LEARNING TRAJECTORIES: FOUNDATIONS FOR EFFECTIVE, RESEARCH-BASED EDUCATION

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Approaches to standards, curriculum development, and pedagogy are remarkably diverse; however, recent years have seen a growing movement to base each of these on learning trajectories. In this paper, I discuss and compare the various terms and conceptions of this construct, present our definition, differentiate between our conception and that of others', and briefly review some of our recent evidence in the area of early childhood mathematics paper.

Throughout history, approaches to standards, curriculum development, and pedagogy have been remarkably diverse. Recent years, however, have seen a growing movement to base each of these on *learning trajectories*. Examples include the National Council of Teachers of Mathematics' *Curriculum Focal Points* (2006) to the National Research Council's report (2009), and most notably the Common Core State Standards (CCSSO/NGA, 2010, for which the "progressions" of a learning trajectory were developed first—the standards followed). Here I compare and contrast different notions of this important concept and summarize results of recent empirical work illustrating its potential.

The term "curriculum" stems from the Latin word for racecourse, referring to the course of experiences through which children grow to become mature adults. Thus, the notion of a path, or trajectory, has always been central to curriculum development and study. In his seminal work, Simon stated that a "hypothetical learning trajectory" included "the learning goal, the learning activities, and the thinking and learning in which the students might engage" (1995, p. 133).

Building on Simon's definition, but emphasizing a cognitive science perspective and a base of empirical research, "we conceptualize learning trajectories as descriptions of children's thinking and learning in a specific mathematical domain, and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain" (Clements & Sarama, 2004, p. 83). The term "learning trajectory" reflects its roots in Simon's constructivist perspective (in emphasizing students' learning). However, although the name appears to focus on learning more than teaching, both Simon's and our definitions clearly involve teaching and instructional tasks. Some interpretations and appropriations of the learning trajectory construct emphasize only the "developmental progressions" of learning (what Simon calls hypothetical learning processes) during the creation of a particular curricular or pedagogical context. That is, they only describe levels of thinking through which students develop, which we believe is but one part of the learning trajectory construct. Some terms, such as "learning progressions" are used ambiguously, sometimes indicating only developmental progressions, and at other times, also suggesting a sequence of instructional activities. Although studying either psychological developmental progressions or instructional sequences separately can be valid research goals, and studies of each can and should inform mathematics education, we believe the power and uniqueness of the learning trajectories construct stems from the inextricable interconnection between these two aspects. Both these aspects (developmental progressions of thinking and instructional sequences) serve the most important, but often least discussed, aspect

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of learning trajectories—the goal. Our learning trajectories base goals on both the expertise of mathematicians and research on students' thinking about and learning of mathematics (Clements, Sarama, & DiBiase, 2004; Fuson, 2004; Sarama & Clements, 2009a). This results in goals that are organized into the "big" or "focal" ideas of mathematics: overarching clusters and concepts and skills that are mathematically central and coherent, consistent with students' (often intuitive) thinking, and generative of future learning (Clements, Sarama, et al., 2004). Once the mathematical goals are established, research is reviewed to determine if there is a natural developmental progression (at least for a given age range of students in a particular culture) that can be identified within theoretically- and empirically-grounded models of children's thinking, learning, and development (Carpenter & Moser, 1984). That is, researchers build a cognitive model of students' learning that is sufficiently explicit to describe the processes involved in students' progressive construction of the mathematics described by the goal across several qualitatively distinct structural levels of increasing sophistication, complexity, abstraction, power, and generality.

What, if Anything, is "New" in the Learning Trajectories Construct?

When we discuss learning trajectories, some (commendably) skeptical colleagues ask what is really different. If curricula have always been "courses" or paths (and frequently "horse races" through them), and if psychological and educational theories always postulated series of goals, then is this not simply renaming old (and palpable) ideas? At certain simple levels, the answer is positive. Most of these notions describe or dictate a series of educational goals. All have some theoretical perspective on why one goal might follow another.

In contrast, these theories often differ markedly on the details, and the learning trajectories construct as we define it builds upon theories and research of years past, as any theory should, but is distinct from previous formulations and constitutes a substantive contribution to theory, empirical research, and praxis. For example, early educational psychology considered educational series or sequences on the accumulation of connections. "We now understand that learning is essentially the formation of connections or bonds between situations and responses, that the satisfyingness of the result is the chief force that forms them, and that habit rules in the realm of thought as truly and as fully as in the realm of action" (Thorndike, 1922, p. v). Thus, curricular sequences could be logically arranged to establish connections between simple situations (addends) and responses (sum) and then later connect these and other bonds to complete more difficult tasks (e.g., multidigit addition) and even to develop mathematical reasoning. However, conceptual, meaningful learning was not the focus, but rather simple paired or associated learning. Also, potential differences and nuances of learning in different subject matter domains were not considered.

Bloom's taxonomy of educational objectives and Robert Gagné's "conditions of learning" and "principles of instructional design" (Gagné, 1965; Gagné & Briggs, 1979) postulated that Thorndike's theory was too simple and that there were "types of learning" and that certain types, such as stimulus-response learning (e.g., Thorndike's "bonds") were prerequisite to other types (e.g., discrimination learning, concept learning, rule learning, and last, problem solving). For a specific topic, or a specific domain within a topic such as mathematics, these could be assembled in "learning hierarchies"—sequences of pairs consisting of a subordinate skill whose acquisition is hypothesized to facilitate the learning of a higher-level skill. These, then, specified a "learning route"—certainly one early form of a learning trajectory. Such routes would be determined by logical analysis (logically identifying what subordinate competence is required by a superordinate competence) and empirical task analysis (Gagné, 1965/1970).

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In a similar manner, others continued to promote task analysis as a way to develop complex hierarchies of skills. Some researchers similarly based these hierarchies on logical and task analyses, but gave more weight to extant findings in educational—and especially psychological—research to perform "cognitive" or "rational" analyses—with follow-up empirical validation studies whenever possible (Resnick & Ford, 1981). Work from this perspective increasingly used the computer metaphor (i.e., information-processing theories), and often actual computer models, in their analyses (Hoz, 1979; Klahr & Wallace, 1976).

These approaches determined hierarchies of educational goals and were the basis of many "scope and sequences" in the educational literature (see Baroody, Cibulskis, Lai, & Li, 2004, for an extended discussion and somewhat different perspective). The view of learning of the earlier approaches was generally that of knowledge acquisition, with the environment providing input that was "received"—that is, imitated and mentally recorded by the student.

Other researchers attended more to students' thinking and cognitive development. Some devised developmental learning theories in attempts to integrate structural views such as those of Piaget with views based on task analysis and information-processing models. Later theoretical efforts in cognitive science extended these efforts to focus on the importance of domain-specific learning and development (Davis, 1984; Karmiloff-Smith, 1992).

In historical parallel, several theories, from Piagetian (Piaget & Szeminska, 1952) to field theories (Brownell, 1928; Brownell & Moser, 1949) and later developmental and cognitive science theories (Case & Okamoto, 1996) emphasized students as makers of meaning. Similarly, cognitively- and constructivist-oriented research programs explicated the concepts and skills children build as they move from one level to the next within a mathematical domain (Baroody, 1987; Carpenter & Moser, 1984; Steffe & Cobb, 1988). Unfortunately, those applying these studies practically often oversimplified and misconstrued their results and implications, emphasizing laissez–fair or outdated "discovery" approaches (Clements, 1997).

Learning trajectories as we have defined them (and our overaching theory of Hierarchic Interactionalism, see Clements & Sarama, 2007a; Sarama & Clements, 2009a) owe much to these previous efforts, which have progressed to increasingly sophisticated and complex views of cognition and learning. However, the earliest applications of cognitive theory to educational sequences tended to feature simple linear sequences based on accretion of numerous facts and skills. This was reflected in their hierarchies of educational goals and the resultant scope and sequences. Learning trajectories include such hierarchies, but are not as limited as these early constructs to sequences of skills or "logically" determined prerequisite pieces of knowledge. Learning trajectories are not lists of everything children need to learn, as are some scope and sequence documents; that is, they do not cover every single "fact" or skill. Most important, they describe children's *levels of thinking*, not just their ability to correctly respond to a mathematics question. They can not be summarized by stating the mathematical definition, concept, or rule (cf. Gagné, 1965/1970). So, for example, a single mathematical problem may be solved differently by students at different (separable) levels of thinking in a learning trajectory. Levels of thinking describe *how* students think about a topic and *why*—including the cognitive actions-on-objects that constitute that thinking.

Further, the ramifications for instruction from earlier theories were often based on transmission views, which hold that these facts and skills are presented and then passively absorbed. In comparison, learning trajectories have an interactionalist view of pedagogy.

To further elaborate these differences, consider the three components of learning trajectories.

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Goal

The explication of the *goal* is important and distinguished from previous theories of learning that tended to either (a) apply the same theories and procedures to all domains, ignoring subject matter, or (b) accept the goal as arbitrary or "given" by existing standards or curriculum. In contrast, as stated, our learning trajectories base goals on both the expertise of mathematicians and research on students' thinking about and learning of mathematics. Thus, in contrast to earlier approaches, both domain-specific expertise and research on students' thinking and learning in that domain play a fundamental role in determining the mathematical goal—the first component of learning trajectories.

Developmental Progression

The *developmental progressions* of learning trajectories are much more than linear sequences based on accretion of numerous facts and skills. They are based on a progression of levels of thinking that (as does the goal) reflects the cognitive science view of knowledge as interconnected webs of concepts and skills. It is important to describe the nature of these levels and differentiate them from 'stages' (such as Piaget's).

A *level* is a period of time of qualitatively distinct cognition, as are stages; however, there are at least four important distinctions between levels and stages. First and most important, they do not apply across domains but only within a *specific* domain. Second, the period of time is generally far shorter, and can be months or days (especially given efficacious instruction), rather than a period of years for stages.

Third, although—like Piaget— Hierarchic Interactionalism postulates that subsequent levels are built upon earlier levels, there are two important differences. (a) The order of magnitude of difference in durations indicates a distinctly different cognitive "distance" between successive states. Informally, the "jump" between contiguous levels is far smaller than the jump between Piagetian stages (admittedly, measuring such distances, for this distinction and related theoretical notions such as Vygotsky's Zone of Proximal Development, remains an open problem). (b) The Hierarchic Interactionalism theory of levels makes no commitment (as does the Piagetian theory of stages) that the actions-on-objects of level $n + 2$ must be built from those of level $n + 1$. In Piagetian theory (Piaget & Szeminska, 1952), for example, stages are long periods of development characterized by cognition across a variety of domains qualitatively different from that of both the preceding and succeeding stages. Further, in Piagetian theory, stage $n + 2$ necessitated passing through stage $n + 1$ *because* stage n + 1 constructed the elements from which stage $n + 2$ would be built.

Levels in Hierarchic Interactionalism are not "stages." Rather, in many cases the cognitive material may be present at level n, requiring only a greater degree of construction or generalization to construct the pattern of thinking and reasoning defining level $n + 2$. We return to this issue when we discuss students "skipping" a level or "jumping ahead."

Fourth, although levels of thinking can be theoretically viewed as nonrecurrent (Karmiloff-Smith, 1984), students not only can, but frequently do, "return" to earlier levels of thinking in certain contexts. Therefore, Hierarchic Interactionalism postulates the construct of nongenetic levels (Clements, Battista, & Sarama, 2001), which has two special characteristics. (a) Progress through nongenetic levels is determined more by social influences, and specifically instruction, than by age-linked development. (At this point, this only implies that progression does not occur by necessity with time, but demands, in addition, instructional intervention, although certain levels may develop under maturational constraints.) (b) Although each higher nongenetic level builds on the knowledge that constitutes lower levels, its nongenetic nature does not preclude the instantiation and application of earlier levels in certain contexts (often, but not necessarily limited to, especially demanding or stressful contexts or tasks). There exists a probability of evoking each level depending on circumstances. Again, Figure 1 illustrates that earlier levels do not “disappear”; people do not “jump” from one type of thinking to a separate type, but rather build new ways of thinking upon the previous patterns of thinking. This process is codetermined by the probabilities of instantiation and conscious metacognitive control, which increases as one moves up through the levels, allowing more intentional application of various cognitive strategies. Therefore, students have increasing choice to override the default probabilities. The use of different levels is environmentally adaptive; thus, the adjective “higher” should be understood as a higher level of abstraction and generality, without the implication of either inherent superiority or the abandonment of lower levels as a consequence of the development of higher levels of thinking. Nevertheless, the levels would constitute veridical qualitative changes in thinking and behavior.

Each level in Hierarchic Interactionalism’s developmental progressions is characterized by specific mental objects (e.g., concepts) and actions (processes) (e.g., Clements, Wilson, & Sarama, 2004; Steffe & Cobb, 1988). Specification of these actions-on-objects allows a degree of precision not achieved by previous theoretical and empirical works. Further, the research methods that generate and test these mental models are distinct from methods used in earlier research. Strategies such as clinical interviews are used to examine students' knowledge of the content domain, including conceptions, strategies, intuitive ideas, and informal strategies used to solve problems. The researchers set up a situation or task to elicit pertinent concepts and processes. Once an initial model has been developed, it is tested and extended with teaching experiments, which present limited tasks and adult interaction to individual children with the goal of building models of children’s thinking *and learning*—that is, transitions between levels are the *crux* of these studies—which is another way learning trajectories differ from many earlier research programs. Once several iterations of such work indicate substantive stability, it is accepted as a working model. Thus, the developmental progressions’ levels of thinking and explication of transitions between levels describe in detail the following: (a) what students are *able* to do, (b) what they are *not yet able* to do *but should be able to learn,* and (c) *why*—that is, *how* they think at each level and how they learned these levels of thinking. This distinguishes learning trajectories’ developmental progressions from earlier efforts to develop educational sequences that, for example, often used reductionist techniques to decompose a targeted competence level only into subskills, based on an adult’s perspective.

Instructional Tasks

The *instructional tasks* of learning trajectories are much more than didactic presentations or external “models” of the mathematics to be learned. They often include these elements, but they are fine-tuned to develop the level of thinking that a particular student needs. Learning trajectories differ from instructional designs based on task (or “rational”) analysis because they are not a reduction of the skills of experts but are models of students’ learning that include the unique constructions of students and require continuous, detailed, and simultaneous analyses of goals, pedagogical tasks, teaching, and children’s thinking and learning. Such explication allows the researcher to test the theory by testing the curriculum (Clements & Battista, 2000).

This early interpretive work evaluates components using a mix of model (or hypothesis) testing and model generation strategies, including design experiments, as well as grounded theory, microgenetic, microethnographic, and phenomenological approaches. The goal is to understand the meaning that students give to the instructional objects and tasks. The focus is on the *consonance* between the actions of the students and the learning trajectory; that is, does the instruction task engender, in a student at level *n*, the cognitive actions-on-objects that are described as accounting for the type of thinking and problem-solving at level $n + 1$. If not, other tasks can be tried, based on a detailed account of the students' responses. (Discrepancies may also reveal a need to alter the developmental progression.) Questions such as the following direct the inquiry. Do students use the tools provided (e.g., manipulatives, tables or graphs, software tools or features) to perform the actions, either spontaneously or only with prompting? If prompting is necessary, which type is successful, and does this differ for different students? Are students' *observable* actions-on-objects enactments of the desired cognitive operations in the way the model posits, or merely trial-and-error manipulation? Are there indications of an internalization of these; that is, indications that students are building mental actions-on-objects and thus developing $n + 1$ level of thinking? In this way, the developer/researcher creates more refined models of the thinking of particular groups of students (the developmental progression) and describes what elements of the instructional tasks, including specific scaffolding strategies, are observed as having contributed to student learning. The objective is to connect the developmental progression with the instructional tasks.

The tightly interwoven and interacting connections among the three components of a learning trajectory—goal, developmental progression, and instructional tasks—encompassing levels from the microscopic and individual student's cognition to the cultural surround, are a major distinguishing features of the learning trajectory construct. There are not two different paths (see footnote 1)—a learning path and a teaching path—but one *learning trajectory* with three components borne of the same theoretical and empirical parents.

Scientific experiments that examine, evaluate, and extend these connections and components include conceptual analyses and theories. They are tested *and iteratively revised* in progressively expanding social situations, which results in greater contributions to both educational theory and practice (Clements, 2007).

Empirical Support

We initially reviewed research in early mathematics because we believed that learning trajectories should be the backbone of our *Building Blocks* research-and-development curriculum project (Clements & Sarama, 1998), which was developed based on a Curriculum Research Framework (Clements, 2007) that itself puts learning trajectories at the core. Our work in that and several subsequent projects convinced us of the usefulness of the construct, with effect sizes from .72 to 2.12 (Clements & Sarama, 2007b; Sarama & Clements, 2009b; Sarama, Clements, Starkey, Klein, & Wakeley, 2008). The effect size of the *Building Blocks Pre-K* curriculum was .72. Longitudinal analyses with follow-up interventions focused *only* on learning trajectories (i.e., the teachers in kindergarten and first grade used their regular curriculum, but studied the research-based learning trajectories) continues these gains (Sarama, Clements, Wolfe, & Spitler, 2011). We believe these results indicate that the use of learning trajectories in curriculum development and professional develop have consistent, substantial, benefits.

References

Baroody, A. J. (1987). *Children's mathematical thinking*. New York: Teachers College.

Baroody, A. J., Cibulskis, M., Lai, M.-l., & Li, X. (2004). Comments on the use of learning trajectories in curriculum development and research. *Mathematical Thinking and Learning, 6*, 227-260.

Brownell, W. A. (1928). *The development of children's number ideas in the primary grades*. Chicago: Department of Education, University of Chicago.
Brownell, W. A., & Moser, H. E. (1949). *Meaningful vs. mechanical learning: A study in grade III subtraction*. Durham, NC: Duke University Press.
Carpenter, T. P., & Moser, J. M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education, 15*, 179-202.
Case, R., & Okamoto, Y. (1996). *The role of central conceptual structures in the development of children's thought*. Monographs of the Society for Research in Child Development.
CCSSO/NGA. (2010). Common core state standards for mathematics. Washington, DC: Council of Chief State School Officers and the National Governors Association Center for Best Practices. Retrieved from http://corestandards.org/
Clements, D. H. (1997). (Mis?)Constructing constructivism. *Teaching Children Mathematics, 4*(4), 198-200.
Clements, D. H. (2007). Curriculum research: Toward a framework for 'research-based curricula'. *Journal for Research in Mathematics Education, 38*, 35–70.
Clements, D. H., & Battista, M. T. (2000). Designing effective software. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 761-776). Mahwah, NJ: Lawrence Erlbaum Associates.
Clements, D. H., Battista, M. T., & Sarama, J. (2001). Logo and geometry. *Journal for Research in Mathematics Education Monograph Series, 10*.
Clements, D. H., & Sarama, J. (1998). *Building Blocks—Foundations for Mathematical Thinking, Pre-Kindergarten to Grade 2: Research-based Materials Development [National Science Foundation, grant number ESI-9730804; see www.gse.buffalo.edu/org/buildingblocks/]*. Buffalo, NY: State University of New York at Buffalo.
Clements, D. H., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6*, 81-89.
Clements, D. H., & Sarama, J. (2007a). Early childhood mathematics learning. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 1, pp. 461-555). New York: Information Age Publishing.
Clements, D. H., & Sarama, J. (2007b). Effects of a preschool mathematics curriculum: Summative research on the *Building Blocks* project. *Journal for Research in Mathematics Education, 38*, 136-163.
Clements, D. H., Sarama, J., & DiBiase, A.-M. (2004). *Engaging young children in mathematics: Standards for early childhood mathematics education*. Mahwah, NJ: Lawrence Erlbaum Associates.
Clements, D. H., Wilson, D. C., & Sarama, J. (2004). Young children's composition of geometric figures: A learning trajectory. *Mathematical Thinking and Learning, 6*, 163-184.
Davis, R. B. (1984). *Learning mathematics: The cognitive science approach to mathematics education*. Norwood, NJ: Ablex.
Fuson, K. C. (2004). Pre-K to grade 2 goals and standards: Achieving 21st century mastery for all. In D. H. Clements, J. Sarama & A.-M. DiBiase (Eds.), *Engaging young children in mathematics: Standards for early childhood mathematics education* (pp. 105-148). Mahwah, NJ: Lawrence Erlbaum Associates.
Gagné, R. M. (1965). *The conditions of learning*. New York: Holt, Rinehart & Winston.

Gagné, R. M. (1965/1970). *The conditions of learning*. New York: Holt, Rinehart, and Winston, Inc.
Gagné, R. M., & Briggs, L. J. (1979). *Principles of instructional design*. New York: Holt, Rinehart & Winston.
Hoz, R. (1979). The use of heuristic models in mathematics teaching. *International Journal of Mathematical Education in Science and Technology, 10*, 137-151.
Karmiloff-Smith, A. (1984). Children's problem solving. In M. E. Lamb, A. L. Brown & B. Rogoff (Eds.), *Advances in developmental psychology (Vol. 3)* (pp. 39-90). Mahwah, NJ: Lawrence Erlbaum Associates.
Karmiloff-Smith, A. (1992). *Beyond modularity: A developmental perspective on cognitive science*. Cambridge, MA: MIT Press.
Klahr, D., & Wallace, J. G. (1976). *Cognitive development: An information-processing view*. Mahwah, NJ: Lawrence Erlbaum Associates.
National Council of Teachers of Mathematics. (2006). *Curriculum focal points for prekindergarten through grade 8 mathematics: A quest for coherence*. Reston, VA: Author.
National Research Council. (2009). *Mathematics in early childhood: Learning paths toward excellence and equity*. Washington, DC: National Academy Press.
Piaget, J., & Szeminska, A. (1952). *The child's conception of number*. London: Routledge and Kegan Paul.
Resnick, L. B., & Ford, W. W. (1981). *The psychology of mathematics for instruction*. Hillsdale, NJ: Lawrence Erlbaum Associates.
Sarama, J., & Clements, D. H. (2009a). *Early childhood mathematics education research: Learning trajectories for young children*. New York, NY: Routledge.
Sarama, J., & Clements, D. H. (2009b, April). *Scaling up successful interventions: Multidisciplinary perspectives.* Paper presented at the American Educational Research Association, San Diego, CA.
Sarama, J., Clements, D. H., Starkey, P., Klein, A., & Wakeley, A. (2008). Scaling up the implementation of a pre-kindergarten mathematics curriculum: Teaching for understanding with trajectories and technologies. *Journal of Research on Educational Effectiveness, 1*, 89-119.
Sarama, J., Clements, D. H., Wolfe, C. B., & Spitler, M. E. (2011). Longitudinal evaluation of a scale-up model for teaching mathematics with trajectories and technologies. *Submitted for publication*.
Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.
Steffe, L. P., & Cobb, P. (1988). *Construction of arithmetical meanings and strategies*. New York: Springer-Verlag.
Thorndike, E. L. (1922). *The psychology of arithmetic*. New York: Macmillan.