First-Year University Students' Difficulties with Mathematical Symbols: The Lecturer/Tutor Perspective

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School and university mathematics: Do they speak the same language? Our university mathematics students come from amongst the successful school mathematics students, but what difficulties with symbols do their lecturers and tutors observe? In this paper, we report on data from interviews with 21 first-year lecturers and tutors from four universities. Key emerging themes focused on mathematical communication: the importance of comprehending mathematical symbols and of composing a mathematical narrative consisting of both explanatory words as well as symbols.

Mathematics comes with its own language, a mix of words and symbols, each of which is infused with meaning that is agreed upon by the community of practice. This sounds straightforward, but we know that, in practice, symbols that look the same may have different meanings in different contexts, or symbols that look different may, in different contexts, be used to mean the same thing. The "rules" of syntax may change as we learn to operate in different domains, and the meaning to be assigned to a symbol also depends on the domain in question. At school and university, students learn to do mathematics by studying it in increasingly complex contexts and extended domains. While the Australian curriculum (Australian Curriculum, Assessment and Reporting Authority, n.d.) promotes the broad mathematical proficiencies of understanding, fluency, problem-solving, and reasoning, the pressures of "fair" high-stakes senior secondary school assessment result in agreed use of symbols and a narrow range of questions. Concerns have been expressed that students are choosing not to proceed to higher mathematics (Chubb, Findlay, Du, Burmester, & Kusa, 2012). One hypothesis under investigation is that a change in symbols load or complexity between school and university may disturb students' confidence to proceed (Bardini & Pierce, 2015). In this paper, we consider the research questions: What do university teaching staff recall of first-year students' difficulties with symbols? What issues do they perceive that students have reading and writing the language of mathematics? We begin by revisiting some of the issues previously reported in the literature by researchers and by summarizing the framework we are using to discuss the different aspects of symbols. This is followed by the methodology of this part of our project, the findings, and finally some implications.

Background and Framework

Symbols in the Literature

Students' misconceptions with the meaning of letters in algebra have been well documented over a long period of time (Küchemann, 1981; Stacey & MacGregor, 2000) and recent research (Bardini, Vincent, Pierce, & King, 2014) suggests that some fundamental uncertainties, like feeling that a different letter must indicate a different number or relate to a different process, exist for some university level students.

Researchers over past decades have agreed that too often students have learned to respond/react to certain patterns of symbols without paying attention to the domain,

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context, or purpose of the mathematics in focus. Twenty years ago, Barbeau (1995) articulated a common concern that some students with technical proficiency (who no doubt would be able, through practise, to score well on high-stakes examinations) may lack insight and only view variables as placeholders for numbers. Arcavi (1994, 2005) offered a way forward for the pedagogy of mathematics by drawing attention to the importance of promoting symbol sense. This, he said, includes an understanding of and an aesthetic feel for the power of symbols, an ability to manipulate and also to "read through" symbolic expressions, the ability to use symbols to represent problem situations, and the realisation that symbols can play different roles in different contexts. Pierce and Stacey (2001), motivated by the introduction of computer algebra systems (CAS) to mathematics classrooms, identified components of "algebraic insight" (i.e., the understanding required when the focus can shift from memorising and automating procedures handled more efficiently by CAS). This, they said, involved both "algebraic expectation" and "ability to link representations". They coined the term "algebraic expectation" to summarise the notions of recognition of conventions and basic properties: "identification of structure" and identification of key features". They suggested that teachers should encourage the development of their students' algebraic expectation by using this framework as guide to thoughtfully consider any algebraic expression before embarking on any processes.

In analysing some difficulties apparent in university students' mathematical work, Pierce and Bardini (2016) identified instances of what Tall (2008) calls "met-befores". He defines a met-before to be "a current mental facility based on specific prior experiences of the individual" that can be "sometimes consistent with the new situation and sometimes inconsistent" (p. 6). Pierce and Bardini note that these met-befores may be encountered at various stages along the mathematical journey, but at university, where the mathematical domain in focus may change, memorised processes, applied efficiently but unthinkingly to seemingly familiar patterns, can lead to illogical results.

Framework

In our study, we have taken the principles of the work of Serfati (2005) for our framework for analyzing mathematical symbols and discussing the implications for students' learning. This framework challenges us to consider symbols from three aspects:

- *Materiality* (i.e., what they look like): This includes whether they are a Latin or Greek letter, or an operator, and their physical attributes.
- *Syntax* (i.e., their position and the conventions associated with this): For example, an = sign must have some symbol or expression on either side of it. We expand this to consider *syntax templates* where a sequence of symbols typically forms a block that is used as a "template" for a particular mathematical action.
- *Meaning* (i.e., the meaning of the symbol within the domain of interest and the context of the problem)

Methodology

For this study, interviews were undertaken with 21 first-year mathematics and/or statistics lecturers and tutors at four Australian universities. Sampling was purposeful. This combination of universities was selected for location and student mix. They represented a mix of urban and regional locations with students selected on the basis of their tertiary entrance rank and others offering more open access. Within each university, first-year mathematics or statistics teaching staff (lecturers and tutors) were identified and asked to participate. Participation was voluntary, but coordinators did encourage their staff.

The semi structured, face-to-face interviews were audio-recorded and later transcribed. Any writing or drawing by the participant was retained and digitised. These transcripts were then subject to thematic analysis, with two researchers separately analyzing all 21 scripts and then discussing the emerging themes. Close agreement between the researchers was achieved on the first reading and further refined by discussion about how best to "name" these themes. In this paper, we report findings from the opening question that we asked of first-year students' university lecturers and tutors: "Just thinking about students' use of symbols. Can you recall any particular difficulties where they might struggle?"

Findings

The over-riding theme that emerged from the analysis of interview transcripts was communication. This theme essentially had two aspects. First and most emphasized by lecturers and tutors was the importance of students communicating – being able to compose a mathematical narrative including their mathematical understandings or statistical findings and the related, underpinning reasoning. The second aspect to which lecturers and tutors drew attention was the difficulty that students have in comprehending certain symbols or material variations of symbols they have met previously at school. We will address these in reverse order.

Comprehension of Symbolic Statements

Certain symbols were reported consistently as causing difficulty. T9, T12, T13, T15, and T17 provided the examples included below. If we analyse their observations in terms of our framework, we see that both symbols with unfamiliar materiality and symbols with familiar materiality but changed syntax or meaning can present stumbling blocks for students.

T9 reported that students have difficulty with the meaning of the square root symbol. This is an example of a symbol, introduced early in the secondary school curriculum, whose materiality remains unchanged but whose associated syntax and meaning depend on the domain of interest and the context of the question. T9 also draws attention to students' fundamental understanding of the meaning of letters in algebra, as illustrated by students' ability to solve a problem with x and y but who are bewildered by the same problem written with a different pair of letters. In this case, the materiality appears different to the student who does not recognize that the syntax and meaning have not changed.

Several tutors reported that students stumbled at the use of capital Greek sigma to signify the operation of summation. The unfamiliar materiality of this symbol may be an issue (T12, T13). At this stage of their mathematical learning, this is the only example of a letter being used to signify an operation. Perhaps both unfamiliarity with the shape of the Greek letter and an "unusual" meaning assigned to a letter present difficulties for the novice. It is also possible that, since most students are unfamiliar with Greek letters, Σ is just seen as a squiggle to be copied thus any potential mental link between Sigma, S, and summation is lost.

T15 and T17 provide examples of the use of symbols, familiar in their materiality but taking new meanings in new contexts, in this case vectors. The dot and cross product symbols "·"and "×" look familiar to operators in arithmetic and algebra but are associated with new syntax and new meanings when working with vectors. Similarly, both points and vectors may be written as ordered triples but the appropriate meaning must be inferred from the context of the question (T15).

> T9: The square root symbol is one of the most common difficulties. They do not understand what the square root means. They have difficulties going from solving equations, so things like $x^2 = 16$ they can tell me that $x = \pm 4$ but when they write down $\sqrt{16}$ they don't know what that actually means. Is this two roots? One root? … if it's got a square root … especially if the square root comes out to be the square root of a negative number and they want a complex root and then they get *i* so then this is a major, major problem.
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> …Using a different letter upsets them, you know a different parameter… if you start with y as a function of x and you change the problem to x is a function of t, you're in trouble, and if I change to p is a function of q or something that's really, really not common, it's like they cannot do the question anymore…they know how to solve a problem with y in terms of x.
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> T12: The one that does seem to hold them up is the sign sigma, capital sigma…I remember my stats students saying things like "What's that big E-looking thing?"
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> T13: They don't have a concept of x representing anything…if you give them numbers they can add it up but if you say, "Okay, that's the sum of *Xi* i is from 1 to n" and they say, "What's that, it's so scary that I can't do this, I can't do this."
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> T15: (*x, y, z*) - (*a, b, c*) If it says, "What is the vector connecting this to this?", then they're both points because you've got to read the language around it. Or if it says you've got to minus (*a, b, c*) or something like that then they have to be vectors because points can't be sort of added. So, you've got to think about what it is actually. You've got to read the context of the question to understand what the issue was… They see something written down and think they know what it is and so that sends them down the wrong path and then you go part way down there and then it's really hard to come back from that, because you've already started to build a picture.
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> T17: Another problem is distinguishing shorthand notation for dot and cross products … we write the divergence of F is equal to del dot F. And again, for them to understand what that translates to in mathematical notation becomes difficult for a lot of students. Because this one here, they don't understand that the order is important with these two here. Like this one is not equal to F dot del, which is something quite different. $\text{div}\, F = \Delta\ F \neq F\ \Delta$

Composing Mathematical/Statistical Narratives

The most common theme across the interviews was that written mathematics should make sense and that it should communicate clearly to the reader. This is exemplified in the statements from T2, T3, T4, T6, T9, T12, and T13 included below. These first-year university lecturers and tutors consistently reported that students were reluctant to use words. Instead, they provided mathematical solutions as strings of symbols that might or might not make sense to the reader. Not only is poor communication due to a lack of words but also to the misuse of symbols, particularly $=; \Rightarrow;$ and $\therefore$, with these symbols being liberally scattered in students' work without apparent attention to their meaning (T2, T9) Concerningly, T9 suggested that students' lack of proficiency with the logical connectors of mathematical notation not only impacts coherent writing and explanation of mathematical workings, but it also creates a barrier to learning. In addition to valuing words, as well as symbols, to elucidate reasoning throughout a problem, statistics staff look for a concluding statement that includes an interpretation of the results of any symbolic or numeric process (T12, T13).

University teachers agreed that it is important for them to model the composing of mathematical narratives and that marking schemes for assignments, at least, should reflect this expectation in students' work. They also commonly reported that they were not aware of the expectations set at the secondary school level but guessed that externally marked high-stakes examinations would be likely to reward correct final answers rather than encouraging full logical mathematical communication.

> T6: …just thinking about symbols. I mean one of the sort of things that I find …I really want students to do is like develop mathematical reasoning and communication skills and I find that often you see what is just a blind sort of statement of symbols one after the other and one of those things that I sort of try to get students to do is to basically write less symbols, write more English. Not necessarily less symbols but write more words to connect them….

> T3: …their general layout and the way they present their mathematics on a page. You and I will write it down in a linear fashion, each using ideas, explaining a point mentioning what we're about to do next then do the calculation, then continue, so there's an interplay between English and mathematics together in a consistent story. They don't think English is appropriate; they don't put any reasoning in their line of argument …

> T4: There are so many things that they're missing, it's the communication. They don't seem to appreciate who is going to read what they've written and how that person might actually understand what was written.

> T2: They just don't write down what they're doing, they don't explain. It's just literally, they think they just need to write a page of equations with each [of] these funny little symbols joining everything together and they'll think they're done.

> T9: …I'm finding that all the problems they have with understanding the notation, the symbols, the way you use the logical connectors – this is actually affecting their ability to understand what we're doing in first-year basic calculus, and then to be able to explain it to me and write something coherent that I understand.

> T13: …when you do the actual workings through the maths part, you do really need the symbol, but the interpretation, you need the language, you need the English to interpret it.

> T12: I do say that at the end there should always be a sentence, at the end you know, summarising what you've done. Their idea of a sentence and mine don't always tally.

Discussion and Implications

Value of the Framework

First, the framework provides us with a systematic way to analyse students' difficulties with symbolic notation. Mathematics teachers at all school and university levels need to explicitly alert students to a change in the accepted meaning of a materially familiar symbol, or a change of symbol to be used with familiar syntax and meaning. Concurrently, we need students to be alert to the domain (e.g., natural numbers, integers, real numbers, complex numbers) or context of each problem (e.g., points, vectors, planes) before they engage with mathematical processes. Instead of just noting a symbol that seems to cause an issue, considering the materiality, syntax, and meaning associated with that symbol can bring in to focus the aspect of the symbol to be attended to when teaching new symbols, or when changing the use of a familiar symbol. The framework can also structure our thinking when we pose questions to students to prompt their thinking about a symbol and so help them overcome barriers to their mathematical progression.

Tension between Trying to Reduce Cognitive Load and Developing Flexible Thinking

The student difficulties noted by first-year university lecturers and tutors have implications for teaching both at the school and university level. At both levels, there is a tension between helping students to achieve fluency in algebraic manipulations and routine problem-solving by limiting what could be seen as extraneous variation (e.g., by using the same letters). This is done with good intention – to help students focus on the process. However, for some students, even some of our "best" secondary school students, this can create a barrier when they perceive that a change of letter for parameter or variable indicates a new and different problem.

Clear Communication must be Valued by Assessment

It is commonly accepted that assessment drives learning. While assigning marks for clear communication in assignments, tutors and lecturers typically told us that exam marking is not so strict. They said things like "if you read this literally, it does not make sense; we can see that the student knows what they are doing (especially if the final answer is "correct")". In fact, markers impute what they expect the student was thinking based on their own understanding of the problem and its solution. Students who arrive at a "wrong" answer also expect that they may be awarded marks for "working" while markers tend to assume that a correct final answer comes as a result of correct mathematical thinking to solve the problem. Here we see the importance of a clear narrative involving words and symbols so that a marker may indeed see if the student has understood the problem and made a slip due to the time pressure of an examination or, alternatively, been lucky to arrive at the preferred result.

Mathematical Reasoning must be Paired with Clear Mathematical Communication

University mathematics and statistics lecturers and tutors expressed the concern that students focused on symbols rather than clear communication, which requires a combination of words and symbols. The Australian mathematics curriculum sets mathematical reasoning as a key proficiency to be developed. However, students apparently too often take the narrow view that the symbolic processes are the total of the mathematics. The habit of communicating mathematical reasoning in words, symbols, and other representations as appropriate needs to be demonstrated and encouraged throughout schooling. Perhaps students need to peer review each other's work to see if they can follow the reasoning.

What Issues do First-Year University Staff Perceive that Students have Reading and Writing the Language of Mathematics?

University lecturers and tutors recalled commonly observing students having difficulties with particular symbols: sometimes because the symbol shape and syntax were unfamiliar, but often because a symbol that was materially familiar, or a syntax template that looked like one they had met before took on a new meaning or range of meanings on an extended domain or new context. Most of these staff had come to realise that this comprehension of symbols, so familiar to them, needs to be explicitly taught to novices.

Staff also expressed concern that students focused on processes rather than clear mathematical communication. It was common for students to write a series of disconnected or incorrectly connected results from applying algebraic algorithms. Students need to learn to make their reasoning explicit by composing mathematical narratives using words and symbols.

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