Preparing for the Final Examination in Abstract Algebra: Student Perspectives and Modus Operandi

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This study focuses on the undergraduate mathematics students' perceptions and applied techniques for the preparation for their final examination in Abstract Algebra. The results of this study suggest that the revision for the final examination involves, firstly, the review of the lecture notes, followed by the solution of the coursework together with the use of model solutions and the solution of the past papers. The order of the last two activities varies. An often-occurring revision technique involves, instead of a linear succession of the aforementioned activities, a 3-dimensional spiral approach towards revision, with the three activities interchanging until the students who apply it feel that they have achieved adequate object-level and metalevel learning.

Many studies have reported on undergraduate mathematics students' difficulty with Abstract Algebra (Ioannou, 2012). It "is the first course in which students must go beyond 'imitative behavior patterns' for mimicking the solution of a large number of variations on a small number of themes" (Dubinsky et al., 1994, p268). A typical first Abstract Algebra course requires deep understanding of the abstract notions involved, as well as the application of techniques in the preparation of coursework and final examination. An important element that causes students' difficulty with Abstract Algebra is its 'abstract' nature (Hazzan, 1999). The deductive way of teaching Abstract Algebra is unfamiliar to students and, in order to achieve mastery of the subject, it is necessary to "think selectively about its entities, paying attention to those aspects consistent with the context and ignoring those that are irrelevant" (Barbeau, 1995, p140). In addition, Gueudet (2008) suggests that many pedagogical issues emerging in undergraduate Mathematics Education are based on the transition from secondary to tertiary Mathematics, which can still occur in their second year. In fact, student difficulties in Abstract Algebra may be an indication of problematic transition, mainly due to the particular nature of this course (Ioannou, 2012). The aim of this study is to investigate the student perspectives and applied study skills for the preparation for the final examination, an essential part of their learning process and assessment. For the purposes of this study, I will use the Commognitive Theoretical Framework (CTF) (Sfard, 2008).

Literature Review

Research in the learning of Abstract Algebra (Theory of groups and rings) is relatively scarce compared to other university Mathematics fields. Even more limited is the commognitive analysis of conceptual and learning issues (Nardi et al., 2014). The first reports on the learning of Abstract Algebra appeared in the early 1990's. Several studies, following mostly a constructivist approach, and within the Piagetian tradition of studying the cognitive processes, examined students' cognitive development and analysed the emerging difficulties in the process of learning certain group-theoretic notions.

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Furthermore, the construction of the newly introduced abstract algebraic notions is often an arduous task for novice students and causes serious difficulties in the transition from the informal secondary education mathematics to the formalism of undergraduate mathematics (Nardi, 2000). Students' difficulty with the construction of these concepts is partly grounded on historical and epistemological factors: "the problems from which these concepts arose in an essential manner are not accessible to students who are beginning to study (expected to understand) the concepts today" (Robert & Schwarzenberger, 1991).

Nowadays, the presentation of the 'fundamental concepts' of Group Theory, namely group, subgroup, coset, quotient group, etc. is "historically decontextualized" (Nardi, 2000, p169), since historically the fundamental concepts of Group Theory were permutation and symmetry. Moreover, this chasm of ontological and historical development proves to be of significant importance in the learning of Abstract Algebra for novice students. From a more participationist perspective, CTF can prove an appropriate and valuable tool in our understanding of the learning of Abstract Algebra due both to its ontological characteristics, as well as its epistemological tenets.

Research suggests that students' understanding of the notion of group proves often primitive at the beginning, predominantly based on their notion of a set. Students often have the tendency to consider group as a 'special set', ignoring the role of binary operation. Iannone and Nardi (2002) suggest that this conceptualisation of group has two implications: the students' occasional disregard for checking associativity and their neglect of the inner structure of a group. An often-occurring confusion amongst novice students is related to the order of the group G and the order of its element g. This is partly based on student inexperience, their problematic perception of the symbolisation used and of the group operation. The use of semantic abbreviations and symbolisation can be particularly problematic at the beginning of their study. Nardi (2000) suggests that there are both linguistic and conceptual interpretations of students' difficulty with the notion of order of an element of the group. The role of symbolisation is particularly important in the learning of Abstract Algebra, and problematic conception of the symbols used probably causes confusion in other instances. In addition, an important means for coping with the level of abstraction in the context of Abstract Algebra is the use of visual images. In fact, their use plays a significant role, since they serve as a meaning-bestowing tool (Ioannou & Nardi, 2009a).

Theoretical Framework

CTF is a coherent and rigorous theory for thinking about thinking, grounded in classical Discourse Analysis. It involves a number of different constructs such as metaphor, thinking, communication, and commognition, as a result of the link between interpersonal communication and cognitive processes (Sfard, 2008). In mathematical discourse, objects are discursive constructs and form part of the discourse. Mathematics is an autopoietic system of discourse, i.e. "a system that contains the objects of talk along with the talk itself and that grows incessantly 'from inside' when new objects are added one after another" (Sfard, 2008, p129). Moreover, CTF defines discursive characteristics of mathematics as the word use, visual mediators, narratives, and routines with their associated metarules, namely the how and the when of the routine. In addition, it involves the various objects of mathematical discourse such as the signifiers, realisation trees, realisations, primary objects and discursive objects. It also involves the constructs of object-level and metadiscursive level (or metalevel) rules. Thinking "is an individualised version of (interpersonal) communicating" (Sfard, 2008, p81). Contrary to the acquisitionist approaches, participationists' ontological tenets propose to consider thinking as an act (not necessarily interpersonal) of communication, rather than a step primary to communication (Nardi et al. 2014).

Mathematical discourse involves certain objects of different categories and characteristics. Primary object (p-object) is defined as "any perceptually accessible entity existing independently of human discourses, and this includes the things we can see and touch (material objects, pictures) as well as those that can only be heard (sounds)" (Sfard, 2008, p169). Simple discursive objects (simple d-objects) "arise in the process of proper naming (baptizing): assigning a noun or other noun-like symbolic artefact to a specific primary object. In this process, a pair <noun or pronoun, specific primary object> is created. The first element of the pair, the signifier, can now be used in communication about the other object in the pair, which counts as the signifier's only realization. Compound discursive objects (d-objects) arise by "according a noun or pronoun to extant objects, either discursive or primary." In the context of this study, groups are an example of compound d-objects. The (discursive) object signified by S in a given discourse is defined as "the realization tree of S within this discourse." (Sfard, 2008, p166). The realization tree is a "hierarchically organized set of all the realizations of the given signifier, together with the realizations of these realizations, as well as the realizations of these latter realizations and so forth" (Sfard, 2008, p300).

Sfard (2008) describes two distinct categories of learning, namely the object-level and the metalevel discourse learning. "Object-level learning […] expresses itself in the expansion of the existing discourse attained through extending a vocabulary, constructing new routines, and producing new endorsed narratives; this learning, therefore results in endogenous expansion of the discourse" (Sfard, 2008, p253). In addition, "metalevel learning, which involves changes in metarules of the discourse […] is usually related to exogenous change in discourse. This change means that some familiar tasks, such as, say, defining a word or identifying geometric figures, will now be done in a different, unfamiliar way and that certain familiar words will change their uses" (Sfard, 2008, p254).

Methodology

This study is part of a larger research project, which conducted a close examination of Year 2 mathematics students' conceptual difficulties and the emerging learning and communicational aspects in their first encounter with Abstract Algebra. The module was taught in a research-intensive mathematics department in the United Kingdom, in the spring semester of a recent academic year. This module was mandatory for Year 2 mathematics undergraduate students, and a total of 78 students attended it. The module was spread over 10 weeks, with 20 one-hour lectures and three cycles of seminars in weeks 3, 6 and 10 of the semester. The role of the seminars was mainly to support the students with their coursework. There were 4 seminar groups, and the sessions were each facilitated by a seminar leader, a full-time faculty member of the school, and a seminar assistant, who was a doctorate student in the mathematics department. The module assessment was predominantly exam-based (80%). In addition, the students had to hand in a threefold piece of coursework (20%) by the end of the semester.

The gathered data includes the following: Lecture observation field notes, lecture notes (notes of the lecturer as given on the blackboard), audio-recordings of the 20 lectures, audio-recordings of the 21 seminars, 39 student interviews (13 volunteers who gave 3 interviews each), 15 members of staff's interviews (5 members of staff, namely the lecturer, two seminar leaders and two seminar assistants, who gave 3 interviews each), student coursework, markers' comments on student coursework, and student examination scripts. For the purposes of this study, the collected data of the 13 volunteers has been scrutinised. Naturally all sources of data have been appropriately analysed, and the conclusions of the data analysis have been triangulated.

Finally, all emerging ethical issues during the data collection and analysis, namely, issues of power, equal opportunities for participation, right to withdraw, procedures of complain, confidentiality, anonymity, participant consent, sensitive issues in interviews, etc., have been addressed accordingly.

Data Analysis

Data analysis suggests that for the revision for the final examination, twelve of the thirteen students (12/13) study the lecture notes, solve the coursework using the model solutions given by the lecturer and solve a various number of past papers (Student O is the exception). Preparation for the final examination, and the final examination per se, is the final stage in the students' learning process and at this stage students are invited to resolve any commognitive conflicts[3]. As the following excerpt suggests, usually, the first step for revising is the *study of the lecture notes*. Students' approaches vary, but their predominant aim is to go through the definitions and theorems, both to improve their object-level learning but also to memorise the ones that will possibly be asked to state. In addition, five of the thirteen students (5/13) students produce their own revision notes, which help them to improve their object-level learning and assist them in memorizing easier.

> I normally write out my notes, a lot... Hmm, yeah like I make revision notes, and I do revision cards. And I normally just sit and rewrite out the definitions a million times and the theorems a million times, and just like – do the revision cards and get people to test me and I'll write them down, and then I'll work through past papers and all the problem sheets. Student A

Seven of the thirteen (7/13) students study their lecture notes without producing revision notes. This is usually the first step for their revision. Studying the lecture notes for the final exam requires a different, all-inclusive, approach from the preparation of the coursework. Studying the lecture notes for the exam is a 'renewed task' leading to improved learning of the theory. As the following excerpt suggests, having a holistic picture of the entire theory, and consequently having already, up to a certain extent, created realizations of the involved d-objects and realization trees, makes the task of revision and objectification a different experience.

> Usually, like the coursework… we start from the lecture notes…and usually I am trying to understand everything… not like when we prepare a coursework. For the coursework we do not have much time so we are going for the exercises… I believe that if you do not understand something, then you cannot understand what it follows as well… In the past, I used to make my own notes, but since it was time consuming, I decided to stop that… I study the notes and I highlight the important things… Something that I need to see again… I study only from the notes… Student B

The above excerpt is a representative example of all thirteen students' awareness regarding the different approach that should follow for the examination revision. Student B expresses her desire to change her study approach and wishes to improve her learning. She identifies that solving a mathematical task without studying the related narratives and routines is a faulty approach. For her, studying the lecture notes as part of the final revision is a task that has to be faced anew. Experience has led her to prioritise efficiency in her

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[3] Commognitive conflict is defined as a "situation that arises when communication occurs across incommensurable discourses". (Sfard, 2008, p 296)

study skills and approaches, as well as the awareness of the demands of examination revision.

The next step in the twelve of the thirteen (12/13) students' revision is usually the *solution of coursework and past exam papers*. There are two distinct categories of students regarding which task is undertaken first: six of the thirteen (6/13) students are studying the coursework first and six of the thirteen (6/13) students start with past exam papers. Studying the coursework first, together with the given model solution, is an important step in the learning process. As the following excerpt suggests, this revision approach allows students to have the chance to exactly locate their weakness and improve their object-level learning of the definitions of certain d-objects. Consequently, this process will allow them to successfully cope with the level of abstraction, improve the structure of the realization trees of these d-objects and objectify them, something that it will permit them to enhance their metalevel learning.

> Um, probably with the questions that we've been given, and with the solutions, I'm hoping to like – help teach myself how to do it… and then I learn by doing past exam papers, mainly, […] I tend to do like quite a few years back, like do all of them, and once I've done them, go back, and like the questions that I […] wasn't able to do before, I try and do it again, cos I've hoped that I've taught myself. Student C

Student C considers working with the coursework and the model solution as a means to 'teach herself' the *how* and *when* of the routines involved. It is a chance to correct and/or improve her object-level and metalevel learning, application of metarules and solving techniques, and consequently overcome any knowledge gaps resulted in the learning of this new mathematical discourse. Using the solutions, the particular students will be able to observe the metalevel rules of Abstract Algebra in practice and learn how they should be applied. For these students, model solutions are apparently an indispensable tool that can be used in order to resolve any preexisting commognitive conflicts and improve the realization trees. These students will possibly have the chance to realize not only the metadiscursive level rules, but it will also allow them to understand how they should approach a mathematical task in general, namely, specifying the routine prompts[4], applying the decided course of action, and successfully completing the task.

Another benefit from working with the model solutions while revising the coursework exercises is the improvement of self-confidence. Although only Student D expressed so overtly this perception, it is important to be highlighted, since other students have implied it as well. When this task is completed, he then works with the past papers.

> I don't generally look at the exam papers… only slightly towards the end – only because they can freak you out if you – I like spending a few days building up your confidence just reading through lectures notes and that sort of thing – examples of the course sheets I like looking through them for a while then go... […] You need confidence. If I have confidence I am quite good. I can actually generally breeze through even if I don't actually know the answers entirely. Student D

Another revision technique is by studying the past papers first and then the coursework. Student E is planning to start by working with the past papers, identifying the demands of the examination as well as his weaknesses.

> Um, well I'll definitely be looking at past papers. […] Then go to lecturers and just get feedback on what I've done, and then they'll help me say like oh no don't do this, or yeah, you're doing all right in this bit. So any kind of gaps in my knowledge hopefully they'll – help fill in. […] I kind of look at what would come up on the exam […] have a little look at lecture notes, maybe a few problem

[4] Elements of situations whose presence increases the possibility of the routine's performance.

> sheets, then maybe get and attempt another one, with a bit more knowledge. Actually I do one, that I kind of do with my lecture notes open really, then try – as I'm getting a little bit better, try and do it without the lecture notes, cos obviously that's gonna be what's happening in the exam. Student E

This technique allows the students to identify the difficulties they will possibly face in their examination, identify the expected types of questions they will probably need to solve and therefore adjust their revision in order to overcome the new demands and revise the appropriate mathematical routines. Student E is the only student that is overtly willing to ask assistance from the lecturer in solving past papers.

Regarding the overall process of revision for the final examination, five of the thirteen (5/13) students have clearly stated the interchange of the three activities, namely studying the lecture notes, reworking the coursework and solving the past papers. This approach can be described as a *3-dimensional spiral approach towards revising*. Students that follow this revision approach, work with the lecture notes, coursework and past papers in an interchangeable way until: first, they have overcome any commognitive conflicts caused by the nature of Abstract Algebra, and second, they have improved their metalevel learning. In each spiral cycle of revision their level of comprehension improves.

Unlike Student E, Student F does not require any assistance from the lecturer but instead he is marking his solutions of the past papers by himself.

> And then start past exam papers, and get the solution and see what they're looking for in the exam questions… So do a few of that, and do a proper exam conditions, and… […] Mark it – no – I done it first, mark it, and then look at the marks scheme, yeah, mark myself and see how much I get? And then, after a few days, redo the paper again, to see how much I improve, or which area I still don't understand or something. Student F

Student F's approach towards revision has some very useful and interesting elements, such as the solution of past papers under exam conditions, or the repetitive, spiral like, approach already encountered in Student E's case. His active approach to revision is manifold with repetitive cycles that possibly allow him to better objectify the material and enrich his experience. Relying solely on his marking, though, without asking for any external control might jeopardise his learning. Student F's examination results (50%) do not show that his revision scheme has led to the expected outcome.

The weakest students, Students G and H, expressed a similar perspective regarding what makes a good examiner, according to which good examiner is the one whose papers are the same every year: *Last year we had a very good lecturer… his papers were exactly the same every year, but with different numbers…* Student G. This statement suggests that these students adopt a 'utilitarian' perspective of learning Mathematics at the university level. This statement, as well as their performance, suggests a difficulty in the transition from secondary education towards university Mathematics education and its demands. It indicates that their mathematical thinking is, according to Sierpinska (2000), only practical, based on *prototypical* examples that they need to see in the past papers, and not *theoretical*.

Finally, Student O is the only student who does not revise by using the three elements of revision, namely the revisit of lecture notes, the solution of coursework making use of the model solutions, and the solution of past papers. He rather uses only the first two. In particular, Student O makes use of books, in parallel with the lecture notes and coursework, and on many occasions places special emphasis on the way he reads the books and the relaxed pace of his reading.

> It's a matter of revising what you have done and revising your coursework answers, going through books just being relaxed. I am sure most other people would have a different approach would be do past questions, but I prefer to be more relaxed more laid back about it. […] I don't want to go through

it at top speed, just go through it normally. Hopefully it will sink in. But then I read it and then close the book and try to reproduce what they have... Student O

Student O's perception is quite distinct, indicating his effort to, not only, approach revision in a superficial way and get a good mark, but rather as a chance to improve his object-level and metalevel learning in the discourse of Abstract Algebra. He considers exams as an opportunity to widen his object-level and metalevel skills, overcome any possible commognitive conflicts that occurred in the coursework and hopefully achieve endogenous discursive expansion. The last is encapsulated in his phrase "sink in".

The above excerpt indicates maturity in his way of reading a mathematical text. Student O has realised that reading a mathematical text is fruitful only if the pace of reading is not fast, but rather compatible with the difficulty of the test and the speed in which an individual grasps the various aspects of the discourse. His approach is overall mature, indicating successful transition towards university Mathematics and its norms.

Conclusion

The above analysis suggests that the revision for the final examination is a 'renewed contract' for mathematical learning, during which students need to develop and/or apply certain techniques. They are invited to revisit what they have been taught, localise the conceptual gaps and overcome the remaining misconceptions. The majority of students adopt a similar approach towards revision for the final examination. Usually the revision process initiates by revisiting the lecture notes. The predominant aim is to engage again, after having acquired more experience, with the various mathematical narratives, namely definitions, theorems, lemmas and proofs, both to improve their object-level learning and memorise the ones that are most likely to appear in the examination paper. The second step of revision is either to the study of the coursework questions in parallel to the given model solutions or attempt to solve past papers. Regarding the solution of coursework using the model solutions, the discussion above indicates that for many students it is an important step in their learning process. Students have the opportunity to compare their solutions with the model solutions and precisely localise their errors. This will enable them to resolve any misconceptions related to these errors, by improving their object-level learning regarding the involved d-objects and will also help them to resolve problems with the governing metalevel rules and, more generally, with proof production. This process requires autodidactical skills (self-teaching) that will enable them to teach themselves, among other things, the *how* and the *when* of the involved routines, and to correct and/or improve their learning and solving techniques. Regarding the solution of past papers, many students at this stage try to specifically identify the definitions, theorems and proofs that are likely to be included in the examination paper, to pinpoint possible mathematical tasks that they may be asked to prove or solve, to extend their experience by solving the past papers as such, and, moreover, to have an opportunity to apply their solving skills, knowledge and understanding to a variety of tasks. The revision process is often nonlinear, and students use the three elements interchangeably until they feel that they have achieved adequate object-level and metalevel learning.

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