Perceiving and Reasoning about Geometric Objects in the Middle Years

Rebecca Seah
RMIT University
<rebecca.seah@rmit.edu.au>

Marj Horne
Australian Catholic University
<Marj.Horne@acu.edu.au>

Although spatial thinking and reasoning is recognised as a key component for promoting STEM discipline, very little research has been done on its promotion in middle years in Australia. How do students perceive three dimensional geometrical objects? Are they able to recognise the objects from different perspectives and explain their reasoning for their drawing of the object? In surveying and analysing 709 students' responses, this study found that classroom experiences can enhance students' spatial and reasoning skills.

There is a common assumption that students have distinctive learning styles, that some students are verbal learners and other visual learners. This ill-informed belief led many to see spatial ability as fixed, with clear gender differences that are hardwired biologically (Newcombe & Stieff, 2011). Evidence shows that males and females use different strategies when visualising objects (Kozhevnikov, Kosslyn, & Shephard, 2005). Gender differences could also be caused by socio-cultural factor (Yilmaz, 2009) and seems to only appear in adults (Li, 2014). Spatial ability is certainly a trainable skill. Living in a three-dimensional (3D) world, the capacity to reason spatially is crucial for human existence. Spatial reasoning plays a critical role for developing Science, Technology, Engineering and Mathematics related disciplines (Wai, Lubinski, & Benbow, 2009). Early spatial skill training such as visualising and analysing shapes with different orientations has been found to improve subsequent arithmetic competence and predict success in engaging in mathematics reasoning task in the middle years (Casey, Lombardi, Pollock, Fineman, & Pezaris, 2017; Verdine, Golinkoff, Hirsh-Pasek, & Newcombe, 2017). Despite its importance, the construct of spatial reasoning is not as well understood. There is a lack of agreement on the process and steps of its development (Yilmaz, 2009). How students learn to cultivate such ability is unclear. Although the Australian Curriculum: Mathematics has integrated spatial reasoning as part of the construct of numeracy, questions remain on how to nurture such an ability.

The *Trends in International Mathematics and Science Study* (TIMSS) reported that Australian students are particularly weak in the content areas of geometry, a discipline that builds on spatial reasoning and visualisation ability (Thomson, Wernert, O'Grady, & Rodrigues, 2017). In particular, very little research is done on how students work with 3D objects. This negatively impact students' understanding of measurement concepts as much of students' difficulties with volume and surface area measurement are due to an inability to visualise and reason 3D objects (Lieberman, 2009). There is clearly a need to investigate the relationships between visualisation, language and representation in the construction of 3D knowledge.

Literature Review

Spatial reasoning is the ability to make sense of spatial relationships between shapes and objects. This thinking encompasses an understanding of the feature, size, orientation, location, direction or trajectory of geometric shapes and objects and being able to make

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 677-684. Auckland: MERGA.

spatial transformations. Spatial reasoning is not a monolithic construct and there is a lack of consensus about its components due to inconsistent naming of the factors (Yilmaz, 2009). Initial psychometric studies viewed spatial abilities as consisting of two main factors: *Spatial visualisation* - the ability to imagine the rotating, folding, or any changes made to the position of objects and *spatial orientation* – one's ability to imagine or 'view' an object from different perspectives (McGee, 1979). Additional factors such as speed, flexibility and spatial relations were added subsequently by other researchers (see Yilmaz, 2009 for a review on this).

From a mathematics education research perspective, the lack of standardisation of how terms should be defined makes researching and teaching spatial ability problematic (Ramful, Lowrie, & Logan, 2017). For Ramful and his colleagues, spatial ability is best viewed in terms of mental rotation, spatial orientation and spatial visualisation to capture much of the middle school mathematics curricula requirements. Mental rotation refers to one's ability to imagine how 2D and 3D objects would appear after they are turned around. Spatial orientation is the ability to imagine how an object looks from a different vantage point. Spatial visualisation, for them, refers to any spatial tasks that do not involve mental rotation or orientation. While the definitions provided useful boundaries for designing multiple choice test items. There may be other skills at play that the test did not address.

Indeed, the concept of spatial visualisation may be the most difficult to define because of its lack of specificity. For Phillips, Norris and Macnab (2010), visualisation is a cognitive process in which objects are interpreted within the person's existing network of beliefs, experiences, and understanding. It takes place when an image is viewed and interpreted for the purpose of understanding something other than the object itself. For example, looking at a net consisting of four equilateral triangles and identifying it as a tetrahedron. To visualise the object requires individuals to introspect possible images similar to a visualisation object and interpret it within the person's existing network of beliefs, experiences and understanding. Since our visual sensory input is constantly bombarded with different imageries, our visual cognition makes a distinction between spatial images (relating to information about the location, size, and orientation of an image) and visual images (such as shapes, colour and depth) (Sima, Schultheis, & Barkowsky, 2013). These two distinct types of visualising style reflect different ways of the brain generate mental images and process visual-spatial information. Those who focus on visual images, the object visualisers, tend to encode images globally as a single perceptual unit based on actual appearances (Kozhevnikov et al., 2005). They generate detailed pictorial images of objects and process the information holistically. They are faster and more accurate when performing recognition and memory tasks. When asked to interpret and reconstruct 3D objects using 2D format, object visualisers often reproduce images that resembles the actual object. Conversely, those who focus heavily on spatial images, the spatial visualisers, tend to encode and process images analytically, using spatial relations to generate schematic and abstract images from what they see. These individuals are better able to interpret and analyse abstract representations.

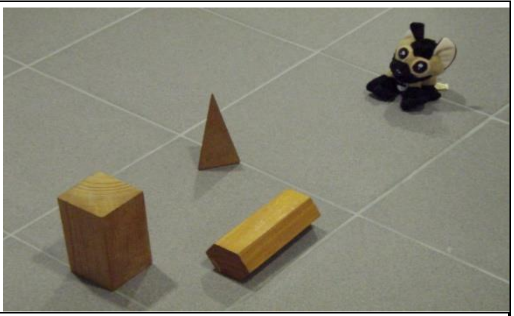
Language, also plays an important role in influencing our visual spatial perception. How one sees an object is influenced by ones' definitions of that object. For example, if a student defines a hexagon as a shape with many corners, or 'roundish', s/he is likely to call an octagon a hexagon. Similarly, if one's sole experience with 3D objects is prism, s/he is likely to call a triangular pyramid a triangular prism. The way a student perceives and talks about geometric visual representations reveals their thought processes and shapes their thinking (Sfard, 2008). Changing how students visualise 3D object necessitates changing their discourse. As stated earlier, investigation of spatial ability tends to rely on multiple choice items to test specific spatial factors defined by the researchers. This makes the identification of other possible spatial abilities difficult. How students apply their spatial skills use in problem situations is unclear. Accordingly, in this study, we present students with a picture showing a dog facing three geometric objects. The students were asked first to name the objects, then draw what the dog sees and explain their reasoning. We ask: Can students comprehend the concept of left, and right? And can they name the 3D objects in the picture? But more importantly, what spatial skills did they employ to produce an image of what the dog sees? And what was their rationale?

The descriptive data from the question will show students' use of *keywords* to name the object and its components and the *narratives*, written utterances, students made to justify their reasoning. The diagrams allow researchers to determine what information best captured individual students' attention when visualising the objects, whether they focus on the spatial or visual images. By comparing their drawing with the reasoning, we seek to identify other possible spatial abilities beyond mental rotation and mental orientation.

Method

[GPERS1] The dog in the picture is looking at three wooden shapes.

a. [GPERS1] Name the wooden shape
   - On your left ________________
   - On your right _______________
   - In the middle at the top _______
b. [GPERS2] Draw what the dog sees
c. [GPERS3] Explain how you decided what to draw and where to draw it.



Score	Description (GPERS1)
0	No response or irrelevant response
1	Names at least two objects in terms of their faces (e.g., square, rectangle, hexagon or triangle), may name one object correctly (e.g., cuboid or rectangular prism) or names faces or objects from dog's perspectives
2	Names at least two objects correctly relative to the students' perspective
3	All objects named correctly relative to the student's perspective (i.e., cuboid or rectangular prism, hexagonal prism and triangular pyramid)
Score	Description (GPERS2)
0	No response or irrelevant response
1	Draws shapes (2D) or objects (3D) from the student's perspective
2	Draws at least one correct shape from the dog's perspective (see below)
3	Draws three correct shapes (see below) but may not be correctly positioned or oriented from the dog's perspective.
4	Draws three correct shapes in the correct position from the dog's perspective (i.e., a hexagon on the left, a rectangle in the middle and a triangle on the right)
Score	Description (GPERS3)
0	No response or irrelevant response
1	Explanation relates to what the student sees
2	Explanation relates to correct shapes or correct position but not both
3	Reasonable explanation provided for naming all shapes and their position

Figure 1. The dog's perspective task (GPERS) and marking rubric.

This study is part of a larger study, Reframing Mathematical Futures II (FMFII), where we have been developing a learning assessment framework to assist teachers to teach reasoning in geometric measurement. It is based on the premise that an evidence-based validated set of an assessment tools and learning tasks can be used to nurture students mathematical reasoning ability (Siemon et al., 2017). Figure 1 shows the item GPERS with its marking rubric, designed to assess students' ability to visualise geometric objects from different perspective. The item is part of the assessment forms designed to assess students' geometric and spatial reasoning. The data collected contributed to the identification of eight distinct thinking zones through Rasch analysis (see Siemon et al., 2017 for more details).

The participants were middle-years students from across Australia States and Territories. Two groups of cohorts were involved. The first set of data – the trial data, was taken from 436 Year 4 - 10 students from three primary and seven high schools across social strata and States to allow for a wider spread of data being collected. The teachers were asked to administer the assessment tasks and return the student work. The trial results were marked by two markers and validated by a team of researchers to ascertain the usefulness of the scoring rubric and the accuracy of the data entry. The second set of data – the project data, was taken from 273 Year 8-10 students from six high schools situated in lower socioeconomic regions with diverse populations. The project school teachers were asked to mark and return the raw score instead of individual forms to the researchers. The project school teachers received two 3 days face-to-face professional learning sessions on spatial and geometric reasoning prior to the implementation of the assessment tasks. They also had access to a bank of teaching resources and four on-site visits to support their teaching effort.

Findings

Table 1 show the overall percentage breakdown of student responses for GPERS and Table 2 show the breakdown according to each year level. No Year 7 data had been obtained from the project schools at the time when this data was analysed. The project schools clearly outperform trial school students in a number of areas. This is very encouraging as it shows that the professional learning the teachers in project schools received may have contributed to better awareness and attention given to the teaching of geometry in school. While gender difference was not the aim of our investigation, we nonetheless found no significant differences at the $p < 0.05$ level.

Table 1
Overall results expressed as percentages for the Perspective Task GPERS.

Score	Trial Schools (n=436)			Project schools (n=273)		
	GPERS1	GPERS2	GPERS3	GPERS1	GPERS2	GPERS3
0	12.6	15.4	37.6	9.9	1.1	35.9
1	42.9	18.6	46.3	28.9	14.3	30.4
2	17.7	17.7	14.2	15	7.7	16.9
3	26.8	30.5	1.8	46.2	12.1	16.9
4		17.9			64.8	

Overall, students' knowledge of 3D objects was poor as 55.5% trial schools and 38.8% project school students were unable to correctly name the 3D objects in the photo (GPERS1). This difficulty was due to a lack of experience rather than based just on year level (e.g., see Table 2 Year 10's result between trial schools and project school).

Table 2

Percentage breakdown for GPERS1, GPERS2, and GPERS3 according to year level.

	Trial Schools						Project Schools		
Score	Yr 4 *n* =31	Yr 5 *n* = 59	Yr 7 *n* = 111	Yr 8 *n* = 74	Yr 9 *n* = 79	Yr 10 *n* = 82	Yr 8 *n* = 87	Yr 9 *n* = 93	Yr 10 *n* = 93
	GPERS1								
0	0	8.5	7.2	12.2	15.2	25.6	4.6	11.8	12.9
1	58.1	61	43.2	25.7	342	47.6	23	44.1	19.4
2	19.4	17	22.5	16.2	15.2	14.6	26.4	6.5	12.9
3	22.6	13.6	27	46	35.4	12.2	46	37.6	54.8
	GPERS2								
0	12.9	6.8	10.8	13.5	12.7	32.9	2.3	0	1.1
1	29	33.9	23.4	10.8	2.5	19.5	24.1	18.3	1.1
2	32.3	22	18	10.8	20.3	12.2	5.8	15.1	2.2
3	25.8	28.8	28.8	31.1	40.5	25.6	18.4	12.9	5.4
4	0	8.5	18.9	33.8	24.1	9.8	49.4	56.8	90.3
	GPERS3								
0	38.7	25.4	29.7	40.5	34.2	57.3	43.7	36.6	28
1	61.3	62.7	47.8	39.2	48.1	31.7	33.3	25.8	32.3
2	0	11.9	18	18.9	15.2	11	9.2	8.6	32.3
3	0	0	4.5	1.4	2.6	0	13.8	29	7.5

Except for Year 10 in the project schools, more than 50% of the students in both cohort were unable to name the three objects correctly (GPERS1). Students who score 1 may have named the objects based on the dog's perspective. Analysis of the trial school data showed that 46.6% of the students name the squared/rectangular based prism correctly; a further 12% wrote rectangle. 45% named hexagonal prism correctly and 8.7% wrote hexagon. The triangular based pyramid presented the most challenge with 24% named it correctly, a further 14% used the term pyramid and 2% wrote square based pyramid. For some, this may have been a difficulty in their knowledge of left and right, however, only 3% gave a complete reversal, mixing the right and the left.

Analysis of *keywords* used in the trial schools showed that 13% of the cohort did not respond to the question. Around 47% named the rectangular prism, 45% named the hexagonal prism, and 24% name the triangular based pyramid correctly. Misspelling words such as 'prisim' 'prizem', 'prymand', 'pryrimid', 'peyment', were not counted as errors. Some students used 2D shape names for the 3D objects (17% rectangle/square; 15% hexagon and triangle) and 10% named triangular pyramid as triangular prism. Others named the objects by joining known terms, such as 'hexagonal cilender', 'rectangular hexagon', 'rectangular square', 'pentagon cilender', and 'rectangular cylinder'.

When drawing the objects from the dog's perspective (GPERS2), the project schools showed a clear improvement through Year 8, 9 and 10 with 90% of the Year 10 students able to successfully complete the task. The greatest error was that the students drew the objects from their perspective but claimed that that was what the dog saw or reversed the order saying the dog was seeing the objects from the other side. This account for between 25% and 40% of the trial school cohort and 5 - 18% of the project data.

Students have varying degrees of experience with drawing 3D objects. Many trial schools' students tried to incorporate all the components in their drawing. Eight students (2

Yr 4, 4 Yr 5 and 1 each from Yr 8 and 10) drew a bird eye view of the scene including the dog, similar to the drawing on the left in Figure 2. The drawing on the right shows that the student may have seen drawings of a square or rectangular prism before but little experience with the other objects. Even with the prism, this student has included all faces although the dog would be unable to see them all. The hexagonal prism shows both ends for the same reason. With the triangular pyramid he knows that there are three triangular faces on the side but was unable to depict it in 3D.

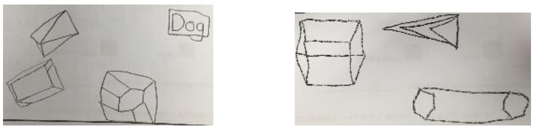


Figure 2. Students' representation of objects showing all components.

When comparing the drawings by year levels, Year 4 students tended to produce a wider range of drawings, from 2D shapes or 3D objects drawn on one plane, that show depth, to drawing an octagon as a hexagon and mixing the positions of the objects. From Year 5 onwards, depth and dimensionality became important features. While the rubric did not specify 'depth' as a criterion (placing the rectangular prism to the rear), many trial schools students demonstrated this in their drawing. Three students (1 Yr 4 and 2 Yr 7) included the tile lines although only the Year 7 students provided an explanation that they were used to either get the proportions or position correct (see row 3 in Figure 2).

Indeed, analysing trial schools' students' drawing and their explanations led to six strategies: mental rotation, physical rotation of page, mental reflection, perspective drawing, position and depth, and 2D perspective of 3D object. Figure 3 shows examples of student drawing and their explanations on how they drew what the dog saw. Some samples used more than one strategy.

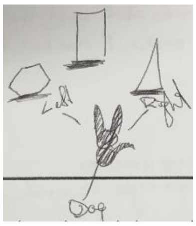
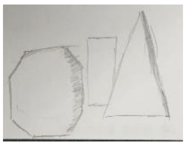
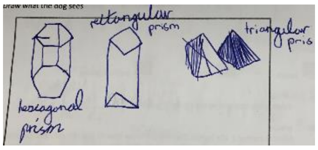
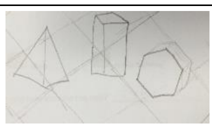
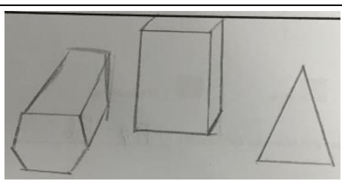
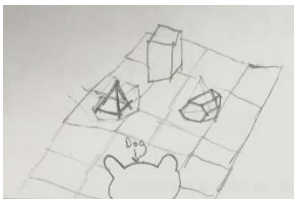
Student's drawing	**Strategy used and student's justification**
i.  ii. 	Mental rotation (with position and depth) i. I did it as if the dog was me. I put myself in the dog position and figured it out as if the shapes were right in front of me. ii. I put myself in the dogs positon and he can only see straight so they look 2D (*sic*) (with 2D perspective of 3D object).
	Physically rotate page I turned the paper around and figured what shapes it would look like from the dogs perspective (*sic*).
	Mental reflection I just flipped it around to make a reverse picture. (incorrectly reflected)
	Perspective drawing (mental or physical rotation) I decided to draw some of the shapes 3D because the dog would be able to see the top of the rectangular prism a little bit and the dog would be able to see the top and right side of the hexagonal prism, but the triangle he would only be able to see the face.

Figure 3. Students' representation of the dog's perspective.

There is a difference in the sophistication of the drawing and explanations as students attempt to indicate depth and dimensionality, as seen in row 1 and 4 of Figure 3. The student's explanation of the perspective drawing (row 4) is correct as the dog would be able to see the top and right side of the hexagonal prism, and at least two sides of the rectangular prism but only one face of the pyramid. Equally, sophistication of drawing may not necessarily reflect the thinking that showed in the explanation (Figure 4).



Correct justification, incorrect drawing

If the hexagonal prisim is on my right, the dog is facing the other way so it is on its left, this means that the pyramid that is on my left would be on its right and tha rectangular prisim stays in the same position (*sic*).

Figure 4. A correct justification with incorrect drawing of the dog's perspective.

Students do not seem to be used to justifying their action (GPERS3) even when they were able to successfully complete the drawing. Many did not response but for those who did, the explanation tended to be superficial such as 'I imagined I was the dog', 'I decided to draw it like that because'. Less than 5% of the trial school cohort and between 7 - 29% of project school students were able to give an adequate explanation.

Discussion

In this study, we presented the data collected on a task design to determine students' ability to reason about a situation involving 3D objects. Given the scarcity of research conducted in this area among Australian students, the data should provide valuable information to shape instructional design and future research direction.

With regards to whether the students can comprehend the concept of left and right, it appears that this was not an issue for most students, although 3% of the students reversed the objects on the left and right demonstrating some confusion. They were spread across year levels. Students' attempts at naming the objects revealed a lack of knowledge of the correct geometric terms as less than half of the cohort were able to correctly name at least one object. Spelling was also an issue together with a confusion between prisms, pyramids, cylinders and cones and between hexagon, pentagon and octagon. Many students showed a willingness to create new terms using a mixture of words they knew (rectangular square).

The spatial skills students used to assist them in drawing what the dog saw included: mental rotation, physical rotation of page, mental reflection, perspective drawing, recognition of position and depth, and a 2D perspective of 3D object. These showed in both the student drawing and their explanations to justify their actions. While the literature refers to mental rotation (see Yilmaz, 2009 for a review on this), it is of interest here that many students in their explanation actually use other strategies including physical rotation to answer a mental rotation question. Research in this area has often used multiple choice type tasks to assess spatial skills such as mental rotation. Yet it is only in situations where students are asked to explain their actions that one can fully appreciate their use of spatial skills. However, we acknowledge that verbal explanation does not always match action as we found with some students in this study. Combining both the drawing and the explanation allows students' spatial skills to be better understood.

The data show that students in the project schools were far more successful in completing the drawing from the dog's perspective (over 50% compared to under 35%). When asked to explain their reasoning, a large number of students did not respond (37% overall) or gave very superficial reasoning (46% in trial schools and 30% in project schools). Only 2% (trial schools) and 17% (project schools) of students gave complete reasoning. This suggest that classroom experiences in handling 3D objects and reasoning about them can assist in contributing to students' development of spatial skills. Similarly, a classroom culture where explanation and reasoning are required contributed to students developing reasoning abilities. This is not a sociocultural artefact since the project schools were from low socioeconomic backgrounds whereas the trial schools were from a diverse range of economic areas.

There is much still to be learned about visualisation and its impact on spatial reasoning. Clearly, more research is needed in this area of spatial skills and reasoning with specific focus on the impact of classroom experiences on learning, teacher knowledge, and the contribution made by the classroom discourse and the culture of the classroom and how these contribute to the development of visualisation and spatial reasoning.

References

Casey, B. M., Lombardi, C. M., Pollock, A., Fineman, B., & Pezaris, E. (2017). Girls' spatial skills and arithmetic strategies in first grade as predictors of fifth-grade analytical math reasoning. *Journal of Cognition and Development, 18*(5), 530-555. doi:10.1080/15248372.2017.1363044

Kozhevnikov, M., Kosslyn, S., & Shephard, J. (2005). Spatial versus object visualizers: A new characterization of visual cognitive style. *Memory & Cognition, 33*(4), 710-726. doi:10.3758/BF03195337

Li, R. (2014). Why women see differently from the way men see? A review of sex differences in cognition and sports. *Journal of Sport and Health Science, 3*(3), 155-162. doi:10.1016/j.jshs.2014.03.012

Lieberman, J. (2009). Using lesson study to develop an appreciation of and competence in task design. In B. Clarke, B. Grevholm, & R. Millman (Eds.), *Tasks in primary mathematics teacher education: Purpose, use and exemplars. Mathematics teacher education 4*. New York: Springer Science+Business Media, LLC.

McGee, M. G. (1979). Human spatial abilities: Psychometric studies and environmental, genetic, hormonal, and neurological influences. *Psychological Bulletin, 86*(5), 889-918. doi:10.1037/0033-2909.86.5.889

Newcombe, N. S., & Stieff, M. (2011). Six myths about spatial thinking. *International Journal of Science Education, 34*(6), 955-971. doi:10.1080/09500693.2011.588728

Phillips, L. M., Norris, S. P., & Macnab, J. S. (2010). *Visualization in matheamtics, reading and science education*. Dordrecht: Springer.

Ramful, A., Lowrie, T., & Logan, T. (2017). Measurement of spatial ability: Construction and validation of the spatial reasoning instrument for middle school students. *Journal of Psychoeducational Assessment, 35*(7), 709-727. doi:10.1177/0734282916659207

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses and mathematizing*. Cambridge: Cambridge University Press.

Siemon, D., Horne, M., Clements, D. H., Confrey, J., Maloney, A., Sarama, J., . . . Watson, A. (2017). Researching and using learning progressions (trajectories) in mathematics education. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 109-136). Singapore: PME.

Sima, J. F., Schultheis, H., & Barkowsky, T. (2013). Differences between spatial and visual mental representations. *Frontiers in Psychology, 4*. doi:10.3389/fpsyg.2013.00240

Thomson, S., Wernert, N., O'Grady, E., & Rodrigues, S. (2017). *TIMSS 2015: Reporting Australia's results*. Melbourne: Australian Council for Educational Research.

Verdine, B. N., Golinkoff, R. M., Hirsh-Pasek, K., & Newcombe, N. S. (2017). Spatial skills, their development, and their links to mathematics *Monographs of the Society for Research in Child Development, 82*(1), 7-30. doi:10.1111/mono.12280

Wai, J., Lubinski, D., & Benbow, C. P. (2009). Spatial ability for STEM domains: Aligning over 50 years of cumulative psychological knowledge solidifies its importance. *Journal of Educational Psychology, 101*(4), 817-835. doi:10.1037/a0016127

Yilmaz, H. B. (2009). On the development and measurement of spatial ability. *International Electronic Journal of Elementary Education, 1*(2), 83-96.