EXPLAINING STUDENT PERFORMANCE THROUGH INSTRUCTION

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Lack of satisfaction with the quality of the mathematical knowledge of non-mathematics majors has led to the design of new curricular materials. In this study, a mixed methods explanatory design was used to compare the performances of two groups of engineering majors enrolled in two courses on differential equations and investigate their written work in light of differing curricular and instructional approaches.

Introduction

Calls for curriculum and instructional reform have been expanded to include collegiate mathematics education (American Mathematical Society, 2011). Lack of satisfaction with the quality of mathematical knowledge of students completing service courses in the mathematics department has raised considerable concerns regarding how these courses are taught. Among many such courses, Differential Equations (DEs) entertains a particularly prominent role since it caters to a variety of clients from engineering areas. Skills fostered in a traditional theory-driven DE course have been claimed to be of little value to these degree programs. For instance, in a survey, Varsavsky (1995) found that engineering faculty value skills like modeling over technical competencies like differentiation and integration. Indeed, engineering and physical science faculty recommend that service courses in mathematics be made more relevant to their students and suggest incorporating an engineering viewpoint (Czocher, 2010; Pennell, Avitable, & White, 2009; Varsavsky, 1995). These perspectives have motivated the design of new curricular materials that aim to frame student learning in meaningful contexts (eg, Rasmussen and King, 2000). Despite such curriculum development efforts, little is known about the actual impact of such curricula on student learning. The primary motive of the present study was to address this gap. To this end, I compared the performances of two groups of students enrolled in two different sections of an introductory course on DEs to see what differences, if any, existed in their work on various types of tasks due to exposure to differing curricular and instructional approaches. One section followed a standard commercial textbook while the other used a reform-based curriculum with a conceptual orientation that was built around contexts central to engineering fields.

Context and Background: The Case of Differential Equations

The study of DEs is a unique point in the trajectories of engineering and physical science majors. In some instances, it serves as a capstone to the calculus sequence. On the other hand, an introductory course on DEs might be the first time that these students are exposed to material that is specialized for their disciplines. Many introductory undergraduate courses strip the equations of their natural contexts in order to treat the equations abstractly and then treat DEs deductively. That is, a general equation is presented, its solution is derived, and applications from the physical sciences related to the target equation can then be handled by manipulating the generic solution.

Researchers have argued that curriculum and instruction must align in order to support students' transitions to studying advanced mathematical topics. The study of DEs poses special challenges to building and implementing supportive curricula since the "switch from

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conceptualizing solutions as numbers to conceptualizing solutions as functions is nontrivial for students" (Rasmussen, 2001, p. 67), and students' documented difficulties with function extend to the form of the initial conditions (ICs) (Raychaudhuri, 2008). Others have shown a related aversion to the use of boundary conditions (Black and Wittmann, n.d.).

"Unifying" views of introductory DEs range from the use of linear operators to physical approaches that emphasize formulation (see West, 1994) to models and derivations from first principles (Myers, Trubatch, and Winkel, 2008). Scholars have studied student performance on procedural and conceptual components of DE knowledge (Arslan, 2010) and on contextual and de-contextualized problems (Upton, 2004), whose analyses graded student responses as either "right" or "wrong;" a method too coarse to account for different heuristics or representational schemes potentially nurtured in instruction. Donovan (2004), in a pair of case studies, demonstrated the variety in students' conceptualizations of linear DEs and their representations, but did not address the bases for student reasoning. In contrast, Black and Wittmann (n.d.) studied students' reasoning strategies and identified important connections between the physical and mathematical interpretations of the DE model that influence student performance. Bingolbali, Monaghan, & Roper (2007) found that the teacher's proclivity for a particular interpretation of derivative was adopted by students. Indeed, it is likely that the student is heavily influenced by the values placed on types of knowledge (Hedegaard, 1998).

These considerations guided the development of the curricular materials whose impact on learning was examined in this work. The reformed curricula, Baker's (n.d.) text, supports a unifying solution strategy with physical reasoning by showing how a guess-and-check heuristic for formulating the solution mimics the physical system's response to the forcing term in the DE. Additional considerations, regarding the contextual domains provided for students' investigations, motivated the development of examples and contexts used in the text, as well as how content was sequenced. The curricula are described more fully in a later section of this article.

Methods

Participants

The participants were 51 undergraduate students enrolled in an introductory DE course and the two postdoctoral lecturers who taught them. The course is intended for non-mathematics majors and all 51 students were engineering majors. Of the 51 volunteers, 30 were in Lecture 1 (L1) which used a custom edition of *Elementary Differential Equations and Boundary Value Problems* by Boyce and DiPrima and 21 were in Lecture 2 (L2) which used "An Introduction to Differential Equations for Scientists and Engineers," a set of course notes written by a member of the math department faculty. There were 25 males and 5 females in L1 and 15 males and 6 females in L2. Typically, students were in the end of their freshman or sophomore years, depending on their level of high school mathematics preparation. However, many of the students had earned enough college credit to have junior standing. The students carried, on average, between 10 and 11 credit hours in addition to their course on DEs, the equivalent of two or three additional classes. The mean grade point average (GPA) for the participants was 3.29 (SD = 0.44) and their mean mathematics GPA was 3.14 (SD = 0.61). Their collegiate mathematical preparation, in terms of coursework, was uniform. All had completed single- and multivariable calculus, but not linear algebra. In addition, all had completed at least the first two quarters of their engineering and physics sequences. The two groups had similar backgrounds relative to the number of incoming credit hours, the number of quarters enrolled, the number of credit hours earned, the number of credit hours carried, overall GPA, math GPA, and the number of math, science, and engineering credit hours taken.

Both lecturers held postdoctoral positions and both were familiar with their respective curricula. L2 had some mathematics education experience and coursework during his graduate studies. L1 was a nonnative speaker of English.

Data Collection Instrument

Three tasks were created toward the end of the observation period from material common to both lectures and embedded in the groups' respective final exams. To establish content validity, the items were drafted and revised five times with input from both lecturers, and finally from the course coordinator. This type of negotiation when designing data collection instruments that measure student learning has precedence in mathematics education (Boaler, 2008). The values of parameters were different for the two classes since final exams were administered on different days. The problems were written with the conventions used in each class, eg, derivatives were denoted by y' in L1 and by dy/dt in L2. Values for parameters were selected in order to simplify calculations while maintaining structural similarity between the matched problems. Students in both classes were allowed graphing calculators. A brief description of each problem is offered below.

Problem 1 (P1): A first-order linear, constant coefficient, nonhomogeneous mixing problem.

Part (a) asked the students to find the amount of contaminant in a tank for any time *t*, supposing that initially the tank was full of pure water. In part (b), the flow of the contaminant is turned off at time $t = \tau$. The students were asked to find the amount of contaminant for times $t > \tau$.

L1 wanted his students to derive the differential equation, while L2 did not want to test that knowledge. Both lecturers decided to abbreviate P1, cutting out a third part that asked the students to interpret their solutions in terms of the physical situation. Thus, this problem is contextually situated, but is not an application of DEs to an engineering context. Since these tasks were not identical for the two groups, I do not compare the students' scores on this item beyond including it in the total score for the tasks. However, some information, such as students' handling of the ICs, is intact and relevant to the analyses presented here.

Problem 2 (P2): A second-order linear, constant coefficient, nonhomogeneous DE:

$4u'' + u = 4e^{-t/2}$. Given the DE, part (a) asked students to find the general solution. In part (b), students were asked to suppose that $u(0) = u_0$ and that $u'(0) = 1$ and then to find a value of u_0 so that the amplitude of the steady state solution was 5.

The model in P2 represents a system that oscillates with no dampening, subject to forcing that is like an exponentially decreasing pulse. The focus is on the connection between ICs and long-term behavior of the system.

Problem 3 (P3): Separation of variables. Students were asked to use the separation of variables method to replace $m(t)u_t - n(x)u_{xx} = 0$ with a pair of ordinary DEs.

In both lectures, PDEs were most commonly treated with constant coefficients. P3 was designed to indicate whether students were able to use the method of separation of variables in a novel setting or if their knowledge was limited to a sequence of steps specific to the case $m(t) = n(x) = 1$.

Data Collection and Analysis

Since this study followed a mixed methods explanatory design (Creswell & Plano Clark, 2011), quantitative analysis of the exam tasks was followed with qualitative analysis of the

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students' responses. Quantitative data were generated through the grading of exams for accuracy. Tasks were graded independently from the lecturers so that data transformation decisions would not affect students' grades. The quantitative design was a prospective causal-comparative study and an ANCOVA was selected to analyze the total scores on the constructed tasks while controlling for the students' prior mathematics achievement, as measured by their math GPAs. Other numerical transformations involve frequencies and percentages. Qualitative data were generated through the observation of 24 (out of 29) sessions of each lecture and the detailed inspection of the students' responses to the tasks.

Student responses were examined first to establish what kinds of solutions they attempted. For each problem, a rubric was created to account for the level of difficulty of each stage of the solution process, and each stage was subdivided into steps. For example, one step of solving the DE by the method of integrating factor would require evaluating an integral. This step would not be broken down to diagnose the student's performance on integration by parts in order to avoid assessing prerequisite knowledge.

Each step of a student's response was assigned a value of "correct" (1) or "incorrect" (0), but responses were graded so that step $n+1$ was graded for consistency with step n. This decision prevented minor mistakes from compounding and so errors in student thinking could be diagnosed instead of tabulating wrong answers. There were a total of 14 steps in P1, 13 steps in P2, and 7 steps in P3. The value of each task, and so its parts, were scaled to 14 points to assign equal weight to each in the total score.

24 class sessions of each section was observed over one academic term. Lectures were 48-minutes long and met three days a week. Two 48-minute recitation sections were held on the other two days. Field notes included the instructors' and the students' comments, the instructors' board work, and my overall impressions of how lessons progressed. In analyzing observational records, I focused on lesson content, context, and pedagogy (Saroyan & Snell, 1997), noting lesson structure, the number and quality of examples used in each session, the number of connections made among topics made within and outside of mathematics, interactivity among the instructor and the students, and the mathematical behaviors that were modeled by the instructor.

Curriculum and Instruction

Both instructors used a traditional lecture format and both classes treated the same set of topics. L1 used Boyce and DiPrima's text (T1), which intended to provide exposure to the theory of differential equations with "considerable material on methods of solution, analysis, and approximation" (Boyce & DiPrima, 2009, p. vii). T1 follows an exposition-example-exercise format, where new topics are introduced formally through definitions and formulae, organized around analytic techniques with topics grouped into modules to allow for flexibility in usage. Theorems are stated precisely in symbols and are sometimes proved rigorously and sometimes through example. T1 does not assume familiarity with linear algebra, but many theorems rely on linear operators. Exercises tend to drill for procedural fluency and theoretically oriented problems have step-by-step directions or ask only for verification. In either case, the solution path is evident from the problem statement. Separate sections are devoted to applications, such as mechanical vibrations, and these are placed after the sections that develop techniques for solving the relevant DEs.

L2 used Baker's (n.d.) text (T2) which uses a modeling approach and the author's goal is to draw on common problems in the practice of science and engineering to motivate the creation and solution of DEs. It follows an example-exposition-exercise format where physical considerations precede mathematical formalism in each section. T2 does not contain "theorems," but instead offers "principles" which are written in English. Some principles are proven rigorously and some are justified through examples. Among the rigorously proved theorems, T2 uses English explanations while T1 relies on symbolic proof. There are few truly concrete examples or exercises, as most have at least one parameter. In comparison to T1, there are fewer exercises and these tend to be less technically difficult and less computationally oriented. Only one solution technique is presented throughout T2: guess-and-substitute (G&S), which is similar to the method of undetermined coefficients.

L1 followed the development of topics from abstract to concrete. The lectures were content-driven and L1 made every effort to convey the material on the syllabus. He drew examples from the text's exposition or from exercises that were similar to the assigned homework problems, with a focus on computation. Between one and seven examples were presented each lecture, with an average of between three and four per lecture. The examples followed presentation of theory. L1 held a goal orientation toward problem solving in that examples were considered complete when an answer was reached. He selected examples to maximize variety in technical complexity and sequenced them logically so that the adjustments in parameters from one example to the next were evident. The steps taken in each example were clearly labeled in order to provide guidelines to help structure student thinking about the problems. Taken together, these pedagogical choices yielded high intra-lesson coherence. Mathematics was communicated through symbolic representations and summary formulas. Students were encouraged to ask questions, and they did so regularly but infrequently.

L2 followed the development of topics through the text, but he did not regularly devote class time to explicit discussions of theory. Lectures were context-driven, in that all abstractions were derived through applications. Class time was spent on the structure of the physical problem, recognizing and articulating assumptions, deriving a model from first principles, and justifying its appropriateness, so the sessions had high inter-lesson continuity. Each example modeled a simplified real-life physical problem and each took between one-half and three sessions. L2 focused on building students' awareness of symbols and formulae as tools to represent quantities and relationships. The pace of the session was driven by L2's questioning, but the students were highly interactive, both posing and answering questions, while pursuing systematic exploration of the relationships between physical properties and their reflections in the model. Examples rarely ended in formulae; instead L2 would ask metacognitive questions. Thus, L2 held a process-orientation toward modeling. He wrote very little on the board, instead communicating mathematics verbally.

Findings

Numerical Results

P1 was missing from one student's exam in L1 and so his data are not included in the numerical analyses. Students in L1 scored a mean of 27.55 (SD=1.76) out of the 42 points available (Range: 3.08, 42). The mean score in L2 was 31.89 (SD = 1.90, Range: 6.38,42). A fixed effects ANCOVA model was selected with Lecture as the independent variable and Math GPA as the covariate. All statistical tests were performed at the $\alpha = .05$ level and the data satisfied all assumptions for the model. A homogeneity of slopes test revealed no significant interaction between Lecture and Math GPA. The ANCOVA summary is shown in Table 1. The main effect for Math GPA is statistically significant $(F_{GPA} = 27.191, df = 1{,}47, p < .001)$ and the adjusted main effect for Lecture is also significant $(F_{Lec} = 5.972, df = 1{,}47, p = .018)$ with a moderate effect size and showing only moderate power (partial $\eta^2 = .113$, observed power = .668). Thus, when taking prior mathematics achievement into account, the L2 student performed better on constructed tasks.

Source	Sum of Squares	*df*	Mean Square	*F*	Sig.	Partial η^2	Observed Power
Lecture	324.366	1	324.366	5.972	0.018	0.113	0.668
Math GPA	1476.928	1	1476.928	27.191	0.000	0.367	0.999
Error	2552.881	47	54.317				
Corrected Total	4259.339	49					

Table 1. ANCOVA summary table

Solution Strategies

Three techniques can be used to solve the DE in P1: the methods of integrating factor, separation of variables, and G&S. The G&S method is the only method appropriate to solving the DE in P2. In L1, the method of undetermined coefficients was presented in the context of second-order linear DEs of the form**Error! Not a valid embedded object.**. After two lessons, *f(t)* was allowed to be nonzero and L1 explained the method symbolically and procedurally. The students, in the text and in class, were given the form of the particular solution as $t^s e^{\omega t}\left[\sum A_k t^k \cos\beta t + \sum B^k t_k \sin\beta t\right]$, in which they had to set the parameters s, ω, and β to match the form of each of the summands of *f,* and then apply the DE to determine the weights A_k and B_k. In L2, the method was explained physically as follows: when written in standard form, the right hand side of the DE represents the external force applied to the physical system (the "forcing term") and that the system responds to the forcing (the "response") in kind. Thus, the form of the response (ie, the solution) must match the form of the forcing. Table 2 illustrates the frequencies of solution strategies used by the students, from left to right, they are: method of integrating factor, separation of variables, guess-and-substitute, and did not attempt (DNA).

	Int. Factor		Separation		G&S		Other		DNA	
	L1	L2	L1	L2	L1	L2	L1	L2	L1	L2
App (a)	26	–	2	1	–	20	1	–	–	–
App (b)	4	–	16	–	–	18	1	–	6	3
Concept (a)	2	–	–	–	28	18	–	–	–	3

Table 2. Chosen solution strategies

Almost all of the L2 students used the G&S strategy. This is not surprising since it was the only strategy they were shown, but they did use it more successfully than L1 students used the other methods. Again, this is not surprising since the L2 students had more practice with the method in various settings. What is surprising is that the L2 students were able to successfully adapt the G&S strategy to the procedural problem, more often than did the L1 students. Many L1 students either set $m(t) = n(x) = 1$ or else incorrectly applied the equation to the guess $u(x,t) = X(x)T(t)$ indicating that the "substitute" part of the strategy was not as prominent for the L1 students. The G&S method highlights the relationship between the solution and the DE by framing the DE as a condition on the solution in much the same way as an algebraic equation is a condition upon its solution. This perspective may be lost with techniques that reduce the relationship between the DE and the solution to a sequence of steps.

Initial Conditions

In P1 part (b), the IC does not correspond to the level of contaminant at time zero, but rather to when the flow of contaminant is halted. Thus, the solution function is piecewise differentiable. Nine students (31%) in L1 handled the ICs correctly in this case as compared with 13 (62%) of the students in L2. The most common mistake committed was to neglect the constant of integration and was made solely by L1 students. The second most common mistake was not ensuring that the pieces of the solution were matched at τ, caused by using $t = 0$. In P2, the ICs determine specific properties of the long-term behavior of the solution so that amplitude is formulated in terms of the unknown ICs. Sixteen students (53%) of the students in L1 handled the ICs correctly as opposed to 19 students (90%) in L2. Many students who arrived at one correct IC did not find the other since they did not extract both roots of the condition squared. However, the most common mistake, among both groups, was to apply the ICs only to the homogeneous solution. The second most common mistake was to apply the ICs to the steady state solution. This points to an interesting conception, since it amounts to setting $t = 0$ after letting $t \to \infty$, but it is not pursued further here.

Both lectures introduced ICs as "another condition on the solution to the DE." ICs are an important part of the DE model, since they contain parameters that determine how the transient solution adjusts to the steady state solution. One key difference between how the two lectures treated ICs was that in L2, ICs were driven by context and each time were derived from the physical situation during the derivation or the discussion of the model. In contrast, the ICs in L1 were either stated outright at the beginning of the initial value problem or were selected by considering the neatness of the solution.

Discussion and Conclusions

The students in L2 were more successful with the G&S strategy, which can be attributed to a number of reasons. First, there were fewer strategies from which they could choose when solving problems. In L2, guessing was an allowed heuristic. Lastly, the G&S was developed on physical grounds, which allowed for alternative ways of understanding the solution strategy (Harel & Koichu, 2010). One could also argue that since G&S does not require memorization then the strategy is easier to access and implement. In a test-taking situation, the take-the-best heuristic (see Gigerenzer, 2008) might be the only accessible approach to the test taker. In this light, L1 students needed a greater amount of time to search through all possible integration techniques in order to recognize a discriminating cue. The fast-and-frugal nature of the G&S heuristic favors the L2 students whose allowable solution-space includes the G&S method, which mimics the familiar educated-guess-and-check strategies.

In P1 and P2, students needed to attend to the ICs in novel ways. In P1, the ICs join solutions from distinct, non-overlapping time periods. The difficulty exhibited here by L1 students may be less an indicator of correctly applied ICs and may instead be symptomatic of their discomfort with piecewise functions. In comparison, L2 students' success in handling ICs suggests that contextual treatment of ICs both places them appropriately within the solution procedure and strengthens conceptual connections among the components of the mathematical model in a sensible way. Alternatively, L2 would close each example, not with a formula, but by modeling how to justify the solution process by matching properties of the solution with assumptions made at the outset of the example. The looking-back heuristic thus reinforces ICs as relevant to the model. The application orientation of L2 and its text provides a foundation for relational understanding (Skemp, 1987) and legitimizes physical intuition as a way of understanding (Harel & Koichu, 2010) when working with ICs.

Overall, the students in L2 performed better on the constructed tasks than did the students in L1. While there may be a number of extraneous factors contributing to this finding, evidence from a closer analysis of the students' responses reveals two main reasons for the difference in performance. The students in L2 were more adept at treating ICs and they were more successful in selecting and applying solution techniques. While results merit further investigation, such as a larger sample size and more varied instrumentation, they do point to the potential of conceptually- and contextually-oriented curriculum and instruction for enhancing student learning of DEs.

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