LEARNING TRAJECTORIES AND KEY INSTRUCTIONAL PRACTICES

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As learning trajectories gain traction in mathematics education, we seek to understand the ways in which teachers may use them in interactions with students. This paper reports on one group of elementary teachers' use of their emerging knowledge of a learning trajectory to examine key pedagogical practices. Findings suggest that a learning trajectory helps teacher place student thinking at the center of these practices.

In this paper we examine the role a learning trajectory (LT) that characterizes students' progression from less to more sophisticated levels of understanding plays in teachers' analysis of classroom discourse orchestration. In the introduction of the *Common Core State Standards* (CCSSO, 2010), the authors of the document highlighted the role LTs played in the initial development of the new standards and reported that the development of the standards began with research-based trajectories.[1] LTs have been demonstrated effective in assessment design (Battista, 2004), curriculum development (Clements, Wilson, & Sarama, 2004), and as a tool for teachers to identify students' ideas in instruction (Furtak, 2009; Mojica, 2010; Wilson, 2009). By providing a framework for aligning evidence of student cognition with research findings on likely tendencies for the development of children's mathematical conceptions, LTs support teachers in locating students' ideas within a range of conceptual development and offer a theory of how mathematical ideas develop overtime (Wilson, 2009).

The recent release and adoption by many states of the Common Core State Standards recalls the question about the relationship between standards, professional development, and instructional practice. Cohen and Hill (2001) showed that in the absence of focused professional development on the content and the ways in which students learn that content, it is unlikely for new standards to result in changes in teachers' practices or increases in student achievement. Further, these authors indicated that professional development related to changes in standards should provide teachers with opportunities to explore and understand new standards through examining student thinking, work samples, and curricular connections if meaningful change is to occur. Thus, we propose that professional development focused on LTs can provide opportunities for teachers to examine both changes in content and the ways in which students think about that content in light of the new standards. Further, we seek to understand how participation in professional development that attends to LTs supports participants' examination of classroom mathematics instruction during whole class discussions.

Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.

Perspectives

Learning trajectories are descriptions of the qualitatively different levels of reasoning sophistication through which students' concepts pass as they evolve from informal ideas to complex understandings. With roots in Simon's (1995) hypothetical learning trajectory, LTs bring clarity to the intermediate understandings a student may have of a mathematical concept. LTs outline how those intermediate understandings relate to previous ideas as well as to those that will evolve from them. In our work we use Confrey et al.'s (2009) definition of a learning trajectory: "a researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e., activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time" (p. 347). In this definition, the role of instruction is highlighted, making teachers and the practices they employ to create learning environments for students central in the development of students' understanding of mathematical concepts.

Researchers have recently begun to investigate the ways in which teachers may use LTs in their practice. For instance, Wilson and Mojica (2010) reported that knowledge of an LT supported both practicing and prospective teachers in constructing models of students' thinking. Furtak (2009) concluded that knowledge of an LT helped teachers in viewing students' conceptions along a continuum of proficiency rather than simply "right" or "wrong." Wilson (2009) described how LTs assisted teachers in interpreting evidence of cognition when analyzing student work and in interacting with students during instruction. He suggested that LTs sensitize teachers to a variety of strategies used by students when gaining proficiency with a mathematical idea. He also conjectured that LTs provide a framework for teachers to organize students' work in class discussions. The research reported in this paper investigated this last conjecture further.

This study is part of a larger project that aims to conceptualize the notion of *Learning Trajectories Based Instruction*, broadly defined at the outset of the project as the ways in which teachers use their own knowledge of a LT to organize their instructional practices and participate in their professional communities. As the context for our investigation, we are working with one LT: the learning trajectory for equipartitioning (EPLT), which is the set of cognitive behaviors that have the goal of producing equal-sized groups (from collections) or equal-sized parts (from continuous wholes), or equal-sized combinations of wholes and parts, such as is typically encountered by children initially in constructing 'fair shares' for each of a set of individuals (Confrey et al., 2009). The EPLT describes how children's informal experiences with creating fair shares evolve over time to a robust understanding of partitive division. The proficiency levels of the EPLT document the strategies, practices, emerging relationships, generalizations and the misconceptions children experience through instruction as they mature to an understanding of $a \div b$ as creating b equal-sized parts of a.

As students engage in equipartitioning tasks, the EPLT describes how they work to coordinate three equipartitioning criteria: (1) students must create the *correct number* of groups or parts; (2) students must create *equal-sized groups or parts*; and (3) students must *exhaust the whole or collection*. Also, the EPLT outlines how different parameters affect the level of difficulty of the task, such as sharing a whole among eight people is less difficult than sharing a whole among five people. The trajectory goes on to characterize the practices of naming and justifying answers to equipartitioning problems. For example, children proceed from nonmathematical names for each share such as parts or pieces, to counts and then to numbers in relation to a unit. Three important emerging relationships are also considered in the trajectory: compensation, the idea that producing more equal-sized parts decreases their size; composition of splits, the idea that one split across another split affects the size of every part in a multiplicative way; and transitivity, the idea that non-congruent parts resulting from the same split on the same whole are of equal size.

To frame our examination of connections between teachers' experiences with EPLT and pedagogical practices that support the orchestration of productive mathematics discourse in the classroom, we draw on the work of Stein, Engle, Smith, & Hughes (2008). These authors proposed a set of five practices to examine teachers' work in supporting discourse: anticipating, monitoring, selecting, sequencing, and connecting. In this paper, we focus on the last three of these practices. *Selecting* is the practice of choosing particular students to share their work with the rest of the class to get certain ideas on the table while remaining in control of which students present and what the mathematical content of the discussion will likely be. *Sequencing* is the practice of making decisions about how to order the students' presentations to maximize the chances that the mathematical learning goals for task are achieved. *Connecting* is the practice of helping students develop relations among presentations and judge the consequences of different approaches for various problems, making sure students' presentations build on each other to develop powerful mathematical ideas.

Methods

The specific research question addressed in this study is: In what ways do teachers use their understandings of EPLT to select, sequence and connect students' mathematical ideas for whole class discussion? We conducted a design experiment within a school-based professional development setting. Design experiments are used to provide "systematic and warranted knowledge about learning and to produce theories to guide instructional decision making" (Confrey, 2006, p. 136). They "entail both 'engineering' particular forms of learning and systematically studying those forms of learning within the context defined by the means of supporting them" (Cobb, Confrey, diSessa, Lehrer, and Schauble, 2003, p. 9). The data set for the study is comprised of videos of professional development meetings and audiotapes (with transcripts) of small group discussion during the professional development. As it is usual for design experiments, data analysis consisted of constant comparison methods (Glaser, 1992; Strauss & Corbin, 1998), which allows for the creation of emerging categories in the data analysis and the refinement of these categories as they are contrasted with new information. The various sources of data were also used for triangulating information (Miles & Huberman, 1994) in search of both confirming and disconfirming evidences. As we examined the ways in which teachers engaged with one particular professional learning task, we coded the data for teachers' use of their knowledge of particular aspects of the EPLT to justify their arguments about how to select, sequence, and connect students' responses in whole group classroom discussion.

Participants

We partnered with one elementary school in a mid-size urban area in the southeast. Twenty-four K-5 teachers from this school volunteered to participate in the professional development. Only three of these teachers had less that 3 years of teaching experience and about half of the teachers had more than 10 years of experience. Many of the teachers have been at this school for a few years. Overall, this school has a yearly turnover teacher rate of 4% and a student population of approximately 600 students. The student population in the 2010-2011 school year was 35% Caucasian, 29% Hispanic, 25% African American, 7% Asian, and 4% other, with 54% of the children on free or reduced lunch.

Professional Development Design and Professional Learning Task

The professional development was designed to offer a total of 96 hours of work on the concept of EPLT over a period of 12 months. The initial 30 hours of professional development were offered as a Summer Institute, followed by 54 hours of face-to-face and web-based activity during the school year, ending with 12 hours the following summer. This report is based on the work conducted during the Summer Institute. The 30 hours of work for the institute were spread over 6 days, immediately before the 2010-2011 school year started. We worked with the teachers for 6 hours during the first 3 days, and then for 4 hours during the following 3 days. All meetings took place at the school, giving teachers the opportunity to also work on organizing their classrooms after the professional development meetings.

The research reported in this paper concerns a professional learning task that was posed to the teachers on the fourth day of the Summer Institute. On Days 1–3 teachers had investigated the concept of equipartitioning, the proficiency levels of the EPLT, and the parameters affecting the difficulty of tasks at a particular level. Through analysis of student work and video clips of children solving various equipartition tasks for collections and single wholes, teachers had discussed the equipartitioning criteria, naming practices, and emerging relations such as compensation, composition of splits, and transitivity.

We began Day 4 of the Summer Institute with a task that aimed to provide teachers with an opportunity to make explicit connections between their learning in the professional development and their classrooms instruction. In this task, we purposefully asked participating teachers to select, sequence and connect student work. Teachers were given a scenario where students in a second grade classroom had worked on the task of fairly sharing a rectangular cake among four people and the second grade instructor saw four different solutions to the task (represented in figure 1, labeled A, B, C, and D). In cross-grade-level groups, teachers were asked to discuss the following question: "How would you select and sequence the presentation of these solutions for a whole class discussion? Why?"

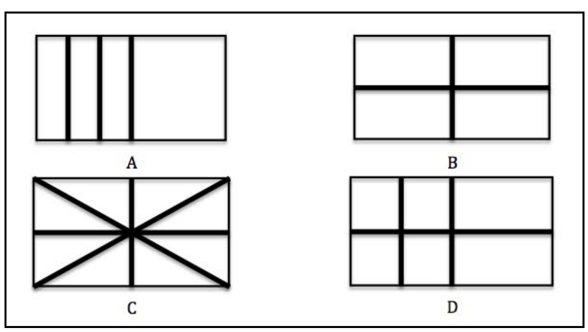

Figure 1. Representations of the approaches to sharing a whole among four.

After the small group discussions, the teachers viewed an 18-minute video recording of the actual second grade classroom. In the video the classroom instructor starts by having students share solution C. The students indicated that each person would get two pieces and that each piece was one-eighth. Next, a student shared approach A, and the instructor guided the students to see that each of the four pieces needed to be the same size and that if they were of equal size, then each could be called one-fourth. Approach D was shared next and students told the class that they realized that although they had not used all of the cake. The instructor concluded the sequence by having a pair of students share approach B, and made connections to the two-eights created in approach C and the parts created in approach D.

In the whole group discussion of the video, teachers examined the instructor's choices in selecting, sequencing, and connecting her students' responses. After each solution was shared with the second grade class in the video, the recording was paused and the teachers were asked to discuss the mathematical issues that arose as students shared a particular solution. They conjectured what they would do at that particular moment in the lesson if they were teaching it and examined the connections the actual instructor made.

Results

Selecting and Sequencing

Selecting and sequencing student work was not a trivial task for participating teachers. It elicited a rich discussion in the small groups and only one of the cross-grade-level small groups actually agreed on a sequence for presenting students' answer: D, A, B, and then C. One group reached consensus that they should begin with approach A, and another group agreed they would begin their sequence with approach D. The two other groups failed to reach a consensus, with one not being able to solve whether they would begin with A or C and the other debating whether to begin with A or D. In the process of these discussions, three aspects of the EPLT emerged in teachers' justifications for how they would select and sequence these particular students' solutions: the three equipartitioning criteria, students' misconceptions, known strategies for creating equal parts.

All of the small groups made reference to the three equipartitioning criteria as they worked to make sense of the four approaches. The teachers used the criteria both in understanding solutions A through D and in examining how they might sequence these solutions for discussion. In selecting a response to begin a classroom conversation, one teacher told her group, "I would start this one [*D*] just because I think this is the problem that…that they're not using the whole. They are thinking it's a cake and they've made four pieces but they've not used the whole." In another group, a teacher stated, "I would probably share the top left corner [A] also and then talk about, you know, first that you did use a whole; that you do have the right amount of pieces, and then really look into 'are the pieces equal?'" In trying to decide which of the criteria she should address first, one teacher questioned her group, "Did we ever establish that there is necessarily a hierarchy of those three criteria?" In this, the teacher was not only recalling the 3 criteria from the EPLT but was also adopting a perspective of student thinking as a continuum of least to most sophisticated. These examples are representative of the teachers' comments across the small groups. Being a pre-requisite for obtaining fair shares, all teachers considered that these criteria needed to be explicitly addressed in the classroom as they made sense of various students' responses.

Concerning the three criteria, approach C generated interesting discussion as one teacher questioned whether the student had created the correct number of parts. Other teachers wanted to use approach C to think about a ratio relationship. "The reason I would do it [C] is because it seems like you can turn that into a higher order, you know, solution. You can say, Okay, I see this one has eight wedges, whatever. Eight triangles. There are eight pieces and we have four [people]." For these teachers, although approach C showed eight pieces, it represented an opportunity to explore equivalent fractions. In another group, a teacher noted the same idea: "So that would be a really good teaching point if you want it. I mean, that… showing the eight could be a really powerful thing."

Teachers also made use of known misconceptions considered in the EPLT as they discussed how they would select and sequence students' solutions in their classrooms. Four of the small groups referred to the misconception that when creating equal-sized parts of a whole some children believe the number of cuts they make is the same as the number of parts they wish to create, usually in the context of making parallel splits. Though the work samples used in the activity did not depict this exact misconception, teachers recalled it in their group conversations. In particular, teachers were concerned that approach C could foster such misconception, as one teacher argued: "One of the dangers is them equating making just four cuts and then to always…it'll always work." Later she repeated: "there is a risk in this [C] if they're going to think 'oh, make four cuts, no matter what'." Similar exchanges happened in three other small groups. As teachers discussed their sequences of approaches for class discussion, erroneous conceptions documented by the EPLT influenced their analyses and considerations of how to chose and order the students' approaches.

In relation to students' equipartitioning strategies, two of the small groups explicitly discussed but all groups made at least some reference to students' use of "halving" and "repeated halving" as a strategy, as well as the relative ease of equipartitioning wholes into two equal parts. In the previous days of the Summer Institute, teachers had learned a variety of different strategies students use to equipartition a whole described in the EPLT, but in examining the given sample of student work, they repeatedly related to halving and repeated halving. As one teacher observed, "I guess given what we talked about last week, halving and halving is easiest. But I mean, like, you know, what would my reasoning be for sharing the easiest?"

Connecting

When watching the actual classroom video, the teachers drew on several ideas from the EPLT to describe the connections the instructor made and to conjecture others. As in the selecting and sequencing small-group exercise, teachers referred to the three equipartitioning criteria to understand the instructor's sequencing and to describe the relationships among the approaches. For instance after sharing the first approach [C], one teacher suggested that the teacher should focus on the number of parts created in subsequent discussion. In response to a question from the researcher about what the instructor may be thinking and where she could take the discussion, the teacher said, "Counting, like 'how many pieces did you make? Let's count them' and like see that there's more than four." After viewing the instructor's discussion of approach D in relation to C, and how two of the pieces created in C matched into one of the pieces created in D, one teacher commented, "And it was really nice how she made the connection back to the previous work to show the change." The three equipartitioning criteria helped teachers notice connections made by the instructor as well as suggest other connections that could have been made.

Another way that the EPLT informed connections teachers noticed was by calling attention to the issues of naming equal-sized parts in relation to the original whole. As the teachers observed the connections the classroom instructor was making in the video, they noted the different naming practices and commented on the instructor's support of its refinement. For example in approach A, a student first referred to a share as "a piece." One teacher commented, "Well she saw that you got a, uh, qualitative answer. We saw that before. A piece. And then she was trying to go for the quantitative one." Another teacher followed up on this comment about naming by pointing out the clarity of the instructor's language. She stated, "And she made a kind of subtle distinction too, because the child who attempted to answer what the second one was, said, 'A second fourth?' And so, she used that but she said, 'a second one-fourth.' Just to kind of clarify that." Thus, the teachers' exploration of the subtleties and challenges of naming parts of wholes supported their noticing of the connections the instructor made.

Perhaps the most profound way that the EPLT affected the connections the teachers suggested was their use of emerging relationships to examine links across students' approaches. While the instructor's connections in the video centered on the three equipartitioning criteria and on naming, participating teachers also focused on emerging relationships. For instance, regarding transitivity, after viewing how two of the small pieces in approach D produced a fourth, one teacher suggested posing a question relating these results to the eights generated from approach C. She stated, "It would have been interesting, I mean, if we had said, okay, well are these…are these two pieces the same [D]? Is it the same fair share as these two pieces [C]?" This led to a discussion among the teachers and PD facilitators about how to show that non-congruent parts are actually the same. Also, the teachers connected the issue of two-eighths and one-fourth to the ideas of compensation and composition of splits. In referring to approach B, one teacher summarized, "Composition of splits - so, if you make a new cut then it affects all the other pieces - but it also is compensation because now there are more people sharing the smaller one." This teacher was noticing that by splitting along the diagonals in approach B, the single split acts on multiple parts, the essence of composition. Further, it creates more equal-sized parts but that each of the parts is smaller. By providing teachers with a description of these relationships that emerge from engaging in equipartitioning tasks, the EPLT assisted the teachers in connections that may support students in gaining proficiency with these emerging relationships.

Discussion

In this paper we examined the ways in which teachers who participated in a 30-hour Summer Institute on the EPLT used their understandings of this particular LT to engage with the three key pedagogical practices of selecting, sequencing and connecting students' mathematical ideas. We noted that teachers used the three criteria for equipartitioning to support their work in all of these three key pedagogical practices and, in fact, these criteria guided most of teachers' decisions when engaged in these practices. Teachers highlighted the importance of supporting students as they develop these criteria by proposing to select and sequence tasks that highlight the criteria and also to use the criteria as a venue for connecting the selected strategies. Teachers also suggested ways to assist students in coordinating the criteria. We also noted that teachers used known misconceptions and splitting strategies when selecting and sequencing students' solutions, whereas they used their knowledge of naming and emerging mathematical relations to examine connections across solutions. Teachers suggested ways to confront students' misconceptions as well as to support the development of emerging relations, which represent more sophisticated levels of thinking in the EPLT.

We started our design experiment using Wilson's (2009) revised conjecture that LTs provide a framework for teachers to organize students' work in class discussions. By examining how teachers used the EPLT in three specific pedagogical practices, we can now further refine the initial conjecture. We contend that LTs allow teachers to put students' mathematics at the center of their instructional decision, bringing specificity to the ways in which teachers' engage with the three key pedagogical practices of selecting, sequencing, and connecting (Stein et al. 2008) student work to generate productive mathematics discourse. The trajectory allows teachers to focus on students' solutions and thought processes to orchestrate discourse, instead of attending to non-mathematical ideas such as what the majority of the class did or what the teachers wanted students to do. Additionally, LTs support teachers in drawing connections among students' ideas and in identifying emergent relationships that build toward more sophisticated understanding.

Endnotes

1. Although the CCSS used the term learning progression, we take learning trajectories and progressions to be synonymous and we have opted to consistently use the latter in our work.

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Authors' Note: This repot is based upon work supported by the National Science Foundation under grant number DRL-1008364. Any opinions, findings, and conclusions or recommendations expressed in this report are those of the authors and do not necessarily reflect the views of the National Science Foundation.