Sense-Making in Mathematics: Towards a Dialogical Framing

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This paper presents a new theoretical viewpoint blended from the perspectives that mathematical meaning is extracted (from objects falling under a particular concept) and that mathematical meaning is given (to objects that an individual interacts with). It is elaborated that neither uni-directional framing (whether involving extracting meaning or giving meaning) provides a comprehensive account of the complex emergence of evolving forms of meaning. It is argued for a framing that construes sense-making in mathematics as dialogical: where what meaning one extracts is a function of what meaning is given to, and vice versa.

Sense-making in mathematics has been a critical theme in research on mathematics knowing, learning, and teaching. Schoenfeld (1992), for instance, discussed mathematics as an act of sense-making and underlined sense-making activities as vital for students coming to understand and use mathematics in meaningful ways. Von Glasersfeld (1995), on the other hand, regarded students as active sense-makers in mathematical concept formation, that is, students actively seek comprehensibility of a mathematical concept. Though consideration of sense-making in mathematics has a long tradition in, and is undoubtedly an essential topic of, mathematics education, the notion of sense-making is somewhat ambiguous, often framed in opposing perspectives. Two of those perspectives are the substance of this paper that are grounded in a division of two strands of mathematical concept formation (i.e., abstraction-from-actions approaches and abstraction-from-objects approaches). Recently, Scheiner (2016) moved the discussion from simple comparison towards a synergy of theoretical frameworks that acknowledges the complementarity of the two strands of mathematical concept formation. Specifically, Scheiner (2016) blended theoretical frameworks on two fundamental kinds of abstraction (reflective abstraction and structural abstraction) and their respective forms of sense-making (extracting meaning and giving meaning). This blending argues strongly against dismissing abstraction from objects as irrelevant for mathematical concept formation, and instead aims to overcome misleading dichotomies of abstraction from actions and abstraction from objects, as Piaget (1977/2001) put forth.

This paper contributes to the current conversation of the relation between extracting meaning and giving meaning. The paper makes a case for a dialogical framing of these two forms of sense-making that has the potential to account for the complex dynamics involved in mathematical concept formation, dynamics which cannot be accounted for considering extracting meaning and giving meaning separately. In doing so, some theoretical assertions are outlined that orient the general discussion of concept formation and sense-making. Afterwards, explicit and implicit assumptions underlying the respective forms of sense-making are examined. Then, the dialogical framing of extracting meaning and giving meaning is delineated, revealing the complex dynamics involved in mathematical concept formation.

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Theoretical Orientations

The theoretical foundation for coordinating the two strands of mathematical concept formation, as presented in Scheiner (2016), relies on and projects several theoretical insights revealed by Frege (1892a, 1892b) that have informed a variety of theoretical perspectives on mathematical knowing, thinking, and learning (see Arzarello, Bazzini, & Chiappini, 2001; Duval, 2006; Radford, 2002). In particular, the theoretical foundation in Scheiner (2016) shares Frege's (1892a) assertion that a mathematical concept is not directly accessible through the concept itself but only through objects that act as proxies for it. However, mathematical objects (unlike objects of natural sciences) cannot be apprehended by human senses (we cannot, for instance, 'see' the object), but only via some 'mode of presentation' (Frege, 1892b) – that is, objects need to be expressed by using signs or other semiotic means such as a gestures, pictures, or linguistic expression (Radford, 2002).

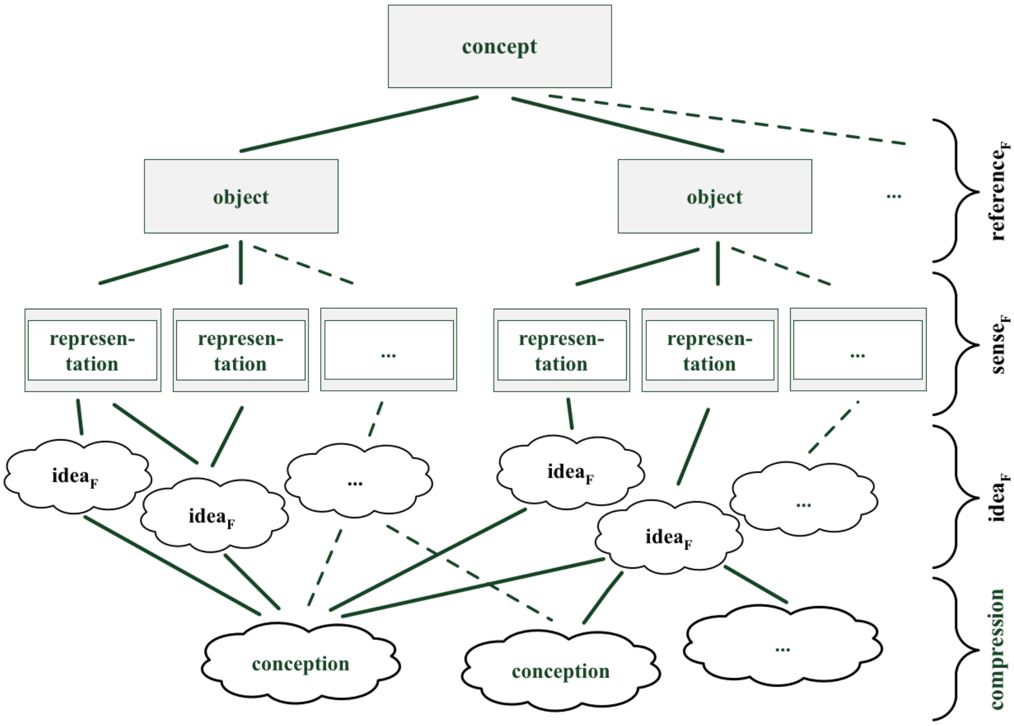

Figure 5. On $reference_F$, $sense_F$, and $idea_F$ (reproduced from Scheiner, 2016, p. 179).

The 'mode of presentation' (or 'way of presentation') of an object is to be distinguished from the object that is represented, as individuals often confuse a $sense_F$ ('Sinn') of an expression (or representation) with the $reference_F$ ('Bedeutung') of an expression (or representation) (the subscript F indicates that these terms refer to Frege, 1892b). The $reference_F$ of an expression is the object it refers to, whereas the $sense_F$ is the way in which the object is given to the mind, or in other words, it is the thought ('Gedanke') expressed by the expression (or representation) (Frege, 1892b). The expression '$a = b$', for instance, is informative, in contrast to the expression '$a = a$', as the $sense_F$ of 'a' differs from the $sense_F$ of 'b'. The upshot of this is, $senses_F$ capture the epistemological and cognitive significance of expressions. This implies one of Frege's decisive assertions, that an object can only be apprehended via a $sense_F$ of an expression (or representation): the $sense_F$ orients how a person thinks of the object being referred to. Thus, it seems reasonable to understand Frege's (1892b) notion of an idea$_F$ ('Vorstellung') as the manner in which a person makes sense$_F$ of the world. Ideas$_F$ can interact with each other and form more compressed knowledge structures, called conceptions. A general outline of this view is provided in Figure 1.

There are several ways that individuals can make sense of a mathematical concept; the focus here is on Pinto's (1998) distinction between extracting meaning and giving meaning with respect to sense-making of a formal concept definition: "Extracting meaning involves working within the content, routinizing it, using it, and building its meaning as a formal construct. Giving meaning means taking one's personal concept imagery as a starting point to build new knowledge." (Pinto, 1998, pp. 298-299)

Gray, Pinto, Pitta, and Tall (1999) stated that in giving meaning a person attempts to build from their own perspective, trying to give meaning to mathematics from current cognitive structures. Tall (2013) elucidated that these two approaches are related to a 'natural approach', that builds on the concept image, and a 'formal approach', that builds formal theorems based on the formal definition. Scheiner (2016) linked extracting meaning to the manipulation of objects and reflection on the variations in modes of presentation when objects are manipulated. These cognitive processes are often associated with Piaget's (1977/2001) reflective abstraction, that is, abstraction through coordination of actions on mental objects (e.g., Dubinsky, 1991). Giving meaning, on the other hand, was related to attaching an idea$_F$ to a mode of presentation. That is, an individual gives meaning to the objects one interacts with from the perspective an individual has taken.

On Extracting Meaning

A common assumption is that the meaning of a mathematical concept is an inherent quality of objects that fall under a particular concept, and that this quality is to be extracted. This extraction of meaning is realised through the manipulation of objects and reflection of variations of senses$_F$ when objects are manipulated. These cognitive processes are often associated with reflective abstraction, that is, reflecting on the coordination of actions on mental objects (see Piaget, 1977/2001). Similarly, Duval (2006) argued that via systematic variation of one representation of an object and reflecting on resulting variations in another representation of the same object, an individual can recognise what is mathematically relevant and separate the sense$_F$ of a representation from the reference$_F$ of a representation. Such a view asserts that individuals internalise extracted mathematical structures and relations associated with their actions and reflections of their actions on objects. It gives the impression that individuals construct mental models (ideas$_F$ or conceptions) that correspond to an ideal realm (objects or concepts), though it might be read as taking a 'trivial constructivist' position (von Glasersfeld, 1989): the view that a necessary condition of knowledge is that individuals construct, constitute, make, or produce their own understanding (see Ernest, 2010). More importantly, such a view seems to suggest a 'conception-to-concept direction of fit' (see Scheiner, 2017) that is, mathematical concept formation is regarded as individuals constructing conceptions that best reflect a (seemingly given) mathematical concept (see Figure 2).

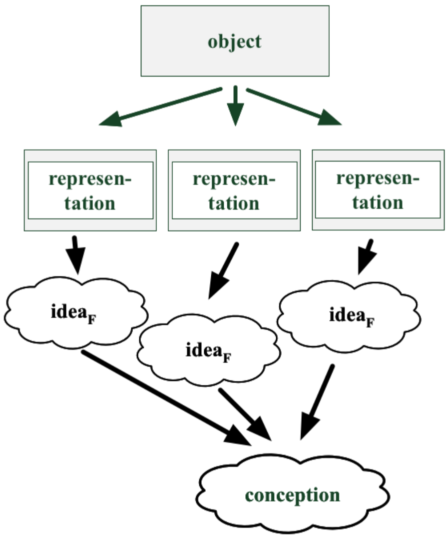

Figure 2. From object to idea$_F$ to conception.

On Giving Meaning

In the attempt to coordinate abstraction-from-actions and abstraction-from-objects approaches, a new understanding of abstraction emerged: abstraction is not so much the extraction of a previously unnoticed meaning of a concept (or the recognition of structure common to various objects), but rather a process of giving meaning to the objects an individual interacts with from the perspective an individual has taken. Abstraction, as such, is more focused on "the richness of the particular [that] is embodied not in the concept as such but rather in the objects that falling under the concept [...]. This view gives primacy to meaningful, richly contextualised forms of (mathematical) structure over formal (mathematical) structures" (Scheiner, 2016, p. 175). This is to say, individuals give meaning to the objects they interact with by attaching ideas$_F$ to objects or, more precisely, by attaching ideas$_F$ to the senses$_F$ expressed by the representations in which an object actualises. Recent research investigating the contextuality, complementarity, and complexity of this sense-making strategy (see Scheiner & Pinto, 2018) asserted that in contrast to Frege (1892b), who construed sense$_F$ in a disembodied fashion as a way an object is given to an individual, an individual assigns sense$_F$ to object. However, what sense$_F$ is assigned to an object is a function of what idea$_F$ is activated in the immediate context. In this view, ideas$_F$ direct forming the modes of presentation under which an individual refers to an object. As such, it is a person's complex system of ideas$_F$ that directs forming a sense$_F$, rather than merely the object a representation refers to. This research also indicated that individuals might even give meaning to objects that are yet to become. This means that although an object does not have a being prior to the individual's attempts to know it, an individual might create a new idea$_F$ that directs their thinking to potential objects, or more precisely: an individual might create an idea$_F$ that allows assigning a new sense$_F$ to objects that are yet to become. That is, individuals might ascribe meaning beyond what is apparent. It is proposed that the creation of such ideas$_F$ is of the nature of what Koestler (1964) described as 'bisociation', and Fauconnier and Turner (2002) elaborated as 'conceptual blending'; that is, to construct a partial match between frames from established conceptual domains, in order to project selectively from those domains into a novel hybrid frame, comprised of unique (emergent) structure of its own (see Figure 3).

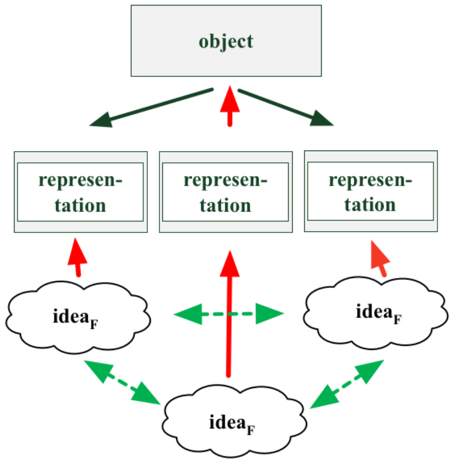

Figure 3. Transforming ideas$_F$ to give (new) meaning to an object.

The key insight here is that unrelated ideas$_F$ can be transformed into new ideas$_F$ that allow 'setting the mind' not only to actual instances, but also to potential instances that might become 'reality' in the future. In such cases, conceptual development is not merely meant to reflect an actual concept, but rather to create a concept: a view that suggests a 'concept-to-conception direction of fit' (see Scheiner, 2017) that is, mathematical concept formation is regarded as individuals creating a concept that best fits their conceptions. Similarly, Lakoff and Johnson (1980), drew attention to the power of (new) metaphors to create a (new) reality rather than simply to provide a way of conceptualising a pre-existing reality: "changes in our conceptual system do change what is real for us and affect how we perceive the world and act upon those perceptions" (pp. 145-146.). It is reasonable to assume that students transform ideas$_F$ to express a yet-to-be-realised state of a concept. This accentuates Tall's (2013) assertion that the "whole development of mathematical thinking is presented as a combination of compression and blending of knowledge structures to produce crystalline concepts that can lead to imaginative new ways of thinking mathematically in new contexts" (p. 28).

Towards a Dialogical Framing

Each of the previous two sections articulated a particular form of sense-making: extracting meaning from objects (via manipulating objects and reflecting on the variations) and giving meaning to objects (via attaching existing and new ideas$_F$ to objects). These two forms of sense-making seem to differ in their directions of fit: extracting meaning involves individuals' attempts to construct conceptions that aim to fit a concept (conception-to-concept direction of fit), whereas giving meaning involves individuals' attempts to create a concept that aims to fit their conceptions (concept-to-conception direction of fit) (for a detailed discussion, see Scheiner, 2017).

Instead of construing extracting meaning and giving meaning as independent processes that point in two opposing directions, it is argued here for a bi-directional theoretical framing of mathematical concept formation. Specifically, it is argued for a dialogical framing of extracting meaning and giving meaning, asserting that extracting meaning and giving meaning are interdependent (rather than independent): what meaning one extracts is very much a function of what meaning is given to, and vice versa (see Figure 4). This dialogical framing can better account for the complex emergence of evolving forms of meaning:

meaning not only emerges (from Latin emergere, 'to become visible') via reflection on manipulations of objects, but also evolves (from Latin evolvere, 'to become more complex') via transforming previously constructed ideas$_F$ (see Scheiner, 2017).

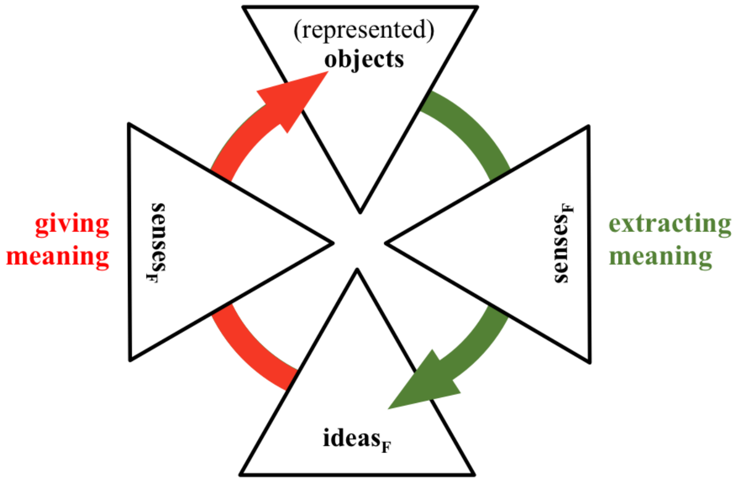

Figure 4. On the dialogue of extracting meaning and ascribing meaning.

The dialogical framing of extracting meaning and giving meaning acknowledges the complex emergence of evolving forms of meaning that cannot be accounted for by viewing extracting meaning or giving meaning as separate. Extracting meaning and giving meaning, though they have value in their own right, are restricted, and restricting, in their accounts of mathematical concept formation. This is due to the 'hidden determinisms' inherent in the two approaches: extracting meaning assumes that what dictates meaning is the concept itself; while giving meaning advocates an individual's conceptions as the determinants of all meaning. The dialogical framing, in contrast, is not deterministic but bi-directional: mathematical concept formation involves processes that direct from conception to concept as much as it involves processes that direct from concept to conception. As such, the dialogical framing is more than a matter of recasting the concept-conception divide: it underlines that concept and conception are not static and apart but fluid and co-specifying.

Figure 5 is an alternative to the reductionist view taken in respective approaches of extracting meaning (see Figure 2) and giving meaning (see Figure 3), both being rather uni-directional and deterministic in orientation. The dialogical framing provides new interpretative possibilities regarding the complex dynamics in mathematical concept formation, allowing for a move beyond simplistic assertions about linearity and determinism (that were transposed from analytical science and analytical philosophy onto discussions of mathematical concept formation). Figure 5 attends to the complexity in mathematical concept formation and speaks to the nonlinear, emergent characters of evolving forms of mathematical meaning (see Pirie & Kieren, 1994; Schoenfeld, Smith, & Arcavi, 1993).

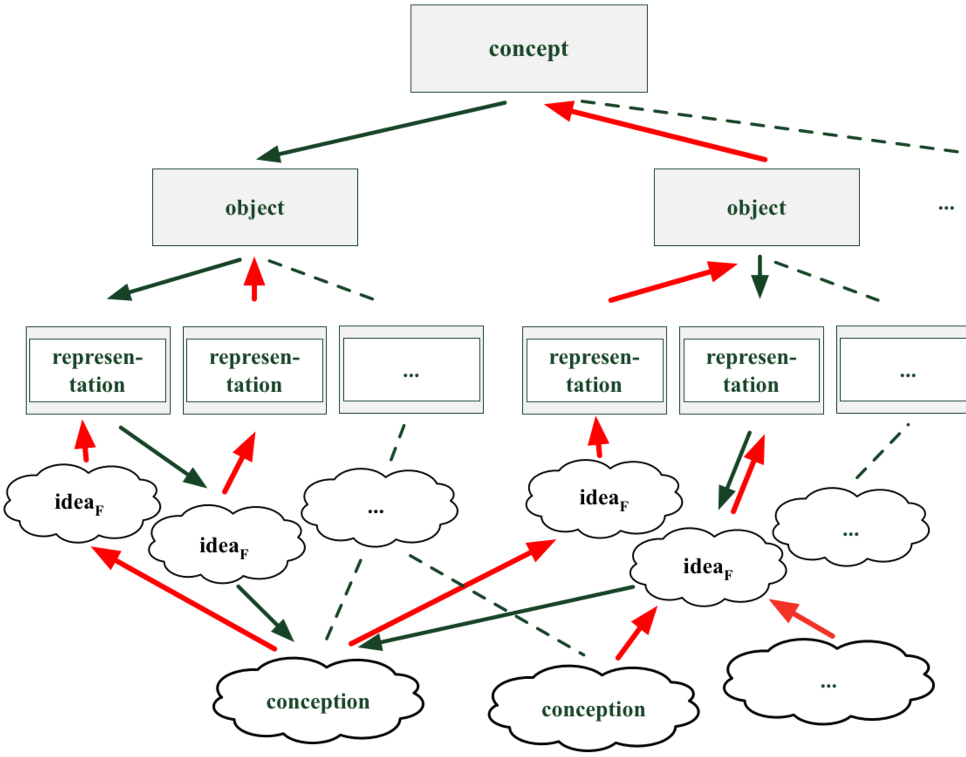

Figure 5. A complexivist frame: on the complex interaction between concept and conception.

Conclusion

This paper presents a new theoretical perspective blended from the existing perspectives that mathematical meaning is extracted (from objects falling under a particular concept) and that mathematical meaning is given (to objects that an individual interacts with by that individual). This blending seeks to frame mathematical concept formation as bi-directional (where what meaning one extracts is a function of what meaning is given to, and vice versa) and to recast the concept-conception divide (by viewing concept and conception as fluid and co-specifying instead of static and apart). In doing so, the dialogical framing presents a view of mathematical concept formation that is complex, dynamic, non-linear, and possessed of emergent characteristics.

This theoretical contribution makes the case that neither uni-directional framing of mathematical concept formation (whether involving extracting meaning or giving meaning) provides a comprehensive account of the complex emergence of evolving forms of meaning. It is argued for an alternative framing that acknowledges mathematical concept formation as both directed from concept to conception and from conception to concept. Mathematical concept formation, then, is construed as an ongoing, intertwined process of extracting meaning and giving meaning, in which conceptions shape, and are shaped by, the concepts with which an individual interacts.

This dialogical framing brings a greater insight: that any attempt to frame cognition in terms of mind over matter or matter over mind is misleading, as cognition is bi-directional: from the outside in (mind- to-world direction of fit) and from the inside out (world-to-mind direction of fit). That is, mind and world are engaged in a co-creative interaction: mind is shaped by the world and mind shapes the world. Such a world is subjectively articulated, in that its objectivity is relative to how it has been shaped by the knower (see Reason, 1998).

Such a dialogical framing is not so much a unification of any monism (that sees, for instance, mind as situated within its world), nor of any dualism (that sees mind apart from the world), but rather is an acknowledgment that mind is an integral part of the world, and as such both mind and world are in a constant state of flux, changing in the ever-unfolding process of extracting meaning and giving meaning.

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