Exploring Undergraduate Mathematics Students' Difficulties with the Proof of Subgroup's Closure under Operation

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This study aims to explore undergraduate mathematics students' difficulties in their initial encounter with the subgroup test, and in particular in the proof of closure under operation. Subgroup test is one of the first results in an introductory course of Group Theory where students need to cope with the characteristic, for novice students, high level of abstraction. For the purposes of this study, the researcher has used the Commognitive Theoretical Framework. Analysis suggests that students' difficulties are due to various reasons, including the formalism of the definition of group, incomplete metaphors from other mathematical discourses, confusion of the involved structures, and the proof process per se.

Subgroup Test is, more often than not, one of the first routines undergraduate mathematics students need to cope with in their first engagement with Group Theory, where they need to prove its three conditions, namely, non-emptiness, closure under operation and closure under inverses. Often, though, this apparently simple task proves to be an arduous endeavour, partly due to the abstract nature of Group Theory (Hazzan, 2001). A typical first Group Theory course requires a deep understanding of the abstract concepts involved, namely group, subgroup, coset, quotient groups etc. In addition, the deductive way of teaching Group Theory is unfamiliar to students and, in order to achieve mastery of the subject, it is necessary to "think selectively about its entities, paying attention to those aspects consistent with the context and ignoring those that are irrelevant" (Barbeau, 1995, p. 140). Gueudet (2008) suggests that many pedagogical issues emerging in undergraduate Mathematics Education are based on the transition from secondary to tertiary Mathematics, which can still occur in their second year. In fact, student difficulties in Abstract Algebra may be an indication of problematic transition, mainly due to the particular nature of this module (Ioannou, 2012). The aim of this study is to investigate the undergraduate mathematics students' difficulties with the concept of subgroup, and in particular in proving closure under operation, during their first encounter with Group Theory. For the purposes of this study, there has been used the Commognitive Theoretical Framework (CTF) (Sfard, 2008), due to its great potential to investigate mathematical learning in both object level and meta-discursive level (Presmeg, 2016).

Theoretical Framework

CTF is a coherent and rigorous theory for thinking about thinking, grounded in classical Discourse Analysis. It involves a number of different constructs such as metaphor, thinking, communication, and commognition, as a result of the link between interpersonal communication and cognitive processes (Sfard, 2008). In mathematical discourse, objects are discursive constructs and form part of the discourse. Mathematics is an *autopoietic system* of discourse, namely "a system that contains the objects of talk along with the talk itself and that grows incessantly 'from inside' when new objects are added one after another" (Sfard, 2008, p. 129). Moreover, CTF defines discursive characteristics of mathematics as the *word use*, *visual mediators*, *narratives*, and *routines* with their associated metarules, namely the *how* and the *when* of the routine. In addition, it involves

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the various objects of mathematical discourse *such as the signifiers, realisation trees, realisations, primary objects and discursive objects. It also involves the constructs of object-level and metadiscursive level (or metalevel) rules.* Thinking “is an individualized version of (interpersonal) communicating” (Sfard, 2008, p. 81). Contrary to the acquisitionist approaches, participationists’ ontological tenets propose to consider thinking as an act (not necessarily interpersonal) of communication, rather than a step primary to communication (Nardi, Ryve, Stadler, & Viirman, 2014).

Mathematical discourse involves certain objects of different categories and characteristics. *Primary object* (p-object) is defined as “any perceptually accessible entity existing independently of human discourses, and this includes the things we can see and touch (material objects, pictures) as well as those that can only be heard (sounds)” (Sfard, 2008, p. 169). *Simple discursive objects* (simple d-objects) “arise in the process of proper naming (baptizing): assigning a noun or other noun-like symbolic artefact to a specific primary object. In this process, a pair <noun or pronoun, specific primary object> is created. The first element of the pair, the signifier, can now be used in communication about the other object in the pair, which counts as the signifier’s only realization. *Compound discursive objects* (d-objects) arise by “according a noun or pronoun to extant objects, either discursive or primary.” In the context of this study, groups are an example of compound d-objects. The (discursive) object signified by S in a given discourse is defined as “the realization tree of S within this discourse.” (Sfard, 2008, p. 166). The *realization tree* is a “hierarchically organized set of all the realizations of the given signifier, together with the realizations of these realizations, as well as the realizations of these latter realizations and so forth” (Sfard, 2008, p. 300).

Sfard (2008) describes two distinct categories of learning, namely the *object-level* and the *metalevel discourse learning*. “Object-level learning […] expresses itself in the expansion of the existing discourse attained through extending a vocabulary, constructing new routines, and producing new endorsed narratives; this learning, therefore results in endogenous expansion of the discourse” (p. 253). In addition, “metalevel learning, which involves changes in metarules of the discourse […] is usually related to exogenous change in discourse. This change means that some familiar tasks, such as, say, defining a word or identifying geometric figures, will now be done in a different, unfamiliar way and that certain familiar words will change their uses” (Sfard, 2008, p. 254).

CTF has proved particularly appropriate for the purposes of this study, since, as Presmeg (2016) suggests, it is a theoretical framework of unrealised potential, designed to consider not only issues of teaching and learning of mathematics per se, but to investigate “the entire fabric of human development and what it means to be human” (p. 423).

Literature Review

Research in the learning of Group Theory is relatively scarce compared to other university mathematics fields, such as Calculus, Linear Algebra or Analysis. The first reports on the learning of Group Theory appeared in the early 1990’s. Several studies, following mostly a constructivist approach, and within the Piagetian tradition of studying the cognitive processes, examined students’ cognitive development and analysed the emerging difficulties in the process of learning certain group-theoretic concepts.

The construction of the newly introduced concept of group is often an arduous task for novice students and causes serious difficulties in the transition from the informal secondary education mathematics to the formalism of undergraduate mathematics (Nardi, 2000). Students’ difficulty with the construction of the Group Theory concepts is partly grounded on historical and epistemological factors: “the problems from which these concepts arose in an essential manner are not accessible to students who are beginning to study (expected to understand) the concepts today” (Robert & Schwarzenberger, 1991, p. 129). Nowadays, the presentation of the fundamental concepts of Group Theory, namely group, subgroup, coset, quotient group, etc. is “historically decontextualized” (Nardi, 2000, p. 169), since historically the fundamental concepts of Group Theory were permutation and symmetry (Carspecken, 1996). Moreover, this chasm of ontological and historical development proves to be of significant importance in the metalevel development of the group-theoretic discourse for novice students.

From a more participationist perspective, CTF can prove an appropriate and valuable tool in our understanding of the learning of Group Theory due both to the ontological characteristics of Group Theory, as well as the epistemological tenets of CTF (Ioannou, 2012). Group Theory can be considered as a metalevel development of the theory of permutations and symmetries, and CTF allows us to consider the historical and ontological development of a rather “historically decontextualized” modern presentation of this Theory.

Research suggests that students’ understanding of the concept of group proves often primitive at the beginning, predominantly based on their conception of a set. An important step in the development of the understanding of the concept of group is when the student “singles out the binary operation and focuses on its function aspect” (Dubinsky, Dautermann, Leron, & Zazkis, 1994, p. 292). Students often have the tendency to consider group as a “special set”, ignoring the role of binary operation. Iannone and Nardi (2002) suggest that this conceptualisation of group has two implications: the students’ occasional disregard for checking associativity and their neglect of the inner structure of a group. These last conclusions were based on students’ encounter with groups presented in the form of group tables. In fact, students when using group tables adopt various methods for reducing the level of abstraction, by retreating to familiar mathematical structure, by using canonical procedure, and by adopting a local perspective (Hazzan, 2001).

An often-occurring confusion amongst novice students is related to the order of the group G and the order of its element g. This is partly based on students’ inexperience, their problematic perception of the symbolisation used and of the group operation. The use of semantic abbreviations and symbolisation can be particularly problematic at the beginning of their study. Nardi (2000) suggests that there are both linguistic and conceptual interpretations of students’ difficulty with the notion of order of an element of the group. The role of symbolisation is particularly important in the learning of Group Theory, and problematic conception of the symbols used probably causes confusion in other instances.

Introduction of Abelian group is also an important milestone in the learning of Group Theory. Ioannou (2016a) suggests that students’ engagement with this concept is generally satisfactory. Yet students face certain difficulties, due to the concept of commutativity in the context of the newly introduced discourse, and in the application of the relevant metarules, with particular focus on the how of the routine. In addition, according to a preliminary investigation of students’ application of metarules, it has been identified that in the first steps of Group Theory learning, disengagement with the ‘how’ of metarules occurs quite often (Ioannou, 2016b). Finally, the characteristics of student responses towards learning Group Theory vary, in accordance to their emotions, beliefs and attitudes (Ioannou, 2016c).

A distinctive characteristic of university mathematics is the production of rigorous and consistent proofs. Proof production is far from a straightforward task to analyse and identify the difficulties students face. These difficulties have been extensively investigated for various levels of student expertise. Weber (2001) categorises student difficulties with proofs into two classes: the first is related to the students' difficulty to have an accurate and clear conception of what comprises a mathematical proof, and the second is related to students' difficulty to understand a mathematical proposition or a concept and therefore systematically misuse it.

Methodology

This study is part of a larger research project, which conducted a close examination of Year 2 mathematics students' conceptual difficulties and the emerging learning and communicational aspects in their first encounter with Group Theory. The module was taught in a research-intensive mathematics department in the United Kingdom, in the spring semester of a recent academic year.

The Abstract Algebra (Group Theory and Ring Theory) module was mandatory for Year 2 mathematics undergraduate students, and a total of 78 students attended it. The module was spread over 10 weeks, with 20 one-hour lectures and three cycles of seminars in Weeks 3, 6, and 10 of the semester. The role of the seminars was mainly to support the students with their coursework. There were 4 seminar groups, and the sessions were each facilitated by a seminar leader, a full-time faculty member of the school, and a seminar assistant, who was a doctorate student in the mathematics department. All members of the teaching team were pure mathematicians.

The lectures consisted largely of exposition by the lecturer, a very experienced pure mathematician, and there was not much interaction between the lecturer and the students. During the lecture, he wrote self-contained notes on the blackboard, while commenting orally at the same time. Usually, he wrote on the blackboard without looking at his handwritten notes. In the seminars, the students were supposed to work on problem sheets, which were usually distributed to the students a week before the seminars. The students had the opportunity to ask the seminar leaders and assistants about anything they had a problem with and to receive help. The module assessment was predominantly exam-based (80%). In addition, the students had to hand in a threefold piece of coursework (20%) by the end of the semester.

The gathered data included the following: Lecture observation field notes, lecture notes (notes of the lecturer as given on the blackboard), audio-recordings of the 20 lectures, audio-recordings of the 21 seminars, 39 student interviews (13 volunteers who gave three interviews each), 15 staff members' interviews (five members of staff, namely the lecturer, two seminar leaders and two seminar assistants, who gave three interviews each), student coursework, markers' comments on student coursework, and student examination scripts. For the purposes of this study, the collected data of the 13 volunteers has been scrutinised. Finally, all emerging ethical issues during the data collection and analysis, namely, issues of power, equal opportunities, right to withdraw, procedures of complain, confidentiality, anonymity, participant consent, sensitive issues in interviews, etc., have been addressed accordingly.

Data Analysis

The analysis that follows, aims to explore undergraduate students' difficulties with the proof of subgroup's closure under operation. The application of this condition for a set to be a subgroup was problematic in six (6/13) students' coursework solutions, namely, Dorabella, Leonora, Manrico, Musetta, Francesca, and Tamino's (all pseudonyms), in the solution of the following mathematical tasks:

1. Using the usual test for being a subgroup, prove that for any $n \in \mathbb{N}$, the sets $\{g \in GL(n, \mathbb{R}) : Det(g) = 1\}$ and $\{g \in GL(n, \mathbb{R}) : gg^T = I_n\}$ are subgroups of $GL(n, \mathbb{R})$.
2. Suppose X is a non-empty set and $G \leq Sym(X)$. Let $a \in X$ and $H = \{g \in G : g(a) = a\}$. Prove that H is a subgroup of G.
3. Suppose $(G, \cdot)$ is a group and H, K are subgroups of G. Show that $H \cap K$ is a subgroup of G.

The first difficulty was related to an incomplete object-level learning regarding the distinction between the element of a group and a subgroup. This possibly indicates unresolved problems regarding the definition of the group and its axioms and, moreover, the properties of the elements of a group. There are indications of an incomplete endorsement of the notation used in the exercises, for instance the subgroup $Sym(X)$ and the set X. The unfavourable effect of the problematic metaphors from Linear Algebra regarding the inverse and the transpose of the matrices, and the impact they have in the application of the routine and the solution of the exercises are obvious. For instance, in her solution of Task 1, as it can be seen in Figure 1, Leonora has applied the metalevel rules accurately, since she has applied the test appropriately, showing that she has grasped the applicability and closure conditions of this particular routine as well as its course of action.

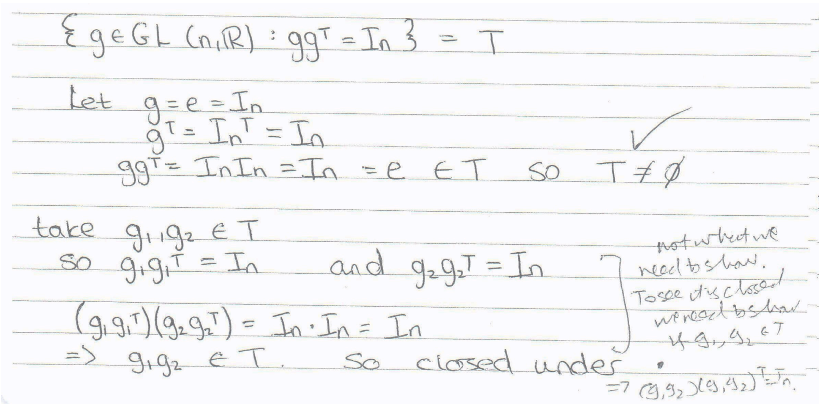

Figure 1. Leonora's solution of Task 1.

She presents her solution in a comprehensive way, using verbal explanations on some occasions. The only inaccuracy occurred in the use of symbolisation in the second example, while proving closure under operation. Instead of proving that $(g_1 g_2)(g_1 g_2)^T = I_n$, she proved that $(g_1 g_1^T)(g_2 g_2^T) = I_n$. This possibly suggests incomplete object-level learning of transposition as well as of the definition of group and the group axioms in particular. In addition, this probably suggests that she may not have realised that $g_1 g_2$ is another element of the group and not a subgroup, and that it has to be considered as such.

Furthermore, in Leonora's solution of Task 2, as seen in Figure 2, there are also indications of an incomplete object-level learning. The first one is revealed in the use of notation, which may have deeper roots relating to the essence of understanding of the elements of the group and their properties. Additionally, she finds it difficult to define the different operations in the different structures. For example, she writes the expression $g(a_1)g(a_2)$, which uses elements of the set X but under operation, which does not operate in X. She has a vague view of what is H and what is X, i.e. that X is a non-empty set and that H is a subgroup of G with a certain condition. At some point, she also writes $a \in H$, which is not true since a is an element of X.

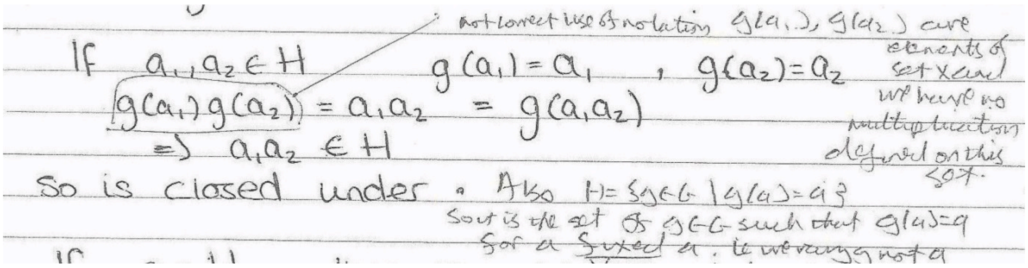

Figure 2. Leonora's solution of Task 2.

Her incomplete object-level learning regarding the concepts involved in this exercise is clearly expressed in the following interview excerpt.

> I found quite hard, because... I got a bit confused with this um… *Sym (X)* and stuff, but – so I don't - I started it but then I weren't sure, whether I was doing it right, so I kind of have stopped, and I'm gonna go ask for help. To like – because I – I don't like, if I'm doing something and I'm not sure if it's right, I don't like to carry on because I don't want to do it all wrong.

Another example of problematic proof of closure under operation occurred in Manrico's solution of Task 1. In the second example for closure under operation he does not prove what he is supposed to prove. As the solution in Figure 3 suggests, he rather concludes that $g_1 g_2 \in GL(\mathrm{n}, \mathbb{R})$, instead of proving $(gh)^T = (gh)^{-1}$.

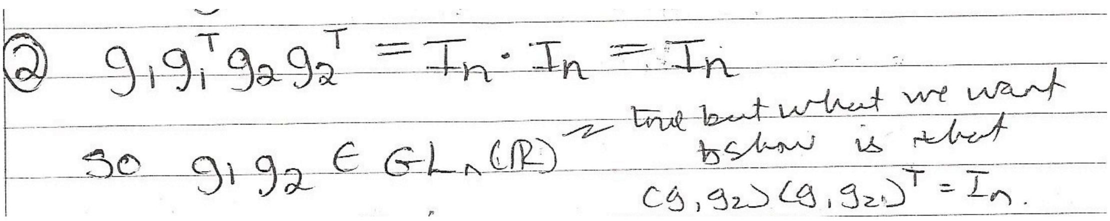

Figure 3. Manrico's solution of Task 1

In contrast to Leonora's case, the above excerpt reveals problematic application of metarules, since the inaccuracy is not a result of incomplete object-level learning. The algebraic manipulations are correct in general, yet inappropriate for the context of this task. I would suggest that this inaccuracy is grounded on incomplete metalevel learning, since it is a result of inaccurate consideration of the applicability conditions of the particular routine as well as the "course of action".

Moreover, regarding Task 3, Manrico's solution also demonstrates further inaccuracies, as shown in Figure 4. The first relates to the expression $h_1 \cap k_1$ and $h_2 \cap k_2$. There are indications of incomplete object-level learning of the d-object of subgroup as well as the elements of the subgroup.

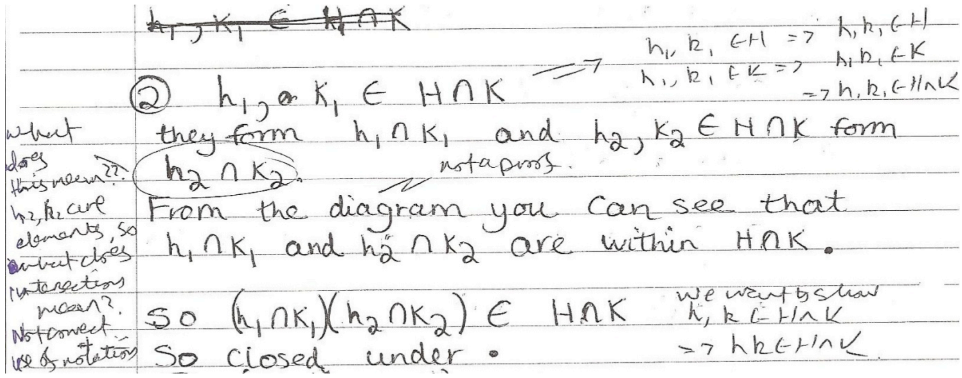

Figure 4. Manrico's solution of Task 3.

In addition, there are problems with the application of metarules (the well-defined and established, among the mathematical community norms of proving), regarding the use of visual mediators. As Figure 5 shows Manrico has based his proof entirely on visual mediators (in this case Venn diagrams, used as metaphor from Set Theory), a tactic that is not acceptable by the markers.

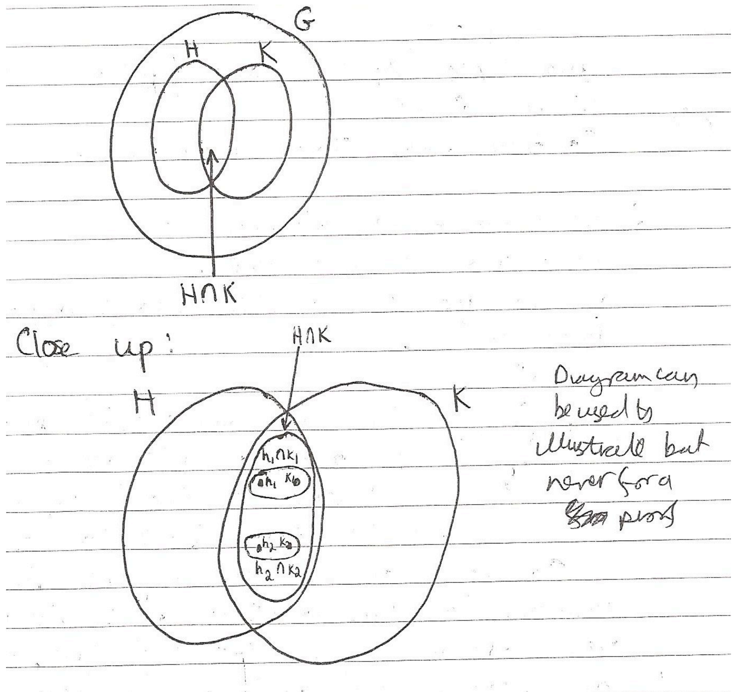

Figure 5. Manrico's solution of Task 3.

Regarding the use of visual mediators as core part of the solutions, there are indications that such use is often linked with lack of confidence or certainty about the quality of the algebraic reasoning. In three students' cases, namely Manrico, Calaf, and Tamino, they make use of visual means of representation, such as Venn diagrams. The use of such visual mediators is not supportive; instead when such approach to solution is applied, these students tend to base the core of their solution on them.

Conclusion

This study's aim was to explore undergraduate mathematics students' difficulties in proving closure under operation, in their initial encounter with the subgroup test. Data analysis suggests that students' difficulties are due to four general reasons. In agreement with Nardi (2000), the formalism of the definition of group requires "decoding" by novice students, due also to the abstract nature of Group Theory (in agreement with Hazzan, 2001). Another difficulty is caused by the problematic metaphors from other mathematical discourses, such as Set Theory and Linear Algebra. Similar to Dubinsky et al. (1994) and Iannone and Nardi (2002), this study highlights students' difficulty to distinguish the different characteristics and requirements that the involved structures have, namely sets, groups, subgroups and their elements. Finally, the last student difficulty that this study reports is related to the process of proof per se, due to incomplete metalevel learning of the relevant metarules that govern the applicability and closure conditions of the subgroup test.

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