USING CONCEPTUAL BLENDING TO ANALYZE STUDENT PROVING

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We explore ways that university students handle proving statements that have the overall structure of a conditional implies a conditional, i.e., $(p \rightarrow q) \Rightarrow (r \rightarrow s)$. We structure our analysis using the theory of conceptual blending. We find conceptual blending useful for describing the creation of powerful new ideas necessary for proof construction as well as for describing the creation of blends that slow or hinder student efforts at proof construction.

Introduction

The purpose of this paper is to illustrate the power of the theory of conceptual blending to clarify issues that students have in proving statements having the overall structure of a conditional implies a conditional, i.e., $(p \rightarrow q) \Rightarrow (r \rightarrow s)$. This logical structure occurs often in statements to be proven at the university level. For example, since the definition of A is a subset of B ($A \subseteq B$) is a conditional statement ($x \in A$ implies $x \in B$), then a simple set theory statement such as "If $A \subseteq B$, then $A \cup B \subseteq B$ " has this logical form.

The research literature indicates that students' misunderstanding of logical rules and misinterpretation of logical statements result in their difficulty with structuring their proofs (Brown, 2003; Duran-Guerrier, 2003; Harel, 2001; Roh, 2010; Selden & Selden, 1995). Students tend to structure their proofs in terms of the chronological order of their thought processes instead of rearranging it with careful consideration of proper implications (Dreyfus, 1999). The literature also shows that students are often unable to bring useful syntactic knowledge to mind. Such knowledge includes formal definitions (Knapp, 2006) as well as theorems and properties (Weber & Alcock, 2004) of the mathematical concepts. Likewise, research calls attention to various forms of personal knowledge of mathematical concepts. Such knowledge is internally meaningful to an individual student (Pinto & Tall, 2002; Vinner, 1991), and helps a student recall conceptual ideas to apply when attempting to construct a proof (Knapp & Roh, 2008). Because of its private and informal nature, students' personal knowledge is often insufficient for them to know how to get started on a proof (Moore, 1994). Raman (2003) suggested the *key idea* as a means of connecting personal intuitive ideas and procedural knowledge when constructing a proof. When students possess a key idea for a proof, it gives them conviction and the basis for the formal mathematical proof.

Theoretical Background: Conceptual Blending

Fauconnier and Turner (2002) posit conceptual blending as a powerful unifying theory to describe how people think across multiple domains. They argue that blending "makes possible … diverse human accomplishments … [in] language, art, religion, [and] science [as well as being] indispensable for basic everyday thought" (p. vi). This theory has begun to be used to describe student understanding of mathematical concepts (Gerson & Walter, 2008; Megowan & Zandieh, 2005; Núñez, 2005). In this section we give a brief example describing three of the main mechanisms of the theory of conceptual blending.

Conceptual blending is a subconscious process that entails the blending of two or more mental spaces (inputs) to form a new stable conceptual model for use in reasoning (See Figure 1). A mental space consists of an array of elements and their relationships to one another, being

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activated as a single unit. Two (or more) mental spaces are activated and crucial elements of each are integrated and *mapped* to a third space to form a blended space. As part of *completing* the blend, a conceptual frame may be recruited to help organize the information in the blend. Once the blend is complete it can be manipulated to make inferences or answer questions. This manipulation is referred to as *running* the blend. The blended concept is treated as a simulation that can be run imaginatively according to principles and properties that the input spaces bring to the blend.

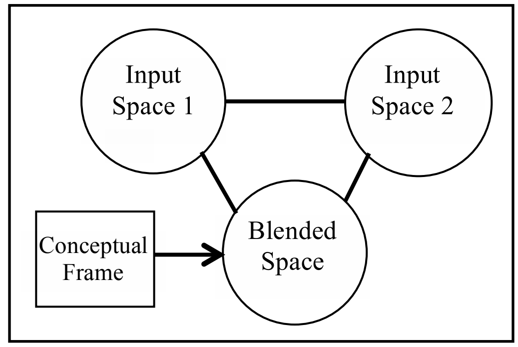

Figure 1. Generic blending diagram

For example Coulson and Oakley (2001) consider the nursery rhyme "the cow jumps over the moon." Children easily comprehend this statement by blending an input space of animals which includes cows, a second input space for the moon and sky, and a conceptual frame of jumping. In the blended space the cow is mapped to the thing that jumps and the moon is mapped to an object which is jumped over. Whereas children easily construct this blend, adults might have to inhibit their notions of reality and instead bring to bear a "nursery rhyme" frame which allows them to think of real things, the cow and moon, in impossible situations. Running the blend might include imagining the cow taking off from the ground, being over the moon, and landing on the ground on the other side of the moon.

Methods and Setting

The data for this study was originally collected as part of a semester long teaching experiment (Cobb, 2000) in an upper division geometry course at a university in the USA. Data consisted of videotape recordings of each 75 minute class session as well as copies of student written work. For the purpose of this paper we chose to analyze one day of class where we recognized something powerful was happening with student reasoning. Maher and Martino (1996) refer to such occasions as "critical events." The class period consisted of a brief introduction of the problem by the teacher (the first author), followed by small group work on the problem and whole class discussion. For the purpose of this paper we focus on the small group consisting of students we call Andrea, Nate, Paul and Stacey. The curriculum consisted of a series of activities in which students would need to define, conjecture, and prove results in geometry on the plane and the sphere (Henderson, 2001). This study focuses on one day late in the semester in which students were asked to prove either Euclid's Fifth Postulate (EFP) implies Playfair's Parallel Postulate (PPP) or PPP implies EFP.

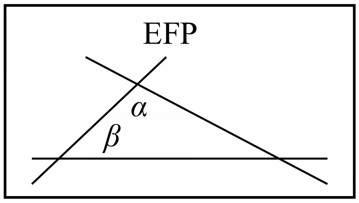

Figure 2. Euclid's Fifth Postulate

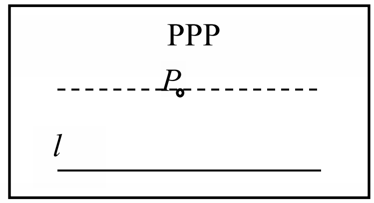

Figure 3. Playfair's Parallel Postulate

Henderson (2001) states EFP as, "If a straight line intersecting two straight lines makes the interior angles on the same side less than two right angles, then the two lines (if extended indefinitely) will meet on that side on which are the angles less than two right angles," (p. 123) and PPP as, "For every line *l* and every point *P* not on *l,* there is a unique line *l'* which passes through *P* and does not intersect (is parallel to) *l*" (p. 124). The instructor told the students that the two postulates are equivalent and gave them the option to "*use*" EFP in order to prove PPP or vice versa. In her introduction of the task the instructor drew the two figures shown (see Figure 2 and Figure 3) while explaining each of the postulates. The instructor's initial drawing showed only the part of the statement that was given. For example, for the PPP picture (see Figure 3), she initially just drew the bottom line, *l*, and a point, *P*, not on that line. However, when she explained the conclusion of each statement she completed the picture and these completed pictures were left on the board for students to reference. A second visual reference was available to students. The two pictures in the book for these two statements were also completed pictures similar to what the teacher had drawn.

Results and Analysis

In our first pass through the data, we noticed that elements of conceptual blending occurred both in students structuring of their proof and in their combining of the pictures and statements of EFP and PPP. We then read through the data looking to specify what blends the students were creating. We noticed the students were creating three types of blends: structural, geometric, and a combination of the two. We also noticed that the same blends occurred whether students were attempting to prove EFP implies PPP or PPP implies EFP. To better illuminate when each blend occurred in the data, each of the authors color coded a portion of the data and then all three authors came to a consensus on the coding for each of the following aspects:

- How students were blending the pictures associated with EFP and PPP: the key geometric blend (KGB) used by most students or Stacey's geometric blend (SGB).
- The logical construct that the students were using to frame their proof: a simple proving frame (SPF) or a conditional implies conditional proving frame (CICF).
- The direction of the proof: EFP implies PPP (EtoP) or PPP implies EFP (PtoE).

As we coded the data it became clear that there were four combined blends each of which could be described as an episode. In Figure 4 we summarize the evolution of student thinking through the four episodes, highlighting the three aspects of the combined blend: a structural blend (SPF or CICF), a geometric blend (KGB or SGB) and the direction of the implication (EtoP or PtoE). In addition to the main blend, we note a secondary blend if there were contrasting remarks or questions from other students during the episode that seemed to refer to a different combined blend.

Episode		*Main Blend*				*Secondary Blend*			
	Time	*Structure*	*Direction*	*Geometry*	*Presenter*	*Structure*	*Direction*	*Geometry*	*Presenter*
E1	9:00-	SPF	EtoP	KGB	Paul	CICF	EtoP	KGB	Nate
E2	15:13-	SPF	EtoP	SGB	Stacey				
E3	17:48-	SPF	PtoE	KGB	Andrea	CICF	EtoP	KGB	Nate
E4	24:38-	CICF	EtoP	KGB	Nate	SPF	EtoP	KGB	Paul

Figure 4: Summary of the progression of student proving ideas.

As illustrated in Figure 4, we found conceptual blending useful for describing the evolution of student thinking while proving. Using the three aspects of the combined blends, we were able to track the progression of the main thrust of the small group discussion as well as the contrasting voices in those discussions. In a longer paper (Zandieh, Roh, & Knapp, 2010) we describe the blends involved in each episode. Here we focus on three salient examples that illustrate the power of blending to describe student reasoning that moves the proof construction forward as well as student reasoning that slows proof construction.

The Key Geometric Blend (KGB)

From the beginning of Episode 1, students looked for a way to blend together the picture and statement of the two postulates geometrically and conceptually. They did this by creating the key geometric blend (KGB) which turned out to be the key idea of the proof for the students.

Stacey: Because no matter what we can put any P out there [reaching out her arms to touch her finger tips together] at our point of intersection [pointing to Andrea's notebook]. [Nate: That's a good point.] And then we know that, that it is unique [tracing a line with her pen], that it is not going to come back and intersect somehow. [...]

(Silence 1 minute -- Paul flips pages, Stacey flips pages (x2))

Paul: Well, if you assume the first one [EFP], would there be three cases that $\alpha + \beta < \pi$, $\alpha + \beta = \pi$, *or* $\alpha + \beta > \pi$? *And then the uniqueness part of it would be proved by the* $\alpha + \beta = \pi$ *and in that case they wouldn't meet.*

As students flipped pages between the picture and statement for EFP and the picture and statement for PPP they began *mapping to a blend.* The two figures in the text functioned as input spaces for a blended space (see Figure 5). Paul's three cases take the EFP picture and lay it on the PPP picture such that the bottom line from each of the input spaces (*m* from EFP and *l* from PPP) is mapped into the bottom line of the blended space. The transversal (*n*) from the EFP input space is included in the blended space, and the top line from each space is mapped into a line in the blended space. Finally, the point *P* from the PPP space is mapped onto the intersection of the transversal and the top line from EFP in the blended space (see Figure 5). To *complete the blend* the students brought to bear their previous geometric knowledge of lines and angles. Notice that Paul imagined three different possible locations of the top line in the blended space and coordinated the different geometric positions with different sums for $\alpha + \beta$. We would say Paul's idea of three cases comes from *running the blend*. After Paul's comment, Andrea and Nate also contributed to running the blend by imagining that if $\alpha + \beta > \pi$, then the lines would intersect on the other side.

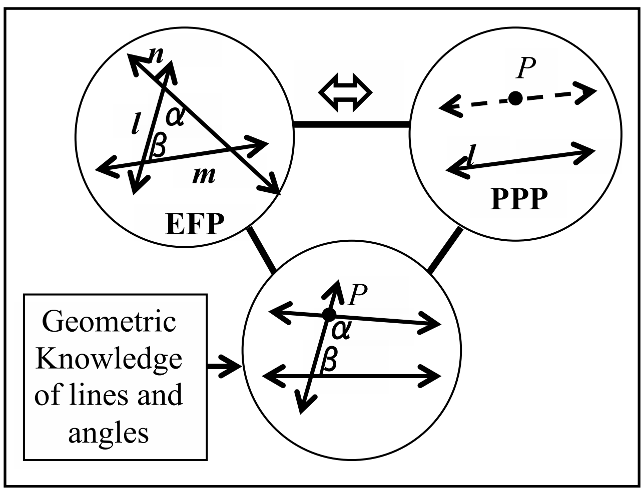

Figure 5. Key Geometric Blend (KGB)

The KGB was also involved similarly when the students attempted to prove the other direction, PPP implies EFP, in Episode 3. The main difference for the students in Episode 3 was that the blending of the two pictures was created by a slightly different mechanism. Andrea started with the PPP picture and constructed a transversal to create line *l* of the EFP picture. This construction does not change the basic content of the KGB, but it is significant in that it allowed students to see additional relationships in the geometric blend focusing on the case when the two lines are parallel transports of each other.

Andrea: If we assume this [points to PPP picture] and draw a transversal through here, through P and through this line. So we know that since these are parallel that they add to 180, right? Because they are supplementary, whatever.

Having agreed upon the sum of the angles being 180 degrees, Andrea then made her full argument more clearly as follows:

Andrea: So say that we rotate this line so that it makes it less than 180, the sum less than 180. [Nate: Yeah]... Oh, so, right! So that since we know that this is a unique line because we've assumed that there is only one line. So we rotate it so that it is a different line, then we know that it is going to intersect because there is only the one line that won't intersect.

The students continued to discuss the issues involved in Andrea's proof idea for a few more minutes, but ultimately returned to trying to prove EFP implies PPP. Although Andrea's proof was not completed, the KGB was fundamental to this discussion as it was in Episode 1 when Paul introduced the idea. In addition, both discussions of the KGB (in Episodes 1 and in Episode 3) played into Nate's idea for the proof that was discussed in Episode 4. Examples of this occur in the next two sections. The KGB was fundamental to discussions of the proof in Episodes 1, 3, and 4 and in the final write up of the proof. The KGB was the central idea of the proof, the key idea of the proof in the sense of Raman (2003). So the result of the blend was powerful for proving. In addition, the students ran the KGB over and over imaginatively as they worked out issues in the proof.

Simple Proving Frame (SPF) vs Conditional Implies Conditional Frame (CICF)

We introduce and compare two proving structures that the students used in their proofs of statements of the form $(p \to q) \Rightarrow (r \to s)$: the Simple Proving Frame (SPF) and the Conditional Implies Conditional Frame (CICF). By SPF, we refer to a proving frame where there is a given statement (premise), then a series of implications, then a conclusion (see Figure 6). There is nothing inherently wrong with the SPF or trying to apply it to a conditional implies a conditional statement. However, unless a student has particular theorems to work with that allow a direct proof from $(p \to q)$ to $(r \to s)$, then a simple proving frame (SPF) may be inadequate.

Generic SPF	Case 1: $(p \to q)$	Case 2: $(p \to q) \Rightarrow (r \to s)$
Given $\boxed{}$ … Series of Implications … Then $\boxed{}$	Given p … Series of Implications … Then q	Given $(p \to q)$ … Series of Implications … Then $(r \to s)$

Figure 6: Simple Proving Frame (SPF)

The students began to work on the task by looking at the pictures and statements of the two postulates described on two sequential pages in their book. As they began to think about which direction might be easier to prove and how to prove it, three of the four students each flipped back and forth between the two pages multiple times. We describe their deliberations in terms of a simple blend in which the two input spaces are the two postulates (see Figure 7). The students then were bringing the SPF to bear on the problem by putting EFP in the place of what is given and PPP in the place of the conclusion. To the extent that there is resolution in this early discussion the students seem to have created a blended space that is structured by the SPF with EFP as the given and PPP in the conclusion.

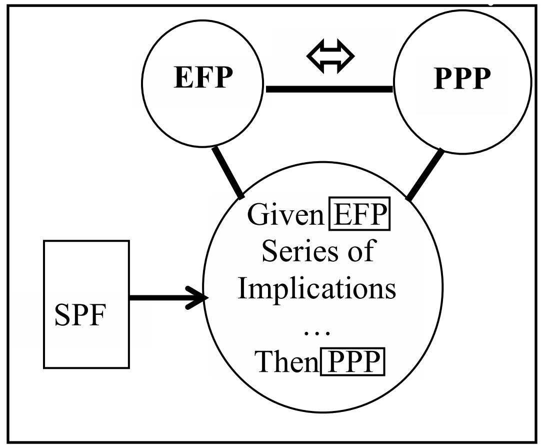

Figure 7. A structural blend of SPF from EFP to PPP

On the other hand, the CICF is substantially different from the SPF that the students used in Episodes 1-3. By CICF we refer to proofs of statements of the form $(p \to q) \Rightarrow (r \to s)$, where one starts with r and uses a series of implications including $p \to q$, to reach the conclusion, s (see Figure 8). In Episode 4 Nate began to lay out his case for the CICF more directly.

Nate: *[…] If we assume this [points to a drawing of EFP] is true? [Andrea: Okay, so this way?] This [EFP] is true for a moment. [Andrea: Okay.] Now we have our little point over there and we draw this line. We know that if this line is such that the angles on the one side are less than 180, that it is not parallel based on this assumption [EFP]. We know that if they are greater than 180, we can apply this assumption again and show that they do [intersect] on the other [side]. Can we use our parallel transport proof to show that the boundary condition when they are equal to 180, that this angle is congruent to this angle and therefore they are parallel and therefore they don't intersect?*

Nate first established that he was assuming EFP is true and therefore was proving PPP. Next, he hinted at the premise of PPP, "we have our little point over there and we draw this line," suggesting that he was starting with r of the $(p \to q) \Rightarrow (r \to s)$. He then stated how he could use EFP in the middle of the proof, "we can apply this assumption." This is the use of $(p \to q)$ in the series of implications. He also hinted at what else may be needed to get to the conclusion of PPP. As we will see in the next section, Paul, Stacey, and Andrea initially struggled with Nate's proof structure. Towards the end of the discussion they began to think Nate's idea might work. This was aided by the teaching assistant visiting the group and being supportive of Nate's idea. Following her departure the group prepared a presentation for the class based on Nate's idea. As the group began work on their presentation the three dissenters made contributions indicating that they understood Nate's blend.

Generic CICF	For the case of $(p \to q) \to (r \to s)$
Given $\boxed{}$	Given r …
…	Then p
Use $\boxed{}$	Since p and $(p \to q)$
…	Then q …
Thus $\boxed{}$	Thus s

Figure 8. The CICF

Blending the Premise and Conclusion

As explained above, students initially, from Episode 1, wanted to use a simply proving frame (SPF) that would start with EFP and end with PPP. Since the SPF is a legitimate proof technique in many situations and one that students were very familiar with, this is understandable, and might have even led to a proof if the students had appropriate theorems for this. However, in this case, the problem with using SPF was compounded by students blending the premise and conclusion of EFP in a way that lost the implication structure. We illustrate this with a transcript from Episode 4. As explained in the previous section, Nate had suggested his proof idea which used CICF. Other students in this group saw a similarity between their and Nate's idea in the sense of using three cases (KGB). However, they struggled with Nate's proof structure.

Paul: So, we don't even need to necessarily have the three cases, do we? Just we need to prove that one case – uniqueness on that one. Because we are assuming they meet on this side so I don't think it really matters if. [Stacey: Exactly. Agreed.] […]

Nate: I disagree because you are applying EFP to your specific cases. You're not assuming that α and β are less than. In order to apply it, you have to show that α and β are less than. […]

Paul: I think you are assuming that.

Stacey: Yeah, it assumes. Because we are assuming the whole EFP.

Andrea: We're assuming EFP.

Stacey: We're saying if we got two lines that are going to intersect on this one side, the interior ones right there are going to be, they have to be less than π.

Nate: Right, right. But now we are drawing a picture where we're going to say that α and β are less than and therefore we can apply EFP to show that they intersect. […]

Paul: If we're assuming that, then how can we say that they are going to be equal to π? Do you know what I am saying?

Nate: Well, that's my point. You have to draw three cases. You have to draw when they sum less than, when they sum equal, and when they sum greater than. And you have to apply EFP to two of those cases.

For Paul, Stacey and Andrea assuming EFP meant that they were assuming both the premise and the conclusion of EFP. In part, we see this as a faulty use of the SPF proof structure, the notion that we are starting with "all" of EFP and we will end the proof with the statement of PPP. In addition to the problem of using the SPF structure there is more specifically the idea that assuming "all" of EFP causes EFP to lose its implication structure. We describe this as a blending of the premise and the conclusion of EFP. Using blending theory we would say that there is a blend with two input spaces, (1) the premise of EFP ($\alpha + \beta < \pi$) and (2) the conclusion of EFP (the two lines intersect on the same side as α and β). The relationship between these two input spaces is that of an implication. However, when the two spaces are mapped to the blend, they are mapped to the same diagram with the implication having been compressed to an "and" or a simple coexistence without any implication structure. When the students were running this blend in the context of proving EFP to PPP using the SPF, they concluded that both the premise and conclusion of EFP were given, so it was not necessary to consider the cases when $\alpha + \beta > \pi$ or the lines didn't intersect, since students were assuming as part of EFP that the lines intersected and $\alpha + \beta < \pi$.

Summary

We find conceptual blending useful for describing the creation of powerful new ideas necessary for proof construction as well as for describing the creation of blends that slow or hinder student efforts at proof construction. We noted how students blended the two pictures of EFP and PPP and ran that blend in ways that allowed them to create a key idea (KGB) for the proof. This blending continued to serve as the foundation for the proof even through its final configuration. In addition, Nate's use of CICF was eventually blended with the KGB to construct the final, correct proof that this group presented to the class. On the other hand, there were two cases of student blending that served to hinder or slow their proving. In the first case we saw that students' initial use of an SPF structural blend hampered their efforts to structure their proof. The students did not have the necessary theorems to complete a proof of this conditional implies conditional statement using an SPF. In a second case we saw students treating “all” of EFP as given. EFP was treated as a collection of parts without maintaining the appropriate implication structure between these parts. As a result, the students’ conceptual blends led them to blend the premise and conclusion in ways that obscured the implication relation between them. Consequently, their heavy reliance on this proof frame in the initial discussions slowed their efforts. It is our suggestion that instruction should be more explicit in contrasting the use of SPF and CICF.

Acknowledgments

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