OPPORTUNITIES TO CONCEPTUALIZE LINEAR RELATIONSHIPS IN UNITED STATES MATHEMATICS TEXTBOOKS: BEYOND Y=MX+B

Linda Arnold
The University of Tennessee
larnold5@utk.edu

Ji-Won Son
The University of Tennessee
sonjiwon@utk.edu

This study examined opportunities provided for students to conceptualize linear relationships, as reflected in five United States mathematics textbooks. Texts represented a broad spectrum of types: commercial, so-called "back to basics", and NSF funded. Analysis of allocation, topic choice, presentation, context, and cognitive level was completed. Analysis results revealed that students are being asked to grapple with linear relationships at increasingly younger ages, limits in the models they are asked to use, limits on discussion of concepts and connections, lack a real world context for most problems, and lower levels of cognitive expectation. The results indicate a significant gap between learning goals from intended curricula, and the potentially implemented curricula contained in many current U.S. textbooks. Conclusions suggest ways in which the present curriculum may be transformed.

Linear relationships are important because they measure a basic way in which one quantity changes in relation to another. Frequently expressed as equations, graphs of lines and tables, almost all are functions. Linear functions are one of the foundational types of functions for students to understand in mathematics. As early as the 1920s, it was recognized that "without functional thinking there can be no real understanding or appreciation of mathematics" (Breslich, 1928, p.42). In the present day, calls for reform recognize that "the concept of function is an important unifying idea in mathematics (NCTM, 1989, p. 154).

Along the same line, recently published Common Core State Standards (2010) makes it clear that 8th graders should be able to: "Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values" (Common Core Standards, 2010, mathematics, grade 8, function, para. 4).

This standard is noticeably written in terms of concepts: "rate of change", "initial value", and "situation it models" and puts the emphasis on connections between descriptions, tables, and graphs. Reasoning and sense making, highlighted here, are cornerstones of mathematics (NCTM, 2009), and so must be integral to learning about linear relationships. Then how should linear relationships be presented in classrooms?

Despite a robust link between instructional attention to concepts and students' level of understanding, Hiebert and Grouws (2007) reported that "typical classrooms in the United States focus on low-level skills and rarely attend explicitly to the important mathematical relationships"(p. 2). Students tend to regurgitate "y = mx + b" and manipulate equations mechanically. Is it possible that textbooks have contributed to this problem by limiting expectations and opportunities for students? Have textbooks provided opportunities for students to learn about linear relationships in conceptual, connected terms?

Textbooks, as potentially implemented curricula, give messages of what students might know

Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.

and be able to do. Although various factors influence student learning and teachers deviate their textbooks, Donavan, Bransford & Pelligrino (1999) reported that students' learning, in fact, is highly correlated with curricular treatment of topics. Many researchers generally agree that the curriculum, especially the curriculum embodied in textbooks, has a large influence on learning and teaching (Son & Senk, 2010; Valverde, Bianchi, Wolfe, Schmidt & Houang, 2002). Therefore, recognizing the influence of textbooks on student learning, we sought to examine the following question: "What opportunities for students to learn about linear relationships are presented in five commonly utilized United States textbooks?" Various types of mathematics textbooks, such as commercial, so-called "back to basics", and NSF funded were examined. Answering this question will help us find a better way to enhancing student learning about linear relationships through textbooks.

Theoretical Framework

A growing body of research has analyzed textbooks in order to understand their potential effect on students' mathematical learning. While some studies focus exclusively on content analysis (e.g., Fuson, Stigler, & Bartsch, 1988), other researchers examined problems presented in textbooks (e.g., Li, 2002). This study examined both the content and problems presented in textbooks. In analyzing the content, we looked at allocation, and topics. In analyzing problems, we looked at context, response type and cognitive level, which were identified by researchers (e.g., Li, 2002; Son & Senk, 2010). In particular, to analyze cognitive level, we built on Webb (2000)'s framework as shown in Table 1, which will be discussed in Methods.

Level	**Characteristics**
1: Recall	Recall of a fact, information, or procedure
2: Skill/Concept	Use information or conceptual knowledge, two or more steps etc.
3: Strategic thinking	Requires reasoning, developing plan or a sequences of steps, more than one possible answer
4. Extended thinking	Requires investigations, time to think and process multiple conditions of problems

Table 1. Depth of Knowledge Levels

Methods

In this study, we analyzed five textbooks, as addressed in Table 2. Similar to Slavin and Lake's framework (2007), three categories of textbooks were selected: (a) "Back to Basics," which originated in the 1970s and early 1980s, (b) "National Science Foundation (NSF) funded" texts, which originated in the 1990s in keeping with mathematics reform standards and (c) "Commercial Texts", which are typically reissued every five to seven years. In addition we selected, (d) a historic text, which was which originated in the 1960s. We felt these categories allowed us to examine representative types of texts currently available to students and the historic text allowed comparison. Each textbook was part of a series and represented the point in the series where linear graphs were first introduced.

Code Name	Origins	Date of Edition	Grade Level	Categorical Description (quotes: Slavin & Lake, 2007)
Historic	1960s	1965	9th	Commonly used Algebra text: 1960s & '70s
Back to Basics	Early 1980s	2001	8th	"A back-to-the-basics curriculum that emphasizes building students' confidence and skill in computations & word problems."
NSF Funded	1990s	2003	9th	"Textbooks developed under funding from the National Science Foundation (NSF), that emphasize constructivist philosophy… problem solving, manipulatives, and concept development …with relative de-emphasis on algorithms."
Commercial Text One	various	2005	8th	"Commercial textbook programs which include computational fluency but also are written to help students develop concepts."
Commercial Text Two	various	2005	8th	"Commercial textbook programs which include computational fluency but also are written to help students develop concepts."

Table 2. Textbooks Examined

- Historic = Dolciani, M. P. (1965). *Modern Algebra Book 1.* Boston, MA: Houghton Mifflen.
- Back To Basics = Saxon, J. H. (2001). *Pre-Algebra.* Norman, OK: Saxon Publishers, Inc.
- NSF Funded = Fendel, Resek, Alper, & Fraser. (2003). *Interactive Mathematics Program: Integrated High School Mathematics: Year 1.* Emeryville, CA: Key Curriculum Press.
- Commercial Text One = Larson, Boswell, Kanold, & Stiff. (2005). *Pre-Algebra.* Evanston, IL: Houghton Mifflen.
- Commercial Text Two = Malloy, Boswell, Willard, & Sloan. (2005). *Pre-Algebra.* New York, NY: Glencoe/McGraw Hill.

Analysis involved two broad foci: content analysis and problem analysis, as presented in Table 3. Content analysis included examination of (a) allocation and (b) topic presentation. Problem analysis included examination of (a) contextual features, (b) mathematical features, and (c) cognitive performance requirements.

Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.

Analysis Category	Description
Content	
● Allocation	(a) What are the number of problems allocated to the topic? (b) What percent of the textbook does the topic occupy? (c) How much time do students spend on the topic?
● Topics	(a) Sub-topics presented including slope and y-intercept (b) Breadth of presentation of further topics
Problem	
● Context	(a) Real world context (b) Exclusively numeric exercises
● Response Type	(a) Single response number, point, or vocabulary word (b) Equation only (c) Graph only (d) Table only (e) Extended response, such as "explain" or "describe" (f) Mixed response, i.e., some combination of the above
● Cognitive Level	(a) Recall (b) Skill/Concept (c) Strategic Thinking (d) Extended Thinking

Table 3. Analytical Foci of the Study

Summary of Results

Content Analysis Results

Results from Analysis of Allocation

We examined allocation, which involved the number of problems, as well as the proportion of time and text devoted to the topic. Table 4 shows both trends and some interesting constants from historic to current textbooks.

Text	Historic	Back to Basics	NSF Funded	Commercial Text One	Commercial Text Two
Grade Level	9th	8th	9th	8th	8th
Total Number of problems	315	44	44	304	305
Percent of text devoted to linear relationships	7.3%	1.1%	8.7%	8.0%	9.6%
Approximate class time	Under 3 weeks	N/A	About 3 weeks.	About 3 weeks.	Just under 3 weeks.

Table 4. Results from Analysis of Allocation

The historic textbook and two commercial texts have much in common with respect to the total number of problems given, percent of text devoted to the topic and approximate class time. Each used a very similar number of problems; about 310, and similar timing: about 3 weeks. In fact, all but the "back to basics" text introduced linear relationships by spending about 3 weeks on them, devoting an entire unit (chapter) to the topic. The percent of material showed a slight general increase: from 7.3% in 1965 to 9.6% in 2005. This may indicate growing emphasis on linear relationships in modern and more technological times.

The introduction to graphing linear equations has been "moved down" to the 8th grade level from historic to current textbooks. This seems to be part of a general trend. Indeed, Common Core Standards now expect that students will see the material in 8th grade.

Compared to other textbooks, the back to basics text did not use chapters, but reviewed many topics each day. It was not possible to determine the exact amount of time focused on linear relationships, hence the "N/A" in the table above.

Results from Analysis of Topics

Table 5 summarizes the various separate objectives pertaining to linear relationships in the five textbooks. We further analyzed topics presented in textbooks, the breadth of presentation by answering the following questions: (1) Were a large number of sub-topics presented, stretching beyond such things as equation, slope, intercept, graphs and tables? (2) If so, this might affect the depth to which those more foundational conceptual topics could be covered.

Textbook	**Objectives beyond points, equation, slope, intercept, graphs & tables**
Historic Text	Equation of a line through two points, a line with a given slope through a given point, and a line parallel and/or perpendicular to a given line through a given point. Points where lines intersect. Graphing parabolas.
Back to Basics	Students do not go beyond plotting three points and sketching a line.
NSF Funded	Students do not go beyond the topics above, though they explore these in conceptual depth.
Commercial Text One	Equation of a line through two given points, a line though one point, though a given parallel and/or perpendicular line. The equation of a line of best fit. Graphing inequalities.
Commercial Text Two	Direct variation and inverse variation. Lines of best fit. Equation of a line through two points and equation of a line from a table. Graphing systems of equations, and inequalities.

Table 5. Breadth of Presentation: Topics

As shown in Table 5, large numbers of sub-topics (objectives) were included in a majority of the texts. The back to basics text, however asked only that students produce a table with three points, graph the points, and draw a line through the graph in their initial foray into linear relationships. The commercial and historic textbooks included numerous sub-topics.

The United States curriculum has often been criticized as "a mile wide and an inch deep" (Common Core State Standards, 2010). We examined how this tendency toward broadness remains from historic to current textbooks. Surprisingly, a very large number of objectives are presented in the commercial texts, just as in the historic text of about forty to fifty years ago. A plethora of topics was not attempted by the back to basics and NSF funded text, however.

Problem Analysis Results

Results from Analysis of Context

Table 6 shows the frequency of two categories of contextual features. Contextual features involve whether a setting is given for the problem, or whether it is devoid of real world context and merely in the form of an "exercise".

Text	With Context	Without Context
Historic	1%	99%
Back to Basics	0%	100%
NSF Funded	100%	0%
Commercial #1	15%	85%
Commercial #2	12%	88%

Table 6. Problem Context

There was a great deal of variation between texts when it comes to providing real world context. The back to basics and historic texts essentially did not include it. In more recently published commercial texts, a middle ground approach was utilized, with an average of 12.5% of problems including some context. In contrast, all problems in NSF funded reform texts were presented in regard to a real-life situation. In this case, the situation involved one over-arching unit theme problem. Thus, students could relate all problems in this text to real life applications. This indicates that students with the NSF funded text will learn linear relationship differently from students with other textbooks.

Results from Analysis of Response Type

Response types were categorized as single number/word, equation, graph, table, written word, or mixed. Single point coordinates were coded with single numbers. Written words (plural) in response might have required a phrase or a sentence to explain or justify, but sometimes called for a longer response in the form of sentences or paragraphs. A mixed response was some combination of responses, for example, "Make a table, then plot a graph".

Text	Single Number	Equation	Graphic	Tabular Response	Written Words	Mixed Response
Historic	50%	17%	19%	0%	6%	7%
Back to Basics	0%	0%	39%	0%	0%	61%
NSF Funded	34%	21%	9%	2%	30%	5%
Commercial #1	37%	20%	14%	1%	24%	4%
Commercial #2	48%	11%	18%	0%	11%	11%

Table 7. Response Types

Regarding limits on these models, it was striking that single numbers were the most common type of response in most of the texts. Equations and graphs tended to be the second most common. Tables were rare as an exclusive response type, but were often included in a mixed response. The NSF funded text allowed for the greatest number of written word responses, which were almost entirely absent in historic and back to basics texts, and comparatively rare in at least one of the commercial texts. Written word responses and mixed responses, the response types most oriented toward multiple representations and connections between models, were generally limited, except as noted. Kilpatrick, Swafford and Findell (2001) stress that multiple representations and connections are important for fostering understanding. Analysis of response type indicates that students with the NSF funded textbook will be provided more opportunities to learn linear relationships with multiple representations and connections.

Results from Analysis of Cognitive Expectation

Cognitive Expectation refers to the type of problem solving required which consists of four levels: recall facts (level 1), concepts (level 2), strategic thinking and evidence (level 3), and extended thinking (level 4). Table 8 shows the percentage of each cognitive expectation in each textbook.

Text	Level 1: Recall	Level 2: Skill/Concept	Level 3: Strategic thinking	Level 4: Extended thinking
Historic	83%	11%	7%	0%
Back to Basics	100%	0%	0%	0%
NSF Funded	0%	0%	0%	100%
Commercial #1	61%	20%	16%	3%
Commercial #2	71%	11%	17%	1%

Table 8. Results on Cognitive Expectation

The large numbers of problems devoted to recall were linked to a procedural presentation of how to do these problems in textbook examples. Thus, the NSF funded textbook was able to attempt greater depth of presentation in comparison with other texts. It should be noted that all 44 problems in the NSF funded text were grouped into a single theme, and so they were coded as one, single, extended problem with 44 parts. However, none of problems in the historic and back to basics texts were at Level 4, and thematic connections between problems, such as those in the NSF funded text, were rare.

Regarding Level 3 "Strategic Thinking," there was an increase in problems of this level from the 1960s to the present. This shows that we may presently be moving toward problems of somewhat higher cognitive demand. Even though an average of only about 34% of problems were conceptual and strategic in the commercial texts, these problems take more time than mere recall problems. Thus, while it is impossible to say for sure, it is possible that students in commercial texts have the opportunity to spend well over half their time or more on conceptually oriented problems.

It is striking, however, that a large percentage of problems in the four non-NSF funded texts were categorized into Level 1, demanding no more than 'recall'. Also, it is comparatively rare for students to work on extended thinking problems if teachers use the historic, back to basics, or commercial textbooks.

Discussion and Implications

This study presents different student opportunities to understand linear relationships in a majority of texts examined from historic to current textbooks. There were numerous similarities between the historic textbook and the present day commercial texts. These included the number of problems, and the time devoted to the topic. The method of presentation was either largely or moderately procedural in these texts. The NSF funded text, on the other hand, showed the greatest difference from the historic text. All of its problems involved real world context and were geared toward extended thinking, in the form of a project, something the historic text did not include.

Regarding content, students using commercial texts who are just being introduced to linear equations are expected to master a broad array of objectives, such as "find the equation of a line between two points" and "find the equation of a line through a given point with a given slope". A deeper curriculum, providing more time for actual understanding of concepts such as slope and y-intercept, may be in order. This finding suggests that students may be limited with regard to models, timing, topics and expectations. In contrast, however, students may be receiving an overly broad a treatment of the topic, a treatment which lacks sufficient depth.

In particular, this lack of depth can be seen in problem presentation. The back to basics and historic texts showed a high degree of orientation toward mere procedural presentation. The NSF funded text, in contrast, was highly oriented toward conceptual problems, with commercial texts somewhere in between. Similarly, regarding real world problem context, historic and back to basics texts tended to provide little, commercial texts a moderate amount, and the NSF funded text provided real world context for all problems. Finally, continuing this theme, cognitive expectations tended to be largely procedural in the historic and back to basics text, to include some mix of procedural and conceptual orientation in commercial texts, and to be almost entirely conceptually oriented in the NSF funded text.

This study also revealed that there was a difference between a high degree of cognitive expectation and a high degree of symbol manipulation. A problem on finding the equation of a line between two points, for example, requires lengthy manipulation of equations, but can be done with little or no conceptual basis. Are we failing to give students enough problems with high cognitive expectation, all the while "feeling good" because we ask more of them in terms of what they do with symbols? It is important for future researchers to address such questions. None of the texts examined appeared to sufficiently address all five strands of mathematical proficiency articulated by Kilpatrick, Swafford and Findell (2001). The NSF sponsored text came the closest, addressing adaptive reasoning, strategic competence, conceptual understanding and productive disposition. However, it failed to address procedural fluency.

This study has implications for curriculum developers, school administrators and teachers. All three groups can help students gain better opportunities to understand linear relationships. This study stresses the important role of teachers. Teachers need to recognize the gap between the intended and potentially intended curriculum and modify their textbook. They can use supplementary materials for the NSF text, for example, providing practice with procedures. Supplementary materials for commercial texts might provide practice with extended thinking problems and additional real world context.

Curriculum developers also should recognize the gap between the intended deeper curriculum in such documents as the Common Core State Standards and the potentially implemented curricula in current textbooks. They should try to minimize this gap by providing context, attending to depth of conceptual topic presentation, calling for a balance of response types, including sufficient procedural practice, and calling for appropriately deep levels of cognitive expectation.

School administrators need to consider the types of textbooks they select. If they want to increase student understanding, reasoning and sense making, they need to choose textbooks which emphasize all five strands of mathematical proficiency, where possible, like those suggested above.

Future research may reveal optimal blends for the factors examined in this study. For example, while the study reports on the percent of problems with a real world context, it is up to future research to reveal whether or not it is helpful for all problems to be set in a context.

To sum up, the data revealed some limited learning opportunities. They indicate that though students are being asked to grapple with linear relationships at increasingly younger ages, that they are limited in the models they are asked to use for linear relationships, that discussion of concepts and connections is often limited, that the majority of problems lack a real world context, and that a majority of problems address lower levels of cognitive expectation. Often, linear relationships are presented in textbooks in a way that, while very broad, in keeping with detailed and lengthy individual state standards, does not aim for depth of understanding. The results therefore revealed a significant gap between learning goals as set forth in intended curricula, and the potentially implemented curricula contained in many current U. S. textbooks.

References

Breslich, E. R. (1928). Developing functional thinking in secondary school mathematics. In J. Clark, & W. D. Reeve (Eds.), *Selected Topics in the Teaching of Mathematics (National Council of Teachers of Mathematics Yearbook)* (pp. 42-56). New York: Teachers' College-Columbia University.

Common Core State Standards Initiative. (2010). *Common Cores Standards for Mathematics.* Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Donavan, M., Bransford, J., & Pellegrino, J. (1999). *How people learn: Bridging research and practice.* Washington D. C.: The National Academies Press.

Fuson, K., Stigler, J., & Bartsch, K. (1988). Grade placement of addition and subtraction topics in Japan, Mainland China, the Soviet Union, Taiwan and the United States. *Journal for Research in Mathematics Education , 19*, 449-456.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding It Up: Helping Students Learn Mathematics.* Washington, DC: National Academies Press.

Li, Y. (2000). A comparison of problems that follow selected content presentations in American and Chinese mathematics textbooks. *Journal for Research in Mathematics Education , 31* (2), 234-241.

National Council of Teachers of Mathematics (NCTM). (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics (NCTM). (2009). *Focus in high school mathematics: Reasoning and sense making.* Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Slavin, R., & Lake, C. (2007). Effective programs in elementary mathematics: A best evidence synthesis. *Review of Educational Research , 78* (3), 427-515.

Son, J., & Senk, S. (2010). How reform curricula in the USA and Korea present multiplication and division of fractions. *Educational Studies in Mathematics , 74* (2), 117-142.

Valverde, G., Bianchi, L., Wolfe, R. G., Schmidt, W., & Houang, R. (2002). *According to the Book. Using TIMSS to investigate the translation of policy into practice through the world of textbooks.* Dordrecht: Kluwer Academic Publishers.

Webb, N. L. (2002). Technical issues in large-scale assessment. Council of Chief State School Officers.

Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.