The Prevalence of the Letter as Object Misconception in Junior Secondary Students

Zarina Akhtar
International Islamic University Islamabad
<zarina.akhtar@iiu.edu.pk>

Vicki Steinle
University of Melbourne
<v.steinle@unimelb.edu.au>

In this study, we investigated students' thinking about the use of letters in algebra. Responses from over 1,400 Australian secondary school students to a set of three algebra items were analysed to determine the prevalence of the "letter as object" misconception. We estimate that 50% to 80% of Year 7 students bring this misconception to their initial learning of algebra. Over 50% of Year 8 students and over 40% of Year 9 students in the sample also selected responses consistent with this misconception.

If you speak to a group of adults about their learning of school mathematics, you are likely to find adults who comment that they found mathematics easy until they met algebra. There has been about three decades of research into difficulties/misconceptions that students experience when they learn algebra; some of this research focusses more on the manipulations required to solve equations while other research (such as this paper) focusses on students' thinking about the meaning of algebraic notation.

Steinle, Gvozdenko, Price, Stacey, and Pierce (2009) distinguish between two groups of misconceptions in algebra: *numerical* and *non-numerical*. The first group consists of student thinking regarding the numerical values which letters stand for; e.g. some students reject the solution $x = 8$ and $y = 8$ to $x + y = 16$, as they believe that different letters should be replaced by different numbers. Another numerical misconception is the *alphabetical value,* where $a = 1$, $b = 2$, etc. (see, for example, MacGregor & Stacey, 1997).

The second group of misconceptions that Steinle et al. (2009) refer to is the *non-numerical* misconceptions, which includes the *letter as object* misconception (using the terminology of Küchemann, 1981) where, for example, students think that *a* stands for *apples* rather than the *number of apples*. Clement (1982) noted this error when he used the Students and Professors problem (below) in a set of problems given to a sample of first year engineering students.

> Write an equation, using the variables *S* and *P* to represent the following statement: "At this university there are six times as many students as professors". Use *S* for the number of students and *P* for the number of professors.

Given that these students were enrolled in a mathematics-related degree, Clement was surprised by the considerable number of students who were unable to provide the correct equation (6P = S), instead writing 6S = P. Clement concluded that schools appeared to be "more successful in teaching students to manipulate equations than they have in teaching students to formulate equations in a meaningful way" (p. 28).

In this paper, we are focussing on this *letter as object* misconception, noting that there are several variations in the terminology used in the literature; for example, *letter as abbreviation*, and *letter as unit*. Akhtar and Steinle (2013) reported the prevalence of this misconception in a preliminary study of 850 students, and this paper builds on this earlier work with a larger sample of students.

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Literature Review

One of the foundational studies into students' understanding and skills in mathematics was the large-scale CSMS study in the U.K. Küchemann (1981) reported on the algebra items in this study and described six ways that students interpreted letters. One of the items (referred to here as Pencils) was as follows (p. 107):

> Blue pencils cost 5 pence each and red pencils cost 6 pence each. I buy some blue and some red pencils and altogether it costs me 90 pence. If b is the number of blue pencils bought and if r is the number of red pencils bought, what can you write down about b and r?

The correct answer ($5b + 6r = 90$) was provided by 10% of the 14 year olds in the study. Just under 20% answered $b + r = 90$, and another 6% answered $6b + 10r = 90$ or $12b + 5r = 90$. The last two equations are consistent with "6 blue pencils and 10 red pencils cost 90 pence" and "12 blue pencils and 5 red pencils cost 90 pence", respectively, which indicates that the letters are representing objects rather than numbers. Note that these two equations involve coefficients which are possible solutions to the problem, that is (6, 10) and (12, 5), rather than the information given in the worded problem.

The Students and Professors problem has been used by various researchers on samples of both secondary and tertiary students in various countries (e.g., Rosnick, 1981; Clement, 1982). The incorrect use of a letter in algebra to indicate an object or abbreviation is widespread. Rosnick (1981) indicated that this tendency is "deeply entrenched" (p. 419), and Warren (1998) noted that, "Even students who were considered by their teacher to be very capable of understanding algebraic concepts, included the letter standing for an object in a number of their responses" (p. 666). More recent research indicates that this incorrect use of letters still exists. Egodawatte (2011) used the following problem with Grade 11 students in Canada: Shirts cost s dollars each and pants cost p dollars a pair. If I buy 3 shirts and 2 pairs of pants, explain what $3s + 2p$ represents? One of the students interviewed (Colin) stated that "3s would equal to 3 shirts and 2p would equal to 2 pairs of pants. So, this would represent the total amount of items he bought… So, in total, there would be 5 items" (p. 119). Colin then proceeded to use the same logic on the next question to state that B stands for "Blue cars".

Textbooks in Australia (Chick, 2009; MacGregor & Stacey, 1997) have been found to contain explanations which use "fruit salad algebra", that is, a stands for apples and b for bananas. MacGregor and Stacey concluded that the students at one of the schools in their study (School C) were adversely affected by the use of a textbook that stated that letters can be used as abbreviated words and labels. Chick (2009) used a page from an Australian Year 8 mathematics textbook in her study of teachers' pedagogical content knowledge. The teachers in this project were asked to comment on two explanations of the distributive law, the first explanation used images of apples and bananas to show that $2(3a + 2b)$ was equal to $6a + 4b$. Of the 32 teachers who responded to the question about the fruit salad explanation, over 70% indicated that they would use this in the future. About one quarter of the group indicated that they had concerns with this explanation as it would reinforce the *letter as object* misconception.

Akhtar and Steinle (2013) analysed a sample of 850 students and reported that 50% to 70% of Year 7 students brought the *letter as object* misconception to their initial learning of algebra and this decreased to about half of Year 8 students and about one quarter of Year 9 students in the sample. The goal of this paper is to determine if this prevalence is confirmed for a larger sample.

Methodology

The instrument used in this study was a "SMART test" containing three algebra items, two of which were based on the work of Küchemann (1981). SMART tests (Specific Mathematics Assessments that Reveal Thinking) are designed to identify students' misconceptions in particular mathematics topics; in this case, about the use of letters in algebra. SMART tests are short, online, diagnostic tests which are automatically marked so that teachers have instant access to the results. Wherever possible, the tests are based on research findings. We intend that teachers pre-test their students before teaching a topic so that they can use this formative assessment to inform their teaching to better meet their students' learning needs.

Figure 1 contains the text (but not images) of the three items in this SMART test *Letters for numbers or objects?* (www.smartvic.com/smart/index.htm). Note that the multiple-choice options, listed here as dot points, appear in drop-down boxes in the test.

The data for this study came from 26 schools in Melbourne where teachers have chosen to use this SMART test with their students. Of the 1,449 Year 7, 8, and 9 students who attempted this test during 2015, 16 students did not complete the three items, and hence their data was removed. This left 1,433 students in total: 648 students from Year 7, 651 from Year 8, and 134 from Year 9. The Year 7 and 8 sample sizes are larger than our previous study (Akhtar & Steinle, 2013), but the Year 9 sample is of similar size. While this sample is not randomly chosen, we have no reason to believe that it is not representative.

Doughnuts (item 2377)	Garden (item 2387)	Wheels (item 2391)
Lucy bought 6 doughnuts for 12 dollars. She wanted to work out how much each doughnut cost. She wrote the equation 6*d*=12. In Lucy's equation, *d* stands for: • one doughnut • dollars • the number of doughnuts • doughnuts • the cost of one doughnut	For my garden, I bought **r** red roses and **g** white gardenia bushes. The roses cost $4 each. The gardenias cost $5 each. Choose the equation that says that the total cost was $70: • 4r + 5g = 70 • 10r + 6g = 70 • r + g = 70	At a bike shop there are **b** bikes (2 wheels each) and **t** trikes (3 wheels each). Choose the equation that says that there are a total of 100 wheels: • 2b + 3t = 100 • b + t = 100 • 35b + 10t = 100

Figure 1. Items from algebra SMART test: Letters for numbers or objects?

Results and Discussion

Figure 2 illustrates the distribution of students' responses across the five options in Doughnuts. The correct option is last; 15% of Year 7 and about 30% of Year 8 and Year 9 chose this option. The option which was chosen most often, however, was the fourth option which is a very clear indication of the *letter as object* misconception; students thinking that "*d* stands for doughnuts". Over 40% of Year 7 students chose this option, while just over 30% of the Year 8 and Year 9 students chose this.

Recall that SMART tests are intended to be used in advance of teaching and, if teachers are using them for this purpose, the data for the Year 7 students should be regarded as pre-test rather than a post-test data. The Year 8 and 9 students, however, have met pronumerals before and the large numbers of students choosing the *letter as object* (LO) option instead of the correct option is of concern.

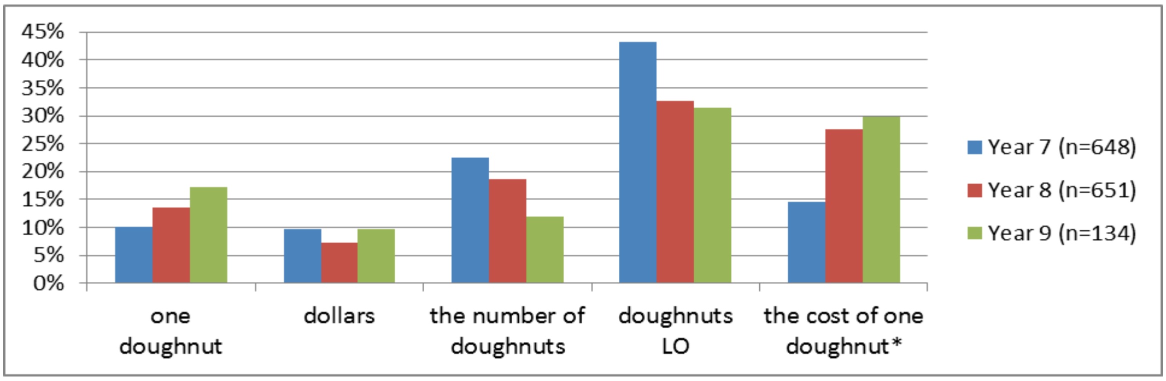

Figure 2. Distribution of responses on Doughnuts (*correct, LO: Letter as Object).

Figure 3 contains the distribution of students' responses across the three options in (a) Garden and (b) Wheels. These are designed to be parallel items and are discussed below. Note the rearrangement of the order of the multiple-choice options in (b) Wheels to match (a) Garden.

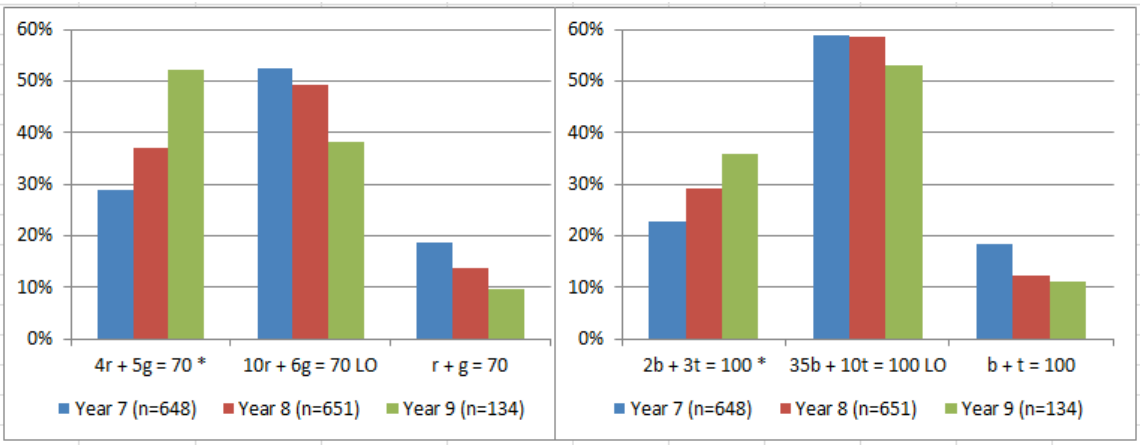

Figure 3. Distribution of responses on (a) Garden and (b) Wheels (*correct, LO: Letter as Object).

Figure 3 indicates that, within each item, there is an increasing trend in the facility from Year 7 to Year 9, but only 52% of the Year 9 students were correct on Garden and only 36% on Wheels. Küchemann (1981) reported that 10% of the 14-year-olds answered the Pencils item correctly; of the Year 8 students in this sample, 37% were correct on Garden and 29% on Wheels. The higher facility in this study is most likely to be due to the multiple-choice format of these items, compared to pen and paper tests reported in Küchemann.

Of the Year 8 students in this sample, 14% chose $r + g = 70$ in Garden and 12% chose $b + t = 100$ in Wheels, similar to the 17% Küchemann noted who wrote $b + r = 90$.

Küchemann reported 6% of the sample wrote $6b + 10r = 90$ or $12b + 5r = 90$ which are indicative of the *letter as object* misconception. In this study, considerably more Year 8 students chose $10r + 6g = 70$ in Garden (49%) and chose $35b + 10t = 100$ in Wheels (59%). The higher prevalence of a response indicative of the *letter as object* misconception in this study is (again) most likely due to the multiple-choice format. In a pen and paper test, students who believe that they need to *solve* the problem *before* they can write the equation are making the task much more difficult for themselves. It will take these *solvers* longer to complete such problems and they might even give up due to the difficulty. In a multiple-choice test, however, these *solvers* can consider each of the given options and choose the one that fits their interpretation (such as $10r + 6g = 70$ means "10 roses and 6 gardenias cost \$70"). Hence, it is reasonable that providing multiple choice options will increase the likelihood of detecting students who think this way.

Garden and Wheels were designed to be parallel items. Figure 4 provides data on students' responses on these two items; the axes are arranged so that students choosing consistently on these items are on the diagonal from left to right.

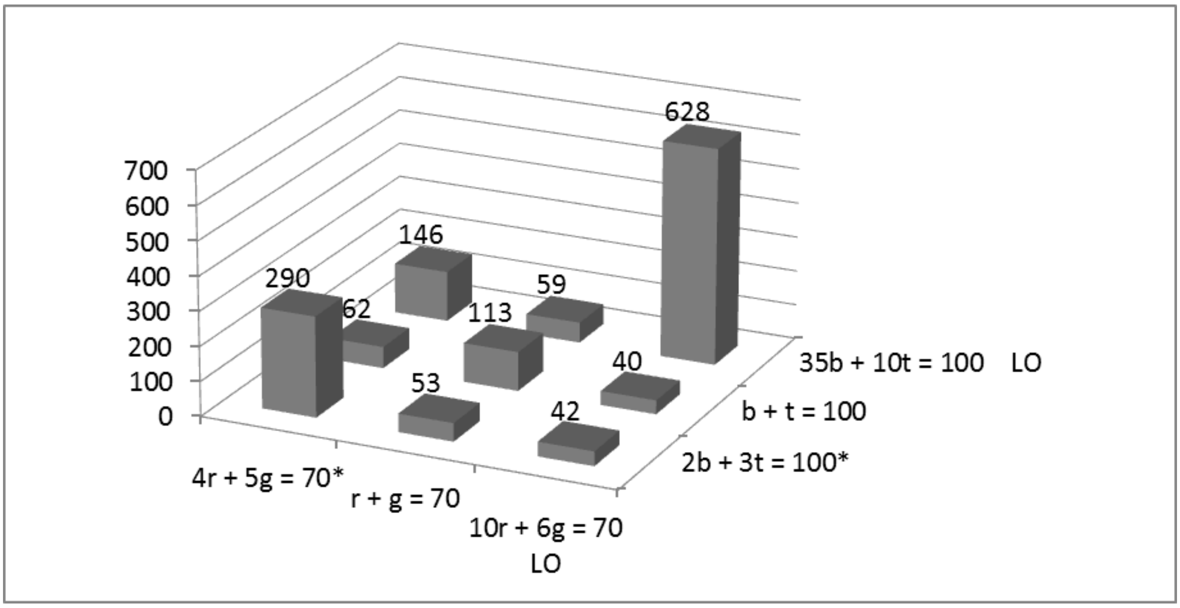

Figure 4. Students' combined responses to Garden and Wheels (n = 1,433).

While 290 students (20%) chose correctly on both items (the left-most column in Figure 4), the right-most column shows that over 620 students (44%) chose $10r + 6g = 70$ and $35b + 10t = 100$ which are the LO options discussed above. Breaking this down by year level; this is 46% of the Year 7 students, 44% of the Year 8 students and 34% of the Year 9 students. Taking account of the earlier comments about the Year 7 data, it is noteworthy that only 22% of the Year 8 students and 32% of Year 9 were correct on both items.

In order to follow students across all three items, a "pattern recognition script" was used on the data. This script has been created to detect common responses by students to a set of items. This data-driven procedure has been found to provide interesting insights into student thinking; see, for example, Steinle et al. (2009).

The six most common response patterns found in this data are listed in decreasing order of frequency in Table 1. The last row (Pattern 6) indicates that only 79 students (i.e. 6% of the sample) chose the correct response on each of these three items. The most common response pattern is Pattern 1; 234 students (i.e., 16% of the sample) chose the three options discussed above indicating *letter as object* misconception. The last column of Table 1 contains the ratio of observed frequency to expected frequency if all students chose randomly on the three items. Pattern 1 has occurred more than seven times what would be expected if choices were random. Pattern 4 (96 students) has correct answers on the last two items, but LO on the first item, indicating that even students with good knowledge seem to be tempted to think about letters as objects occasionally (as found by Warren, 1998).

Table 1
Most Common Response Patterns

Patt-ern	Doughnuts (2,377)	Garden (2,389)	Wheels (2,391)	Freq.	Ratio
1	doughnuts^{LO}	$10r + 6g = 70^{LO}$	$35b + 10t = 100^{LO}$	234	7.3
2	the cost of one doughnut*	$10r + 6g = 70^{LO}$	$35b + 10t = 100^{LO}$	148	4.6
3	the number of doughnuts	$10r + 6g = 70^{LO}$	$35b + 10t = 100^{LO}$	105	3.3
4	doughnuts^{LO}	4r + 5g = 70*	2b + 3t = 100*	96	3.0
5	one doughnut	$10r + 6g = 70^{LO}$	$35b + 10t = 100^{LO}$	87	2.7
6	the cost of one doughnut*	4r + 5g = 70*	2b + 3t = 100*	79	2.5

*correct option, LO: Letter as Object

Conclusion and Implications for Teaching

The purpose of this paper was to determine the current incidence of the *letter as object* misconception in algebra in a sample of Australian students some 30 years after the seminal work by Küchemann (1981) in the U.K. The test used was a three-item, multiple-choice, computerised test containing items adapted from Küchemann. Student performance improved from Year 7 to Year 9 on each of the three test items, however, the average facility on these three items was only 40% for the Year 9 students.

If teachers are using this test for formative assessment, then the Year 7 data reported here needs to be interpreted with caution; it will contain some students who have not yet received formal instruction in algebra and hence provides an indication of the thinking that Year 7 students bring to their first algebra lessons in Australia. Exactly 20% of the Year 7 students did not choose any of the responses associated with the *letter as object* misconception, leaving 80% with at least one such response on the three items. Just over 50% of these students had either two or three such responses. Hence, we conclude that between 50% and 80% of Year 7 students bring the *letter as object* misconception to their learning of algebra, which indicates that the 50% to 70% range found in the previous smaller sample was a slight underestimate.

The Year 8 and Year 9 data, on the other hand, provides an indication of the post-teaching prevalence of the *letter as object* misconception. Based on the same criteria as above, between 50% and 75% of Year 8 students and between 40% and 70% of Year 9 students in this sample have this misconception. Thus, our previous estimates (about one half of Year 8 students and about one quarter of Year 9) appear to be underestimates.

It is interesting to note that students who have chosen the options suggesting the *letter as object* misconception in two of these three items, are not choosing an equation based on the *information* given in the question, but are choosing an equation which seems to represent the *solution* to the problem. When teachers provide worded problems for their students to solve, there are likely to be some students who attempt to write an *initial equation* based on the solution to the problem. As noted by Stacey and MacGregor (2000), such students have not grasped the power of algebra to *find* the solutions. Küchemann (1981) noted that this confusion occurred "even with children who did well on the test as a whole" (p. 107).

There is evidence to show that some teachers and textbooks use the letter as object analogy (e.g., Chick, 2009; MacGregor & Stacey, 1997). If teachers believe that it is an appropriate analogy and it is also found in textbooks, then it is likely that students will retain their initial beliefs about letters in algebra standing for objects rather than numbers.

> The SMART test system was designed to make the results of mathematics education research readily available to teachers. As well as the diagnostic information about each of their students in the specific topic, teachers are provided with explanations of the diagnoses and teaching suggestions for dealing with misconceptions and for taking students to the next level of understanding. We expect that this information, in the context of the results of their own students, will increase teachers' pedagogical content knowledge in the particular topic. As Holmes, Miedema, Nieuwkoop, and Haugen (2013) note in their study with teachers: "All in all, identifying and correcting misconceptions, not mistakes, is a skill well worth developing" (p. 40). Likewise, Russell, O'Dwyer, and Miranda (2009) conclude, "this study suggests that the use of diagnostic assessment systems, such as the DAAS, promises to enhance teaching and learning by enabling teachers to more effectively assess student understanding in a timely manner, diagnose misconceptions, and then help students develop their understanding so that a given misconception is no longer held" (p. 423).

Support for explicit classroom discussion of incorrect student work (including misconceptions) is provided by Booth, Lange, Koedinger, and Newton (2013). They compared the progress of students receiving various instruction and noted,

> The present study…suggests that receiving incorrect examples can be beneficial regardless of whether it is paired with correct examples. This finding is especially important to note because when examples are used in classrooms and in textbooks, they are most frequently correctly solved examples. In fact, in our experience, teachers generally seem uncomfortable with the idea of presenting incorrect examples, as they are concerned their students would be confused by them and/or would adopt the demonstrated incorrect strategies for solving problems. Our results strongly suggest that this is not the case, and that students *should* work with incorrect examples as part of their classroom activities. (p. 32)

Preliminary evidence of the success of the SMART test system is provided in Steinle and Stacey (2012). Teachers are requested to complete surveys after using a SMART test. One of the multiple-choice survey questions is: As a result of using this quiz have you learned something useful for you as a teacher? Of the 127 responses to this question, 92% answered either "Yes, very valuable learning" or "Yes, useful learning". Another question probed the effect on teaching practice: Did you adjust your teaching plan as a result of the diagnostic information? Of the 124 responses to this question, 87 (70%) indicated that they did adjust their teaching. We expect that teachers' use of this system will lead to improved teaching and learning as they take steps to either avoid misconceptions (such as not using unhelpful analogies) or to help students to leave them behind (by discussing them explicitly in classrooms).

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