Explicitly Connecting Mathematical Ideas: How Well Is It Done?

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Multiplicative thinking is a critical stage of mathematical understanding upon which many mathematical ideas are built. The myriad aspects of multiplicative thinking and the connections between them need to be explicitly developed. One such connection is the relationship between place value partitioning and the distributive property of multiplication. In this paper, we explore the extent to which students understand partitioning and relate it to the distributive property and whether they understand how the property is used in the standard multiplication algorithm.

Multiplicative thinking is one of the "big ideas" of mathematics (Hurst & Hurrell, 2014; Siemon, Bleckley, & Neal, 2012) and it underpins many important mathematical concepts required beyond primary school years. Multiplicative thinking could be described as a complex set of concepts which are interrelated and linked in various ways (Hurst & Hurrell, 2016). The Australian Curriculum, Assessment, and Reporting Authority (2017) describes the proficiency of Understanding in terms of a clear and strong knowledge of "adaptable and transferable mathematical concepts" which enables students to make connections between concepts that are related. In short, it could be said that the development of multiplicative thinking depends largely on knowing about the links and relationships between ideas in order to understand why procedures work as they do.

Several researchers (Clark & Kamii, 1996; Siemon, Breed, Dole, Izard, & Virgona, 2006) have noted that an inability to think multiplicatively greatly hinders the development of higher level concepts such as fractions, proportional reasoning, and algebra. This underlines the need to understand what constitutes multiplicative thinking and to identify how key elements are linked so that they can be developed in a conceptual and connected way across all school years.

Siemon et al. (2006) defined multiplicative thinking in the following terms:

- a capacity to work flexibly and efficiently with an extended range of numbers (e.g., larger whole numbers, decimals, common fractions, ratio and percent);
- an ability to recognise and solve a range of problems involving multiplication or division including direct and indirect proportion;
- the means to communicate this effectively in a variety of ways (e.g., materials, words, diagrams, symbolic expressions and written algorithms).

If students are to work "flexibly" with a range of numbers, we believe that there must be explicit teaching of the many connections within the broad idea of multiplicative thinking. Specifically, in this paper, we explore the link between partitioning based on place value, and the distributive property of multiplication.

Background: Place Value, Partitioning, and the Distributive Property

The distributive property of multiplication could be considered as the basis of the vertical multiplication algorithm that is taught in a range of ways by teachers. The importance of this property cannot be under-estimated and Kinzer and Stafford (2013) note the importance of partitioning, stating that "this kind of reasoning is the first step in moving beyond repeated addition and using the distributive property to make sense of

(2017). In A. Downton, S. Livy, & J. Hall (Eds.), *40 years on: We are still learning! Proceedings of the 40th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 325-332). Melbourne: MERGA.

multiplication" (p. 304). Kinzer and Stafford (2013) also underline the importance of the array in developing an understanding of the distributive property and note that "the distributive property helps students understand what multiplication means, how to break down complicated problems into simpler ones, and how to relate multiplication to area by using array models" (p. 308). Their view supports the Common Core State Standards for Mathematics (NGA Centre, 2010) which also underlines the importance of the link between multiplication and the array, and hence the distributive property.

The importance of the distributive property and the link with partitioning is emphasised by Jacob and Willis (2003) who noted that "part-part whole reasoning with groups also enables children to use the distributive property of multiplication over addition" (p. 7). Also, Norton and Irvin (2007) said that "critical concepts underpinning algebra (e.g., equal concepts, integer study, fractions, the distributive law and general arithmetic computational competency) need to be emphasised in the primary years" (p. 559). The quality of understanding about multiplication that results from knowing about the distributive property is further noted by Young-Loveridge and Mills (2009) who said that multiplication strategies based on partitioning and the distributive property are more advanced than those based on other ideas such as repeated addition.

Kaminski (2002) studied how a group of pre-service primary teachers used the distributive property in flexible ways when solving multiplication exercises. He made an interesting observation that "It was clear that while many students had heard of the distributive law, many were still not clear on its application" (p. 141). Kaminski's sample consisted of pre-service teachers, as opposed to in-service teachers but his observation begs the question as to whether or not the majority of in-service teachers would be clear about how the distributive property can be applied. It also follows that the same situation might apply to realising (or not realising) the connection between place value partitioning and the distributive property.

Methodology

This paper reports on a study that developed from a larger and on-going study into multiplicative thinking of children from 9 to 11 years of age. The original study has been conducted for over three years in Western Australian primary schools and has gathered data from over 1,000 children in eight schools. Two data gathering instruments – a written Multiplicative Thinking Quiz, and a semi-structured interview – have been developed and refined during that time and are used in this current study involving two primary school classes at one school in the south-west of the United Kingdom. The quiz was administered to both classes on the same day under identical conditions. The framework for the analysis of data is based on connections between place value partitioning, the distributive property of multiplication, and the standard written algorithm for multiplication, in order to determine if students have an understanding of those connections, and are able to articulate that understanding. The framework is depicted in Figure 1.

In the Multiplicative Thinking Quiz (MTQ), students were asked a total of 18 questions, five of which are based on aspects of the framework (see Table 1). We wanted to find out the extent to which students demonstrated an understanding of partitioning, were able to identify when the distributive property was correctly applied, and whether they were able to explain why the property worked in terms of partitioning. In short, we wanted to see the extent to which they connected the ideas and then how they used the written multiplication algorithm during the semi-structured interview.

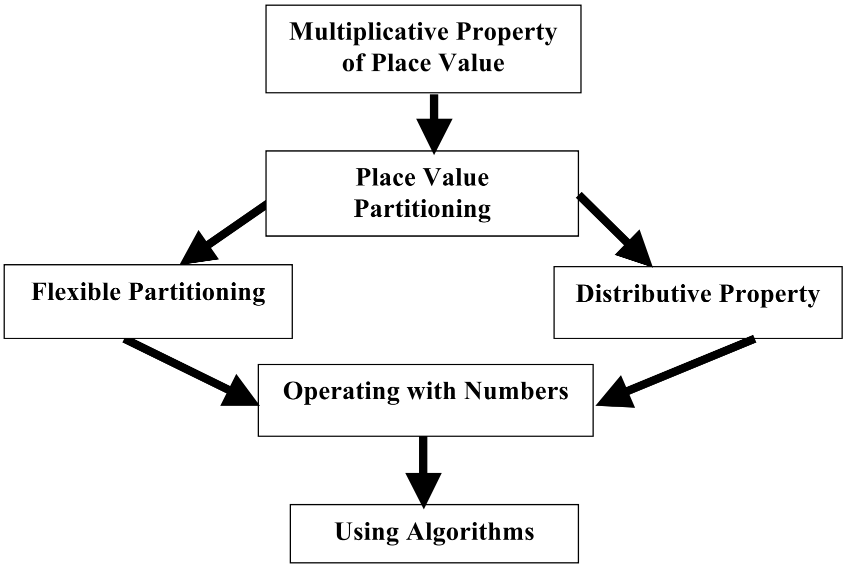


Figure 1. Framework for analysing data from Multiplicative Thinking Quiz.

Data and Analysis

Table 1 presents the responses to the relevant questions from the MTQ. Class A was a Year 5 class (n = 29) and Class B was a Year 6 class (n = 27). Descriptive statistics are used to show the percentage of each class that responded correctly for each question. Several observations can immediately be made from Table 1.

First, while approximately two thirds of the total sample were able to mentally calculate the answer to 6 × 17 (Question 1), a smaller percentage were able to explain their calculation in terms of place value partitioning (Question 2), which is the basis of the written algorithm. However, a similar proportion of students who performed a correct mental calculation were able to use a written algorithm to solve 9 × 15, based on the standard place value partition (Question 3). In the analysis of the quiz responses for Question 3, students needed to indicate that they had 'carried a four' to qualify as a correct response. Second, a much smaller proportion of students were able to identify both correct responses to the question about the distributive property. The interesting aspect of this observation is that the mathematical understanding that underpins Questions 2 and 3 is the same as for Questions 4 – partitioning based on place value. Third, a comparatively small proportion of students could explain their choices of answers (Question 5) in terms of what they had already seemed to understand from their responses to Questions 1, 2, and 3. In other words, the majority of students were able to use place value partitioning either mentally or in a two by one digit algorithm, but many of them were unable to connect the same idea of partitioning to identify when the distributive property was correctly applied, and even less could explain that in terms of partitioning. All of the seven students who explained the fifth question in terms of partitioning used partitioning to explain their answers to Questions 2 and 3.

Table 1

Summary of Responses to Selected Questions from the Multiplicative Thinking Quiz

Question from Multiplicative Thinking Quiz	Class A	Class B
1. Used mental computation to obtain correct answer for 6 × 17	62%	67%
2. Explanation of mental computation for 6 × 17 is based on place value partition	45%	59%
3. Use of standard algorithm is correct and shows place value partitioning (i.e., the “carried 4”) to solve 9 × 15	66%	70%
4. Identifies both (80 × 3 + 9 × 3) and (90 × 3) – (1 × 3) as the only correct options giving the same answer as 89x3 (Distributive Property)	34%	26%
5. Explanation of above question (about Distributive Property) is based on place value partitioning	10%	15%

The following samples from Student Wesley are indicative of responses for the MTQ questions.

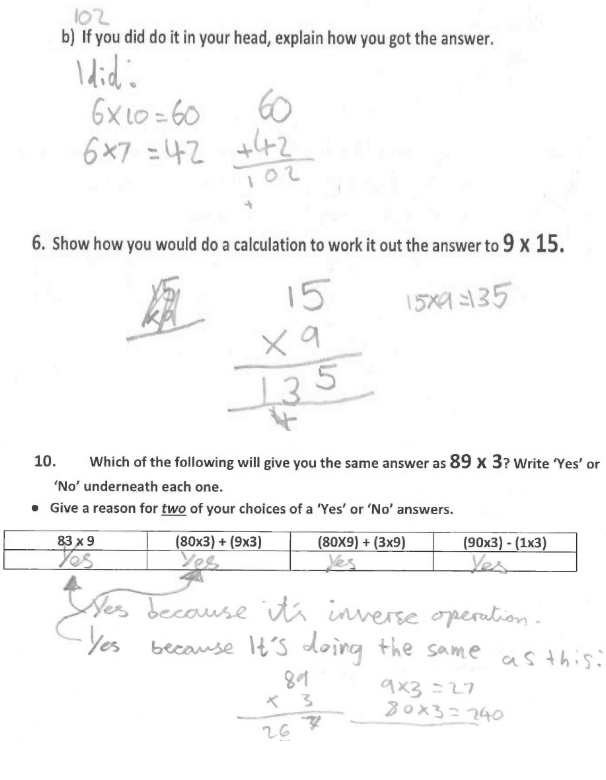
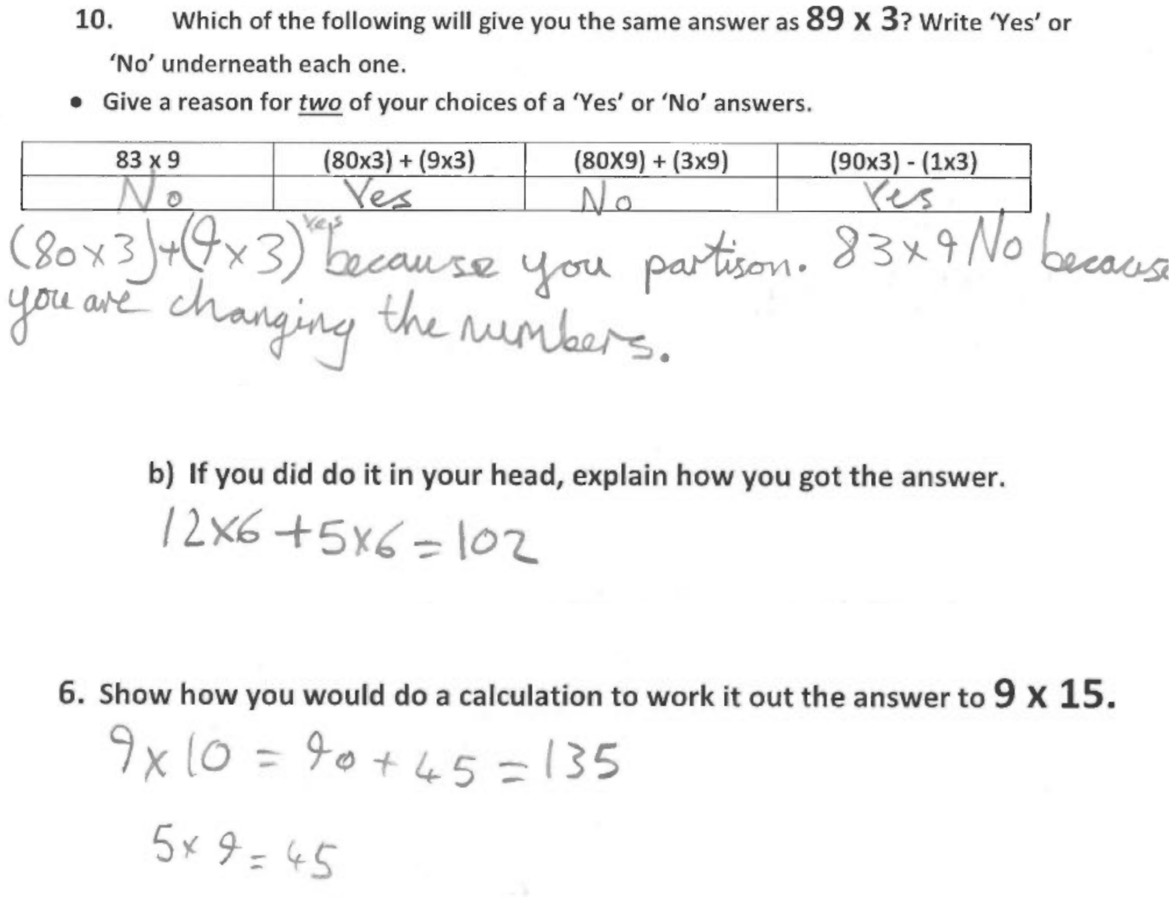


Figure 2. Samples from student Wesley.

Wesley appears to have an understanding of place value partitioning and has given sound examples of it for the first two questions. However, when the question is presented in a different context, he seems quite confused and has mistakenly identified all options as being correct. Wesley has also confused the idea of 'inverse operations' a term that he would have heard at some stage but not fully understood. As well, Wesley did not seem to trust the idea of partitioning as he has used an algorithm to work out the answer to 89 × 3 when there was really no need to do so, if he understood how the property works. During the interview, Wesley used a four-line algorithm to solve 29 × 37. This seems to indicate that he understands how to apply the distributive property as he has identified that there are four elements to the multiplication.

In contrast to the explanations of students who were unable to explain the fifth question in terms of partitioning, the following sample from Student Callum is presented as an example of a satisfactory explanation. Student Callum also displayed some flexibility in his thinking by solving the first example with non-standard partitioning as shown in the second part of the sample.


10. Which of the following will give you the same answer as **89 X 3**? Write 'Yes' or 'No' underneath each one.

- Give a reason for *two* of your choices of a 'Yes' or 'No' answers.

83 x 9	(80x3) + (9x3)	(80X9) + (3x9)	(90x3) - (1x3)
No	Yes	No	Yes

(80×3)+(9×3) Yes because you partison. 83×9 No because you are changing the numbers.

b) If you did do it in your head, explain how you got the answer.

12×6 + 5×6 = 102

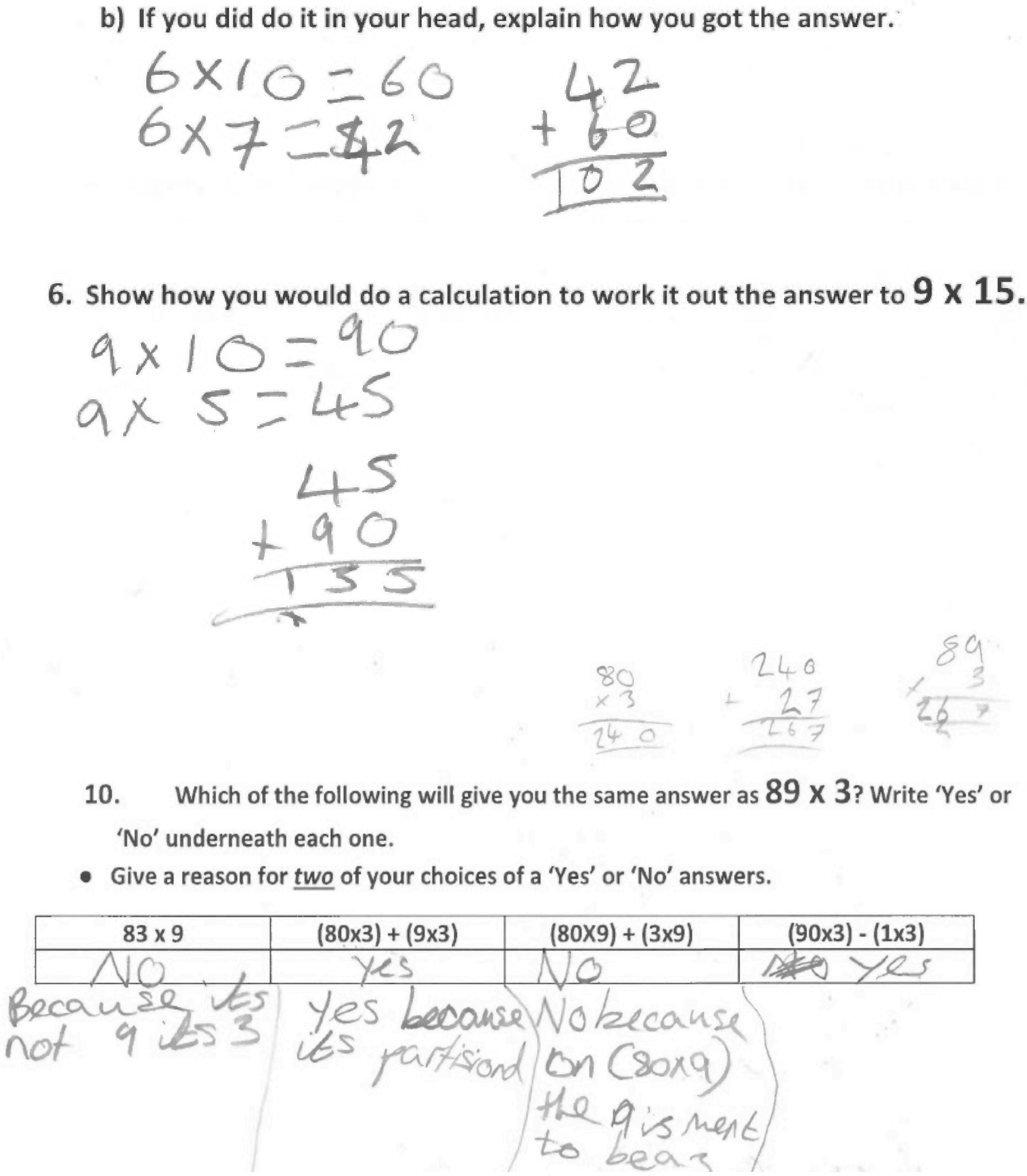
6. Show how you would do a calculation to work it out the answer to **9 X 15.**

9 × 10 = 90 + 45 = 135

5 × 9 = 45

Figure 3. Sample from student Callum.

Another point of interest is how some students who used place value partitioning for both the questions about 6 × 17 and 9 × 15, and who also identified the correct choices for the question about the distributive property, still found it necessary to calculate the answer for (80 × 3) + (9 × 3), despite saying that it would give the same answer as 89 × 3. There seem to be a couple of possible explanations for this, as exemplified by the sample from Student Izzy (Figure 4). First, it could be that students who did that did so as a matter of course or habit, in that they accept that they need to use an algorithm for such calculations irrespective of whether they actually need to do so or not. Second, it may be that their understanding is not sufficiently robust – perhaps they need to calculate with an algorithm to prove to themselves that the partition actually works.



b) If you did do it in your head, explain how you got the answer.

6x10 = 60
6x7 = 42

42
+ 60
102

6. Show how you would do a calculation to work it out the answer to **9 x 15.**

9 x 10 = 90
9 x 5 = 45

45
+ 90
135

80 x 3 = 240 240 + 27 = 267 89 x 3 = 267

10. Which of the following will give you the same answer as **89 x 3**? Write 'Yes' or 'No' underneath each one.

- Give a reason for *two* of your choices of a 'Yes' or 'No' answers.

83 x 9	(80x3) + (9x3)	(80X9) + (3x9)	(90x3) - (1x3)
NO	yes	NO	~~No~~ yes

Because its not 9 its 3

yes because its partisiond

No because on (80x9) the 9 is ment to bea 3

Figure 4. Samples from student Izzy.

It is worth considering the work of a student who, in general, did not respond well to the five MTQ questions, as shown in Table 1. Student Francis made an incorrect calculation for the question about 6 x 17, *did* use an algorithm to correctly work out the answer for 9 x 15, but was unable to identify the correct choices for the question about the distributive property. During the interview, the following exchange occurred [with notes by the interviewer]:

> I: [Francis said that (80 × 3) + (9 × 3) would give the same answer as 89 × 3 but when explaining how it worked, he had to actually work out the two parts and took prompting to arrive at the correct answers for each part. He wrote it as a vertical addition]. "Do you need to work it out to prove it?"
>
> F: "Yes".
>
> I: [He was shown the example (50 × 6) + (3 × 6)] "What would it be the same as?"
>
> F: "Fifty-three times . . . twelve . . . no . . . times six".
>
> I: [Francis was shown (70 × 4) + (6 × 4)] "Do you need to work them out or are you happy that they will give the same answer as 76 × 4"?
>
> F: "Yes".

There is a considerable degree of uncertainty about the answers offered by Francis. While he made a computational error in the 6 × 17 question, he did use place value partitioning for that question and also the question about 9 × 15. However, he was unable to apply that knowledge to the questions about the distributive property, both in the MTQ and the interview. This suggests that he has developed partial understanding of the mathematics involved but has certainly not been able to connect the idea of place value partitioning to the explanation of how and why the distributive property is applied.

Conclusion

On the basis of the analysis of data from the MTQ and the interview, it would seem that there are several levels of understanding shown by students in the sample. These could broadly be described as follows:

- Students who understand place value partitioning, use it when calculating answers to multiplication examples (either mentally or written), understand the distributive property, and explain the latter in terms of partitioning.
- Students who understand place value partitioning, use it when calculating answers to multiplication examples (either mentally or written), correctly identify examples of the distributive property, but do not trust the partitioning and need to calculate a product as proof.
- Students who understand place value partitioning, use it when calculating answers to multiplication examples (either mentally or written), but do not apply it to explain how and why the distributive property works.
- Students who demonstrate a partial understanding of aspects of the above three characteristics but whose understanding is incomplete and not consistently applied.

Hence, we believe that there are some clear implications for teaching. First, teaching should focus on establishing the link between standard place value partitioning and the distributive property and this could be successfully developed through the use of the multiplicative array. Second, the written algorithm for multiplication needs to be developed from the grid method, which is based on standard place value partitioning and the array. Third, the specific mathematical language related to 'partitioning' should be incorporated when developing students' understanding of the distributive property. As well, we think it is important for teachers to encourage students to trust the fact that ideas like the distributive property will work when applied correctly. Helping students to make such connections should situate them better when learning how the distributive property informs aspects of algebraic reasoning.

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