STUDYING MATHEMATICS CONCEPTUAL LEARNING: STUDENT LEARNING THROUGH THEIR MATHEMATICAL ACTIVITY

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This paper describes a program of research aimed at better understanding mathematics conceptual learning. Rather than studying the products of learning (conceptual steps), it focuses on the process (mechanism) of students' learning – learning that occurs through their mathematical activity in the context of a sequence of mathematical tasks. Intensive design research investigates subtle shifts in thinking that lead to the development of new abstractions.

Introduction

In Simon (1995), I introduced the idea of a "hypothetical learning trajectory (HLT)" to characterize important aspects of pedagogical thinking. The HLT consists of (1) the educator's goal for the students' learning, (2) the mathematical tasks that will be used to promote the students' learning, and (3) hypotheses about the process by which the students will learn. The paper emphasized the interdependency of the learning tasks and the learning process. Much research is still needed to support all of the three components. However, my colleagues[1] and I have embarked on a research program to investigate particular aspects of the conceptual learning process, which also implies careful attention to tasks. Whereas considerable important work has been going on investigating social interactive aspects of learning that take place during class discussions, our research focuses on cognitive learning that can take place when students are engaged in a series of mathematical tasks in small groups and individually. In this paper, I describe this emerging research program.

What Do We Mean by "Studying Learning?"

> In spite of the animated theoretical debate that has taken place on the nature of abstraction, little experimental research is available. … We surmise that the lack of experimental evidence is due to the difficulty of observing the processes of abstraction (as opposed to the products, for which there is more evidence). (Hershkowitz et al., 2001, p. 197)

Because our research focuses on learning through interaction with a task, we use primarily cognitive (constructivist) constructs as the basis for this work.[2] The focus of this work is on the process by which abstractions are developed.[3] Much of the important research in mathematics education over the last 30 years has been referred to as "research on mathematics learning." Given that this language is already widespread, it is difficult to denote the differences between our research program and the research that has been labelled "research on mathematics learning." I will try to articulate two key distinctions. First, most of the work subsumed under this heading has characterized static mathematical student understandings (e.g., Steffe, 2003) or a hierarchy of student understandings (schemes, classroom practices) (e.g., Cobb, McClain, & Gravemeijer, 2003). Whereas this work has been foundational to our work, it is not what we mean by "studying learning." Our focus is in understanding the subtle shifts in thinking that account for the transition between two consecutive schemes (or understandings or classroom practices). Our goal is to explain

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how the scheme or understanding has come about as students interact with a sequence of mathematical tasks. Second, when learning is explained, it is often explained using broad concepts such as reflective abstraction, generalizing assimilation or negotiation of meaning. For us, these are not sufficiently nuanced to be useful for instructional design. Siegler (1996) argued,

> The standard labels for hypothesized transition processes: assimilation, accommodation, and equilibration; change in M-space; conceptual restructuring; differentiation and hierarchic integration; are more promissory notes, telling us that we really should work on this some time, than serious mechanistic accounts. (p. 223)

And diSessa & Cobb (2004, p. 81) pointed out, “Piaget’s theory is powerful and continues to be an important source of insight. However, it was not developed with the intention of informing design and is inadequate, by itself, to do so deeply and effectively.” Our research program is aimed at moving from Piagetian constructs as a source of insight and broad hypothesis toward an explication of students’ mathematics learning through engagement with tasks that can support instructional design.

What is the Nature of a Mathematical Concept?

Understanding the nature of mathematical concepts and understanding how they are learned are necessarily intertwined. Each has implications for the other. Thus, investigating how mathematical concepts are learned requires a useful characterization of the nature of concepts, but as will become clear, the nature of a concept involves considering aspects of how it is learned. To make some key distinctions about mathematical concepts, I use the following example presented in Simon (2006, pp. 4-5):

> In a fourth-grade class [students 11-12 yrs old], I asked the students to use a blue rubber band on their geoboards to make a square of a designated size, and then to put a red rubber band around one half of the square. Most of the students divided the square into two congruent rectangles. However, Mary cut the square on the diagonal, making two congruent right triangles. The students were unanimous in asserting that both fit with my request that they show half of the square. Further, they were able to justify that assertion by explaining that each of the parts was 1 of 2 equal parts and that the two parts made up the whole. I then asked, “Is Joe’s (rectangular) half larger; is Mary’s half larger, or are they the same size?” Approximately a third of the class chose each option.[4]

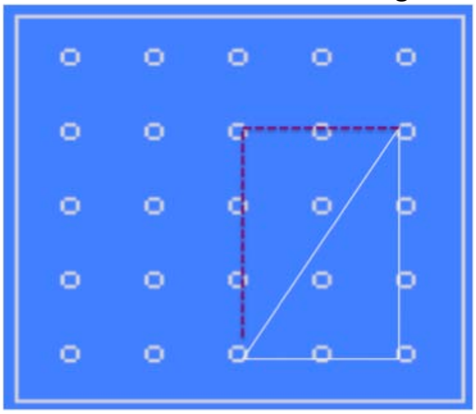
When I share this scenario and ask current and prospective mathematics educators about potential instructional interventions for the two thirds of the class who did not recognize the equality of the area of the two parts, the overwhelming majority suggest having students cut up the triangular half and superimpose it on the rectangular half (and additional activities of this type). Let us examine this response.

No matter how many such experiences are created for the student, these experiences will never teach the students the critical concept involved. Readers of this paper likely know that the areas of the two different shaped halves are equal. They know this, not because they have cut up and superimposed one half on the other half; they know that the two halves *must* be equal. They know the logical necessity of their equality. Having students cut and superimpose the halves may convince the students *that* the halves have equal area, but the activity does nothing to help them learn that the halves *must* have the same area.

The cutting up and superimposing activity could be done with different fractions and different shaped parts. Students could come to believe that the same fraction with a different shape has the same area. This is what I have called an "empirical learning process" (Simon, 2006). In an *empirical learning process*, the student introduces a number of inputs and sees a pattern in the outputs. However, an empirical learning process *never* produces a mathematical concept. A mathematical concept involves knowledge of the logical necessity of an idea. I have used Piaget's (2001) term *reflective abstraction* to identify learning processes that lead to mathematical concepts. The choice of this term will become clearer in the subsequent discussion.

Let's consider a second example (Heinz et al, 2000). Ivy is a sixth-grade teacher (students 11-12 yrs old) who is committed to her students learning mathematics with understanding. In this episode, she was beginning the teaching of the area of triangles. Following is an outline of her lesson as it unfolded:

1. Students worked in groups to find the area of a right triangle (legs 2 and 3 units) on a geoboard.
2. The class discussed their strategies. Completing the rectangle was a popular strategy.



3. Students worked in groups to find areas of other right triangles on their geoboards. Following Ivy's instructions, they recorded measures of the base, height, and area for each triangle in a chart.
4. Ivy convened the class and recorded in a large chart the base, height, and area of the different triangles contributed by the students.
5. Ivy then directed,

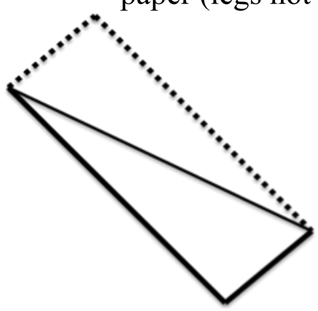
> Look at how these numbers are in this chart with our areas . . . and see if you can figure out a pattern that you can use every time using the numbers [measures of base and height] to come up with the area. . . . There is something that you can do to these [measures of] the bases and the heights to get the area. (Heinz et al., 2000, p. 94)

Ivy's encouragement of students to find a pattern in the "numbers" is a promotion of an empirical learning process. In her lesson, she treated the geoboard work as if it was a black box that turned inputs (measures of the legs of the triangles) into outputs (measures of the areas) for the purpose of finding a pattern in the numbers. Finding a pattern in this way does not result in knowing the logical necessity of the relationships expressed in the formula for the area of a triangle.

At this point, I anticipate the following objection, "Denoting patterns is a key aspect of doing mathematics." Mathematics has been called "the science of patterns" (Steen, 1988; Devlin, 1996). We do not want to eliminate attention to patterns in mathematics classrooms. However, the discussion here is about explaining the process of learning mathematical concepts, not of doing mathematics more broadly. Empirical identification of a pattern never, in itself, results in conceptual understanding. It may be the trigger for some other activity that results in such learning. I argue that identifying patterns empirically is neither necessary nor sufficient for conceptual learning. Further, in some situations it might not even be appropriate. In our example, looking for a pattern among the legs of a triangle and the area is neither optimal for learning nor does it provide a useful model of mathematics.

Consider now this modification of Ivy's lesson. I begin the lesson as Ivy did in order to highlight particular contrasts in students' opportunities to learn, even though more effective lessons might be designed for this subject matter. The sequence follows:

1. Students work in groups to find the area of a right triangle (legs 2 and 3 units) on a geoboard.
2. The class discusses their strategies. Completing the rectangle is a popular strategy.
3. Students work in groups to find areas of other right triangles on their geoboards. *No recording chart*.
4. Students are given a ruler and asked to find the area of right triangles drawn on plain paper (legs not parallel to sides of paper).



5.
6. Students are given the measurement of the legs of right triangles involving larger numbers for the dimensions. They are asked to find the area of each triangle (without drawing) by mentally running the process they did on paper.
7. Students are asked to write a generalization (algorithm) for how to calculate the area of a right triangle given the measures of the sides.

What do we see in this example? If we accept that the lesson sequence could promote reinvention of an algorithm for the area of a right triangle, we can see that it does so *without* employing an empirically generated pattern. The sequence is designed so that students can explain the appropriateness of each step in the algorithm based on the relationship between a right triangle and a particular related rectangle. I will engage in further analysis of this lesson in the next section.

Evolving Constructs for Studying Learning during Engagement with Mathematical Tasks

If we are convinced that empirical learning processes alone do not result in conceptual learning, then the important question is, *how might we think about learning that develops conceptual understanding?* The modified lesson discussed above is not an exemplary lesson and the intended learning is not particularly impressive. However, the simplicity of the lesson, allows it to be used for exemplification of particular ideas.

Students are able, without difficulty, to find the area of right triangles on the geoboard by completing the rectangle and taking half of it. For the purpose of this paper, we are not concerned with where the idea came from to complete the rectangle, only that carrying out the strategy with

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understanding is not beyond their current conceptual knowledge. The geoboard work developed an activity from which the learning could take place. It became a *process* in the APOS sense (Dubinsky & McDonald, 2001). That is, students reached a point where they were able to anticipate the whole process, not needing to stop and determine each next step. [5]

Note that for an elementary student, adding a rubber band to complete the rectangle, determining the number of little boxes contained in the rectangle (perhaps by counting) and taking half of that number is still pretty far from knowing how to compute the area of a right triangle given the measures of its legs. In the modified lesson, the diagrams of right triangles are given next. This changes the context, putting additional demands on the student. First, the student needs to draw the rectangle that produces two congruent triangles. When the geoboard was used, the choice of rectangle probably occurred without discrimination, because the geoboard is arranged as an orthogonal array and the triangles were arranged with the legs parallel to the sides of the geoboard. When working with the diagram, the students are able to produce appropriate rectangles based on their anticipation of a rectangle made up of two congruent triangles. Once again their strategy becomes a process as they reach a point where they do not need to think about how to draw the additional lines to complete the rectangle. The second difference in the paper context is the unit squares do not appear on the paper. Therefore, the students need to use their prior knowledge of measurement and computation of the area of a rectangle.

The mental run problems offer a third context, again changing the demands of the task. These problems are aimed at fostering abstraction of the relationships involved. On the geoboard or on paper, the student could determine the length of the rectangular sides using visual clues after having completed the rectangle. For the mental run problems, the students have no rectangle to visually examine. They need to anticipate the dimensions of the rectangle based on the dimensions of the triangle. They are able to do this based on an anticipation of how they constructed the rectangle (on paper) from the triangle. At this point they should have the abstractions that can be represented by $A = \frac{1}{2} l_1 l_2$, where l_1 and l_2 are the legs of the right triangle.

What can we take from this example?

1. This was not a problem solving activity. That is, at no point were the students facing a real problem. They moved smoothly through the tasks. Problem solving is an important part of mathematics. However, there are ways of teaching concepts that do not depend on students spontaneously solving the "next" problem.
2. Related to #1, this lesson sequence was not constructed to foster cognitive conflict. Rather, it was anticipated that the students would have the knowledge to operate at every point in the lesson when they reached those points.
3. The lesson sequence was designed to foster conceptual learning without hints or leading questions on the part of the teacher.
4. The learning was not the result of an empirical learning process. Students were not involved in finding a pattern in the numbers. Students came to understand the logical necessity of the relationships represented by the formula above.

If the learning is not the result of empirical learning, problem solving, or cognitive conflict, and if the teacher did not lead the student to the idea, how can we understand the learning process that was hypothesized in this lesson? (See Simon, et al, 2010 for the analysis of a teaching experiment involving division of fractions.) I will provide a brief, rudimentary explanation. We use three constructs adapted from Piaget's (2001) description of reflective abstraction: *goal-directed activity*, which can be mental and/or physical, *reflection*, which we understand as humans' innate ability (and tendency) to recognize (not necessarily consciously) commonality in their experience (von Glasersfeld's, 1995), and *abstraction*, which involves a learned anticipation. Using these constructs we view the students' hypothesized learning as the development of abstractions through reflection on their goal-directed activity (consistent with Piaget's reflective abstraction). If you return to the description above, at each point the students came to anticipate a process that coordinated particular aspects of their extant knowledge. In particular, the students came to anticipate:

1. the right triangle as half a rectangle divided diagonally,
2. the requisite rectangle formed by creating sides at right angles to the legs of the triangle,
3. the sides of the resultant rectangle having the dimensions of the legs of the right triangle,
4. the area of the triangle determined using a known formula for the area of a rectangle and halving the result.

But what does it mean that they "came to anticipate?" The student, faced with a task, engages in activity to meet their goal. Initially, the student engages step by step, determining the next action as they complete the preceding one. As they encounter similar situations, they at some point begin to notice a commonality (reflection) in their *activity* used to meet their goal. This results in anticipation of the needed actions. The sequence of actions no longer need to be assembled, the whole activity sequence is anticipated in response to recognizing the situation. Finally, the anticipation of the process reaches a point at which not all of the activity needs to be carried out in order to arrive at the intended result. The student no longer needs to create or imagine the rectangle to find the area of the right triangle.

Vision of a More Scientific use of Tasks for Engendering Conceptual Learning

I offered the example above to provide a concrete basis for discussing the particular vision for mathematics education that underlies our research program. Overall, my colleagues and I support the shared goals of the mathematics education community: equal opportunity, deep understanding, strong problem solving abilities, and competence in communicating mathematical ideas. However, our research program is focused on a specific goal that can advance the larger goal of deep understanding of mathematics. This particular goal is an improved ability to engineer task sequences that can foster particular understandings for a diverse set of students. We use the term "foster" to emphasize that teaching cannot *cause* learning. However, mathematics teaching has the potential to make conceptual learning likely.

To consider the vision, I invite engagement in the following thought experiment. A highly competent teacher gives her mathematics class a problem whose solution requires the mathematics to be learned. The students are asked to work in small groups, provided with a rich set of representations to work with, and asked after considerable work time to discuss their work with the class as a whole. A couple groups generate solutions to the problem, but the majority are not able to generate solutions (although there is considerable variability in their efforts). During the class discussion, different students talk about their work and one of the students who solved the problem presents her solution. After some questions and paraphrasing by the teacher, many of the students who did not solve the problem, seem to understand the solution presented.

I have probably described many classes with which you are familiar. Now I ask you to make a prediction. Who will more likely retain the new mathematics, apply it appropriately, and build on it – a student who independently generated the solution or a student who came to understand it in the context of the class discussion?[6] I think most mathematics educators would predict that the student who generated the solution to be more successful with the concept in the future. There is a difference in the cognitive demands of generating a solution versus understanding a solution (analogous to active versus passive knowledge of a language). This difference is not generally discussed.[7] Further, one can see this distinction as an equity issue. Who are the students who usually solve the mathematical tasks during small group time? It is the students who are more advanced in their mathematical knowledge. In the common scenario I described, the more-advanced students continue to work novel problems, while the less-advanced students struggle to follow the explanations of solutions given by the more-advanced students. This phenomenon can work against narrowing the gap between less-advanced and more-advanced students and even contribute to increasing the gap.

The hypothetical class described is not an unusual one. Particularly when a difficult new idea is the focus, generally few students generate the mathematics on their own. Sometimes no one in the class is able to generate a solution prior to the class discussion. This is where our research program comes in. What if we had the research-based knowledge so that typically 80% or more of the students could generate the new mathematical ideas through their engagement with the mathematical tasks? Would this improve the quality of student learning? Would this improve the quality and impact of the class discussion? Would this increase access to high-quality learning for more students? – an issue of equity.

The vision behind our research program is just that: understand how students learn through their mathematical activity, so that pedagogical design principles can be created that guide task sequence design. As a result, task sequences would be designed to engage students in the particular activity that allows them to abstract the intended new ideas. Currently, mathematical tasks often discriminate in favor of those who have the appropriate prior experience and against those who do not have it. The vision is to provide instructional task sequences that are designed to build up the requisite experience.

One might ask, “Don’t we do this already?” Let’s consider a couple of frequently used approaches. The use of manipulative materials and other physical or iconic representations is often thought to create the experiential base for conceptual learning. These tools are useful and have potential beyond their current uses. However, there is insufficient theory guiding the use of physical and iconic representations. Teachers often have students initially solve problems using these representations. The students produce solutions that are isomorphic, *in the mind of the knowledgeable adult*, to formal solutions. However, these “concrete solutions” in themselves do not produce conceptual understanding and do not embody the vision articulated in the last paragraph.
A second approach is the pedagogical use of a *series* of problems. This is an attempt to create a series that allows students to generate ideas, but where the steps from one problem to the next are within the students’ grasp, that is novel problems that are only a small change from the prior one. Although there is not much theory to guide the development of such series, these series at their best

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might approximate what is described by the vision presented. However, it seems that in many cases, key problems in the sequence are only solved by a minority of the students.

In contrast, in our work, the intention is to identify (goal-directed) activities in which students are already capable of engaging, from which they predictably can abstract the new ideas. There is no point at which we *hope* that they can do or see something that is key to the new mathematics. There is no point at which their learning is based on solving a novel problem that they might not be able to solve. The vision involves a shift from provoking learning to engineering learning (albeit not in a deterministic sense).

One of the basic tenets of this vision is that children are born with an ability to learn concepts and they do this through their activity. Let us take two well-known examples. First, most people have probably observed that children, after learning the rules and basic moves of a game (e.g., chess, video games), often develop sophisticated concepts for playing the game without instruction from anyone. They are certainly more sophisticated than learning simply what move has a positive effect and what move has a negative one. Do we understand how these concepts are developed? Could this ability be harnessed?

This second example involves mathematics learning. One of the greatest mathematical learning achievements in the life of any individual is the development of a concept of number. This is generally accomplished without professional teachers although parents figure prominently. There are many types of activities in which parents can and do engage their young children. However, there is one that is ingenious – counting. Counting activities, in which children engage in coordinating the touching of an object with the reciting of the next number name in the learned number name sequence, result in major conceptual gains. The child is not trying to solve a problem or even trying to learn something mathematical. They are most likely trying to imitate the adult, enjoy the play, and gain the adult's approval. Counting is a specific activity that leads to important abstractions of number concepts. Our vision is to understand the process underlying learning to count and an array of other mathematical concepts and to develop pedagogical design principles based on this understanding.

How Does This Work Potentially Contribute to and Fit into Mathematics Instruction?

In classrooms in which students initially work in small groups (and/or individually) and then convene in whole-class discussions, we envision our work informing the design of sequences of tasks for students' small group and individual work. Envisioned task sequences would provide the experience (requisite activity and opportunities for abstraction) that would allow students to have significant insight into the mathematics at hand. The purpose of the classroom discussion would not be to *bring* students to the intended ideas, but rather for students to share their insights developed through their work with the tasks, make new ideas explicit, provide justifications, and develop shared language and symbols for the ideas (as in *situations of institutionalization* (Brousseau, 1997). Teachers, in addition to leading class discussions, can interact with small groups of students who are in the process of abstracting a new idea.

It may not be necessary to employ this type and level of didactical engineering to all conceptual learning in the classroom. However, there are particular mathematical ideas taught in elementary and secondary school that many students never learn or never learn well. It is towards these pedagogical challenges that this approach is particularly directed.

I have described this approach to fostering learning as not involving problem solving. Certainly, the mathematics classroom must promote and engage students regularly in problem solving. However, having an approach to foster new concepts that does not rest on each student's ability to solve novel problems would be a useful pedagogical tool.

Current Project

In our last project, we developed a methodology for studying students learning. The methodology is an adaptation of teaching experiments with individual students (Simon, et al., 2010). We are now embedding that methodology in a 5-year project, Measurement Approach to Rational Number (MARN).[8] The project combines two main goals, continuing our inquiry into students' learning through activity and developing approaches to teach some of the most intractable concepts in the domain of fractions and proportional reasoning. Therefore we are studying learning in the context of fractions and decimals, and we are using our emerging understanding of how students learn through their activity to design tasks for promoting fraction and ratio understanding. The design of instruction builds on a foundation of measurement concepts modifying and extending the approach of Davydov and his colleagues (c.f., Davydov, & Svetkovich, 1991). Our hypothesis is that measurement-based activities can provide the specific basis for students' abstractions in these key mathematical areas.

MARN Methodology

The MARN design research will occur in two phases, individual teaching experiments and whole-class teaching experiments. The purpose of the first phase is to develop knowledge of promoting intended abstractions through task sequences. The resulting task sequences and hypothesized learning processes will constitute a significant part of the hypothetical learning trajectories for the whole-class teaching experiment.

As mentioned, the methodology we developed for studying students' learning through their mathematical activity involves individual teaching experiments. The design of the project is also predicated on the potential value of the individual teaching experiments to inform the whole class work on learning fractions and ratio. Our expectation of this potential is not shared by many in the mathematics education research community.[9] The MARN Project gives us a chance to examine this methodological issue empirically.

Rationale for Individual Teaching Experiments

> How do changes in children's thinking occur? Focusing on change . . . will require reformulation of our basic assumptions about children's thinking, the kinds of questions we ask about it, our methods for studying it, the mechanisms we propose to explain it, and the basic metaphors that underlie our thinking about it. (Siegler, 1996, p. 218)

Our goal for studying learning (understanding the mechanism by which conceptual understanding develops) is seen by many as impossible. Indeed it is both difficult and the success of any study is uncertain. The methodology for data collection is critical to producing a set of data from which warranted inferences about the learning process can be made.

We have found that the most important feature of the data collection is that it produces reasonably continuous evidence of the student's activity (mental and physical). Whereas observing physical activity is relatively straight forward, inferring mental activity requires attention to physical activity in conjunction with encouraged regular verbalization on the part of the student. Inference of students' mental activity has been part of qualitative research for several decades. What is different in this research is trying to infer the ongoing evolution of the thinking as opposed to key snapshots.

The main obstacle to continuous evidence of students' activity is instances when the students are listening to someone else who affects their subsequent activity. For this reason, we found that we could not use even 2 students together in a teaching experiment. When one person watched and listened to the other pursuing an idea, it often changed the flow of the observing student; however the researchers had no evidence of what went on in the mental activity of that student. For similar reasons the researcher avoids hints, suggestions, and leading questions. This data collection approach also allows us to focus on what aspect of the abstraction process can take place without extensive interaction.

Conclusions

The research program I have described is based on a simple idea: the more we understand about the learning of mathematical concepts, the better we will be able to promote conceptual learning. Bransford, Brown, and Cocking (2000, p. 221) asserted, "A scientific understanding of learning. . . provides the fundamental knowledge base for understanding and implementing changes in education. Our research program focuses on students' learning through their mathematical activity in the context of series of mathematical tasks. It rests on two important claims. First, learning can be studied directly and to productive ends Second, learning can be engineered in ways that do not depend on novel problem solving on the part of students. Third, engineering learning opportunities that allow students to build abstractions upon their mathematical activity (engendering the requisite activity) would increase equitable access to high-quality conceptual learning. Because this research program pushes the boundaries of what is generally believed to be possible, this program of research, meets with skepticism. In particular, the following questions are raised about the research.

1. Can data be produced to make useful inferences about the learning process?
2. Can empirically based constructs be developed which are useful for a variety of mathematical concepts for a variety of students in a variety of contexts?
3. Can design principles for task and task sequence creation be developed across concepts?
4. Can analyses of individual concept learning usefully inform design for classroom lessons?

Our research program represents a conjecture that the answers to these questions are, "Yes." As mentioned this is an extremely difficult and uncertain undertaking. Let me speculate (optimistically) about the potential payoff of this endeavor.

Contribution to Research

Design research is a productive and increasingly used methodology in mathematics education. It figures prominently in the generation of learning progressions, an exciting and growing area of current research. The problem with design research is that researchers' ability to study the learning of particular concepts is limited by their ability to promote the learning they wish to study. A greater understanding of students' learning through their mathematical activity (developing abstractions) and concomitant design principles for generative task sequences could provide a basis for more consistently productive design research.

Contribution to Curriculum Development

Whereas recent curricula tend to increase students' opportunities to learn, they tend to be uneven (within each curriculum) in quality. Part of the unevenness may be the variation among the authors, but much of the unevenness seems to be due to lack of task design principles grounded in an understanding of student learning processes. Strong lessons reflect an author's insight into a particular area. However, there is insufficient theory to guide the majority of lessons.

Contribution to Teaching and Teacher Education

In the context of ongoing mathematics education reform efforts, teachers seem to have derived both constraints on and recommendations for their teaching. As a result they try to refrain from telling students what they want them to learn and giving answers. They tend to use a variety of strategies: classroom discussions, small groups, non-routine problems, multiple representations, manipulatives and computer tools. However, none of this gives them a clear understanding of how to help students learn a new concept. As a result, teachers often fall back on strategies that do not differ *fundamentally* from traditional teaching, such as asking leading questions (trying to get students to say what the teacher would say if they were not avoiding telling) or having the student who already has the concept tell the others. Classroom discussion is a powerful tool. However teachers are too often working from a vague plan of conducting a classroom discussion so the majority of the students, who do not understand the new concept, understand it. Teacher education aims at strategies, skills, and dispositions, but neglects, for the most part, to help prospective and practicing teachers understand fundamentally new ways to promote the learning of particular concepts.

The research program described is aimed at providing a partial basis for filling the voids in each of these areas. However, while remaining optimistic, some tempering is in order. It is reasonable to expect that what is learned through basic research should be useable by researchers (e.g., in design research). However, the effect on teaching and teacher education will require additional research and development. What a researcher learns after 10+ years of intensive investigation is not necessarily accessible to practitioners.[10] We must ask of every product of basic research, "What part or version of this can practicing teachers understand with a realistic level of support?" And another set of questions not often asked about adult learners (e.g., teachers, doctoral students) is, "How can we understand the learning trajectory with respect to these new ideas (support experiences and learning process)?"

Finally, there is good reason why few research groups endeavor to study learning directly. It is difficult and success in any particular study is uncertain (Simon et al, 2010). The ambitious program of research that I have outlined will require the efforts of many research groups who take part in shaping the program and contributing results. Further, as a community, we will need to continue to make a strong case for support of basic research in mathematics education in an era dominated by a demand for "results now."

Endnotes

1. Ron Tzur and I collaborated on the initial work of this research program. We built upon his dissertation work and my prior research. Other collaborators have included Luis Saldanha, Evan McClintock, Tad Wattanabe, Gulseren Karagoz Akar, Ismail Zembat, Karen Heinz, and Margaret

Kinzel. Our current project includes Barbara Dougherty, Zaur Berkaliev, Arnon Avitzur, Nicora Placa, and Jessica Tybursky.

2. We use a social lens to examine interactive aspects of the classroom and the constructs of social and sociomathematical norms to examine opportunity for learning

3. Focus on *abstraction* is often seen as the province of constructivist theoretical approaches. However, Russian activity theory, derived from the theory of Vygotsky, uses abstraction as a key construct (c.f., Davydov, 1990).

4. In case the reader is sceptical about whether there was a shared meaning for "bigger," the same problem has been done in interviews in which the squares represented cookies that the interviewee liked and the question was "Which piece would you rather eat?" and the reasons for the answer were probed. The same phenomenon was observed.

5. This process could be investigated and explicated further, but it is not within the aims of this paper.

6. Answering this thought question empirically would present significant methodological challenges.

7. There is widespread conviction that engaging students in developing new mathematical ideas is more effective than giving them the ideas in a lecture. However, given classrooms in which students are engaged in generating ideas, there is little or no attention to the differential contributions of students and the effect on the quality of their learning.

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9. Case in point was some protracted arguments on this point during our first project advisory board meeting.

10. Often other researchers struggle to deeply understand the results of lengthy, focused investigations.

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