GSA. However, these authors note that GSA does not delineate the ways in which gesture can be beneficial for speakers or for listeners, and this is a limitation of the framework.

Thus Nathan (2017) proposed an extension of GSA to account for the influences of motor activity the learner is induced to perform (e.g., directed actions) on cognition. In this reciprocating model, learners' actions and movements serve as inputs capable of driving the cognition-action system toward associated cognitive states through a bi-directional process. In other words, in addition to cognitive states giving rise to physical actions, directing learners to engage in physical motions may give them new ideas and insights relevant to understanding and solving tasks. Gerofsky (2010) implemented a teaching experiment where middle school students participated in multimodal experiences where they gestured the shape of graphs from a character (i.e., first-person) perspective. She found that this experience prompted students' engagement with graphs from a first-person "being the graph" perspective, and that students performed well on a post-test administered one year later, often activating their previous gestures.

Similarly, Nathan et al. (2014) found that students directed to perform arm movements that were designed to be relevant to the solution of two mathematical tasks – one involving a triangle and one involving a system of gears – were more likely to realize the key insight behind those tasks than students performing random arm motions. This result occurred in spite of the fact that participants did not report being consciously aware that their arm movements related to the mathematical problems. While Nathan's (2017) extension to GSA accentuates how directing the learner to motion in particular ways can trigger new cognitive and embodied states, here we emphasize how observing and embodying the actions and gestures of others, through collaborative activity and joint sense-making, can trigger new cognitive and gestural states in learners. We take this to be an important case of distributed cognition, which we discuss next.

2.2 Distributed Cognition and Extended Cognition

Professional practice involves the coordination of many different inscriptions and representational technologies by differently-positioned actors whose actions occur across a range of social and physical spaces (Goodwin, 1995; Hutchins, 1995). Through joint, coordinated activity, cognition becomes distributed over a patchwork of discontinuous spaces and representational media. Lave (1988) describes how "'Cognition' observed in everyday practice is distributed—stretched over, not divided among—mind, body, activity and culturally organized settings (which include other actors)" (p. 1). In this conceptualization of *distributed cognition*, the environment is a key element that should be considered to understand cognitive processing (Wilson, 2002). For example, when a child is learning to read, the responsibility for this transformation of knowledge states does not exist merely in the students' head – it is distributed over a complex system of teachers, other students, cultural artifacts designed to diagnose and advance the child's understanding, and rules and roles for interpersonal interactions (Cole & Engestrom, 1993). Distributed cognition observes that joint activity can be best understood with the system as the unit of analysis and accentuates how representations that exist inside and outside of the minds of individuals are used and transformed, as the system works to achieve a shared goal or plan. Also important to distributed cognition is understanding how people within the system align themselves and coordinate their action around internal and external representations and artifacts (Nardi, 1996), like gesture.

Theories of *extended cognition* argue that the social and physical environment of learners is actually constituent of their cognitive system (Clark & Chalmers, 1998). The implication is that cognition, rather than existing in the head of an individual, is instead distributed over the bodies of multiple learners and the environment around them as they interact. External resources like media, gestures, social networks, and cultural institutions provide capacity for cognition that acts in concert with individual's neural systems. In this way, cognitive processes are partially constituted by such external resources which directly contribute to cognitive processing, rather than cognitive processes simply being interactively coupled with these external resources (Sutton, Harris, Keil, & Barnier, 2010).

If we take gestures as an important representational form that exist externally to a learner, then theories of distributed cognition would argue that gestures made by others are inherently linked to each individual's cognitive states and act as key representational media in a distributed system of joint activity. Theories of extended cognition would argue that such gestures, when shared and exchanged, represent a form of group-level cognitive processing of mathematical ideas that exists outside of the head of any individual. This is an important extension to GSA theory which accentuates the emergence of gestures as an individual cognitive process. In this alternate view, the entire system of activity could have its own co-developed "gesture threshold," along with imagery and simulations that are jointly-constituted and gestural norms and practices that belong to the system, rather than existing in one individual's mind.

Gesture during Collaboration

Much of the prior research on gesture has not focused on the use of gestural exchanges as a tool for collaboration and shared cognition. Important gestural moves mathematics teachers use to establish common ground with students have been identified, including using pointing gestures to show relatedness between mathematical entities and using repeated gestures representing mathematical entities to link ideas over time (Nathan & Alibali, 2011). Theories of gesture use in mathematics have also described how gestures arise from a blended space in which the affordances of the hands, arms, and physical environment becomes combined with the mental image one wishes to convey (e.g., Edwards, 2009). The *grounded blends* that are created when these two spaces come together are gestured representations of mathematical meaning. However, such accounts do not describe how *multiple* agents use gestures interactively to compare, explain, and clarify each others' reasoning to establish common ground.

An important theoretical idea for considering gestures during collaborative exchanges is a *semiotic bundle* (Arzarello, Paola, Robutti, & Sabena, 2009). A semiotic bundle consists of the set of multimodal resources – including spoken and written words, gestures, tools, diagrams, and manipulatives – that groups of learners are using to solve problems in any given moment. Crucially, the semiotic bundle is conceptualized as being shared, with its contents accessible to all collaborators. The gestures, drawings, verbal statements, etc. made by collaborators are dynamically added to the semiotic bundle as problem-solving unfolds, and different collaborators can access and use various parts of the semiotic bundle, by, for example, re-using a gesture that another learner has previously made. Arzarello, Robutti, and Thomas (2015) further describe how learners must coordinate free-form mental images with sequential and hierarchically-structured language to communicate mathematical ideas, which creates an instability that needs to be resolved. This resolution often happens through co-speech gestures, which are theorized to emerge from a common internal process -- a “growth point” for the speaker (McNeill, 2000) -- which mediates the production of both speech and gestures. As conceptualized by McNeill (2000; McNeill & Duncan, 2000), growth points reflect the internal mental processes of an individual speaker that are made external through their expression in multimodal language production, such as speech and co-speech gestures. Once publicly expressed, these may contribute to the semiotic resources that make up a collaborative discourse. Furthermore, the growth point can be reinstated over time and across contexts through a series of *gestural catchments* (McNeill & Duncan, 2000), which explains the recurrence of gestural forms that take on a greater signifying role as the catchments become increasingly less depictive of their original iconic form and more idealized, even as they are intended to reinvoke the earlier referent among group members. This is one way some scholars (e.g., Azarello et al., 2015) have observed catchments refer to increasingly abstract concepts.

Yoon, Dreyfus, and Thomas (2011; 2014) also use the idea of grounded blends, but describe how during collaboration, learners will coordinate their gesture spaces, and may be able to interpret gestures in similar ways in their grounded blends are similar. In their use, derived in part from Goodwin (2000) and Haviland (2000), *gesture spaces* encompass both the physical space in which learners gesture, and the socially constructed mathematical meaning this space becomes endowed with through grounded blends. However, this differs from a more restricted account of gesture space as described by McNeill (1992), where it refers to the space occupied physically by the one producing the gesture without regard for the audience. Yoon et al. (2014) also frame gestural catchments as a kind of gesture mimicry, where repeated gestures indicate a cohesiveness of mathematical ideas among collaborators, and where one learner can appropriate the gestures and grounded blends of another.

Several studies outside of mathematics have demonstrated how learners and the teacher access gestures in the shared semiotic bundle and establish coordinated gesture spaces. Enyedy (2005) in a study of elementary students discusses how learners and the teacher use repeated hand gestures as a collaborative resource to form a shared problem space and bind resources together when learning about contour mapping. Koshmann and LaBarron (2002), in an analysis of medical students' communication, highlight how repeating another speaker's gestures can create *gestural cohesion* – a link between different elements of a conversation - across an interaction. Another study on students learning origami identified *collaborative gestures* as gestures through which a learner interacts with the gestures of a communicative partner (Furuyama, 2000). These gestures involved a student pointing to or manipulating a teacher's gestures about origami folds.

Here we reimagine the idea of collaborative gestures to be relevant to learner-learner interactions around mathematical sense-making and take such gestures to be a case of extended cognition. We thus next turn to a discussion of how gestures and actions have been studied in the context of mathematical reasoning.

2.3 Gesture, Embodiment, and Mathematical Proving

One type of gesture that has been identified as important to mathematical reasoning is *dynamic gestures* (Göksun et al., 2013; Uttal et al., 2012; Walkington et al., 2014). These are gestures where learners use their bodies to physically formulate and then transform or manipulate mathematical objects. For example, a learner might make a triangle with their thumbs and forefingers, and then make that triangle grow, shrink, rotate, flip, etc. These kinds of gestures might be particularly important to examine when learners are engaging in mathematical proving or "the process employed by an individual to remove or create doubts about the truth of an observation" (Harel & Sowder, 1998, p. 241). When learners successfully justify mathematical arguments, their reasoning can be *transformational* in nature, where they perform valid mathematical operations on objects in order to build a logical deductive chain of reasoning that transcends particular cases and applies to all mathematical objects under consideration. This kind of transformational proof scheme (Harel & Sowder, 1998) may be closely coupled with both mentally simulated and physically enacted gesture and action (Nathan et al., 2014). Much of this work has been conducted in laboratory settings, where gestures are used by individuals working on mathematical tasks in the presence of a researcher or interviewer. Like the research on gesture, the research on dynamic gestures and proving activities often accentuates cognitive operations carried out by and existing within an individual mind – rather than as the product of a collaborative distributed system of activity that includes multiple bodies in motion.

Research has shown that dynamic gestures arise during the mathematical reasoning of experts (Marghetis, Edwards, & Núñez, 2014) and that dynamic gestures have a strong association with students formulating valid proofs to geometrical conjectures (Walkington et al., 2014; Nathan & Walkington, 2017). The professional practice of proving itself has been described as "a richly embodied practice that involves inscribing and manipulating notations, interacting with those notations through speech and gesture, and using the body to enact the meanings of mathematical ideas" (Marghetis et al., 2014, p. 243). The multimodal nature of proving is particularly evident in classroom settings, as students' proofs often take on oral and gestural forms, as opposed to formal, written ones (Healy & Hoyles, 2000). For example, Nathan et al. (2017) describe a high school geometry classroom exploring conjectures about circles. The teacher repeatedly traces arcs with her fingers, and this gesture is later taken up by students and becomes a key element of their understanding. Similarly, Arzarello and Paola (2007) described how teachers use "semiotic games," where they will often repeat the gesture of students, but couple that gesture with mathematical explanations that are in more formal language. Flood (2018) describes how an instructor took a complex series of semiotic resources (including gesture) a student used to describe a mathematically proportional relationship, and then decomposed, reused, and transformed those semiotic resources, particularly the gestures the student used. Flood refers to this phenomenon as "multimodal revoicing," and argues that the instructor used the student's original gestures and multimodal resources to create a culturally refined and scientific version of her explanation.

Collaborative gestures occur between learners as well. Yoon et al. (2014) used the idea of gesture mimicry to show how one learner used another learner's gestures to come to a new understanding of the concept of antiderivative in calculus. Hall, Ma, and Nemirovsky (2015) describe activities where student groups used global-positioning system devices draw geometric constructions while walking in open spaces. Participants jointly created new forms of embodied mathematical activity that involved the development of tools and interactive routines. Rasmussen, Stephan, and Allen (2004) show how college students learning differential equations can use repeated gesture/argumentation dyads collaboratively to support the evolution of classroom mathematical practices and ideas. Similarly, Flood, Harrer, and Abrahamson (2016) show how repeated gestures regarding proportional reasoning allow for intersubjectivity between speakers and the joint establishment of mathematical objects. Yoon et al. (2011) also identify a kind of gesture that is conceptually similar to the origami collaborative gestures, but in the domain of calculus reasoning. They describe how one learner gestured the shape of a graph, and then a second learner both pointed to and used gesture to extend this graph that existed only through past gestural activity.

These studies provide useful illustrations how the body might be used during joint, collaborative activity around a common goal, but a systematic study of the different ways in which gestural exchanges can serve this function has been more elusive. What is more common is systematic examination of the oral activity or *talk moves* that students engage in as they participate in mathematical discussions – we turn to this research next.

2.4 Mathematical Discourse and Discussion

Learning mathematics can be conceptualized as becoming enculturated into particular patterns of *discourse*, defined as the full set of communication activities that are practiced in a given community (Sfard, 2000). Academic mathematical discourse involves particular modes of argumentation, including precision, brevity, and logical coherence. Learners engaging in mathematical discourse seek to make claims, to generalize, to imagine mathematical objects and transformations, and to search for certainty (Moschkovich, 2007). This conceptualization fits well with more general frameworks for academic "accountable talk" (Michaels et al., 2007), where students are held accountable to their learning community to build on each others' ideas, ask each other questions, make concessions, and provide reasons for their claims. This kind of accountable talk involves students trying to understand the premises or assumptions of others and backing claims or explanations with evidence from publicly accessible and trustworthy sources of accepted knowledge, like texts.

Saxe et al. (2009; Saxe, 2002) provide a framework for understanding how ideas emerge, transform, and travel in mathematics classrooms through discourse. They define *microgenesis* as the moment-to-moment ways in which students construct mathematical representations and ideas, and *sociogenesis* as the manner in which these representational forms are reproduced and altered through social interaction among students. Sociogenic propagation involves the uptake of ideas as they travel through a community like a classroom, to potentially generate collective practices. Sociogenesis is accomplished through coordinated speech, action, and use of artifacts (i.e., coordinated use of the semiotic bundle), and can be facilitated by teachers.

One way to enculturate students into mathematical discourse practices and to facilitate sociogenesis is through mathematical discussions. Mathematical discussion "provides opportunities for students to reason, defend, and prove their conceptions to one another" (Hufferd-Ackles, Fuson, & Sherin, 2004, p. 113). Such a *math-talk learning community* involves students asking questions of one another, taking a position on mathematical ideas and supporting and defending their answers and methods, articulating information to other students in response to probes, and providing clear and full explanations of their thinking. It also involves students comparing and contrasting different ideas, being able to explain other students' thinking in their own words and seeking to clarify or correct other students' ideas to better their own understanding and assist others. Along these same lines, a popular framework by Chapin et al. (2009) identifies productive talk moves for teachers, aimed at allowing students to more deeply come to understand mathematical ideas and create a community of learners in the mathematics classroom. Teachers can *revoice* student contributions, repeating the contribution to verify whether it is understood correctly. Teachers can also use a *repeating* talk move where they ask students to restate another student's reasoning in their own words. Finally, teachers can use *reasoning* talk moves where they ask students if they agree or disagree with someone else's reasoning, and why.

In sum, prior work has focused on the talk moves that students use to make claims, back claims with evidence, and explore and refute each others' reasoning, but often underemphasizes a consideration of the role of the body and of gesture in these communicative practices. Our goal in the present research is to foreground the role of gesture as students engage in collaborative mathematical communication, and to identify *gesture moves* that students make while explaining or responding to others' reasoning that complement the "talk moves" prior research has identified. Gesturing has some clear advantages over oral communication – for example, it can more easily show dynamic and spatial transformations in real time and it can be utilized by students who struggle with the English language generally and with academic-style discourse specifically. Attending to gestures during student-student discussion exchanges can open up new possibilities for how teachers consider supporting students in engaging in meaningful discussions and interactions with their peers.

2.5 Research Questions

Prior work has identified how gestures arise, why gestures are important, the close connection between embodied action and mathematical proving, and the centrality of gestures that show dynamic transformations. In addition, prior research has examined how productive discourse can facilitate students' learning of mathematical ideas during classroom discussions. Here we extend this work by looking at how gestures as embodied actions are used as part of a distributed cognitive system through collaborative activity across multiple learners as they engage in mathematical discussions. We address the following research questions:

1) In what ways are gestures used collaboratively during mathematical discussions?
2) How often are collaborative gestures used during mathematical discussions, and how can this vary by teaching experience and type of collaborative gesture? How is this associated with successful mathematical problem-solving?
3) How do learners engage differently with collaborative gestures across individuals and groups?

3. Method

3.1 Participants

Participants included 20 pre-service and 34 in-service teachers, enrolled in one of five different courses at a private university. They were either enrolled in an undergraduate elementary mathematics methods course for pre-service teachers, a graduate elementary mathematics methods course for pre-service teachers, a graduate course for in-service middle school mathematics teachers in their first year, a graduate course for high school mathematics teachers in their first year, or a graduate mathematics teacher course for in-service teachers who are generalists interested in mathematics or who are secondary mathematics teachers. We refer to teachers in the latter group as *experienced* teachers, to differentiate them from the pre-service teaches or the in-service teachers who are in their first year. However, their amount of teaching experience varied. A total of 64 students across the five classes were recruited to participate, however 3 were absent on the day of the study and 7 did not consent to participate, for a final sample of 54. Forty-six participants were female, while 8 were male; 38 identified as Caucasian, 6 as African-American, 5 as Asian, 3 as Hispanic or Latino/a, and 2 as Other race/ethnicity.

The teachers' prior knowledge of mathematics and mathematics teaching, as well as their self-image relating to mathematics and mathematics teaching, likely varied considerably both between and within classes. These variables were not directly measured, except through the use of a short pre-test of basic geometry knowledge of terms and definitions. The pre-service teachers averaged 76% correct on this quiz, while the novice first-year teachers averaged 86% and the experienced teachers averaged 90%. In prior work, we have found that a general population of undergraduates at the study site scored 81% on average, while high school students from the surrounding school district score at around 70%.

3.2 Procedure

Participants were placed in groups of 3 to 6 and were asked to prove or disprove eight geometric conjectures (see Table 1). Participant groups were standing while the conjecture text was projected on a laptop screen in front of them. Six conjectures were mathematically true, while two were false. Participants were directed to take turns for the person responsible for narrating the group's "final proof" (after discussion), which was audio-recorded by the computer, such that each person was responsible for giving a proof at least once. Two groups only experienced 6 of the 8 conjectures, and one group only experienced seven of the eight conjectures. One group had technical issues in which the computer crashed near the beginning of their time (before they had a chance to completely prove any conjectures). They joined another group mid-way through to assist this group in proving their final 4 conjectures, creating a large group of 12 students.

Table 1.

Conjectures participant groups were asked to prove or disprove

Conjecture
1. The sum of the lengths of any two sides of a triangle must be greater than the length of the remaining side.
2. Given that you know the measure of all 3 angles of a triangle, there is only one unique triangle that can be formed with these 3 angle measurements.
3. The area of a parallelogram is the same as the area of a rectangle with the same base and height.
4. Diagonals of a rectangle are always congruent.
5. If one angle of a triangle is larger than a second angle, than the side opposite the first angle is longer than the side opposite the second angle.
6. The measure of the central angle of a circle is twice the measure of any inscribed angle intersecting the same two endpoints on the circumference of the circle.
7. Reflecting a point over the x-axis is the same as rotating the point 90 degrees about the origin.
8. If you double the length and width of a rectangle, the area is exactly doubled.

Participants were instructed to not use pencil or paper to assist them in proving or disproving the conjectures and were told to work together. Not having access to paper and pencil was a purposeful design constraint that was intended to promote body-based reasoning. Here our focus is on the gestures that learners formulated themselves to help them understand, reason through, and collaborate around proving each conjecture. The conjectures were presented to students in the context of an early version of a videogame for learning geometry; however, the game context seemed to have little influence for most groups. This early version of the game did not provide learners with any explicit supports for proving of the conjectures.

3.3 Analysis

Video was captured of each group using one camera focused on the group member's faces and upper bodies, which could be moved if group members' positions shifted. The video was transcribed in the Transana Analysis software (Woods & Fassnacht, 2012). Videos were clipped such that one clip represented one student group proving one conjecture. Each clip was then coded for each gesture sequence that arose in the clip. A *gesture sequence* was defined as all of the hand gestures made by a single person from the time their hands rose until the time their hands returned to rest. There were two exceptions to this definition of a gesture sequence. First, if gesturers froze their hands while raised for a prolonged period (perhaps while someone else spoke and gestured), and then began gesturing again, this was considered a new sequence, even though the hands did not actually fall. Second, when gesturers engaged in a special category of gestures we call *joint gestures* (defined later), we counted any pause or reorganization in the joint gesture as a new gesture sequence. This is because this particular type of gesture also often involved participants keeping their hands raised for prolonged periods of time through complex group interactions.

Gesture sequences were coded as being *individual* if the gesturer was making a gesture that was not triggered by or related to the gestures of others, and *collaborative* if they were. Note that in gesture sequences coded as individual, gesturers could still be building on or responding to the *speech* of other learners – gesture sequences were only coded as collaborative if learners were gesturing in response to the *gestures* of other learners. Collaborative gesture sequences were coded for both the learner that was performing them (i.e., actually making the gesture) and for the learner that the gesturer was responding to or intending to collaborate with through their gestures. Collaborative gestures were then separated into different categories using a grounded, bottom-up approach of constant comparisons (Glaser & Strauss, 1976).

The coding process proceeded as follows. First, the entire corpus was coded for collaborative gestures by 3 different coders, engaging in weekly discussions about issues that arose with their coding and collaboratively viewing clips of gestures. This process was used to iteratively develop a codebook with a full set of categories for collaborative gesture. During this first round of coding, 44% of the corpus was double-coded by at least two coders, such that discussions could take place about discrepancies and such that comparisons between coding practices could be made. The double-coding ended when it seemed like a consensus on categories and definitions had been reached. The clips for double coding were chosen either arbitrarily or because one coder flagged them as clips they were unsure about or had questions about. After the first round of coding the corpus, a final codebook with collaborative gesture categories and accompanying definitions for each category was settled upon. The codebook was then used by one of the three coders to code the entire corpus a second time, in weekly consultation with one of the other coders. In all, a total of 83 clips of 12 groups from across the five classes were coded for gesture sequences.

Each clip was also coded for whether the group came to a correct proof for the conjecture. A codebook that captured the elements of a valid proof for each conjecture, used in previous studies (Nathan & Walkington, 2017; Walkington, Woods, Nathan, Chelule, & Wang, under review) was used to code the oral proofs given in the 83 clips. The operationalization of correct proofs in this codebook was based on the three elements of valid proof given by Harel and Sowder (1998): (1) the proof must be *generalizable* and hold for all cases under consideration, (2) the proof must utilize *logical inference* progressing through a inferentially sound chain of reasoning, and (3) the proof must utilize *operational thought* where conclusions are drawn from valid premises. These three elements were operationalized for each conjecture, with most conjectures having multiple valid proofs that met these criteria. For example, for the conjecture that the sum of the lengths of any two sides of a triangle must be greater than the remaining side, a valid oral proof would involve observing that the shortest distance between two points is a straight line, and thus given a base length, any other set of two segments that formed a triangle must be longer than the base or the triangle would not close. This proof is generalizable in that it applies to all types of triangles, it involves operational thought where operations are performed on the three line segments and valid conclusions are drawn about those operations, and it involves logical inference where the implications of the line being the shortest distance between two points are deduced step-by-step. See the Appendix for a copy of the codebook. Groups formulated correct proofs approximately 51.8% of the time, indicating that the tasks were of an appropriate difficulty.

4. Results and Discussion

4.1 RQ1: In what ways are gestures used collaboratively during mathematical discussions?

Analyses revealed four main categories of collaborative gestures, detailed in Table 2, as well as 6 sub-categories. We first briefly describe each major category, and then give examples from the corpus that illustrate the gesture types in more detail.

In ***echoing gestures***, one gesturer would repeat or take up a gesture that another gesturer had previously made that they had observed, often while explaining their reasoning in their own words. In ***mirroring gestures***, one gesturer would initiate gesturing about a problem task, and then another would watch the gesturer and attempt to make matching or similar gestures at the same time as that gesturer in order to physically follow that student's reasoning using their own hands. The difference between echoing and mirroring gestures is that mirroring gestures occur approximately at the same time with two gesturers matching or anticipating each other's motions often while only one person is talking, while echoing gestures occur one after another with a significant time lapse in between, with both learners explaining their reasoning.

In ***alternating gestures***, one gesturer had previously shown their understanding using an arm or hand gesture, and then another learner who had been observing that gesture and listening to their speech proposes an alternate gesture to advance the group's mathematical line of thinking. The difference between alternating gestures and echoing and mirroring gestures is that when making alternating gestures, the gestures made by different learners are not structurally similar and are often not conceptually similar either. Finally, in ***joint gestures***, multiple learners use their hands to formulate one single mathematical object or system together, that is built and represented only through their bodies working in conjunction. Such gestures often involved students physically placing their hands together to form mathematical objects that would be difficult to represent with only one set of hands. We next present and discuss photo transcripts with examples of each of these gesture categories in turn.

Table 2.

Categories of collaborative gestures

Main Category	Sub-Category	Description
Echo	Simple Echo	One learner makes a gesture, and then a second learner makes the same gesture (or a very similar gesture) afterwards. Must be evidence that the person doing the echoing was looking at the original gesturer when the original gesturer originally made the gesture.
	Echo and Build	One learner makes a gesture, and then a second learner makes part or all of the same gesture (or a very similar gesture) afterwards. However, the second learner also changes or adds to the gesture in some way, in addition to echoing the original gesture.
Mirror	Simple Mirror	One learner makes a gesture, and then a second learner makes the same gesture (or a very similar gesture) nearly at the same time. Only the second gesturer is coded as collaborative. Must be evidence that the person doing the mirroring was looking at the original gesturer.
	Anticipation	One learner is gesturing, and then a second person anticipates a gesture they are about to do (correctly or incorrectly), and makes that gesture slightly ahead of them or at the exact same time.
Alternate	Alternate and Build	One learner gestures their understanding, and then another learner follows up, building upon or extending their reasoning using a different gesture.
	Alternate and Redirect	One learner gestures their understanding, and then another learner follows up, showing how their reasoning is different using a different gesture, potentially refuting the original learner's reasoning
Joint		Multiple learners manipulate, point to, or formulate a single mathematical object or a unified set of interacting mathematical objects. Each gesturer who is included in the joint gesture is coded as doing a collaborative gesture.

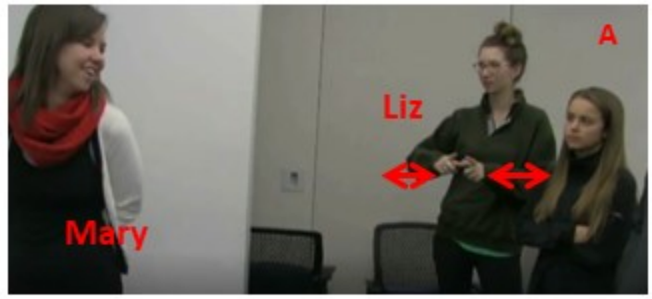
4.1.1 Simple Echoing Gestures

Our first case is taken from a group of pre-service elementary teachers proving the conjecture, “If you know the measure of all three angles of a triangle, there is only one unique triangle that can be formed with these three angle measurements.” Mary (bottom image, Figure 1) was in charge of formulating the group’s final proof, with the other students, including Liz (top image, left, Figure 1), assisting her.

1 Mary: Okay, given that you know the measure of all three angles of a triangle, there is only one unique triangle that can be formed with these three angle measurements.

2 Liz: Because you can always…you have the…you can always scale it once you have those three lengths, like the three angles you scale it.

((A. Liz places her index fingers and thumbs together and moves them apart and together.))



3 Mary: Okay, so I say false because you have certain angles, but you can bring them in and out to make the side lengths bigger or smaller.

((B. Mary places her flat palms together at the heel of her hands, and then moves her hands apart and together.))

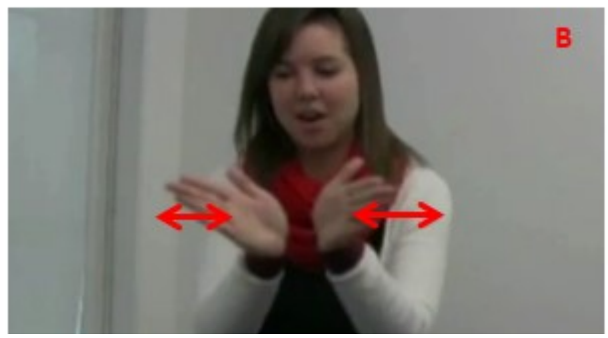


Figure 1. Transcript of simple echoing gestures

Once Mary reads the conjecture (Line 1), Liz explains why the conjecture must be false, and uses a dynamic gesture where she formulates a triangle with her thumb and index fingers, making it grow and shrink (Line 2). Mary seems to immediately understand and take up this gesture, repeating the gesture herself, and putting Liz's explanation into her own words (Line 3). Mary performed an echoing gesture, where one person made a dynamic gesture, and then a second person repeated that gesture while making the accompanying oral reasoning her own. Other literature has identified gestural catchments as repeated similar or identical hand gestures used by a single gesturer (usually an instructor) to convey similarity of or highlight important conceptual connections (McNeill & Duncan, 2000). However, other literature has identified how gestures can be repeated across multiple learners, and how these repeated gestures can build gestural cohesion across turns of talk (Koshmann & LeBarron, 2002), creating and maintain shared meaning (Enyedy, 2005; Rasmussen et al., 2004).

Echoing gestures were not always this kind of *simple echo* – sometimes one learner would echo another's gesture, and then add on their own gesture onto that gesture – performing a category of collaborative gestures we call *echo and build*. A transcript in a later section gives an example of such an instance.

4.1.2 Simple Mirroring Gestures

Our second example, shown in Figure 2, is taken from a group of pre-service elementary teachers proving the conjecture, "Reflecting a point over the *x*-axis is the same as rotating the point 90 degrees about the origin." The transcript begins with Mary (top image, left) setting up the mathematical system for the group to consider using gesture, by formulating the *x*-axis with one arm (Line 1), and then the *y*-axis with the other arm (Line 2). Laura (top image, right) gesturally echoes her formation of the *x*-axis (Line 3), and then in effect "takes over" making the mathematical argument orally. Laura narrates and shows a reflection over the *x*-axis ("You reflect it") using her other hand to show the reflecting motion. As Laura speaks and performs the reflection motion, Mary mirrors her gesture, making the same reflection gesture on her own *x*-axis she had made with her arm previously.

1 Mary: You have the x-axis right here.

((A. Mary holds out horizontal flat left arm, and then with her right hand motions back and forth across the arm's horizontal length.))

2 Mary (at the same time): This is the y…

((B. Mary makes right arm vertically cross her horizontal; left arm))

3 Lisa (at the same time): This is the x-axis…

((B. Lisa makes horizontal right arm motion))

4 Lisa: …you reflect it

((C. Lisa uses left hand to show movement from in front of her horizontal right arm to behind it))

((C. Mary copies Lisa's same motion without speech))

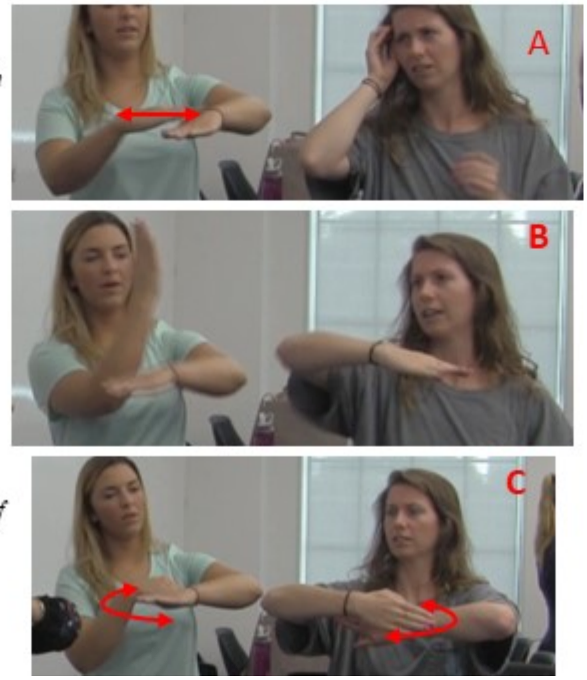


Figure 2. Transcript of simple mirroring gesture

Mary and Laura performed mirroring gestures as they were gesturing at nearly the same time in response to the same line of reasoning and jointly formulating a mathematical argument. In addition, at times their gestures were structurally identical. Other literature has identified *return gestures* as being identical or similar gestures used by recipients to acknowledge and show attentiveness to the speaker (Koshmann & LeBarron, 2002). However, in our data, mirroring gestures also seemed to be used by the recipient to directly aid their own understanding, as they used the gesture to "follow along" with what the original gesturer was doing and understand the relationships in their own terms, even if that person was not attending to them. This is related to Kim, Roth, and Thom's (2010) observation that second graders in their study often silently gestured to themselves with no apparent communicative purpose to physically model geometric concepts that the teacher or a peer was talking about. These researchers also described instances where learners would mirror each other in this way, but the learner who was not speaking would use slightly different gestures than the original gesturer, demonstrating their own unique understanding of what was unfolding. They termed these *co-emerging gestures*. Both return gestures and co-emerging gestures would fall under our definition of mirroring gestures. Subdividing mirroring gestures into these two categories in our coding system was not productive for us, as mirroring gestures could serve both purposes simultaneously, and making this distinction reliably would require understanding what a student is thinking from moment-to-moment even though they are often not speaking.

Echoing and mirroring gestures can both reveal how learners can reflect on and adopt each other's mathematical thinking and reasoning, jointly accessing the semiotic bundle containing gestures relating to the group's past and present communication.

4.1.3 Alternating and Echo & Build Gestures

In the example in Figure 3, a group of pre-service elementary teachers are discussing their reasoning for the conjecture "Diagonals of a rectangle are always congruent." Tanya and Karen initially represent their understanding, with Tanya gesturing horizontal sides with her hands and Karen gesturing crossed diagonals. In their speech, Tanya focuses on the two sides being equal (Line 1), while Karen focuses on the diagonal's opposing relationship to each other (Line 2). Lisa performs an *alternate and redirect* gesture to redirect the conversation to another idea – that the sides are parallel (Line 3). She then integrates Tanya's equality gesture into her explanation by echoing it and saying "So they're the same" (Line 5). Karen then echoes both Tanya's equality gesture and Lisa's parallel gesture and builds by adding on her crossed arms gesture at the end of the sequence (Lines 6 and 8), thus performing an *echo and build* gesture sequence. Lisa assists her in formulating this explanation by repeating her own parallel line gesture, then echoing Karen's crossed arm gesture (Line 7). Lisa then sweeps her diagonal hands in and out to accentuate the idea of congruency, *echoing and building* upon Karen's original gesture (Line 7).

1 Tanya (at the same time): I think so because two sides have to be. The two sides are equal.
((A. Flat hands horizontally in air, parallel to each other))
2 Karen (at the same time): Yeah if they were the opposite.
((A-B. Forearms crossed in front of body))
3 Lisa: The two sides are parallel.
((B. Flat hands moved up and down vertically in the air, parallel to each other))
4 Tanya: Parallel.
5 Lisa: So they're the same.
((C. Flat hands moved horizontally in the air, parallel to each other))
6 Karen: Wait, the two sides are parallel are the height and the width.
((D-E. Flat hands vertically, horizontally front of body))
7 Lisa: So diagonally they'd have to be the same.
((F-G. Flat hands placed vertically in front of body, then hands are crossed and moved up and down.))
8 Karen: The height and width are parallel so the diagonals have to be the same.
((H-J. Forearms crossed, then flat hands placed horizontally with thumb out and upper hand folded, then forearms crossed))

Figure 3. Alternating and Echo & Build Gestures

This transcript demonstrates how different aspects of understanding the same problem become embodied through different gestures, and how learners take up and build upon each other's' gestures to extend their understanding. Karen is able to adopt the gestures of Lisa and

Tanya in order to create a mathematical argument that brings in all three of their ideas – parallel sides, congruent sides, and diagonals.

Although this gesture sequence showed an alternating gesture that was meant to redirect collaborators' reasoning, alternating gestures can also build upon collaborators' reasoning without directly mirroring or echoing a collaborator. We show an example of this next.

4.1.4 Alternating and Anticipation Gestures

An example of alternating and anticipation gestures is shown in Figure 4. In this transcript, John, Cynthia, Carole, and Bree from a course for first-year in-service middle school mathematics teachers are discussing the conjecture "The area of a parallelogram is the same as the area of a rectangle with the same base and height." Cynthia begins by describing and physically modelling with her whole body by leaning how a parallelogram is just a slanted rectangle (Line 1). John agrees (Line 2), but then Carole challenges Cynthia's reasoning (Line 3) using an alternate and redirect gesture. In this gesture, she shows and then distorts a quadrilateral formed with her thumb and forefingers. Bree then alternates and builds on Cynthia's gesture (she was not observing Carole's gesture), agreeing with her and making a new gesture relating to the area formula that outlines the shape of a parallelogram (Line 4). John then alternates and builds on Bree's gesture in Line 5, proposing the idea of cutting off a triangle. The first half of this transcript demonstrates how learners exchange ideas through alternation gestures, often using alternate and build gestures when they agree, and alternate and redirect gestures when they disagree. Sometimes, like in Carole's case, their argument for why they agree or disagree is expressed entirely through gesture, with no accompanying speech to explain their mathematical argument. Koshmann and LeBaron (2002) describe a similar situation where one speaker contrasts a collaborator's gesture with a new gesture in order to reveal a misunderstanding.

1 Cynthia: Yeah because a parallelogram is just slanted.

((A. Cynthia draws right arm horizontally across her body while leaning right, then repeats leaning motion))

2 John: Yeah.

3 Carole: Is it the same?

((B. Carole holds up two thumbs and index fingers at 90 degree angles and then twists them slightly))

4 Bree: Yeah cause a parallelogram is length times width.

((C. Bree draws finger across horizontally then vertically, then horizontally. She did this same gesture silently while Carole was talking above, and was not observing Carole.))

5 John: Yeah because basically if you move… like if you cut off a triangle

((D. John makes a vertical cutting motion in the air.))

6 Cynthia: Oh put it on the other side.

((E. Cynthia makes upside-down "U" motion, right hand))

7 John: And like put it on the other side, it would be the same.

((F. John makes upside-down "U" motion with right hand))

Figure 4. Alternating and Anticipation gestures

As the transcript continues, John has described his idea of cutting off and moving a triangle from one side of a parallelogram, making a cutting motion in the air (Line 5). Cynthia then anticipates his next gesture where the triangle is moved to the other side of the parallelogram, making a motion indicating the path of the triangle's movement (Line 6). Cynthia correctly anticipated John's gesture and line of reasoning, and John makes the same moving gesture as Cynthia a moment later (Line 7). A sub-category of mirroring gestures, *anticipation gestures* are instances where one learner is following the gestures of another closely in time. However, in anticipation gestures, the second gesturer may "get ahead" of the first, anticipating and making the gesture they believe they are about to make. Sometimes this anticipation is correct and the original gesturer does the same motion, while sometimes they mis-anticipate and must reorganize their reasoning when they see the gesture that actually follows by the original gesturer. Both alternating and anticipation gestures represent new additions to the group's semiotic bundle, rather than direct re-use of gestures that have already occurred.

4.1.5 Joint Gestures

Figure 6 shows an example of perhaps the most interesting category of collaborative gestures in our dataset – joint gestures. A group from the experienced mathematics teacher class are discussing the conjecture, "The measure of the central angle of a circle is twice the measure of any inscribed angle intersecting the same two endpoints on the circumference of the circle." Diana has been explaining her understanding using a gesture of two angles, one formed with her two index fingers (the inscribed angle) and one formed with her thumbs (the central angle – see Figure 5). Kyla has been somewhat removed from the conversation, and brings herself in by posing the question to Diana "Where is the circle?" (Line 1). Since it would be physically impossible for Diana to represent both angles and a circle with her hands alone, Diana responds "You have to visualize it" (Line 2). In the meantime, Kyla walks over to Diana and traces a circle around the outside of her angles, and then points to the vertex of the inscribed angle, which she believes is the outside of the circle. A third group member, Sophie, responds to the need to represent the circle in a more permanent manner by offering her hands to represent the circle, placing them under Diana's hands (Line 3). Kyla and the fourth group member, Carl, then proceed to discuss and point to various parts of the embodied diagram that Sophie and Diana have formed (Lines 5-10).

This transcript demonstrates how gestural activity can become distributed over multiple learners as they represent a single, shared mathematical system that is jointly embodied and that they can collaboratively refer to and discuss. This bears some relation to the instance described in Yoon et al. (2011) where one learner points to and extends a graph that was gestured by another learner – we would consider this to be a joint gesture as well. Here, the cognitive work of exploring the mathematical system is distributed over all four of their bodies, and is coupled with the gestures and speech they engage in. Yoon et al. (2011; Edwards, 2009) discuss how gestures are conceptual blends between learners' mental images and the affordances of their bodies and the physical environment. In this transcript, the emergence of joint gestures shows how learners use the affordances of *other bodies* to gesture as well, and to coordinate and build new shared images of mathematical ideas.

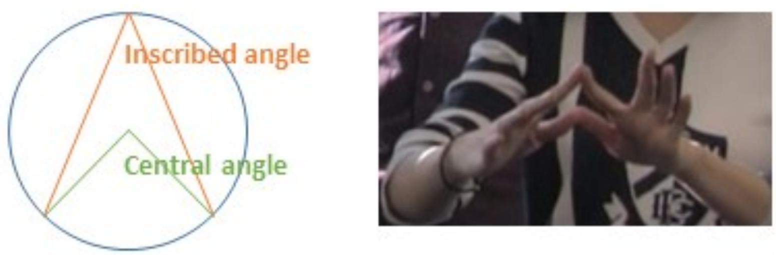


Figure 5. (Left) Inscribed and central angles of a circle (Right) Diana's initial gesture showing these angles

1 Kyla: So where's your circle?
2 Diana: You have to visualize it.
((A. Diana makes gesture shown in Fig 5, Kyla comes over and traces a circle around it using right hand))
3 Kyla: So this is the point on your circle. Right there?
((B. Kyla points to the tip of Diana's joined index fingers; Sophie comes over and cups hands into circle underneath Diana's gesture))
4 Sophie: Yeah this-
5 Kyla: This is the outside of the circle, this is the middle of the circle. So this is where the diameter goes, right between these points.
((C. Kyla points to the top and bottom of Diana's gesture, and then traces finger along diameter of Sophie's circle))
6 Carl: Right, right.
7 Diana: Right.
8 Kyla: So what angle are we talking about?
9 Carl: This one and these two-
((D. Carl points to 3 different locations on gesture.))
10 Kyla: Are the same-
((E. Kyla points to each location after Carl does))
11 Sophie: Are the same, right?

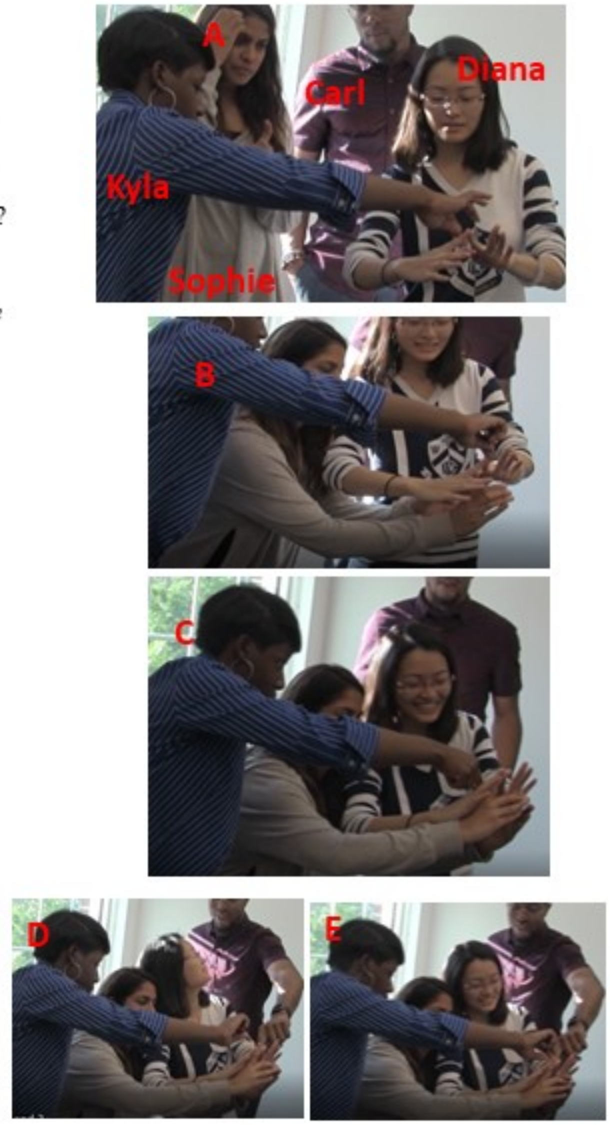


Figure 6. Joint Gestures

4.1.6 Summary

Overall, we found a variety of ways in which gestures could be used collaboratively in the sense that gestures became distributed over multiple learners. Mirroring and echoing gestures showed one learner taking up the mathematical reasoning of another, using a similar physical model for the mathematical concept. Echo and build gestures and alternate gestures, on the other hand, showed learners extending and refuting each other's reasoning using embodied action, adding on to the mathematical models that the group was confronting. Joint gestures allowed multiple learners to use their bodies in concert to build and transform an embodied mathematical system together, creating a complex representation of dynamic mathematical ideas. These collaborative gestures did not simply accompany students' mathematical arguments or their collaborative activities – they were an intrinsic part of learners' reasoning processes.

4.2 RQ2: How often are collaborative gestures used, how can this vary by experience and type of collaborative gesture, and how is this associated with successful problem-solving?

A total of 443 gesture sequences were coded across the corpus, of which 218 (49.2%) were collaborative. This came to approximately 2.6 collaborative gestures per student group proving attempt (*SD* = 4.1). However, the number of collaborative gestures used by different groups varied widely – from 0.67 collaborative gestures per proving attempt for one group, to 5.9 collaborative gestures per proving attempt for another group. The use of collaborative gestures seemed to vary by teaching experience – teachers from the two pre-service classes used an average of 1.5 collaborative gestures per conjecture (*SD* = 2.4), while teachers from the two first-year in-service classes used an average of 3.5 (*SD* = 5.4), and teachers from the experienced teacher class used an average of 2.8 (*SD* = 3.3). Thus, although the pre-service elementary teachers had the lowest collaborative gesture rates, the highest actually came from the less-experienced in-service mathematics teachers. When comparing this to how likely each group of teachers was to give a correct proof for the conjecture, the pre-service teachers had a 32.1% success rate at giving a valid proof to the conjectures, while the novice first-year teachers had a 54.8% success rate and the experienced teachers had a 70.8% success rate. It might be that because of the greater knowledge of the experienced teachers, collaborative gestures were less necessary in the establishment of shared valid proofs among their group, compared to novice in-service teachers. Also note the high standard deviation in collaborative gestures rates for the first-year novice teachers – there was one student group in this category who used an especially high number of collaborative gestures for two conjectures (34 and 29 collaborative gestures) - the highest frequency in the corpus.

As can be seen in Table 3, of the 218 collaborative gestures, echoing gestures were most common, followed by alternating, mirroring, and joint. Of the 110 echoing gestures, 81 (73.6%) were simple echoes while 29 (26.4%) were echo and build gestures. Of the 56 alternating gestures, 37 (66.1%) were alternate and build gestures while 19 (33.9%) were alternate and redirect. Of the 28 mirroring gestures, 24 (85.7%) were simple mirrors and 4 (14.3%) were anticipation gestures. Thus, the most common kind of collaborative gesture was to repeat someone else's gesture after they had performed it, often without changing the nature of the gesture, but sometimes while adding on gestural elements. Responding to a collaborator's gesture with your own different gesture was the next most common category, while following someone's gesture yourself in real time and engaging in the joint construction of a single gesture with one or more partners was rarer. It could be argued that mirroring and joint gestures are the most difficult collaborative gestures to perform, as they require close, real-time coordination between multiple people.

In terms of the types of collaborative gestures the three categories of teachers tended to make, for all three, echoing gestures made up approximately 50% of the group's collaborative gestures (see Table 3). Experienced teachers were most likely to use mirroring gestures, while novice first year teachers were most likely to use alternating and joint gestures. Overall there did not seem to be compelling differences between categories in the types of collaborative gestures made – although the sample size is small.

Table 3.

Percentage of total collaborative gestures that are of each type, broken down by teacher experience

	All Participants	**Pre-service**	**Novice In-Service**	**Experienced**
Echo	50.5%	54.7%	47.3%	50.7%
Alternate	25.7%	23.8%	28.6%	20.9%
Mirror	12.8%	11.9%	9.8%	17.9%
Joint	12.4%	9.5%	14.3%	10.4%

As a final exploratory analysis, we examined how making collaborative gestures during problem-solving was associated with getting the correct proof. Results are shown in Figure 7 below. As can be seen from the figure, groups had the lowest success rate (37.5% correct) when there were no collaborative gestures made during the proving attempt. Making one or two collaborative gestures was associated with much higher success rate (70% and 75%, respectively), but making 3 or more collaborative gestures was associated with success rates falling to 40-50%. Interpretation of this data is difficult as it is correlational and does not take into account the importance of students' prior knowledge. In the videos, making no collaborative gestures often was related to a shallow proving attempt that appealed to authority or that represented a well-known, convincing mathematical misconception that was immediately accepted by the group with little discussion. On the other hand, making lots of collaborative gestures was sometimes associated with the group floundering and being out of their depth at being able to reason about the mathematical conjecture in a sensible way – sometimes because of a fundamental misunderstanding of the conjecture or a lack of necessary prior knowledge.

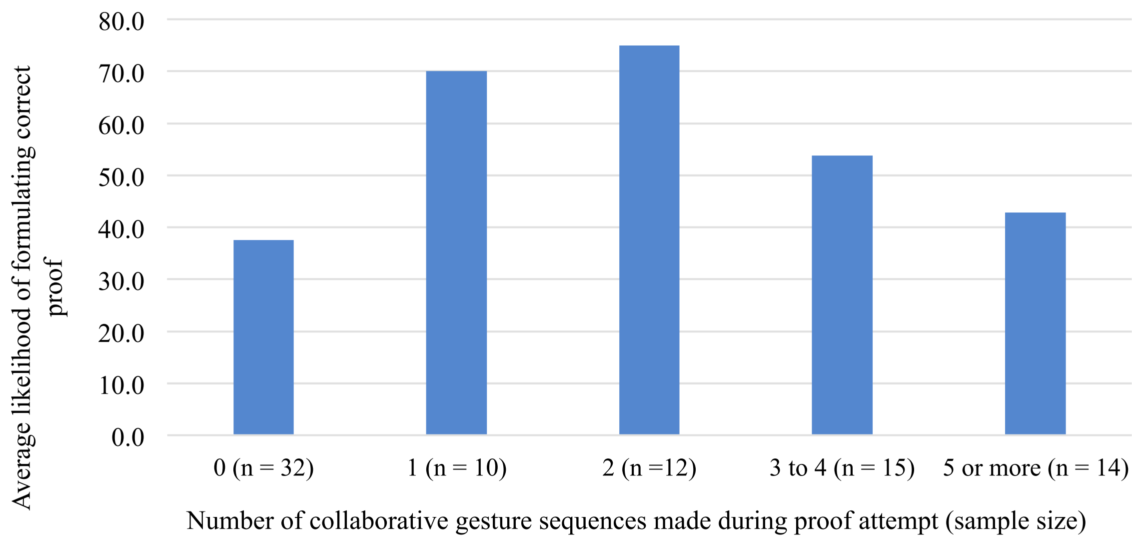


Figure 7. Bar graph showing relationship between number of collaborative gesture sequences that groups made during each proving attempt and the likelihood of attaining the correct proof

4.3 RQ3: How were learners engaging differently with collaborative gestures?

Figure 8 shows how often individual participants initiated versus received collaborative gestures. Each gray box represents one group, and each oval-shape inside of a gray box represents one participant. The left side of the oval shows how many collaborative gestures per conjecture the participant performed (e.g., when they echoed or mirrored another learner). The right side of the oval shows how many collaborative gestures per conjecture were directed at the participant (e.g., someone echoing or mirroring them). Red coloring indicates 0 collaborative gestures given/received by the learner, yellow coloring indicates an average less than 1 but more than 0 collaborative gestures given/received per conjecture by the learner, and green coloring indicates an average of 1 or more collaborative gestures given/received per conjecture. The top row of the figure is ovals representing the pre-service teacher classes, the middle row is the in-service first year teacher classes, and the bottom row is the experienced teacher class.

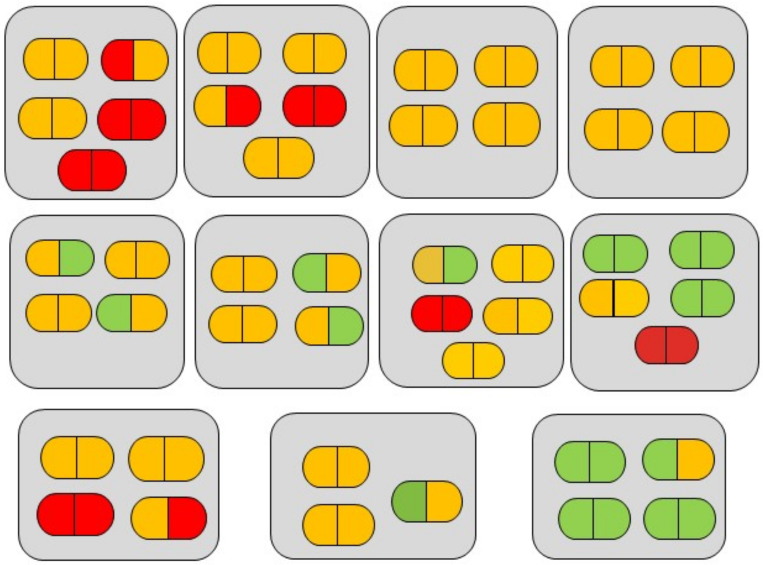


Figure 8. Distribution of collaborative gesture giving/receiving across 11 participant groups. Participants in the group that joined another group mid-way through the study due to technical problems are omitted.

As can be seen from Figure 8, across levels of teaching experience, there were participants whose ovals had red coloring who stayed on the fringes of collaborative gesture activity and did not engage in this kind of joint reasoning. There were also some groups of highly collaborative participants with yellow and green coloring as well, particularly among the in-service teachers. Most participants seemed to give versus receive a similar number of collaborative gestures, although there were some exceptions. Participants who received a lot of collaborative gestures but gave few often began the gestural sequence in the clip with an individual gesture showing their understanding, and then group members would continually build on that. In other cases, group members would give quite a few collaborative gestures that would get little attention from their group in terms of being considered or built upon. However, both of these cases were somewhat rare, as can be seen from Figure 8. When examining how many group members typically participated in collaborative gestures for each conjecture, we found that overall, 38.6% of the time no learners were involved (giving or receiving) in collaborative gestures, 22.9% of the time 2 learners were involved in collaborative gesturing, 24.1% of the time 3 learners were involved in collaborative gesture, and 13.3% of the time 4 or more learners were involved in the collaborative gestures. Thus, collaborative gestures were not always simply exchanged between two people during a conjecture - 3 or 4 group members could get involved.

5. Implications and Significance

While the importance of gestures to student learning has been established in a variety of studies, less work has been done detailing how gestures allow for cognition to be physically distributed over multiple learners when exchanged during collaborative discourse. Documenting this process and how it comes about with different groups of learners in different learning environments is a first step in understanding how learners can engage in joint, embodied reasoning to solve and learn from complex tasks. It also contributes to an understanding of how learners can use accountable talk and academic mathematical discourse in conjunction with gestures to exchange ideas and improve their mathematical communication and reasoning skills. We close this paper by considering the significance and implications of this research for the three areas identified in our theoretical framing: gestures, distributed and extended cognition, and mathematical discussion and proving activities.

5.1 Implications for Gesture Theory and Research

Prior research and theoretical work on gestures (e.g., Hostetter & Alibali, 2008) has focused on how gestures arise in response to both individual characteristics and characteristics of the task and the environment. Here we detail how gestures can arise in response to the gestures of others, with the gestural activities of others directly influencing the ways in which learners create and use their own gestures. This is shown in the echoing and mirroring gestures in Figures 1 and 2, where learners adopt a gesture they observed another student making. In addition, when extending or refuting another students' argument, students may be more likely to respond with a gesture if the original argument was made coupled with a gesture. Students' gesture threshold may be influenced by the degree of gestural activity they see going on around them. This may be in part due to there being more gestural resources present for use in the group's semiotic bundle. In the joint gesture transcript in Figure 6, Carl initially hung back from the joint gesture, but eventually, perhaps given its proximity, centrality to the group's reasoning processes, and the length of time it had been in place, he decided to join in the gestural activity and become part of the collaborative gesture sequence.

Prior research on how gestures arise in mathematics classroom (Alibali & Nathan, 2012; Arzarello & Paoloa, 2007; Nathan & Alibali, 2011; Nathan et al., 2017) has also provided significant insight into how teachers use gestures to guide instructional activities and clarify mathematical ideas, as well as how students use gestures to accompany their speech during mathematical explanations. Here we delineate ways in which collaborative gestural exchanges – rather than individual gestures coupled with speech – arise during group activity. These collaborative gestures can be key mediators of the flow of ideas between students (i.e., sociogenesis) and in the development of the group's cognitive processing. Indeed, when collaborating, there are times like in case of Carole in Figure 4 where a student might present an argument *only* through gesture – rather than through oral means. In addition, in a case like that of Tanya, Karen, and Lisa in Figure 3, a mathematical explanation might be effectually "cobbled together" from the gestures that different learners contributed as they built on each others' transformational and spatial reasoning through physical action. Enyedy (2005) calls attention to the importance of gestures that are repeated among different learners as an indication of the stability of an idea in the classroom system – however, here we reveal a broader phenomenon where different gestures can be combined together, or can build upon each other, as a way of establishing joint meaning.

This research also builds on prior theory on the access of semiotic bundles – i.e., the multimodal resources students bring to bear on a task, including gestures - during group activity (Arzarello et al., 2008). As Arzarello's work uses a Vygotskian paradigm, it necessarily foregrounds the role of the teacher in accessing and using the semiotic bundle to promote student learning, while placing less emphasis on how students use gestures to build ideas off of one another in the absence of a teacher. The rise of a collaborative gesture may occur as one learner attempts to reconcile the speech and gesture of another learner with their own mental image, and then package this proposed reconciliation into co-speech gesture. Such a moment might be indicative in a conceptual turn in the development of the group's mathematical reasoning, as evidenced by Figure 7. We also build on the work of Yoon and colleagues (2011; 2014) who illustrated some important collaborative gestures, by systematically studying their emergence over a variety of groups and problem tasks.

Finally, prior research (Nathan et al., 2014; Havas, Glenberg, Gutowski, Lucarelli, & Davidson, 2010; Novack et al, 2014; Thomas & Lleras, 2007) has begun to explore the transductive nature of body movements – the idea that one's physical state or actions such as gestures influence cognition and enable new insights for actors. However, observing another's gesture and then physically enacting that same gesture through a collaborative gesture move like an echo or a mirror suggests a particularly important kind of transduction that promotes intersubjectivity and collaboration. In the corpus, there were a number of examples of mirroring gestures where one gesturer silently followed another gesturer as the other learner spoke and gestured, as seen at the end of the transcript in Figure 2. This suggests that there is something about physically enacting the gesture yourself – rather than simply observing another's hand movement – that might be particularly powerful in influencing one's own mathematical reasoning. This follows a finding from Petrick and Martin (2012) that suggests that physically embodying dynamic geometric relations – rather than simply observing them – can improve learning outcomes. Other kinds of collaborative gestures like echoing, alternating, and joint gestures – that are facilitated by a system where gestural exchange becomes common – might be powerful transducers as well, giving students new ideas about mathematics. During joint gestures particularly, the learner physically becomes "part" of the mathematical system as it is transformed and discussed – and then must learn to coordinate both their perspective as an individual piece of the system and their overall understanding of the system as a whole.

5.2 Implications for Distributed and Extended Cognition

Distributed cognition theories posit that actors, resources, and the environment together constitute a complex cognitive system with its own goals and norms, where minds, bodies and activity all function jointly. Extended cognition takes this proposition further by positing that cognitive processing actually takes place at the system level, which cannot be simply understood as the sum of the cognitive processes of its parts. Collaborative gestures represent a fascinating case of how cognition becomes distributed or extended over multiple bodies who are working jointly in a social system. The embodied activity of making echoing or mirroring gestures can only be understood respective to the activity of other bodies of other actors in the system – if considered on an individual basis, it loses much of its meaning. Similarly, alternating gestures take place within the joint discursive, communicative space formed by multiple actors and their bodies, and seek to advance the group's thinking towards a mathematical goal – in this case, to prove or disprove a geometry conjecture.

Joint gestures are a particularly striking case of the extended nature of cognition. In the transcript in Figure 6, the learners making the joint gestures are not simply performing actions directed by a leader or facilitator who has the "big picture" of the proof in their head. Rather, they are spontaneously making a physical, embodied representation together that constitutes both the progression of their individual thinking and the progression of the group's reasoning as a whole, asking for clarification and enacting elaboration as the gesture develops. This gesture exists as an embodied artifact of the system itself that arises through joint coordination, rather than as the creation of any individual learner. Joint gestures could be seen as evidence of cognitive processing taking place at the system level, where jointly-constituted mental simulations give rise to system-level motor activity.

5.3 Implications for Mathematical Discussion and Proving Activities

Prior research has delineated important ways in which students can use talk moves to interact and exchange mathematical ideas during group and classroom discussions. An interesting result from this study is that conceptualizing these discursive patterns as *talk moves* might be underemphasizing the role that gestures and the body can take during this exchange. As Kim et al. (2010) observe, "it is not common practice for mathematics teachers to pay attention to children's bodily engagement as a way of knowing" (p. 234). Indeed, although theoretical frameworks like Saxe et al. (2009; Saxe, 2002) include gesture in their account of how ideas propagate through sociogenesis, there has been little research that has focused specifically on the different ways in which gesture can support or constrain the spread and uptake of mathematical ideas in a distributed system. In the present study, we have identified some important *gesture moves* that might be an important complement to, and sometimes function even in the absence of, important mathematical talk moves. As such, we conceptualize turn-taking in mathematical discussions as involving overarching *multimodal moves*, that include key elements like speech and gesture, but also other relevant resources from the semiotic bundle.

Mirroring gestures allow students to carefully follow and interpret the reasoning of another student that is being expressed through gesture. Echoing gestures may serve as a body-based manner in which students can then actively “revoice” each other’s’ contributions, although “re-enact” may be a more appropriate term when considering a multimodal perspective. Echo and build gestures may both allow students to both re-enact another’s’ reasoning, and to extend or clarify their mathematical premise or argument and to explain the other students’ thinking in their own way. They can also be used to gesturally represent or explain why one student agrees with another student’s reasoning. Alternate and build gestures do not directly re-enact, but instead focus on building on each other’s reasoning, comparing different ideas, and explaining another students’ thinking in your own way using an alternative representation. Alternate and redirect gestures allow students to ask each other critical questions about their thinking, contrast different ideas, and show disagreement with another’s reasoning expressed via gesture. Finally, joint gestures allow students to directly build upon each other’s’ reasoning, and to create a joint system that allows them to clarify and see underlying reasons for the ways in which others are interpreting mathematical ideas.

Yoon et al. (2011) give some further ideas as to why gestures may be particularly powerful for mathematics learning. Gestures encourage experimentation, and as they leave no permanent record of students' mathematical thinking, may be less threatening for learners. They also have a low demand for exactness and accuracy, as these kinds of gestures often are intended to get bigger ideas and relationships across in a general way, and therefore may lighten the cognitive load of exploring new ideas. However, alternately, this low demand for accuracy may promote misconceptions. Another potential pitfall is that when gesturing collaboratively, collaborators must orient their gesture spaces such that they correctly overlap from each person's perspective (e.g., if two people are facing each other, "left" is a different direction for each).

Finally, using these collaborative "gesture moves" in a *dynamic* manner may be especially powerful. Recall that dynamic gestures were defined as the motion-based transformation of a mathematical object through multiple states. These kinds of gestures leverage the affordances of the physical body for showing complex spatial actions and have some clear advantages for this purpose over speech that attempts to convey such transformations through only descriptions of actions. In addition, using these gesture moves when engaging in mathematical justification and proving activities might be particularly critical to formulating valid, generalizable, logical mathematical arguments. Collaborative and dynamic gestures allow learners to build a chain of reasoning together that is mutually elaborated upon and subject to critical discussion and reflection from the learners who are present, while also being embodied and action-based. This may facilitate the formation of strong mathematical arguments, and the sociogenesis of strong mathematical arguments through the classroom community.

5.4 Future Considerations

This study took place with pre-service and in-service mathematics teachers, so an important question is whether this kind of gestural activity would generalize to more typical K-12 student populations. In current work, we are having groups of ninth and tenth grade high school students from a diverse urban school prove similar conjectures in groups and have found similar collaborative gesture categories emerge (see Walkington, Wang, & Nathan, under review). Although this research suggests that the prevalence, type, and spread of collaborative gestures may be influenced by mathematics experience, the basic premise that the categories identified here are key ways in which mathematical reasoning is jointly embodied seems to hold across learner populations. In addition, although the larger classification system for collaborative gestures (e.g., echo, mirror joint) was well established in this study, making the second-order differentiations of sub-categories of these gestures (e.g., echo and build) may need further fleshing out.

Another extension of this work to consider is that in typical mathematical activity, gestures often occur in conjunction with other resources in the semiotic bundle – such as words and symbols that are written down, mathematical diagrams, physical designed objects, manipulatives, digital simulations, and measurement tools. Gestures, particularly pointing gestures, as well as other physical cues – like gaze and body position – are critical to understanding how these semiotic resources can be used collaboratively. These resources also have unique affordances for distributing cognition across multiple actors, and actors likely position themselves collaboratively using these media in a variety of different ways. One could imagine that some of the categories we have identified here might be relevant to thinking about collective use of other semiotic resources - for example, students might position manipulatives jointly, in concert, to form a single representation, or one learner may physically follow and mirror another's technique for measuring in real time. Indeed, Smith (2014) presents an instance of one learner gesturing over a diagram drawn by another learner, and likens it to Furuyama's (2000) definition of a collaborative gesture, which we call a joint gesture. However, novel categories may arise when considering collaborative use of other media, as well.

A variety of other questions were raised by this study, some of which we have begun to explore but are beyond the scope of this paper, while others are ripe for further investigation. These include: (1) How are collaborative gestures paired with speech processes implicating different argumentation moves? (2) How are collaborative gestures associated with learners successfully overcoming trouble spots and formulating valid mathematical arguments? Are certain types or sequences of collaborative gesture more effective than others? (3) What are the tradeoffs of using gestures as a collaboration tool rather than, for instance, a shared written or digital workspace? How do collaborative gesture interact with other semiotic resources, and are other semiotic resources used collaboratively in similar ways? and (4) Can collaborative gestures be directed or taught, particularly using motion capture technologies that can detect multiple bodies in motion? For example, a number of mathematics games and simulations use body tracking to detect the motion of one learner's body or hands (Nathan & Walkington, 2017; Abrahamson & Trninic, 2015; Smith, King, & Hoyte, 2014); however, collaborative body movements, using a device like the Kinect that can track multiple skeletons, may be more effective.

This study was exploratory and thus raises more questions than it answers, but here we explore a phenomenon that seems both powerful and important to understanding and intervening upon students' mathematical learning. This phenomenon has the potential to be leveraged in digital learning environments as motion-based technologies become more common and feasible in classrooms. It also has the potential to allow teachers to better guide students in ways they can meaningfully participate in mathematical discussions with their peers.

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Appendix. Codebook for coding whether oral proofs to conjectures were correct or incorrect

Conjecture	**Criteria for valid proof**
1. The sum of the lengths of any two sides of a triangle must be greater than the length of the remaining side.	States the conjecture is true because either: (1) triangle wouldn't be able to close, sides would not meet or (2) You would just have two straight lines on top of each other, (3) the shortest distance between any two points is a straight line
2. Given that you know the measure of all 3 angles of a triangle, there is only one unique triangle that can be formed with these 3 angle measurements.	States the conjecture is false because: (1) although angles are held constant, side lengths can change (creating triangles that are similar but not congruent). (2) gives a specific counter example
3. The area of a parallelogram is	States the conjecture is true because either: (1) a right triangle

the same as the area of a rectangle with the same base and height.	can be cut off of one side of the parallelogram and added to the other side, making a rectangle, or (2) a right triangle can be cut off of one side of the rectangle and added to the other side, making a parallelogram, or (3) all rectangles are parallelograms, thus the formula for the area of a parallelogram would hold for a rectangle as well, or (4) the rectangles and the parallelogram can be divided into two triangles with the diagonals, and the product of the base and height are the same, thus they the area would be the same.
4. Diagonals of a rectangle are always congruent.	States the conjecture is true because the diagonals form two right triangles - these right triangles are congruent (by SAS), since opposite sides of a rectangle are congruent, so their hypotenuses must be congruent
5. If one angle of a triangle is larger than a second angle, than the side opposite the first angle is longer than the side opposite the second angle.	States the conjecture is true because either (1) as one angle of a triangle becomes larger, it forces the opposite to open up, making the side longer. Or (2) And as one angle becomes smaller, it forces the opposite side to compress, making the side shorter. Or (3) law of cosines or law of sines or (4) angles and sides are in correlational relationship
6. The measure of the central angle of a circle is twice the measure of any inscribed angle intersecting the same two endpoints on the circumference of the circle.	States the conjecture is true because the central angle opens wider, while the inscribed angle is narrower, so it would make sense that the central angle has to be twice as large as the inscribed angle.
7. Reflecting a point over the x-axis is the same as rotating the point 90 degrees about the origin.	States the conjecture is false and either (1) gives a specific counter-example or (2) says that reflecting over the x-axis reverses the sign of the y-coordinate, while rotating 90 degrees impacts the coordinates differently (reverses x and y and changes the sign of y)
8. If you double the length and width of a rectangle, the area is exactly doubled.	States the conjecture is false by either (1) giving a specific counter-example, or (2) explaining that if the length and width are multiplied by 2, and area is length times width, area would be multiplied by 4