STUDENTS' COMBINATORIAL REASONING: THE MULTIPLICATION OF BINOMIALS[1]

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Three clinical interviews were conducted with each of 15 sixth grade students to test conjectures about the relationship between their level of multiplicative reasoning and their solution of combinatorics problems that could involve single and multi-digit multiplication. The problems that involved multi-digit multiplication were designed with the intent of investigating whether students engaged in binomial multiplication. The conjectures about the relationship between students' multiplicative reasoning and their solution of combinatorics problems that could involve multi-digit multiplication were refined as a result of students' problem solving activity: Students who had not constructed the most advanced multiplicative concept were able to engage in a form of binomial multiplication.

Curriculum writers have responded to recommendations to incorporate combinatorics problems into K-12 curricula (e.g., Lappan, Fey, Fitzgerald, Friel, & Phillips 2002). These recommendations have been based on arguments that such problems have the potential to support both process and content standards outlined by the National Council of Teachers of Mathematics (Srirman & English, 2004). In response to these changes in curricula, researchers have produced a small body of research that has investigated students' combinatorial reasoning (Jones, Langrall, & Mooney, 2007). However, this body of research remains small, and so relatively little is known about when particular problems are appropriate to introduce to students, how teachers can support students' understanding of these problems, and how such problems are compatible with extant goals of the curricula. Therefore, research that aims to address these issues is critical for successfully and coherently incorporating such problems into extant curricula.

The study reported on in this paper addresses these issues by investigating how 15 6th grade students at three different levels of multiplicative reasoning solved two-dimensional combinatorics problems and how they used such problems to reason about multi-digit multiplication. The study involved three clinical interviews—one unrecorded selection interview and two hour-long video recorded interviews. Each interview was conducted one-on-one with study participants. In the first video recorded interview, students solved two-dimensional combinatorics problems that could involve single digit multiplication like the Outfits Problem.

> *The Outfits Problem:* You have three pairs of pants and four shirts. An outfit is one shirt and one pair of pants. How many different outfits could you make?

In the second video recorded interview, students solved combinatorics problems that could involve multi-digit multiplication like the Card Problem.

> *Card Problem:* You have the ace through king of hearts (13 cards). Your friend has the ace through king of clubs (13 cards). Use an array to show all of the possible 2-card hands you could make that consist of one heart and one club. On your array show the number of 2-card hands that have two face cards (Jack, Queen, King), that have exactly one face card, and that have no face cards. Use the sections of your array to determine the total number of 2-card hands you can make.

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In both interviews, students were encouraged to use *arrays* to symbolize the solution of these problems. The following research questions guided the study:

1) How do students at different levels of multiplicative reasoning reason about two-dimensional combinatorics problems that can involve single digit multiplication?
2) How do students at different levels of multiplicative reasoning reason about multi-digit multiplication in the context of solving combinatorics problems?

Literature Review

English (1991, 1993) has studied kindergarten through 8th grade students' understanding of two-dimensional combinatorics problems like the Outfits Problem. She concluded that many 6th grade students were ready to begin analyzing the structure of such problems, whereas younger students were less ready to do so, even though they could solve the problems with concrete materials. However, English has exclusively studied how students could use two-dimensional combinatorics problems as a basis for reasoning about single digit multiplication.

Researchers who have studied students' understanding of multi-digit multiplication have primarily used repeated groups problems not combinatorics problems (e.g., Ambrose, Baek, & Carpenter, 2003). Moreover, they frequently have *not* used array representations for these problems (Verschaffel, Greer, & De Corte, 2007), even though array representations are often included in researchers' classification of situations involving multiplication (e.g., Greer, 1992). Izsak (2004) is one of the few researchers to investigate elementary grade students' use of arrays to represent multi-digit multiplication problems (although he did not use combinatorics problems). Izsak found that students in the 4th grade classroom he studied were more successful in solving multi-digit multiplication problems than U.S. students in earlier large-scale studies (e.g., Mullis, Martin, Beaton, Gonzalez, Kelly, Smith, 1997). This finding suggests that array representations may be fruitful for helping students develop an understanding of multi-digit multiplication. However, because Izsak's study is one of the few that has investigated how students use arrays to understand multi-digit multiplication, further research is needed (Vershcaffel, Greer & De Corte).

A second finding of Izsak's (2004) study, which is supported by Ambrose, Baek, & Carpenter's (2003) findings, is that students tended not to partition *both* the multiplier and multiplicand when computing multi-digit multiplication problems. For example, to solve 13 x 13, some students partitioned either the multiplier or multiplicand into 10 and 3, but they did not partition both 13s into (10 + 3). When students partition both numbers into two parts, they are multiplying two binomials together, as opposed to multiplying a monomial times a binomial. Problems like the Card Problem were designed for this study to open the possibility for students' to solve the problems using the multiplication of two binomials.

Methodological and Analytic Framework

Methodology

The study used clinical interview methodology (Clement, 2000). Clinical interviews allow "the ability to collect and analyze data on mental processes at the level of a subject's authentic ideas and meanings, and to expose hidden structures and processes in the subject's thinking that could not be detected by less open-ended techniques" (Clement, p. 547). When using this methodology, a researcher formulates conjectures about these mental processes (Confrey & LaChance, 2000; Steffe & Thompson, 2000). These conjectures are formulated based on conceptual analysis of the mathematical domain and prior research with students in this domain.

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Problem sequences are then carefully designed so that they allow for testing these conjectures (Cobb & Gravemeijer, 2008). During data collection and analysis, conjectures are refuted or not refuted based on how students operate to solve problems. When a conjecture is refuted, the conjecture becomes open to being refined so that it can be tested in future studies.

Analytic Tools

The two primary analytic constructs used to characterize students' mental processes were *schemes* and the *mental operations* that constitute these schemes. A scheme is a goal-directed way of operating that has three parts—an assimilatory mechanism, an activity, and a result (von Glasersfeld, 2001). When a student is presented with a problem situation, the problem situation may trigger records of prior operating, and in doing so the student may come to recognize the situation (assimilate it) as one that involves a particular type of activity. The activity of a scheme involves mental operations like partitioning, disembedding, and uniting (three mental operations, which will be discussed below). Finally, the student's activity produces a result (i.e. a solution to a problem).

Framework for Participant Selection

Steffe (1994) has identified three distinctly different levels of multiplicative reasoning for elementary grade students. These different levels of multiplicative reasoning are a result of learning and development, and they open possibilities for, as well as constrain how, students operate mathematically. The different levels of multiplicative reasoning have been used as a framework to study students' reasoning as it pertains to the solution of problems involving both whole numbers and fractions. However, these levels have not been used as a basis for studying students' combinatorial reasoning. In this section, I give a brief overview of how students operating at the different levels typically solve repeated groups multiplication problems.

Students operating at the first level of multiplicative reasoning are able to coordinate two levels of units in activity. That is, to determine the number of doughnuts a person has if the person has 4 packages, with 8 doughnuts in each package, these students coordinate two counts—one count tracks the number of doughnuts and the other the number of packages. This coordination of two levels of units often involves double counting like the following: one, two, three, four, five, six, seven, eight, that is one package; nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, that is two packages, etc. The two units that these students are able to coordinate are the number of doughnuts and the number of packages, but they have to make this coordination as part of the activity they use to solve a problem.

Students operating at the second level of multiplicative reasoning are able to take the coordination of two levels of units as given, which means that they do not have to make a coordination between two counts as part of their activity. Instead, a number word like 8 automatically means both 1 package and 8 doughnuts. Students who operate with the second multiplicative concept are likely to solve the doughnut problem by reasoning that 8 and 8 is 16 because 8 and 2 is 10 and 6 more is 16. Here the number word 16 would mean 16 doughnuts and 2 packages. In reasoning in this way, these students are able to operate on 8 by partitioning it into two parts (i.e., breaking 8 into 2 and 6), and disembedding both of the parts in order to strategically unite them with another group of 8 (e.g., 8 and 2 makes one group of 10, and 6 more makes one group of 10 with 6 ones). To finish solving the problem they would likely continue this sequential process of adding 8 more onto the previous amount, using strategic ways of partitioning the numbers to help them calculate the total amount.

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Students operating at the third level of multiplicative reasoning are no longer constrained to sequentially combining amounts and using addition to solve repeated groups multiplication problems. Instead these students might determine the number of doughnuts a person has if the person has 12 packages of 8 doughnuts by reasoning that 10 packages of doughnuts would be 80 doughnuts and that 2 more packages of doughnuts would be 16 doughnuts, which would yield a total of 96 doughnuts. In solving the problem in this way, these students are able to reason that 12 groups of 8 is composed of 10 groups of 8 and 2 groups of 8, and they can use that to determine the total amount. In doing so, they treat the 12 groups of 8 as a unit of 12 units each containing 8 units. That is, they treat the 12 groups of 8 as itself a unit that can be operated on, which means that they are reasoning with a third level of unit. Reasoning with the third level of unit enables them to partition 12 groups of 8 into two parts, a unit of 10 units each containing 8 units and a unit of 2 units each containing 8 units. Then they disembed each part, evaluate each part using multiplication (i.e., ten 8s is 80 and two 8s is 16), and subsequently unite the two parts together (i.e., 80 and 16 is 96).

Students' Schemes for Solving Two-Dimensional Combinatorics Problems

In a prior study with three eighth grade students, all of whom were operating at the third level of multiplicative reasoning, Tillema (under review a) identified a scheme that students used to solve basic two-dimensional combinatorics problems. The scheme entailed students *assimilating* such situations using *two composite units* (*two input quantities*). In the case of the Outfits Problem, the two composite units were 4 shirts and 3 pants. The *activity* of their scheme involved two key mental operations—ordering and pairing. Students used an *ordering* operation when they supplied a qualitative property that they used to differentiate the units of a composite unit. For example, in the Outfits Problem when a student used colors, a qualitative property, to differentiate among the three shirts, the student ordered the shirts. Students then used a *pairing* operation when they created a correspondence between one unit of each composite unit and applied their unitizing operation to this correspondence. That is, to be engaged in pairing a student created a correspondence between a shirt and pants and then applied her unitizing operation to this correspondence to create an outfit, an output unit.

Students produced the output units by following a lexicographic ordering (cf. English's, 1991, 1993 odometer strategy). A lexicographic ordering is similar to a dictionary ordering—the word "aa" appears before the word "ab" and all words that begin with "a" appear before all words that begin with "b". So, for example, in the Outfits Problem students followed a lexicographic ordering when they created the outfit that contained the first shirt and first pants prior to creating the outfit that contained the first shirt and the second pants, and they created all outfits that contained the first shirt prior to creating any outfits that contained the second shirt. For these reasons, students' schemes for solving these problems were called a *lexicographic units pairing scheme* (LUPS).

To differentiate the extent to which students carried out the pairing operations as part of the activity of their scheme, two ways that students treated the units of an input quantity were defined. A student treated the units of an input quantity as *particular units* when she repeated the pairing operations that she used with one of the units with all of the other units of that quantity. For example, in the Outfits Problem, a student treated the pants as particular units when she paired the first shirt with the first pants, the first shirt with the second pants, and the first shirt with the third pants because she repeated the pairing operations between the first shirt and each particular pants. In contrast, when a student did not need to repeat the pairing operations with

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one or both input quantities, the student operated with a *representative unit*. For example, a student might operate as described above, and then say, "you could do that with all of the other shirts." In this case, the student treated the first shirt as representative of how all of the other shirts would function, which enabled her *not* to carry out all of the pairing operations with the other shirts. Based on this distinction between particular and representative units, four different levels of the LUPS were identified.

Study Design and Guiding Conjectures

Selection Interviews

The selection interviews were used to identify at least three 6^{th} grade students at each of the three different levels of multiplicative reasoning, and involved a total of fifteen students. The students all attended a magnet school in an urban school district in the Midwest. They were selected from a pool of 65 possible students all of whom had the same 6^{th} grade teacher.

During the selection interviews, I presented problems to students with the intent of determining whether a student were able to coordinate two levels of units in activity (the first level of multiplicative reasoning), take the coordination of two levels of units as a given (the second level of multiplicative reasoning), or take the coordination of three levels of units as given (the third level of multiplicative reasoning). The interview protocols involved both whole number and fraction problems that were intended to elicit this information, but none of the problems involved any type of combinatorics problem.

Design of the First Interview

During the first interview, students were presented with two-dimensional combinatorics problems like the Outfits Problem. They introduced themselves or the researcher introduced to them three ways—lists, tree diagrams, and arrays—to represent these initial problems. During students' solution of these initial problems, the researcher emphasized the use of arrays as a way to symbolize such problems. The intent of posing these problems was to test three conjectures about the relationship between the different levels of multiplicative reasoning and the different levels of the LUPS.

Conjecture 1: Students operating at the first level of multiplicative reasoning will be constrained to the second level of the lexicographic units pairing scheme (LUPS).

Conjecture 2: Students operating at the second level of multiplicative reasoning will be constrained to the third level of the LUPS.

Conjecture 3: Students operating at the third level of multiplicative reasoning will operate at the fourth level of the LUPS.

Design of the Second Interview

During the second interview, students were initially presented with problems like the Restaurant Problem.

The Restaurant Problem: A meal at a local restaurant consists of one salad and one entrée. The restaurant serves 6 different kinds of salad and 14 different kinds of entrées. 10 of the entrees have meat. Illustrate with an array the total number of meals that are vegetarian and the total number of meals that are non-vegetarian.

These problems all involved one two-digit number (14 in the case of the Restaurant Problem) and one one-digit number (6 in the case of the Restaurant Problem), and again the researcher

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emphasized using arrays as a way to symbolize these problems. These problems were used to test the following conjecture.

Conjecture 4: Students operating at the third level of multiplicative reasoning will be the only students to solve these problems using distributive reasoning.

If students were able to solve problems like the Restaurant Problem, then they were presented with problems that had the potential to involve the multiplication of two binomials like the Card Problem, which is symbolized below using an array (Figure 2).

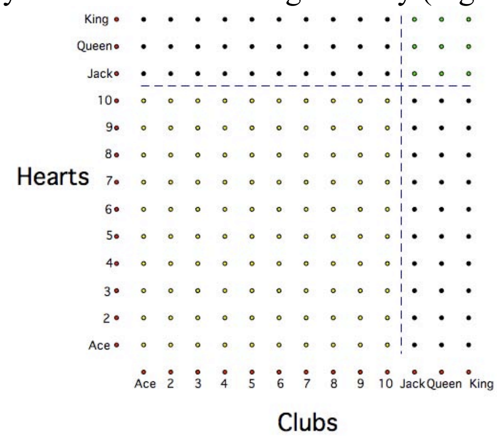

Figure 2: An array for the Card Problem.

In the Card Problem, 13^2 can refer to the total number of 2-card hands a person could make, while $(10 + 3)^2$ refers to this same total once a person has thought about breaking the hearts and clubs into non-face and face cards. Similarly, each of the following, 10^2, (10 x 3), (3 x 10), 3^2, refers to parts of the array: 0 face cards, 1 face card, or 2 face cards. This reasoning can lead to the development of the following equivalences: $(13)^2 = (10 + 3)^2 = 10^2 + (10 \text{ x } 3) + (3 \text{ x } 10) + 3^2 = 10^2 + 2 \text{ x } (10 \text{ x } 3) + 3^2 = 100 + 30 + 30 + 9 = 169$. Problems like the Card Problem were used to test the following conjecture.

Conjecture 5: Students operating at the third level of multiplicative reasoning are the only students who will experience these types of problems as involving the multiplication of two binomials.

Initial Findings and Discussion

Based on preliminary data analysis from this study there was no evidence that refuted conjectures 1, 2, 3. That is, all students in the study operated as the conjectures predicted they would operate. However, conjectures 4 and 5 needed to be refined as a result of this study. Providing data exemplars of students' reasoning that provides evidence for these claims will comprise a major component of the presentation of this paper. Here, I present exemplars of data to discuss how conjecture 5 has been refined as a result of the study.

Conjecture 5 stated that students operating at the third level of multiplicative reasoning are the only students who will experience the Card Problem as involving the multiplication of two binomials. The findings from the study indicate that students at both the second and third level of multiplicative reasoning were able to solve problems like the Card Problem. However, there were qualitative differences between the students' solutions of the Card Problem depending on whether they were operating at the second or third level of multiplicative reasoning.

Students who were operating at the second level of multiplicative reasoning solved the problem by first pairing a particular card (e.g., the two of clubs) with a representative card from the other suit (e.g., the two of hearts), and could take that as indication of producing the first

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thirteen pairs (i.e., all pairs that could be made with the two of clubs). They repeated the pairing operation with each of the thirteen clubs (i.e., treated the clubs as particular units), and could then state that the total number of two-card hands would be 13 x 13 and symbolize this multiplication problem using an array. Students could then find in their array the four sections shown in Figure 2, but to quantify each of these sections they had to re-engage in pairing operations for each section of the array. So, for example to determine the number of two-card hands that contained only face cards the students had to pair a particular face card (e.g., the king of clubs) with a representative face card from the other suit (e.g., the king of hearts), which they took as indication of producing the first three two-card hands that contained only face cards. They engaged in this pairing operation again for the remaining two clubs that were face cards to establish that there would be a total of 3 x 3 or 9 two-card hands that contained only face cards. They had to repeat these operations to establish the multiplication problems for the other three parts of the array. In doing so, they established that the array could be quantified using one multiplication problem (13 x 13) or four multiplication problems (3 x 3, 10 x 10, 3 x 10, and 10 x 3), but these two ways of quantifying the array remained two separate ways to view the array and were not integrated into a single structure (scheme).

Students who were operating at the third level of multiplicative reasoning solved the Card Problem by pairing a representative unit (e.g., the two of clubs) with a representative unit (e.g., the two of hearts). Because they treated both one club as a representative unit and the one heart as a representative unit, they were able to take a single pairing operation as implying all of the two-card hands that they could make. That is, they reasoned, for example, that the two of hearts was representative of any of the hearts that could be paired with the two of clubs, and so 13 two-card hands could be made with the two of clubs. Subsequently, they treated the two of clubs as representative of any of the clubs in the deck and so could take it as indicating the number of times that they would produce 13 two-card hands without actually having to make these two-card hands by pairing cards together. They could symbolize this pairing operation using an array. Once they produced the array they were able to determine the four sections of the array without having to engage in any further pairing operations. Rather they simply partitioned the two composite units they used in assimilation, 13 and 13, into two parts, and this implied to them partitioning the array into four sections. Moreover, they could identify a multiplication problem for each section of the array without having to use pairing operations to re-establish the multiplication problem for each part of the array. This enabled them to generate the equivalence that 13 x 13 = (10 + 3) x (10 + 3) = 10 x 10 + 3 x 10 + 10 x 3 + 3 x 3, and see these two ways of quantifying the array as part of a single structure (scheme).

In the presentation, there will be a discussion of each of the 5 conjectures and video data will be used to support the statement of the conjecture or to discuss how the conjecture was refined as a result of the study.

Endnotes

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