Generalising Fraction Structures as a Means for Engaging in Algebraic Thinking

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In this paper, we report on how Year 5 and 6 students (10 to 13 years old) solve reverse fraction problems; that is, where students are required to find the quantity of an unknown whole given a known partial quantity and its equivalent fraction of the unknown whole. To what extent do students' solutions generalise fraction structures that indicate algebraic thinking? Which solution strategies to reverse fraction problems seem to promote generalisation and which appear to hold students back?

The links between fractional knowledge and readiness for algebra have been highlighted by many researchers (e.g., Empson, Levi, & Carpenter, 2011; Jacobs, Franke, Carpenter, Levi, & Battey, 2007; Siegler et al., 2012; Wu, 2001). The current study builds on the research of Lee (2012) and Lee and Hackenberg (2014), who investigated students' quantitative reasoning with fractions and algebraic reasoning in writing and solving equations. Their research showed that fractional knowledge appeared to be closely related to establishing algebra knowledge in the domains of writing and solving linear equations. They concluded: "Teaching fraction and equation writing together can create synergy in developing students' fractional knowledge and algebra ideas" (p. 9).

After analysing the data, Lee (2012) constructed models to determine the fraction schemes used by students and their reasoning about unknowns and writing equations. These two studies focused on whether students were able to execute specific algebraic performances, namely, to write and solve an algebraic equation to represent a multiplicative relationship between two unknown quantities. Our study involves students in the final two years of primary schools, and our research question focuses on identifying indicators of generalised fraction thinking as students attempt to find an unknown whole when presented with one known quantity representing a known fraction of the whole.

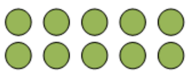
Reverse Fraction Task 1	Reverse Fraction Task 2	Reverse Fraction Task 3
This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.  a. How many counters did I start with? b. Explain how you decided that your answer is correct.	Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? ______ *Show all your working.*	This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with.  a. How many counters did I start with? b. Explain how you decided that your answer is correct.

Figure 1. The three reverse thinking fraction questions.

These three reverse fraction tasks (Figure 1) formed the core of our earlier study (Pearn & Stephens, 2015). Unlike the Lee and Hackenburg (2014) paper, which described an extended interview of one student, our study looks at the performances of whole-class

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groups on these and related tasks, through a written test and selected interviews, and aims to develop criteria to identify the emergence of algebraic thinking and to evaluate students' different solution strategies. To address our research question, we employed an interview protocol using reverse fraction tasks similar to those shown in Figure 1 but with progressive levels of abstraction to capture students' ability to generalise.

The Australian Curriculum Context

According to the rationale given for the Australian Curriculum: Mathematics (ACM; Australian Curriculum, Assessment and Reporting Authority [ACARA], 2016),

> The curriculum focuses on developing increasingly sophisticated and refined mathematical understanding, fluency, reasoning, and problem-solving skills. These proficiencies enable students to respond to familiar and unfamiliar situations by employing mathematical strategies to make informed decisions and solve problems efficiently.

The Australian Curriculum: Mathematics (ACARA, 2016) presents fractions as an important topic across all years with emphasis in Years 5 to 8 on fraction operations. The focus at Year 6 is on finding fractional parts of a known whole and at no stage directs the attention of teachers to finding the whole when given a known fractional part. Year 7 students are expected to solve problems involving addition and subtraction but this appears to exclude multiplicative solutions to fraction problems, especially those involving an unknown whole. Table 1 implies that the link between fractions and algebra is limited to number patterns and sequences involving fractions. However, in Year 7, when students are introduced to the concept of variables, we argue that the bridge between fractional knowledge and algebraic thinking needs to be more explicit. This connects to the synergy emphasised by Lee and Hackenburg (2014) above and forms a major focus of this paper.

Table 1
Content Descriptors from Australian Curriculum: Mathematics (ACARA, 2016)

Year	Fractions and Decimals	Patterns and Algebra
5	Investigate strategies to solve problems involving addition and subtraction of fractions with the same denominator (ACMNA103)	Describe, continue and create patterns with fractions, decimals and whole numbers resulting from addition and subtraction (ACMNA107)
6	Find a simple fraction of a quantity where the result is a whole number, with and without digital technologies (ACMNA127)	Continue and create sequences involving whole numbers, fractions and decimals. Describe the rule used to create the sequence (ACMNA133)
7	Solve problems involving addition and subtraction of fractions, including those with unrelated denominators (ACMNA153)	Introduce the concept of variables as a way of representing numbers using letters (ACMNA175)
8	Carry out the four operations with rational numbers and integers, using efficient mental and written strategies and appropriate digital technologies (ACMNA183)	Simplify algebraic expressions involving the four operations (ACMNA192)

This Study

At the end of 2016, 46 students from Years 5 and 6 from an inner-city Melbourne primary school completed two paper and pencil tests: The Fraction Screening Test (Pearn & Stephens, 2015) and The Algebraic Thinking Questionnaire (Pearn & Stephens, 2016). Three days later, 17 students were interviewed to gain further insights into the strategies they used to solve the three reverse fraction tasks. Their selection will be described later.

Results of Paper and Pencil Assessment

The students' written responses to three reverse fraction tasks from the Fraction Screening Test were analysed to determine the types of strategies that students were using. These responses were then classified according to six categories: Incomplete, Visual, Additive/Subtractive, Mixed, Multiplicative, and Advanced Multiplicative.

Table 2
Methods Used for the Three Reverse Fraction Tasks (Figure 1)

	Incomplete	Visual	Additive/ subtractive	Mixed	Multiplicative	Advanced multiplicative
Year 5 (*n* = 26)	6	3	4	10	3	0
Year 6 (*n* = 20)	6	1	5	4	2	2
Total (*n* = 46)	12	4	9	14	5	2

Incomplete refers to students who did not attempt more than one of the reverse fraction tasks in Figure 1 and who provided no explanations. Visual refers to students who showed explicit partitioning of the diagrams (Figure 1) for Task 1 and Task 3 before using additive or subtractive strategies. Some students created a diagram for Task 2 to represent the 12 CDs before solving the task. Additive refers to students who used additive or subtractive methods without explicit partitioning of the given diagram, or a new diagram, to find the whole. These students could find the number of objects needed to represent the unit fraction and then added or subtracted objects correctly to make the whole.

Mixed refers to students who used multiplicative strategies to solve at least one task while still using additive/subtractive strategies to solve at least one other task. Multiplicative refers to students who only used multiplicative reasoning to solve at least two questions successfully. Generally, these students found the quantity represented by the unit fraction and then scaled up or down to find the whole. Advanced multiplicative describes students who used either algebraic notation to find the whole, or used a one-step method to find the whole by dividing the given quantity by the known fraction. It needs to be noted that 6 of the 12 students who were unable to complete more than one of the reverse fraction tasks were in Year 6 and about to transition into secondary school.

Developing an Interview Protocol and Selection of Students

The follow-up interview was designed to respond to questions raised by Kieran (personal communication, 2016) who encouraged the researchers to vary the quantities associated with each of the given fractions. This would provide stronger evidence of generalised thinking if students were consistently able to use the same methods in similar reverse fraction questions where the quantities were changed but the fractions remained the same. This suggestion is supported by Marton, Runesson, and Tsui's (2004) research, which shows how numbers can be varied to foster a generalisable pattern. Could students treat different given fractions as "quasi-variables" (Fujii & Stephens, 2001)?

Figure 2 gives an example of one of the three extended fraction questions used in the interview based on the original tasks in Figure 1 using 2/3, 4/7, and 7/6. For each fraction, there are two changes. *Change of number 1* is a change of the given number of counters. *Change of number 2, and Unknown number* gives a new number of counters and then asks the student to find the whole if given any number of counters.

Original tasks	**Change of number 1**	**Change of number 2, and Unknown number**
1. This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.  a. How many counters did I start with? b. Explain how you decided that your answer is correct.	**1.** Imagine that I gave you 12 counters which is $\frac{2}{3}$ of the number of counters I started with. a. How many counters did I start with? b. Explain your thinking.	**4.** a. If I gave you 18 counters, which is $\frac{2}{3}$ of the number of counters I started with, how would you find the number of counters I started with? b. If I gave you any number of counters, which is also $\frac{2}{3}$ of the number I started with, what would you need to do to find the number of counters I started with?

Figure 2. Example of an extended fraction question based on Figure 1.

If students completed the related questions for 4/7 (Questions 2 and 5) and for 7/6 (Questions 3 and 6), a final question, Question 7, was given:

> Think about the tasks you have just done.
> What if I gave you any number of counters, and they represented any fraction of the number of counters I started with, how would you work out the number of counters I started with?
> a. Can you tell me what you would do?
> b. Please write your explanation in your own words.

When the two changes shown in Figure 2 were used with different known quantities, our aim was to see if students' solution approaches replicated those they used in the written test or whether the interview questions induced them to change from additive/subtractive methods to generalisable multiplicative methods. In particular, we needed to ascertain whether students who had relied on additive or subtractive methods, with or without a diagram, were able to use multiplicative methods once diagrams were no longer provided.

Subsequent interview questions involving any number were designed to make additive and/or subtractive strategies less attractive and less easy to use and so to push students to generalising the fractional structure. Students were asked to find an unknown whole if they had any number of counters that represented a specific fraction of the whole. Finally, students were given the general question about any quantity with any fraction and asked how they would find the whole. This question was designed to provide conclusive evidence that students could generalise their solution method (e.g., by dividing the unknown quantity by the numerator and then multiplying by the denominator). This question also allowed more confident multiplicative thinkers to use algebraic notation to represent the unknown quantity and its accompanying fraction,

Students to be interviewed were chosen from the 32 students who successfully solved and explained their solutions to at least two of the three reverse fraction tasks. Initially, 19 students were interviewed but two interviews were terminated as students went "off-task". The final sample is shown in Table 3.

Table 3
The Sample of Students Interviewed

Year Level	Boys	Girls	Total
Year 5	5	4	9
Year 6	5	3	8
Total	10	7	17

Referring to Table 2, three of four students described as using visual strategies were interviewed, five of nine students who used additive strategies, 4 of 14 students who used a mix of multiplicative and additive methods, three of five students who used only multiplicative methods, and the two students who used advanced multiplicative strategies were both interviewed. The subgroups of students' methods are shown in Table 4.

Table 4
Methods used by interviewed students to solve the three Figure 1 tasks (n = 17)

Year Level	Visual	Additive	Mixed	Multiplicative	Advanced multiplicative	Total
5	2	3	2	2	0	9
6	1	2	2	1	2	8
Total	3	5	4	3	2	17

Administration of the Interview

The record of interview consisted of a three-page document that included the questions, and subsequent space for students to explain their thinking and record their answer and explanation. Each interview took approximately 15 minutes. Interviewers encouraged students to first verbalise and then record their thinking. Students could leave the interview at any point. The written records from the interviews were analysed by two researchers independently, using the following scoring framework shown in Table 5.

Table 5
The Scoring Framework for Interview Questions 1 to 7

Level	Description
0	Not able to successfully complete any questions
1	Completed some or all questions with known fractions and a given quantity (Questions 1-3, 4a, 5a, and 6a) but unable to answer any other questions.
2	Completed all questions with known fractions and a given quantity (Questions 1-3, 4a, 5a, and 6a). Relied on additive methods to solve Questions 4b, 5b, and 6b. Could not give a suitable multiplicative response to Question 7.
3	Completed all Questions 1-6 using multiplicative and/or mixed methods. Gave an appropriate multiplicative response to Question 7.
4	Completed all Questions 1-6 using consistent multiplicative methods. Used suitable algebraic notation to give a multiplicative response to Question 7.

Interview Results

No student was scored at Level 0, an unsurprising result given that all selected students had successfully answered two of the three reverse fraction tasks from the written test. Two of the three students who used visual methods in the written test were able to use additive methods for the interview with no diagram, and were deemed to be at Level 2. The third student who had used visual methods was scored at Level 1 as shown in Table 6.

Table 6
Evidence of Generalising Fraction Structures as a Result of the Interview

Reverse Fraction Tasks (Figure 1)		Interview Score (Table 5)			
Written Test Methods	Number	Level 1	Level 2	Level 3	Level 4
Visual	3	1	2	-	-
Additive	5	1	2	2	-
Mixed	4	-	-	4	-
Multiplicative	3	-	-	3	-
Advanced Multiplicative	2	-	-	-	2

Two students using additive methods in the written test were also at Level 2 in the interview, and one was at Level 1. Two students who used additive methods in the written test converted to fully multiplicative methods and were deemed to be at Level 3. Four students who used mixed methods in the written test successfully solved all other interview questions and solved Question 7 multiplicatively. These students were scored at Level 3.

All three students who used multiplicative methods in the written test were scored at Level 3 in the interview. These students continued to apply generalisable multiplicative methods in the interview. Two students who used advanced multiplicative methods for the written test answered all questions multiplicatively in the interview and responded to Question 7 using appropriate symbolic notation and were deemed to be at Level 4. The numbers highlighted in Table 6 show that 11 of the 17 students, including two who had used only additive methods on the written test, scored at either Level 3 or Level 4 when interviewed, thus demonstrating an ability to generalise a procedure that is independent of a particular fraction or quantity. At Level 3, this was typically expressed as "divide by the numerator and multiply by the denominator".

Illustrative Examples

Students scoring at Level 2 or below were restricted to using additive methods when it was no longer appropriate or useful. This is illustrated in Figure 3 by Student G's attempted solution to Question 7, which only appears to be generalizable since it depends on knowing how many times the quantity equivalent to the unit fraction needs to be added or subtracted.

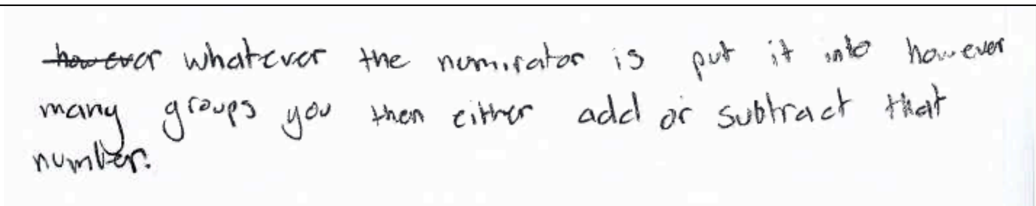

Figure 3. Student G's additive response to Question 7b.

Student E used algebraic notation in what looks like a generalised solution, but only works for specific cases where the given fraction is one part more (or less) than the whole. This student confidently solved all preceding interview tasks, but she was scored at Level 2 because she was unable to give an appropriate multiplicative response to Question 7.

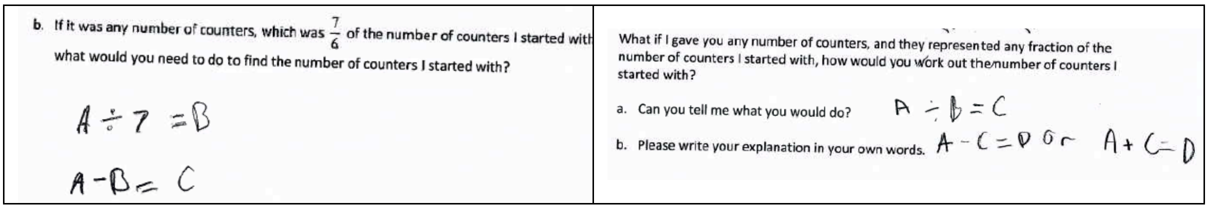

Figure 4. Student E's responses to Question 6b and 7b.

Student K used a similar additive/subtractive method as Student E to answer Question 6b. When presented with Question 7, Student K suddenly realised that the preceding method would no longer work, and then gave a fully multiplicative answer to Question 7. Student K was scored at Level 3.

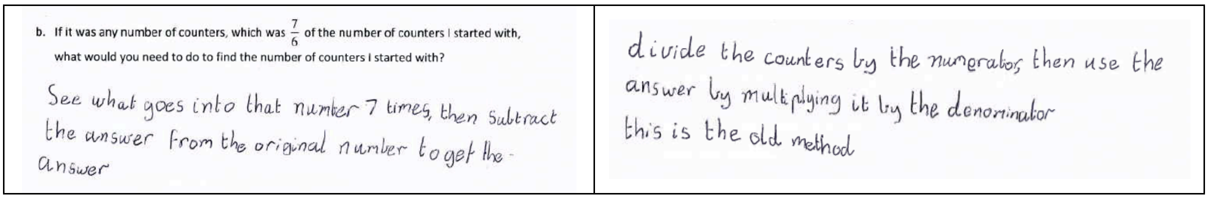

Figure 5. Student K's additive response to Question 6b followed by Question 7b.

Student J consistently solved all previous interview questions by dividing whatever quantity was given by its equivalent fraction. In Question 7, Student J represented all quantities and fractions algebraically in a fully generalised solution, scored at Level 4.

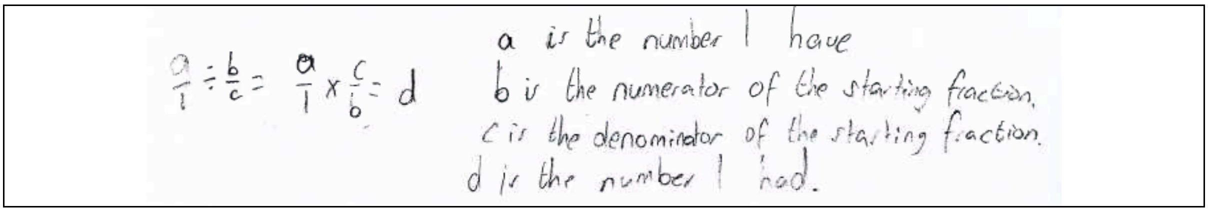

Figure 6. Student J's symbolic multiplicative response to Question 7b.

Discussion and Conclusion

All nine students who used either mixed methods or multiplicative methods to solve the three reverse fraction tasks on the written test showed in the interview that they were able to deal with variations in both fractions and corresponding quantities and to generalise their methods. One student, having successfully solved the first three questions, explained subsequent solutions as "same as I did before", but later gave an explicit symbolic response for Question 7.

The interview format exemplified in Figure 2 allowed these nine students to treat variations in the given fractions as "quasi-variables" (i.e., recognising that the same multiplicative operations applied regardless of the fraction). These students appear well-positioned for formal algebra as expected in Year 7 as given in the Australian Curriculum:

Mathematics (ACARA, 2016) where they will be introduced to "the concept of variables as a way of representing numbers using letters (ACMNA175)". The two students in this group who were scored at Level 4 showed even stronger evidence of being able to create and "simplify algebraic expressions involving the four operations" (ACMNA192), as recommended for Year 8 (Table 1).

The data collected from the interview demonstrated that students who relied on visual methods or additive methods experience difficulty in adopting a multiplicative approach and describing a rule as implied in ACMNA133 (Table 1). These students appear to be most at risk in subsequent years when meeting linear equations involving rational numbers also in relation to proportional reasoning.

The interview task design acted as a scaffolding mechanism for two students who had relied previously on additive methods for all three reverse fraction tasks in the written test to use multiplicative and generalizable methods to solve questions that presented either an unknown quantity or, in the case of Question 7, with both an unknown fraction and unknown quantity. This has clear implications for teaching in helping those students who are dependent on visual or additive methods to move to more generalisable approaches. Visual and additive methods have limited power because every problem has to be treated afresh. This study suggests that by using careful scaffolding more students can be helped to recognise and use repeatable multiplicative methods. Being able to generalise fraction structures is vital for all students to achieve success.

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