Collaborative Gesture as a Case of Distributed Mathematical Cognition

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Abstract: Gestures have been shown to play a key role in mathematical reasoning and be an indicator that mathematical reasoning is *embodied* – inexorably linked to action, perception, and the physical body. Theories of extended cognition accentuate looking beyond the body and mind of an individual, thus here we examine how gestural embodied actions become distributed over multiple learners confronting mathematical tasks. We identify several ways in which gesture can be used collaboratively and explore patterns in how collaborative gestures seem to arise in a learning environment involving a motion capture game for geometry. Learners use collaborative gestures to extend mathematical ideas over multiple bodies as they explore, refine, and extend each other's reasoning.

Introduction

Gestures – movements that accompany speech – have been found to be a powerful component of reasoning in a variety of domains (Alibali, Spencer, Knox, & Kita, 2011; Beilock & Goldin-Meadow, 2010; Glenberg, Gutierrez, Levin, Japuntich, & Kaschak, 2004), including mathematics. The gestures learners formulate can reveal information not expressed in speech (Church & Goldin-Meadow, 1986), can show an association with conceptual performance (Goldin-Meadow, 2005; Cook & Goldin-Meadow, 2006), can be manipulated to give students new actionable ideas (Goldin-Meadow, Cook, & Mitchell, 2009; Nathan et al., 2014; Novack, Congdon, Hemani-Lopez, & Goldin-Meadow, 2014), and, when prevented, can impair reasoning (Hostetter, Alibali, & Kita, 2007; Nathan & Martinez, 2015). In addition, teachers use gestures to communicate ideas to students in a multimodal manner (Alibali & Nathan, 2012; Valenzeno, Alibali, & Klatzky, 2003), which may be particularly important in mathematics classrooms where gestures can make spatial and relational aspects of mathematical concepts come alive (Nathan et al., 2011).

While considerable research has been conducted on how teachers and learners use gestures during mathematical reasoning (see Alibali & Nathan, 2012 for a review), less work has looked at gesturing as a collaborative activity that is part of mathematical discussion and argumentation between different learners. Here we define *collaborative gestures* as gestures that physically encompass multiple learners. We propose that learners engaging in mathematical reasoning can use their bodies, particularly their hands, in collaborative ways to reinforce, extend, and redirect the mathematical ideas of others, which were also expressed through physical movement. Theories of *embodied cognition* (e.g., Wilson, 2002) posit that learners process and understand ideas through their bodies and their senses, and that the mind and body, rather than being separate entities, have a bi-directional relationship. A complementary theory, *extended cognition* (Clark & Chalmers, 1998), accentuates the idea that a student's cognitive system extends beyond their own minds and bodies, into their environment and those around them. Collaborative gestures are a fascinating area of study for the learning sciences, because they show how cognitive processes become extended over the embodied experiences of multiple learners. In addition, collaborative gestures provide a window into understanding how the body can be leveraged to help students understand mathematical ideas as they work jointly on challenging tasks. In the present paper, we examine collaborative gestural activity that groups engage in while proving geometry conjectures.

Theoretical framework

Gesture as Simulation Action

The theory of Gesture as Simulated Action (GSA; Hostetter & Alibali, 2008) provides an empirical, embodied cognition account of how the multimodal production of gestures comes about. Gestures in the GSA framework arise during speaking when pre-motor activation, formed in response to motor or perceptual imagery, is activated beyond a speaker's current gesture threshold. The threshold is the level of motor activation needed for a simulation to be expressed in overt gesture; this threshold can vary depending on factors such as the current task demands (e.g., strength of motor activation when processing spatial imagery), individual differences (e.g., level of spatial skills), and situational considerations (e.g., social contexts). Nathan (2017) proposed an extension of GSA to account for the influences of motor activity the learner is induced to perform (e.g., directed actions) on cognition. In this reciprocating model, learners' actions and movements serve as inputs capable of driving the cognition-action system toward associated cognitive states through a bi-directional process. In other words, in addition to cognitive states giving rise to actions, directing learners to engage in physical motions may give them new ideas and insights relevant to understanding and solving tasks. While Nathan's (2017) extension to GSA accentuates how directing the learner to motion in particular ways can trigger new cognitive and embodied states, here we emphasize how *observing* and *embodying* the actions and gestures of others, through collaborative activity and joint sense-making, can trigger new cognitive and gestural states in learners. We take this to be an important case of *distributed cognition*, which we discuss next.

Distributed cognition and collaborative gesture

Professional practice involves the coordination of many different inscriptions and representational technologies by differently-positioned actors whose actions occur across a range of social and physical spaces (Goodwin, 1995; Hutchins, 1995). Through joint, coordinated activity, cognition becomes distributed over a patchwork of discontinuous spaces and representational media. Lave (1988) describes how "'Cognition' observed in everyday practice is distributed—stretched over, not divided among—mind, body, activity and culturally organized settings (which include other actors)" (p. 1). Theories of *extended cognition* further argue that the social and physical environment of learners is actually constituent of their cognitive system (Clark & Chalmers, 1998). The implication is that cognition, rather than existing in the head of an individual, is distributed over the bodies of multiple learners and the environment around them as they interact. One way cognition can be extended across learners is through the use of gestures that extend over multiple persons.

Prior research on learning origami has identified *collaborative gestures* as gestures through which a learner interacts with the gestures of a communicative partner (Funiyama, 2000). In the context of this past research, these gestures often involved a student pointing to or manipulating a teacher's gestures about origami folds. Here we reimagine the idea of collaborative gestures to be relevant to learner-learner interactions around mathematical sense-making and take such gestures to be a case of extended cognition. We next turn to a discussion of how gestures and actions have been studied in the context of mathematical reasoning.

Gesture, embodiment, and mathematical proof

One type of gesture that has been identified as important to mathematical reasoning is *dynamic gestures* (Göksun et al., 2013; Uttal et al., 2012). These are gestures where learners use their bodies to physically formulate and then transform or manipulate mathematical objects. For example, a learner might make a triangle with their thumbs and forefingers, and then make that triangle grow, shrink, rotate, flip, etc. These kinds of gestures might be particularly important and revealing when learners are engaging in mathematical proving or "the process employed by an individual to remove or create doubts about the truth of an observation" (Harel & Sowder, 1998, p. 241). When learners successfully justify mathematical arguments, their reasoning can be transformational in nature, where they perform valid mathematical operations on objects in order to build a logical deductive chain of reasoning that transcends particular cases and applies to all mathematical objects under consideration. This kind of *transformational proof scheme* (Harel & Sowder, 1998) may be closely coupled with both mentally simulated and physically enacted gesture and action (Nathan et al., 2014).

Research has shown that dynamic gestures arise during the mathematical reasoning of experts (Marghetis, Edwards, & Núñez, 2014) and that dynamic gestures have a strong association with students formulating valid proofs to geometrical conjectures (Nathan et al., 2014; Nathan & Walkington, 2017). The professional practice of proof itself has been described as "a richly embodied practice that involves inscribing and manipulating notations, interacting with those notations through speech and gesture, and using the body to enact the meanings of mathematical ideas" (Marghetis, Edwards, & Núñez, 2014, p. 243). The multimodal nature of proof is also evident for novice students in classrooms, as proofs often take on verbal and gestural forms, as opposed to formal, written ones (Healy & Hoyles, 2000), and teachers and students use gestures to track the development of ideas when exploring conjectures (Nathan et al., 2011; Nathan & Walkington, 2017).

Research questions

Prior work has identified how gestures arise, why gestures are important, the close connection between embodied action and mathematical proof, and the centrality of gestures that show dynamic transformations. Here we extend this work by looking at how these embodied actions are used as part of a distributed cognitive system through collaborative activity across multiple learners. We address the following research questions:

1) How are gestures used collaboratively during geometric proof activities?

2) How often were collaborative gestures used, and how did this vary by expertise and type of gesture?
3) How were learners engaging differently with collaborative gestures across individuals and groups?

Methods

Participants

Participants included 20 pre-service and 34 in-service teachers, enrolled in one of five different courses at a private university. They were either enrolled in an undergraduate elementary math methods course for pre-service teachers, a graduate elementary math methods course for pre-service teachers, a graduate course for in-service middle school math teachers in their first year, a graduate class for high school math teachers in their first year, or a graduate *master math teacher* course for in-service teachers who are generalists interested in mathematics or who are secondary mathematics teachers. A total of 64 students across the five classes were recruited to participate, however 3 were absent on the day of the study and 7 did not consent to participate, for the final sample of 54. Forty-six participants were female, while 8 were male; 38 identified as Caucasian, 6 as African-American, 5 as Asian, 3 as Hispanic/Latin@, and 2 as Other race/ethnicity.

Procedure

Participants were placed in groups of 3 to 6 to play a Kinect-based video game for learning geometry, called *The Hidden Village* (Nathan & Walkington, 2017). The game included 8 tasks where players perform directed motions with their arms and then prove or disprove related geometric conjectures (Table 1). Participants were directed to take turns of the person controlling the game for each task such that everyone in the group controlled the game at least once. Due to technical issues, some groups only experienced six of the eight game conjectures. Participants were instructed to not use pencil or paper to assist them in proving the conjectures, and were told to work together. Although the body movements they had been directed to perform in the game were intended to give them insights about the proofs (e.g., for a conjecture about similar triangles they were directed to make growing triangles with their arms), the directed movements were somewhat rarely explicitly re-enacted in participants' discussions. Here our focus is on the gestures that these learners formulated themselves to help them understand, reason through, and collaborate around the proof to each conjecture.

Table 1: Conjectures participant groups proved

1. The sum of the lengths of any two sides of a triangle must be greater than the length of the remaining side.
2. Given that you know the measure of all 3 angles of a triangle, there is only one unique triangle that can be formed with these 3 angle measurements. (False)
3. The area of a parallelogram is the same as the area of a rectangle with the same base and height.
4. Diagonals of a rectangle are always congruent.
5. If one angle of a triangle is larger than a second angle, than the side opposite the first angle is longer than the side opposite the second angle.
6. The measure of the central angle of a circle is twice the measure of any inscribed angle intersecting the same two endpoints on the circumference of the circle.
7. Reflecting a point over the x-axis is the same as rotating the point 90 degrees about the origin. (False)
8. If you double the length and width of a rectangle, the area is exactly doubled. (False)

Analysis

Video was captured of groups playing the game, and was transcribed in the Transana software (Woods & Fassnacht, 2012). Videos were clipped such that one clip was one group proving one conjecture. Each clip was coded for each gesture sequence that arose – a gesture sequence was defined as all of the hand gestures made by a single person from the time their hands rose until the time their hands returned to rest. Gesture sequences were coded as being individual if the gesturer was making a gesture that was not triggered by or related to the gestures of others, and collaborative if they were. For individual gestures, gesturers could still be building off of the speech of other learners – gesture were only coded as collaborative if learners were gesturing in response to the *gestures* of other learners. Collaborative gesture sequences were coded for both the learner that was performing them and the learner the performer was responding to or collaborating with. Collaborative gestures were then separated into different categories using a grounded, bottom-up approach of constant comparisons (Glaser & Strauss, 1976). As new collaborative gesture categories emerged, prior clips were revisited and recoded. Two coders completed all gesture sequence coding. Fort-four percent of the corpus was double-coded by both coders, with discrepancies, issues, and ideas discussed as coding was compared and new categories were formulated. A total of 87 clips of 12 groups across the five classes were coded for gesture sequences.

Results and discussion

RQ1: In what ways were gestures used collaboratively?

Results revealed five categories of collaborative gestures (Table 2). In the example of **mirroring** gestures in Figure 1, two learners are making the same triangle gesture at the same time when proving Conjecture 5 in Table 1. In the example of **echoing** gestures in Figure 1, one learner scales her hands in and out to make similar triangles for Conjecture 2 in Table 1, and then another learner takes up her explanation while repeating her gesture. These gestures reveal how learners can reflect on and adopt each other's mathematical reasoning.

Table 2: Categories of collaborative gestures

Gesture Type	Description
Mirror	Learners make same gesture at same time, while purposefully following each other's movement.
Echo	One learner purposefully repeats the gesture made by another learner.
Echo & Build	One learner repeats all or part of another learners' gesture, but then modifies it or builds on it in some way to show a progression of reasoning.
Alternate	One learner gestures their understanding, another learner responds with a different gesture showing their understanding, either *building* or *redirecting* from the original gesture.
Joint	Two or more learners make a single gestural representation together.

Figure 1. Mirroring (left) and echoing (right) gestures.

In the extended example in Figure 2, participants are discussing their reasoning for Conjecture 4 in Table 1. Tanya and Karen initially represent their understanding, with Tanya gesturing horizontal sides with her hands and Karen gesturing crossed diagonals. In their speech, Tanya focuses on the two sides being equal (Line 1), while Karen focuses on the diagonal's opposing relationship to each other (Line 2). Lisa performs an **alternating** gesture to redirect the conversation to another idea – that the sides are parallel (Line 3). She then integrates Tanya's equality gesture into her explanation by echoing it and saying "So they're the same" (Line 5). Karen then **echoes** both Tanya's equality gesture and Lisa's parallel gesture, and **builds** by adding on her crossed arms gesture at the end of the sequence (Lines 6 and 8). Lisa assists her in formulating this explanation by repeating her own parallel line gesture, then echoing Karen's crossed arm gesture (Line 7). Lisa then sweeps her diagonal hands in and out to accentuate the idea of congruency, building upon Karen's gesture (Line 7). This transcript demonstrates how different aspects of understanding the same problem become embodied through different gestures, and how learners take up and build upon each other's' gestures to extend their understanding. Karen is able to adopt the gestures of Lisa and Tanya in order to create a mathematical argument that brings in all three of their ideas – parallel sides, congruent sides, and diagonals.

The second extended example in Figure 4 shows an example of **joint** gestures. Participants are discussing Conjecture 6 in Table 2, "The measure of the central angle of a circle is twice the measure of any inscribed angle intersecting the same two endpoints on the circumference of the circle." Diana has been explaining her understanding using a gesture of two angles, one formed with her two index fingers (the inscribed angle) and one formed with her thumbs (the central angle – see Figure 3). Kyla has been somewhat removed from the conversation, and brings herself in by posing the question to Diana "Where is the circle?" (Line 1). Since it would be physically impossible for Diana to represent both angles and a circle with her hands alone, Diana responds "You have to visualize it" (Line 2). In the meantime, Kyla walks over to Diana and traces a circle around the outside of her angles, and then points to the vertex of the inscribed angle, which she believes is the outside of the circle. A third group member, Sophie, responds to the need to represent the circle in a more permanent manner by offering her hands to represent the circle, placing them under Diana's hands (Line 3). Kyla and the fourth group member, Carl, then proceed to discuss and point to various parts of the embodied diagram that Sophie and Diana have formed (Lines 5-10). This transcript demonstrates how gestural activity can become distributed over multiple learners as they represent a single, shared mathematical system that is jointly embodied and that they can collaboratively refer to and discuss. The cognitive work of the mathematical system is distributed over all four of their bodies, and the gestures and speech they engage in.

1 Tanya (at the same time): I think so because two sides have to be. The two sides are equal.
((Gesture of two equal horizontal sides with hands))
2 Karen (at the same time): Yeah if they were the opposite.
((Crossed arms gesture))
3 Lisa: The two sides are parallel.
((Gesture of two straight vertical sides by sweeping hands))
4 Tanya: Parallel.
5 Lisa: So they're the same.
((Gesture of two straight horizontal sides by sweeping hands))
6 Karen: Wait, the two sides are parallel are the height and the width.
((Gestures of parallel vertical and horizontal sides))
7 Lisa: So diagonally they'd have to be the same.
((Gestures of two vertical sides, then crossed hands swept up and down.))
8 Karen: The height and width are parallel so the diagonals have to be the same.
((Gestures crossed arms, then horizontal sides with thumb out and upper hand folded, then crossed arms))

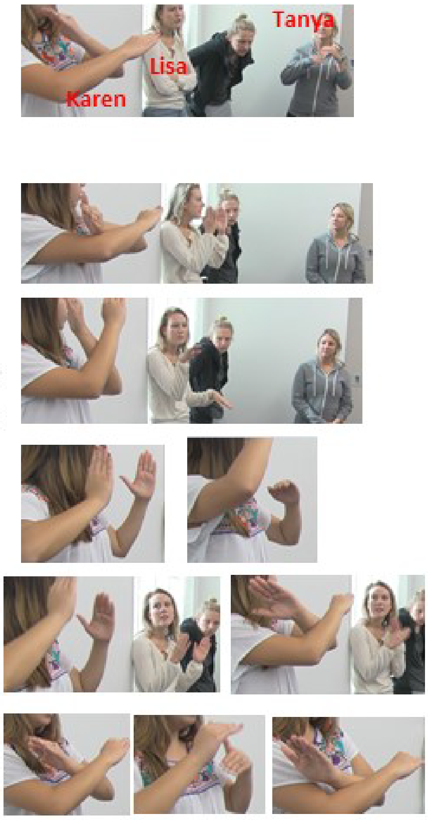

Figure 2. Alternating and Echo & Build Gestures.

Overall, we found a variety of ways in which gestures could be used collaboratively in the sense that gestures became distributed over multiple learners. This was an initially surprising and unexpected phenomenon to arise as these students engaged in mathematical reasoning. Mirroring and echoing gestures showed one learner taking up the mathematical reasoning of another, while echo & build gestures and alternate gestures showed learners extending and refuting each other's reasoning using embodied action. Joint gestures allowed multiple learners to use their bodies in concert to build and transform an embodied mathematical system together. These gestures did not simply accompany their mathematical arguments or their collaborative activities – they were an inexorable part of their reasoning processes.

Figure 3. (Left) Inscribed and central angles of a circle (Right) Diana's initial gesture showing these angles.

RQ2: How often were collaborative gestures used?

A total of 443 gesture sequences were coded across the corpus, of which 218 (49.2%) were collaborative. This came to approximately 2.6 collaborative gestures per student group proof attempt. However, the number of collaborative gestures used by different groups varied widely – from 0.67 collaborative gestures per proof for one group, to 5.9 collaborative gestures per proof for another group. The use of collaborative gestures seemed to vary by teacher expertise level – teachers from the two pre-service classes used an average of 1.5 collaborative gestures per conjecture, while teachers from the two first-year in-service classes used an average of 3.5, and teachers from the master teacher class used an average of 2.8.

1 Kyla: So where's your circle?

2 Diana: You have to visualize it.

((Diana making double angle gesture with thumbs and index fingers, Kyla comes over and traces a circle around it))

3 Kyla: So this is the point on your circle. Right there?

((Kyla points to the tip of Diana's top angle; Sophie comes over and makes a circle underneath Diana's angles with cupped hands))

4 Sophie: Yeah this-

5 Kyla: This is the outside of the circle, this is the middle of the circle. So this is where the diameter goes, right between these points.

((Kyla points to the top angle vertex and lower angle vertex formed by Diana, and then traces the diameter across Sophie's circle))

6 Carl: Right, right.

7 Diana: Right.

8 Kyla: So what angle are we talking about?

9 Carl: This one and these two-

((Carl points to three different positions on the geometric object))

10 Kyla: Are the same-

((Kyla points to each point on the object after Carl does))

11 Sophie: Are the same, right?

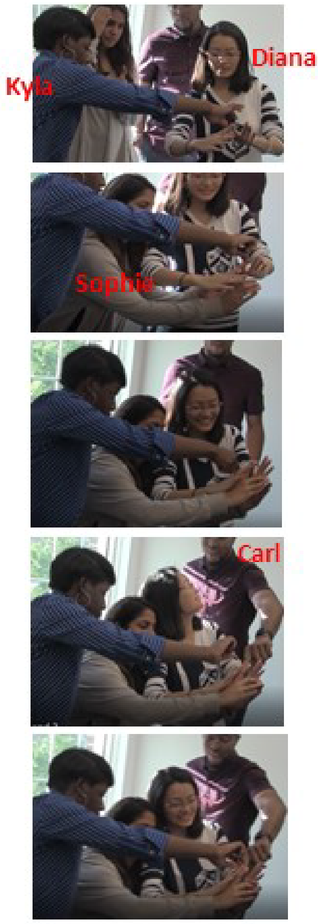

Figure 4. Joint Gestures.

Of the 218 collaborative gestures, 110 (50.5%) were echoing gestures, 56 (25.7%) were alternating gestures, 28 (12.8%) were mirroring gestures, and 27 (12.4%) were joint gestures. Of the 110 echoing gestures, 81 (73.6%) were simple echoes while 29 (26.4%) were echo and build gestures. In terms of the types of collaborative gestures the three categories of teachers tended to make, for all three teacher groups, echoing gestures made up approximately 50% of the group's collaborative gestures (pre-service: 54.7%; novice in-service: 47.3%; master: 50.7%). Master teachers were most likely to use mirroring gestures (pre-service: 11.9%; novice in-service: 9.8%; master: 17.9%), while novice first year teachers were most likely to use alternating and joint gestures (alternating: pre-service: 23.8%; novice in-service: 28.6%; master: 20.9%; joint: pre-service: 9.5%; novice in-service: 14.3%; master: 10.4%). Overall there did not seem to be compelling differences between categories in the types of collaborative gestures made – although the sample size is small.

RQ3: How were learners engaging differently with collaborative gestures?

Figure 5 shows how often students initiated versus received collaborative gestures. Each gray box represents one group, and each pill-shape inside of a gray box represents one student. The left side of the pill shows how many collaborative gestures per conjecture the student performed (e.g., when they echoed or mirrored another learner). The right side of the pill shows how many collaborative gestures per conjecture were directed at the student (e.g., someone echoing or mirroring them). Red indicates 0 collaborative gestures by the learner, yellow indicates less than 1 but more than 0 collaborative gestures given/received per conjecture by the learner, green indicates greater than one but less than two collaborative gestures given/received per conjecture. The top row of the figure is the pre-service classes, the middle row is the in-service first year classes, and the bottom row is the master teacher class.

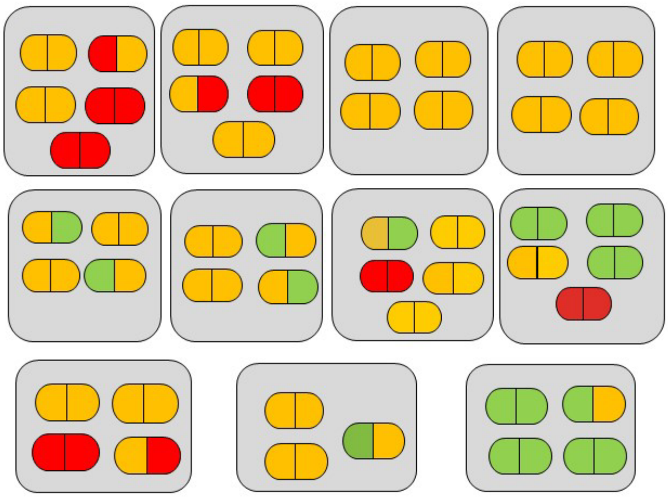

Figure 5. Distribution of collaborative gesture giving/receiving across 11 student groups. One group is omitted because technical problems caused them to join another student group part way through the study.

As can be seen from Figure 5, across levels of expertise, there were yellow and red students who stayed on the fringes of collaborative gesture activity. There were groups of highly collaborative green students as well. Some students tended to do a lot of individual gestures, but had group members who would echo or mirror or alternate with them regularly. This, however, was somewhat rare – most students tended to give and receive a similar number of collaborative gestures. In other cases, group members would give quite a few collaborative gestures that would get little attention from their group in terms of being considered or built upon. This figure demonstrates how cognition was embodied and distributed across student groups, and how certain learners seemed to be more central to these processes than others.

Significance

While the importance of gestures to student learning has been established in a variety of studies, less work has been done detailing how gestures allow for cognition to be physically distributed over multiple learners. Documenting this process and how it comes about with different groups of learners in different learning environments is a first step in understanding how learners can engage in joint, embodied reasoning to solve and learn from complex tasks. A variety of questions were raised by this study, some of which we've addressed but were beyond the scope of this paper, while others are ripe for further investigation. These include: (1) How are collaborative gestures paired with speech processes implicating different argumentation moves? (2) How are collaborative gestures associated with learners successfully overcoming trouble spots and formulating valid mathematical arguments? Are certain types or sequences of collaborative gesture more effective than others? (3) What are the tradeoffs of using gestures as a collaboration tool rather than, for instance, a shared written or digital workspace? and, (4) Can collaborative gestures be directed or taught, particularly using motion capture technologies that can detect multiple bodies in motion? This study was exploratory and thus raises more questions than it answers, but here we identify a phenomenon that seems both powerful and important to understanding and intervening upon students' mathematical learning. This phenomenon has the potential to be leveraged in digital learning environments as motion-based technologies become more feasible in classrooms.

References

Alibali, M., & Nathan, M. (2012). Embodiment in mathematics teaching and learning: Evidence from students' and teachers' gestures. *Journal of the Learning Sciences, 21*(2), 247–286.

Alibali, M. W., Spencer, R. C., Knox, L., & Kita, S. (2011). Spontaneous gestures influence strategy choices in problem solving. *Psychological Science*, *22*(9), 1138-1144.

Beilock, S. L., & Goldin-Meadow, S. (2010). Gesture changes thought by grounding it in action. *Psychological Science*, *21*(11), 1605–1610.

Church, R. B., & Goldin-Meadow, S. (1986). The mismatch between gesture and speech as an index of transitional knowledge. *Cognition*, *23*, 43-71.

Clark, A., & Chalmers, D. (1998). The extended mind. *Analysis*, 7-19.

Cook, S. W., & Goldin-Meadow, S. (2006). The role of gesture in learning: Do children use their hands to change their minds?. *Journal of cognition and development*, *7*(2), 211-232.
Funiyama, N. (2000). Gestural interaction between the instructor and the learner in origami instruction. In D. McNeill *Language and gesture*, pp. 99-117.
Glaser, B., & Strauss, A. (1967). *The Discovery of Grounded Theory*. New Brunswick: Aldine.
Glenberg, A. M., Gutierrez, T., Levin, J. R., Japuntich, S., & Kaschak, M. P. (2004). Activity and imagined activity can enhance young children's reading comprehension. *Journal of Educational Psychology*, *96*(3), 424–436.
Goldin-Meadow, S. (2005). *Hearing gesture: How our hands help us think*. Cambridge, MA: Harvard University Press.
Goldin-Meadow, S., Cook, S. W., & Mitchell, Z. A. (2009). Gesturing gives children new ideas about math. *Psychological Science*, *20*(3), 267–272.
Göksun, T., Goldin-Meadow, S., Newcombe, N., & Shipley, T. (2013). Individual differences in mental rotation: What does gesture tell us? *Cognitive processing*, *14*(2), 153–162.
Goodwin, C. (1995). Seeing in depth. *Social Studies of Science*, *25*, 237–274.
Harel, G., & Sowder, L. (1998). Students' proof schemes. In E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), *Research on collegiate mathematics education* (Vol. III, pp. 234–283). Providence, RI: American Mathematical Society.
Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education*, *31*(4), 396–428.
Hostetter, A. B., & Alibali, M. W. (2008). Visible embodiment: Gestures as simulated action. *Psychonomic bulletin & review*, *15*(3), 495–514.
Hutchins, E. (1995). *Cognition in the Wild*. Cambridge, MA: MIT Press.
Lave, J. (1988). *Cognition in practice: Mind, mathematics and culture in everyday life*. Cambridge Univ. Press.
Marghetis, T., Edwards, L. D., & Núñez, R. (2014). More than mere handwaving: Gesture and embodiment in expert mathematical proof. In L.D. Edwards, F. Ferrara, & D. Moore-Russo (Eds.), *Emerging perspectives on gesture and embodiment in mathematics* (pp. 227–246). Charlotte, NC: Information Age Publishing.
Nathan, M. J. (2017). One function of gesture is to make new ideas: Evidence for reciprocity between action and cognition. In R. B. Church, M. W. Alibali, & S. D. Kelly, (Eds.), *Why gesture? How the hands function in speaking, thinking and communicating* (pp. 175-196). John Benjamins Publishing.
Nathan, M. J., & Martinez, C. V. (2015). Gesture as model enactment: the role of gesture in mental model construction and inference making when learning from text. *Learning: Research and Practice*, *1*, 4–37.
Nathan, M. & Walkington, C. (2017). Grounded and Embodied Mathematical Cognition: Promoting Mathematical Insight and Proof Using Action and Language. *Cognitive Research: Principles and Implications, 2*(1), 9.
Nathan, M. J., Walkington, C., Boncoddo, R., Pier, E. L., Williams, C. C., & Alibali, M. W. (2014). Actions speak louder with words: The roles of action and pedagogical language for grounding mathematical proof. *Learning and Instruction, 33,* 182–193.
Nathan, M., Walkington, C., Srisurichan, R., & Alibali, M. (2011). Modal Engagements in Pre-College Engineering: Tracking Math and Science Concepts Across Symbols, Sketches, Software, Silicon, and Wood. In *Proceedings of the 118th American Society of Engineering Education Annual Conference and Exposition* (pp. 22.1070.1 - 22.1070.32). Vancouver, CA.
Uttal, D. H., Meadow, N. G., Tipton, E., Hand, L. L., Alden, A. R., Warren, C., & Newcombe, N. S. (2012). The malleability of spatial skills: A meta-analysis of training studies. *Psychological Bulletin*, *13*(2), 352–402.
Valenzeno, L., Alibali, M. W., & Klatzky, R. (2003). Teachers' gestures facilitate students' learning: A lesson in symmetry. *Contemporary Educational Psychology*, *28*(2), 187–204.
Wilson, M. (2002). Six views of embodied cognition. *Psychonomic Bulletin and Review*, *9,* 625–636.
Woods, D., & Fassnacht, C. (2012). Transana v2.52. http://transana.org. Madison, WI: The Board of Regents of The University of Wisconsin System.

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