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ENGENDERING MULTIPLICATIVE REASONING IN STUDENTS WITH LEARNING DISABILITIES IN MATHEMATICS[1]: SAM'S COMPUTER-ASSISTED TRANSITION TO ANTICIPATORY UNIT DIFFERENTIATION-AND-SELECTION

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We examined how a student with learning disabilities (SLD) in mathematics constructed a scheme for differentiating, selecting, and properly operating on/with units that constitute a multiplicative situation, namely, singletons ('1s') and composite units (abbreviated UDS). Conducted as part of a larger teaching experiment in a learning environment that synergizes human and computer-assisted teaching, this study included 12 videotaped teaching episodes with a 5th grader (pseudonym-Sam), analyzed qualitatively. Our data provide a window onto the conceptual transformation involved in advancing from absence, through a participatory, to an anticipatory stage of a UDS scheme—a cognitive root for the distributive property. We postulate this scheme as a fundamental step in SLDs' learning to reason multiplicatively, and highlight the transfer-empowering nature of constructing it at the anticipatory stage.

Introduction

This study examined how students with learning disabilities (SLD) may construct a scheme for reasoning about multiplicative situations. Learning to reason multiplicatively is a major feat for elementary age children (Harel & Confrey, 1994; Kamii & Clark, 1996; Sowder, et al., 1998;

Steffe, 1994), whether or not they are identified as having learning disabilities/difficulties in math. Fostering it in all children is critical, because it provides a basis for understanding and properly using not only multiplication and division, but also fractional, proportional, and algebraic reasoning (NCTM, 2000, 2006). Lacking such a basis seems a key hurdle in SLDs' progress toward the latter, advanced concepts (Xin, 2008; Xin, Wiles, & Lin, 2008) and a cause for being and feeling 'stuck' in mathematics.

This study, a part of the NSF-funded *Nurturing Multiplicative Reasoning in Students with Learning Disabilities*[1] (NMRSD) project, examined SLDs' construction of a scheme for operating on, and coordinating, not just one but two sets of composite units (Steffe & Cobb, 1998) at a stage conducive to solving novel tasks ('transfer'). Such tasks may call for finding the sum or difference between two quantities (e.g., 'You have 7 boxes, 8 crayons each; I have 3 boxes, 8 crayons each; How many more crayons do you have?'). To solve such tasks in the absence of tangible 1s, a solver must identify and coordinate the units involved additively and multiplicatively. She either first multiplies (7x8=56, 3x8=24) to find the total of 1s in each set and then subtracts (56-24=32)—a *Totals-First* method, or first subtracts to find the difference in *composite units* and then multiplies the resulting, new set of composite units by the number of 1s in each (7-3=4; 4x8=32)—a *Difference-First* method. Our research question was how SLD may learn to differentiate, select, and operate on those quantities while forming a cognitive basis for what adults refer to as the distributive property (e.g., 8x(7-3) = 8x7 - 8x3).

Conceptual Framework of this Study

The NMRSD project is developing a software that draws on three research-based frameworks: a constructivist view of learning from mathematics education, generalization of word-problem underlying structures ('story-grammar') from special education, and machine (or statistical) learning from computer sciences. For this study the constructivist scheme theory (Piaget, 1970, 1985; von Glasersfeld, 1995) and its recent extension into the reflection on activity-effect relationship (*Ref*AER*) account (Simon et al., 2004; Simon & Tzur, 2004; 2004; Tzur, 2007, 2008) provided the cognitive lens. In particular, we used Tzur & Simon's (2004) distinction between the participatory and anticipatory stage in the construction of a new mathematical conception for designing instructional tasks and for assessing Sam's ways of operating. This stage distinction drew on von Glasersfeld's 3-part notion of scheme: (a) recognition of a situation, which sets the student's goal, (b) a mental activity associated with that situation and goal, and (c) an expected result. At the participatory stage, a learner forms a novel anticipation—an invariant relationship between the activity and a newly noticed/linked effect it brings forth. However, this *AER* is yet to be linked to a scheme's first part, and the learner can only access it if prompted for the activity, which regenerates the link (see Roig & Llinares, 2009). At the anticipatory stage, the learner links the novel *AER* with a host of 'structurally similar' situations, thus abstracting spontaneous (prompt-less) access to it across contexts (see Woodward, et al., 2009). We built on Tzur and Lambert's (in press) recent articulation of prompt features to guide decisions about sequencing tasks that could foster, and help assessing, transition to the participatory and then anticipatory stage of UDS.

The content-specific constructs of our framework draw on the *Initial*, *Tacitly Nested*, and *Explicitly Nested Number Sequences* (Steffe & Cobb, 1988)—three schemes a child uses in assimilating and operating on abstract composite units (CU). In the latter, the child conceives of smaller CUs as embedded within larger CU. For example, 3 CU of 8 singletons (3 boxes of crayons) and 4 CU of 8 singletons (4 boxes) are embedded within 7 CU of 8. Coordinating such same-size CU is key for early multiplicative reasoning in general and for children's construction of a UDS scheme in particular.

Methodology

Within the NMRSD project, this case study consisted of 12 videotaped teaching episodes in which Sam, a 5th grader, was jointly taught by the software and the second author. Sam was a strategic case, because RTI-like, problem-based pre-intervention assessment by the software and researchers indicated his 'readiness' for UDS. He had formed anticipatory stages of operating on CU multiplicatively to find the total of 1s (e.g., How many cubes are in 6 towers of cubes, 3 cubes each? Denoted $6T_3$ here) and additively to find CU sums/differences (e.g., You have $7T_3$; I have $4T_3$; how many more T_3 do you have?).

The software was designed based on results of previous (Tzur et al., 2009; Woodward et al., 2009; Xin et al., 2009; Zhang et al., 2009) and on-going teaching experiments with SLD. It presents students with problems in an interactive environment, and collects and analyzes data from the student's work (e.g., prompts needed, solution processes used) to select and present the next problem. This includes assessing if a student is ready to start the next level of problems. During our work with Sam, human teaching played a significant role to gain insight into software improvement—studying its impact (or lack thereof) on SLD learning of the intended mathematics.

Typically, software tasks commence within a context of producing, actually and/or mentally, a CU in the form of a tower made of several cubes (e.g., $7T_8$). One type of UDS problems asks the child to compare two different sets of same-size towers (e.g., $7T_8$ and $3T_8$). For example, the software may initially present those towers, then cover them, and pose the tasks: "*Jacqueline has a collection of 7 towers with 8 cubes in each. Mercedes has a collection of 3 towers with 8 cubes in each. How are these collections similar? How are they different? Who has more cubes and how many more?* A second type of problems switches the quantities, so two equal sets are made of different-size CU (e.g., $7T_8$ and $7T_3$). In teaching sessions in which the software was the primary source of UDS tasks, we often suggested to Sam to use one of the mini-tools available in the software (e.g., a calculator, a simulation of fingers that depict 'double-counting'). We also frequently asked Sam about the referent units of his solutions and how he figured these out.

We also used two interventions outside the software. First, Sam's work indicated that he did not have a conventional meaning for "*more*". For him, this term referred not to the difference between two quantities, but to the total of units in the larger quantity. Thus, we asked Sam to place two groups of Lego cubes (12 and 9), and then pair-off (1-to-1) as many cubes as possible. The researcher then clarified that (a) the number of non-paired cubes is the difference between the two sets (which Sam knew already) and (b) "more" refers to this number. This intervention seemed to create a shared, proper meaning for "more". The second intervention, which preceded Excerpt 2 (see Results), took place during the third UDS episode, as the software continued posing the first type of UDS problems. Sam seemed to struggle with operating on the various units, so we asked him to draw the two sets of towers on a paper. We said that a drawn tower could show each individual cube or just a schematic line/bar that Sam would label with the number of cubes it signified. Importantly, Sam felt the need to draw each cube in every tower (i.e., producing 1s), a need that was later diminished.

Data analysis occurred throughout the study. In our on-going analysis we discussed and recorded significant events after each teaching episode. We paid close attention to possible indicators (prompts needed) that Sam was in the participatory or anticipatory stage of UDS, and used our observations to plan for the next episode(s). In our retrospective analysis we began with those teaching episodes in which we previously identified segments when Sam was working in either stage of UDS. We transcribed and analyzed those episodic "snapshots" line-by-line to articulate advancements in Sam's mathematical thinking.

Results

In this section, we first briefly summarize data pertaining to Sam's lack of unit differentiation-and-selection (UDS) scheme. We then present and analyze data in support of claiming his transition to (a) the participatory stage and (b) the anticipatory stage of UDS. This includes his learning of a conventional meaning for 'more' or 'less', and a task in which he applied ('transferred') anticipatory UDS to a novel context.

Absence of a UDS Scheme

Once the NMRSD software determined that Sam has constructed an anticipatory stage of multiplication, and of *additively* operating on the *same* composite units (SUC scheme, e.g., you have $7T_8$ and I have $5T_8$; how many T_8 do we have altogether?), it moved to a UDS task: "Jacqueline has a collection of 12 towers with 8 cubes in each. Mercedes has a collection of 7 towers with 8 cubes in each. How are these collections similar?" He selected 'They both have 8 cubes in each tower'. Then, asked about difference, he selected 'Jacqueline (J) has more towers'. The software asked, "Altogether, who has more cubes?" Sam responded not with a name, but with an attempt to compute how many cubes J has (calculator). First, he multiplied 7x7=49, followed by 49x8=392, and noted "That's not right." The researcher (R) prompted, "Why did you do 49 times 8?" Sam calculated 7x8=56 (his *available* multiplication scheme) and stated that M has 56 cubes *in each tower* (unit conflation). Then, R said that the task asks *who* has more cubes, J or M. Sam responded by finding (calculator) 12x7=96 and stating, "Jacqueline." R asked if this makes sense; Sam nodded (yes), "Because J has more towers".

The software proceeded, "How many more cubes does Jacqueline have?" and R said that Sam could use the calculator. Sam re-read the task, stated he does *not* see how the calculator could be used, and after some thinking said, while inserting the numbers into the computer: "96 divided by fifty … (looks at R) Was it 58 or 56?" R, after saying "56," asked Sam why dividing. Sam said that this will give him the number he's looking for (clarifying this was "how many more cubes," and that multiplication would not work as it gives too high-a-number). R noted that Sam could have also used addition or subtraction. After a short pause to think of these options Sam keyed 96÷56, saw the answer (1.7142…) and responded in big surprise: "Whoa!" He noted that multiplication and division would not work, contemplated "96 minus … No," and added (calculator) while saying: 96+56=152. R asked him again why adding would be proper and explained to Sam the importance of his *reasoning* about the operation. In response, Sam stated that multiplication, addition, and division (!) gave him too large a number, calculated 96-56=40, and claimed this seemed correct.

The entire exchange indicated to us that Sam was not using an operation based on a reasoned choice (about difference). Rather, he tried one operation after another and judged the answer's plausibility on the basis of size/form. He also did not indicate any consideration of finding the difference in CU (towers) as a first step to be followed by multiplication (e.g., $12T_8$-$7T_8$=$5T_8$, 5x8=40 cubes). Thus, R began working with Sam on such a method and its link to the transposed one (e.g., 12x8–7x8), that is, on constructing UDS.

Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.

Participatory Stage of UDS

To foster Sam's construction of UDS, we realized he would first need to attribute a proper meaning—*difference*—to the terms 'more' and 'less'. An alternative meaning he seemed to attribute was evident in his responses to SUC tasks. For example, in the above situation (J has $12T_8$, M has $7T_8$), if asked how many more *towers* did J have, Sam would most often respond '12'. We briefly note that (a) such a meaning was found in other SLDs during the larger study and (b) it made sense to him because J has more *and* she has 12 so '12-is-the-more'. Sam quickly adopted the conventional meaning that R introduced through tasks about Lego singletons (1s). Thereafter, he consistently applied the terms to the difference between two quantities, either singletons or composite units, and strategically selected subtraction to find it. He seemed ready to construct UDS, which was fostered via tasks with 'easy' numbers.

The software stated: "John has a collection of 12 towers with 3 cubes in each, Sarah has a collection of 8 towers with 3 cubes in each." Sam playfully built the $12T_3$; he was asked and properly responded that both John and Sarah have 3 cubes in each of their towers. Asked how many towers each child had, he responded 12 and 8 respectively, responded that John and Sarah do not have the same number of towers, and that John has more cubes-in-all than Sarah. Excerpt 1 below provides what transpired next (C=computer, S=Sam, R=researcher).

Excerpt 1 (September 22, 2010)

S: [Reads the task in C out loud] "So, how many more cubes does John have than Sarah has?" and continues: Okay, so, take [away] 8. That's going to leave me with 4.

R: Four what?

S: Four towers; [and] 4 times 3 is 12.

R: You know that one, right?

[A little later, after R asked S to open the software Toolbox and use the calculator.]

R: If you do 12 x 3 and 8 x 3 … just to check the other way.

S: (Inserts 12 x 3 = while saying) 12 times 3 equals 36.

R: That's how many cubes - who has? [Pause] That's John, right?

S: (Inserts 8 x 3 = while saying) 8 times 3 equals 24. (Appears to think what's next) Twen … Twenty … Twenty-four (Inserts 24 and the *symbol for multiplication*, '*', while saying) times … [no] subtract, minus … [Clears calculator screen.]

[At this point, after R suggested to subtract the smaller number from the larger, S found 36-24=12 and, with a 'high-five' from R, they noted that it is, and must be, the same answer as Sam obtained by first subtracting the towers and then multiplying.]

Excerpt 1 indicates Sam's learning to operate in a UDS situation. He (a) differentiated the 1s (cubes) and unit-rate (3 cubes/tower) from the CU (towers), (b) selected the latter and operated on it to find the difference (via subtraction), (c) selected the unit-rate and used it as operand for (d) multiplying by the difference (4 towers) to figure out the number of cubes (12) that John had more than Sarah. Once prompted for solving the task in the alternative method, Sam's attempt to use multiplication indicated he was yet to establish UDS.

We argue that Sam's construction of UDS was at the participatory stage for a threefold reason. First, his execution of the *Difference-First* method, while proper and independent, followed R's teaching of using such a method earlier in that session. Thus, theoretically, the claim about anticipatory stage is not possible. Second, a fully established (anticipatory) UDS would consist of adeptly using either method. Excerpt 1 provides evidence that, initially, Sam thought of multiplying the totals in each collection (36x24) instead of subtracting. Third, data from the following two sessions showed that, when the software opened an episode with a UDS task Sam was unable to solve it via the *Difference-First* method without being prompted by R, although Sam himself stated it was easier. Excerpt 2 shows his choice of initial method (*Totals-First*) and incorrect operation (multiplying totals) after properly answering what's similar/different about J's and A's collections ($13T_9$ and $3T_9$ respectively). Because Sam's *single*, wrong method/answer was sufficient to *self-prompt* him for shifting to the alternative, *Difference-First* method, we consider his work as indication of high participatory stage.

Excerpt 2 (October 20, 2010)
C: Altogether who has more cubes?
S: (Selects 'Jacqueline'.)
C: How many more cubes does Jacqueline have?
S: (Pulls up the calculator tool. Inputs: 13 x 9 = 117; 3 x 9 = 27; 117 x 27 = 3159. He then clears the calculator screen and inputs: 13 – 3 = 10; 10 x 9 = 90.) Ninety!

Anticipatory Stage of UDS

We culminate with data that show Sam's UDS at the anticipatory stage. As *Ref*AER* stresses, a participatory and anticipatory stage differ not in the nature a scheme's parts (situation/goal, activity, effect), but rather in the learner's access to it—with or without prompt. Excerpt 3 shows Sam's prompt-less solution to a task presented outside the computer at the start of an episode 3 weeks after he indicated the high participatory stage (Excerpt 2). In the previous episode, he solved a tower/cubes UDS task without a prompt; here, we show how he applied ('transferred') it to a novel, realistic word problem with larger numbers.

Excerpt 3 (November 10, 2010)
S: (Reads the problem): "Evan's daughter (Sarah) needs to prepare 18 birthday bags. In each bag she plans to put 8 candies. Ron's daughter (Lihi) needs to prepare 7 birthday bags. She also plans to put 8 candies in each bag. Who needs more candies, Sarah or Lihi? How many more candies does that daughter need?" (He writes, 'Sarah needs')
R: Do you need a calculator?
S: (Nods yes): Hm-hmm. (Opens the calculator, inserts 18x8=144, comments it is the same as 12x12, clears the calculator and uses it for 18-7=11.) Eleven? Hmm. They are candies (clears screen). So what do I need to do here? I found the difference, which is 11. (Pauses, then shifts method again) Seven times … (calculates 7x8=56, then 18x8=144, and finally 144-56=88, so he exclaims): Eighty-eight!!
R: [After asking Sam to explain why he solved it this way, what did he find in each calculation, and Sam's writing 'Sarah needs 88 more candies' and stating he subtracted to find the difference in candies] Is there a different way to solve the problem?
S: [Appears to be thinking] Hmm …
R: I think you almost started one … Anything in your mind that you could have done?
S: Well, I did 18 minus 7, which is 11, and then 8 times 11 is 88! (Looks at R proudly.)

Excerpt 3 provides evidence that Sam has established the UDS scheme as an invariant way of operating in multiplicative situations where two quantities of the same unit rate (e.g., 8 candies per bag) are compared to find the total difference in singletons (1s). Establishing UDS at the anticipatory stage allowed assimilation of and operation on quantities given in a realistic situation. Interestingly, he considered solving the novel situation via the *Difference-First* method. However, he seemed to lose track of the second step (as he later explained to R's question). Yet, the anticipatory scheme empowered his resourceful, independent shift to and

Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.

successful completion of the *Total-First* method. In turn, he could return to and successfully complete the alternative method, with proud awareness of the answer identity.

Discussion

This study makes a twofold contribution. First, it portrays cognitive changes in forming a unit differentiation-and-selection (UDS) scheme. UDS was articulated while studying how multiplicative reasoning evolves in students with learning disabilities/difficulties (SLD) in mathematics. For these students, and likely for their normal achieving peers, UDS serves as an intermediate cognitive step. It builds on a student's assimilatory scheme of multiplicative double counting, used for the goal of quantifying a total of 1s via distributing the given numerosity (unit-rate) of a composite unit over a number of such units (Steffe, 1994; Steffe & Cobb, 1998). It is constructed through and applied to solving problems in which the learner's goal is to *compare two such quantities*, each consisting of so many composite units (e.g., How many more marbles are in 11 bags with 7 each than in 6 bags with 7 each?).

Articulating UDS shows the need to foster SLDs' intentional identification of how two quantities are similar/different in terms of 3 types of units of a multiplicative situation. Sam, and often other SLD, could initially operate on just one level of units, that is, 1s. However, to strategically and effectively employ mental activities for finding the difference in total, one must distinguish singletons from composite units and operate on the latter while using two or three levels of units (Steffe & Cobb, 1988). The UDS scheme involves anticipating and coordinating *Difference-First* and *Total-First* methods. In the former, the child first selects and compares the two quantities in terms of the *numerical difference between composite units* (e.g., 5 more bags in the situation above). To this end, as Sam *taught us*, a child may have to re-learn a proper meaning for 'more' and 'less'. Then, the child has to re-select the numerosity of each composite unit (unit-rate) and anticipate multiplicatively distributing it over the difference found (e.g., 7 marbles/bag x 5 bags of the difference only = 35 singletons in the difference). This process requires simultaneous operation on at least two levels of units. In the latter, *Total-First* method, the child first selects and multiplicatively computes the total of 1s in each quantity—a step that *may* involve two levels of units (e.g., 7x11=77 marbles, 7x6=42 marbles). This turns the situation into an additive comparison, for which subtraction is called upon as a second step (e.g., 77-42=35 marbles). Our study shows that the anticipatory, well-coordinated stage of a UDS scheme empowers its application (transfer) to novel situations. We postulate it as a conceptual root of what knowledgeable adults call 'the distributive property of multiplication over addition' (e.g., 7x(11-6) = 7x11-7x6).

Second, our study shows that computer-assisted learning opportunities can be designed by using ongoing analysis of student understandings as a basis for task selection. In the NMRSD software, tailoring tasks to student assimilatory schemes is achieved by operationalizing constructs of the *Ref*AER* framework (Simon, et al., 2004; Tzur & Simon, 2004). In particular, software programming draws on Tzur and Lambert's (in press) work of linking the participatory stage to Vygotsky's notion of ZPD and identifying three parameters of a prompt: (a) its *locus*—whether self-generated within a learner's mental processes or by an outside entity, (b) its *focus*/*essence*—ranging along the continuum from generic (e.g., could you solve the problem in a different way) to specific (e.g., could you begin by finding how many more towers does J have?), and (c) *number*—ranging from one, to a few, to many prompts. Using these parameters, as well as the *time* it takes a student to solve each problem and the *computer actions* she or he is using (e.g., typing 11 and then hovering with the mouse over '*' before typing '-' and '6' followed by '='), the NMRSD software successfully determined and fostered Sam's progress. He started when seemingly not yet having constructed a basic scheme of multiplicative double counting. The software taught him that scheme and determined he had constructed it at the anticipatory stage, as well as a scheme for additively coordinating same-size composite units (SUC). It then shifted to UDS, gradually changing the prompts given to Sam to engender his progress to participatory and then anticipatory stage (including situations with different unit-rates, such as comparing $7T_3$ with $7T_5$, or all units differing, such as $7T_3$ and $9T_2$). While the software 'progressed' based on assessing Sam's thinking, its work was synergized with human teaching (e.g., detect non-conventional meaning for 'more', introduce novel, realistic tasks, provide social-emotional support). This synergy empowered not only transfer to novel situations, but also Sam's quite effortless learning of a more advanced, pre-algebraic scheme (e.g., "You have $9T_7$; you receive 28 more cubes; how many towers will you have once all cubes are put into T_7?").

Endnotes

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